Urs Schreiber on joint work with Hisham Sati:

an invitation to the monographs:

-*Equivariant Principal ∞-Bundles*, CUP (2025, in print)
-*Geometric Orbifold Cohomology*, CRC (2026, to appear)

Geometric Orbifold Cohomology in Singular-Cohesive ∞-Topoi

talk at *ItaCa Fest 2025*

17 June 2025

CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS

@

جامعة نيويورك أبوظبي
NYU | ABU DHABI

Cohomology – Motivation.

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in math: topology via algebra $\left(\begin{array}{l}\text{Hopf-, Kervaire-invrnt,}\\ \text{chromatic nilpotence, ...}\end{array}\right)$

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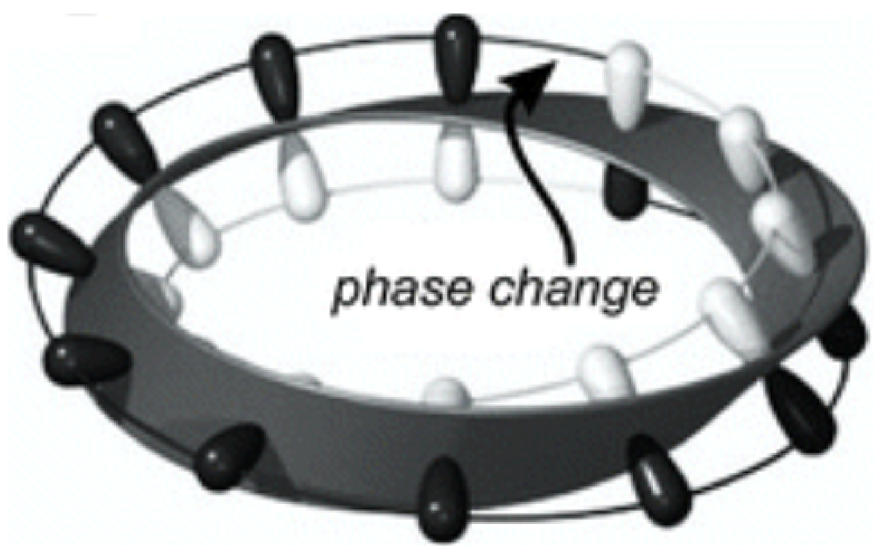

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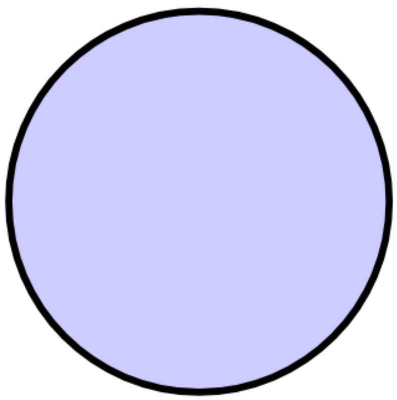

local chart
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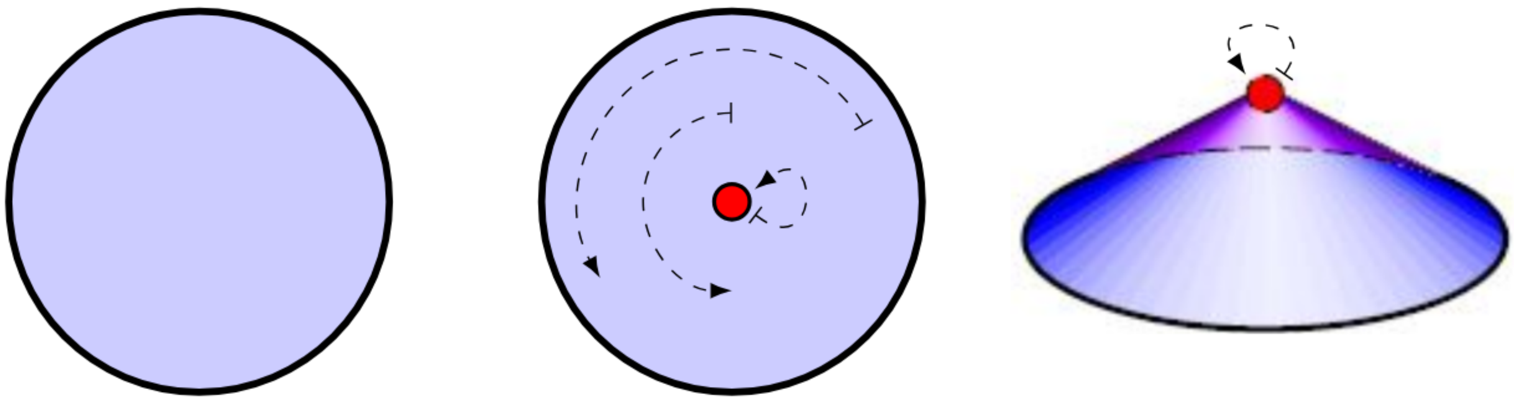

local chart of a manifold

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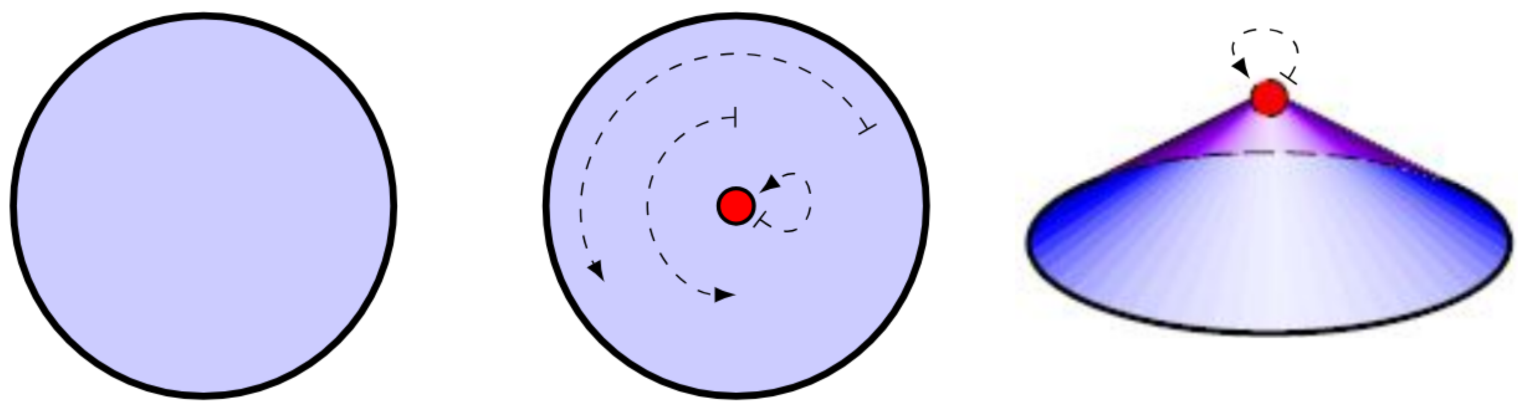

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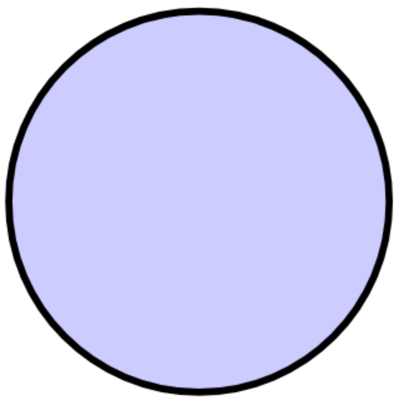

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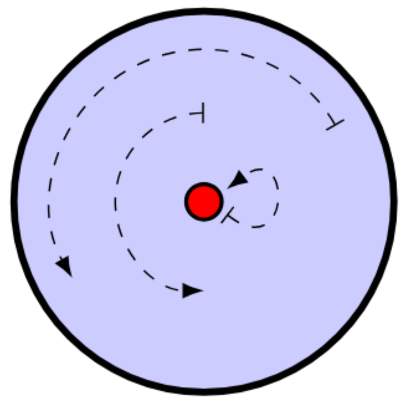

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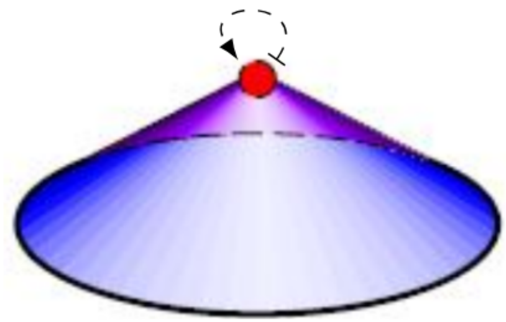

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orbifolds $=$ smooth manifolds with identifications of points

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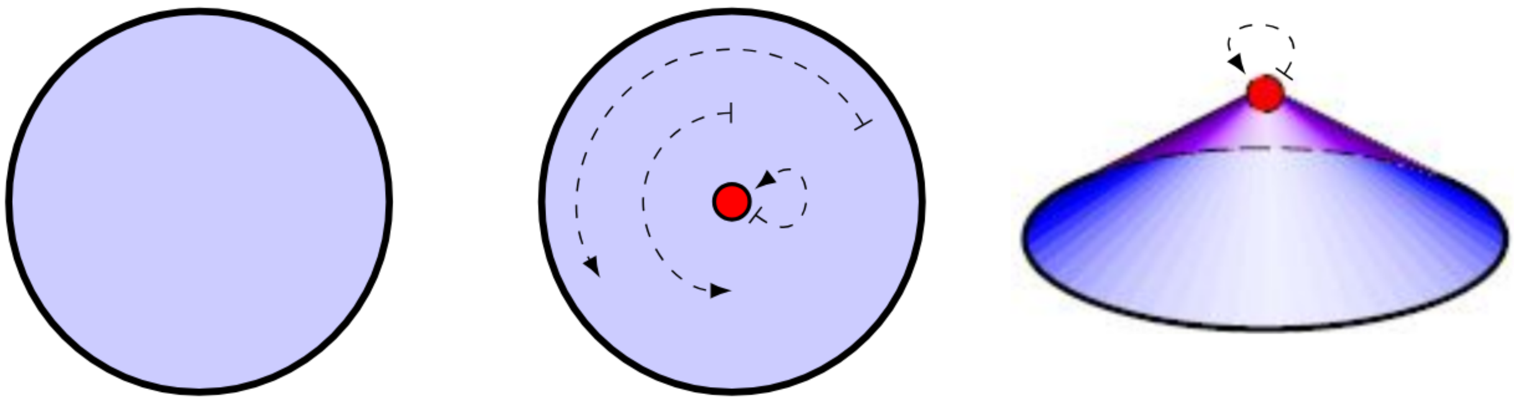

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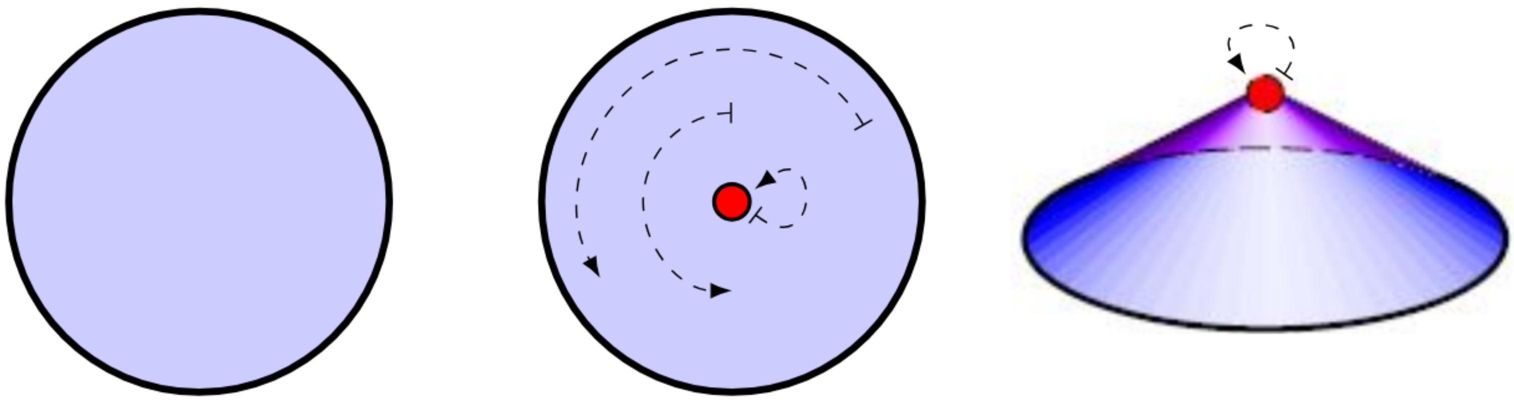

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$$
\begin{aligned}
\text{orbifolds} &= \text{smooth manifolds with identifications of points} \\
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&= \textbf{smooth stacks!} \text{ (étale) if regarded intrinsically}
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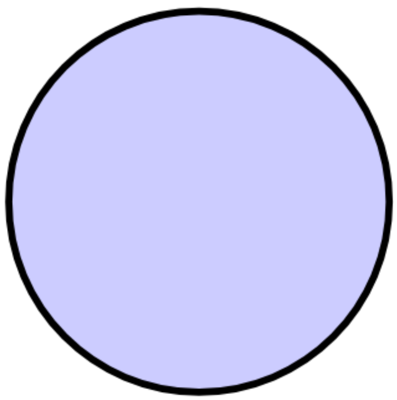

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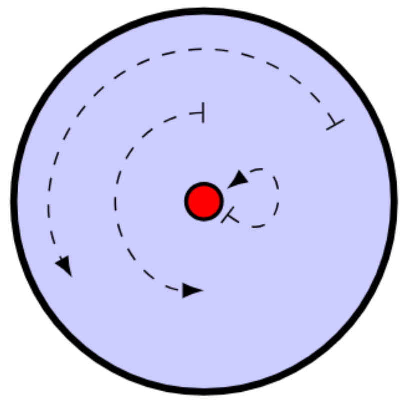

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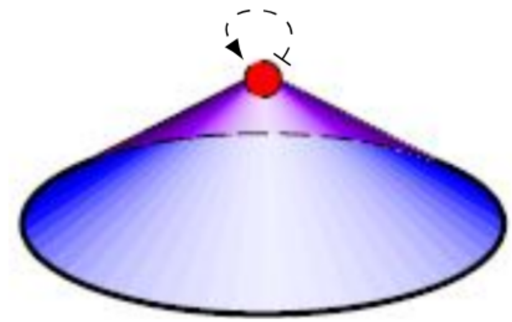

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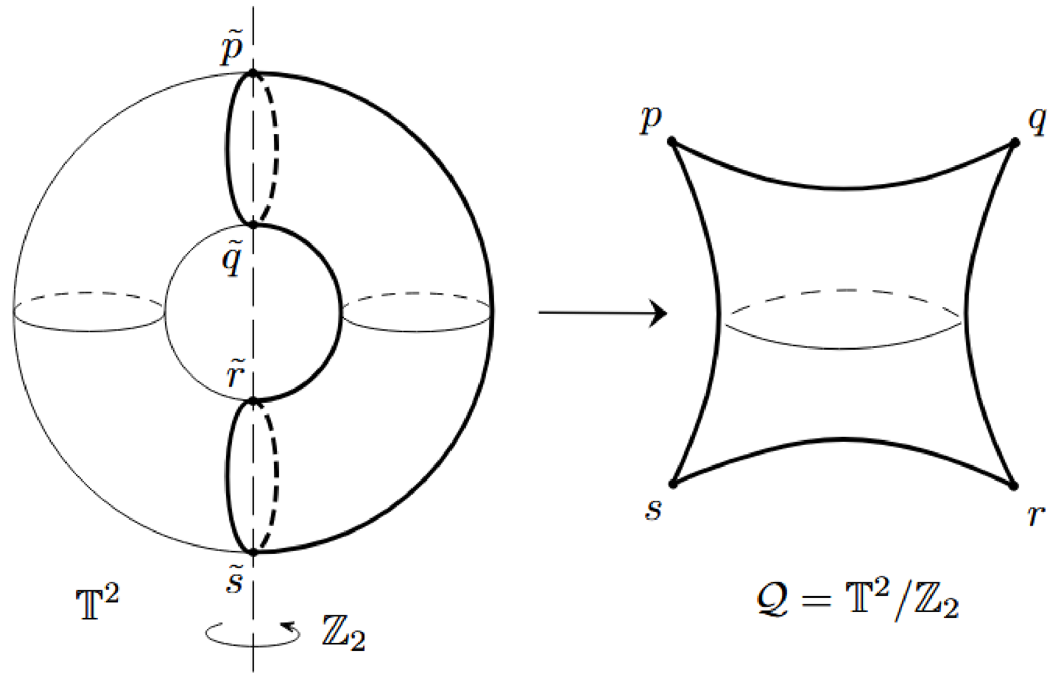

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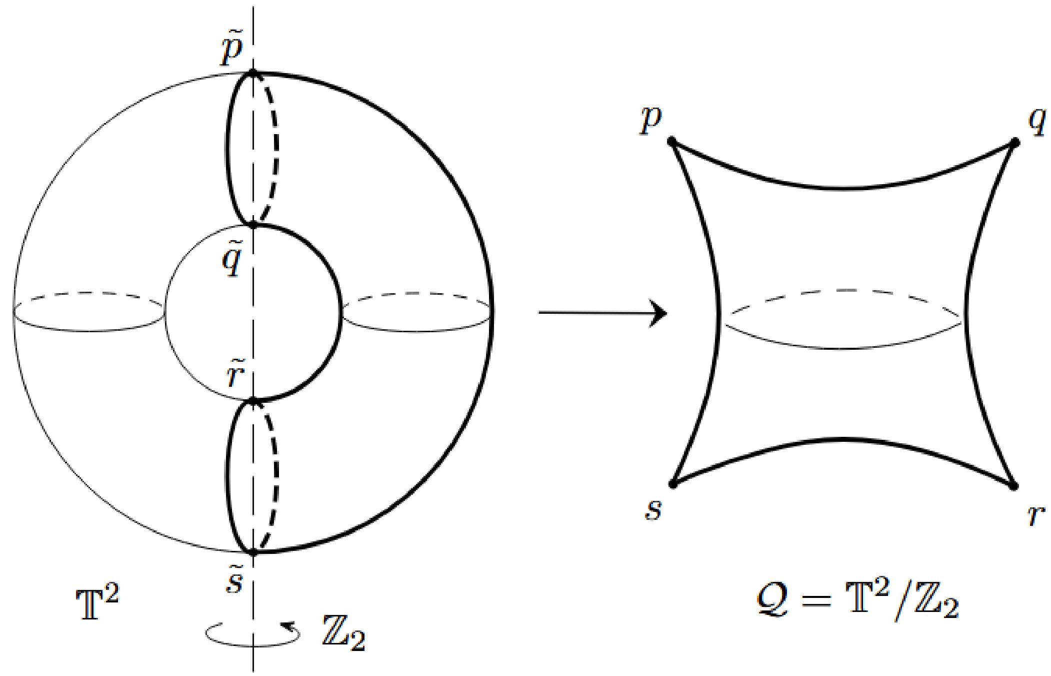
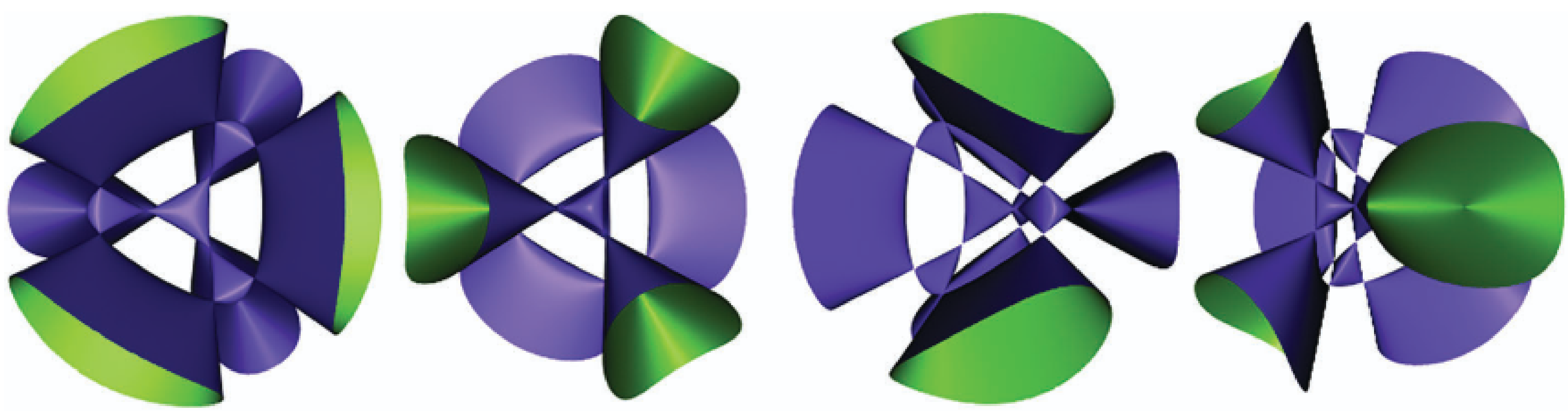


or the *Kummer surface* $\mathbb{T}^4/\!\!/\mathbb{Z}_2$:



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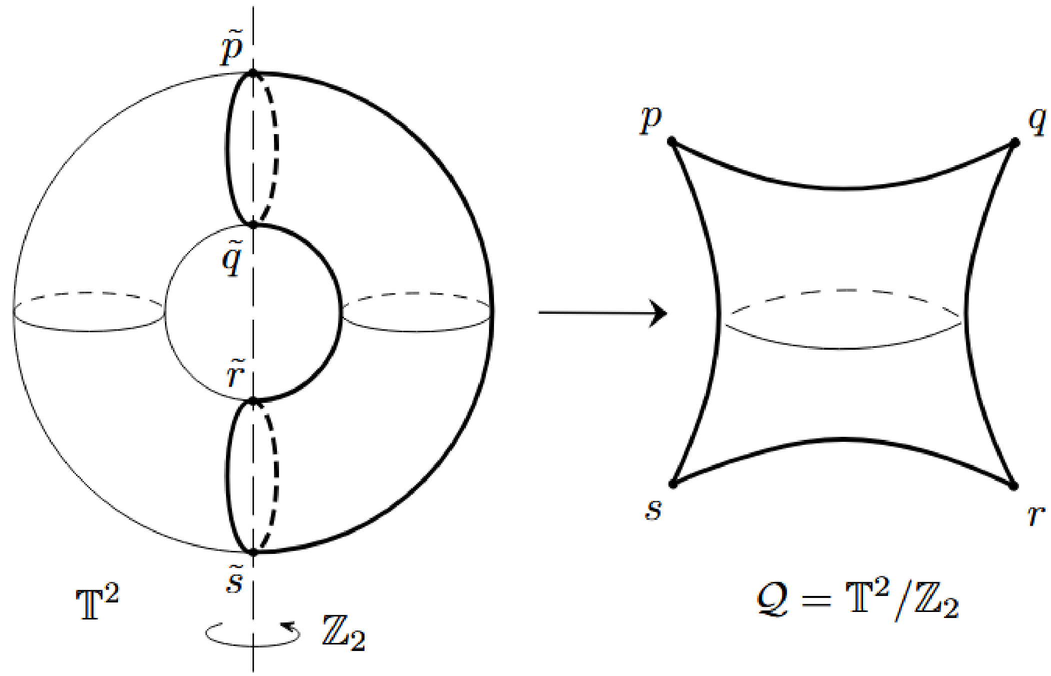

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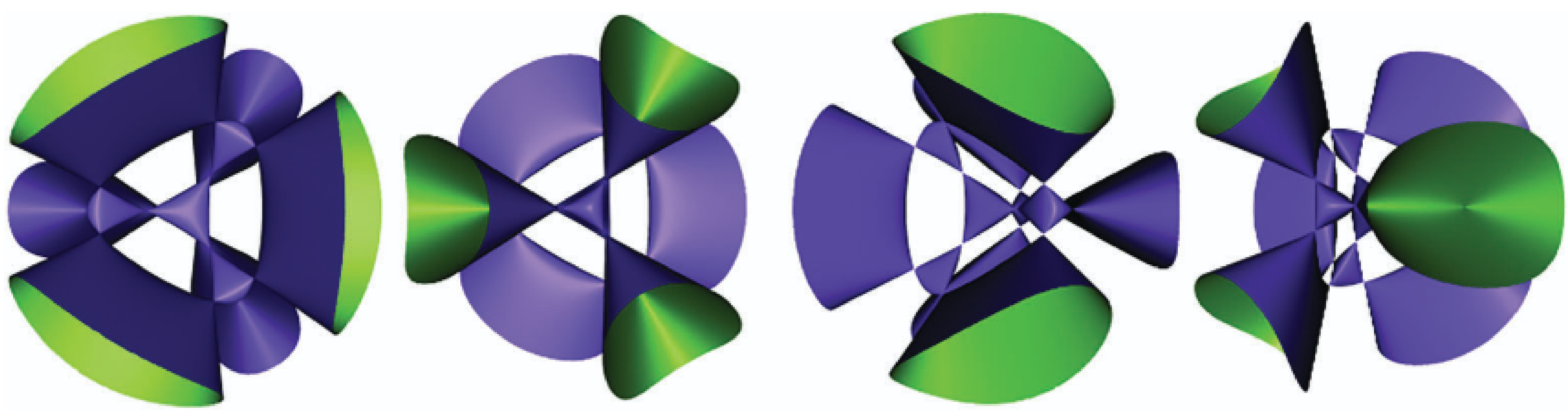

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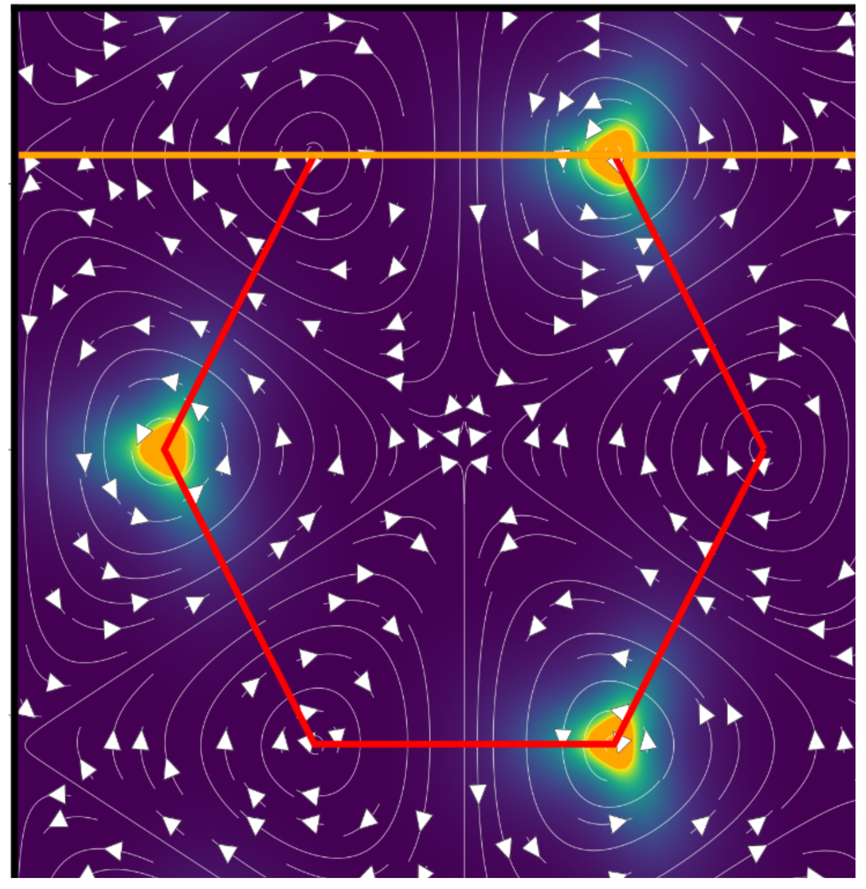

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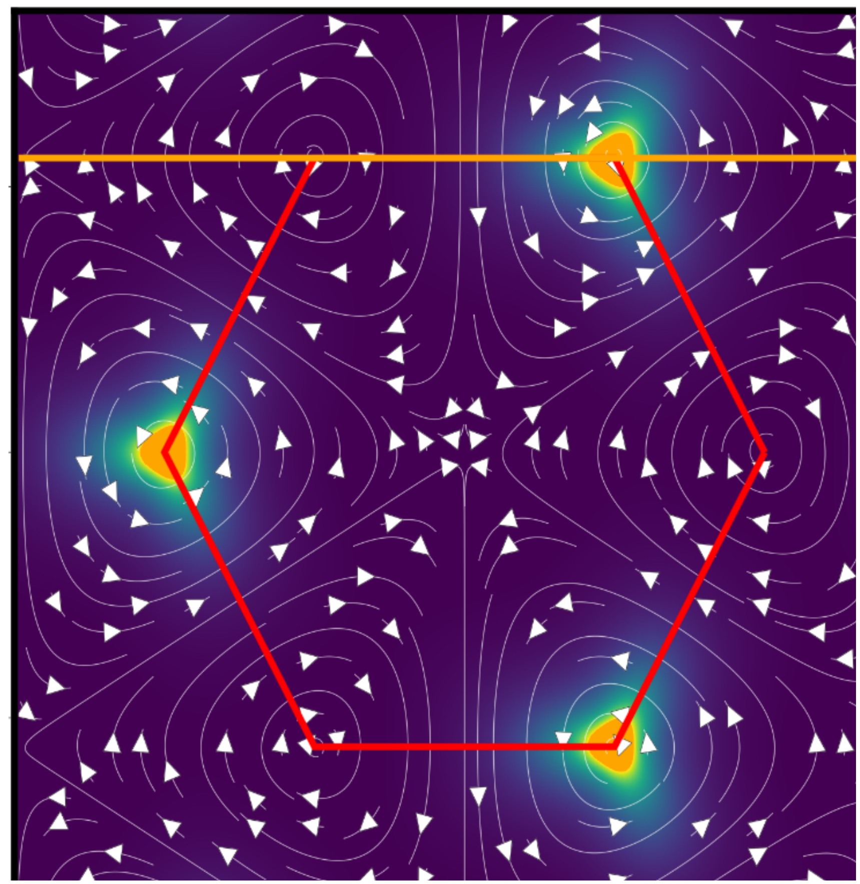

fine detail of physical fields!

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ncatlab.org/schreiber/show/Equivariant+Principal+infinity-Bundles

[EPB]

- Hisham Sati and Urs Schreiber:

  Equivariant Principal ∞-Bundles §4.5

  Cambridge University Press (2025, in print)

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[GOC]

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- Hisham Sati and Urs Schreiber:

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  CRC Press (2026, to appear)

Paradigm: Cohomology is Mapping Classes

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tplgcl group

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$\mathcal{A} \equiv E_n$	$E^n(X)$	extra-ordinary abelian cohomology

stage of
a spectrum
of spaces

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Paradigm: Cohomology is Mapping Classes

$$\textit{domain}\text{ space} \quad X \dashrightarrow \mathcal{A} \quad \textit{classifying}\text{ space}$$

$$H^1\big(X;\, \Omega\mathcal{A}\big) \;\equiv\; \pi_0\,\mathrm{Map}\big(X,\, \mathcal{A}\big)$$

$\mathcal{A} \equiv BG$	$H^1(X;\, G)$	ordinary non-abelian cohomology
$\mathcal{A} \equiv B^n A$	$H^n(X;\, A)$	ordinary abelian cohomology
$\mathcal{A} \equiv E_n$	$E^n(X)$	extra-ordinary abelian cohomology
$\mathcal{A} \equiv \mathcal{A}$	$H^1(X;\, \Omega A)$	extra-ordinary non-abelian cohomology

e.g., $H^1\big(X;\, \Omega S^n\big)$ is *Cohomotopy* (Borsuk, Pontrjagin 1930s)

Paradigm: Cohomology is Mapping Classes

Paradigm: Cohomology is Mapping Classes

twisted cohomology

Paradigm: Cohomology is Mapping Classes

twisted cohomology

$$\begin{tikzcd} & \mathcal{A} \arrow[r, hook] & \mathcal{E} \arrow[d, two heads] \\ X \arrow[urr, dashed] \arrow[rr, "\tau"] & & B\Gamma \end{tikzcd}$$

classifying fibration

Paradigm: Cohomology is Mapping Classes

twisted cohomology

$$
\begin{tikzcd}
 & \mathcal{A} \arrow[r, hook] & \mathcal{E} \arrow[d, two heads] \\
X \arrow[urr, dashed] \arrow[rr, "\tau"] & & B\Gamma
\end{tikzcd}
$$

classifying fibration

$$H^\tau(X;\mathcal{A}) \;\equiv\; \pi_0 \,\mathrm{Map}\big(X,\,\mathcal{E}\big)_{B\Gamma}$$

slice mapping space

Paradigm: Cohomology is Mapping Classes

twisted cohomology

$$
\begin{tikzcd}
 & \mathcal{A} \arrow[r, hook] & \mathcal{E} \arrow[d, two heads] \\
X \arrow[urr, dashed] \arrow[rr, "\tau"] & & B\Gamma
\end{tikzcd}
$$

classifying fibration

$$H^{\tau}\big(X;\,\mathcal{A}\big) \;\equiv\; \pi_0\,\mathrm{Map}\big(X,\,\mathcal{E}\big)_{B\Gamma}$$

equivariant cohomology

Paradigm: Cohomology is Mapping Classes

twisted cohomology

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classifying fibration

$$H^\tau\big(X;\,\mathcal{A}\big) \;\equiv\; \pi_0\,\mathrm{Map}\big(X,\,\mathcal{E}\big)_{B\Gamma}$$

---

equivariant cohomology

$$G \circlearrowright X \dashrightarrow G \circlearrowright \mathcal{A}$$

classifying G-space

Paradigm: Cohomology is Mapping Classes

twisted cohomology

$$
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$$H^{\tau}\big(X;\, \mathcal{A}\big) \;\equiv\; \pi_0\,\mathrm{Map}\big(X,\, \mathcal{E}\big)_{B\Gamma}$$

equivariant cohomology

$$G \circlearrowright X \dashrightarrow G \circlearrowright \mathcal{A}$$

classifying G-space

$$H^0_G\big(X;\, \mathcal{A}\big) \;\equiv\; \pi_0\,\mathrm{Map}\big(X,\, \mathcal{A}\big)^G$$

equivariant mapping space

Paradigm: Cohomology is Mapping Classes

twisted cohomology

$$
\begin{tikzcd}
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X \arrow[urr, dashed] \arrow[rr, "\tau"] & & B\Gamma
\end{tikzcd}
$$

classifying fibration

$$H^\tau(X;\, \mathcal{A}) \;\equiv\; \pi_0\, \mathrm{Map}(X,\, \mathcal{E})_{B\Gamma}$$

---

equivariant cohomology

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Paradigm: Cohomology is Mapping Classes

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geometric
cohomology

Paradigm: Cohomology is Mapping Classes

geometric cohomology

$X \dashrightarrow \mathbf{A}$ classifying ∞-stack in ∞-topos $\mathbf{H}$

Paradigm: Cohomology is Mapping Classes

geometric cohomology

$X \dashrightarrow \mathbf{A}$ classifying ∞-stack in ∞-topos $\mathbf{H}$

$$H^0\big(X;\, \mathbf{A}\big) \;\equiv\; \pi_0\, \mathbf{H}\big(X,\, \mathbf{A}\big)$$

Paradigm: Cohomology is Mapping Classes

geometric cohomology

$$X \dashrightarrow \mathbf{A}$$

classifying ∞-stack

in ∞-topos $\mathbf{H}$

$$H^0(X;\, \mathbf{A}) \;\equiv\; \pi_0\, \mathbf{H}(X,\, \mathbf{A})$$

$\mathbf{A} \equiv \mathbf{B}^n \underline{A}$ $\quad H^n(X;\, \underline{A})$

sheaf of abelian groups

ordinary abelian sheaf cohomology

Paradigm: Cohomology is Mapping Classes

geometric cohomology

$$X \dashrightarrow \mathbf{A}$$

classifying ∞-stack

in ∞-topos $\mathbf{H}$

$$H^0\big(X;\,\mathbf{A}\big) \;\equiv\; \pi_0\,\mathbf{H}\big(X,\,\mathbf{A}\big)$$

$\mathbf{A} \equiv \mathbf{B}^n \underline{A}$ $\quad H^n(X;\,\underline{A})$

ordinary abelian sheaf cohomology

$\mathbf{A} \equiv H\underline{A}_\bullet$ $\quad H^0(X;\,\underline{A}_\bullet)$

hyper abelian sheaf cohomology

sheaf of chain complexes

Paradigm: Cohomology is Mapping Classes

geometric cohomology

$$X \dashrightarrow \mathbf{A}$$

classifying ∞-stack

in ∞-topos $\mathbf{H}$

$$H^0(X;\,\mathbf{A}) \;\equiv\; \pi_0\,\mathbf{H}(X,\,\mathbf{A})$$

$\mathbf{A} \equiv \mathbf{B}^n \underline{A}$	$H^n(X;\,\underline{A})$	ordinary abelian sheaf cohomology
$\mathbf{A} \equiv H\underline{A}_\bullet$	$H^0(X;\,\underline{A}_\bullet)$	hyper abelian sheaf cohomology
$\mathbf{A} \equiv \mathbf{A}$	$H^0(X;\,\mathbf{A})$	extra-ordinary non-abelian geometric cohomology

any ∞-stack!

Paradigm: Cohomology is Mapping Classes

geometric cohomology

$$X \dashrightarrow \mathbf{A}$$

classifying ∞-stack in ∞-topos $\mathbf{H}$

$$H^0\big(X;\,\mathbf{A}\big) \;\equiv\; \pi_0\,\mathbf{H}\big(X,\,\mathbf{A}\big)$$

$\mathbf{A} \equiv \mathbf{B}^n \underline{A}$	$H^n(X;\,\underline{A})$	ordinary abelian sheaf cohomology
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$\mathbf{A} \equiv \mathbf{A}$	$H^0(X;\,\mathbf{A})$	extra-ordinary non-abelian geometric cohomology

e.g., $H^0\big(X;\Omega^1_{\mathrm{dR}}(\text{-};\,\mathfrak{g})/\!\!/G\big)$ is nonab diff cohomology

Paradigm: Cohomology is Mapping Classes

geometric cohomology

$$X \dashrightarrow \mathbf{A}$$

classifying ∞-stack in ∞-topos $\mathbf{H}$

$$H^0\big(X;\,\mathbf{A}\big) \;\equiv\; \pi_0\,\mathbf{H}\big(X,\,\mathbf{A}\big)$$

$\mathbf{A} \equiv \mathbf{B}^n \underline{A}$ $\quad H^n(X;\,\underline{A})$ ordinary abelian sheaf cohomology

$\mathbf{A} \equiv H\underline{A}_\bullet$ $\quad H^0(X;\,\underline{A}_\bullet)$ hyper abelian sheaf cohomology

$\mathbf{A} \equiv \mathbf{A}$ $\quad H^0(X;\,\mathbf{A})$ extra-ordinary non-abelian geometric cohomology

e.g., $H^0\big(X;\Omega^1_{\mathrm{dR}}(\text{-};\,\mathfrak{g})/\!\!/G\big)$ is nonab diff cohomology

A really convenient category of spaces.

A really convenient category of spaces.

Now to combine all this!

A really convenient category of spaces.

Now to combine all this!

first, pass to the
really convenient category of spaces

$$\mathrm{SmthGrpd}_\infty := \mathrm{Sh}_\infty(\mathrm{CrtSp})$$

smooth
∞-groupoids

A really convenient category of spaces.

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really convenient category of spaces

$$\mathrm{SmthGrpd}_\infty := \mathrm{Sh}_\infty(\mathrm{CrtSp})$$

smooth
∞-groupoids

modeled on

probe
spaces

$$\mathrm{CrtSp} := \left\{ \mathbb{R}^n \xrightarrow{\text{smooth}} \mathbb{R}^{n'} \;\middle|\; n \in \mathbb{N} \right\}$$

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$$\mathrm{SmthGrpd}_\infty := \mathrm{Sh}_\infty(\mathrm{CrtSp})$$ **smooth ∞-groupoids**

$$\cup\!\!\!\!\;\mid$$

$$\mathbf{X} \quad : \quad \mathbb{R}^n \longmapsto \begin{array}{c}\text{plain } \infty\text{-groupoid} \\ \text{of smooth maps} \\ \mathbb{R}^n \to \mathbf{X}\end{array}$$

modeled on

probe spaces $$\mathrm{CrtSp} := \left\{ \mathbb{R}^n \xrightarrow{\text{smooth}} \mathbb{R}^{n'} \;\middle|\; n \in \mathbb{N} \right\}$$

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$$\Cup$$

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modeled on

probe spaces $$\mathrm{CrtSp} := \left\{ \mathbb{R}^n \xrightarrow{\text{smooth}} \mathbb{R}^{n'} \;\middle|\; n \in \mathbb{N} \right\}$$

$\mathrm{SmthGrpd}_\infty$

a cohesive ∞-topos

$$\mathrm{SmthGrpd}_\infty \xrightarrow{\quad \mathrm{Pnt} \quad} \mathrm{Grpd}_\infty \qquad \textbf{points}$$

a cohesive ∞-topos

$$
\begin{tikzcd}[column sep=huge]
\mathrm{SmthGrpd}_\infty \arrow[r, "\mathrm{Pnt}"', ""{name=P}] & \mathrm{Grpd}_\infty \arrow[l, hook', bend right=0, shift right=3ex, "\mathrm{Dsc}"', ""{name=D}] \arrow[from=D, to=P, phantom, "\perp"]
\end{tikzcd}
$$

discrete

points

a cohesive ∞-topos

$$\begin{tikzcd}[column sep=huge] \mathrm{SmthGrpd}_\infty \arrow[r, shift left=6, "\mathrm{Shp}"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, shift right=6, "\mathrm{Pnt}"'] & \mathrm{Grpd}_\infty \arrow[l, hook', "\mathrm{Dsc}" description] \end{tikzcd}$$

shape

discrete

points

a cohesive ∞-topos

$$
\begin{tikzcd}[column sep=huge]
\mathrm{SmthGrpd}_\infty
\arrow[r, shift left=9, "\mathrm{Shp}"]
\arrow[r, phantom, shift left=6, "\perp"]
\arrow[r, leftarrow, shift left=3, "\mathrm{Dsc}"]
\arrow[r, phantom, shift left=0, "\perp"]
\arrow[r, shift right=3, "\mathrm{Pnt}"]
\arrow[r, phantom, shift right=6, "\perp"]
\arrow[r, leftarrow, shift right=9, "\mathrm{Cht}"]
& \mathrm{Grpd}_\infty
\end{tikzcd}
$$

shape

discrete

points

chaotic

a cohesive ∞-topos

$$
\begin{tikzcd}[column sep=huge]
\mathrm{SmthGrpd}_\infty
\arrow[r, shift left=12, "\mathrm{Shp}"]
\arrow[r, phantom, shift left=8, "\perp"]
\arrow[r, leftarrow, hook', shift left=4, "\mathrm{Dsc}"]
\arrow[r, phantom, shift left=0, "\perp"]
\arrow[r, shift right=4, "\mathrm{Pnt}"]
\arrow[r, phantom, shift right=8, "\perp"]
\arrow[r, leftarrow, hook', shift right=12, "\mathrm{Cht}"]
& \mathrm{Grpd}_\infty
\end{tikzcd}
$$

shape

discrete

points

chaotic

a cohesive ∞-topos

$$
\mathrm{SmthGrpd}_\infty
\begin{array}{c}
\xrightarrow{\quad \mathrm{Shp} \quad} \\
\perp \\
\xleftarrow{\quad \mathrm{Dsc} \quad} \\
\perp \\
\xrightarrow{\quad \mathrm{Pnt} \quad} \\
\perp \\
\xleftarrow{\quad \mathrm{Cht} \quad}
\end{array}
\mathrm{Grpd}_\infty
\qquad
\begin{array}{c}
\textbf{shape} \\
\\
\textbf{discrete} \\
\\
\textbf{points} \\
\\
\textbf{chaotic}
\end{array}
$$

pure shape aspect

$$\int \;:=\; \mathrm{Dsc} \circ \mathrm{Shp}$$

a cohesive ∞-topos

$$
\mathrm{SmthGrpd}_\infty
\begin{array}{c}
\xrightarrow{\;\;\mathrm{Shp}\;\;} \\
\perp \\
\xhookleftarrow{\;\;\mathrm{Dsc}\;\;} \\
\perp \\
\xrightarrow{\;\;\mathrm{Pnt}\;\;} \\
\perp \\
\xhookleftarrow{\;\;\mathrm{Cht}\;\;}
\end{array}
\mathrm{Grpd}_\infty
\qquad
\begin{array}{c}
\textbf{shape} \\
\\
\textbf{discrete} \\
\\
\textbf{points} \\
\\
\textbf{chaotic}
\end{array}
$$

pure shape aspect	$\int$	$:=$	$\mathrm{Dsc} \circ \mathrm{Shp}$
	$\perp$		
purely discrete aspect	$\flat$	$:=$	$\mathrm{Dsc} \circ \mathrm{Pnt}$

a cohesive ∞-topos

$$
\begin{tikzcd}[row sep=small]
 & \xrightarrow{\ \mathrm{Shp}\ } & \\
 & \bot & \\
 & \xleftarrow{\ \mathrm{Dsc}\ } & \\
\mathrm{SmthGrpd}_\infty & \bot & \mathrm{Grpd}_\infty \\
 & \xrightarrow{\ \mathrm{Pnt}\ } & \\
 & \bot & \\
 & \xleftarrow{\ \mathrm{Cht}\ } &
\end{tikzcd}
$$

shape

discrete

points

chaotic

pure shape aspect	$\int$	$:=$	$\mathrm{Dsc} \circ \mathrm{Shp}$
	$\bot$		
purely discrete aspect	$\flat$	$:=$	$\mathrm{Dsc} \circ \mathrm{Pnt}$
	$\bot$		
purely continuous aspect	$\sharp$	$:=$	$\mathrm{Cht} \circ \mathrm{Pnt}$

a cohesive ∞-topos

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\mathrm{SmthGrpd}_\infty
\arrow[rr, shift left=18pt, "\mathrm{Shp}"]
\arrow[rr, leftarrow, hook', shift left=6pt, "\mathrm{Dsc}" description]
\arrow[rr, shift right=6pt, "\mathrm{Pnt}" description]
\arrow[rr, leftarrow, hook', shift right=18pt, "\mathrm{Cht}"']
& & \mathrm{Grpd}_\infty
\end{tikzcd}
$$

Shp $\perp$ Dsc $\perp$ Pnt $\perp$ Cht

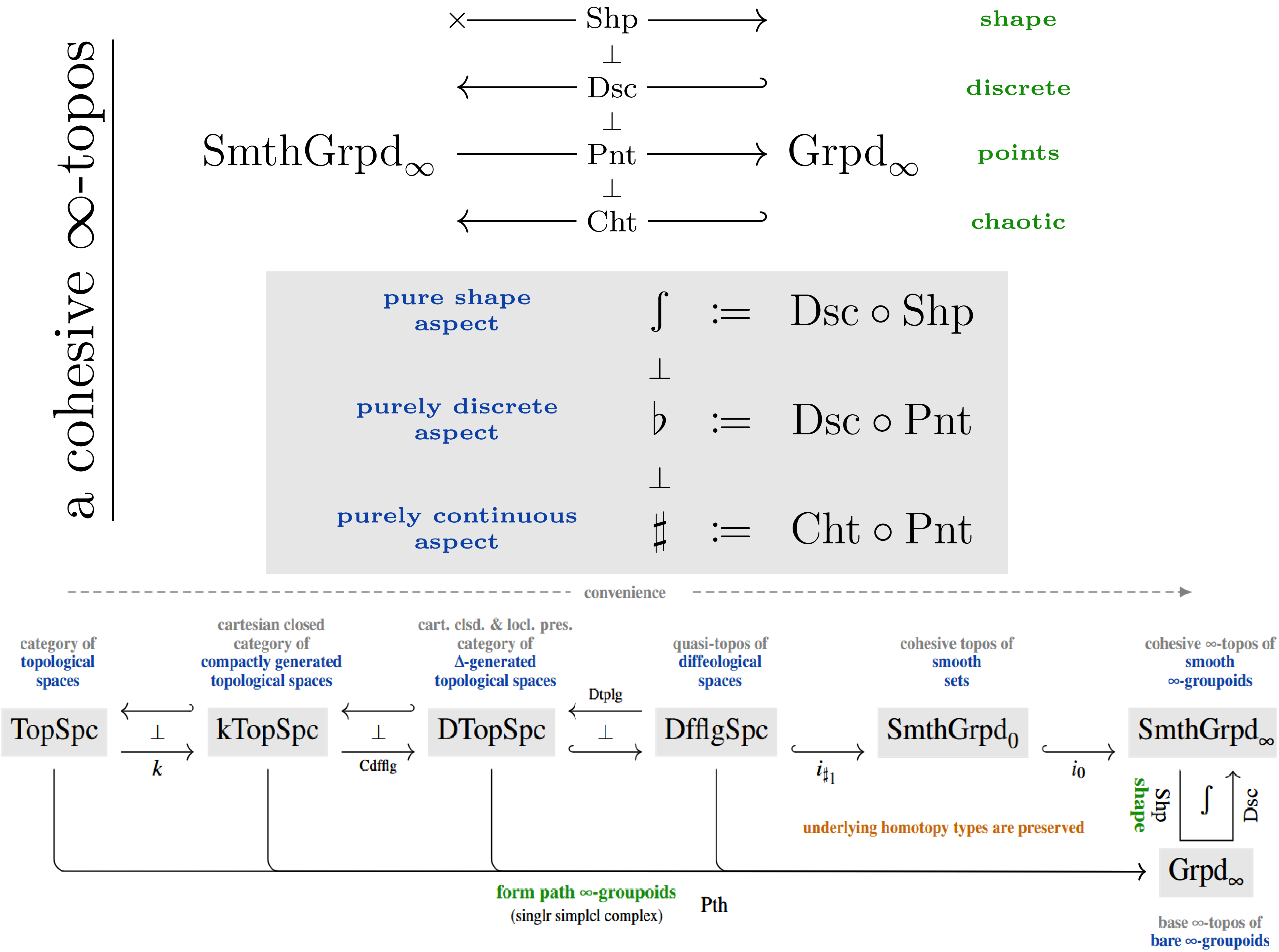
shape

discrete

points

chaotic

pure shape aspect	$\int$	$:=$	$\mathrm{Dsc} \circ \mathrm{Shp}$
	$\perp$		
purely discrete aspect	$\flat$	$:=$	$\mathrm{Dsc} \circ \mathrm{Pnt}$
	$\perp$		
purely continuous aspect	$\sharp$	$:=$	$\mathrm{Cht} \circ \mathrm{Pnt}$


a cohesive ∞-topos
SmthGrpd∞
Shp
Dsc
Pnt
Cht
Grpd∞
shape
discrete
points
chaotic
pure shape aspect
∫ := Dsc ∘ Shp
purely discrete aspect
♭ := Dsc ∘ Pnt
purely continuous aspect
♯ := Cht ∘ Pnt
convenience
category of topological spaces
cartesian closed category of compactly generated topological spaces
cart. clsd. & locl. pres. category of Δ-generated topological spaces
quasi-topos of diffeological spaces
cohesive topos of smooth sets
cohesive ∞-topos of smooth ∞-groupoids
TopSpc
kTopSpc
DTopSpc
DfflgSpc
SmthGrpd0
SmthGrpd∞
k
Cdfflg
Dtplg
i♯1
i0
shape
Shp
∫
Dsc
underlying homotopy types are preserved
Grpd∞
form path ∞-groupoids
(singlr simplcl complex)
Pth
base ∞-topos of bare ∞-groupoids

In $\mathrm{SmthGrpd}_\infty$

... exist the homotopy quotients $\mathrm{X}/\!\!/G$:

In $\mathrm{SmthGrpd}_\infty$

... exist the homotopy quotients $\mathrm{X}/\!\!/G$:

as $\mathrm{X} \;\equiv\; \left\{ x \;\middle|\; x \in \mathrm{X} \right\}$ topological space

so $\mathrm{X}/\!\!/G \;\equiv\; \left\{ \begin{tikzcd} & g\cdot x \arrow[dr, "g'", "\sim"'] & \\ x \arrow[ur, "g", "\sim"'] \arrow[rr, "g'\cdot g", "\sim"'] & & g'\cdot g\cdot x \end{tikzcd} \;\middle|\; \begin{array}{c} x \in \mathrm{X} \\ g \in G \end{array} \right\}$

topological groupoid/stack

In $\mathrm{SmthGrpd}_\infty$

... exist the homotopy quotients $\mathrm{X} /\!\!/ G$:

$$\text{as } \mathrm{X} \;\equiv\; \left\{ \, x \;\middle|\; x \in \mathrm{X} \right\}$$

topological space

$$\text{so } \mathrm{X} /\!\!/ G \;\equiv\; \left\{ \begin{tikzcd} & g \cdot x \arrow[dr, "g'", "\sim"'] & \\ x \arrow[ur, "g", "\sim"'] \arrow[rr, "g' \cdot g"', "\sim"] & & g' \cdot g \cdot x \end{tikzcd} \;\middle|\; \begin{array}{c} x \in \mathrm{X} \\ g \in G \end{array} \right\}$$

topological groupoid/stack

$$\text{eg. } \mathbf{B}G \;\equiv\; \ast /\!\!/ G \;\simeq\; \left\{ \begin{tikzcd} \bullet \arrow[loop above, "g"] \end{tikzcd} \;\middle|\; g \in G \right\}$$

delooping groupoid

In $\mathrm{SmthGrpd}_\infty$

... exist the homotopy quotients $\mathrm{X} /\!\!/ G$:

$$\text{as } \mathrm{X} \;\equiv\; \Big\{\, x \;\Big|\; x \in \mathrm{X} \Big\}$$

topological space

$$\text{so } \mathrm{X} /\!\!/ G \;\equiv\; \left\{ \begin{tikzcd} & g \cdot x \arrow[dr, "g'", "\sim"'] & \\ x \arrow[ur, "g", "\sim"'] \arrow[rr, "g' \cdot g", "\sim"'] & & g' \cdot g \cdot x \end{tikzcd} \;\middle|\; \begin{array}{c} x \in \mathrm{X} \\ g \in G \end{array} \right\}$$

topological groupoid/stack

$$\text{eg. } \mathbf{B}G \;\equiv\; \ast /\!\!/ G \;\simeq\; \Big\{ \begin{tikzcd} \bullet \arrow[loop above, "g"] \end{tikzcd} \;\Big|\; g \in G \Big\}$$

delooping groupoid

Basic idea of twisted orbifold cohomology

Basic idea of twisted orbifold cohomology

$\mathrm{SmthGrpd}_\infty$ is the place to speak about twisted orbifold cohomology.

Basic idea of twisted orbifold cohomology

$\mathrm{SmthGrpd}_\infty$ is the place to speak about twisted orbifold cohomology.

In the following, square brackets mean *concordance classes* of maps hence *geometric homotopy classes*:

$$\left[\ \mathbf{X} \dashrightarrow \mathbf{A}\ \right] \;:=\; \pi_0 \textstyle\int \mathrm{Map}\big(\mathbf{X}, \mathbf{A}\big)$$

Basic idea of twisted orbifold cohomology

$\mathrm{SmthGrpd}_\infty$ is the place to speak about twisted orbifold cohomology.

In the following, square brackets mean *concordance classes* of maps hence *geometric homotopy classes*:

$$\Big[\ \mathbf{X} \dashrightarrow \mathbf{A}\ \Big] \;:=\; \pi_0 \underbrace{\textstyle\int}_{\text{shape}} \underbrace{\mathrm{Map}}_{\text{internal hom}}\big(\mathbf{X},\, \mathbf{A}\big)$$

Basic idea of twisted orbifold cohomology

Basic idea of twisted orbifold cohomology

first recall equivariant cohomology

Basic idea of twisted orbifold cohomology

equivariant cohomology

$$H_G(X;A) = \left[\begin{tikzcd} & A \arrow[loop right, "G"] \\ X \arrow[ur, dashed] \arrow[loop right, "G"] & \end{tikzcd} \right]$$

for G-spaces $G \circlearrowright X$ and $G \circlearrowright A$

Basic idea of twisted orbifold cohomology

equivariant cohomology

$$H_G(\mathrm{X};\, \mathrm{A}) = \left[\begin{tikzcd} & \mathrm{A} /\!\!/ (\;G\;) \arrow[dr] & \\ \mathrm{X} /\!\!/ G \arrow[ur, dashed] \arrow[rr] & & \mathbf{B}G \end{tikzcd} \right]$$

equivalently, maps
of quotient stacks
sliced over $\mathbf{B}G$!

for G-spaces $G \circlearrowright \mathrm{X}$ and $G \circlearrowright \mathrm{A}$

Basic idea of twisted orbifold cohomology

equivariant cohomology

$$
H_G(\mathrm{X};\, \mathrm{A}) = \left[
\begin{tikzcd}
 & \mathrm{A} /\!\!/ (\;G\;) \arrow[d, bend left] \\
\mathrm{X} /\!\!/ G \arrow[ur, dashed] \arrow[r, bend right] & \mathbf{B}G
\end{tikzcd}
\right]
$$

equivalently, maps
of quotient stacks
sliced over $\mathbf{B}G$!

for G-spaces $G \circlearrowright \mathrm{X}$ and $G \circlearrowright \mathrm{A}$

Basic idea of twisted orbifold cohomology

equivariant cohomology

but this admits
natural generalization

$$H_G(\mathrm{X};\,\mathrm{A}) = \left[\begin{tikzcd} & \mathrm{A}/\!\!/(\;\;G\;\;) \arrow[dr] & \\ \mathrm{X}/\!\!/G \arrow[ur, dashed] \arrow[rr, bend right] & & \mathbf{B}G \end{tikzcd}\right]$$

Basic idea of twisted orbifold cohomology

twisted equivariant cohomology

natural generalization

$$
H^{\tau_G}_G(\mathrm{X};\,\mathrm{A}) = \left[
\begin{tikzcd}
 & \mathrm{A}/\!\!/(\Gamma \rtimes G) \arrow[d] \arrow[ddr, bend left] & \\
\mathrm{X}/\!\!/G \arrow[ur, dashed] \arrow[r, dashed, "\tau_G"] \arrow[rrd, bend right] & \mathbf{B}(\Gamma \rtimes G) \arrow[dr] & \\
 & & \mathbf{B}G
\end{tikzcd}
\right]
$$

Basic idea of twisted orbifold cohomology

twisted equivariant cohomology

$$
H_G^{\tau_G}(\mathrm{X};\,\mathrm{A}) = \left[
\begin{tikzcd}
 & \mathrm{A}/\!\!/(\Gamma \rtimes G) \arrow[d] \arrow[ddr, bend left] & \\
\mathrm{X}/\!\!/G \arrow[ur, dashed] \arrow[r, dashed, "\tau_G"] \arrow[drr, bend right] & \mathbf{B}(\Gamma \rtimes G) \arrow[dr] & \\
 & & \mathbf{B}G
\end{tikzcd}
\right]
$$

classes of sections of $\left\{ \begin{array}{l} G\text{-equivariant} \\ \Gamma\text{-structured} \\ \text{A-fiber bundle} \end{array} \right.$

Basic idea of twisted orbifold cohomology

twisted equivariant cohomology

$$H_G^{\tau_G}(\mathrm{X};\, \mathrm{A}) = \left[\begin{tikzcd} & \mathrm{A}/\!\!/(\Gamma \rtimes G) \arrow[d, two heads] \arrow[ddr, bend left] & \\ \mathrm{X}/\!\!/G \arrow[ur, dashed] \arrow[r, "\tau_G"] \arrow[ddrr, bend right] & \mathbf{B}(\Gamma \rtimes G) \arrow[dr] & \\ & & \mathbf{B}G \end{tikzcd}\right]$$

$$\begin{array}{r} \text{classes of} \\ \text{sections of} \end{array} \left\{ \begin{array}{l} G\text{-equivariant} \\ \Gamma\text{-structured} \\ \mathrm{A}\text{-fiber bundle} \end{array} \right.$$

Basic idea of twisted orbifold cohomology

twisted equivariant cohomology

$$
H_G^{\tau_G}(\mathrm{X};\,\mathrm{A}) = \left[
\begin{tikzcd}
 & \mathrm{A}/\!\!/(\Gamma \rtimes G) \arrow[d, ->>] \arrow[ddr, bend left] & \\
\mathrm{X}/\!\!/G \arrow[ur, dashed] \arrow[r, "\tau_G"] \arrow[drr, bend right] & \mathbf{B}(\Gamma \rtimes G) \arrow[dr] & \\
 & & \mathbf{B}G
\end{tikzcd}
\right]
$$

G-equivariance is absorbed into the twisting!

Basic idea of twisted orbifold cohomology

twisted equivariant cohomology

$$
H^{\tau_G}_G\big(\mathrm{X};\, \mathrm{A}\big) = \left[
\begin{tikzcd}
 & \mathrm{A}/\!\!/(\Gamma \rtimes G) \arrow[d, ->>] \arrow[ddr, bend left] & \\
\mathrm{X}/\!\!/G \arrow[ur, dashed] \arrow[r, "\tau_G"] \arrow[drr, bend right] & \mathbf{B}(\Gamma \rtimes G) \arrow[dr] & \\
 & & \mathbf{B}G
\end{tikzcd}
\right]
$$

key principle:

G-equivariance is absorbed into the twisting!

Basic idea of twisted orbifold cohomology

twisted equivariant cohomology

$$
H^{\tau_G}_G(\mathrm{X};\,\mathrm{A}) = \left[
\begin{tikzcd}
 & \mathrm{A}/\!\!/(\Gamma \rtimes G) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\text{(pb)}"] \arrow[ddr, bend left] & \mathrm{A}/\!\!/\mathcal{G} \arrow[d, two heads, "\substack{\text{universal}\\ \text{coefficient}\\ \text{bundle}}"] \\
\mathrm{X}/\!\!/G \arrow[ur, dashed] \arrow[r, dashed, "\tau_G"] \arrow[drr, bend right] & \mathbf{B}(\Gamma \rtimes G) \arrow[r] \arrow[dr] & \mathbf{B}\mathcal{G} \arrow[d] \\
 & & \mathbf{B}G
\end{tikzcd}
\right]
$$

key principle:

suppose coefficient bundle is
pullback of universal one

Basic idea of twisted orbifold cohomology

twisted equivariant cohomology

$$
H^{\tau_G}_G(\mathrm{X};\,\mathrm{A}) = \left[
\begin{tikzcd}
 & \mathrm{A}/\!\!/(\Gamma \rtimes G) \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] \arrow[ddr, bend left] & \mathrm{A}/\!\!/\mathcal{G} \arrow[d] \\
\mathrm{X}/\!\!/G \arrow[r, dashed, "\tau_G"] \arrow[ur, dashed] \arrow[drr, bend right] & \mathbf{B}(\Gamma \rtimes G) \arrow[r] \arrow[dr] & \mathbf{B}\mathcal{G} \\
 & & \mathbf{B}G
\end{tikzcd}
\right]
$$

universal coefficient bundle

suppose coefficient bundle is pullback of universal one

Basic idea of twisted orbifold cohomology

twisted equivariant cohomology

$$
H_G^{\tau_G}(\mathrm{X};\, \mathrm{A}) = \left[
\begin{tikzcd}
 & & \mathrm{A} /\!\!/ \mathcal{G} \arrow[d, ->>] \\
\mathrm{X} /\!\!/ G \arrow[rru, dashed] \arrow[r, dashed, "\tau_G"] \arrow[rrd, bend right] & \mathbf{B}(\Gamma \rtimes G) \arrow[r] \arrow[rd] & \mathbf{B}\mathcal{G} \\
 & & \mathbf{B}G
\end{tikzcd}
\right]
$$

universal property of the pullback

Basic idea of twisted orbifold cohomology

twisted equivariant cohomology

$$H_G^{\tau_G}\big(\mathrm{X};\, \mathrm{A}\big) = \left[\begin{tikzcd} & & \mathrm{A} /\!\!/ \mathcal{G} \arrow[d] \\ \mathrm{X} /\!\!/ G \arrow[r, dashed, "\tau_G"] \arrow[urr, dashed, bend left] \arrow[drr, bend right] & \mathbf{B}(\Gamma \rtimes G) \arrow[r] \arrow[dr] & \mathbf{B}\mathcal{G} \\ & & \mathbf{B}G \end{tikzcd} \right]$$

Basic idea of twisted orbifold cohomology

twisted cohomology of ho-quotient

$$H^{\tau}\big(\mathrm{X} /\!\!/ G;\ \mathrm{A}\big) = \left[\begin{tikzcd} & & \mathrm{A} /\!\!/ \mathcal{G} \arrow[d] \\ \mathrm{X} /\!\!/ G \arrow[urr, dashed] \arrow[r, dashed, "\tau_G"] \arrow[rr, bend right, "\tau"', color=blue] & \mathbf{B}(\Gamma \rtimes G) \arrow[r] & \mathbf{B}\mathcal{G} \end{tikzcd} \right]$$

equivalent universal twist

Basic idea of twisted orbifold cohomology

twisted cohomology of ho-quotient

$$H^{\tau}\big(\mathbf{X};\, \mathrm{A}\big) = \left[\begin{tikzcd} & \mathrm{A} /\!\!/ \mathcal{G} \arrow[d] \\ \mathbf{X} \arrow[ur, dashed] \arrow[r, bend right, "\tau"'] & \mathbf{B}\mathcal{G} \end{tikzcd} \right]$$

Basic idea of twisted orbifold cohomology

twisted cohomology of ho-quotient

$$H^{\tau}(\mathbf{X};\, \mathrm{A}) = \left[\begin{tikzcd} & \mathrm{A} /\!\!/ \mathcal{G} \arrow[d, two heads] \\ \mathbf{X} \arrow[ur, dashed] \arrow[r, bend right, "\tau"'] & \mathbf{B}\mathcal{G} \end{tikzcd} \right]$$

choice of $G \circlearrowright \mathrm{X}$
disappeared!

Basic idea of twisted orbifold cohomology

twisted cohomology of ho-quotient

$$H^{\tau}\big(\mathbf{X};\, \mathrm{A}\big) = \left[\begin{tikzcd} & \mathrm{A} /\!\!/ \mathcal{G} \arrow[d, ->>] \\ \mathbf{X} \arrow[ur, dashed] \arrow[r, bend right, "\tau"'] & \mathbf{B}\mathcal{G} \end{tikzcd} \right]$$

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Basic idea of twisted orbifold cohomology

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Basic idea of twisted orbifold cohomology

twisted orbifold cohomology

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Basic idea of twisted orbifold cohomology

in
conclusion
so far

Basic idea of twisted orbifold cohomology

in conclusion so far

twisted
orbifold cohomology

$$H^{\tau}(\mathbf{X};\, \mathrm{A})$$

Basic idea of twisted orbifold cohomology

in conclusion so far

twisted orbifold cohomology

$$H^{\tau}\big(\mathbf{X};\,\mathrm{A}\big)$$

$\Vert$ if $\mathbf{X}=X$ is manifold

$$H^{\tau}\big(\mathrm{X};\mathrm{A}\big)$$

twisted cohomology

Basic idea of twisted orbifold cohomology

in conclusion so far

twisted orbifold cohomology

$$\begin{tikzcd}
 & H^{\tau}(\mathbf{X};\, \mathrm{A}) \arrow[dl, equal, "\text{if } \mathbf{X}{=}X \text{ is manifold}"'] \arrow[dr, equal, "\text{if } \mathbf{X}{=}X/\!\!/G \text{ is } G\text{-quotient}"] & \\
H^{\tau}(\mathrm{X};\mathrm{A}) & & H_G^{\tau_G}(\mathrm{X};\, \mathrm{A})
\end{tikzcd}$$

twisted cohomology

Basic idea of twisted orbifold cohomology

in conclusion so far

twisted orbifold cohomology

$$\begin{tikzcd}
& H^{\tau}(\mathbf{X};\,\mathrm{A}) \arrow[dl, equal, "\text{if } \mathbf{X}=X \text{ is manifold}"'] \arrow[dr, equal, "\text{if } \mathbf{X}=X/\!\!/G \text{ is } G\text{-quotient}"] & \\
H^{\tau}(\mathrm{X};\,\mathrm{A}) & & H_G^{\tau_G}(\mathrm{X};\,\mathrm{A}) \arrow[dl, equal, "\text{if } \mathbf{B}G \text{ factors } \tau"] \\
& H_G(\mathrm{X};\,\mathrm{A}) &
\end{tikzcd}$$

twisted cohomology ($H^{\tau}(\mathrm{X};\,\mathrm{A})$)

equivariant cohomology ($H_G(\mathrm{X};\,\mathrm{A})$)

Basic idea of twisted orbifold cohomology

in conclusion so far

$$
\begin{tikzcd}
& \overset{\textbf{twisted orbifold cohomology}}{H^{\tau}(\mathbf{X};\, \mathrm{A})} \arrow[dl, equal, "\text{if } \mathbf{X}{=}X \text{ is manifold}"'] \arrow[dr, equal, "\text{if } \mathbf{X}{=}X /\!\!/ G \text{ is } G\text{-quotient}"] & \\
\underset{\textbf{twisted cohomology}}{H^{\tau}(\mathrm{X};\, \mathrm{A})} & & H^{\tau_G}_{G}(\mathrm{X};\, \mathrm{A}) \arrow[dl, equal, "\text{if } \mathbf{B}G \text{ factors } \tau"'] \\
& \textbf{equivariant cohomology}\quad H_G(\mathrm{X};\, \mathrm{A}) &
\end{tikzcd}
$$

Examples of twisted orbifold cohomology.

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$\mathrm{A} \equiv \mathrm{Fred}(\mathcal{H})$ — $\mathbb{Z}_2$-graded Fredholm operators

$\mathcal{G} \equiv \mathrm{PU}^{\mathrm{gr}}(\mathcal{H}) \rtimes \mathbb{Z}_2$ — $\mathbb{Z}_2$-graded projective unitaries

on $\mathbb{Z}_2$-graded Hilbert space

$\Rightarrow$ twisted orbifold K-theory

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$\mathcal{G} \equiv \mathrm{O}(n+1)$ orthogonal group

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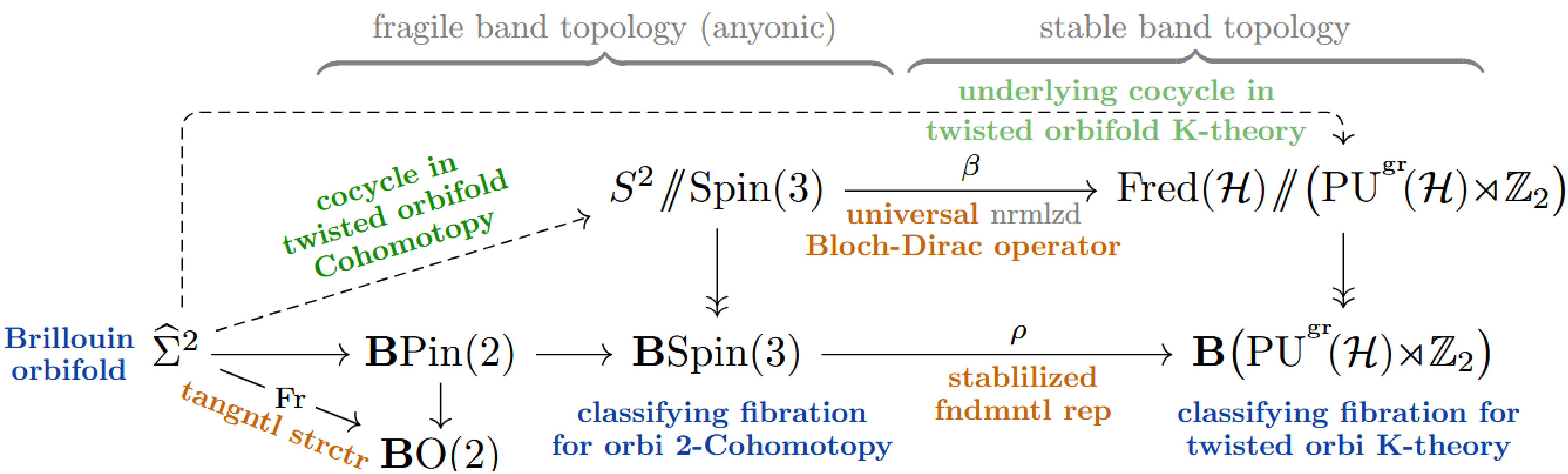

The need for singular cohesion.

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while this is nice, we are not done yet:

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mapping classes considered so far are *concordances*

$$H^{\tau}\big(\mathbf{X};\, \mathrm{A}\big) \;=\; \pi_0 \underbrace{\textstyle\int}_{\text{shape}} \underbrace{\mathrm{Map}}_{\text{internal hom}}\big(\mathbf{X},\, \mathrm{A} /\!\!/ \mathcal{G}\big)_{\mathbf{B}\mathcal{G}}$$

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shape

internal hom

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external hom

points

so we need *Oka principles* to
"take the shape inside the mapping space"

The need for singular cohesion.

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The need for singular cohesion.

[EPB 3.3.53]

Smooth Oka principle:

For $\mathrm{X} \in \mathrm{SmthMfd}$, $\mathbf{A} \in \mathrm{SmthGrpd}_\infty$:

$$\int \mathrm{Map}(\mathrm{X}, \mathbf{A}) \;\simeq\; \flat\, \mathrm{Map}\big(\int \mathrm{X}, \int \mathbf{A}\big)$$

goes back to [Pavlov 2014]

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[EPB 4.5.2]

Elmendorf theorem – recast:

For G-spaces X, A:

$$\int \mathrm{Map}(\mathrm{X} /\!\!/ G, \mathrm{A} /\!\!/ G)_{\mathbf{B}G}$$

$$\simeq \cup\, \mathrm{Map}(\int \curlyvee \mathrm{X} /\!\!/ G, \int \curlyvee \mathrm{A} /\!\!/ G)_{\int \curlyvee \mathbf{B}G}$$

following [Rezk 2014]

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works with another modality appearing: "orbi-singular"

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Where it takes place: Singular-Cohesive ∞-Topoi

$\flat$

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Where it takes place: Singular-Cohesive ∞-Topoi

$$\mathbf{H} := \mathrm{Sh}_\infty\big(\mathrm{CrtSp} \times \mathrm{Snglrt}\big)$$

singular-cohesive ∞-topos

Where it takes place: Singular-Cohesive ∞-Topoi

$$\mathrm{CrtSp} \;\;:=\;\; \left\{ \; \mathbb{R}^n \xrightarrow{\;\;\text{smooth}\;\;} \mathbb{R}^{n'} \;\middle|\; n \in \mathbb{N} \right\}$$

$$\mathrm{Snglrt} \;\;:=\;\; \left\{ \; \ast /\!\!/ G \begin{tikzcd} {} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & {} \end{tikzcd} \ast /\!\!/ G' \;\middle|\; G \text{ finite} \right\}$$

$$\mathbf{H} := \mathrm{Sh}_\infty\big(\mathrm{CrtSp} \times \mathrm{Snglrt}\big)$$ singular-cohesive ∞-topos

Where it takes place: Singular-Cohesive ∞-Topoi

probes
smooth
nhbrhds

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Where it takes place: Singular-Cohesive ∞-Topoi

probes smooth nhbrhds $\quad \mathrm{CrtSp} \;:=\; \left\{ \mathbb{R}^n \xrightarrow{\text{smooth}} \mathbb{R}^{n'} \;\middle|\; n \in \mathbb{N} \right\}$

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singular-cohesive ∞-topos

$$\simeq \mathrm{Sh}\big(\mathrm{Snglrt}, \mathrm{Sh}_\infty(\mathrm{CrtSp})\big)$$

globally equivariant smooth homotopy theory

Where it takes place: Singular-Cohesive ∞-Topoi

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$$\mathbf{H} := \mathrm{Sh}_\infty\big(\mathrm{CrtSp} \times \mathrm{Snglrt}\big)$$ singular-cohesive ∞-topos

$$\simeq\; \mathrm{Sh}\big(\mathrm{Snglrt},\, \mathrm{Sh}_\infty(\mathrm{CrtSp})\big)$$ globally equivariant smooth homotopy theory

$$\simeq\; \mathrm{Sh}\big(\mathrm{CrtSp},\, \mathrm{Sh}_\infty(\mathrm{Snglrt})\big)$$ smooth gbl equivariant homotopy theory

Where it takes place: Singular-Cohesive ∞-Topoi

probes smooth nhbrhds $\mathrm{CrtSp} \;:=\; \left\{ \mathbb{R}^n \xrightarrow{\text{smooth}} \mathbb{R}^{n'} \;\middle|\; n \in \mathbb{N} \right\}$

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$$\mathbf{H} := \mathrm{Sh}_\infty\big(\mathrm{CrtSp} \times \mathrm{Snglrt}\big)$$

singular-cohesive ∞-topos

$$\simeq \mathrm{Sh}\big(\mathrm{Snglrt},\, \mathrm{Sh}_\infty(\mathrm{CrtSp})\big)$$

globally equivariant smooth homotopy theory

$$\simeq \mathrm{Sh}\big(\mathrm{CrtSp},\, \mathrm{Sh}_\infty(\mathrm{Snglrt})\big)$$

smooth gbl equivariant homotopy theory

$$\simeq \mathrm{Sh}_\infty(\mathrm{CrtSp}) \underset{\mathrm{Topos}_\infty}{\times} \mathrm{Sh}_\infty(\mathrm{Snglrt})$$

smooth & globally equivariant homotopy theory

Where it takes place: Singular-Cohesive ∞-Topoi

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$$\simeq \mathrm{Sh}\big(\mathrm{Snglrt}, \mathrm{Sh}_\infty(\mathrm{CrtSp})\big) \qquad \begin{array}{c}\text{globally equivariant}\\\text{smooth homotopy theory}\end{array}$$

$$\simeq \mathrm{Sh}\big(\mathrm{CrtSp}, \mathrm{Sh}_\infty(\mathrm{Snglrt})\big) \qquad \begin{array}{c}\text{smooth}\\\text{gbl equivariant homotopy theory}\end{array}$$

$$\simeq \mathrm{Sh}_\infty(\mathrm{CrtSp}) \underset{\mathrm{Topos}_\infty}{\times} \mathrm{Sh}_\infty(\mathrm{Snglrt}) \qquad \begin{array}{c}\text{smooth \& globally equivariant}\\\text{homotopy theory}\end{array}$$

Where it takes place: Singular-Cohesive ∞-Topoi

$\mathbf{H}$

Where it takes place: Singular-Cohesive ∞-Topoi

$$\mathbf{H} \xrightarrow{\quad \mathrm{Smth} \quad} \mathrm{SmthGrpd}_{\infty} \qquad \textbf{smooth}$$

Where it takes place: Singular-Cohesive ∞-Topoi

$$
\begin{tikzcd}[column sep=huge]
\mathbf{H}
  \arrow[r, shift right=2, "\mathrm{Smth}"']
  \arrow[r, phantom, "\perp"]
&
\mathrm{SmthGrpd}_\infty
  \arrow[l, hook', shift right=2, "\mathrm{Spc}"', color=blue]
\end{tikzcd}
$$

spatial

smooth

Where it takes place: Singular-Cohesive ∞-Topoi

$$
\begin{tikzcd}
{} & {\color{blue}\mathrm{Cncl}} \arrow[l, blue, dash] \arrow[r, blue] \arrow[d, phantom, "\perp"] & {} & {\color{blue}\textbf{conical}} \\
{} & \mathrm{Spc} \arrow[l] \arrow[r, hook'] \arrow[d, phantom, "\perp"] & {} & {\color{green}\textbf{spatial}} \\
\mathbf{H} \arrow[r, dash] & \mathrm{Smth} \arrow[r] & \mathrm{SmthGrpd}_{\infty} & {\color{green}\textbf{smooth}}
\end{tikzcd}
$$

Where it takes place: Singular-Cohesive ∞-Topoi

$$
\begin{tikzcd}[column sep=6em]
\mathbf{H}
\arrow[r, shift left=12, "\mathrm{Cncl}" description]
\arrow[r, phantom, shift left=9, "\perp"]
\arrow[r, leftarrow, shift left=6, "\mathrm{Spc}" description]
\arrow[r, phantom, shift left=3, "\perp"]
\arrow[r, "\mathrm{Smth}" description]
\arrow[r, phantom, shift right=3, "\perp"]
\arrow[r, leftarrow, shift right=6, "\mathrm{Sngl}" description]
& \mathrm{SmthGrpd}_\infty
\end{tikzcd}
$$

conical

spatial

smooth

singular

Where it takes place: Singular-Cohesive ∞-Topoi

$$
\mathbf{H}
\begin{array}{c}
\xrightarrow{\;\;\mathrm{Cncl}\;\;} \\
\perp \\
\xhookleftarrow{\;\;\mathrm{Spc}\;\;} \\
\perp \\
\xrightarrow{\;\;\mathrm{Smth}\;\;} \\
\perp \\
\xhookleftarrow{\;\;\mathrm{Sngl}\;\;}
\end{array}
\mathrm{SmthGrpd}_{\infty}
\qquad
\begin{array}{c}
\textbf{conical} \\
\\
\textbf{spatial} \\
\\
\textbf{smooth} \\
\\
\textbf{singular}
\end{array}
$$

Where it takes place: Singular-Cohesive ∞-Topoi

$$
\begin{array}{cccccc}
 & \times\!\!\longleftarrow & \mathrm{Cncl} & \longrightarrow & & \textbf{conical} \\
 & & \perp & & & \\
 & \longleftarrow & \mathrm{Spc} & \hookrightarrow & & \textbf{spatial} \\
 & & \perp & & & \\
\mathbf{H} & \longrightarrow & \mathrm{Smth} & \longrightarrow & \mathrm{SmthGrpd}_\infty & \textbf{smooth} \\
 & & \perp & & & \\
 & \longleftarrow & \mathrm{Sngl} & \hookrightarrow & & \textbf{singular}
\end{array}
$$

purely conical aspect

$$\vee \;:=\; \mathrm{Spc} \circ \mathrm{Cncl}$$

Where it takes place: Singular-Cohesive ∞-Topoi

$$\mathbf{H} \quad \begin{array}{c} \times\!\!\relbar\!\!\relbar\!\!\relbar\; \mathrm{Cncl} \longrightarrow \\ \perp \\ \longleftarrow \mathrm{Spc} \hookrightarrow \\ \perp \\ \relbar\!\!\relbar\!\!\relbar\; \mathrm{Smth} \longrightarrow \\ \perp \\ \longleftarrow \mathrm{Sngl} \hookrightarrow \end{array} \quad \mathrm{SmthGrpd}_\infty$$

$$\begin{array}{l} \textbf{conical} \\ \textbf{spatial} \\ \textbf{smooth} \\ \textbf{singular} \end{array}$$

purely conical aspect $\quad \vee \;:=\; \mathrm{Spc} \circ \mathrm{Cncl}$

$\perp$

purely smooth aspect $\quad \cup \;:=\; \mathrm{Spc} \circ \mathrm{Smth}$

.

Where it takes place: Singular-Cohesive ∞-Topoi

$$
\mathbf{H}
\begin{array}{c}
\times\!\!\longleftarrow \mathrm{Cncl} \longrightarrow \\
\perp \\
\longleftarrow \mathrm{Spc} \hookrightarrow \\
\perp \\
\relbar\!\!\relbar \mathrm{Smth} \longrightarrow \\
\perp \\
\longleftarrow \mathrm{Sngl} \hookrightarrow
\end{array}
\mathrm{SmthGrpd}_\infty
\qquad
\begin{array}{c}
\textbf{conical} \\
\\
\textbf{spatial} \\
\\
\textbf{smooth} \\
\\
\textbf{singular}
\end{array}
$$

purely conical aspect	$\vee$	$:=$	$\mathrm{Spc} \circ \mathrm{Cncl}$
	$\perp$		
purely smooth aspect	$\cup$	$:=$	$\mathrm{Spc} \circ \mathrm{Smth}$
	$\perp$		
purely orbi-singular aspect	$\curlyvee$	$:=$	$\mathrm{Snglr} \circ \mathrm{Smth}\,.$

Where it takes place: Singular-Cohesive ∞-Topoi

$$
\begin{array}{cccccc}
 & \times\!\!\longleftarrow\!\!\!-\!\!\!-\!\!\!- & \mathrm{Cncl} & -\!\!\!-\!\!\!-\!\!\!\longrightarrow & & \textbf{conical} \\
 & & \perp & & & \\
 & \longleftarrow\!\!\!-\!\!\!-\!\!\!- & \mathrm{Spc} & -\!\!\!-\!\!\!-\!\!\!\hookrightarrow & & \textbf{spatial} \\
 & & \perp & & & \\
\mathbf{H} & -\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!- & \mathrm{Smth} & -\!\!\!-\!\!\!-\!\!\!\longrightarrow & \mathrm{SmthGrpd}_\infty & \textbf{smooth} \\
 & & \perp & & & \\
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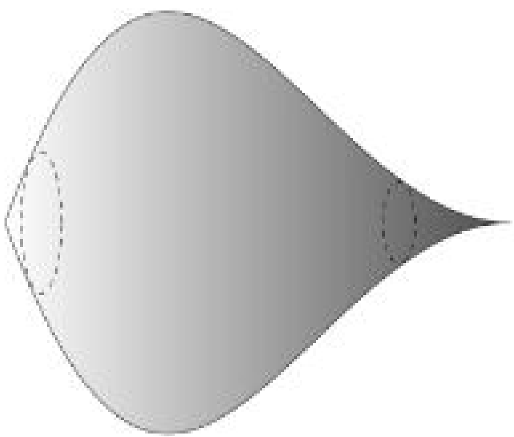
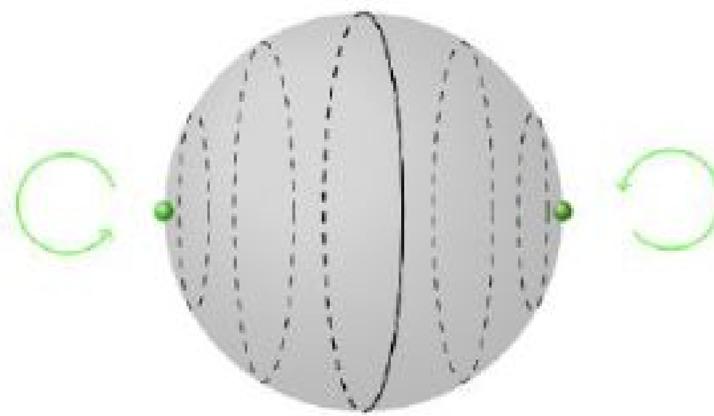
\end{array}
$$

Where it takes place: Singular-Cohesive ∞-Topoi

Singular quotient



Smooth homotopy quotient



orbi-singularity

$\prec^{G}$

project onto purely singular aspect $\curlyvee$

project onto purely smooth aspect $\cup$

opposite extreme aspects of orbifold singularities

$\curlyvee^{G} = * / G = *$

singular quotient

$\cup^{G} = * /\!\!/ G = BG$

smooth homotopy quotient

Where it takes place: Singular-Cohesive ∞-Topoi

Here we have a yet better Oka principle,

true to "absorbing equivariance into the twist":

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Twisted Elmendorf theorem

For G-space X, and $(\Gamma \rtimes G)$-space A:

$$\int \mathrm{Map}\big(\mathrm{X} /\!\!/ G,\ \mathrm{A} /\!\!/ (\Gamma \rtimes G)\big)_{\mathbf{B}(\Gamma \rtimes G)}$$

$$\simeq\ \cup \flat\, \mathrm{Map}\big(\int \curlyvee \mathrm{X} /\!\!/ G, \int \curlyvee \mathrm{A} /\!\!/ (\Gamma \rtimes G)\big)_{\int \curlyvee \mathbf{B}(\Gamma \rtimes G)}$$

[EPB 4.5.3]

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$$\equiv \; \pi_0 \textstyle\int \mathrm{Map}\big(\mathbf{X},\, \mathrm{A}/\!\!/\mathcal{G}\big)$$

by definition

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by *choice* of G-presentation

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by twisted Elmendorf

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Outlook: Differential twisted orbifold cohomology

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From this point on one can define
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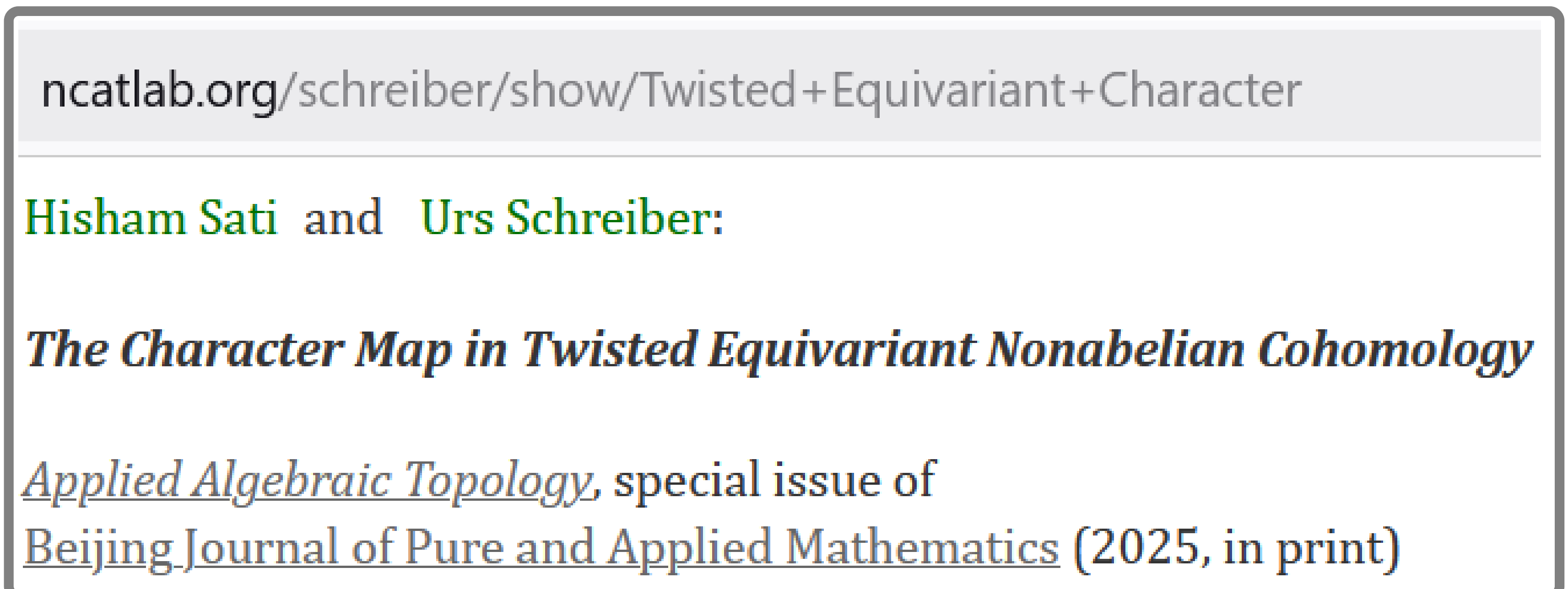
From this point on one can define *differential* twisted orbifold cohomology via the twisted equivariant character of the coefficient object $\int \curlyvee (\mathrm{A} /\!\!/ (\Gamma \rtimes G))$


ncatlab.org/schreiber/show/Twisted+Equivariant+Character

Hisham Sati and Urs Schreiber:

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ncatlab.org/schreiber/show/The+Character+Map

- Domenico Fiorenza, Hisham Sati, and Urs Schreiber:

  The Character Map in Nonabelian Cohomology
  – Twisted, Differential, and Generalized

  World Scientific (2023)
  doi:10.1142/13422

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Urs Schreiber on joint work with Hisham Sati:

an invitation to the monographs:

-*Equivariant Principal ∞-Bundles*, CUP (2025, in print)
-*Geometric Orbifold Cohomology*, CRC (2026, to appear)

Geometric Orbifold Cohomology in Singular-Cohesive ∞-Topoi

talk at *ItaCa Fest 2025*

17 June 2025

The End.

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