Super-Exceptional Geometry for 11D Supergravity

Urs Schreiber[a]

December 9, 2024

remote talk at

HEP Seminar @ MPI for Grav Physics
Albert Einstein Institute, Potsdam

December 9, 2024
seminars.aei.mpg.de/seminar/26232

on joint work with
G. Giotopoulos & H. Sati

primarily arXiv:2411.03661

These talk notes are available for download at:
ncatlab.org/schreiber/show/Super-Exceptional+Geometry+for+11D+Supergravity

[a] Mathematics, Division of Science; and
Center for Quantum and Topological Systems,
NYUAD Research Institute,
New York University Abu Dhabi, UAE.

CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS

The unreasonable effectiveness of Super-Tangent Spaces. Consider the 11D super-tangent space

$$\underset{\text{super-Minkowski}}{\mathbb{R}^{1,10\,|\,\mathbf{32}}} \hookrightarrow \underset{\text{super-Poincaré}}{\mathrm{Iso}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big)} \twoheadrightarrow \underset{\text{Lorentz}}{\mathfrak{so}(1,10)}$$

with its super-invariant 1-forms

$$\mathrm{CE}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big) \;\simeq\; \underset{\text{super-transl. invar. forms}}{\Omega^\bullet_{\mathrm{dR}}\big(\mathbb{R}^{1,10|\mathbf{32}}\big)^{\mathrm{li}}} \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{l}(\Psi^\alpha)_{\alpha=1}^{32} \\ (E^a)_{a=0}^{10}\end{array}\right] \Big/ \left(\begin{array}{lcl}\mathrm{d}\,\Psi^\alpha &=& 0 \\ \mathrm{d}\,E^a &=& \big(\overline{\Psi}\,\Gamma^a\,\Psi\big)\end{array}\right)$$

Remarkably, the 11D quartic Fierz identities entail that

$$\left.\begin{array}{lcl} G_4^0 &:=& \tfrac{1}{2}\big(\overline{\Psi}\,\Gamma_{a_1a_2}\,\Psi\big)\,E^{a_1}E^{a_2} \\ G_7^0 &:=& \tfrac{1}{5!}\big(\overline{\Psi}\,\Gamma_{a_1\cdots a_5}\,\Psi\big)\,E^{a_1}\cdots E^{a_5}\end{array}\right\} \in \underset{\text{fully super-invariant forms}}{\mathrm{CE}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big)^{\mathrm{Spin}(1,10)}} \quad \text{satisfy :} \quad \begin{array}{l}\mathrm{d}\,G_4^0 \,=\, 0 \\ \mathrm{d}\,G_7^0 \,=\, \tfrac{1}{2}G_4^0\,G_4^0\end{array}$$

To globalize this situation, say that an **11D super-spacetime** X is a super-manifold equipped with a super-Cartan connection, locally on an open cover $\widetilde{X} \twoheadrightarrow X$ given by

$$\left.\begin{array}{l}(\Psi^\alpha)_{\alpha=1}^{32} \\ (E^a)_{a=0}^{10} \\ \big(\Omega^{ab} = -\Omega^{ba}\big)_{a,b=0}^{10}\end{array}\right\} \in \Omega^1_{\mathrm{dR}}\big(\widetilde{X}\big) \quad \begin{array}{l}\text{such that the} \\ \text{super-torsion} \\ \text{vanishes}\end{array} \quad \mathrm{d}\,E^a - \Omega^a{}_b\,E^b \;=\; \big(\overline{\Psi}\,\Gamma^a\,\Psi\big)\,,$$

and say that **C-field super-flux** on such a super-spacetime are super-forms with these co-frame components:

$$\boxed{\begin{array}{lclcl} G_4^s &:=& G_4 + G_4^0 &:=& \tfrac{1}{4!}(G_4)_{a_1\cdots a_4}E^{a_1}\cdots E^{a_4} + \tfrac{1}{2}\big(\overline{\Psi}\,\Gamma_{a_1a_2}\,\Psi\big)E^{a_1}\,E^{a_2} \\ G_7^s &:=& G_7 + G_7^0 &:=& \tfrac{1}{7!}(G_4)_{a_1\cdots a_7}E^{a_1}\cdots E^{a_7} + \tfrac{1}{5!}\big(\overline{\Psi}\,\Gamma_{a_1\cdots a_5}\,\Psi\big)E^{a_1}\cdots E^{a_5}\end{array}}$$

Theorem [JHEP07(2024)082]: On an 11D super-spacetime X with C-field super-flux (G_4^s, G_7^s):

$$\begin{array}{c}\textit{The duality-symmetric} \\ \textit{super-Bianchi identity}\end{array} \left\{\begin{array}{lcl}\mathrm{d}\,G_4^s &=& 0 \\ \mathrm{d}\,G_7^s &=& \tfrac{1}{2}\,G_4^s\,G_4^s\end{array}\right\} \begin{array}{c}\textit{is equivalent to} \\ \textit{the full 11D SuGra} \\ \textit{equations of motion!}\end{array}$$

Next consider the involution $\Gamma_{012345} \in \mathrm{Pin}^+(1,10)$ with super-fixed subspace $\mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}_+} \xhookrightarrow{\phi_0} \mathbb{R}^{1,10\,|\,\mathbf{32}}$
Since $\overline{\Gamma_{012345}} = -\Gamma_{012345}$ it follws that, simply:

$$H_3^0 \;:=\; 0 \;\in\; \mathrm{CE}\big(\mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}_+}\big)^{\mathrm{Spin}(1,5)} \qquad \text{satisfies :} \qquad \mathrm{d}\,H_3^0 \;=\; \phi_0^*\,G_4^0$$

To globalize this situation, say that a super-immersion $\Sigma^{1,5\,|\,2\cdot\mathbf{8}_+} \xrightarrow{\phi_s} X^{1,10\,|\,\mathbf{32}}$ is ½**BPS M5** if it is "locally like" [1] ϕ_0, and say that **B-field super-flux** on such an M5-probe is a super-form with these co-frame components:

$$\boxed{H_3^s \;:=\; H_3 + H_3^0 \;:=\; \tfrac{1}{3!}(H_3)_{a_1a_2a_3}e^{a_1}\,e^{a_2}\,e^{a_3} + 0} \qquad \big(e^{a<6} \;:=\; \phi_s^*E^a\big)$$

Theorem [JHEP10(2024)140]: On a super-immersion ϕ_s with B-field super-flux H_3^s:

$$\begin{array}{c}\textit{The} \\ \textit{super-Bianchi identity}\end{array} \quad \big\{\mathrm{d}\,H_3^s \;=\; \phi_s^*G_4^s\big\} \quad \begin{array}{c}\textit{is equivalent to} \\ \textit{the ½BPS M5} \\ \textit{equations of motion.}\end{array}$$

In particular, the (self-)duality conditions on the ordinary fluxes are *implied*: $G_4 \leftrightarrow G_7$ and $H_3 \leftrightarrow H_3$.

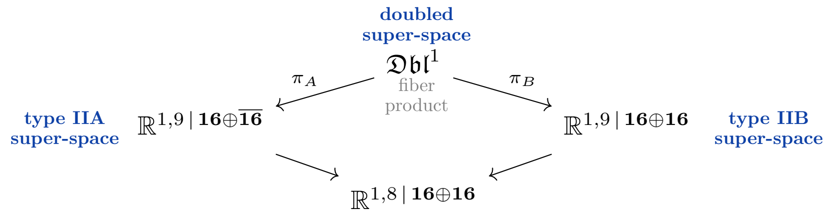
These results witness a strong form of Cartan geometry (globalized/curved Kleinian geometry). As slogans:

11D SuGra is the globalization of the super-tangent space $\mathbb{R}^{1,10\|\mathbf{32}}$ *including* its super-flux content.	M5-probes are the globalization of the immersion $\mathbb{R}^{1,5\|2\cdot\mathbf{8}_+} \hookrightarrow \mathbb{R}^{1,10\|\mathbf{32}}$ *including* its super-flux content.

This motivates looking for more hidden structure on more super-tangent spaces.

[1] The technical condition on a super-immersion to be ½BPS is that it admits the super-analog of a *Darboux coframe*, see §2.2 in JHEP10(2024)140.

More hidden structure in Super-Space. After reduction to 10D it turns out [ATMP22(2018)5] that the whole structure of (topological) T-duality is preconfigured in the super-fluxes on super-tangent spaces:



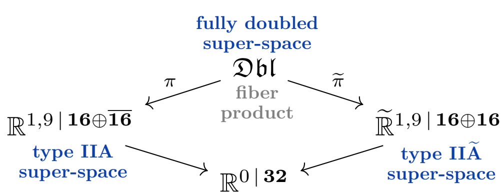
$$
\begin{aligned}
H_3^A &:= \big(\overline{\Psi}\,\Gamma_a \Gamma_{10}\,\Psi\big)E^a\\
F_{\leq 0} &:= 0\\
F_2 &:= \big(\overline{\Psi}\,\Gamma_{10}\,\Psi\big)\\
F_4 &:= \tfrac{1}{2}\big(\overline{\Psi}\,\Gamma_{a_1 a_2}\,\Psi\big)E^{a_1}E^{a_2}\\
F_6 &:= \tfrac{1}{4!}\big(\overline{\Psi}\,\Gamma_{10}\Gamma_{a_1\cdots a_4}\,\Psi\big)E^{a_1}\cdots E^{a_4}\\
F_8 &:= \tfrac{1}{6!}\big(\overline{\Psi}\,\Gamma_{a_1\cdots a_6}\,\Psi\big)E^{a_1}\cdots E^{a_6}\\
F_{10} &:= \tfrac{1}{8!}\big(\overline{\Psi}\,\Gamma_{10}\Gamma_{a_1\cdots a_8}\,\Psi\big)E^{a_1}\cdots E^{a_8}\\
F_{12} &:= \tfrac{1}{10!}\big(\overline{\Psi}\,\Gamma_{a_1\cdots a_{10}}\,\Psi\big)E^{a_1}\cdots E^{a_{10}}\\
F_{\geq 14} &= 0
\end{aligned}
\quad \in \mathrm{CE}\big(\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}\big)
\quad \text{s.t.} \begin{cases} \mathrm{d}\,H_3^A = 0 \\ \mathrm{d}\,F_{2\bullet+2} = H_3^A\,F_{2\bullet}\,, \end{cases}
$$

$$
\int_{\pi_B} e^{P_2^1}\cdot \pi_A^*(-) \quad \text{(Fourier-Mukai transform)}
$$

super-Poincaré form $\underbrace{P_2^1 := e_B^9\, e_A^9}_{\in\, \mathrm{CE}(\mathfrak{Dbl}^1)}$ s.t. $\mathrm{d}\,P_2^1 = \pi_A^* H_3^A - \pi_B^* H_3^B$

$$
\begin{aligned}
H_3^B &:= \big(\overline{\Psi}\,\Gamma_a^B\,\Gamma_{10}\,\Psi\big)E^a\\
F_{\leq 1} &= 0\\
F_3 &:= \big(\overline{\Psi}\,\Gamma_a^B\Gamma_9\,\Psi\big)E^a\\
F_5 &:= \tfrac{1}{3!}\big(\overline{\Psi}\,\Gamma^B_{a_1a_2a_3}\Gamma_9\Gamma_{10}\,\Psi\big)E^{a_1}E^{a_2}E^{a_3}\\
F_7 &:= \tfrac{1}{5!}\big(\overline{\Psi}\,\Gamma^B_{a_1\cdots a_5}\Gamma_9\,\Psi\big)E^{a_1}\cdots E^{a_5}\\
F_9 &:= \tfrac{1}{7!}\big(\overline{\Psi}\,\Gamma^B_{a_1\cdots a_7}\Gamma_9\Gamma_{10}\,\Psi\big)E^{a_1}\cdots E^{a_7}\\
F_{11} &:= \tfrac{1}{7!}\big(\overline{\Psi}\,\Gamma^B_{a_1\cdots a_9}\Gamma_9\,\Psi\big)E^{a_1}\cdots E^{a_9}\\
F_{\geq 13} &:= 0
\end{aligned}
\quad \in \mathrm{CE}\big(\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}}\big)
\quad \text{s.t.} \begin{cases} \mathrm{d}\,H_3^B = 0 \\ \mathrm{d}\,F_{2\bullet+1} = H_3^B\,F_{2\bullet-1}\,, \end{cases}
$$

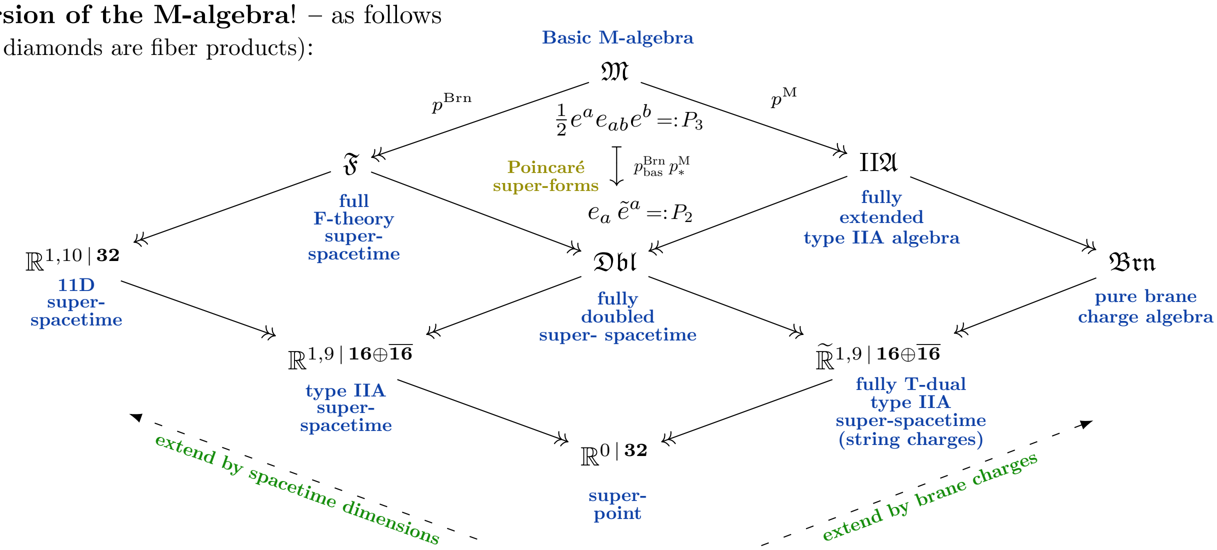
This generalizes all the way to **T-duality along <u>all</u> 10 space-time directions**, where we have [`arXiv:2411.10260`]



$$
\big(H_3^A,\, F_{2\bullet}\big) \xrightarrow{\ \int_{\widetilde{\pi}} e^{P_2}\cdot\pi^*(-)\ } \big(H_3^{\widetilde{A}},\, \widetilde{F}_{2\bullet}\big) \quad \text{(Fourier-Mukai transform)}
$$

full super-Poincaré form $\underbrace{P_2 := e^a\,\widetilde{e}_a}_{\in\,\mathrm{CE}(\mathfrak{Dbl})}$ s.t. $\mathrm{d}\,P_2 = \pi^* H_3^A - \widetilde{\pi}^* H_3^{\widetilde{A}}$

But the fully doubled $\mathfrak{Dbl}$ is a 10D **version of the M-algebra!** – as follows (all diamonds are fiber products):



The M-Algebra is the super-Lie algebra which is the maximal central extension of the $\mathcal{N}=32$ super-point:

$$\begin{array}{c}\mathfrak{M}\\ \downarrow\\ \mathbb{R}^{0|\mathbf{32}}\end{array} \quad \text{hence} \quad \mathrm{CE}(\mathfrak{M}) \;\simeq\; \mathbb{R}\Big[\underbrace{(\Psi^\alpha)_{\alpha=1}^{32}}_{\deg=(1,\mathrm{odd})},\; \underbrace{(E^a)_{a=0}^{10}}_{\deg=(1,\mathrm{evn})},\; \underbrace{\big(E_{a_1 a_2}=E_{[a_1 a_2]}\big)_{a_i=0}^{10}}_{\deg=(1,\mathrm{evn})},\; \underbrace{\big(E_{a_1\cdots a_5}=E_{[a_1\cdots a_5]}\big)_{a_i=0}^{10}}_{\deg=(1,\mathrm{evn})}\Big],$$

with differential on generators given equivalently by [`arXiv:2411.11963`]:

$$\left.\begin{aligned} \mathrm{d}\,\Psi &= 0\\ \mathrm{d}\,E^a &= +\big(\overline{\Psi}\,\Gamma^a\,\Psi\big)\\ \mathrm{d}\,E_{a_1a_2} &= -\big(\overline{\Psi}\,\Gamma_{a_1a_2}\,\Psi\big)\\ \mathrm{d}\,E_{a_1\cdots a_5} &= +\big(\overline{\Psi}\,\Gamma_{a_1\cdots a_5}\,\Psi\big)\end{aligned}\right\} \xleftrightarrow[\text{by Fierz decomposition}]{E^{\alpha\beta} := \frac{1}{32}\left(E^a\,\Gamma_a^{\alpha\beta} + \frac{1}{2}E^{a_1a_2}\,\Gamma_{a_1a_2}^{\alpha\beta} + \frac{1}{5!}E^{a_1\cdots a_5}\,\Gamma_{a_1\cdots a_5}^{\alpha\beta}\right)} \left\{\begin{aligned}\mathrm{d}\,\Psi^\alpha &= 0\\ \mathrm{d}\,E^{\alpha\beta} &= \Psi^\alpha\,\Psi^\beta\,.\end{aligned}\right.$$

The Fierz-form on the right shows that $\mathrm{Aut}(\mathfrak{M}) \simeq \mathrm{GL}(32)$ (West '99: "brane-rotating symmetry") $\supset \mathrm{Spin}(1,10)$.

It is suggestive to **Hodge-dualize temporal components** in order to identify actual (probe-)brane charges (Hull 1997).

$$\begin{aligned}\mathfrak{M}_{\mathrm{bos}} &\simeq \mathbb{R}^{1,10} \oplus \wedge^2(\mathbb{R}^{1,10})^* \oplus \wedge^5(\mathbb{R}^{1,10})^*\\ &\simeq \underbrace{\mathbb{R}^{0,1}}_{\text{time}} \oplus \underbrace{\mathbb{R}^{10}}_{\text{space}} \oplus \underbrace{\wedge^2(\mathbb{R}^{10})^*}_{\text{M2-brane charges}} \oplus \underbrace{\wedge^9\mathbb{R}^{10}}_{\text{“9-brane” charges}} \oplus \underbrace{\wedge^5(\mathbb{R}^{10})^*}_{\text{M5-brane charges}} \oplus \underbrace{\wedge^6\mathbb{R}^{10}}_{\text{“6-brane” charges}}\end{aligned}$$

The **reduction** of the M-algebra to 10D is the fully extended II$\mathfrak{A}$ SuSy algebra, with:

$$\left(\begin{aligned} \mathrm{d}\,\Psi &= 0\\ \mathrm{d}\,E^a &= +\big(\overline{\Psi}\,\Gamma^a\,\Psi\big)\\ \mathrm{d}\,\tilde{E}_a &= -\big(\overline{\Psi}\,\Gamma_a\Gamma_{10}\,\Psi\big)\\ \mathrm{d}\,E_{a_1a_2} &= -\big(\overline{\Psi}\,\Gamma_{a_1a_2}\,\Psi\big)\\ \mathrm{d}\,E_{a_1\cdots a_4} &= +\big(\overline{\Psi}\,\Gamma_{a_1\cdots a_4}\Gamma_{10}\,\Psi\big)\\ \mathrm{d}\,E_{a_1\cdots a_5} &= +\big(\overline{\Psi}\,\Gamma_{a_1\cdots a_5}\,\Psi\big)\end{aligned}\right)$$

$$\begin{aligned}\mathrm{II}\mathfrak{A}_{\mathrm{bos}}\\ &\simeq_{\mathbb{R}} \mathbb{R}^{1,9} \oplus (\mathbb{R}^{1,9})^* \oplus \wedge^2(\mathbb{R}^{1,9})^* \oplus \wedge^4(\mathbb{R}^{1,9})^* \oplus \wedge^5(\mathbb{R}^{1,9})^*\\ &\simeq_{\mathbb{R}} \underbrace{\mathbb{R}^{1,9}}_{\text{space-time}} \oplus \underbrace{(\mathbb{R}^{1,9})^*}_{\text{fully T-dual space-time / string charges}} \oplus \underbrace{\wedge^2(\mathbb{R}^9)^*}_{\text{D2-brane charges}} \oplus \underbrace{\wedge^8(\mathbb{R}^9)}_{\text{D8-brane charges}} \oplus \underbrace{\wedge^4(\mathbb{R}^9)^*}_{\text{D4-brane charges}} \oplus \underbrace{\wedge^6(\mathbb{R}^9)}_{\text{D6-brane charges}} \oplus \underbrace{\wedge^5(\mathbb{R}^{1,9})^*}_{\text{NS5-brane charges}}\end{aligned}$$

The full doubling of the 10D super-space is by the wrapped M2-brane charges in the M-algebra!
[`arXiv:2411.10260`]

	$\mathfrak{M}$	$\xrightarrow{\text{M-circle fibration}}$	II$\mathfrak{A}$	$\xrightarrow{\text{forget brane charges}}$	$\mathfrak{Dbl}$
	Ψ	$\longleftarrow$	Ψ	$\longleftarrow$	Ψ
	E^a	$\longleftarrow$	E^a	$\longleftarrow$	E^a
wrapped M2-brane charges	$E_{a\,10}$	$\longleftarrow$	$\tilde{E}_a$	string charges / dbld spacetime	$\tilde{E}_a$
	$E_{a_1a_2}$	$\longleftarrow$	$E_{a_1a_2}$		
	$E_{10\,a_1\cdots a_4}$	$\longleftarrow$	$E_{a_1\cdots a_4}$		
	$E_{a_1\cdots a_5}$	$\longleftarrow$	$E_{a_1\cdots a_5}\,.$		

The **restriction** of the bosonic spatial part of the M-algebra to $\mathbb{R}^n \hookrightarrow \mathbb{R}^{10}$ for $n \in \{4,5,6,7\}$ yields the traditional *exceptional tangent spaces* of Hull 2007:

n	$\dim\big(\mathbb{R}^n$	$\oplus \wedge^2(\mathbb{R}^n)^*$	$\oplus \wedge^5(\mathbb{R}^n)^*$	$\oplus \wedge^6\mathbb{R}^n$	$\oplus \wedge^9\mathbb{R}^n\big)$		basic rep of	excptnl Lie alg
4	4	+ 6				$\rightsquigarrow$	**10**	$\mathfrak{sl}_{5(5)}$
5	5	+ 10	+ 1			$\rightsquigarrow$	**16**	$\mathfrak{so}_{5,5}$
6	6	+ 15	+ 6	+ 1		$\rightsquigarrow$	$\mathbf{27}\oplus\mathbf{1}$	$\mathfrak{e}_{6(6)}$
7	7	+ 21	+ 21	+ 7		$\rightsquigarrow$	**56**	$\mathfrak{e}_{7(7)}$

Traditional discussion stops here at $n=7$ because **this pattern breaks** for $n \geq 8$.

There are *not enough M-brane charges* to carry the basic $\mathfrak{e}_{\geq 8}$-rep!

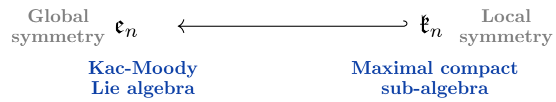
But our identification of the M-algebra as an M-theoretic incarnation of the fully doubled super-spacetime, supporting super-space T-duality, suggests that in some sense:

The M-algebra is the full super-exceptional tangent space, after all.

resolution on next page...

The M-Algebra completes the hierarchy of Exceptional Tangent Spaces [arXiv:2411.03661] as follows, by appeal to these two observations by Nicolai, Kleinschmidt, et al.:

(i) Local hidden symmetry: While the Kac-Moody Lie algebras $\mathfrak{e}_n$ reflect the expected *global* hidden symmetry, it is only their "maximal compact" (or "involutory") subalgebras, which reflect the corresponding *local* hidden symmetry.



(ii) Spinorial hidden symmetry: In contrast to the Kac-Moody algebras $\mathfrak{e}_n$ themselves, their maximal compact $\mathfrak{k}_n$ have non-trivial *finite-dimensional* representations. Among these is a spinorial **32** both for $\mathfrak{k}_{10}$ as well as for $\mathfrak{k}_{1,10}$, which lifts the familiar Majorana spinor representation of 11D SuGra.

$$\begin{array}{ccccc} \mathfrak{so}_{1,10} & \longrightarrow & \mathfrak{k}_{1,10} & \longleftarrow & \mathfrak{k}_{10} \\ \mathbf{32} & \longleftarrow\!\!\shortmid & \mathbf{32} & \shortmid\!\!\longrightarrow & \mathbf{32}\,. \end{array}$$

Thereby we find this shaded completion of the hierarchy, as explained below:

n	$\dim\big(\ \mathbb{R}^n$	$\oplus\ \wedge^2(\mathbb{R}^n)^*$	$\oplus\ \wedge^5(\mathbb{R}^n)^*$	$\oplus\ \wedge^6\mathbb{R}^n$	$\oplus\ \wedge^9\mathbb{R}^n\big)$		basic rep	of (max cmpt sub-alg	of)	excptnl Lie alg
4	4	+ 6				$\rightsquigarrow$	**10**			$\mathfrak{sl}_{5(5)}$
5	5	+ 10	+ 1			$\rightsquigarrow$	**16**			$\mathfrak{so}_{5,5}$
6	6	+ 15	+ 6	+ 1		$\rightsquigarrow$	$\mathbf{27} \oplus \mathbf{1}$			$\mathfrak{e}_{6(6)}$
7	7	+ 21	+ 21	+ 7		$\rightsquigarrow$	**56**			$\mathfrak{e}_{7(7)}$
8	8	+ 28	+ 56	+ 28		$\rightsquigarrow$	**120**	$\mathfrak{so}_{16}$	$\subset$	$\mathfrak{e}_8$
9	9	+ 36	+ 126	+ 84	+ 1	$\rightsquigarrow$	**256**	$\mathfrak{k}_9$	$\subset$	$\mathfrak{e}_9$
10	10	+ 45	+ 252	+ 210	+ 10	$\rightsquigarrow$	**527**	$\mathfrak{k}_{10}$	$\subset$	$\mathfrak{e}_{10}$
1+10	$\dim\Big(\mathbb{R}^{1,10} \oplus \wedge^2(\mathbb{R}^{1,10})^* \oplus \wedge^5(\mathbb{R}^{1,10})^*\Big)$					$\rightsquigarrow$	**528**	$\mathfrak{k}_{1,10}$	$\subset$	$\mathfrak{e}_{11}$

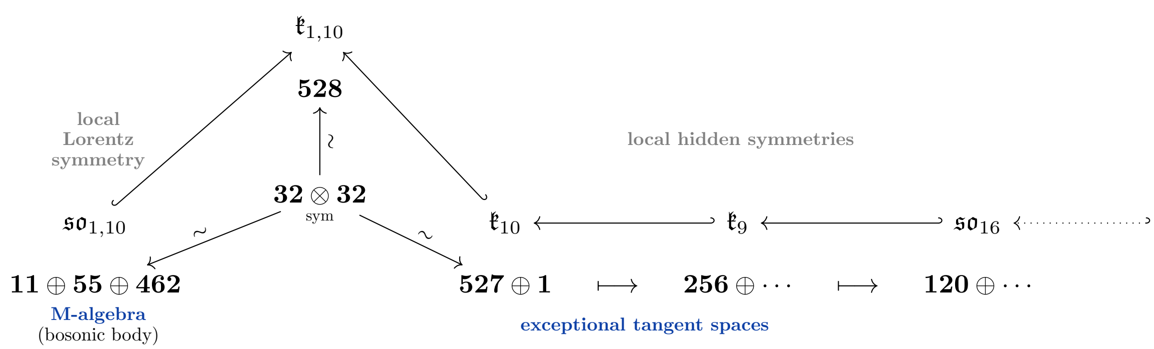
- $\mathbf{n = 8}$: the **248** of $\mathfrak{e}_8$ branches as $\mathbf{120} \oplus \mathbf{128}$ of the maximal compact $\mathfrak{so}_{16}$ — as a representation-theoretic statement this is classical, but as part of a change in pattern from $\mathfrak{e}_n$ to $\mathfrak{k}_n$ this may not have been appreciated.

- $\mathbf{n = 9}$: the (infinite-dimensional) basic rep of $\mathfrak{e}_9$ branches as $\mathbf{256} \oplus$higher-parabolic-levels under $\mathfrak{k}_9$ — this was only very recently shown by König 2024;

- $\mathbf{n = 10}$: remarkably, there is an irrep **527** of $\mathfrak{k}_{10}$, and it appears in the symmetric square of a spinorial **32** irrep as: $\mathbf{32} \otimes_{\mathrm{sym}} \mathbf{32} \;\simeq\; \mathbf{1} \oplus \mathbf{527}$ [Damour, Kleinschmidt & Nicolai 2006 p 37], which exactly matches the interpretation here, where the bosonic dimension of the M-algebra is the same expression $\dim(\mathbf{32} \otimes_{\mathrm{sym}} \mathbf{32})$ — the remaining **1** is the first summand (the time axis);

- $\mathbf{n = 1 + 10}$: re-including this temporal component and hence going back to the unbroken bosonic M-algebra we need an irrep **528** of $\mathfrak{k}_{1,10}$; this also exists [Gomis, Kleinschmidt & Palmkvist 2019 p 29] and it is isomorphic to the symmetric square $\mathbf{32} \otimes_{\mathrm{sym}} \mathbf{32} \;\simeq\; \mathbf{528}$ [Bossard, Kleinschmidt & Sezgin 2019 §D] of the original **32**.

In summary this means that the **528** of $\mathfrak{k}_{1,10}$ is the root of the hierarchy of exceptional tangent spaces, while at the same time exactly unifying **11**-dimensional spacetime with the **55** M2- and **462** M5-brane charges:



Finally, the infinite-dimensional $\mathfrak{k}_{1,10}$ must act through a finite-dimensional quotient on **528**, and this turns out [Bossard, Kleinschmidt & Sezgin 2019 p. 42] to be just the "brane-rotating symmetry" $\mathrm{SL}(32) \subset \mathrm{Aut}(\mathfrak{M})$, so that: All this lifts to the M-algebra $\mathfrak{M}$, thus identified as the "**super-exceptional tangent space**"!

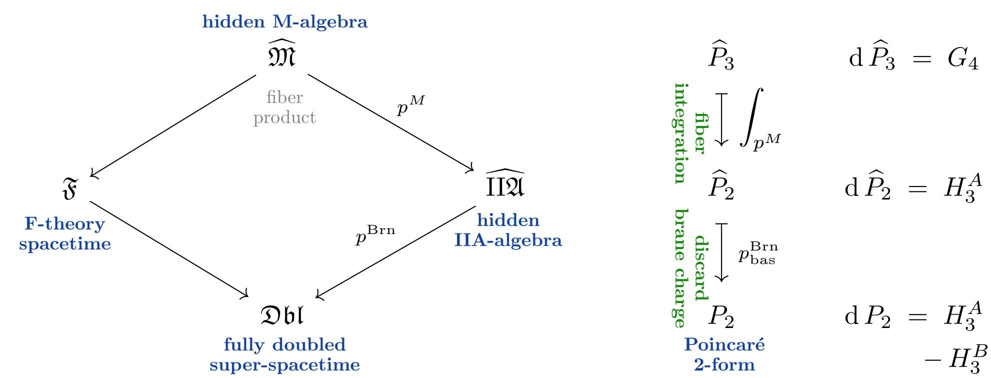
T-Duality on Super-exceptional space?

The SL(32)-symmetry on $\mathfrak{M}$ which is generated by Γ_{10} is as shown on the right. [arXiv:2411.11963 Ex. 2.4]

$$
\begin{array}{ccc}
\mathfrak{M} & \xleftarrow{\exp\left(r\,\Gamma_{10}\right)} & \mathfrak{M} \\
E^{10} & \longmapsto & E^{10} \\
E^{a} & \longmapsto & \cosh(2r) E^{a} - \sinh(2r)\, E^{a\,10} \\
E^{a\,10} & \longmapsto & \cosh(2r)\, E^{a\,10} - \sinh(2r) E^{a} \\
E^{ab} & \longmapsto & E^{ab} \\
E^{a_1 \cdots a_5} & \longmapsto & \cosh(2r)\, E^{a_1 \cdots a_5} + \sinh(2r) \star E^{a_1 \cdots a_5} \\
E^{a_1 \cdots a_4\,10} & \longmapsto & E^{a_1 \cdots a_4\,10}
\end{array}
\qquad
\begin{array}{l}
r \in \mathbb{R} \\
a_i < 10
\end{array}
$$

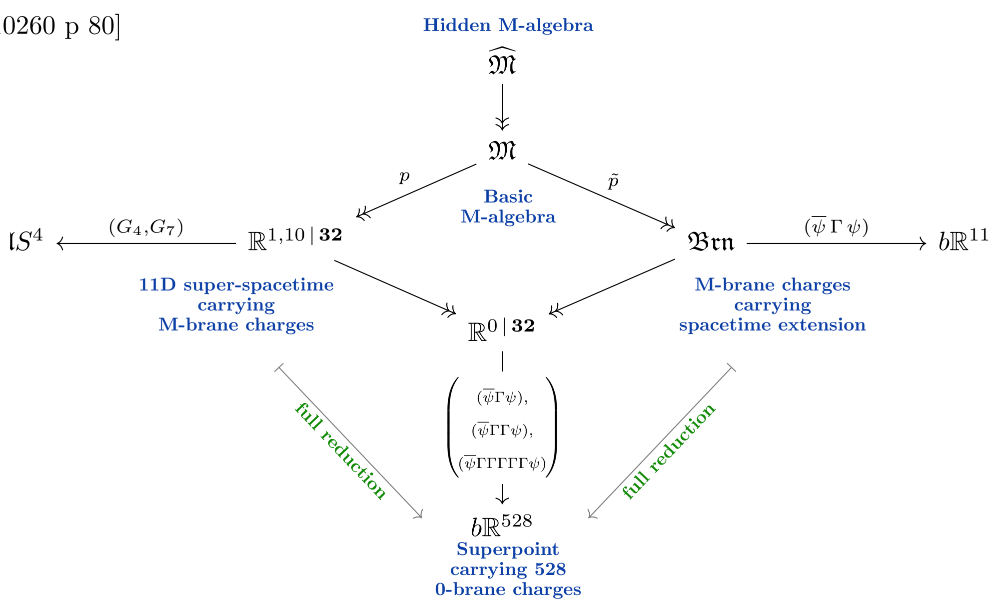
This *mixes* the 10D spacetime directions E^a with their T-duals $E^{a\,10}$ but *never swaps* them.
Hence T-duality is *not* among the "brane-rotating symmetries" $\mathrm{GL}(32) \simeq \mathrm{Aut}(\mathfrak{M})$.
Indeed, we already saw it must instead be a kind of Fourier-Mukai transformation lifted to $\mathfrak{M}$.
For this there ought to be an M-theoretic Poincaré 3-form which reduces to the Poincaré 2-form.

Proposition [arXiv:2411.11963, §2.2.3]: There exists a *fermionic extension* $\widehat{(-)}$ (not changing the bosonic body) of $\mathrm{II}\mathfrak{A}$, and hence of $\mathfrak{M}$, on which the Poincaré 2-form P_2 (controlling super-space T-duality) lifts as follows:



(Here $\widehat{\mathfrak{M}}$ is the "hidden" extension for parameter $s = -1$ of D'Auria & Fré 1982 and Bandos et al. 2004.)

So while in 10D the Poincareé 2-form cancels the difference between the dual B-field fluxes,
in M-theory the Poincaré 3-form cancels the C-field flux itself.
This makes sense, because the M-algebra correspondence absorbs all fluxes into the exceptionalized geometry!
[arXiv:2411.10260 p 80]



Flux quantization. One upshot of these results is their implication on flux-quantization [EncMathPhys4(2025)281]:

Underappreciated Fact:

A C-field configuration is more than a differential 3-form C_3 (and a 6-form C_6).
This is only the data on a *single chart* $\mathbb{R}^{1,10\,|\,\mathbf{32}} \simeq U_i \xhookrightarrow{\iota_i} X$.
A global C-field configuration is instead:
C_3 & C_6 on charts of an open cover of spacetime X,
& gauge transformations on double intersections of charts
& gauge-of-gauge-transformation on triple intersections of charts
& higher gauge transformations on higher intersections of charts
all subject to *some* flux- or charge-quantization law.

Case of electromagnetic field. This is familiar from the electromagnetic field
which is a 1-form A_i on each chart
with gauge transformations $\lambda_{ij} : A_j = A_i + \mathrm{d}\lambda_{ij}$ on each double intersection
and charge quantization $\lambda_{ij} + \lambda_{jk} - \lambda_{ik} = n_{ijk}$ on each triple intersection
making a cocycle in *ordinary differential cohomology*
(equivalently to a principal U(1)-bundle with connection).
This flux quantization *stabilizes the solitons* of electromagnetism: Dirac monopoles and Abrikosov vortices.

Case of NS/RR-field. The analogue is famous for the NS/RR-fields in IIA:
the *duality symmetric* fluxes $\mathrm{d}\,F_{2\bullet} = F_{2\bullet-2}\,H_3$ have the form of the image of the Chern-character on K-theory
hence one may ask that the RR-fields are globally cocycles in differential K-theory.
Doing so stabilizes certain non-supersymmetric D-branes.

Case of C-field in 11D bulk [JHEP07(2024)082].
Similarly the C-field may be flux-quantized in any generalized cohomology theory
whose character image is of the form $\mathrm{d}\,G_4 = 0,\ \mathrm{d}\,G_7 = \frac{1}{2} G_4\, G_4$ (e.g.: 4-Cohomotopy).
except for one issue: this does not *seem* to account for the constraint $G_7 = \star\, G_4$
resolution: on superspace this constraint is already implied by $\mathrm{d}\,G_4^s = 0,\ \mathrm{d}\,G_7^s = \frac{1}{2} G_4^s\, G_4^s$

Case of the Self-dual field on M5 [JHEP10(2024)140].
Similarly the tensor field on M5 probes $\Sigma \xrightarrow{\phi} X$ may be flux-quantized in any generalized cohomology theory
whose character image is in addition of the form $\mathrm{d}\,H_3 = \phi^* G_4$ (e.g. bulk-twisted 3-Cohomotopy).
except for one issue: this does not *seem* to account for the non-linear self-duality constraint
resolution on superspace this constraint is already implied by $\mathrm{d}\,H_3^s = \phi^* G_4^s$

In summary: On super-space, the Bianchi-identities on the super-fluxes
determine the admissible flux-quantization laws and hence
determine the possible global completions of the SuGra field content.

Vista.
On the other hand, exceptional geometry seems to provide *local* completion of SuGra field content.
Hence the full completion of 11D SuGra ("M-Theory") seems to require super-exceptional geometry:

Flux Quantization:	Global completion	} of 11D Sugra field content {	via	super-geometry
E_{10}, E_{11}-Program:	Local completion		via	exceptional-geometry
suggests:	Full completion	of 11D Sugra	via super-exceptional geometry ?	

Where super-exceptional geometry should be Cartan geometry locally modeled on the M-algebra $\mathfrak{M}$.