QUANTUM GAUGE FIELD THEORY
IN
COHESIVE HOMOTOPY TYPE THEORY

URS SCHREIBER AND MICHAEL SHULMAN

Abstract. We implement in the formal language of *homotopy type theory* a new set of axioms called *cohesion*. Then we indicate how the resulting *cohesive homotopy type theory* naturally serves as a formal foundation for central concepts in quantum gauge field theory. This is a brief survey of work by the authors developed in detail elsewhere [48, 45].

Contents

1. Introduction

The observable world of physical phenomena is fundamentally governed by *quantum gauge field theory* (QFT) [16], as was recently once more confirmed by the detection [40] of the Higgs boson [9]. On the other hand, the platonic world of mathematical concepts is fundamentally governed by *formal logic*, as elaborated in a foundational system such as axiomatic set theory or type theory.

Quantum gauge field theory is traditionally valued for the elegance and beauty of its mathematical description (as far as this has been understood). Formal logic is likewise valued for elegance and simplicity; aspects which have become especially important recently because they enable formalized mathematics to be verified by computers. This is generally most convenient using type-theoretic foundations [34]; see e.g. [15].

However, the mathematical machinery of quantum gauge field theory — such as differential geometry (for the description of spacetime [35]), differential cohomology (for the description of gauge force fields [24, 21]) and symplectic geometry (for the description of geometric quantization [11]) — has always seemed to be many levels of complexity above the mathematical foundations. Thus, while automated proof-checkers can deal with fields like linear algebra [15], even formalizing a basic differential-geometric *definition* (such as a principal connection on a smooth manifold) seems intractable, not to speak of proving its basic properties.

We claim that this situation improves drastically by combining two insights from type theory. The first is that type theory can be interpreted "internally" in a wide variety of categories (see e.g. [14]), including categories of smooth spaces (which contain the classical category of smooth manifolds). In this way, type-theoretic arguments which appear to speak about discrete sets may be interpreted to speak about smooth spaces, with the smooth structure automatically "carried along for the ride". Thus, differential notions can be developed in a simple and elegant axiomatic framework — avoiding the complexity of the classical definitions by working in a formal system whose *basic objects* are already "smooth". This is known as *synthetic differential geometry* [27, 28, 37]. In this paper, we will axiomatize it in a way which does not require that our basic objects are "smooth", only that they are "cohesive" as in [29]. This includes topological objects as well as smooth ones, and also variants of differential geometry such as *supergeometry*, which is necessary for a full treatment of quantum field theory (for the description of fermions).

The second insight is that types in type theory can also behave like *homotopy types*, a.k.a. ∞*-groupoids*, which are not just sets of points but contain higher homotopy information. Just as SDG imports differential geometry directly into type theory, this one imports *homotopy theory*, and as such is called *homotopy type theory* [56, 2, 53].

Combining these insights, we obtain *cohesive homotopy type theory* [48, 45]. Its basic objects (the "types") have both *cohesive structure* and *higher homotopy structure.* These two kinds of structure are independent, in contrast to how classical algebraic topology identifies homotopy types with the topological spaces that present them. For instance, the geometric circle S^1 is categorically *0-truncated* (it has a mere set of points with no isotropy), but carries an interesting topological or smooth structure — whereas the homotopy type it presents, denoted $\Pi(S^1)$ or $\mathbf{B}\mathbb{Z}$ (see below), has (up to equivalence) only *one* point (with trivial topology), but that point has a countably infinite isotropy group. More general cohesive homotopy types can be nontrivial in both ways, such as orbifolds [36, 30] and moduli stacks [57].

Today it is clear that homotopy theory is at the heart of quantum field theory. One way to define an n-dimensional QFT is as a rule that assigns to each closed $(n-1)$-dimensional manifold, a vector space — its space of *quantum states* — and to each n-dimensional *cobordism*, a linear map between the corresponding vector spaces — a *correlator* — in a way which respects gluing of cobordisms. An *extended* or *local* QFT also assigns data to all $(0 \leq k \leq n)$-dimensional manifolds, such that the data assigned to any manifold can be reconstructed by gluing along lower dimensional boundaries. By [4, 33], the case of *extended topological QFT* (where the manifolds are equipped only with smooth structure) is entirely defined and classified by a universal construction in *directed homotopy theory*, i.e. (∞, n)-category theory [8, 52]. For non-topological QFTs, where the cobordisms have conformal or metric structure, the situation is more complicated, but directed homtopy type theory still governs the construction; see [43] for recent developments.

The value of *cohesive* homotopy theory for physics lies in the observation that the QFTs observed to govern our world at the fundamental level — namely, Yang-Mills theory (for electromagnetism and the weak and strong nuclear forces) and Einstein general relativity (for gravity) — are not random instances of such QFTs. Instead (1) their construction follows a geometric principle, traditionally called the *gauge symmetry principle* [39], and (2) they are obtained by *quantization* from ("classical") data that lives in *differential cohomology.* Both of these aspects are actually native to cohesive homotopy type theory, as follows:

(1) The very concept of a collection of quantum field configurations with *gauge transformations* between them is really that of a configuration *groupoid*, hence of a homotopy 1-type. More generally, in higher gauge theories such as those that appear in string theory, the *higher gauge symmetries* make the configurations form higher groupoids, hence general homotopy types. Furthermore, these configuration groupoids of gauge fields have *smooth structure*: they are *smooth homotopy types.*

    To physicists, these smooth configuration groupoids are most familiar in their infinitesimal Lie-theoretic approximation: the (higher) *Lie algebroids* whose function algebras constitute the *BRST complex*, in terms of which modern quantum gauge theory is formulated [22]. The degree-n *BRST cohomology* of these complexes corresponds to the n^{th} *homotopy group* of the cohesive homotopy types.

(2) Gauge fields are *cocycles* in a cohomology theory (sheaf hyper-cohomology), and the gauge transformations between them are its coboundaries. A classical result [10] says, in modern language, that all such cohomology theories are realized in some interpretation of homotopy type theory (in some $(\infty, 1)$-category of $(\infty, 1)$-sheaves): the cocycles on X with coefficients in A are just functions $X \to A$. Moreover, if A is the coefficients of some *differential* cohomology theory, then the *type* of all such functions is exactly the configuration gauge groupoid of quantum fields on A from above. This cannot be expressed in plain homotopy theory, but it can be in *cohesive* homotopy theory.

> Finally, differential cohomology is also the natural context for *geometric quantization*, so that central aspects of this process can also be formalized in cohesive homotopy type theory.

In §2, we briefly review homotopy type theory and then describe the axiomatic formulation of cohesion. The axiomatization is chosen so that if we *do* build things from the ground up out of sets, then we can construct categories (technically, $(\infty, 1)$-categories of $(\infty, 1)$-sheaves) in which cohesive homotopy type theory is valid internally. This shows that the results we obtain can always be referred back to a classical context. However, we emphasize that the axiomatization stands on its own as a formal system.

Then in §3, we show how cohesive homotopy type theory directly expresses fundamental concepts in differential geometry, such as differential forms, Maurer-Cartan forms, and connections on principal bundles. Moreover, by the homotopy-theoretic ambient logic, these concepts are thereby automatically generalized to *higher differential geometry* [38]. In particular, we show how to naturally formulate *higher moduli stacks* for *cocycles in differential cohomology.* Their 0-truncation shadow has been known to formalize gauge fields and higher gauge fields [21]. We observe that their full homotopy formalization yields a refinement of the Chern-Weil homomorphism from secondary characteristic classes to cocycles, and also the action functional of generalized Chern-Simons-type gauge theories with an *extended* geometric prequantization. At present, however, completing the process of quantization requires special properties of the usual models; work is in progress isolating exactly how much quantization can be done formally in cohesive homotopy type theory.

The constructions of §2 have been fully implemented in `Coq` [1]; the source code can be found at [50]. With this as foundation, the implementation of much of §3 is straightforward.

2. COHESIVE HOMOTOPY TYPE THEORY

2.1. Categorical type theory.

A type theory is a formal system whose basic objects are *types* and *terms*, and whose basic assertions are that a term a belongs to a type A, written "$a : A$". More generally, $(x : A), (y : B) \;\vdash\; (c : C)$ means that given variables x and y of types A and B, the term c has type C. For us, types themselves are terms of type Type. (One avoids paradoxes arising from $\mathsf{Type} : \mathsf{Type}$ with a hierarchy of universes.) Types involving variables whose type is not Type are called *dependent types.*

Operations on types include cartesian product $A \times B$, disjoint union $A + B$, and function space $A \to B$, each with corresponding rules for terms. Thus $A \times B$ contains pairs (a, b) with $a : A$ and $b : B$, while $A \to B$ contains functions $\lambda x^A.b$ where $b : B$ may involve the variable $x : A$, i.e. $(x : A) \;\vdash\; (b : B)$. Similarly, if $(x : A) \;\vdash\; (B(x) : \mathsf{Type})$ is a dependent type, its *dependent sum* $\sum_{x:A} B(x)$ contains pairs (a, b) with $a : A$ and $b : B(a)$, while its *dependent product* $\prod_{x:A} B(x)$ contains functions $\lambda x^A.b$ where $(x : A) \;\vdash\; (b : B(x))$.

In many ways, types and terms behave like sets and elements as a foundation for mathematics. One fundamental difference is that in type theory, rather than proving theorems *about* types and terms, one *identifies* "propositions" with types containing at most one term, and "proofs" with terms belonging to such types. Constructions such as $\times, \to, \prod$ restrict to logical operations such as $\wedge, \Rightarrow, \forall$, embedding logic into type theory. By default, this logic is *constructive*, but one can force it to be classical.

Type theory also admits *categorical models*, where types indicate objects of a category $\mathbf{H}$, while a term $(x : A), (y : B) \;\vdash\; (c : C)$ indicates a morphism $A \times B \to C$. A dependent type $(x : A) \;\vdash\; (B(x) : \mathsf{Type})$ indicates $B \in \mathbf{H}/A$, while $(x : A) \;\vdash\; (b : B(x))$ indicates a *section* of B, and substitution of a term for x in $B(x)$ indicates pullback. Thus, we can "do mathematics" internal to $\mathbf{H}$, with any additional structure on its objects carried along automatically. In this case, the logic is usually unavoidably constructive.

In the context of quantum physics, such "internalization" has been used in the "Bohrification" program [23] to make noncommutative von Neumann algebras into internal commutative ones. There are also "linear" type theories which describe mathematics internal to monoidal categories (such as Hilbert spaces); see [6].

2.2. Homotopy type theory.

Since propositions are types, we expect *equality types* $(x : A), (y : A) \;\vdash\; ((x = y) : \mathsf{Type})$. But surprisingly, $(x = y)$ is naturally *not* a proposition. We can add axioms forcing it to be so, but if we don't, we obtain *homotopy type theory*, where types behave less like sets and more like *homotopy types* or *∞-groupoids.*[1] Space does not permit an introduction to homotopy theory and higher category theory here; see e.g. [32, §1.1]. We re-emphasize that in *cohesive* homotopy type theory, simplicial or algebraic models for homotopy types are usually less confusing than topological ones.

---

[1] The associativity of terminology "homotopy (type theory) = (homotopy type) theory" is coincidental, though fortunate!

Homotopy type theory admits models in $(\infty,1)$-*categories*, where the equality type of A indicates its diagonal $A \to A \times A$. Voevodsky's *univalence axiom* [26] implies that the type Type is an *object classifier*: there is a morphism $p : \widetilde{\mathsf{Type}} \to \mathsf{Type}$ such that pullback of p induces an equivalence of ∞-groupoids

$$\mathbf{H}(A, \mathsf{Type}) \simeq \mathrm{Core}_\kappa(\mathbf{H}/A). \tag{1}$$

(The *core* of an $(\infty,1)$-category contains all objects, the morphisms that are equivalences, and all higher cells; Core_κ denotes some small full subcategory.) A well-behaved $(\infty,1)$-category with object classifiers is called an $(\infty,1)$-*topos*; these are the natural place to internalize homotopy type theory.[2]

An object $X \in \mathbf{H}$ is *n-truncated* if it has no homotopy above level n. The 0-truncated objects are like sets, with no higher homotopy, while the (-1)-truncated objects are the propositions. The n-truncated objects in an $(\infty,1)$-topos are reflective, with reflector π_n; in type theory, this is a *higher inductive type* [51, 31, 55]. The (-1)-truncation of a morphism $X \to Y$, regarded as an object of $\mathbf{H}/Y$, is its *image factorization.*

2.3. **Cohesive $(\infty,1)$-toposes.** A *cohesive* $(\infty,1)$-*topos* is an $(\infty,1)$-category whose objects can be thought of as ∞-groupoids endowed with "cohesive structure", such as a topology or a smooth structure. As observed in [29] for 1-categories, this gives rise to a string of adjoint functors relating $\mathbf{H}$ to ∞-$\mathbf{Gpd}$ (which replaces $\mathbf{Set}$ in [29]). First, the *underlying* functor $\Gamma : \mathbf{H} \to \infty$-$\mathbf{Gpd}$ forgets the cohesion. This can be identified with the hom-functor $\mathbf{H}(*, -)$, where the terminal object $*$ is a single point with its trivial cohesion.

Secondly, any ∞-groupoid admits both a *discrete* cohesion, where no distinct points "cohere" nontrivially, and a *codiscrete* cohesion, where all points "cohere" in every possible way. This gives two fully faithful functors $\Delta : \infty$-$\mathbf{Gpd} \to \mathbf{H}$ and $\nabla : \infty$-$\mathbf{Gpd} \to \mathbf{H}$, left and right adjoint to Γ respectively.

Finally, Δ also has a left adjoint Π, which preserves finite products. In [29], Π computes *sets of connected components*, but for $(\infty,1)$-categories, Π computes entire *fundamental ∞-groupoids.* There are two origins of higher morphisms in $\Pi(X)$: the higher morphisms of X, and the cohesion of X. If $X = \Delta Y$ is an ordinary ∞-groupoid with discrete cohesion, then $\Pi(X) \simeq Y$. But if X is a plain set with some cohesion (such as an ordinary smooth manifold), then $\Pi(X)$ is its ordinary fundamental ∞-groupoid, whose higher cells are paths and homotopies in X. If X has both higher morphisms and cohesion, then $\Pi(X)$ automatically combines these two sorts of higher morphisms sensibly, like the Borel construction of an orbifold.

Thus, we define an $(\infty,1)$-topos $\mathbf{H}$ to be *cohesive* if it has an adjoint string

$$\begin{tikzcd} \mathbf{H} \arrow[d, shift right=4.5, "\Pi"'] \arrow[d, shift left=1.5, "\Gamma"] \\ \infty\text{-}\mathbf{Gpd} \arrow[u, shift right=1.5, "\Delta"'] \arrow[u, shift left=4.5, "\nabla"] \end{tikzcd} \tag{2}$$

where Δ and ∇ are fully faithful and Π preserves finite products. Using ∞-sheaves on sites [47], we can obtain such $\mathbf{H}$'s which contain smooth manifolds as a full subcategory; we call these *smooth models.*

Now we plan to work in the internal type theory of such an $\mathbf{H}$, so we must reformulate cohesiveness internally to $\mathbf{H}$. But since Δ and ∇ are fully faithful, from inside $\mathbf{H}$ we see two subcategories, of which the codiscrete objects are reflective, and the discrete objects are both reflective (with reflector preserving finite products) and coreflective. We write $\sharp := \nabla\Gamma$ for the codiscrete reflector, $\flat := \Delta\Gamma$ for the discrete coreflector, and $\Pi := \Delta\Pi$ for the discrete reflector. Assuming only this, if A is discrete and B is codiscrete, we have

$$\mathbf{H}(\sharp A, B) \simeq \mathbf{H}(A, B) \simeq \mathbf{H}(A, \flat B)$$

so that $\sharp \dashv \flat$ is an adjunction between the discrete and codiscrete objects. If we assume that for *any* $A \in \mathbf{H}$, the maps $\flat A \to \flat\sharp A$ and $\sharp\flat A \to \sharp A$ induced by $\flat A \to A \to \sharp A$ are equivalences, this adjunction becomes an equivalence, modulo which $\flat$ is identified with $\sharp$ (i.e. Γ). From this we can reconstruct (2), except that the lower $(\infty,1)$-topos need not be ∞-$\mathbf{Gpd}$. This is expected: just as homotopy type theory admits models in all $(\infty,1)$-toposes, cohesive homotopy type theory admits models that are "cohesive over any base".

We think of $\sharp$, $\flat$, and Π as *modalities*, like those of [3], but which apply to all types, not just propositions. Note also that as functors $\mathbf{H} \to \mathbf{H}$, we have $\Pi \dashv \flat$, since for any A and B

$$\mathbf{H}(\Pi A, B) \simeq \mathbf{H}(\Pi A, \flat B) \simeq \mathbf{H}(A, \flat B)$$

as both ΠA and $\flat B$ are discrete.

---

[2] There are, however, coherence issues in making this precise, which are a subject of current research; see e.g. [54].

2.4. **Axiomatic cohesion I: Reflective subfibrations.** We begin our internal axiomatization with the reflective subcategory of codiscrete objects. An obvious and naïve way to describe a reflective subcategory in type theory is to use Type. First we need, for any type A, a proposition expressing the assertion "A is codiscrete". Since propositions are particular types, this can simply be a function term

$$\mathsf{isCodisc} : \mathsf{Type} \to \mathsf{Type} \tag{3}$$

together with an axiom asserting that for any type A, the type $\mathsf{isCodisc}(A)$ is a proposition:

$$(A : \mathsf{Type}) \;\vdash\; (\mathsf{isCodiscIsProp}_A : \mathsf{isProp}(\mathsf{isCodisc}(A))) \tag{4}$$

Next we need the reflector $\sharp$ and its unit:

$$\sharp : \mathsf{Type} \to \mathsf{Type}. \tag{5}$$

$$(A : \mathsf{Type}) \;\vdash\; (\mathsf{sharpIsCodisc}_A : \mathsf{isCodisc}(\sharp A)) \tag{6}$$

$$(A : \mathsf{Type}) \;\vdash\; (\eta_A : A \to \sharp A)\,. \tag{7}$$

Finally, we assert the universal property of the reflection: if B is codiscrete, then the space of morphisms $\sharp A \to B$ is equivalent, by precomposition with η_A, to the space of morphisms $A \to B$.

$$(A : \mathsf{Type})\,,\, (B : \mathsf{Type})\,,\, (\mathsf{bc} : \mathsf{isCodisc}(B)) \;\vdash\; \big(\mathsf{tsr} : \mathsf{isEquiv}(\lambda f^{\sharp A \to B}. f \circ \eta_A)\big) \tag{8}$$

This looks like a complete axiomatization of a reflective subcategory, but in fact it describes more data than we want, because Type is an object classifer for *all* slice categories. If $\mathbf{H}$ satisfies these axioms, then each $\mathbf{H}/X$ is equipped with a reflective subcategory, and moreover these subcategories and their reflectors are stable under pullback. For instance, if $A \in \mathbf{H}/X$ is represented by $(x : X) \;\vdash\; (A(x) : \mathsf{Type})$, then the dependent type $(x : X) \;\vdash\; (\sharp(A(x)) : \mathsf{Type})$ represents a "fiberwise reflection" $\sharp_X(A) \in \mathbf{H}/X$.

We call such data a *reflective subfibration* [12]. If a reflective subcategory underlies some reflective subfibration, then its reflector preserves finite products, and the converse holds in good situations [49]. If the reflector even preserves all finite *limits*, as $\sharp$ does, then there is a *canonical* extension to a reflective subfibration. Namely, we define $A \in \mathbf{H}/X$ to be *relatively codiscrete* if the naturality square for η:

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d] & \sharp A \arrow[d] \\
X \arrow[r, "\eta_X"] & \sharp X
\end{tikzcd}$$

is a pullback. (This says that A has the "initial cohesive structure" induced from X: elements of A cohere in precisely the ways that their images in X cohere.) For general $A \in \mathbf{H}/X$, we define $\sharp_X A$ to be the pullback of $\sharp A \to \sharp X$ along $X \to \sharp X$. When $\sharp$ preserves finite limits, this defines a reflective subfibration.

Reflective subfibrations constructed in this way are characterized by two special properties. The first is that the relatively codiscrete morphisms are closed under composition. A reflective subfibration with this property is equivalent to a *stable factorization system*: a pair of classes of morphisms $(\mathcal{E}, \mathcal{M})$ such that every morphism factors essentially uniquely as an $\mathcal{E}$-morphism followed by an $\mathcal{M}$-morphism, stably under pullback. (The corresponding reflective subcategory of $\mathbf{H}/X$ is the category of $\mathcal{M}$-morphisms into X.) If a reflective subcategory underlies a stable factorization system, its reflector preserve all pullbacks over objects in the subcategory — and again, the converse holds in good situations [49]; such a reflector has *stable units* [13].

We can axiomatize this property as follows. Axiom (8) implies, in particular, a factorization operation:

$$(\mathsf{bc} : \mathsf{isCodisc}(B))\,,\, (f : A \to B) \;\vdash\; (\mathsf{fact}_\sharp(f) : \sharp A \to B) \tag{9}$$

$$(\mathsf{bc} : \mathsf{isCodisc}(B))\,,\, (f : A \to B) \;\vdash\; (\mathsf{ff}_f : (\mathsf{fact}_\sharp(f) \circ \eta_A = f)) \tag{10}$$

It turns out that relatively codiscrete morphisms are closed under composition if and only if we have a more general factorization operation, where B may depend on $\sharp A$:

$$\Big(\mathsf{bc} : \textstyle\prod_{x:\sharp A} \mathsf{isCodisc}(B(x))\Big)\,,\, \big(f : \textstyle\prod_{x:A} B(\eta_A(x))\big) \;\vdash\; \Big(\mathsf{fact}_\sharp(f) : \textstyle\prod_{x:\sharp A} B(x)\Big) \tag{11}$$

$$\Big(\mathsf{bc} : \textstyle\prod_{x:\sharp A} \mathsf{isCodisc}(B(x))\Big)\,,\, \big(f : \textstyle\prod_{x:A} B(\eta_A(x))\big) \;\vdash\; (\mathsf{ff}_f : (\mathsf{fact}_\sharp(f) \circ \eta_A = f)) \tag{12}$$

This sort of "dependent factorization" is familiar in type theory; it is related to (9)–(10) in the same way that proof by induction is related to definition by recursion.

The second property is that if $g \in \mathcal{E}$ and $gf \in \mathcal{E}$, then $f \in \mathcal{E}$. If a stable factorization system has this property, then it is determined by its underlying reflective subcategory, whose reflector must preserve finite limits. ($\mathcal{E}$ is the class of morphisms inverted by the reflector, and $\mathcal{M}$ is defined by pullback as above.) We can state this in type theory as the preservation of $\sharp$-contractibility by homotopy fibers:

$$(\mathsf{acs} : \mathsf{isContr}(\sharp A)),\ (\mathsf{bcs} : \mathsf{isContr}(\sharp B)),\ (f : A \to B),\ (b : B) \ \vdash\ (\mathsf{fcs} : \mathsf{isContr}\,(\sharp \textstyle\sum_{x:A}(f(x) = b))) \tag{13}$$

This completes our axiomatization of the reflective subcategory of codiscrete objects. We can apply the same reasoning to the reflective subcategory of *discrete* objects. Now Π does not preserve all finite limits, only finite products, so we cannot push the characterization all the way as we did for $\sharp$. But because the target of Π is ∞-**Gpd**, it automatically has stable units; thus the discrete objects underlie *some* stable factorization system $(\mathcal{E}, \mathcal{M})$, which can be axiomatized as above with (13) omitted. (For cohesion over a general base, we ought to demand stable units explicitly.) We do not know whether there is a particular choice of such an $(\mathcal{E}, \mathcal{M})$ to be preferred. In §2.5, we will mention a different way to axiomatize Π which is less convenient, but does not require choosing $(\mathcal{E}, \mathcal{M})$.

2.5. **Axiomatic cohesion II: $\sharp$Type.** We now change notation slightly by writing function-types as $[A, B]$ instead of $A \to B$, to suggest that they denote internal cohesive function-spaces. We would like $A \to B$ to denote instead the external hom-∞-groupoid $\mathbf{H}(A, B)$, re-internalized as a codiscrete object. (Since $\sharp$ preserves more limits than Π, the codiscrete objects are a better choice for this sort of thing.)

To construct such an external function-type, consider $\sharp\mathsf{Type}$. This is a codiscrete object, which as an ∞-groupoid indicates the core of (some small full subcategory of) $\mathbf{H}$. Any type $A : \mathsf{Type}$ has an "externalized" version $\eta_{\mathsf{Type}}(A) : \sharp\mathsf{Type}$, which we denote $[\![A]\!]$. And since $\sharp$ preserves products, the operation

$$[-,-] : \mathsf{Type} \times \mathsf{Type} \to \mathsf{Type} \qquad \text{induces an operation}$$
$$[-,-]^\sharp : \sharp\mathsf{Type} \times \sharp\mathsf{Type} \to \sharp\mathsf{Type}$$

which indicates the internal-hom $[-,-]$ as an operation $\mathrm{Core}(\mathbf{H}) \times \mathrm{Core}(\mathbf{H}) \to \mathrm{Core}(\mathbf{H})$. We now define the *escaping* morphism $\uparrow : \sharp\mathsf{Type} \to \mathsf{Type}$ as follows.

$$\uparrow\! A :\equiv \sum_{B : \sharp \sum_{X : \mathsf{Type}} X} (\sharp(\mathsf{pr}_1)(B) = A)$$

Here $\sum_{X:\mathsf{Type}} X$ is the type-theoretic version of the domain $\widetilde{\mathsf{Type}}$ of the morphism $p : \widetilde{\mathsf{Type}} \to \mathsf{Type}$ from §2.2, with p being the first projection $\mathsf{pr}_1 : \sum_{X:\mathsf{Type}} X \to \mathsf{Type}$. Thus functoriality of $\sharp$ gives $\sharp(\mathsf{pr}_1) : \sharp \sum_{X:\mathsf{Type}} X \to \sharp\mathsf{Type}$, and so $\sharp(\mathsf{pr}_1)(B) : \sharp\mathsf{Type}$ can be compared with A. Now $\uparrow\! A$ is codiscrete, and the composite

$$\mathsf{Type} \xrightarrow{\eta_{\mathsf{Type}}} \sharp\mathsf{Type} \xrightarrow{\uparrow} \mathsf{Type}$$

is equivalent to $\sharp$. Thus, since $\mathbf{H}(A, B) = \Gamma[A, B]$, we can define the external function-type as

$$(A \to B) :\equiv \uparrow\left([A, B]^\sharp\right)$$

Note that this makes sense for any $A, B : \sharp\mathsf{Type}$. If instead $A, B : \mathsf{Type}$, then $([\![A]\!] \to [\![B]\!]) \simeq \sharp[A, B]$. Thus, if $f : [A, B]$, then applying this equivalence to $\eta_{A \to B}(f) : \sharp[A, B]$, we obtain an "externalized" version of f, which we denote $[\![f]\!] : [\![A]\!] \to [\![B]\!]$. Now using dependent factorization for the modality $\sharp$, we can define all sorts of categorical operations externally. We have composition:

$$(A : \sharp\mathsf{Type}),\ (B : \sharp\mathsf{Type}),\ (C : \sharp\mathsf{Type}),\ (f : A \to B),\ (g : B \to C) \ \vdash\ (g \circ f : A \to C)$$

and the property of being an equivalence:

$$(A : \sharp\mathsf{Type}),\ (B : \sharp\mathsf{Type}),\ (f : A \to B) \ \vdash\ (\mathsf{eisEquiv}(f) : \mathsf{Type})\,.$$

Using these external tools, we can now complete our internal axiomatization of cohesion. One may be tempted to define the coreflection $\flat$ as we did the reflections $\sharp$ and Π, but this would amount to asking for a pullback-stable system of *coreflective* subcategories of each $\mathbf{H}/X$, and at present we do not know any way to obtain this in models. Instead, we work externally:

$$\mathsf{eisDisc} : [\sharp\mathsf{Type}, \mathsf{Type}] \tag{14}$$

$$(A : \sharp\mathsf{Type}) \;\vdash\; (\mathsf{eisDiscIsProp}_A : \mathsf{isProp}(\mathsf{eisDisc}(A))) \tag{15}$$

$$\flat : [\sharp\mathsf{Type}, \sharp\mathsf{Type}] \tag{16}$$

$$(A : \sharp\mathsf{Type}) \;\vdash\; (\mathsf{flatIsDisc}_A : \mathsf{eisDisc}(\flat A)) \tag{17}$$

$$(A : \sharp\mathsf{Type}) \;\vdash\; (\epsilon_A : \flat A \to A) \tag{18}$$

$$(A : \sharp\mathsf{Type})\,,\,(B : \sharp\mathsf{Type})\,,\,(\mathsf{ad} : \mathsf{eisDisc}(A)) \;\vdash\; \Big(\mathsf{flr} : \mathsf{isEquiv}(\lambda f^{A \to \flat B}.\, \epsilon_B \circ f)\Big)\,. \tag{19}$$

If we axiomatize discrete objects as in §2.4, we can define $\mathsf{eisDisc}(A) :\equiv \uparrow(\sharp(\mathsf{isDisc})(A))$. But we can also treat discrete objects externally only, with $\mathsf{eisDisc}$ axiomatic and Π defined analogously to $\flat$. This would allow us to avoid choosing an $(\mathcal{E}, \mathcal{M})$ for the categorical interpretation. In any case, (19) implies factorizations:

$$(\mathsf{ad} : \mathsf{eisDisc}(A))\,,\,(f : A \to B) \;\vdash\; (\mathsf{fact}_\flat(f) : A \to \flat B)$$

$$(\mathsf{ad} : \mathsf{eisDisc}(A))\,,\,(f : A \to B) \;\vdash\; (\mathsf{ff}_f : (\epsilon_B \circ \mathsf{fact}_\flat(f) = f))\,.$$

Thus, we can state the final axioms internally as

$$(A : \sharp\mathsf{Type}) \;\vdash\; (\mathsf{sfe} : \mathsf{eisEquiv}(\mathsf{fact}_\sharp(\llbracket \eta_A \rrbracket \circ \epsilon_A))) \tag{20}$$

$$(A : \sharp\mathsf{Type}) \;\vdash\; (\mathsf{fse} : \mathsf{eisEquiv}(\mathsf{fact}_\flat(\llbracket \eta_A \rrbracket \circ \epsilon_A))) \tag{21}$$

This completes the axiomatization of the internal homotopy type theory of a cohesive $(\infty, 1)$-topos, yielding the formal system that we call *cohesive homotopy type theory.*

3. QUANTUM GAUGE FIELD THEORY

We now give a list of constructions in this axiomatics whose interpretation in cohesive $(\infty, 1)$-toposes $\mathbf{H}$ reproduces various notions in differential geometry, differential cohomology, geometric quantization and quantum gauge field theory. Because of the homotopy theory built into the type theory, this also automatically generalizes all these notions to homotopy theory. For instance, a *gauge group* in the following may be interpreted as an ordinary gauge group such as the Spin-group, but may also be interpreted as a *higher* gauge group, such as the String-2-group or the Fivebrane-6-group [44]. Similarly, all fiber products are automatically homotopy fiber products, and so on. (The fiber product of $f : [A, C]$ and $g : [B, C]$ is $A \times_C B :\equiv \sum_{x:A} \sum_{y:B} (f(x) = g(y))$, and this can be externalized easily.)

In this section we will mostly speak "externally" about a cohesive $(\infty, 1)$-topos $\mathbf{H}$. This can all be expressed in type theory using the technology of §2.5, but due to space constraints we will not do so.

3.1. Gauge fields.

The concept of a *gauge field* is usefully decomposed into two stages, the *kinematical* aspect and the *dynamical* aspect bulding on that:

gauge field	**kinematics**	**dynamics**
physics terminology:	instanton sector / charge sector	gauge potential
formalized as:	cocycle in (twisted) cohomology	cocycle in (twisted) differential cohomology
diff. geometry terminology:	fiber bundle	connection
required ambient logic:	homotopy type theory	cohesive homotopy type theory

3.1.1. *Kinematics.* Suppose A is a pointed connected type, i.e. we have $a_0 : A$ and $\pi_0(A)$ is contractible. Then its loop type $\Omega A :\equiv * \times_A * \simeq (a_0 = a_0)$ is a group. This establishes an equivalence between pointed connected homotopy types and group homotopy types; its inverse is called *delooping* and denoted $G \mapsto \mathbf{B}G$.[3] For instance, the *automorphism group* $\mathbf{Aut}(V)$ of a homotopy type V is the looping of the image factorization

$$\begin{tikzcd} * \arrow[r, two heads] \arrow[rr, bend right, "\vdash V"'] & \mathbf{B}\mathrm{Aut}(V) \arrow[r, hook] & \mathrm{Type}\,. \end{tikzcd}$$

---

[3] Currently, we cannot fully formalize completely general ∞-groups and their deloopings, because they involve infinitely many higher homotopies. This is a mere technical obstruction that will hopefully soon be overcome. It is not really a problem for us, since we generally care more about deloopings than groups themselves, and pointed connected types are easy to formalize.

Given a group G and a homotopy type X, we write $H^1(X,G) \coloneqq \pi_0(X \to \mathbf{B}G)$ for the *degree-1 cohomology* of X with coefficients in G. If G has higher deloopings $\mathbf{B}^n G$, we write $H^n(X,G) \coloneqq \pi_0(X \to \mathbf{B}^n G)$ and speak of the *degree-n cohomology* of X with coefficients in G. The interpretation of this simple definition in homotopy type theory is very general, and (if we allow disconnected choices of deloopings) much more general than what is traditionally called *generalized cohomology*: in traditional terms, it would be called *non-abelian equivariant twisted sheaf hyper-cohomology.*

A *G-principal bundle* over X is a function $p : P \to X$ where P is equipped with a G-action over X and such that p is the quotient $P \to P /\!/ G$ (as always, this is a *homotopy quotient*, constructible as a higher inductive type [51, 31, 55]). One finds that the delooping $\mathbf{B}G$ is the *moduli stack* of G-principal bundles: for any $g : X \to \mathbf{B}G$, its (homotopy) fiber is canonically a G-principal bundle over X, and this establishes an equivalence $G\mathrm{Bund}(X) \simeq (X \to \mathbf{B}G)$ between G-principal bundles and cocycles in *G-cohomology.* In particular, equivalence classes of G-principal bundles on X are classified by $H^1(X,G)$.

Conversely, any G-action $\rho : G \times V \to V$ is equivalently encoded in a fiber sequence $\begin{tikzcd} V \arrow[r] & V/\!/G \arrow[r, "\bar{\rho}"] & \mathbf{B}G \end{tikzcd}$, hence in a V-fiber bundle $V/\!/G$ over $\mathbf{B}G$. This is the *universal ρ-associated* V-fiber bundle in that, for g and P as above, the V-bundle $E \coloneqq P \times_G V \to X$ is equivalent to the pullback $g^* \bar{\rho}$. This implies that the (discrete) homotopy type of sections $\Gamma_X(E)$ of E is equivalent to the mapping space $\mathbf{H}_{/\mathbf{B}G}(g, \bar{\rho})$ in the slice. (These external hom-spaces can be defined analogously to how we dealt with the simpler case in §2.5.)

Since all the bundles involved are locally trivial with respect to the intrinsic notion of covers (epimorphic maps) it follows that elements of $\Gamma_X(E)$ are locally maps to V. If V here is pointed connected, and hence $V \simeq \mathbf{B}H$, then E is called an *H-gerbe* over X. In this case a section of $E \to X$ is therefore locally a cocycle in H-cohomology, and hence globally a cocycle in *g-twisted H-cohomology* with respect to the *local coefficient bundle $E \to X$*. Hence twisted cohomology in $\mathbf{H}$ is ordinary cohomology in a slice $\mathbf{H}_{/\mathbf{B}G}$. All this is discussed in detail in [38].

In gauge field theory a group H as above serves as the *gauge group* and then an H-principal bundle on X is the charge/kinematic part of an *H-gauge field* on X. (In the special case that H is a discrete homotopy type, this is already the full gauge field, as in this case the dynamical part is trivial). The mapping type $[X, \mathbf{B}H]$ interprets as the moduli stack of kinematic H-gauge fields on X, and a term of identity type $\lambda : (\phi_1 = \phi_2)$ is a *gauge transformation* between two gauge field configurations $\phi_1, \phi_2 : [X, \mathbf{B}H]$.

It frequently happens that the charge of one G-gauge field Φ *shifts* another H-gauge field ϕ, in generalization of the way that magnetic charge shifts the electromagnetic field. Such shifts are controlled by an action ρ of G on $\mathbf{B}H$ and in this case Φ is a cocycle in G-cohomology and ϕ is a cocycle in Φ-twisted H-cohomology with respect to the local coefficient bundle $\bar{\rho}$. Discussion of examples and further pointers are in [46].

3.1.2. *Dynamics.* Given a G-principal bundle in the presence of cohesion, we may ask if its cocycle $g : X \to \mathbf{B}G$ lifts through the counit $\epsilon_{\mathbf{B}G} : \flat \mathbf{B}G \to \mathbf{B}G$ from (18) to a cocycle $\nabla : X \to \flat \mathbf{B}G$. By the $(\Pi \dashv \flat)$-adjunction this is equivalently a map $\Pi(X) \to \mathbf{B}G$. Since $\Pi(X)$ is interpreted as the *path ∞-groupoid* of X, such a ∇ is a *flat parallel transport* on X with values in G, equivalently a *flat G-principal connection* on X.

Consider then the homotopy fiber $\flat_{\mathrm{dR}} \mathbf{B}G \coloneqq \flat \mathbf{B}G \times_{\mathbf{B}G} *$. By definition, a map $\omega : X \to \flat_{\mathrm{dR}} \mathbf{B}G$ is a flat G-connection on X together with a trivialization of the underlying G-principal bundle. This is interpreted as a *flat $\mathfrak{g}$-valued differential form*, where $\mathfrak{g}$ is the *Lie algebra* of G. By using this definition in the statement of the above classification of G-principal bundles, one finds that every flat connection $\nabla : X \to \flat \mathbf{B}G$ is *locally* given by a flat $\mathfrak{g}$-valued form: ∇ is equivalently a form $A : P \to \flat_{\mathrm{dR}} \mathbf{B}G$ on the total space of the underlying G-principal bundle, such that this is G-equivariant in a natural sense. Such an A is interpreted as the incarnation of the connection ∇ in the form of an *Ehresmann connection* on $P \to X$.

Moreover, the coefficient $\flat_{\mathrm{dR}} \mathbf{B}G$ sits in a long fiber sequence of the form

$$\begin{tikzcd} G \arrow[r, "\theta_G"] & \flat_{\mathrm{dR}} \mathbf{B}G \arrow[r] & \flat \mathbf{B}G \arrow[r, "\epsilon_{\mathbf{B}G}"] & \mathbf{B}G \end{tikzcd} .$$

with the further homotopy fiber θ_G giving a canonical flat $\mathfrak{g}$-valued differential form on G. This is the *Maurer-Cartan form* of G, in that when interpreted in smooth homotopy types and for G an ordinary Lie group, it is canonically identified with the classical differential-geometric object of this name. Here in cohesive homotopy type theory it exists in much greater generality.

Specifically, assume that G itself is once more deloopable, hence assume that $\mathbf{B}^2G$ exists. Then the above long fiber sequence extends further to the right as $\flat\mathbf{B}G \xrightarrow{\epsilon_{\mathbf{B}G}} \mathbf{B}G \xrightarrow{\theta_{\mathbf{B}G}} \flat_{\mathrm{dR}}\mathbf{B}^2G$, since $\flat$ is right adjoint. This means, by the universal property of homotopy fibers, that if $g : X \to \mathbf{B}G$ is the cocycle for a G-principal bundle on X, then the class of the differential form $\theta_{\mathbf{B}G}(g)$ is the *obstruction* to the existence of a flat connection ∇ on this bundle. Hence this class is interpreted as the *curvature* of the bundle, and we interpret the Maurer-Cartan form $\theta_{\mathbf{B}G}$ of the delooped group $\mathbf{B}G$ as the *universal curvature characteristic* for G-principal bundles.

This universal curvature characteristic is the key to the notion of non-flat connections, for it allows us to define these in the sense of twisted cohomology as *curvature-twisted flat cohomology.* There is, however, a choice involved in defining the universal curvature-twist, which depends on the intended application. But in standard interpretations there is a collection of types singled out, called the *manifolds*, and the standard universal curvature twist can then be characterized as a map $i : \Omega^2_{\mathrm{cl}}(-, \mathfrak{g}) \to \flat_{\mathrm{dR}}\mathbf{B}^2G$ out of a 0-truncated homotopy type such that for all manifolds Σ its image under $[\Sigma, -]$ is epi, meaning that $\Omega^2_{\mathrm{cl}}(\Sigma, \mathfrak{g}) \to [\Sigma, \flat_{\mathrm{dR}}\mathbf{B}^2G]$ is an *atlas* in the sense of geometric stack theory.

Assuming such a choice of universal curvature twists has been made, we may then define the moduli of general (non-flat) G-principal connections to be the homotopy fiber product

$$\mathbf{B}G_{\mathrm{conn}} \coloneqq i^*\theta_{\mathbf{B}G} = \mathbf{B}G \times_{\flat_{\mathrm{dR}}\mathbf{B}^2G} \Omega^2(-, \mathfrak{g})\,.$$

In practice one is usually interested in a canonical abelian (meaning arbitrarily deloopable, i.e. E_∞) group A and the tower of delooping groups $\mathbf{B}^nA$ that it induces. In this case we write $\mathbf{B}^nA_{\mathrm{conn}} \coloneqq \mathbf{B}^nA \times_{\flat_{\mathrm{dR}}\mathbf{B}^{n+1}A} \Omega^{n+1}(-, A)$. When interpreted in smooth homotopy types and choosing the Lie group $A = \mathbb{C}^\times$ or $= U(1)$ one finds that $\mathbf{B}^nA_{\mathrm{conn}}$ is the coefficient for *ordinary differential cohomology*, specifically that it is presented by the ∞-stack given by the *Deligne complex.*

Notice that the pasting law for homotopy pullbacks implies generally that the restriction of $\mathbf{B}G_{\mathrm{conn}}$ to vanishing curvature indeed coincides with the universal flat coefficients: $\flat\mathbf{B}G \simeq \mathbf{B}G_{\mathrm{conn}} \times_{\Omega^2(-,\mathfrak{g})} \{*\}$. This means that we obtain a factorization of $\epsilon_{\mathbf{B}G}$ as $\flat\mathbf{B}G \to \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}G$.

Let then G be a group which is not twice deloopable, hence to which the above universal definition of $\mathbf{B}G_{\mathrm{conn}}$ does not apply. If we have in addition a map $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^nA$ given (representing a universal characteristic class in $H^n(\mathbf{B}G, A)$), then we may still ask for *some* homotopy type $\mathbf{B}G_{\mathrm{conn}}$ that supports a *differential refinement* $\mathbf{c}_{\mathrm{conn}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^nA_{\mathrm{conn}}$ of $\mathbf{c}$ in that it lifts the factorization of $\epsilon_{\mathbf{B}^nA}$ by $\mathbf{B}^nA_{\mathrm{conn}}$. Such a $\mathbf{c}_{\mathrm{conn}}$ interprets, down on cohomology, as a *secondary universal characteristic class* in the sense of refined Chern-Weil theory. Details on all this are in [20, 45].

With these choices and for G regarded as a gauge group, a genuine *G-gauge field* on Σ is a map $\phi : \Sigma \to \mathbf{B}G_{\mathrm{conn}}$. For G twice deloopable, the *field strength* of ϕ is the composite $F_\phi : \Sigma \to \mathbf{B}G_{\mathrm{conn}} \xrightarrow{F_{(-)}} \Omega^2_{\mathrm{cl}}(-, \mathfrak{g})$.

Moreover, the choice of $\mathbf{c}_{\mathrm{conn}}$ specifies an exended *action functional* on the moduli type $[\Sigma, \mathbf{B}G_{\mathrm{conn}}]$ of G-gauge field configurations, and hence specifies an actual quantum gauge field theory. This we turn to now.

3.2. σ-Model QFTs.

An n-dimensional ("nonlinear") *σ-model* quantum field theory describes the dynamics of an $(n-1)$-dimensional quantum "particle" (for instance an electron for $n = 1$, a string for $n = 2$, and generally an "$(n-1)$-brane") that propagates in a *target space* X (for instance our spacetime) while acted on by forces (for instance the *Lorentz force*) exerted by a fixed background A-gauge field on X (for instance the electromagnetic field for $n = 1$ or the *Kalb-Ramond B-field* for $n = 2$ or the *supergravity C-field* for $n = 3$). By the above, this background gauge field is the interpretation of a map $\mathbf{c}_{\mathrm{conn}} : X \to \mathbf{B}^nA_{\mathrm{conn}}$.

Let then Σ be a cohesive homotopy type of *cohomological dimension n*, to be thought of as the abstract *worldvolume* of the $(n-1)$-brane. The homotopy type $[\Sigma, X]$ of cohesive maps from Σ to X is interpreted as the moduli space of field configurations of the σ-model for this choice of shape of worldvolume. The (gauge-coupling part of) the *action functional* of the σ-model is then to be the *n-volume holonomy* of the background gauge field over a given field configuration $\Sigma \to X$.

To formalize this, we need the notion of *concreteness.* If X is a cohesive homotopy type, its *concretization* is the image factorization $X \twoheadrightarrow \mathrm{conc}X \hookrightarrow \sharp X$ of $\eta_X : X \to \sharp X$ (7). We call X *concrete* if $X \to \mathrm{conc}X$ is an equivalence. In the standard smooth model, the 0-truncated concrete cohesive homotopy types are precisely the *diffeological spaces* (see [7]). Generally, for models over *concrete sites*, they are the *concrete sheaves.*

Now we can define the *action functional* of the σ-model associated to the background gauge field $\mathbf{c}_{\mathrm{conn}}$ to be the composite

$$\exp(iS(-)) : \; [\Sigma, X] \xrightarrow{[\Sigma, \mathbf{c}_{\mathrm{conn}}]} [\Sigma, \mathbf{B}^n A_{\mathrm{conn}}] \longrightarrow \mathrm{conc}\, \pi_0 [\Sigma, \mathbf{B}^n A_{\mathrm{conn}}] \; .$$

In the standard smooth model, with $A = \mathbb{C}^\times$ or $U(1)$, the second morphism is *fiber integration in differential cohomology* $\exp(2\pi i \int_\Sigma (-))$. For $n = 1$ this computes the line holonomy of a circle bundle with connection, hence the correct gauge coupling action functional of the 1-dimensional σ-model; for $n = 2$ it computes the surface holonomy of a circle 2-bundle, hence the correct "WZW-term" of the string; and so on.

Traditionally, σ-models are thought of as having as target space X a manifold or at most an orbifold. However, since these are smooth homotopy n-types for $n \leq 1$, it is natural to allow X to be a general cohesive homotopy type. If we do so, then a variety of quantum field theories that are not traditionally considered as σ-models become special cases of the above general setup. Notably, if $X = \mathbf{B}G_{\mathrm{conn}}$ is the moduli for G-principal connections, then a σ-model with target space X is a G-gauge theory on Σ. Moreover, as we have seen above, in this case the background gauge field is a secondary universal characteristic invariant. One finds that the corresponding action functional $\exp(2\pi i \int_\Sigma [\Sigma, \mathbf{c}_{\mathrm{conn}}]) : [\Sigma, \mathbf{B}G_{\mathrm{conn}}] \to A$ is that of *Chern-Simons*-type gauge field theories [18, 19], including the standard 3-dimensional Chern-Simons theories as well a various higher generalizations.

More generally, at least in the smooth model, there is a *transgression* map for differential cocycles: for Σ_d a manifold of dimension d there is a canonical map

$$\exp(2\pi i \int_\Sigma [\Sigma, \mathbf{c}_{\mathrm{conn}}]) : \; [\Sigma_d, X] \xrightarrow{[\Sigma_d, \mathbf{c}_{\mathrm{conn}}]} [\Sigma_d, \mathbf{B}^n A_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_\Sigma)} \mathbf{B}^{n-d} A_{\mathrm{conn}}$$

modulating an n-bundle on the Σ_d-mapping space. For $d = n - 1$ and quadratic Chern-Simons-type theories this turns out to be the (off-shell) prequantum bundle of the QFT. See [18] for details on these matters. Thus the differential characteristic cocycle $\mathbf{c}_{\mathrm{conn}}$ should itself be regarded as a *higher prequantum bundle*, in the sense we now discuss.

3.3. **Geometric quantization.** Action functionals as above are supposed to induce n-dimensional *quantum* field theories by a process called *quantization*. One formalization of what this means is *geometric quantization*, which is well-suited to formalization in cohesive homotopy type theory. We indicate here how to formalize the spaces of higher (pre)quatum states that an extended QFT assigns in codimension n.

The *critical locus* of a local action functional – its *phase space* or *Euler-Lagrange solution space* – carries a canonical closed 2-form ω, and standard geometric quantization gives a method for constructing the *space of quantum states* assigned by the QFT in dimension $n - 1$ as a space of certain sections of a *prequantum bundle* whose curvature is ω. This works well for non-extended topological quantum field theories and generally for $n = 1$ (quantum mechanics). The generalization to $n \geq 2$ is called *multisymplectic* or *higher symplectic geometry* [42], for here ω is promoted to an $(n + 1)$-form which reproduces the former 2-form upon transgression to a mapping space. Exposition of the string σ-model ($n = 2$) in the context of higher symplectic geometry is in [5], and discussion of quantum Yang-Mills theory ($n = 4$) and further pointers are in [25]. A homotopy-theoretic formulation is given in [19, 17]: here the prequantum bundle is promoted to a prequantum n-bundle, a $(\mathbf{B}^{n-1}A)$-principal connection as formalized above.

Based on this we can give a formalization of central ingredients of geometric quantization in cohesive homotopy type theory. When interpreted in the standard smooth model with $A = \mathbb{C}^\times$ or $= U(1)$ the following reproduces the traditional notions for $n = 1$, and for $n \geq 2$ consistently generalizes them to higher geometric quantization.

Let X be any cohesive homotopy type. A closed $(n + 1)$-form on X is a map $\omega : X \to \Omega^{n+1}_{\mathrm{cl}}(-, A)$, as discussed in section 3.1.2. We may call the pair (X, ω) a *pre-n-plectic* cohesive homotopy type. The group of *symplectomorphisms* or *canonical transformations* of (X, ω) is the automorphism group of ω:

$$\mathbf{Aut}_{/\Omega^{n+1}_{\mathrm{cl}}(-,A)}(\omega) = \left\{ \begin{tikzcd} X \arrow[rr, "\simeq"] \arrow[dr, "\omega"'] & & X \arrow[dl, "\omega"] \\ & \Omega^{n+1}_{\mathrm{cl}}(-, A) & \end{tikzcd} \right\} ,$$

regarded as an object in the slice $\mathbf{H}_{/\Omega^{n+1}_{\mathrm{cl}}(-,A)}$. A *prequantization* of (X,ω) is a lift

$$\begin{tikzcd}
& \mathbf{B}^n A_{\mathrm{conn}} \arrow[d, "F_{(-)}"] \\
X \arrow[ur, "\mathbf{c}_{\mathrm{conn}}"] \arrow[r, "\omega"] & \Omega^{n+1}_{\mathrm{cl}}(-,A)
\end{tikzcd}$$

through the defining projection from the moduli of $(\mathbf{B}^{n-1}A)$-principal connections. This $\mathbf{c}_{\mathrm{conn}}$ modulates the *prequantum n-bundle.* Since A is assumed abelian, there is abelian group structure on $\pi_0(X \to \mathbf{B}^n A_{\mathrm{conn}})$ and hence we may rescale $\mathbf{c}_{\mathrm{conn}}$ by a natural number k. This corresponds to rescaling *Planck's constant* $\hbar$ by $1/k$. The limit $k \to \infty$ in which $\hbar \to 0$ is the *classical limit.*

The automorphism group of the prequantum bundle

$$\mathbf{Aut}_{/\mathbf{B}^n A_{\mathrm{conn}}}(\mathbf{c}_{\mathrm{conn}}) := \left\{ \begin{tikzcd}
X \arrow[rr, "\sigma"] \arrow[dr, "\mathbf{c}_{\mathrm{conn}}"'] & \arrow[d, phantom, "\Swarrow\simeq"] & X \arrow[dl, "\mathbf{c}_{\mathrm{conn}}"] \\
& \mathbf{B}^n A_{\mathrm{conn}} &
\end{tikzcd} \right\},$$

in the slice $\mathbf{H}_{/\mathbf{B}^n A_{\mathrm{conn}}}$, is the *quantomorphism group* of the system. There is an evident projection from the quantomorphism group to the symplectomorphism group, and its image is the group of *Hamiltonian symplectomorphisms.* The Lie algebra of the quantomorphism group is that of *Hamiltonian observables* equipped with the *Poisson bracket.* If X itself has abelian group structure, then the subgroup of the quantomorphism group covering the action on X on itself is the *Heisenberg group* of the system. An action of any group G on X by quantomorphisms, i.e. a map $\mu : \mathbf{B}G \to \mathbf{B}\mathbf{Aut}_{/\mathbf{B}^n A_{\mathrm{conn}}}(\mathbf{c}_{\mathrm{conn}})$, is a *Hamiltonian G-action* on (X,ω). The (homotopy) quotient $\mathbf{c}_{\mathrm{conn}}/\!\!/G : X/\!\!/G \to \mathbf{B}^n A_{\mathrm{conn}}$ is the corresponding *gauge reduction* of the system.

After a choice of representation ρ of $\mathbf{B}^{n-1}A$ on some V, the space of *prequantum states* is

$$\mathbf{\Gamma}_X(E) := [\mathbf{c}, \mathbf{p}]_{/\mathbf{B}^n A_{\mathrm{conn}}} = \left\{ \begin{tikzcd}
X \arrow[rr, "\sigma"] \arrow[dr, "\mathbf{c}"'] & \arrow[d, phantom, "\Swarrow\simeq"] & V/\!\!/\mathbf{B}^{n-1}A \arrow[dl, "\bar{\rho}"] \\
& \mathbf{B}^n A &
\end{tikzcd} \right\},$$

the space of $\mathbf{c}$-twisted cocycles with respect to the local coefficient bundle $\bar{\rho}$. There is an evident action of the quantomorphism group on $\mathbf{\Gamma}_X(E)$ and this is the action of *prequantum operators* on the space of states.

It remains to formalize in cohesive homotopy type theory the notion of *polarization* of the prequantum bundle: the actual quantum space of states is the subspace of prequantum states which are *polarized.* The full formalization of this step remains under investigation, but we have proposals at least for aspects of the solution and have checked that these give rise to the right structures in examples [41, 17].

In the models one may go further with the construction of full (extended) QFTs. While several of the further steps involved remain to be formalized in the cohesive axiomatics, the collection of gauge QFT notions easily formalized here in cohesive homotopy type theory is already remarkable, emphasizing the value of a formal, logical, approach to concepts like smoothness and cohomology.

References

[1] The Coq proof assistant. `http://coq.inria.fr/`.

[2] S. Awodey. Type theory and homotopy. `http://www.andrew.cmu.edu/user/awodey/preprints/TTH.pdf`, 2010. To appear.

[3] S. Awodey and L. Birkedal. Elementary axioms for local maps of toposes. *Journal of Pure and Applied Algebra*, 177(3):215–230, 2003.

[4] J. Baez and J. Dolan. Higher-dimensional algebra and topological quantum field theory. *J. Math. Phys.*, 36(11):60736105, 1995.

[5] J. Baez, E. Hoffnung, and L. Rogers. Categorified symplectic geometry and the classical string. *Commun. Math. Phys.*, (293):701–725, 2010. arXiv:0808.0246.

[6] J. Baez and M. Stay. Physics, topology, logic and computation: A Rosetta stone. In Bob Coecke, editor, *New Structures for Physics*, volume 813 of *Lecture Notes in Physics*, pages 95–174. Springer, 2011.

[7] J. C. Baez and A. E. Hoffnung. Convenient categories of smooth spaces. *Trans. Amer. Math. Soc.*, 363(11):5789–5825, 2011.

[8] C. Barwick and C. Schommer-Pries. On the unicity of the homotopy theory of higher categories. arXiv:1112.0040.

[9] J. Bernstein. Spontaneous symmetry breaking, gauge theories, the Higgs mechanism and all that. *Rev. Mod. Phys.*, 46:7–48, 1974.

[10] K. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Transactions of the American Mathematical Society*, 186:419–458, 1973.

[11] J.-L. Brylinski. *Loop Spaces, Characteristic Classes, and Geometric Quantization.* Birkhäuser, 1993.

[12] A. Carboni, G. Janelidze, G. M. Kelly, and R. Paré. On localization and stabilization for factorization systems. *Appl. Categ. Structures*, 5(1):1–58, 1997.

[13] C. Cassidy, M. Hébert, and G. M. Kelly. Reflective subcategories, localizations and factorization systems. *J. Austral. Math. Soc. Ser. A*, 38(3):287–329, 1985.

[14] P. Clairambault and P. Dybjer. The biequivalence of locally cartesian closed categories and Martin-Löf type theories. In *Proceedings of the 10th international conference on Typed lambda calculi and applications*, TLCA'11, pages 91–106, Berlin, Heidelberg, 2011. Springer-Verlag.

[15] T. Coquand. ForMath: Formalisation of mathematics research project. `http://wiki.portal.chalmers.se/cse/pmwiki.php/ForMath/ForMath`.

[16] P. Deligne, P. Etingof, D. Freed, L. Jeffrey, D. Kazhdan, J. Morgan, D.R. Morrison, and E. Witten. *Quantum Fields and Strings, A course for mathematicians.* Amer. Math. Soc., 1999.

[17] D. Fiorenza, C. L. Rogers, and U. Schreiber. ∞-geometric prequantization. `http://ncatlab.org/schreiber/show/infinity-geometric+prequantization`.

[18] D. Fiorenza, H. Sati, and U. Schreiber. Extended higher cup-product Chern-Simons theories. arXiv:1207.5449, 2012.

[19] D. Fiorenza and U. Schreiber. ∞-Chern-Simons theory. `http://ncatlab.org/schreiber/show/infinity-Chern-Simons+theory`.

[20] D. Fiorenza, U. Schreiber, and J. Stacheff. Čech cocycles for differential characteristic classes. *Advances in Theoretical and Mathematical Phyiscs*, 16(1), 2012. arXiv:1011.4735.

[21] D. Freed. Dirac charge quantization and generalized differential cohomology. In *Surveys in Differential Geometry*, pages 129–194. Int. Press, 2000. arXiv:hep-th/0011220.

[22] M. Henneaux and C. Teitelboim. *Quantization of gauge systems.* Princeton University Press, 1992.

[23] C. Heunen, N. P. Landsman, and B. Spitters. Bohrification. In H. Halvorson, editor, *Deep Beauty*, pages 271–313. Cambridge University Press, 2011.

[24] M. Hopkins and I. Singer. Quadratic functions in geometry, topology, and M-theory. *J. Differential Geom.*, 70(3):329–452, 2005.

[25] I.V. Kanatchikov. Precanonical quantization of Yang-Mills fields and the functional Schroedinger representation. *Rept. Math. Phys.*, (53):181–193, 2004. arXiv:hep-th/0301001.

[26] C. Kapulkin, P. L. Lumsdaine, and V. Voevodsky. Univalence in simplicial sets. arXiv:1203.2553, 2012.

[27] A. Kock. *Synthetic differential geometry*, volume 51 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1981.

[28] R. Lavendhomme. *Basic concepts of synthetic differential geometry*, volume 13 of *Kluwer Texts in the Mathematical Sciences.* Kluwer Academic Publishers Group, Dordrecht, 1996. Translated from the 1987 French original, Revised by the author.

[29] W. Lawvere. Axiomatic cohesion. *Theory and Applications of Categories*, 19(3):41–49, 2007.

[30] E. Lerman. Orbifolds as stacks? *L'Enseign. Math.*, 56(3–4):315–363, 2010.

[31] P. L. Lumsdaine. Higher inductive types: a tour of the menagerie. `http://homotopytypetheory.org/2011/04/24/higher-inductive-types-a-tour-of-the-menagerie/`, April 2011.

[32] J. Lurie. *Higher topos theory.* Number 170 in Annals of Mathematics Studies. Princeton University Press, 2009.

[33] J. Lurie. On the classification of topological field theories. *Current Developments in Mathematics*, 2008:129–280, May 2009. arXiv:0905.0465.

[34] P. Martin-Löf. *Intuitionistic type theory.* Studies in Proof Theory 1. Bibliopolis, Naples, 1984.

[35] M.Göckeler and T. Schücker. *Differential geometry, gauge theories, and gravity.* Cambridge monographs on Mathematical Physics. 1987.

[36] I. Moerdijk and D. Pronk. Orbifolds, sheaves and groupoids. *K-theory*, 12:3–21, 1997.

[37] I. Moerdijk and G. E. Reyes. *Models for smooth infinitesimal analysis.* Springer-Verlag, New York, 1991.

[38] T. Nikolaus, U. Schreiber, and D. Stevenson. Principal ∞-bundle – general theory. arXiv:1207.0248.

[39] L. ORaifeartaigh and N. Straumann. Gauge theory: Historical origins and some modern developments. *Rev. Mod. Phys.*, 72:1–23, 2000.

[40] L. Randall. Higgs discovery: The power of empty space, 2012.

[41] C. Rogers. Higher geometric quantization. `http://www.crcg.de/wiki/images/2/21/CR_Higherstruct_2011.pdf`, 2011.

[42] C. Rogers. Higher symplectic geometry. 2011. PhD thesis, arXiv:1106.4068.

[43] H. Sati and U. Schreiber. *Mathematical Foundations of Quantum Field and Perturbative String Theory*, volume 83 of *Proceedings of symposia in pure mathematics.* American Mathematical Society, 2011.

[44] H. Sati, U. Schreiber, and J. Stacheff. Twisted differential string- and fivebrane-structures. *Communications in Mathematical Physics*, 2012. arXiv:0910.4001.

[45] U. Schreiber. Differential cohomology in a cohesive ∞-topos. `http://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos`, 2012.

[46] U. Schreiber. Twisted smooth cohomology in string theory. ESI Lecture series, `http://ncatlab.org/nlab/show/twisted+smooth+cohomology+in+string+theory`, 2012.

[47] U. Schreiber et al. ∞-cohesive site. `http://ncatlab.org/nlab/show/infinity-cohesive+site`.

[48] M. Shulman. Internalizing the external, or the joys of codiscreteness. `http://golem.ph.utexas.edu/category/2011/11/internalizing_the_external_or.html`.

[49] M. Shulman. Reflective subfibrations, factorization systems, and stable units. `http://golem.ph.utexas.edu/category/2011/12/reflective_subfibrations_facto.html`.

[50] M. Shulman. `Coq` code for cohesive homotopy type theory. `https://github.com/mikeshulman/HoTT/tree/master/Coq/Subcategories`, 2011.

[51] M. Shulman. Homotopy type theory IV: Higher inductive types. `http://golem.ph.utexas.edu/category/2011/04/homotopy_type_theory_vi.html`, April 2011.

[52] M. Shulman. Directed homotopy type theory. `http://golem.ph.utexas.edu/category/2012/06/directed_homotopy_type_theory.html`, 2012.

[53] M. Shulman. Minicourse on homotopy type theory. `http://www.math.ucsd.edu/~mshulman/hottminicourse2012/`, 2012.

[54] M. Shulman. The univalence axiom for inverse diagrams. arXiv:1203.3253, 2012.

[55] M. Shulman and P. L. Lumsdaine. Higher inductive types. In preparation.

[56] V. Voevodsky. Univalent foundations of mathematics. `http://www.math.ias.edu/~vladimir/Site3/Univalent_Foundations.html`.

[57] J. Wang. The moduli stack of G-bundles. 2011. arXiv:1104.4828.