Prequantum physics in a cohesive ∞-topos
Talk at Quantum Physics and Logic 2011

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Details and references at

http://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos
Motivation

Cohesive $\infty$-toposes

Geometric action functionals

Addendum – Technical details
Quantum Physics and Logic?
Physicist

Logician
Physicist: I have a theory!

Logician
Physicist

Logician: 

Nice!
Physicist

Logician:

I am an expert on theories.
Physicist: Which one is it?

Logician: \textit{Which one is it?}
Physicist: It has a field $\phi$...

Logician
Physicist:

It has a field $\phi$

with many indices!

Logician
Physicist

Logician: ?
Physicist: And the action functional... 

Logician
Physicist: And the action functional...

$S(\phi) = \cdots$
Physicist:

...has a kinetic term...

\[ S(\phi) = \frac{1}{2} \langle \phi, D\phi \rangle + \cdots \]
Physicist: ...and interaction given by...

\[ S(\phi) = \frac{1}{2} \langle \phi, D\phi \rangle + \cdots \]
Physicist:

...a cubic term...

$$S(\phi) = \frac{1}{2} \langle \phi, D\phi \rangle$$

$$+ \frac{1}{6} \langle \phi, [\phi, \phi] \rangle + \cdots$$
Physicist:  

...and a quartic term...  

$$S(\phi) = \frac{1}{2} \langle \phi, D\phi \rangle$$

$$+ \frac{1}{6} \langle \phi, [\phi, \phi] \rangle$$

$$+ \frac{1}{24} \langle \phi, [\phi, \phi, \phi] \rangle + \cdots$$

Logician:  

???
Physicist:  

...and a quintic term...  

\[ S(\phi) = \frac{1}{2} \langle \phi, D\phi \rangle \]

\[ + \frac{1}{6} \langle \phi, [\phi, \phi] \rangle \]

\[ + \frac{1}{24} \langle \phi, [\phi, \phi, \phi] \rangle \]

\[ + \frac{1}{120} \langle \phi, [\phi, \phi, \phi, \phi] \rangle + \cdots \]
Physicist: \[ S(\phi) = \frac{1}{2} \langle \phi, D\phi \rangle \]

\[ + \sum_{k=2}^{\infty} \frac{1}{(k + 1)!} \langle \phi, [\phi^k] \rangle \]

...and so on.

Logician

\[ ?^{\infty} \]
Physicist

Logician
Physicist: Is that it?

Logician: *Is that it?*
Physicist: *Next I quantize this!*

Logician
Physicist

Logician:

%#!&
Physicist

Enough! Go on!

Logician:

%#!&
Physicist

Logician:

%#!&
Physicist: Let's see...

Logician: *Let’s see...*
Physicist

Logician:

What are the models of your “theory”? 
Physicist: Everything!

Logician
Physicist

Logician: Everything?
Physicist: Yes, in physics:

- electromagnetism,
- Yang-Mills fields,
- gravity,
- electrons,
- quarks,
- gravitinos,
- $B$-fields,
- $C$-fields,
- RR-fields,
- Chern-Simons fields,
- Poisson $\sigma$-model fields,
- Courant $\sigma$-model fields,
- string fields,...
Physicist: Logician

Yes, in physics:
electromagnetism,
Physicist: Logician

Yes, in physics:
electromagnetism, Yang-Mills fields,
Physicist: Yes, in physics: electromagnetism, Yang-Mills fields, gravity,

Logician
Physicist: Logician

Yes, in physics:
electromagnetism, Yang-Mills fields, gravity, electrons,
Physicist:

Yes, in physics:
electromagnetism, Yang-Mills fields, gravity, electrons, quarks,
Physicist:

Yes, in physics:
electromagnetism, Yang-Mills fields, gravity, electrons, quarks, gravitinos,
Physicist: Logician

Yes, in physics:
electromagnetism, Yang-Mills fields, gravity, electrons, quarks, gravitinos, $B$-fields,
Physicist:

Yes, in physics:
electromagnetism, Yang-Mills fields, gravity, electrons, quarks, gravitinos, $B$-fields, $C$-fields,
Physicist: Yes, in physics:
electromagnetism, Yang-Mills fields, gravity, electrons, quarks, gravitinos, $B$-fields, $C$-fields, RR-fields,
Physicist: Logician

Yes, in physics: electromagnetism, Yang-Mills fields, gravity, electrons, quarks, gravitinos, $B$-fields, $C$-fields, RR-fields, Chern-Simons fields,
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Chern-Simons fields,
Poisson $\sigma$-model fields, Courant $\sigma$-model fields,
string fields,...
Physicist

Logician:

Hold it!
Physicist

Logician:

What’s going on here?
Is there a *formal theory* of

1. geometric action functionals;
2. their quantization

that produces the fundamental *physical theories*

of interest?
Notice that **quantized** field theory has been identified with a universal construction in *higher category theory*.
Crash course in higher category theory:
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An \((\infty, n)\)-category is a directed space in which \((k \leq n)\)-dimensional paths need not be reversible.
Crash course in higher category theory:

An \((\infty, n)\)-category is a directed space in which \((k \leq n)\)-dimensional paths need not be reversible.

So an \((\infty, 0)\)-category is an \(\infty\)-groupoid is a space.
Crash course in higher category theory:

An \((\infty, n)\)-category is a directed space in which \((k \leq n)\)-dimensional paths need not be reversible.

So an \((\infty, 0)\)-category is an \(\infty\)-groupoid is a space.

End of the crash course.
again:

Notice that quantized field theory has been identified with a universal construction in higher category theory.
again:

Notice that quantized field theory has recently been identified with a universal construction in higher category theory.

Sure. What?
cobordism theorem, roughly:

$$(\infty, n)\text{Cat}$$

All $(\infty, n)$-categories.
cobordism theorem, roughly:

\[
\text{SymMon}(\infty, n)\text{Cat} \quad (\infty, n)\text{Cat}
\]

Those with symmetric monoidal structure.
cobordism theorem, roughly:

\[ U : \text{SymMon}(\infty, n)\text{Cat} \overset{\text{forget}}{\longrightarrow} (\infty, n)\text{Cat} \]

Forget the structure.
cobordism theorem, roughly:

\[ U : \text{SymMon}(\infty, n)\text{Cat} \overset{\text{free}}{\leftarrow} \overset{\text{forget}}{\rightarrow} (\infty, n)\text{Cat} : F \]

Or generate it freely.
cobordism theorem, roughly:

Example:

\[ \text{Bord}_n \]

\((k \leq n)\)-paths are \(k\)-dimensional cobordisms, \((k > n)\)-paths are diffeomorphisms.
cobordism theorem, roughly:

Example:

\[ n \text{Vect} \]

Points are higher analogs of vector spaces, paths are higher linear maps.
cobordism theorem, roughly:

\[
\text{Bord}_n \rightarrow \mathbb{Z} \rightarrow n\text{Vect}
\]

A topological quantized field theory is a symmetric monoidal functor.
cobordism theorem, roughly:

Lurie (Baez-Dolan):

\[ F(\ast) \xrightarrow{\sim} \text{Bord}_n \xrightarrow{\mathbb{Z}} n\text{Vect} \]

A topological quantized field theory is a symmetric monoidal functor.
cobordism theorem, roughly:

Lurie (Baez-Dolan):

\[
\begin{array}{c}
F(\ast) \cong \text{Bord}_n \xrightarrow{Z} \text{nVect} \\
\ast \xrightarrow{Z(\ast)} \text{U}(\text{nVect})
\end{array}
\]

\[Z(\ast)\text{ is the } n\text{-space of states.}\]
So $n$-dimensional topological QFT is characterized by its $n$-space of states $Z(\ast)$. 
In nature, the space of states of a QFT is not random, but arises from quantization of geometric action functionals, such as

\[ S : \phi \mapsto \frac{1}{2} \langle \phi, D\phi \rangle + \sum_{k=2}^{\infty} \frac{1}{(k+1)!} \langle \phi, [\phi^k] \rangle. \]
Task:

1. formalize differential geometry;
2. formally derive these action functionals;
3. and their quantization.
Task:
1. formalize differential geometry;
2. formally derive these action functionals;
3. and their quantization.

Solution:
By a universal construction in *higher topos theory*...
Cohesive \((\infty, 1)\)-toposes
Set

The category of sets.
A category of geometric structures.
The underlying set.
The discrete (free) geometric structure.
Convention throughout: a morphism on top of another one denotes a left adjoint.

Disc ⊣ Γ
The codiscrete geometric structure.
The set of connected components.
If $\mathcal{H}$ a topos and $\Pi_0(\ast) \simeq \ast$: cohesive topos (Lawvere).
For instance $H = $ sheaves on smooth $\mathbb{R}^n$s: smooth manifolds and diffeological spaces.
We may consider this also in *higher topos theory*. 
The $(\infty, 1)$-category of $\infty$-groupoids ($\simeq$ spaces).
An $(\infty,1)$-category of geometric structures.
The underlying $\infty$-groupoid.
The discrete (free) geometric structure.
The codiscrete geometric structure.
The...
The $\infty$-groupoid of paths.
The $\infty$-groupoid of paths!
If $\mathbf{H}$ an $(\infty, 1)$-topos and $\Pi(\ast) \simeq \ast$: cohesive $(\infty, 1)$-topos.
For instance $\mathbf{H} = \infty$-stacks on smooth $\mathbb{R}^n$s: smooth $\infty$-groupoids.
The intrinsic existence of paths gives rise to an intrinsic dynamics...
III

Geometric action functionals
Reflect paths back to $\mathcal{H}$:

$$\left(\prod \dashv b \right) : \begin{array}{c} \mathcal{H} \\ \Gamma \end{array} \begin{array}{c} \rightarrow \ \infty \text{Grpd} \\ \leftarrow \ \Pi \end{array} \begin{array}{c} \leftarrow \ \text{Disc} \end{array} \begin{array}{c} \rightarrow \ \mathcal{H} \\ \end{array}$$
Reflect paths back to $\mathcal{H}$:

$$(\Pi \dashv \flat) : \mathcal{H} \xrightarrow{\Gamma} \infty \text{Grpd} \xleftarrow{\Pi} \text{Disc}$$

"$\flat$" is pronounced "flat"
A morphism

\[ \phi : \frac{\Pi(X)}{X} \rightarrow A \]

is flat parallel transport with values in \( A \).
Alternative perspective:

\[ \phi : \Pi(X) \rightarrow A \]

\[ X \rightarrow bA \]

is \textit{A-valued field with} vanishing field strength.
For instance we could have

\[ A = U(1) \]

the circle group.
For instance we could have

\[ A = \mathbf{B} U(1) \]

the one-object groupoid with

\[ \text{End}(\ast) = U(1). \]
For instance we could have

\[ A = \mathbf{B}^2 U(1) \]

the one-object 2-groupoid with

\[ \text{End}(\ast) = \mathbf{B} U(1). \]
For instance we could have

\[ A = B^{n+1} U(1) \]

the one-object \((n + 1)\)-groupoid with \(\text{End}(\ast) = B^n U(1)\).
A morphism

\[ X \rightarrow \mathbb{B}^{n+1} U(1) \]

encodes locally a closed \((n+1)\)-form on \(X\)

(and globally a bit more).
$X$ an $(n + 1)$-dim compact manifold:

*volume holonomy* functional

$$[X, \mathcal{B}^{n+1} U(1)] \xrightarrow{\int_X} U(1)$$

field space
Therefore every morphism
\[ c : A \rightarrow B^{n+1} U(1) \]
induces a functional
\[
\exp(i S_{CS_c}) : [\Sigma, bA] \xrightarrow{\int \Sigma b c} U(1)
\]
on flat \( A \)-valued fields.
After generalization to non-flat fields this is the geometric action functional that we are after ("higher Chern-Weil theory").
Refinement to \textit{non-flat} fields; by canonical factorization:

\[
\begin{array}{c}
\mathcal{B}^{n+1} U(1) \\
\mathcal{B}^{n+1} U(1)
\end{array}
\xrightarrow{\text{counit}} \begin{array}{c}
\text{flat fields} \\
\text{underlying cocycles}
\end{array}
Refinement to *non-flat* fields; by canonical factorization:

$$b \mathbf{B}^{n+1} U(1) \rightarrow \mathbf{B}^{n+1} U(1)_{\text{conn}} \rightarrow \mathbf{B}^{n+1} U(1)$$

flat fields \hspace{2cm} general fields \hspace{2cm} underlying cocycles
Refinement to \textit{non-flat} fields; by canonical factorization:

\[
\begin{align*}
\text{flat fields} & \quad \xrightarrow{\text{by canonical factorization}} \quad \text{general fields} \\
\mathbb{B}^{n+1} U(1) & \quad \xrightarrow{\text{underlying cocycles}} \quad \mathbb{B}^{n+1} U(1)_{\text{conn}} \\
& \quad \xrightarrow{\text{explained in Addendum}} \quad \mathbb{B}^{n+1} U(1)
\end{align*}
\]
By naturality, \( c \) gives:

\[
\begin{array}{c}
\mathcal{A} \\
\downarrow \\
\mathcal{B}^{n+1} U(1)
\end{array}
\]

\[
\begin{array}{c}
\mathcal{A} \\
\downarrow \\
\mathcal{B}^{n+1} U(1)
\end{array}
\]
with factorization

\[
\begin{array}{c}
\downarrow \quad \downarrow \\
A & \longrightarrow & B^{n+1} U(1)
\end{array}
\]
then consider

\[
\begin{array}{ccc}
\mathbb{b} A & \xrightarrow{\mathbb{b} \mathbf{c}} & \mathbb{b} \mathbf{B}^{n+1} \mathbf{U}(1) \\
& \downarrow & \\
\mathbf{A}_{\mathrm{conn}} & \xrightarrow{\mathbf{c}} & \mathbf{B}^{n+1} \mathbf{U}(1)_{\mathrm{conn}} \\
& \downarrow & \\
\mathbf{A} & \xrightarrow{\mathbf{c}} & \mathbf{B}^{n+1} \mathbf{U}(1)
\end{array}
\]
A morphism

$$X \rightarrow A_{\text{conn}}$$

encodes $A$-valued fields

(and globally a bit more).
Therefore every morphism
\[ \hat{c} : A_{\text{conn}} \to \mathbb{B}^{n+1} U(1)_{\text{conn}} \]
induces a functional
\[ \exp(iS_{CS_{\hat{c}}}) : [X, A_{\text{conn}}] \xrightarrow{\int_X \hat{c}} U(1) \]
on $A$-valued fields.
Example.

- $\mathfrak{g}$ an $L_\infty$-algebra
- invariant bilinear $\langle -, - \rangle$
- $A := \exp(\mathfrak{g})$
- $c := \exp(\langle -, - \rangle)$

$$S_{\hat{\mathcal{CS}}} (\phi) = \sum_{k=1}^{\infty} \frac{1}{(k + 1)!} \langle \phi, [\phi^k] \rangle.$$
But much more:

\[ \hat{c} : A_{\text{conn}} \to \mathbf{B}^{n+1} U(1)_{\text{conn}} \]

is the full "prequantum circle \((n+1)\)-bundle with connection on the moduli \(\infty\)-stack of fields"
Therefore next: apply higher geometric quantization to $\hat{c}$ to obtain $Z(*)$. 
Therefore next: apply higher geometric quantization to \( \hat{c} \) to obtain \( Z(*) \).

But not today.
End.
Addendum

Some technical details.

On the derivation of geometric action functionals.
A group object in $\mathsf{H}$. 
A group object: "cohesive $\infty$-group"."
A group object: “grouplike cohesive $A_\infty$-space”.
Equivalently its delooping: $\text{Hom}_{BG}(\ast, \ast) \simeq G$. 

$BG$
Equivalently its delooping: $\mathbf{BG}$
For instance for $U(1) := \mathbb{R}/\mathbb{Z}$. 
Since $U(1)$ is abelian, the delooping $B U(1)$ is a group object itself.
Hence there is a second delooping.
$B G$ $B^3 U(1)$

And a third.
\[ B^n U(1) \]

And so on.
Under geometric realization...
\[ BG \xrightarrow{c} K(\mathbb{Z}, n + 2) \]

... this classifies integral cohomology \([c] \in H^{n+2}(BG, \mathbb{Z}).\]
$BG \longrightarrow B^{n+1}U(1)$

$BG \xrightarrow{c} K(\mathbb{Z}, n+2)$

Such $[c]$ is a *universal characteristic class*. 
It takes equivalence classes \([P]\) of \(G\)-bundles to integral cohomology.
Let $c$ be a cohesive refinement.
This takes $G$-bundles $P$...
... to $\mathbb{B}^{n+1}U(1)$-bundles $c(P)$. 
Example: first Chern class of unitary bundles.
Lifted by determinant function.
This takes a unitary bundle $P$...
... to its determinant line bundle $\det(P) \otimes_{U(1)} \mathbb{C}$.
\[ \text{BG} \xrightarrow{\text{c}} \text{B}^{n+1}U(1) \]

\[ \text{BG} \xrightarrow{\text{c}} K(\mathbb{Z}, n + 2) \]

Generally, for any cohesive characteristic map...
\[ \mathbf{B}G \xrightarrow{\mathbf{c}} \mathbf{B}^{n+1}U(1) \]

\[ BG \xrightarrow{\mathbf{c}} K(\mathbb{Z}, n + 2) \]

... we may ask for a further differential refinement...
\[ \mathbf{BG}_{\text{conn}} \xrightarrow{c_{\text{conn}}} \mathbf{B}^{n+1}U(1)_{\text{conn}} \]

\[ \mathbf{BG} \xrightarrow{c} \mathbf{B}^{n+1}U(1) \]

\[ BG \xrightarrow{c} K(\mathbb{Z}, n + 2) \]

... which takes \textit{G-connections} to \textit{B}^{n}U(1)-connections.
To construct this, first differentially refine the coefficient object...
To that end, first consider one more delooping...

$B^{n+1}U(1)$

$B^{n+2}U(1)$
$B^{n+1} U(1)$

$\triangleright B^{n+2} U(1) \longrightarrow B^{n+2} U(1)$

..and the universal map it receives from the flat coefficients.
The homotopy fiber of this...
\[ \ldots \text{classifies flat } \mathbf{B}^{n+1}U(1)\text{-connections} \]
...whose underlying $\mathbb{B}^{n+1}U(1)$-bundle is trivial.
(By the universal property of homotopy pullbacks.)
But this are closed differential \((n + 2)\)-forms \(\omega \in \Omega_{\text{cl}}^{n+2}(X)\)!
Canonical example: pull back further along point inclusion...
\[ \mathbf{B}^{n+1} U(1) \rightarrow b_{dR} \mathbf{B}^{n+2} U(1) \rightarrow \ast \]

\[ \downarrow \]

\[ \mathbf{B}^{n+2} U(1) \rightarrow \mathbf{B}^{n+2} U(1) \]

... to recover \[ \mathbf{B}^{n=1} U(1) \simeq \Omega \mathbf{B}^{n+2} U(1) \]...
...equipped with universal form $\text{curv}$. 
These are the first steps in constructing a long fiber sequence...
\[
\begin{array}{c}
\mathbf{B}^{n+1}U(1) \xrightarrow{\text{curv}} b_{dR}\mathbf{B}^{n+2}U(1) \rightarrow \ast \\
\downarrow \quad \downarrow \\
\ast \rightarrow b\mathbf{B}^{n+2}U(1) \rightarrow \mathbf{B}^{n+2}U(1)
\end{array}
\]

... the next step...
...recovers the flat coefficients of $\mathbf{B}^{n+1} U(1)$. 
We learn: “flat” means \( \text{curv} \simeq 0 \).
Refine this by considering all curvature forms in $\Omega^{n+2}_{\text{cl}}(-)$. 
This finally gives the coefficients for $B^n U(1)$-connections...
With underlying bundle $\eta$ and curvature form $F$. 
In summary, so far, we have *abelian* differential cohomology.
Recall that the total map is still the canonical counit.
Therefore for a cohesive characteristic map $c$...
...we have a canonical refinement to \textit{flat} differential cohomology.
Hence a differential refinement of $c$ should fit into...
\[ \begin{align*}
\mathcal{B} \mathcal{B} G & \xrightarrow{bc} \mathcal{B} B^{n+1} U(1) \\
\mathcal{B} G_{\text{conn}} & \xrightarrow{c_{\text{conn}}} \mathcal{B} B^{n+1} U(1)_{\text{conn}} \\
\mathcal{B} G & \xrightarrow{c} \mathcal{B} B^{n+1} U(1)
\end{align*} \]

... a diagram of this form.
(In total we looked at this situation in the cohesive $\infty$-topos.)
\[ \mathcal{B} G_{\text{conn}} \xrightarrow{c_{\text{conn}}} \mathcal{B}^{n+1} U(1)_{\text{conn}} \]

So given the differential characteristic map...
...we canonically send $G$-connections $\nabla$...
...to $B^n U(1)$-connections $c_{conn}(\nabla)$. 
This statement refines to the full moduli stack \([X, BG_{\text{conn}}]\) of \(G\)-connections.
Consider the projection to the concretified 0-truncation.
If $\dim \Sigma = n + 1$, then this is $\simeq U(1)$...
\[
[\Sigma, B G_{\text{conn}}] \xrightarrow{c_{\text{conn}}} [\Sigma, B^{n+1} U(1)_{\text{conn}}] \xrightarrow{\int_{\Sigma}} U(1)
\]

... and the map computes the holonomy \( \int_{\Sigma} c_{\text{conn}}(\nabla) \).
\[ \exp(iS_{\text{CS}_c}) : \left[ \Sigma, \mathcal{B} G_{\text{conn}} \right] \xrightarrow{c_{\text{conn}}} \left[ \Sigma, \mathcal{B}^{n+1} U(1)_{\text{conn}} \right] \xrightarrow{\int_\Sigma} U(1) \]

... we call the \( \infty \)-Chern-Simons functional induced by \( c \).