Quantum Language via Linear Homotopy Types

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Abstract

It is well-appreciated that (intuitionistic but otherwise) classical (functional, programming) language is essentially the internal logic to cartesian closed categories (of data types), in particular to (higher) toposes — and that epistemology and other modality expressing physical observations and effects are reflected by (idempotent) co/monads on these categories.

We explore how this classical situation naturally extends to subsume quantum logic of quantum systems controlled and measured by classical observers:

Here doubly closed monoidal categories (of entangled quantum data types parameterized by classical data), such as higher tangent toposes, reflect in their linear slices the substructural (no-deleting/no-cloning) quantum coherence, while their base change co/monads between linear slices turn out to know everything about decoherent quantum measurement (wavefunction collapse), including the ancient Born rule as well as contemporary spider-fusion in ZX-calculus string diagrams.

For example, the infamous quantum measurement paradox resolves in the internal logic to the deferred measurement principle which obtains a rigorous proof as the Kleisli equivalence of the quantum necessity modality.

We close with application of this general theory to the concrete question of operating quantum-gates and -measurement on anyonic topological order in fractional quantum Hall systems.

course notes, following [8][9][15],
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Motivation

quantum circuit computing:	coherent arrangement (programming) of *individual* quantum processes on fin-dim Hilbert spaces ("qbits")
universal quantum computing:	this embedded into classical computing backdrop controlling and conditioned on quantum measurement (non-deterministic!)
universal quantum language:	a programming language for these tasks, effectively a formalization of quantum theory with fin-dim state spaces
practical promise:	potentially enormous compute speedups for certain problems like prime factorization; in any case: insights into fundamental physics
practical hurdle – intrinsic tension:	quantum systems amenable to local manipulation and observation are also quickly decohered by local noise
salvation strategy "QEC": quantum error correction:	on a highly redundant quantum register, classical computer continuously measures error syndromes and intervenes accordingly
salvation strategy "TQC": quantum error protection:	utilize non-local "topological" ground states intrinsically protected against noise and operated adiabatically
practice and perspective:	currently almost all credible activity towards QEC, but TQC plausibly inevitable for the real deal
both need much more fundamental development for commercial-value scale quantum computing,	- QEC needs certifiable universal quantum programming languages - TQC needs better formalization of topological quantum systems

Both are issues of **more accurate quantum language**.

Here is a **concrete motivating question** for the following development, whose answer we will have explained by the end of this course:

Topological(ly ordered) quantum materials are effectively governed by topological quantum field theory (TQFT), hence by a form of "generally covariant" QFT (like quantum gravity is expected to be).

In such generally covariant systems "bulk diffeos are gauge symmetries", while "boundary diffeos are physical evolutions".

$$1 \to \mathrm{BulkGaugeDiffeos} \hookrightarrow \mathrm{Diffeos} \twoheadrightarrow \mathrm{BoundaryDiffeos} \to 1$$

Question: *What is the most usefully pedantic* (⇒ programmable!) *description of* operable & measurement quantum gates *that applies to such topological systems?*

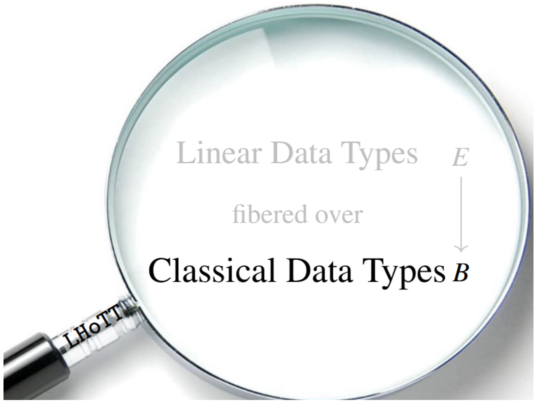
Synopsis

Classical Computational Trinitarianism:
(Constructive but otherwise) Classical Logic
& Functional Programming Laguages
have "operational semantics" in
LCC categories of classical data types.

Question:
What becomes of this statement
as we generalize to allow also
quantum logic & quantum computing?

Answer:

Quantum Computational Trinitarianism:
Classical/Quantum Logic & Universal
Quantum Programming Languages
have operational semantics
in categories of linear data types
fibered over classical data types:
classically controlled quantum data.



Remarkably I:
Systematic unwinding of the
definable monads on such quantum data
reveals all about fundamental quantum effects:
quantum measurement, state collapse, Born rule, ...
(as an incarnation of Grothendieck's motivic yoga!).

Remarkably II:
Then generalizing to
higher classical data types
namely: homotopy types
captures topological quantum phenomena.

whence generally we are dealing
with dependent linear homotopy types
verbalizing topological quantum data.

Next to explain what all this means...

1 Categories of Classical Data Types: Propositions, Quantifiers, Modalities, Effects

The simple but far-reaching **Paradigm of Data Types** (jargon: just *types*):

All data d is to be of some type D

notation $d : D$,
specifying how to construct & read the data.

This makes a **category** Type,
whose objects are data types,
and whose morphisms are programs.

Programs p constructs output- from input data
notation $i : I \;\vdash\; p(i) : O$,
respecting their data type specification.

$$\begin{tikzcd}
\text{input data type} & I \arrow[r, "p", "\text{program}"'] & O & \text{output data type}
\end{tikzcd}$$

Type formation. Given data types L, R:

data of **pair type** $(l, r) : L \times R$ is
constructed by providing $l : L$ and $r : R$,
& extracted by retaining either, so that
$L \times R$ is the **cartesian product** in Type.

data of **function type** $f : L \to R$ is
constructed by providing $f(l) : R$ for $l : L$,
& extracted by evaluating at $l : L$
so that $L \to R$ is the **internal hom** in Type.

$$\text{deduction rule: } \frac{(\gamma, l) : \Gamma \times L \quad \vdash \quad p_\gamma(l) : R}{\gamma : \Gamma \quad \vdash \quad p_\gamma : L \to R} \quad \Leftrightarrow \quad \frac{\Gamma \times L \xrightarrow{p} R}{\Gamma \xrightarrow{\widetilde{p}} [L, R]} \quad \text{product/hom adjointness}$$

This makes Type a **cartesian closed category** (CCC).

Typing paradigm to be applied relentlessly:

Data types D themselves are data and hence of some type, $D : \mathrm{Type}_i$,
hence also $\mathrm{Type}_i : \mathrm{Type}_{i+1}$, and so on.
This makes Type a category with a **hierarchy of universes**.

Hence programs may *output data types*,
jargon: **dependent types**
notation: $\gamma : \Gamma \;\vdash\; D_\gamma : \mathrm{Type}_i$

$$\begin{tikzcd}
D_\gamma \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & D \arrow[r] \arrow[d, ->>] \arrow[dr, phantom, "\text{(pb)}"] & \widehat{\mathrm{Type}_i} \arrow[d, ->>] & \text{dependent types} \\
& & & \text{fibration} \\
\ast \arrow[r, "\gamma"] & \Gamma \arrow[r, "\vdash D"] & \mathrm{Type}_i & \text{base type}
\end{tikzcd}$$

& data of varying type: $\gamma : \Gamma, \;\; i_\gamma : I_\gamma \;\vdash\; p_\gamma(i) : D_\gamma$

This makes CCC **slice categories** $\mathrm{Type}_{/\Gamma}$,
hence makes Type **locally cartesian closed** (LCCC).

$$\begin{tikzcd}
I \arrow[rr, "p", "\text{flagged program}"'] \arrow[dr, ->>] & & D \arrow[dl, ->>] \\
\Gamma\text{-dependent input data} & \Gamma & \Gamma\text{-dependent output data}
\end{tikzcd}$$

Dependent type formation. Now given dependent data types $l : L \;\vdash\; R_l : \mathrm{Type}$:

data of **dependent pair type** $(l, r) : \coprod\limits_{l : L} R_l$

is constructed by providing $l : L$ and $r_l : R_l$

data of **dependent function type** $f : \prod\limits_{l : L} R_l$

is constructed by providing $f(l) : R_l$ for $l : L$

such that these form the left/right adjoint
base change functors between slice categories:

$$\begin{tikzcd}
\mathrm{Type}_{/\Gamma} \arrow[rr, bend left, "\coprod_\Gamma"] \arrow[rr, bend right, "\prod_\Gamma"'] & & \mathrm{Type} \arrow[ll, "(-)\times\Gamma"'] & \begin{array}{c}\text{dependent co-product}\\ \text{forms total space}\\[1ex] \text{pullback forms}\\ \text{trivial fibration}\\[1ex] \text{dependent product}\\ \text{forms space of sections}\end{array}
\end{tikzcd}$$

with $\perp$ between $\coprod_\Gamma$ and $(-)\times\Gamma$, and $\perp$ between $(-)\times\Gamma$ and $\prod_\Gamma$.

The archetypical example is Type := Set with a hierarchy of Grothendieck universes.

Logic of data types.

Certificates for properties P of Γ-data
are data of *sub*-type $P : \mathrm{Type}_\Gamma$
("**propositions are types**")

On such propositional types,
the above type formation rules
implement **first-order logic**
constructively ("BHK interpretation").

Thereby **any program** outputting $p(i) : O$
is also a constructive proof/certificate
that $p(i)$ adheres to the specification O!

$$\begin{tikzcd}
P := \left\{ \gamma : \Gamma \,\middle|\, \gamma \text{ verifies } P \right\} \arrow[d, hook] \\
\Gamma
\end{tikzcd}$$

Propositions as fibrations whose fibers are either empty or singletons.

this data	**certifies that**
$\gamma : P_1 \times P_2$	P_1 and P_2 hold for γ
$\gamma : P_1 \sqcup P_2$	P_1 or P_2 hold for γ
$c : \coprod_\Gamma P$	there exists γ for which P holds
$c : \prod_\Gamma P$	for all γ, P holds
$c : \prod_\Gamma (P_1 \to P_2)$	P_1 implies P_2

Consider then the description of some **measurement** with $W : \mathrm{Type}_i$ of *possible measurement outcomes*, hence of "possible worlds" seen after the measurement; and some proposition $P : \mathrm{Type}_{/W}$ about the outcomes.

measurement outcome / possible world measured $\quad w : W$

proposition about the possible measurement outcomes/worlds $\quad P \in \mathrm{Type}_{/W}$

Recalling that every adjunction induces a monad and a comonad

left adjoint

$$\begin{tikzcd}
\mathcal{C} := L \circ R \arrow[r, phantom] & \mathrm{Type}' \arrow[loop left] \arrow[r, bend right, "R"'] \arrow[r, phantom, "\perp"] & \mathrm{Type} \arrow[l, bend right, "L"'] \arrow[loop right] & R \circ L \equiv: \mathcal{E}
\end{tikzcd}$$

induced co-monad

right adjoint

induced monad

the dependent type formation/base change induces a **pair of adjoint pairs** of (co)monads

This carries rich logical meaning:

possibly **dependent co-product** **randomly**

$$\begin{tikzcd}
\Diamond_W \arrow[r, phantom, "\text{actual data} \quad \perp"] & \mathrm{Type}_{/W} \arrow[loop above, "\Diamond_W"] \arrow[loop below, "\square_W"] \arrow[r, bend left, "\coprod_W"] \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\prod_W"'] & \mathrm{Type} \arrow[l, "(-)\times W" description] \arrow[loop above, "☆_W"] \arrow[loop below, "\bigcirc_W"] \arrow[r, phantom, "\perp \quad \text{potential data}"] & {}
\end{tikzcd}$$

necessarily **dependent product** **indefinitely**

On the left, inspection shows that:

$\square_W P$ means:	P_w means:	$\Diamond_W P$ means:
"*P does* or *is known to* hold **necessarily**" namely, no matter which world w is measured.	"*P does* or *is known* to hold **actually**" namely for the *given* world w measured.	"*P does* or *is known* to hold **possibly**" namely for *some* possibly measured world w.

Moreover, the (co)units $\mathrm{ret}^{\Diamond}$ ($\mathrm{obt}^{\square}$) of these (co)monads reflect the expected logical entailments of these modal propositions:

necessarily P **entails** **actually** P **entails** **possibly** P

$$\square_W P \xrightarrow{\;\mathrm{obt}^{\square_W}_P\;} P \xrightarrow{\;\mathrm{ret}^{\Diamond_W}_P\;} \Diamond_W P$$

$$w : W \;\vdash\; \prod_{w' : W} P_{w'} \xrightarrow{(d_{w'} : W) \,\mapsto\, d_w} P_w \xrightarrow{d_w \,\mapsto\, (w,\, d_w)} \coprod_{w' : W} P_{w'}$$

Logic with such operators $\Diamond \dashv \square$ is known as ("S5") **modal logic**, and the operators are known as **modalities** — read: "modes of being (true)".

The topic of modal logic is ancient and much studied, and yet its above emergence from dependent type formation/base change remains under-appreciated.

But we will see that this perspective is the golden path to proper quantum logic, knowing about quantum measurement.

On the right, $☆_W \dashv \bigcirc_W$ known as *(co)reader (co)monads*, express computations with **potential** data, that remain **indefinite** up to specification of **random** readout (in the sense of RAM): namely of the measurement outcome w.

The monad's type formation: $\quad \begin{array}{ccccc} \bigcirc_W & : & \mathrm{Type} & \longrightarrow & \mathrm{Type} \\ & & D & \longmapsto & [W,\, D] \end{array} \quad$ and its operations:

$$\begin{tikzcd}
{[W,\,[W,\,D]]} \simeq {[W \times W,\, D]} \arrow[r, "\mathrm{join}^{\bigcirc_W}_D", "\equiv {[\mathrm{diag}_W,\, D]}"'] & {[W,\, D]} & {[*,\, D]} \simeq D \arrow[l, "\mathrm{ret}^{\bigcirc_W}_D"', "\equiv \mathrm{const}"] \\
W \times W & W \arrow[l, "\mathrm{diag}_W"] \arrow[r, "\exists !"'] & *
\end{tikzcd}$$

binding of W-indefiniteness effects

$$\mathtt{bind}^{\bigcirc_W}_{D,D'} \;:\; \big(D \to (W \to D')\big) \xrightarrow{\qquad} \big((W \to D) \to (W \to D')\big)$$

$$\mathtt{bind}^{\bigcirc_W}_{D,D'} \;\equiv\; \Big(d \mapsto \big(w \mapsto d'_w(d)\big)\Big) \mapsto \Big(\big(w \mapsto d_w\big) \mapsto \big(\, w \;\mapsto\; d'_w(d_w)\big)\Big)$$

program producing output depending on a global W-parameter

global parameter gets passed to all subsequent programs

We will see in the quantum case that the above adjunction is in fact mondadic (for finite W) whence in **quantum modal logic** left/right perspectives are essentially equivalent, providing two perspectives on measurement: on the left as for parameterized quantum circuits, on the right as formalized in ZX-calculus.

In order to achieve this, all we need to do now is pass to dependent *linear* data types...

2 Categories of Quantum Data Types: Quantization, Classicization, Entanglement

Basic among the rules for handling classical data are the seemingly tautological "*structural rules*" which say that:

Idea	**Syntax**	**Semantics**	
data may be *systematically duplicated*	C $\dfrac{\Gamma,\ p_1 : P,\ p_2 : P \ \vdash\ t_{p_1,p_2} : T}{\Gamma,\ p : P \ \vdash\ t_{p,p} : T}$ Contraction rule	$\dfrac{\Gamma \times P \times P \xrightarrow{\vdash t} T}{\Gamma \times P \xrightarrow{\mathrm{id}_\Gamma \times \mathrm{diag}_P} \Gamma \times P \times P \xrightarrow{\vdash t} T}$ Diagonal map (cloning)	**structural rules for classical data**
data may be *systematically discarded*	W $\dfrac{\Gamma \ \vdash\ P : \mathrm{Type} \qquad \Gamma \ \vdash\ t : T}{\Gamma,\ P \ \vdash\ t : T}$ Weakening rule	$\dfrac{\Gamma \xrightarrow{\vdash t} T}{\Gamma \times P \xrightarrow{\mathrm{pr}_\Gamma} \Gamma \xrightarrow{\vdash t} T}$ Projection map (deletion)	

But a hallmark of **coherent quantum data** is that these rules do *not apply*: the **no-cloning/no-deleting** property.

Computationally this means that coherent quantum programs invoke any input variable $d : \mathcal{H}$

at least once (not discarding it)
at most once (not duplicating it)
hence exactly once: **linearly**!

Therefore the logic of coherent quantum data is known as (*sub-structural*) *linear logic.*

The archetypical category of coherent quantum data is $\mathrm{Mod}_{\mathbb{C}}^{(\mathrm{fd})}$: (finite-dimensional) vector spaces.

Hence the quantum version of the BHK paradigm identifies quantum data certificates/propositions with quantum sub-types – this yields **Birkhoff-vonNeumann quantum logic**:

	proposition	**logical "and"**	**logical "or"**
in abstract slice $\mathrm{QuType}_{/\mathcal{H}}$	sub-object	categorical product	truncated coproduct
diagram	$\begin{tikzcd} \mathcal{P} \arrow[d, hook] \\ \mathcal{H} \end{tikzcd}$	$\begin{tikzcd} \mathcal{P}_1 \arrow[dr, hook] & \mathcal{P}_1 \cap \mathcal{P}_2 \arrow[l] \arrow[r] \arrow[d, hook] & \mathcal{P}_2 \arrow[dl, hook] \\ & \mathcal{H} & \end{tikzcd}$	$\begin{tikzcd} \mathcal{P}_1 \arrow[r] \arrow[dr, hook] & \langle \mathcal{P}_1, \mathcal{P}_2 \rangle \arrow[d, hook] & \mathcal{P}_2 \arrow[l] \arrow[dl, hook] \\ & \mathcal{H} & \end{tikzcd}$
in slice of vector spaces $\mathbb{C}\mathrm{Mod}^{\mathrm{fd}}_{/\mathcal{H}}$	linear subspace	subspace intersection	linear span

Entanglement.
But the above shows that cloning/deleting is encoded in the diagonal/projection map of the cartesian product $\times$ of data types.

Therefore the *coherent product* $\otimes$ of quantum data must be a non-cartesian hence a (symmetric) *tensor product.*

In fact, the cartesian product on $\mathbb{C}\mathrm{Mod}^{\mathrm{fd}}$ coincides with the coproduct, both being the *direct sum* $\times = \oplus$.

cartesian product	$\times = \oplus$	describes **parallelization**	of quantum data
tensor product	$\otimes$	describes **coupling/entangling**	

With respect to $\otimes$, the category $\mathbb{C}\mathrm{Mod}$ is closed, with internal hom $(\text{-}) \multimap (\text{-})$ being the linear space of linear maps:

$$\dfrac{\mathcal{H}_1 \otimes \mathcal{H}_2 \longrightarrow \mathcal{H}_3}{\mathcal{H}_1 \longrightarrow (\mathcal{H}_2 \multimap \mathcal{H}_3)}$$

but it is *not* closed with respect to $\times = \oplus$, hence does *not reflect classical logic* anymore.

Towards combined classical/quantum logic.
But in the vein of Bohr's dictum, one eventually needs classical logic to report on results of quantum logic.

At least, classical and quantum data types are related by a **quantization** $\dashv$ **classicization adjunction**

$$\begin{tikzcd} S \arrow[r, mapsto, "\text{quantization}"] & \mathbb{C}\langle S \rangle \ \text{linear span} \\ \mathrm{Set} \arrow[r, bend left, "Q"] \arrow[r, phantom, "\perp"] & \mathbb{C}\mathrm{Mod} \arrow[l, bend left, "C"] \arrow[loop right, "!"] & \text{exponential modality} \\ \mathrm{Hom}(\mathbb{C}, \mathcal{H}) \ \text{underlying set} & \mathcal{H} \arrow[l, mapsto, "\text{classicization}"] \end{tikzcd}$$

which is suitably monoidal so that the induced monad/modality "!" takes direct sums to tensor products

$$!(\mathcal{H}_1 \oplus \mathcal{H}_2) \ \simeq \ (!\,\mathcal{H}_1) \otimes (!\,\mathcal{H}_2)$$
$$!\,0 \ \simeq \ \mathbb{1}$$

This allows a "hack" where some classical logic is re-imported as "exponentiated quantum logic".

While this approach has found much attention by linear logicians, we will next see a more natural & encompassing approach...

Namely, in general **quantum data is parameterized by classical data**,

notoriously so by the classical measurement outcomes (“worlds”)

$$\begin{tikzcd}
 & \mathcal{H}\otimes\mathrm{QBit} \arrow[rr, "\sum_b |\psi_b\rangle\otimes|b\rangle \longmapsto |\psi_1\rangle\otimes|1\rangle"] \arrow[dd] & & \mathcal{H} \arrow[dd] \\
\mathcal{H}\otimes\mathrm{QBit} \arrow[rr, "\sum_b |\psi_b\rangle\otimes|b\rangle \longmapsto |\psi_0\rangle\otimes|0\rangle"] \arrow[dd] & & \mathcal{H} \arrow[dd] & \\
 & 1 \arrow[rr, equal] & & 1 \\
0 \arrow[rr, equal] & & 0 &
\end{tikzcd}$$

classical worlds

quantum measurement process ⟶

but also by parameters for quantum state preparation

$$\begin{tikzcd}
 & \mathcal{H}\otimes\mathrm{QBit} \arrow[dd] & & \mathcal{H} \arrow[ll, "|\psi\rangle\otimes|1\rangle \longleftarrow\!\shortmid |\psi\rangle"'] \arrow[dd] \\
\mathcal{H}\otimes\mathrm{QBit} \arrow[dd] & & \mathcal{H} \arrow[ll, "|\psi\rangle\otimes|0\rangle \longleftarrow\!\shortmid |\psi\rangle"'] \arrow[dd] & \\
 & 1 \arrow[rr, equal] & & 1 \\
0 \arrow[rr, equal] & & 0 &
\end{tikzcd}$$

classical parameters

⟵ conditional state preparation

This means that coherent quantum data is *fibered* or *bundled* over classical data, with archetypical category that of vector bundles over any sets:

$$\begin{tikzcd}
\mathcal{H}_\bullet \arrow[d] \\
W
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{H}_w \arrow[d] \\
\{w\}
\end{tikzcd}$$

bundle of quantum data

classical worlds / parameters

single quantum data type

over definite world/param.

More generally, quantum data may transform under adiabatic movement of classical parameters. This makes it form (higher) **flat** vector bundles over (higher) groupoids, aka (higher) *local systems*. More on this in §4. For now we have that:

Syntax	**Semantics**	
Types	**Category**	**Morphisms**
ClType classical types	Set sets	$W \xrightarrow{f} W'$ maps
QuType linear types	ℂMod vector spaces	$\mathcal{H} \xrightarrow{\phi} \mathcal{H}'$ linear maps
QuType_W W-dependent linear types	$\mathbb{C}\mathrm{Mod}^W$ W-indexed vector space	$\begin{bmatrix}\mathcal{H}_\bullet\\ \downarrow\\ W\end{bmatrix} \begin{matrix}\xrightarrow{\phi_\bullet}\\ =\!=\!=\end{matrix} \begin{bmatrix}\mathcal{H}'_\bullet\\ \downarrow\\ W\end{bmatrix}$ W-indexed linear maps
Type linear bundle types	$\int\limits_{W:\mathrm{Set}} \mathbb{C}\mathrm{Mod}^W$ Grothendieck construction	$\begin{bmatrix}\mathcal{H}_\bullet\\ \downarrow\\ W\end{bmatrix} \begin{matrix}\xrightarrow{\phi_\bullet}\\ \xrightarrow{f}\end{matrix} \begin{bmatrix}\mathcal{H}'_\bullet\\ \downarrow\\ W'\end{bmatrix}$ map covered by indexed linear map

Remarkably, the category of

parameterized quantum data is *both*:

1. cartesian closed – expressing classical logic
2. tensor-closed – expressing quantum logic

(jargon: *doubly closed monoidal*)

Concretely, as shown on the right,

The **classical product** is the “**external direct sum**”:
the product of classical base types, covered by fiberwise direct sum of quantum types.

The **quantum product** is the “**external tensor product**”:
the product of classical base types, covered by fiberwise tensor product of quantum types.

The internal **quantum hom** “$\multimap$”
is the internal hom of pure quantum data fibered over the hom-set of classical parameters.

But the internal **classical hom** “$\to$”
is surprisingly rich: the
base is set of combined classical/quantum maps & the fibers are pullbacks of the codomain.

We see next that this has
interesting consequences.

(It is an elementary exercise to check all this, but it has not been widely appreciated.)

Pair types $\mathrm{Hom}(X \cdot X',\, X'')$	**Function types** $\simeq\ \mathrm{Hom}\big(X,\, [X', X'']\big)$
$W \times W'$ cartesian product	$W' \to W''$ set of maps
$\bigoplus\limits_S \mathcal{H}'$ direct sum	$\natural(\mathcal{H}' \to \mathcal{H}'')$ set of linear maps
$\mathcal{H} \otimes \mathcal{H}'$ tensor product	$\mathcal{H}' \multimap \mathcal{H}''$ vector space of linear maps
$\bigoplus\limits_S \mathcal{H}'_\bullet$ direct sum	$\prod\limits_w \natural(\mathcal{H}'_w \to \mathcal{H}''_w)$ set of indexed linear maps
$\mathcal{H} \otimes \mathcal{H}'_\bullet$ index-wise tensor product	$\prod\limits_w (\mathcal{H}'_w \multimap \mathcal{H}''_w)$ vector space of indexed linear maps
$\begin{bmatrix}\mathcal{H}_\bullet\\ \downarrow\\ W\end{bmatrix} \times \begin{bmatrix}\mathcal{H}'_\bullet\\ \downarrow\\ W'\end{bmatrix}$ $= \begin{bmatrix}\mathcal{H}_\bullet \oplus \mathcal{H}'_\bullet\\ \downarrow\\ W \times W'\end{bmatrix}$ external direct sum	$\begin{bmatrix}\mathcal{H}'_\bullet\\ \downarrow\\ W'\end{bmatrix} \to \begin{bmatrix}\mathcal{H}''_\bullet\\ \downarrow\\ W''\end{bmatrix} =$ $\begin{bmatrix}\prod_{w'} \mathcal{H}''_{f(w')}\\ \downarrow\\ (f : W' \to W'') \times\\ \prod\limits_{w'} \natural(\mathcal{H}'_{w'} \to \mathcal{H}''_{f(w')})\end{bmatrix}$
$\begin{bmatrix}\mathcal{H}_\bullet\\ \downarrow\\ W\end{bmatrix} \otimes \begin{bmatrix}\mathcal{H}'_\bullet\\ \downarrow\\ W'\end{bmatrix}$ $= \begin{bmatrix}\mathcal{H}_\bullet \otimes \mathcal{H}'_\bullet\\ \downarrow\\ W \times W'\end{bmatrix}$ external tensor product	$\begin{bmatrix}\mathcal{H}'_\bullet\\ \downarrow\\ W'\end{bmatrix} \multimap \begin{bmatrix}\mathcal{H}''_\bullet\\ \downarrow\\ W''\end{bmatrix} =$ $\begin{bmatrix}\prod_{w'} (\mathcal{H}_{w'} \multimap \mathcal{H}''_{f(w')})\\ \downarrow\\ (f\ :\ W' \to W'')\end{bmatrix}$

The categorial structure of parameterized quantum data. Having motivated *parameterized* quantum data (Type) as a natural unification of purely classical data (ClType) & purely quantum data (QuType), we recognize the latter as full (co)reflective subcategories:

purely classical data types

$$\begin{tikzcd}
W & \longmapsfrom & {\begin{bmatrix}\mathcal{H}_\bullet \\ \downarrow \\ W\end{bmatrix}} \\
\mathrm{ClType} \arrow[rr, hook, shift right=2] \arrow[rr, phantom, "\bot\ \top"] & & \mathrm{Type} \arrow[ll, shift right=2] \arrow[loop right, "\natural"] \\
W & \longmapsto & {\begin{bmatrix}0_\bullet \\ \downarrow \\ W\end{bmatrix}}
\end{tikzcd}$$

classically (modality)

purely quantum data types

$$\begin{tikzcd}
\bigoplus_w \mathcal{H}_w & \longmapsfrom & {\begin{bmatrix}\mathcal{H}_\bullet \\ \downarrow \\ W\end{bmatrix}} \\
\mathrm{QuType} \arrow[rr, hook, shift right=2] \arrow[rr, phantom, "\bot"] & & \mathrm{Type} \arrow[ll, shift right=2] \arrow[loop right, "\triangleright"] \\
\mathcal{H} & \longmapsto & {\begin{bmatrix}\mathcal{H} \\ \downarrow \\ *\end{bmatrix}}
\end{tikzcd}$$

quantumly (modality)

Quantum/Classical Data Types		**Quantum/Classical Maps**
General **bundles of linear types**	$\natural \circlearrowright \mathrm{Type} \circlearrowleft \triangleright$ $\begin{bmatrix}\mathcal{H}_\bullet \\ \downarrow \\ W\end{bmatrix}$	$\mathcal{H}_\bullet \longrightarrow \mathcal{H}'_\bullet$ $\begin{bmatrix}\mathcal{H}_\bullet \\ \downarrow \\ W\end{bmatrix} \begin{matrix}\xrightarrow{\phi_\bullet} \\ = \end{matrix} \begin{bmatrix}\mathcal{H}'_{f(\bullet)} \\ \downarrow \\ W\end{bmatrix} \begin{matrix}\rightarrow \\ \xrightarrow{f} \end{matrix} \begin{bmatrix}\mathcal{H}'_\bullet \\ \downarrow \\ W'\end{bmatrix}$
Purely **classical types** (bundles of zeros)	$\mathrm{ClType} \equiv \mathrm{Type}^\natural$ $\begin{bmatrix}0_\bullet \\ \downarrow \\ W\end{bmatrix}$	$W \longrightarrow W'$ $\begin{bmatrix}0_\bullet \\ \downarrow \\ W\end{bmatrix} \begin{matrix}\xrightarrow{0_\bullet} \\ \xrightarrow{f} \end{matrix} \begin{bmatrix}0_\bullet \\ \downarrow \\ W'\end{bmatrix}$
Purely **quantum types** (bundles over point)	$\mathrm{QuType} \equiv \mathrm{Type}^\triangleright$ $\begin{bmatrix}\mathcal{H} \\ \downarrow \\ *\end{bmatrix}$	$\mathcal{H} \longrightarrow \mathcal{H}'$ $\begin{bmatrix}\mathcal{H} \\ \downarrow \\ *\end{bmatrix} \begin{matrix}\xrightarrow{\phi} \\ \longrightarrow \end{matrix} \begin{bmatrix}\mathcal{H}' \\ \downarrow \\ *\end{bmatrix}$

Quantization modality. By composing these adjunctions with those of the doubly-closed monoidal structure we obtain more adjunctions, and something interesting happens:

First, composing the Cartesian hom-adjunction for the tensor unit $\mathbb{1}$ with the classicality-coreflection gives another adjunction between linear bundle types and purely classical types.

$$\begin{tikzcd}
W & & \longmapsto & & {\begin{bmatrix}\mathbb{1}_\bullet \\ \downarrow \\ W\end{bmatrix}} \text{ unit bundle} \\
\mathrm{ClType} \arrow[rr, hook, shift left=2] \arrow[rr, phantom, "\bot"] \arrow[rrrr, bend left=40] & & \mathrm{Type} \arrow[ll, shift left=2, "\natural"] \arrow[rr, shift left=2, "\mathbb{1}\times(-)"] \arrow[rr, phantom, "\bot"] & & \mathrm{Type} \arrow[ll, shift left=2, "\mathbb{1}\to(-)"] \arrow[llll, bend left=40] \\
(w : W) \times \natural(\mathbb{1} \to \mathcal{H}_w) & & \longmapsfrom & & {\begin{bmatrix}\mathcal{H}_\bullet \\ \downarrow \\ W\end{bmatrix}}
\end{tikzcd}$$

set of all vectors in all the fibers

Then, further composing with the reflection of purely quantum types reveals an adjunction between classical and quantum data...
...which recovers the quantization / classicization adjunction and
hence the **exponential modality**!
(In this form the adjunction has an evident generalization to higher quantum structures, where quantization becomes the *suspension spectrum* functor Σ^∞_+.)

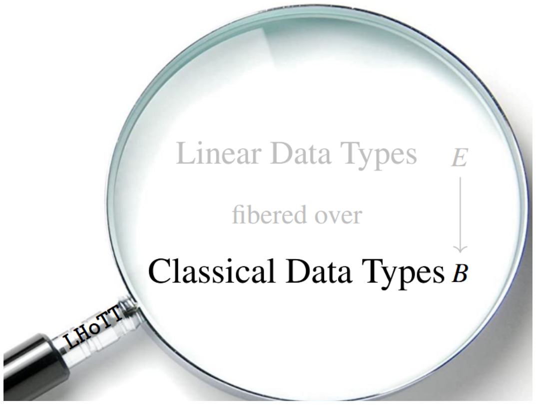
$$\begin{tikzcd}
W & & \longmapsto & & \bigoplus_W \mathbb{1} \\
\mathrm{ClType} \arrow[rr, shift left=2, "\mathbb{1}\times(-)"] \arrow[rr, phantom, "\bot"] \arrow[rrrr, bend left=40, "{\text{quantized } Q \equiv \Sigma^\infty_+ \text{ motive}}"] & & \mathbf{Type} \arrow[ll, shift left=2, "\natural(\mathbb{1}\to(-))"] \arrow[rr, shift left=2, "\triangleright"] \arrow[rr, phantom, "\bot"] & & \mathrm{QuType} \arrow[ll, hook', shift left=2] \arrow[loop right, "!"] \arrow[llll, bend left=40, "{C \equiv \Omega^\infty_+ \text{ classicized}}"] \\
\natural(\mathbb{1} \to \mathcal{H}) & & \longmapsfrom & & \mathcal{H}
\end{tikzcd}$$

exponential modality

Note that while this demonstrates **backwards-compatibility with linear logic**,
we no longer *need* the exponential modality to combine classical with quantum logic – we can now speak about the bold middle part of the above composite adjunction, right away.

Instead of importing classical logic into quantum logic by "exponentiating", we have hereby obtained an ambient classical control-logic around quantum data, naturally reflecting **Bohr's dictum** ("*However far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms*").

In fact, with the above modalities — elementary as they are — we obtain a "**platonic quantum microscope**" that logically resolves quantum properties of superficially classical-looking logical structure:
Taken as a cartesian closed category, Type interprets classical logic – but the modalities $\natural$, $\triangleright$ resolve inside each such superficially classical data type quantum aspects subject to quantum logic. We will see how this is useful...



3 Monads of Quantum Effects:
Quantum Measurement, Collapse, Paradoxes

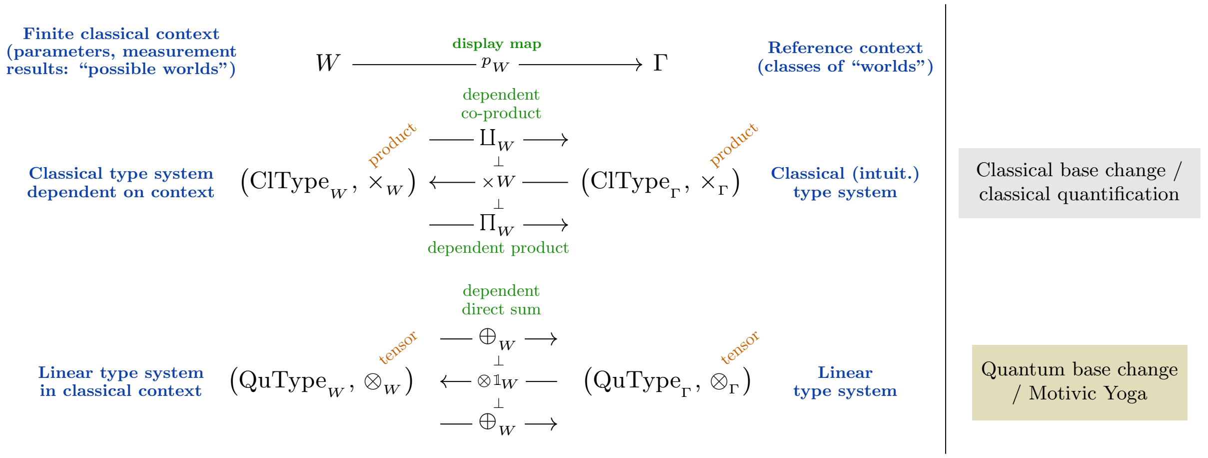
The W-parameterized quantum types QuType_W, for fixed set of measurement outcomes/worlds W, are (readily seen to be):

- **reflective** in the slice over W of all parameterized quantum types,
- **symmetric monoidal** under W-wise tensor product $(-) \otimes_W (-)$,
- **closed** under W-wise formation of spaces $[-,-]_W$ of linear maps,
- **base-changed** along maps $f : W \to \Gamma$ of base sets, by adjoint triples $f_! \dashv f^* \dashv f_*$.

$$\begin{tikzcd}
\left[\begin{array}{c}\bigoplus\limits_{p'(w')=w} \mathcal{H}'_{w'} \\ \downarrow \\ (w\,:\,W)\end{array}\right] & \longmapsfrom & \begin{array}{c}[\mathcal{H}'_\bullet \to W'] \\ \downarrow \quad\;\; \downarrow^{p'} \\ {[0_\bullet \to W]}\end{array} & \\
\mathrm{QuType}_W \arrow[rr, hook, shift right] & \perp & \mathrm{Type}_{/W} \arrow[ll, shift right] \arrow[loop right, "\triangleright_W"] & \text{W-quantumly} \\
\left[\begin{array}{c}\mathcal{H}_\bullet \\ \downarrow \\ W\end{array}\right] & \longmapsto & \begin{array}{c}[\mathcal{H}_\bullet \to W] \\ \downarrow \qquad \downarrow \\ {[0_\bullet \to W]}\end{array} &
\end{tikzcd}$$

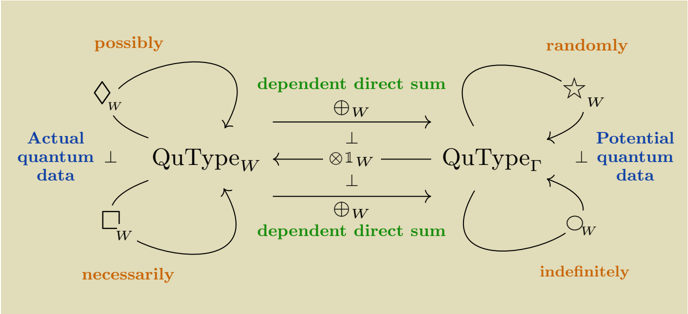
such that these operations jointly satisfy compatibility axioms known as Grothendieck's "**motivic yoga**", a quantum version of local Cartesian closure.

In particular, for *finite* $W \to \Gamma$ (as is the case for any realistic measurement), the left and right quantum base change coincide ("ambidexterity") on the **direct sum** $\oplus_W$, whence we find the following **quantum analog of classical quantification**:



Therefore the system of (co)monads induced from this motivic base change must be expressing the quantum analog of classical epistemic modal logic: **quantum modal logic**.

Indeed, direct analysis shows that these modalities know about the hallmark properties of quantum physics, as previewed in the following diagram:



principle of quantum compulsion

$$\begin{tikzcd}[column sep=small]
\text{necessarily } \mathcal{H}_\bullet & \text{entails} & \text{actually } \mathcal{H}_\bullet & \text{entails} & \text{possibly } \mathcal{H}_\bullet & \text{is} & \text{necessarily } \mathcal{H}_\bullet \\
\Box_W \mathcal{H}_\bullet \arrow[rr, "\mathrm{obt}^{\Box_W}_{\mathcal{H}_\bullet}" description] & & \mathcal{H}_\bullet \arrow[rr, "\mathrm{ret}^{\Diamond_W}_{\mathcal{H}_\bullet}" description] & & \Diamond_W \mathcal{H}_\bullet & \simeq & \Box_W \mathcal{H}_\bullet
\end{tikzcd}$$

$$\begin{tikzcd}
\text{In world} & \text{observe...} & & & & \\
w : W & \vdash & \mathcal{H} \arrow[r, two heads, "\bigoplus_{w'} |\psi_{w'}\rangle \;\mapsto\; |\psi_w\rangle", "\text{measurement collapse}"'] \arrow[rr, bend right, "\text{linear projector onto sub-Hilbert space } \mathcal{H}_w"'] & \mathcal{H}_w \arrow[r, hook, "|\psi_w\rangle \mapsto \bigoplus_{w'} \delta^{w'}_w |\psi_w\rangle", "\text{state preparation}"'] & \mathcal{H}, & \text{where } \mathcal{H} := \bigoplus\limits_{w' : W} \mathcal{H}_{w'}
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
\text{randomly } \mathcal{H} & \text{entails} & \text{potentially } \mathcal{H} & \text{entails} & \text{indefinitely } \mathcal{H} \\
\star_W \mathcal{H} \arrow[rr, "\mathrm{obt}^{\star_W}_{\mathcal{H}}" description] & & \mathcal{H} \arrow[rr, "\mathrm{ret}^{\bigcirc_W}_{\mathcal{H}}" description] & & \bigcirc_W \mathcal{H}
\end{tikzcd}$$

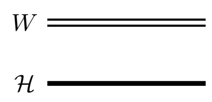
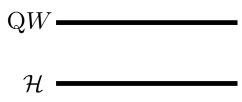
$$\begin{tikzcd}
\bigoplus\limits_{w : W} \mathcal{H} \arrow[r, "\bigoplus_W |\psi_W\rangle \mapsto \sum_W |\psi_W\rangle", "\text{quantum superposition}"'] & \mathcal{H} \arrow[r, "|\psi\rangle \mapsto \bigoplus_{w' : W} |\psi\rangle", "\text{quantum parallelism}"'] & \bigoplus\limits_{b:B} \mathcal{H}
\end{tikzcd}$$

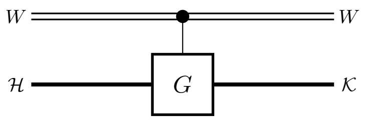
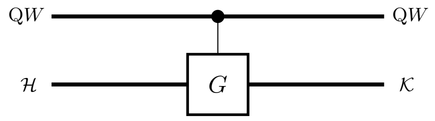
Controlled quantum data.
Modal quantum logic now serves to reason about *quantum circuits*, including

classical wires ═══

quantum wires ━━━

Namely a classical wire carries data of the type W of measurement outcomes of the corresponding quantum wire $\mathrm{Q}W$. Putting this next to a quantum circuit means to make the quantum data *parameterized* by W. Putting instead the quantum wire means to allow W-superpositions.

	Classical/quantum register	**Controlled quantum register**
Symbolic		
Epistemic	actual quantum data $\dfrac{\mathcal{H}_\bullet \;:\; \mathrm{QuType}_W}{w : W \;\vdash\; \mathcal{H}_w \;:\; \mathrm{QuType}}$	potential quantum data $\dfrac{\underset{W}{\Box}\mathcal{H}_\bullet \;:\; \mathrm{QuType}_W}{w : W \;\vdash\; \underset{w'}{\bigoplus}\mathcal{H}_{w'} \;:\; \mathrm{QuType}}$

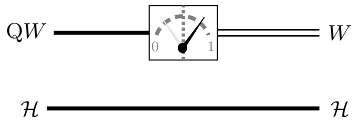
	Classically controlled quantum gate	**Quantumly controlled quantum gate**
Symbolic		
Epistemic	$\mathcal{H}_\bullet \xrightarrow[\text{an actual entailment}]{G_\bullet} \mathcal{K}_\bullet$ $w : W \;\vdash\; \mathcal{H}_w \xrightarrow{G_w} \mathcal{K}_w$	$\underset{W}{\Box}\mathcal{H}_\bullet \xrightarrow[\text{a potential entailment}]{\underset{W}{\Box}G_\bullet} \underset{W}{\Box}\mathcal{K}_\bullet$ $w : W \;\vdash\; \underset{W}{\bigoplus}\mathcal{H}_\bullet \xrightarrow{\underset{W}{\bigoplus}G_\bullet} \underset{W}{\bigoplus}\mathcal{K}_\bullet$

Accordingly, *classically controlled* quantum gates map W-dependent quantum data, while the corresponding *quantumly controlled* quantum gates are the W-superposition of these operations, acting "inside the indefiniteness monad".

Quantum measurement gates *obtain* (= monad counit) classical data from quantum data, while collapsing the quantum state accordingly.

(Noteworthy that all possible outcomes $w : W$ are accounted for: the actual measurement outcome w is available only at run-time, then handled (read-out) as a computational effect.)

This formulation of quantum circuits by our modal logic of quantum gates is natural and seamless. A first result now is a general statement and proof of:

	Quantum measurement gate
Symbolic	
Epistemic	$\underset{W}{\Box}\mathcal{H}_\bullet \xrightarrow[\text{the necessary becomes actual}]{\mathrm{obt}^{\Box_W}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$ $w : W \;\vdash\; \underset{w'}{\bigoplus}\mathcal{H}_{w'} \xrightarrow[\text{quantum state collapse}]{\mathrm{pr}_w} \mathcal{H}_w$ $\underset{w'}{\bigoplus}\lvert\psi_{w'}\rangle \longmapsto \lvert\psi_w\rangle$

The deferred measurement principle states that every quantum circuit with mid-circuit measurement followed by quantum gates controlled by the measurement outcomes is equivalent to a coherent quantum circuit consisting all of quantumly-controlled gates with measurement happening only at the end.

The infamous paradox stories of quantum physics are all but narrations of – and are resolved by – the deferred measurement principle: Schrödinger's cat (1935), Everett's observer A (1957), Wigner's fiend (1961) are all enactors of the intermediate measurement gate.

This traditionally un-proven principle now follows rigorously – it is just the *Kleisli equivalence* for the $\Box$-comonad:

$$\begin{tikzcd}
(\mathrm{QuType}_W)_{\Box_W}(\mathcal{H}_\bullet,\, \mathcal{H}''_\bullet) \arrow[r, "\sim", "\underset{W}{\Box}(\text{-}) \circ \mathrm{dplc}^{\Box_W}_{(\text{-})}"'] \arrow[rr, bend left, "\text{id}", "\text{Kleisli equivalence}"'] & \mathrm{QuType}^{\Box_W}_W(\underset{W}{\Box}\mathcal{H}_\bullet,\, \underset{W}{\Box}\mathcal{H}''_\bullet) \arrow[r, "\sim", "\mathrm{obt}^{\Box_W}_{(\text{-})} \circ (-)"'] & (\mathrm{QuType}_W)_{\Box_W}(\mathcal{H}_\bullet,\, \mathcal{H}''_\bullet) \\
\text{$\underset{W}{\Box}$-Kleisli morphisms} & \text{homomorphisms of free $\underset{W}{\Box}$-coalgebras} & \text{$\underset{W}{\Box}$-Kleisli morphisms} \\
\left(\underset{W}{\Box}\mathcal{H}_\bullet \xrightarrow{F} \underset{W}{\Box}\mathcal{H}'_\bullet \xrightarrow{\mathrm{obt}^{\Box_W}_{\mathcal{H}'_\bullet}} \mathcal{H}'_\bullet \xrightarrow{G_\bullet} \mathcal{H}''_\bullet\right) \arrow[r, mapsto] & \left(\underset{W}{\Box}\mathcal{H}_\bullet \xrightarrow{F} \underset{W}{\Box}\mathcal{H}'_\bullet \xrightarrow{\Box_W G_\bullet} \underset{W}{\Box}\mathcal{H}''_\bullet\right) \arrow[r, mapsto] & \left(\underset{W}{\Box}\mathcal{H}_\bullet \xrightarrow{F} \underset{W}{\Box}\mathcal{H}'_\bullet \xrightarrow{\Box_W G_\bullet} \underset{W}{\Box}\mathcal{H}''_\bullet \xrightarrow{\mathrm{obt}^{\Box_W}_{\mathcal{H}''_\bullet}} \mathcal{H}''_\bullet\right) \\
\text{measurement-controlled quantum gate} & \text{quantum-controlled quantum gate...} & \text{...followed by measurement}
\end{tikzcd}$$

4 Homotopy Quantum Data Dependency: Symmetries, Adiabatic Transport, Holonomy

Our categorical formulation of quantum data admits **higher homotopical generalization**, first by promoting base sets to homotopy 1-types.

homotopy theory jargon: *homotopy 1-types*
category theory jargon: *1-groupoids*
physics-style meaning: “sets with gauge symmetries”

To appreciate this, note the
differing community jargons:

“**topological**” in physics	typically means	“**homotopical**” in math
as in		where topological spaces (yes, but:)
- “topological quantum field theory“		are regarded only up to *homotopy equivalence*
- “topological phases of matter“		hence as representing their *homotopy types*
- “topological quantum computing“		reflected in their *homotopy groups*.

A **1-groupoid** $\mathcal{G}$ (or just: *groupoid*) is
- a set S where all pairs $s_1, s_2 \in S$ of elements (now: “objects”) are equipped with a set $\mathcal{G}(s_1, s_2)$ of *gauge transformations* $s_1 \xrightarrow{\gamma} s_2$ (“morphisms”), invertible with respect to a given associative and unital composition law;

hence:
- a small category whose morphisms are all invertible.

A **map** of groupoids respects this structure (a *functor*):

$$\begin{tikzcd}
\mathcal{G} \arrow[rr, "f"] & & \mathcal{G}' \\
s_1 \arrow[d, "\gamma"] & \longmapsto & f(s_1) \arrow[d, "f(\gamma)"'] \\
s_2 & & f(s_2)
\end{tikzcd}$$

A **homotopy** of maps of groupoids intertwines such functors (*natural transformation*)

$$\begin{tikzcd}
& & \mathcal{G} \arrow[rr, bend left, "f_0"{name=U}] \arrow[rr, bend right, "f_1"'{name=D}] & & \mathcal{G}' \arrow[from=U, to=D, Rightarrow, "\eta"] \\
s_1 \arrow[d, "\gamma"] & \mapsto & f_0(s_1) \arrow[rr, "\eta(s_1)"'] \arrow[d, "f_0(\gamma)"] & & f_1(s_1) \arrow[d, "f_1(\gamma)"] \\
s_2 & \mapsto & f_0(s_2) \arrow[rr, "\eta(s_2)"] & & f_1(s_2)
\end{tikzcd}$$

A **homotopy equivalence** of groupoids, $\mathcal{G} \simeq \mathcal{H}$, is maps

$$\begin{tikzcd}
\mathcal{G} \arrow[r, shift left, "f"] & \mathcal{H} \arrow[l, shift left, "\overline{f}"]
\end{tikzcd}$$

being inverses up to homotopy

$$\begin{tikzcd}
\mathcal{G} \arrow[r, "f"] \arrow[rr, bend left=40, "\mathrm{id}_{\mathcal{G}}"] & \mathcal{H} \arrow[r, "\overline{f}"] \arrow[rr, bend right=40, "\mathrm{id}_{\mathcal{G}}"'] & \mathcal{G} \arrow[r, "f"] & \mathcal{H}
\end{tikzcd}$$

Linear representation of groupoid $\mathcal{G}$
is functor $\rho : \mathcal{G} \longrightarrow \mathrm{Mod}_{\mathbb{C}}$

Intertwiner of grpd reps
is natural transformation

$$\begin{tikzcd}
\mathcal{G} \arrow[r, bend left, "\rho"{name=U}] \arrow[r, bend right, "\rho'"'{name=D}] & \mathrm{Mod}_{\mathbb{C}} \arrow[from=U, to=D, Rightarrow]
\end{tikzcd}$$

⇒ category $\mathrm{Mod}_{\mathbb{C}}^{\mathcal{G}}$ of $\mathcal{G}$-representations

Examples:

$\Pi_1(X)$
fundamental groupoid of a space X
objects: the points of X
morphisms: homotopy-clases of paths in X
composition: concatenation of paths

$S /\!\!/ G$
quotient groupoid of a group action $G \circlearrowright S$
objects: the elements of S
morphisms: group translations $s \xrightarrow{g} g(s)$
composition: group operation

$\mathbf{B}G = \ast /\!\!/ G$
delooping groupoid of a group G
objects: a single one $\bullet$
morphisms: group elements $g \circlearrowright \bullet$
composition: group operation

Homotopy groups:
$\pi_0(\mathcal{G})$:= gauge-equivalence classes:
$\pi_1(\mathcal{G}, s)$:= auto-gauge group of object s
compatible with homotopy groups of spaces

$$\begin{aligned}
\pi_0(X) &= \pi_0\big(\Pi_1(X)\big) \\
\pi_1(X, x) &= \pi_1\big(\Pi_1(X), x\big)
\end{aligned}$$

<u>**Skeleton Theorem**</u> (assuming axiom of choice):
Any groupoid $\mathcal{G}$ is homotopy equivalent to the disjoint union of delooping groupoids of the fundamental groups of its connected components:

$$\underset{\text{any groupoid}}{\mathcal{G}} \;\simeq\; \underset{\text{its “skeleton”}}{\coprod_{[s] \in \pi_0(\mathcal{G})} \mathbf{B}\pi_1(\mathcal{G}, s)}$$

⇒ homotopy-equivalent groupoids have equivalent categories of representations,
⇒ and hence with Skeleton Theorem:

$$\begin{aligned}
\mathrm{Mod}_{\mathbb{C}}^{\mathcal{G}} &\simeq \prod_{[s] \in \pi_0(\mathcal{G})} \mathrm{Mod}_{\mathbb{C}}^{\mathbf{B}\pi_1(\mathcal{G}, s)} \\
&\simeq \prod_{[s] \in \pi_0(\mathcal{G})} \pi_1(\mathcal{G}, s)\mathrm{Rep}_{\mathbb{C}}
\end{aligned}$$

Example (reps of fundamental groupoids):

$$\mathrm{Mod}_{\mathbb{C}}^{\Pi_1(X)} \simeq \begin{array}{c}\text{flat vector bundles} \\ \text{aka: local systems}\end{array} \left[\begin{array}{c} \mathcal{H}_\bullet \\ \downarrow \\ X \end{array}\right]$$

so dependent quantum data
as before, but now
including operator actions

Base change of groupoid reps.
Given $\mathcal{G} \xrightarrow{f} \mathcal{H}$, the precomposition functor

$$\begin{tikzcd}
\mathrm{Mod}_{\mathbb{C}}^{\mathcal{G}} & & \mathrm{Mod}_{\mathbb{C}}^{\mathcal{H}} \arrow[ll, "f^*"']
\end{tikzcd}$$

$$\Big(\mathcal{G} \xrightarrow{f} \mathcal{H} \xrightarrow{\rho} \mathrm{Core}(\mathrm{Mod}_{\mathbb{C}})\Big) \longleftarrow\!\shortmid \Big(\mathcal{H} \xrightarrow{\rho} \mathrm{Core}(\mathrm{Mod}_{\mathbb{C}})\Big)$$

has left & right adjoints
(<u>Kan extension</u>)

$$\begin{tikzcd}
\mathrm{Mod}_{\mathbb{C}}^{\mathcal{G}} \arrow[rr, bend left=40, "f_!"] \arrow[rr, bend right=40, "f_*"'] & & \mathrm{Mod}_{\mathbb{C}}^{\mathcal{H}} \arrow[ll, "f^*"{description}] \\
\end{tikzcd}$$

for subgroups $G \overset{i}{\hookrightarrow} H$,
$f := \mathbf{B}i : \mathbf{B}G \to \mathbf{B}H$, this reduces to (co)induced reps:
$f_!(V) \simeq \mathbb{C}[H] \otimes_G V$
$f_*(V) \simeq \mathrm{Hom}_{\mathbb{C}(G)}\big(\mathbb{C}[H], V\big)$

Fact [9]: Our model of classically-controlled quantum data generalizes to hotype-parameterization: $\mathrm{Type} := \int\limits_{\mathcal{G} \in \mathrm{Grpd}} \mathrm{Mod}_{\mathbb{C}}^{\mathcal{G}}$

here (by the above Skeleton Theorem) quantum data:
- depends on **possible worlds**, as before, but now:
- in each world is acted on by a group of operators
- ⇒ quantum **symmetries** & quantum **evolution**

Symmetries vs. Evolutions.

Given *normal* subgroup $H \xrightarrow{i} G$, set again $f := \mathbf{B}i$.
Consider a G-representation $\mathcal{H} \in G\mathrm{Rep}_{\mathbb{C}} \simeq \mathrm{Mod}_{\mathbb{C}}^{\mathbf{B}G}$
and its restriction to an H-rep

$$\begin{tikzcd} \mathbf{B}H \arrow[r, "f \equiv \mathbf{B}i"] \arrow[rr, bend left, "f^*\mathcal{H}_\bullet"] & \mathbf{B}G \arrow[r, "\mathcal{H}_\bullet"] & \mathrm{Mod}_{\mathbb{C}} \end{tikzcd}$$

Observation 1:
Each $g \in G$ gives a
controlled quantum gate:

$$\begin{tikzcd} f^*\mathcal{H}_\bullet \arrow[r, "\mathcal{H}_g"] \arrow[d] & f^*\mathcal{H}_\bullet \arrow[d] \\ \mathbf{B}H \arrow[r, "\mathbf{B}\mathrm{Ad}_g"] & \mathbf{B}H \end{tikzcd}$$

namely

$$\begin{tikzcd} \mathbf{B}H \arrow[rr, "\mathbf{B}\mathrm{Ad}_g"] \arrow[dr, "f^*\mathcal{H}_\bullet"'] & \arrow[d, phantom, "\overset{\mathcal{H}_g}{\Rightarrow}"] & \mathbf{B}H \arrow[dl, "f^*\mathcal{H}_\bullet"] \\ & \mathrm{Mod}_{\mathbb{C}} & \end{tikzcd} \quad : \quad \begin{tikzcd} * \arrow[d, "h"] & \mapsto & \mathcal{H}_* \arrow[r, "\mathcal{H}_g"] \arrow[d, "\mathcal{H}_h"] & \mathcal{H}_* \arrow[d, "\mathcal{H}_{ghg^{-1}}"] \\ * & \mapsto & \mathcal{H}_* \arrow[r, "\mathcal{H}_g"] & \mathcal{H}_* \end{tikzcd}$$

To appreciate this, to note the
Automorphism 2-group of a group H.
The automorphisms of delooping groupoids form the groupoid

$$\mathrm{Aut}\big(\mathbf{B}H\big) \;\equiv\; \left\{ \begin{tikzcd} \mathbf{B}H \arrow[r, bend left=50, "\mathbf{B}\alpha", ""{name=U, below}] \arrow[r, bend right=50, "\mathbf{B}\alpha'"', ""{name=D}] & \mathbf{B}H \arrow[Rightarrow, from=U, to=D, "\eta"] \end{tikzcd} \right\} \;\simeq\; \mathrm{Aut}(H) /\!\!/_{\mathrm{Ad}} \mathrm{Inn}(H), \quad \text{because} \quad \begin{tikzcd} * \arrow[d, "h"] & \mapsto & * \arrow[r, "\eta_*"] \arrow[d, "\alpha(h)" description] & * \arrow[d, "\alpha'(h)" description] \\ * & \mapsto & * \arrow[r, "\eta_*"] & * \end{tikzcd} \;\Leftrightarrow\; \begin{array}{c} \textbf{group} \\ \textbf{automorphisms} \\ \alpha' = \\ \mathrm{Ad}_{\eta_*} \circ \alpha \end{array}$$

(on objects this is functoriality on morphisms it is naturality).
This groupoid itself has group structure (it's a *2-group*)
by horizontal composition

In particular, inner H-automorphisms deloop
to homotopy-trivial $\mathbf{B}H$-automorphisms:

$$\begin{tikzcd} \mathbf{B}H \arrow[r, bend left=50, "\mathbf{B}\mathrm{Ad}_h", ""{name=U, below}] \arrow[r, bend right=50, "\mathrm{Id}"', ""{name=D}] & \mathbf{B}H \arrow[Rightarrow, from=U, to=D, "h^{-1}"] \end{tikzcd} \qquad \begin{tikzcd} * \arrow[d, "h'"] & \mapsto & * \arrow[r, "h^{-1}"] \arrow[d, "\mathrm{Ad}_h(h')"] & * \arrow[d, "h'"] \\ * & \mapsto & * \arrow[r, "h^{-1}"] & * \end{tikzcd}$$

Whence

Observation 2:
The above controlled quantum gate is
homotopy-trivial/*pure gauge*
whenever $g \equiv h \in H \subset G$
$\Rightarrow$ the evolutionary transfors constitute G/H

$$\begin{tikzcd} \mathbf{B}H \arrow[rr, "\mathbf{B}\mathrm{Ad}_h" description] \arrow[rr, bend left=60, "\mathrm{id}"] \arrow[dr, "f^*\mathcal{H}_\bullet"'] & \arrow[u, Rightarrow, "h^{-1}"] \arrow[d, phantom, "\overset{\mathcal{H}_h}{\Longrightarrow}"] & \mathbf{B}H \arrow[dl, "f^*\mathcal{H}_\bullet"] \\ & \mathrm{Mod}_{\mathbb{C}} & \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbf{B}H \arrow[d, bend right=60, "f^*\mathcal{H}_\bullet"', ""{name=L}] \arrow[d, bend left=60, "f^*\mathcal{H}_\bullet", ""{name=R, left}] \\ \mathrm{Mod}_{\mathbb{C}} \arrow[Rightarrow, from=L, to=R, "\mathrm{id}"] \end{tikzcd}$$

Lesson: For quantum state spaces $f^*\mathcal{H}_\bullet$, the
G-transfors decompose into
symmetries in H &
evolutions in G/H

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, hook, "\text{symmetries}"] & G \arrow[r, two heads, "\text{all transfors}"', "\text{evolutions}"] & G/H \arrow[r] & 1 \end{tikzcd}$$

But this fits squarely into the abstract quantum language:

Quantum measurement in presence of symmetries.
Our quantum language says that of this form $f^*\mathcal{H}$
are quantum state spaces carrying a measurement basis:
given space of H-symmetric basis states $\mathcal{V}_\bullet \in H\mathrm{Rep}_{\mathbb{C}} \simeq \mathrm{Mod}_{\mathbb{C}}^{\mathbf{B}H}$
then measurable quantum states are by $\Box_f$-comonad

$$\Box_f \mathcal{V} \;=\; f^* \underbrace{f_* \mathcal{V}}_{\mathrm{Hom}_H(\mathbb{C}[G],\,\mathcal{V})} \;=:\; f^*\mathcal{H}_\bullet$$

on which quantum measurement is the counit

$$\Box_f \mathcal{V}_\bullet \xrightarrow{\mathrm{obt}^{\Box}_{\mathcal{V}}} \mathcal{V}_\bullet$$

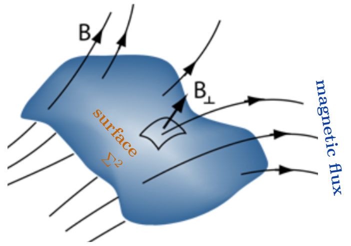
just as discussed in §3:
the abstract monadic language of quantum effects
applies verbatim to the present higher homotopy types.

With all this in hand we are now prepared to
formalize quantum gates and quantum measurement
for topological quantum systems.
A concrete example/application is the topic of the last §5.

5 Application: Quantum states in FQH-systems

We consider now, following [15], the example of quantum states of topological materials called *fractional quantum Hall*-systems, where magnetic flux penetrates a semi-conducting surface Σ^2.

The "gauge group" of the electromagnetic field is $G \equiv \mathrm{U}(1)$ and *ordinarily* such flux is classified by maps to $B\mathrm{U}(1) \simeq \mathbb{C}P^\infty$.



Precisely, when quantum-effects are being resolved, then:

Theorem [11] (Yang-Mills flux quantum observables): *For ordinary gauge fields on a spacetime* $\simeq \mathbb{R}^{1,1} \times \Sigma^2$ *the* ***quantum observables of field flux*** *through* Σ^2 *form the group-convolution* C^**-algebra* $\underset{\text{quantum flux observables}}{\mathbb{C}\Big[C^\infty\big(\Sigma^2,\, G \ltimes (\mathfrak{g}/\Lambda)\big)\Big]}$ for $\Lambda \subset \mathfrak{g}$ an Ad-invariant lattice.

G	Lie group ("gauge group")
$\mathfrak{g}$	its Lie algebra

Commercial-value quantum computing will require **robust** quantum observables, insensitive to local fluctuations, only depending on **topological sectors** of field configurations.

$$\underset{\text{all quantum flux observables}}{\mathbb{C}\Big[C^\infty\big(\Sigma^2,\, G \ltimes (\mathfrak{g}/\Lambda)\big)\Big]} \xleftarrow{[-]^*} \underset{\text{robust topological observables}}{\mathbb{C}\Big[\pi_0\, C^\infty\big(\Sigma^2,\, G \ltimes (\mathfrak{g}/\Lambda)\big)\Big]}$$

$C^\infty(\text{-},\text{-})$	manifold of smooth functions
$(\text{-}) \ltimes (\text{-})$	semidirect product via adjoint
$\mathbb{C}[-]$	group convolution C^*-algebra
$\pi_0(-)$	path-connected components

Proposition [11] (topological sector observables): The topological flux quantum observables form the homology Pontrjagin algebra of maps from space to classifying space. (shown now assuming $\Lambda = 0$, for simplicity):

$$\overset{\text{topological flux quantum observables}}{\mathbb{C}\Big[\pi_0\, C^\infty\big(\Sigma^2,\, G\big)\Big]} \simeq \mathbb{C}\Big[\pi_0\, \mathrm{Maps}\big(\Sigma^2,\, G\big)\Big]$$

$$\simeq \underset{\substack{\text{group algebra of fundamental group}\\ \text{of maps to classifying space}}}{\mathbb{C}\Big[\pi_1\, \mathrm{Maps}\big(\Sigma^2,\, BG\big)\Big]} \simeq \underset{\substack{\text{homology Pontrjagin algebra of}\\ \text{soliton moduli space}}}{H_0\Big(\mathrm{Maps}^*\big((\mathbb{R}^1 \times \Sigma^2)_{\cup\{\infty\}},\, BG\big);\, \mathbb{C}\Big)}$$

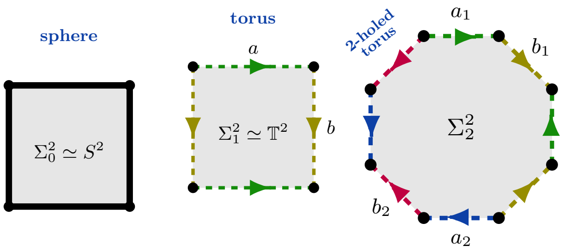
soliton on X = topological field configuration that vanishes at the ends of X

classified by *pointed* map
$\Rightarrow$ $X_{\cup\{\infty\}} \to BG$
from one-point compactification

Example: $\mathbb{C}\big[\pi_0\, \mathrm{Maps}\big(\Sigma^2_g,\, \mathrm{U}(1)\big)\big] \simeq \mathbb{C}\big[H^1(\Sigma^2_g;\, \mathbb{Z})\big] \simeq \mathbb{C}\big[\mathbb{Z}^{2g}\big]$

Σ^2_g orientable surface of genus=g

Effective flux of "fractional quantum Hall systems" (FQH). But, at very low temperature, experiment suggests instead of $\mathbb{Z}^{2g}$ its 2nd integer Heisenberg extension $\widehat{\mathbb{Z}^{2g}}$ being the observables of an "**effective Chern-Simons field**", where the center $\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}^{2g}}$ observes an **anyon braiding phase**.



Question: Is there classifying space $\mathcal{A}$ for this effective CS field?
Answer: Yes! The 2-sphere $S^2 \simeq \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty \simeq B\mathrm{U}(1)$
Theorem [4][7]: The cofiber presentation of the surface

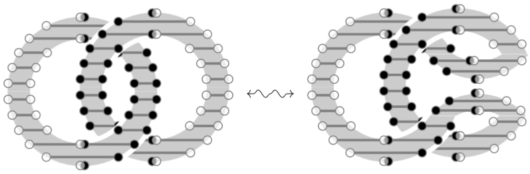
$$S^1 \xrightarrow{\prod_i [a_i, b_i]} \bigvee_g (S^1_a \vee S^1_b) \longrightarrow \Sigma^2_g \longrightarrow S^2$$

induces short exact sequence exhibiting the Heisenberg extension:

$$1 \to \underbrace{\pi_1 \mathrm{Maps}\big(S^2,\, S^2\big)}_{\mathbb{Z}} \to \underbrace{\pi_1 \mathrm{Maps}\big(\Sigma^2_g,\, S^2\big)}_{\widehat{\mathbb{Z}^{2g}}} \to \underbrace{\pi_1 \mathrm{Maps}^*\big(\bigvee_{2g} S^1,\, S^2\big)}_{\mathbb{Z}^{2g}} \to 1$$

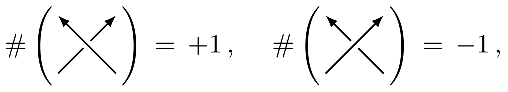
$$\widehat{\mathbb{Z}^{2g}} := \left\{ \begin{array}{c} \big(\vec{a}, \vec{b}, n\big) \in \mathbb{Z}^g \times \mathbb{Z}^g \times \mathbb{Z} \\ \\ \big(\vec{a}, \vec{b}, n\big) \cdot \big(\vec{a}', \vec{b}', n'\big) = \\ \big(\vec{a} + \vec{a}',\, \vec{b} + \vec{b}',\, n + n' + \underset{\substack{\text{twice the unit}\\ \text{central extension}}}{\vec{a} \cdot \vec{b}' - \vec{a}' \cdot \vec{b}}\big) \end{array} \right\}$$

Question: Can we identify the center $\mathbb{Z}$ as arising from braiding?
Answer: Yes!
Theorem [12]: $\mathrm{Maps}^*(S^2, S^2)$ is configurations of charged strings such that $\Omega\mathrm{Maps}^*(S^2, S^2)$ is framed links subject to cobordism, $\pi_1\mathrm{Maps}^*(S^2, S^2)$ generated from framed unknot with 1 braiding



$$\begin{tikzcd} \Omega\mathrm{Maps}^*\big(S^2,\, S^2\big) \arrow[r, "{[-]}"] & \pi_3(S^2) \simeq \mathbb{Z} \\ \underset{\text{framed link}}{L} \arrow[r, mapsto] & \underset{\text{linking + framing number}}{\#L} \end{tikzcd}$$

is CS observable ("Wilson loop")



Ergo: Remarkably, topological quantum observables of effective flux in quantum Hall systems is algebro-topologically described by $\left\{ \begin{array}{rr} \text{replacing the classifying space} & B\mathrm{U}(1) \simeq \mathbb{C}P^\infty \\ \text{with its 2-skeleton} & S^2 \simeq \mathbb{C}P^1 \end{array} \right.$

Question: Is there a deeper rationale for such replacement? **Answer:** Yes [13][14]: *Hypothesis H.*

CS-Level and groupoid reps.
In fact, the moduli space has more connected components:

$\pi_0 \mathrm{Maps}(\Sigma^2_g,\, S^2) \;\simeq\; \mathbb{Z}$ **Hopf degree**

and the fundamental group of the degree=k component is

$$\pi_1\Big(\mathrm{Maps}\big(\Sigma^2_g,\, S^2\big),\, k\Big) \;:=\; \left\{ \begin{array}{c} \big(\vec{a}, \vec{b}, n\big) \;\in\; \mathbb{Z}^g \times \mathbb{Z}^g \times \mathbb{Z}_{2|k|} \\[6pt] \big(\vec{a}, \vec{b}, [n]\big) \cdot \big(\vec{a}', \vec{b}', [n']\big) \;= \\ \big(\vec{a} + \vec{a}',\, \vec{b} + \vec{b}',\, [n + n' + \vec{a} \cdot \vec{b}' - \vec{a}' \cdot \vec{b}]\big) \end{array} \right\}$$

This means that the quantum states of all levels k are unified in a single groupoid representation of the moduli space

$$\mathcal{H}^{\bullet}_{T^2} \;:\; \underbrace{\Pi_1\Big(\mathrm{Maps}\big(\Sigma^2_1,\, S^2\big)\Big)}_{\simeq \coprod_{k \in \mathbb{Z}} \mathbf{B}\pi_1\left(\mathrm{Maps}\left(\Sigma^2_g,\, S^2\right),\, k\right)} \longrightarrow \mathrm{Mod}_{\mathbb{C}}$$

group(oid) of gauge transfors of the gauge field (tplgcl sector)

Hence the **observable algebra** Obs_0 for $g = 1$, $\Sigma^2_1 = T^2$, has generators

$$\left\{ \begin{array}{rcl} W_a & := & (1, 0, [0]) \\ W_b & := & (0, 1, [0]) \\ \zeta & := & (0, 0, [1]) \end{array} \right\} \quad \begin{array}{c} \text{subject to} \\ \text{the relations} \end{array} \left\{ \begin{array}{l} W_a \cdot W_b \;=\; \zeta^2\, W_b \cdot W_a \\ \zeta^{2k} \;=\; 1 \\ [\zeta, -] = 0 \end{array} \right\}.$$

This is just the observable algebra expected [17, (5.28)] for anyonic topological order on the torus described by abelian Chern-Simons theory at level k.

The irreps have dim $= k$

$\mathcal{H}^k_{T^2} := \mathrm{Span}\Big(\big|[n]\big\rangle,\, [n] \in \mathbb{Z}_{|k|}\Big)$

$p \in \{1, 2, \cdots, k\}$

$$\begin{array}{rcl} W_a\big|[n]\big\rangle & := & e^{2\pi i n \nu}\big|[n]\big\rangle \\ W_b\big|[n]\big\rangle & := & \big|[n+1]\big\rangle \\ \zeta\big|[n]\big\rangle & := & e^{\pi i \nu}\big|[n]\big\rangle\,. \end{array}$$

General covariance. In fact, the moduli space is acted on by the (orientation-preserving) diffeomorphism group, as be‘s a generally covariant system, whence the generally covariant quantum states form a groupoid representation

$$\mathcal{H}^{\bullet}_{T^2} \;:\; \Pi_1\Big(\mathrm{Maps}\big(\Sigma^2_g,\, S^2\big) \,/\!\!/\, \mathrm{Diff}^+\big(\Sigma^2_g\big)\Big) \longrightarrow \mathrm{Mod}_{\mathbb{C}}$$

which makes them carry also the semidirect product action of the *mapping class group* (MCG) of the surface

$$\begin{array}{rl} & \pi_1\Big(\mathrm{Maps}\big(\Sigma^2_g,\, S^2\big) \,/\!\!/\, \mathrm{Diff}^+\big(\Sigma^2_1\big),\, k\Big) \\ \simeq & \pi_1\Big(\mathrm{Maps}\big(\Sigma^2_g,\, S^2\big),\, k\Big) \rtimes \underbrace{\pi_0 \mathrm{Diff}^+\big(\Sigma^2_g\big)}_{\mathrm{MCG}(\Sigma^2_g)} \end{array}$$

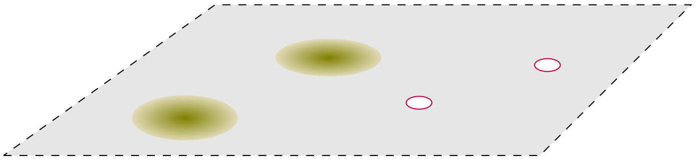
group(oid) of diffeos (gravity) (tplgcl sector)

Question: Does this new model make novel predictions?

With the classifying space identified for known situations, we find its implications for previously inaccessible cases:

Namely generalize now to *n-punctured* surfaces $\Sigma^2_{g,n}$, reflecting n *defect points* in the semiconductor where the magentic field is *expelled* (type-I superconducting spots).

Answer: Yes – *defect anyons* in FQH-systems:

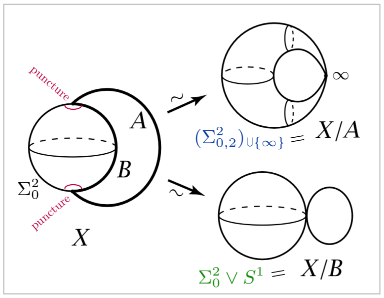


field solitons: Pontrjagin submanifolds

flux-expelling defects: punctures

Proposition.
The observables are, in this generality:

$$\begin{array}{rcl} \mathrm{Obs}_0 & \simeq & \mathbb{C}\Big[\pi_1 \mathrm{Maps}^*\Big(\big(\Sigma^2_{g,n}\big)_{\cup\{\infty\}},\, S^2\Big)\Big] \\ & \simeq & \mathbb{C}\Big[\pi_1 \mathrm{Maps}^*\big(\Sigma^2_g \vee \bigvee_{n-1} S^1,\, S^2\big)\Big] \\ & \simeq & \mathbb{C}\Big[\pi_1 \mathrm{Maps}^*\big(\Sigma^2_g,\, S^2\big) \times \mathbb{Z}^{n-1}\Big] \\ & \underset{g=0}{\simeq} & \mathbb{C}\big[\mathbb{Z}^n\big] \end{array}$$



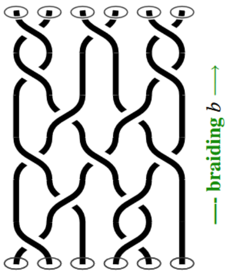
Topology change due to defects! (cf. [5, p 11]) Different to but not unlike the *genon*-proposal [1].

subject to the **diffeomorph. action** *by:*

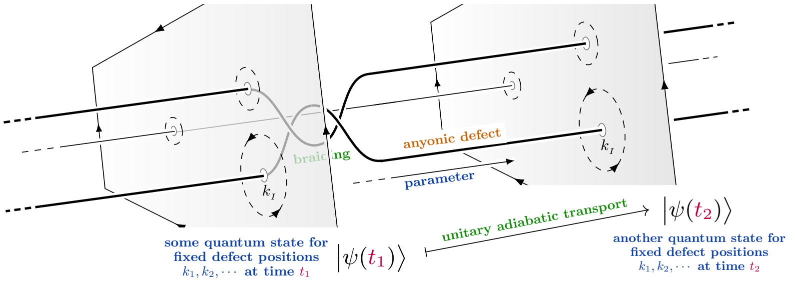
$$1 \to \underset{\text{surface braid group}}{\mathrm{Br}_n(\Sigma^2_g)} \hookrightarrow \underset{\substack{\text{mapping class group}\\ \text{of punctured surface}}}{\pi_0 \mathrm{Homeos}^*_{\mathrm{or}}\big((\Sigma^2_{g,n})_{\cup\{\infty\}}\big)} \twoheadrightarrow \underset{\substack{\text{mapping class group}\\ \text{of plain surface}}}{\mathrm{MCG}(\Sigma^2_g)} \to 1$$

Therefore the equivariant quantum states (jargon: “generally covariant”) on $\Sigma^2_{0,n}$ are representations of the *wreath product of solitonic and defect phases*:

$$\mathbb{Z} \wr \mathrm{Br}_n(\Sigma^2_0) \;=\; \overset{\text{solitonic anyons}}{\mathbb{Z}^n} \rtimes \overset{\text{defect anyons}}{\mathrm{Br}_n(\Sigma^2_0)} \;\twoheadrightarrow\; \mathbb{Z}^n \rtimes \mathrm{Sym}_n$$



Such *braid representations for defects* have not previously been derived for FQH systems –
but are just what is needed for the grand goal of *topological quantum gates*:
programmable unitary transformations of quantum systems,
insensitive to continuous deformations (hence to noise!)



That's the idea, but.

Question: Since the braid group arises here as a *gauge symmetry* (general covariance), will one actually be able to use it as *physical evolution*?

a special case of the general

Question: Groupoid reps encode operator actions which can be both symmetries as well as evolution; how to disentangle these?

Asymptotic symmetries. (cf. [16, §2.10][2])
On spacetimes with
(asymptotic) boundaries,
a normal subgroup of
diffeos trivial on asmpttc bdry
are gauge transformations,
while their cosets are evolutions

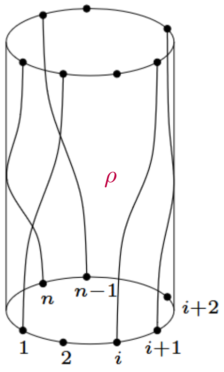
$$1 \longrightarrow \overset{\text{symmetries}}{\mathrm{BlkDiff}} \hookrightarrow \overset{\text{all diffeos}}{\mathrm{Diff}} \twoheadrightarrow \overset{\text{evolutions}}{\mathrm{Diff}/\mathrm{BlkDiff}} \longrightarrow 1$$

Fact [6][3, Rem 1.2] The n-punctured annulus $\Sigma_{0,n,2}$ has

$$1 \longrightarrow \overset{\text{bulk symmetries}}{\mathrm{Br}_n^{\mathrm{aff}}} \hookrightarrow \overset{\substack{\text{diffeomorphisms of}\\ n\text{-punctured annulus}}}{\mathrm{MCG}(\Sigma^2_{0,n,2})} \twoheadrightarrow \overset{\substack{\text{asymptotic}\\ \text{symmetries}}}{\mathbb{Z}\langle\rho,\tau\rangle} \longrightarrow 1$$

where ρ is cyclic permutation of defects along the annulus boundary.

This means that plain braid group plays the role of bulk symmetries with ρ an asymptotic symmetry.



graphics from [3, Fig. 2]

Overall conclusion.

Whith this it follows that the cosets of ρ may serve to implement topological quantum gates

$$\begin{tikzcd}
\mathcal{H}_{\Sigma^2_{0,n,2}} \arrow[r, "\mathcal{H}_\rho"] \arrow[d] & \mathcal{H}_{\Sigma^2_{0,n,2}} \arrow[d] \\
\mathbf{B}\mathrm{Br}_n^{\mathrm{aff}} \arrow[r, "\mathrm{Ad}_\rho"] & \mathbf{B}\mathrm{Br}_n^{\mathrm{aff}}
\end{tikzcd}$$

and that
quantum measurement bases
for these topological states are
given by the base change adjunction
along
$\mathbf{B}\big(\mathrm{Br}_n^{\mathrm{aff}} \hookrightarrow \mathrm{MCG}(\Sigma^2_{0,n,2})\big)$.

This is the kind of answer promised on p. 2.

It combined homotopy-theoretic considerations of quantum language, with algebro-topological insights into topological quantum systems.

We claim that this is of practical/experimental relevance, to be discussed in more detail elsewhere.

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