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(New York University, Abu Dhabi & Czech Academy of Science, Prague)

Microscopic brane physics from Cohomotopy

talk at
M-Theory and Mathematics
NYU AD 2020

based on joint work with
H. Sati
0) Introduction

1) Microscopic Brane Charge

2) Orientifold Tadpole Cancellation

3) $D6 \perp D8$-Brane Intersections

4) Hanany-Witten Theory

5) Chan-Paton Data

6) BMN Matrix Model States

7) M2/M5 Brane Bound States
Introduction

Open Problem M and Hypothesis H
Open problem QCD

Confined QCD

“Millennium problem”
- QCD-cosmology
- nucleosynthesis
- form factors
- ...

Flavored QCD

“Flavor problem”
- cosm. constant
- EW hierarchy
- vacuum stability
- $V_{cb}$-puzzle
- flavour anomalies
- Leptoquark \rightarrow GUT
- ...

Solution

\[ \Rightarrow \]
Open problem QCD

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Solution

AdS/QCD correspondence

braneworld geometric engineering

IIA super-gravity near black intersecting branes

higgs sector

\[ \text{D4 with } N_c \text{ strings} \]

\[ \text{D4-brane} \]

\[ \text{flavor branes} \]

\[ \text{color branes} \]
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Solution

$N_c = 3$ quark colors

D4 with $N_c$ strings

AdS/QCD correspondence

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  - GUT
  - ...

Solution

M-theory

- with microscopic intersecting branes

\[ N_c = 3 \text{ quark colors} \]

\[ \text{baryon} \]

\[ \text{brane world geometric engineering} \]

AdS/QCD correspondence

\[ \text{M-theory with microscopic intersecting branes} \]
Open problem QCD

Confined QCD

“Millennium problem”
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- ...

Open problem M

small $N_c$ \(\prime \)t Hooft doubling

Solution

baryon

braneworld geometric engineering

AdS/QCD correspondence

M-theory with microscopic intersecting branes

D4 with $N_c$ strings

Leptoquark $\Rightarrow$ GUT

Hypothesis H

color branes

flavor branes
Open problem QCD \( \xrightarrow{\text{small } N_c \ 't Hooft doubling} \) Open problem M
Open problem QCD \[\text{small } N_c \quad \text{t'Hooft doubling}\] first consider \[\text{large } g_{YM}^2 N_c\] \[\rightarrow 11d \text{ supergravity}\] Open problem M
Open problem QCD \( \rightarrow \) \( small \ N_c \) 't Hooft doubling

first consider \( large \ g_{YM}^2 N_c \) \( \rightarrow \) 11d supergravity

CovariantPhaseSpace\(_{11d \ SuGra} \)

\[
\{ \text{spacetimes} \}
\]

\text{formalized as:}
\[
G_{ADE} - \text{orbi} \ \mathbb{R}^{10,1|32} - \text{folds} \ (X)
\]

\text{equipped with:}
\[
0) \text{gravity} \quad 1) \text{C-field}
\]

\text{formalized as:}
\[
\text{Pin}^+ - \text{structure} \ (E, \Psi)
\]

\text{differential forms} \ (G_4, G_7)

\text{subject to:}
\[
\text{Einstein equations} \quad G = T(\Psi, G_4, G_7)
\]

\text{Page equation} \quad dG_7 + \frac{1}{2} G_4 \wedge G_4 = 0
Open problem QCD

first consider

Open problem M

11d supergravity

CovariantPhaseSpace\(\text{SUgra}_{11d}\) =

\[
\begin{align*}
\text{spacetimes} & \\
\text{formalized as:} & \quad \text{G}_\text{ADE}-\text{orbi} \quad \mathbb{R}^{10,1|32} - \text{folds} \quad (\mathcal{X}) \\
\text{equipped with:} & \quad 0) \text{gravity} & 1) \text{C-field} \\
\text{formalized as:} & \quad \text{Pin}^+\text{-structure} \ (E, \Psi) & \quad \text{differential forms} \ (G_4, G_7) \\
\text{subject to:} & \quad \text{Einstein equations} & \quad \text{Page equation} \\
\text{equivalent to:} & \quad \text{super-torsion} = 0 & \quad \text{flux is in rationalized J-twisted Cohomotopy} \\
& \quad \text{Candiello-Lechner 93, Howe 97} & \quad \text{Sati 13, Fiorenza-Sati-S. 19a}
\end{align*}
\]
Open problem QCD \( t \) Hooft doubling small \( N_c \)

\& large \( g_{YM}^2 N_c \)

\( 11d \) supergravity charge-quantized in Cohomotopy

\[ \text{CovariantPhaseSpace}_{11d \ SuGra} = \]

\begin{enumerate}
  \item spacetimes
    \begin{enumerate}
      \item formalized as: \( G_{ADE} \)-orbifold \( R^{10,1|32} \)-folds \( (\mathcal{X}) \)
      \item equipped with:
        \begin{enumerate}
          \item gravity
            \begin{enumerate}
              \item super-vielbein
              \item Einstein equations
            \end{enumerate}
          \item C-field
            \begin{enumerate}
              \item differential forms \( (G_4, G_7) \)
            \end{enumerate}
        \end{enumerate}
    \end{enumerate}
  \end{enumerate}

  subject to:

  \begin{enumerate}
    \item Page equation
      \begin{enumerate}
        \item flux is in rationalized J-twisted Cohomotopy
      \end{enumerate}
    \item equivalent to:
      \begin{enumerate}
        \item super-torsion = 0
        \item Candiello-Lechner 93, Howe 97
      \end{enumerate}
  \end{enumerate}

  Sati 13, Fiorenza-Sati-S. 19a
**Hypothesis H**

Sati 13 | Fiorenza-Sati-S. 19b 19c

\[
\text{Covariant Phase Space} = M-\text{Theory}
\]

| Spacetimes | \[ G_{ADE} \text{-orbi } \mathbb{R}^{10,1|32} \text{-folds } (\mathcal{X}) \] |
|------------|----------------------------------------------------------------------------------|
| Formulated as: | \[ G_{ADE} \text{-orbi } \mathbb{R}^{10,1|32} \text{-folds } (\mathcal{X}) \] |
| Equipped with: | 0) Gravity | 1) C-field |
| | super-vielbein | flux densities |
| | Pin\(^+\)-structure \((E, \Psi)\) | differential forms \((G_4, G_7)\) |
| Subject to: | Einstein equations | Page equation |
| | super-torsion = 0 | flux is in rationalized J-twisted Cohomotopy |
| | | FSS 19b 19c | SS 19a 19b 19c |
today:

Compare implications of Hypothesis H to M-folklore.

Salvageable?

Looks like M-theory?

Discrepancies?

Adjust fine-print in Hypothesis H (e.g. differential refinement)

Solve
1) Millennium problem
2) Vacuum selection problem
3) Flavor problem
... (for later)
## Implications of Hypothesis H

<table>
<thead>
<tr>
<th>on curved but smooth spacetimes</th>
<th>on flat but orbi-singular spacetimes</th>
<th>on spacetimes with horizons</th>
</tr>
</thead>
<tbody>
<tr>
<td>FSS 19b, FSS 19c, GS20</td>
<td>BSS 18, SS 19a</td>
<td>SS 19c</td>
</tr>
<tr>
<td><strong>topological anomaly cancellation:</strong></td>
<td><strong>equivariant anomaly cancellation</strong></td>
<td><strong>Dp ⊥ D(p + 2)</strong> worldvolume QFT</td>
</tr>
<tr>
<td>- shifted C-field flux quantization</td>
<td>- M5/MO5 anomaly cancellation</td>
<td>- fuzzy funnels</td>
</tr>
<tr>
<td>- C-field tadpole cancellation</td>
<td>- RR-field tadpole cancellation</td>
<td>- BLG 3-algebras</td>
</tr>
<tr>
<td>- M5 Hopf-WZ level quantization</td>
<td>- no irrational D-brane charge</td>
<td>- BMN matrix model</td>
</tr>
<tr>
<td>- DMW anomaly cancellation</td>
<td></td>
<td>- M2/M5 bound states</td>
</tr>
<tr>
<td>- C-field integral eom</td>
<td></td>
<td>- AdS3-holography</td>
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<tr>
<td>...</td>
<td></td>
<td>- Coulomb branch indices</td>
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<tr>
<td></td>
<td></td>
<td>- Hanany-Witten rules</td>
</tr>
<tr>
<td></td>
<td></td>
<td>- ...</td>
</tr>
</tbody>
</table>

H. Sati’s talk

D. Fiorenza’s talk

my talk

at M-Theory and Mathematics 2020
(1)

Microscopic Brane Charge

implied by

Hypothesis H with Pontrjagin-Thom Theorem

Sati-Schreiber 19a [arXiv:1909.12277]
**Dirac charge quantization** – The topological sector of the electromagnetic field is a cocycle in degree-2 ordinary cohomology, with classifying/coefficient space $BU(1)$.  

\[
\begin{align*}
X &:= \mathbb{R}^1 \times \left( \mathbb{R}^3 \setminus \{0\} \right) \simeq S^2 \\
\text{spacetime around} & \quad \text{a magnetic monopole} \\
c &\quad \text{electromagnetic field} \\
\text{sourced by monopole} & \\
BU(1) &\simeq \mathbb{C}P^\infty \\
\text{classifying space of} & \quad \text{electromagnetic gauge group} \\
\end{align*}
\]

\[
\left[ c \right] \in \left\{ X \longrightarrow BU(1) \right\} \Big/ \sim_{\text{homotopy}} \simeq \mathbb{Z} \\
\text{charge} = \text{homotopy class}
\]
\[ \mathbb{R}^1 \times \left( \mathbb{R}^3 \setminus \{ \bar{x}_1, \ldots, \bar{x}_k \} \right) \subset \mathbb{C}P^1 \]
spacetime around Yang-Mills monopoles

c
nuclear force field sourced by monopole

\[ \mathbb{C}P^1 \]
classifying space of complex Cohomotopy

\begin{align*}
\left[ c \right] & \in \left\{ \mathbb{C}P^1 \rightarrow \mathbb{C}P^1 \right\} / \sim_{\text{homotopy}} \\
\text{charge} & = \text{homotopy class}
\end{align*}

**Atiyah-Hitchin charge quantization** – The moduli space of SU(2) Yang-Mills monopoles is the cocycle space of complex-rational Cohomotopy of any sphere enclosing them.
\[ \mathbb{R}^{5,1} \times (\mathbb{R}^4 \setminus \{ x_1, \ldots, x_k \}) \times \mathbb{R}^1_{/\mathbb{Z}_2^{HW}} \]

spacetime around M5-branes

\[ c \]

C-field sourced by M-branes

??

classifying space of which cohomology theory??

\[ S^4 \]

\[ [c] \in \{ X \longrightarrow \text{??} \} / \sim_{\text{homotopy}} \cong \text{??} \text{ charge lattice} \]

\[ \text{charge} = \text{homotopy class} \]

**Strominger-Witten:** Monopoles are wrapped M5-branes and the elusive non-perturbative Yang-Mills theory is in M-theory.

\[ \sim \text{ Open problem: } \text{Wherein is M5-brane charge quantization?} \]
Hypothesis H (Fiorenza-Sati-Schreiber 19):
$C$-field is charge-quantized in $J$-twisted Cohomotopy theory.
Cohomotopy charge of normally framed submanifolds is represented by the submanifold’s asymptotic distance function, traditionally known as the Pontrjagin-Thom collapse.
Cohomotopy charge of 0-dimensional submanifolds (traditionally known as “electric field map” or scanning map) exhibits net brane/anti-brane charge in $\mathbb{Z}$. 
Brane/anti-brane pair creation & annihilation is exhibited, under Hypothesis H, by normally framed cobordism.
Cohomotopy charge vanishing at $\infty$ on Euclidean $n$-space is equivalently the Cohomotopy charge of the $n$-sphere and hence takes values in homotopy groups of spheres.
Orientifold Tadpole Cancellation

implied by

Hypothesis H with Equivariant Hopf Degree Theorem

Sati-Schreiber 19a [arXiv:1909.12277]
### Euclidean linear representation

\[ Z_2 \Rightarrow \mathbb{R} \times_{\text{sgn}} \]

\[ x_2 = \frac{1}{2} \]
\[ x_2 = 0 \]
\[ x_2 = -\frac{1}{2} \]
\[ x_1 = -\frac{1}{2} \]
\[ x_1 = 0 \]
\[ x_1 = \frac{1}{2} \]

### Representation sphere

\[ Z_2 \Rightarrow S^1 \times_{\text{sgn}} \]

### Representation torus

\[ Z_4 \Rightarrow \mathbb{T}^2 \times_{\text{rot}} \]

---

**Examples of linear representations and induced G-spaces**
**Figure H – Hopf degree in the unstable range** takes values in the set \( \{0, 1\} \) (14), corresponding to the binary choice of there being or not being a unit charge at the single point.
The equivariant Hopf degree theorem says that $\mathbb{Z}_2$-equivariant Cohomotopy charges near singularities are sourced by, possibly, a charge attached to the singularity and any integer number of twice this charge located nearby.
Stabilization & linearization of equivariant Cohomotopy lands in equivariant K-theory. In this approximation virtual $G$-sets of (anti-)branes map to virtual permutation representations.
Equivariant Cohomotopy on toroidal orbifolds glued from local cocycles in the vicinity of singularities. By the equivariant Hopf degree theorem, all global cocycles are obtained this way.
Figure O – Pushforward in equivariant Cohomotopy from the vicinity of a singularity to the full toroidal orientifold is an isomorphism on brane charges and an injection on O-plane charges, by Prop. 3.18. Shown is a case with $G = \mathbb{Z}_4$, as in Figure M.
Equivariant Cohomotopy implies local tadpole cancellation by the combined unstable and stable version of the equivariant Hopf degree theorem.
Super-differential equivariant Cohomotopy implies global tadpole cancellation by forcing the charge to vanish at global Elmendorf stage, and only there.
The following four slides show technical detail of the realization of this mechanism for MO5-planes at ADE-singularities in heterotic M-theory.
Figure T – Subgroup lattice and fixed/singular subspaces in the singularity structure from Table 7. On the left, groups associated to the middle of a sub-simplex are diagonal subgroups inside the direct product of subgroups associated to the vertices, as indicated by the superscripts. On the right, all fixed loci with superscript \((-)^0\) are actually empty, but appear as superficially non-empty (un-charged) singularities after M/IIA KK-reduction (68), e.g. O4\(^0\) (71), O8\(^0\) (74), as on the right of Figure OP. The numbered subscripts (xx) indicate the corresponding expression in the text.
Figure V – Equivariant Cohomotopy of ADE-orbifolds in heterotic M-theory with singularity structure as in Figure S. The resulting charge classification (Cor. 4.4) implies, via the unstable PT isomorphism (§2.1), the $\frac{1}{2}M5 = MO9 \cap MK6$-brane configurations (65) similarly shown in [FLO99, Fig. 1][KSTY99, p. 7][FLO00a, Fig. 1][FLO00b, Fig. 2][FLO00c, Fig. 1][GKST01, p. 4, 68, 71]. This is as in Figure L but with points (M5s) extended to half-line (MK6s), see Remark 4.7 and Table 8.
### Spacetimes on which to measure flux sourced by M5/MO5-charge

<table>
<thead>
<tr>
<th>Definition</th>
<th>$X_{\text{MO5}} \simeq \text{htpy } S(R_{\text{Mga},4\text{ga}})/Z_2^{\text{heli+HW}}$</th>
<th>$X_{1/2\text{M5}} \simeq \text{htpy } S(R_{\text{Mga}})/Z_2^{\text{HW}} \times T^4_{\text{RT}} // Z_2^{\text{refl}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(75)</td>
<td>(79)</td>
<td></td>
</tr>
</tbody>
</table>

#### Illustration
- Smooth but curved
- Singular but flat

#### Cohomological charge quantization

**by Hypothesis H**

<table>
<thead>
<tr>
<th>Cohomology theory</th>
<th>$J$-twisted Cohomotopy $\pi^{TX}(X)$</th>
<th>Equivariant Cohomotopy $\pi^Y(\mathbb{P}^Y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(by Table 4)</td>
<td>[FSS19b][FSS19c]</td>
<td>§3</td>
</tr>
</tbody>
</table>

#### Illustration (Remark 4.7)

#### Charge classification
- $c_{\text{tot}} = 1 - N \cdot 2$  
  \[ (77) \]  
- $c_{\text{tot}} = N_{\text{MO5}} \cdot \text{triv} - N_{\text{M5}} \cdot 2_{\text{reg}}$  
  \[ \text{(Cor. 4.4)} \]  
  $\Rightarrow \frac{|Q_{\text{tot}}|}{N_{\text{M5}}} = 8$  
  \[ \text{(Cor. 4.6)} \]
D6 $\perp$ D8-brane intersections implied by Hypothesis H with May-Segal Theorem by Sati-Schreiber 19c [arXiv:1912.10425]
Cohomotopy cocycle space

\[ \pi^4(X) := \Maps^*(X, S^4) \]

\[ \pi_0(\pi^4(X)) = \left\{ \text{Cohomotopy cohomology classes} \right\} = \pi^4(X) \]

\[ \pi_1(\pi^4(X)) = \left\{ \text{Cohomotopy gauge transformations} \right\} \]

\[ \pi_2(\pi^4(X)) = \left\{ \text{Cohomotopy gauge of gauge transformations} \right\} \]

\[ \vdots \]
Cohomotopy cocycle space
vanishing at $\infty$ on Euclidean 3-space

$\pi^4 \left( (\mathbb{R}^3)^{\text{cpt}} \right) \xrightarrow{\text{hmtpy}} \sim \text{Conf}(\mathbb{R}^3, \mathbb{D}^1)$

May-Segal theorem
configuration space of points
in $\mathbb{R}^3 \times \mathbb{R}^1$
which are:
1) unordered
2) distinct after projection to $\mathbb{R}^3$
3) allowed to vanish to $\infty$ along $\mathbb{R}^1$

\[
\begin{align*}
\begin{cases}
\mathbb{R}^3 \times \{0\} \\
\mathbb{R}^3 \times \{\infty\}
\end{cases}
\end{align*}
\]
hence: a form of \textcolor{blue}{differential Cohomotopy} assigns configuration spaces:

\[
\pi^4 \left( (\mathbb{R}^d)_{\text{cpt}} \wedge (\mathbb{R}^{4-d})^+ \right) \xrightarrow{\text{hmtpy} \simeq} \pi^4_{\text{diff}} \left( (\mathbb{R}^d)_{\text{cpt}} \wedge (\mathbb{R}^{4-d})^+ \right) := \text{Conf}(\mathbb{R}^d, \mathbb{D}^{4-d})
\]
Lemma:

\[ \bigcup_{n \in \mathbb{N}} \text{Conf}(\mathbb{R}^D) \cong \text{Conf}(\mathbb{R}^D, \mathbb{D}^1) \times \text{Conf}(\mathbb{D}^{D+1}) \]

Ordered configurations of points in \( \mathbb{R}^D \)

Un-ordered configurations of points in \( \mathbb{R}^1 \) with labels in \( \mathbb{D}^1 \)

Un-ordered configurations of points in \( \mathbb{R}^D \) with labels in \( \mathbb{D}^D \)

\[
\begin{pmatrix}
\text{Conf}(\mathbb{R}^D, \mathbb{D}^1) & \text{Conf}(\mathbb{R}^1, \mathbb{D}^D) \\
((i_L)^*)_* & ((i_R)^*)_*
\end{pmatrix}
\]
Lemma: \( \bigcup_{n \in \mathbb{N}} \text{Conf}(\mathbb{R}^D) \cong_{\text{hmtpy}} \text{Conf}(\mathbb{R}^D, \mathbb{D}^1) \times \text{Conf}(\mathbb{D}^{D+1}) \)

Ordered configurations of points in \( \mathbb{R}^D \)

\( \square \) Un-ordered configurations of points in \( \mathbb{R}^D \)

with labels in \( \mathbb{D}^1 \)

Un-ordered configurations of points in \( \mathbb{R}^1 \)

with labels in \( \mathbb{D}^D \)

Un-ordered configurations of points in \( \mathbb{D}^{D+1} \)

Un-ordered configurations of points in \( \mathbb{R}^1 \)

\{0\} \times \mathbb{R}^1 \rightarrow \mathbb{R}^3 \times \mathbb{R}^1 \rightarrow \mathbb{R}^3 \times \{0\} \quad \text{induced ordering}
Consequence:

Assuming Hypothesis H:

\[
\pi^4_{\text{diff}} : \bigwedge^3 \mathbb{R}^3 \cup \bigwedge^2 \mathbb{R}^3 \rightarrow \bigcup_{n \in \mathbb{N}} \text{Conf}(\mathbb{R}^3) \]

Differential Cohomotopy cocycle space reflecting $D6 \perp D8$-charges

Transversal space to 3-codim branes hence to $D6$-branes

\[ (\mathbb{R}^3)_{\text{cpt}} \wedge (\mathbb{R}^1)_+ \cup (\mathbb{R}^3)_+ \wedge (\mathbb{R}^1)_{\text{cpt}} \]

Transversal space to 1-codim branes hence to $D8$-branes

\[ \mathbb{R}^1 \cup \mathbb{R}^1 \]

Chan-Paton labels

ordering
(4)

Hanany-Witten Theory

implied by

Hypothesis H with Fadell-Husseini Theorem

Sati-Schreiber 19c [arXiv:1912.10425]
higher co-observables on $D6 \perp D8$-intersections

$$H_\bullet \left[ \bigsqcup \big[ c \big] \Omega_c \pi^4 \text{diff} \right]$$

$$\simeq H_\bullet \left( \bigsqcup_{N_f \in \mathbb{N}} \Omega^{\text{Conf}}(\mathbb{R}^3) \right)$$ (by the above)

$$\simeq \bigoplus_{N_f \in \mathbb{N}} \mathcal{A}^{\text{pb}}_{N_f}$$

are algebra of horizontal chord diagrams

Fadell-Husseini theorem
Horizontal chord diagrams form algebra under concatenation of strands.

This is universal enveloping algebra of the infinitesimal braid Lie algebra (Kohno):

(i) the 2T relations:

(ii) the 4T relations
Consider the subspace of **skew-symmetric co-observables**, denote elements as follows:

\[
\begin{bmatrix}
\end{bmatrix}
= \ t_{45} \wedge t_{35} \wedge t_{25} \wedge t_{15} \wedge t_{14} \wedge t_{24}
\]
In the subspace of skew-symmetric co-observables we find:

the 2T relations become the *ordering constraint*

skew-symmetry becomes the *s-rule*

the 4T relations become the *breaking rule*
these are the rules of **Hanany-Witten theory** for \( \text{NS5} \perp Dp \perp D(p + 2) \)-brane intersections

if we identify horizontal chord diagrams as follows:

(i) strands as \( D(p + 2) \)-branes;

(ii) chords as \( Dp \)-branes, stretching between \( D(p + 2) \)’s;

(iii) green dots as \( \text{NS5} \)-branes;

(iv) gray lines as \( Dp \)-branes, stretching from \( \text{NS5} \) to \( D(p + 2) \).
(5)

Chan-Paton data

implied by

**Hypothesis H** with **Bar-Natan theorem**

Sati-Schreiber 19c [arXiv:1912.10425]
higher co-states on $\text{D6} \perp \text{D8}$-intersections

$$H^\bullet \left( \bigsqcup \Omega_c \, \pi^4_{\text{diff}} \right)$$

$$\simeq H^\bullet \left( \bigsqcup_{N_f \in \mathbb{N}} \text{Conf} \left( \mathbb{R}^3 \right) \right) \quad \text{(by the above)}$$

$$\simeq \bigoplus_{N_f \in \mathbb{N}} \mathcal{W}^\text{pb}_{\text{pf}}$$

are horizontal weight systems.

**Kohno & Cohen-Gitler theorem**
All horizontal weight systems $w : \mathcal{A}^{pb} \to \mathbb{C}$ come from Chan-Paton data:

1) metric Lie representations $\rho$
2) stacks of coincident strands
3) winding monodromies:

Bar-Natan theorem
<table>
<thead>
<tr>
<th>Data of metric Lie representation</th>
<th>Category notation</th>
<th>Penrose notation</th>
<th>Index notation</th>
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<tr>
<td><strong>Lie bracket</strong></td>
<td><img src="image" alt="Diagram" /></td>
<td><img src="image" alt="Diagram" /></td>
<td>$f_{ab}^c$</td>
</tr>
<tr>
<td><strong>Jacobi identity</strong></td>
<td><img src="image" alt="Diagram" /></td>
<td><img src="image" alt="Diagram" /></td>
<td>$f_{ae}^d f_{bc}^e - f_{be}^d f_{ac}^e = f_{ec}^d f_{ab}^e$</td>
</tr>
<tr>
<td><strong>Lie action</strong></td>
<td><img src="image" alt="Diagram" /></td>
<td><img src="image" alt="Diagram" /></td>
<td>$\rho_a^i_j$</td>
</tr>
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<td><strong>Lie action property</strong></td>
<td><img src="image" alt="Diagram" /></td>
<td><img src="image" alt="Diagram" /></td>
<td>$\rho_a^i l \rho_b^j_l - \rho_b^j l \rho_a^i_l = f_{ab}^c \rho_{c}^j_i$</td>
</tr>
<tr>
<td><strong>Metric</strong></td>
<td><img src="image" alt="Diagram" /></td>
<td><img src="image" alt="Diagram" /></td>
<td>$g_{ab}, g^{ab}$</td>
</tr>
<tr>
<td></td>
<td><img src="image" alt="Diagram" /></td>
<td><img src="image" alt="Diagram" /></td>
<td>$k_{ij}, k^{ij}$</td>
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</tbody>
</table>
(6)

BMN Matrix Model States

implied by

Hypothesis H
\[ \rho \in \mathfrak{su}(2)_\mathbb{C} \text{ MetricReps} \] equivalently identified with:

0) configuration of concentric fuzzy 2-spheres
1) fuzzy funnel state in DBI model for \( Dp \perp D(p + 2) \)
2) susy state in BMN matrix model for \( \text{M2/M5} \)

corresponding weight systems \( w_{(\rho,\sigma)} : \mathcal{A}^{pb} \to \mathbb{C} \) are:

0) radius fluctuation amplitudes of fuzzy 2-spheres
1) invariant multi-trace observables in \( \{ \text{DBI model, BMN model} \} \)
0) Radius fluctuation observables on $N$-bit fuzzy 2-spheres $S^2_N$ are $N \in \text{su}(2)_\mathbb{C}$ MetricReps weight systems on chord diagrams:

$$\int_{S^2_N} (R^2)^2 \otimes$$

$$= \frac{4\pi}{\sqrt{N^2-1}} \text{Tr}(X_a \cdot X_b \cdot X^a \cdot X^b)$$

$$\int_{S^2_N} (R^2)^3$$

$$= \frac{4\pi}{\sqrt{N^2-1}} \text{Tr}(X_a \cdot X^a \cdot X_b \cdot X_c \cdot X^b \cdot X^c)$$
1,2) weight system $w_\rho$ is the observable aspect of matrix model state $\rho$:

- linear combinations of finite-dim $\mathfrak{su}_C$-representations
- naive funnel- / susy-states of DBI model / BMN matrix model

Supersymmetric state of BMN matrix model (fuzzy 2-sphere geometry)

$(\mathfrak{su}(2)_C \otimes V \xrightarrow{\rho} V)$

monomial $S^2$-rotation invariant multi-trace observable

$\text{Tr}_V(\rho_a \cdot \rho_d \cdot \rho^a) \text{Tr}_V(\rho_b \cdot \rho^d \cdot \rho_c \cdot \rho_e \cdot \rho^b \cdot \rho_f \cdot \rho^c) \text{Tr}_V(\rho^e \cdot \rho^f)$

Value of multi-trace observable

weight systems on horizontal chord diagrams

$\mathcal{W}^{\text{pb}}$ states of DBI model / BMN matrix mode as observed by invariant multi-trace observables
(7)

M2/M5 Brane Bound States

implied by

Hypothesis H
Given a \textit{sequence} of susy states in the BMN matrix model

\[
(V, \rho) := \bigoplus_i \left( N_i^{(M2)} \cdot N_i^{(M5)} \right) \in \mathfrak{su}(2)_\mathbb{C} \text{MetricReps} / \sim
\]

this is argued to converge to macroscopic M2- or M5-branes depending on how the sequence behaves in the large \(N\) limit:

<table>
<thead>
<tr>
<th>Stacks of macroscopic...</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>M2-branes</strong></td>
</tr>
<tr>
<td>If for all (i)</td>
</tr>
<tr>
<td>with fixed (N_i^{(M2)})</td>
</tr>
<tr>
<td>and fixed (N_i^{(M2)}/N)</td>
</tr>
</tbody>
</table>

\[
\left( \begin{array}{c}
\text{the relevant} \\
\text{large \(N\) limit}
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\text{the number of coincident branes} \\
\text{in the \(i\)th stack}
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\text{the charge/light-cone momentum} \\
\text{carried by the \(i\)th stack}
\end{array} \right)
\]
Given a **sequence** of susy states in the BMN matrix model

\[
(V, \rho) := \bigoplus_i \left( N_i^{(M2)} \cdot N_i^{(M5)} \right) \in \mathfrak{su}(2)_C \text{MetricReps} / \sim
\]

the **large** \( N \)** limit does **not** exist here:

\[
\text{Span} (\mathfrak{su}(2)_C \text{MetricReps}) \xrightarrow[\rho \mapsto w_\rho]{} \mathcal{W}^{pb}
\]

but does exist in weight systems **if we normalize by the scale of the fuzzy 2-sphere geometry:**

\[
\frac{4\pi 2^{2n}}{\left( \left( N^{(M5)} \right)^2 - 1 \right)^{1/2+n}} \cdot w_N^{(M5)}
\]

Single M2-brane state in BMN model (multiple of \( \mathfrak{su}_C \)-weight system)

\[
\in \mathcal{W}^{pb}
\]

states as seen by multi-trace observables (weight systems on chord diagrams)
M2/M5-brane bound states as emergent under Hypothesis H
End.