Urs Schreiber on joint work with Hisham Sati:

invitation to our preprint: [arXiv:2507.00138]

Identifying Anyonic Topological Order in FQAH Systems

CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS

@

جامعة نيويورك أبوظبي

NYU | ABU DHABI

(September 2025) find these slides at: [ncatlab.org/schreiber/show/QMATH16]

Urs Schreiber   on joint work with   Hisham Sati:

invitation to our preprint: [arXiv:2507.00138]

arXiv > cond-mat > arXiv:2507.00138

Search...

Help | Ad

Condensed Matter > Strongly Correlated Electrons

[Submitted on 30 Jun 2025]

Identifying Anyonic Topological Order in Fractional Quantum Anomalous Hall Systems

Hisham Sati, Urs Schreiber

CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS

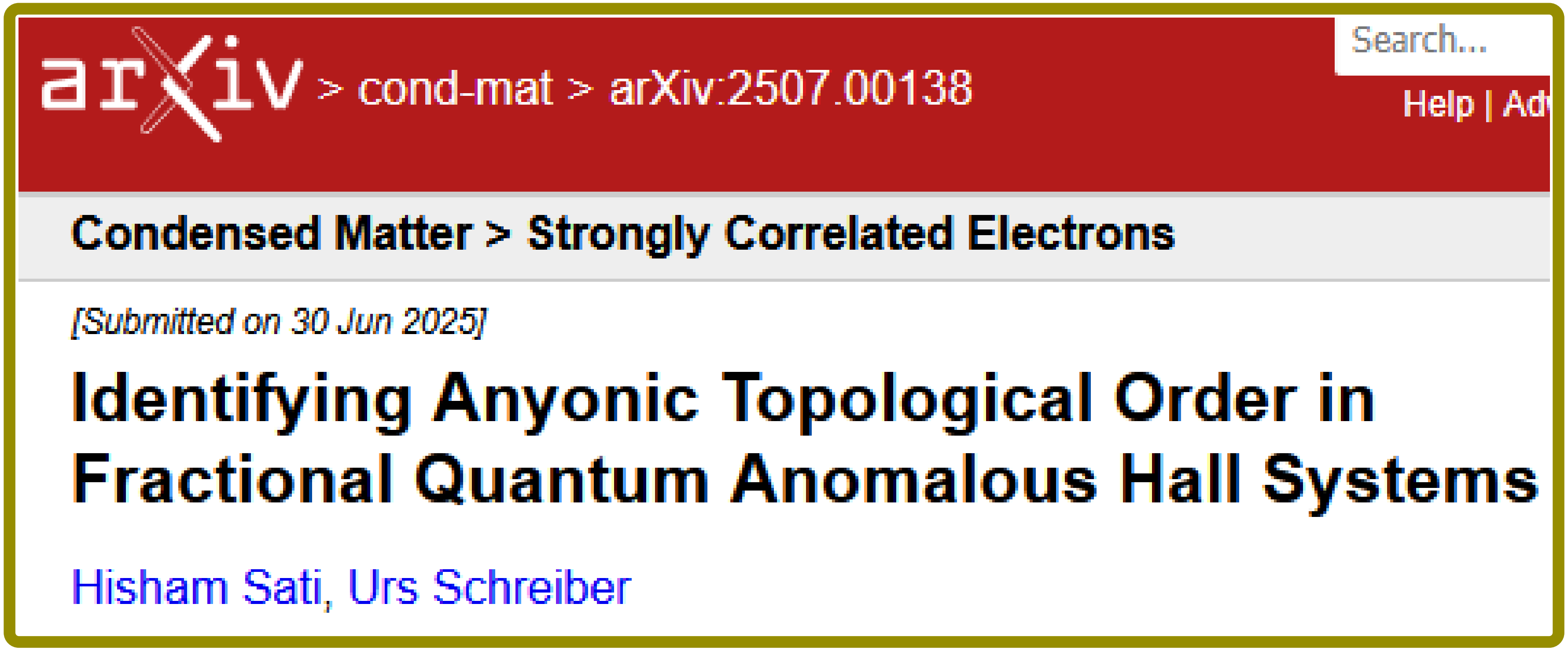
@

جامعة نيويورك أبوظبي

NYU | ABU DHABI

(September 2025) find these slides at: [ncatlab.org/schreiber/show/QMATH16]

Featured at *Quantum Zeitgeist*.



arXiv > cond-mat > arXiv:2507.00138

Search...

Help | Adv

Condensed Matter > Strongly Correlated Electrons

[Submitted on 30 Jun 2025]

Identifying Anyonic Topological Order in Fractional Quantum Anomalous Hall Systems

Hisham Sati, Urs Schreiber

Featured at *Quantum Zeitgeist.*

quantumzeitgeist.com/topological-quantum-hardware-emerges-from-fractional-anomalous-hall-e

QUANTUM ZEITGEIST

QUANTUM COMPUTING TECHNOLOGY NEWS

QUANTUM TECHNOLOGY

Topological Quantum Hardware Emerges From Fractional Anomalous Hall Effect Physics.

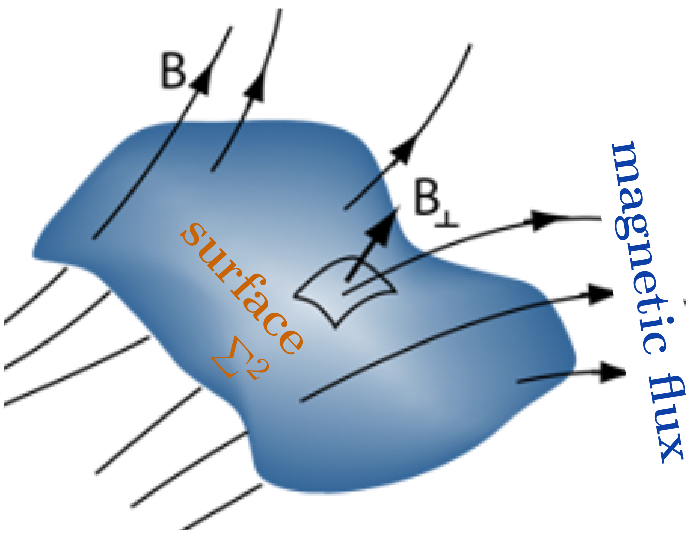
July 2, 2025
BY QUANTUM NEWS

The pursuit of robust quantum computation necessitates the identification and control of exotic states of matter exhibiting topological order, where information is encoded not in local degrees of freedom but in the global properties of the system. Recent observations of fractional anomalous Hall (FQAH) states, characterised by fractionalised quantum Hall effects in the absence of an external magnetic field, present a promising avenue for realising such topological hardware. However, confirming the existence of the crucial anyonic excitations, quasiparticles obeying non-Abelian statistics essential for quantum computation, remains a significant challenge. Hisham Sati and Urs Schreiber, alongside colleagues at the Center for Quantum and Topological Systems at New York University Abu Dhabi, address this issue in their work, “Identifying Anyonic Topological Order in Fractional Quantum Anomalous Hall Systems”. Their research establishes a link between the fragile topology of these systems and the identification of anyons within momentum space, utilising a theorem from algebraic topology dating back to 1980, and providing a framework for understanding symmetry-protected topological order in FQAH systems through computations in equivariant cohomotopy.

Fractional Quantum Hall systems.

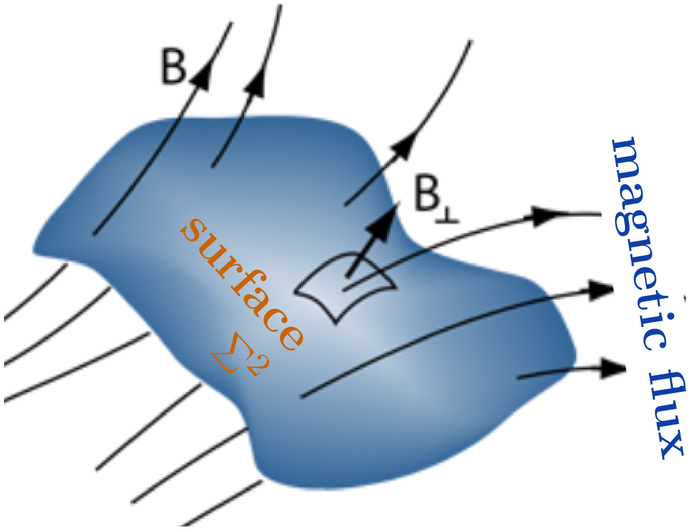
Fractional Quantum Hall systems.

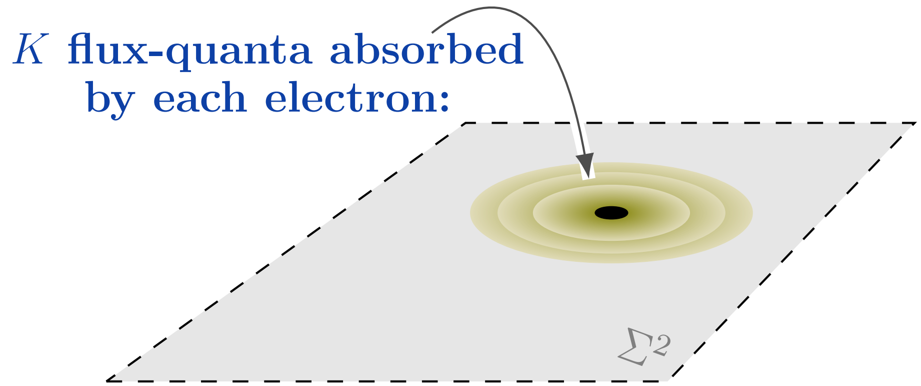
electron gas in 2D semiconductor Σ^2
subject to transverse magnetic field



Fractional Quantum Hall systems.

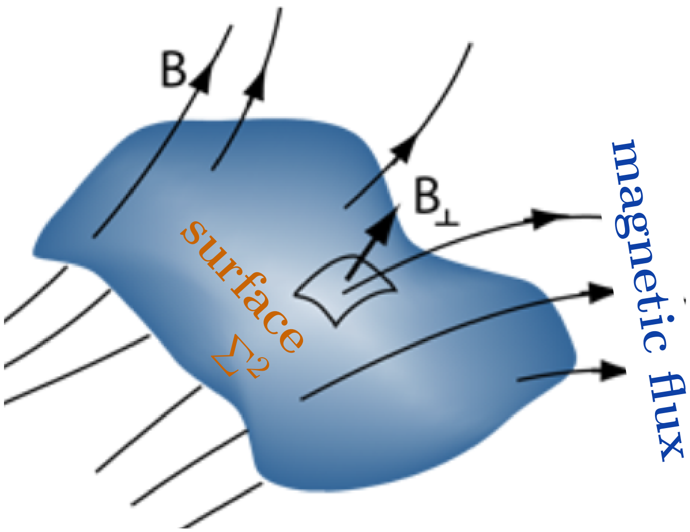
electron gas in 2D semiconductor Σ^2
subject to transverse magnetic field
at *rational filling fraction* of
electrons per magnetic flux quanta

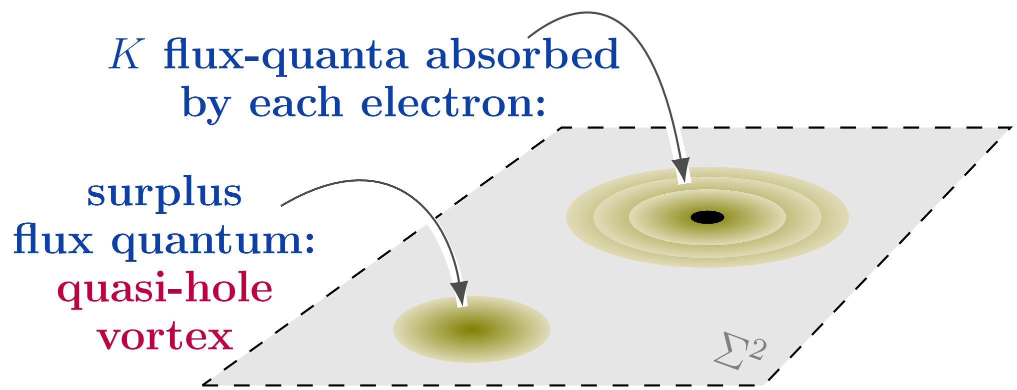




Fractional Quantum Hall systems.

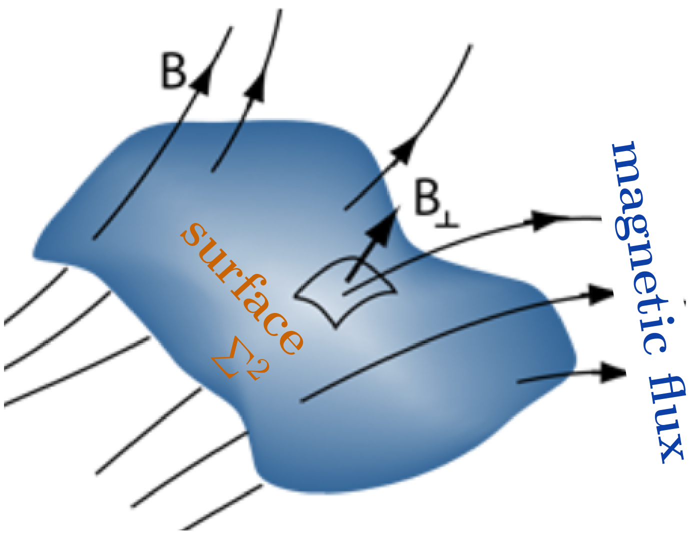
electron gas in 2D semiconductor Σ^2
subject to transverse magnetic field
at *rational filling fraction* of
electrons per magnetic flux quanta

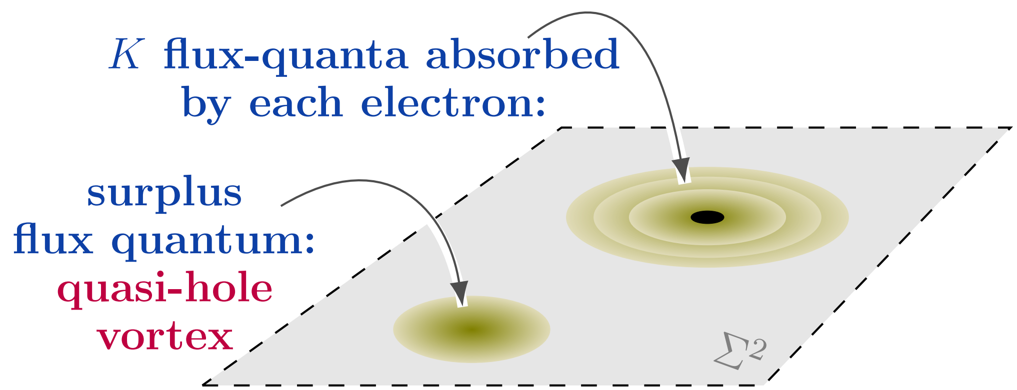




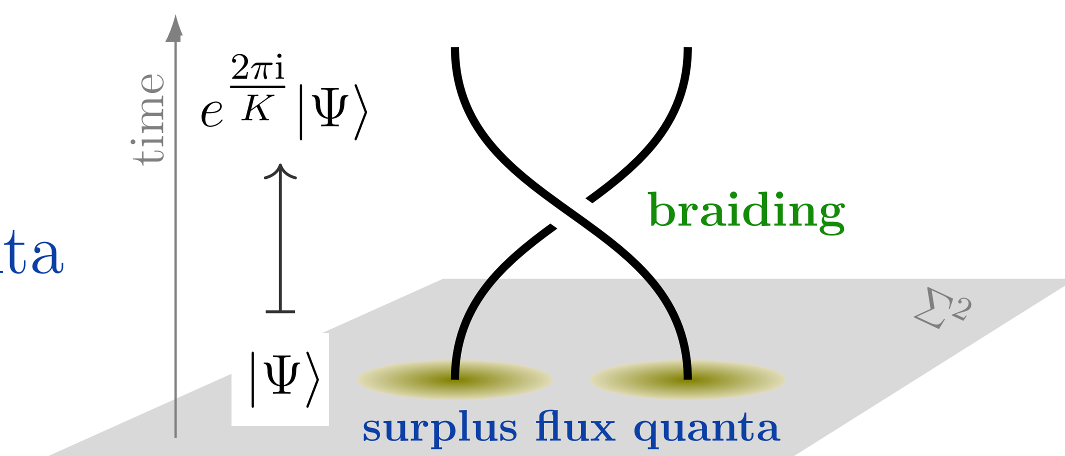
Fractional Quantum Hall systems.

electron gas in 2D semiconductor Σ^2
subject to transverse magnetic field
at *rational filling fraction* of
electrons per magnetic flux quanta



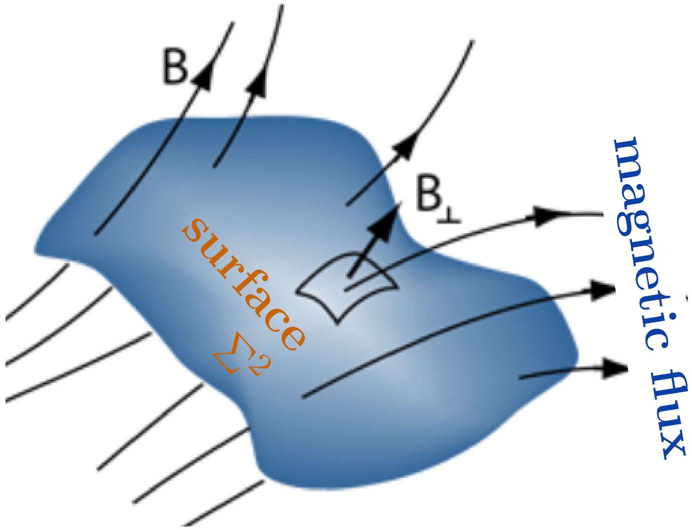


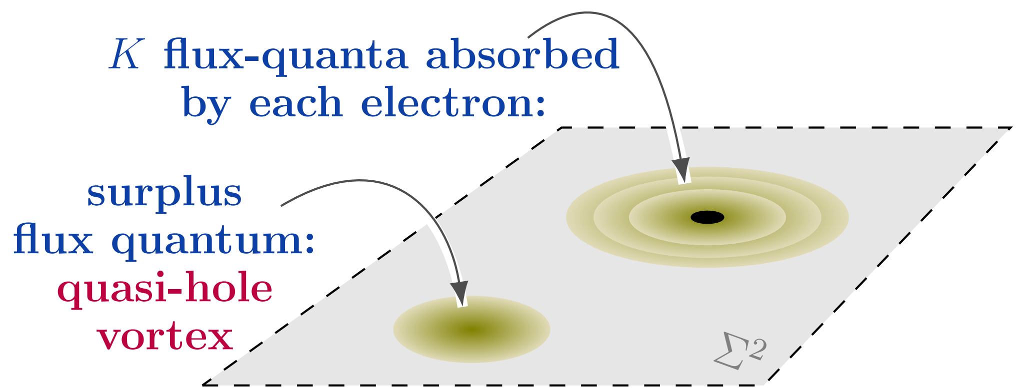
remarkably:
surplus magnetic flux quanta
are solitonic anyons!



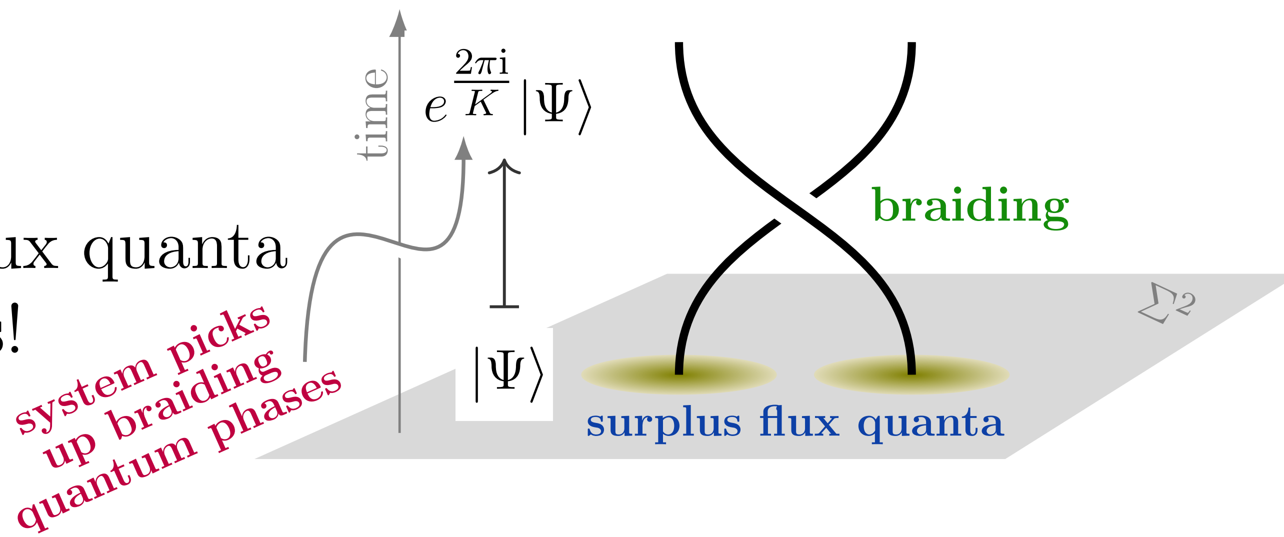
Fractional Quantum Hall systems.

electron gas in 2D semiconductor Σ^2
subject to transverse magnetic field
at *rational filling fraction* of
electrons per magnetic flux quanta





remarkably:
surplus magnetic flux quanta
are solitonic anyons!



Fractional Quantum Hall systems.

Fractional Quantum Hall systems.

in recent years, these FQH anyons
are experimentally observed

Fractional Quantum Hall systems.

in recent years, these FQH anyons
are experimentally observed

[Nakamura et al. 2020]
[Nakamura et al. 2023]
[Ruelle et al. 2023]
[Glidic et al. 2023]
[Kundu et al. 2023]
[Veillon et al. 2024]
[Ghosh et al. 2025]

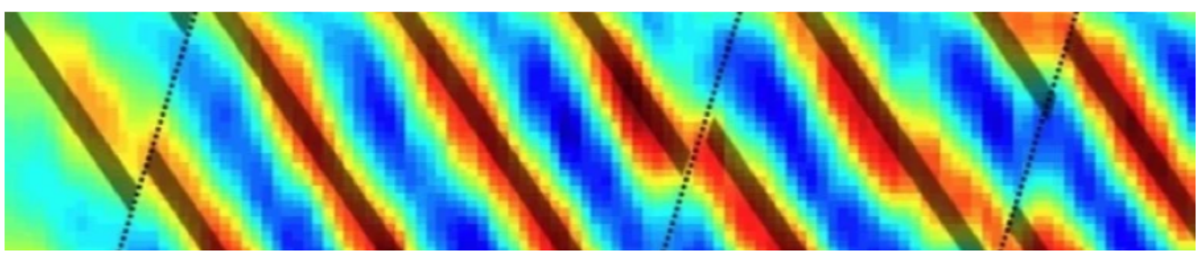
nature

NEWS | 03 July 2020

Welcome anyons! Physicists find best evidence yet for long-sought 2D structures

The 'quasiparticles' defy the categories of ordinary particles and herald a potential way to build quantum computers.

By Davide Castelvecchi



Fractional Quantum Hall systems.

Fractional Quantum Anomalous Hall systems.

Fractional Quantum Anomalous Hall systems.

Very recently also observed:
2D crystals, exhibiting “dual” FQH

Fractional Quantum Anomalous Hall systems.

Very recently also observed:

2D crystals, exhibiting "dual" FQH, where

Berry curvature over momentum space $\widehat{T}^2$

plays the role of flux in position space

Fractional Quantum Anomalous Hall systems.

Very recently also observed:
2D crystals, exhibiting “dual” FQH, where
Berry curvature over momentum space $\widehat{T}^2$
plays the role of flux in position space

Hall effect

Semiclassical electron equation of motion

$$\dot{\vec{k}} = -\partial\epsilon/\partial\vec{x} + \underbrace{\vec{x} \times e\vec{B}}_{\text{Lorentz force}}$$

$$\dot{\vec{x}} = \partial\epsilon/\partial\vec{k} - \underbrace{\vec{k} \times \vec{\Omega}}_{\text{anomalous velocity}}$$

position	**magnetic flux density**
$\vec{x}$	$e\vec{B}$
$\updownarrow$	$\updownarrow$
$\vec{k}$	$\vec{\Omega}$
momentum	**Berry curvature**

Anomalous Hall effect

Fractional Quantum Anomalous Hall systems.

Very recently also observed:
2D crystals, exhibiting "dual" FQH, where
Berry curvature over momentum space $\widehat{T}^2$
plays the role of flux in position space

Hall effect

Lorentz force

Semiclassical electron equation of motion

$$\dot{\vec{k}} = -\partial\epsilon/\partial\vec{x} + \vec{x} \times e\vec{B}$$

$$\dot{\vec{x}} = \partial\epsilon/\partial\vec{k} - \vec{k} \times \vec{\Omega}$$

anomalous velocity

position: $\vec{x}$ $\leftrightarrow$ momentum: $\vec{k}$

magnetic flux density: $e\vec{B}$ $\leftrightarrow$ Berry curvature: $\vec{\Omega}$

Anomalous Hall effect

nature

Article | Published: 17 August 2023

Observation of fractionally quantized anomalous Hall effect

Heonjoon Park, Jiaqi Cai, Eric Anderson, Yinong Zhang, Jiayi Zhu, Xiaoyu Liu, Chong Wang, William Holtzmann, Chaowei Hu, Zhaoyu Liu, Takashi Taniguchi, Kenji Watanabe, Jiun-Haw Chu, Ting Cao, Liang Fu, Wang Yao, Cui-Zu Chang, David Cobden, Di Xiao & Xiaodong Xu

Nature **622**, 74–79 (2023) | Cite this article

Fractional Quantum Anomalous Hall systems.

Very recently also observed:
2D crystals, exhibiting "dual" FQH, where
Berry curvature over momentum space $\widehat{T}^2$
plays the role of flux in position space

Hall effect

Lorentz force

$$\dot{\vec{k}} = -\partial\epsilon/\partial\vec{x} + \vec{x} \times e\vec{B}$$

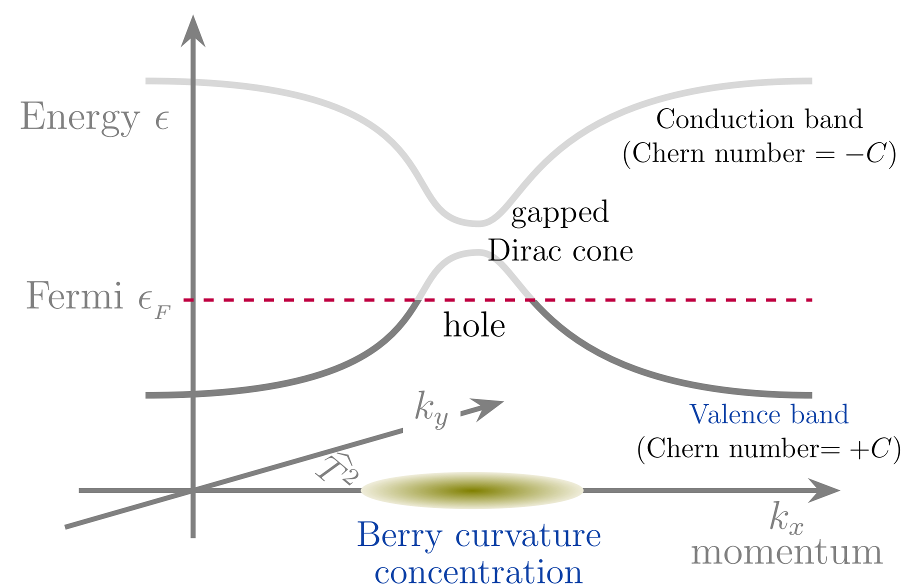
Semiclassical electron equation of motion

$$\dot{\vec{x}} = \partial\epsilon/\partial\vec{k} - \vec{k} \times \vec{\Omega}$$

anomalous velocity

position	magnetic flux density
$\vec{x}$	$e\vec{B}$
$\updownarrow$	$\updownarrow$
$\vec{k}$	$\vec{\Omega}$
momentum	Berry curvature

Anomalous Hall effect



Fractional Quantum Anomalous Hall systems.

Very recently also observed:
2D crystals, exhibiting "dual" FQH, where
Berry curvature over momentum space $\widehat{T}^2$
plays the role of flux in position space

Hall effect

Lorentz force | **position** | **magnetic flux density**

Semiclassical electron equation of motion

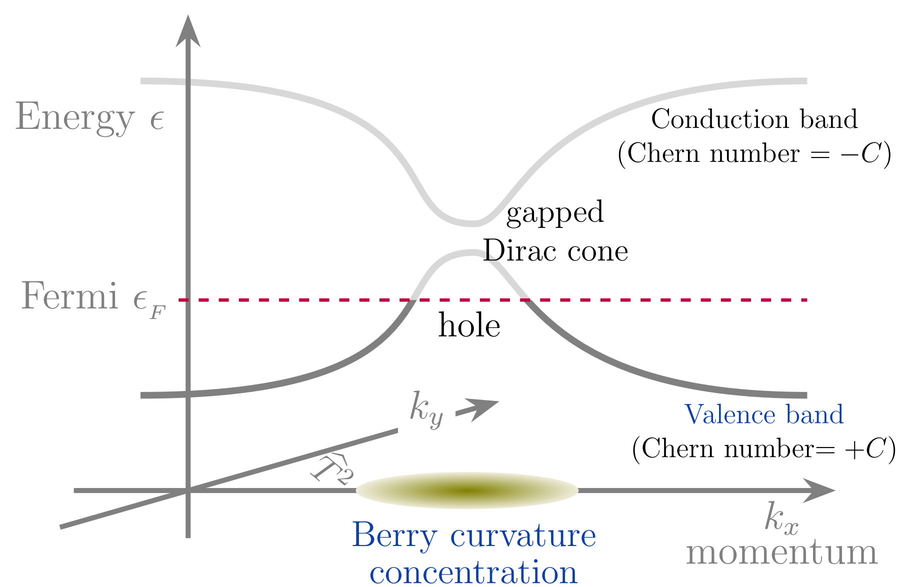
$$\dot{\vec{k}} = -\partial\epsilon/\partial\vec{x} + \vec{x} \times e\vec{B} \qquad \vec{x} \qquad e\vec{B}$$

$$\dot{\vec{x}} = \partial\epsilon/\partial\vec{k} - \vec{k} \times \vec{\Omega} \qquad \vec{k} \qquad \vec{\Omega}$$

anomalous velocity | **momentum** | **Berry curvature**

Anomalous Hall effect

suggests FQAH systems as viable topological quantum hardware!



Fractional Quantum Anomalous Hall systems.

Very recently also observed:
2D crystals, exhibiting "dual" FQH, where
Berry curvature over momentum space $\widehat{T}^2$
plays the role of flux in position space

Hall effect

Lorentz force

Semiclassical electron equation of motion

$$\begin{aligned} \dot{\vec{k}} &= -\partial\epsilon/\partial\vec{x} + \vec{x} \times e\vec{B} \\ \dot{\vec{x}} &= \partial\epsilon/\partial\vec{k} - \vec{k} \times \vec{\Omega} \end{aligned}$$

anomalous velocity

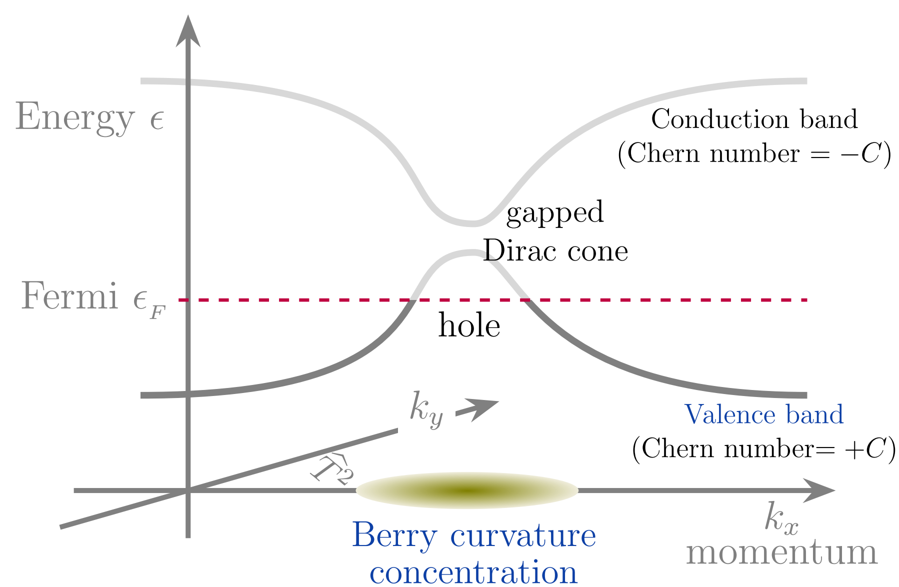
position	**magnetic flux density**
$\vec{x}$	$e\vec{B}$
$\updownarrow$	$\updownarrow$
$\vec{k}$	$\vec{\Omega}$
momentum	**Berry curvature**

Anomalous Hall effect

suggests FQAH systems as viable topological quantum hardware!

but **open question**:

Where & how are FQAH anyons?



Fractional Quantum Anomalous Hall systems.

Very recently also observed:
2D crystals, exhibiting "dual" FQH, where
Berry curvature over momentum space $\widehat{T}^2$
plays the role of flux in position space

Semiclassical electron equation of motion

$$
\begin{aligned}
\dot{\vec{k}} &= -\partial\epsilon/\partial\vec{x} + \underbrace{\vec{x} \times e\vec{B}}_{\text{Lorentz force}} \\
\dot{\vec{x}} &= \partial\epsilon/\partial\vec{k} - \underbrace{\vec{k} \times \vec{\Omega}}_{\text{anomalous velocity}}
\end{aligned}
$$

Hall effect

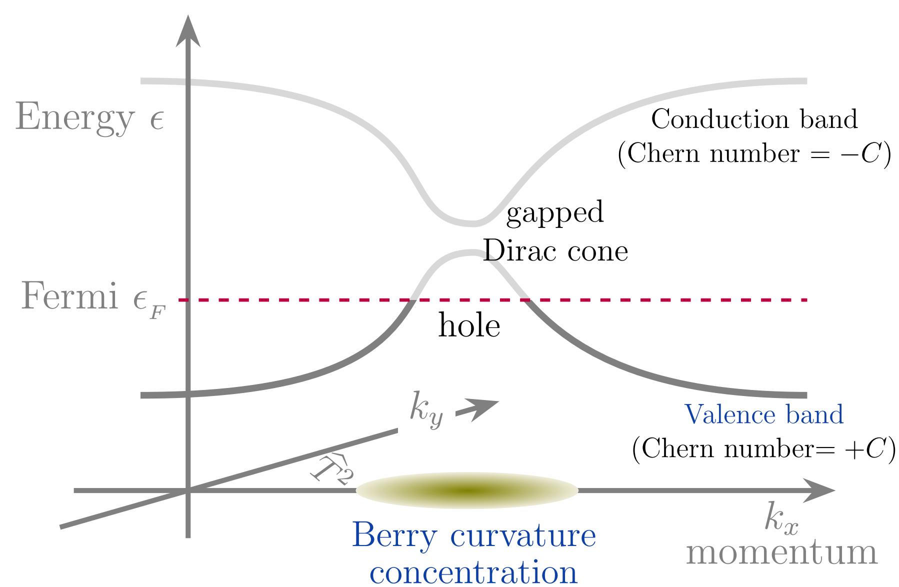
position	magnetic flux density
$\vec{x}$	$e\vec{B}$
$\updownarrow$	$\updownarrow$
$\vec{k}$	$\vec{\Omega}$
momentum	Berry curvature

Anomalous Hall effect

suggests FQAH systems as viable topological quantum hardware!

but **open question**:

Where & how are FQAH anyons?



Answer – in general:

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

topological phase: $\ldots$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big)$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

mapping space

topological phase: $\quad C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big)$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

mapping space

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big)$

the Bloch Hamiltonians are a map
from the Brillouin torus of crystal momenta
to the given space of Hamiltonians

$$\begin{array}{ccc} \widehat{T}^2 & \longrightarrow & \mathcal{A} \\ k & \longmapsto & H_{\mathrm{Blch}}(k) \end{array}$$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

mapping space

Then:

topological phase: $\quad C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big)$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

mapping space

equivariant maps

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big)$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

mapping space

equivariant maps

topological phase: $\quad C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big)$

equivariance is the mathematical term
for what in condensed matter is:

$$\boxed{H_{\mathrm{Blch}}\big(g \cdot k\big) \;=\; U_g \circ H_{\mathrm{Blch}}(k) \circ U_g^{-1}}$$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

mapping space

equivariant maps

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big)$

equivariance is the mathematical term
for what in condensed matter is:

$$\boxed{H_{\mathrm{Blch}}\big(g \cdot k\big) \;=\; U_g \circ H_{\mathrm{Blch}}(k) \circ U_g^{-1}}$$

crystal symmetry

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

topological phase: $\quad C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big)$

mapping space

equivariant maps

equivariance is the mathematical term
for what in condensed matter is:

$$\boxed{H_{\mathrm{Blch}}\big(g\cdot k\big) \;=\; U_g \circ H_{\mathrm{Blch}}(k) \circ U_g^{-1}}$$

crystal symmetry

band symmetry

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

topological phase: $\color{blue}{C} \in \pi_{\color{purple}{0}}\Big(\underbrace{\mathrm{Map}}_{\text{mapping space}}\big(\widehat{T}^2, \mathcal{A}\big)^{\overbrace{G}^{\text{equivariant maps}}} \Big)$

equivariance is the mathematical term
for what in condensed matter is:

$$\boxed{H_{\mathrm{Blch}}\big(\underbrace{g}_{\text{crystal symmetry}} \cdot k\big) \;=\; \underbrace{U_g}_{\text{band symmetry}} \circ H_{\mathrm{Blch}}(k) \circ U_g^{-1}}$$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

mapping space

equivariant maps

Then:

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big)$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

topological phase: $\quad C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big)$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big)$

connected components give the topological deformation classes of Bloch Hamiltonians

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big)$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

topological phase: $C \in \underbrace{\pi_0}_{\text{connected components}}\Big(\underbrace{\mathrm{Map}}_{\text{mapping space}}\big(\widehat{T}^2, \mathcal{A}\big)^{\overset{\text{equivariant maps}}{G}}\Big)$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

topological phase: $\quad {\color{blue}C} \in \pi_{\color{purple}0}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \;{\color{blue}\equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)}$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $\quad C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $\quad C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

Examples:

classifying space	$\mathcal{A}$	phases

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

Then:

topological phase: $\quad C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

Examples:

classifying space	$\mathcal{A}$	phases
$\mathrm{Gr}_v\big(\mathbb{C}^{v+c}\big) \;=$	$\frac{\mathrm{U}(c+v)}{\mathrm{U}(c)\times\mathrm{U}(v)}$	

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $\quad C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

Examples:

classifying space	$\mathcal{A}$	phases
$\mathrm{Gr}_v\big(\mathbb{C}^{v+c}\big) \; =$	$\frac{\mathrm{U}(c+v)}{\mathrm{U}(c)\times\mathrm{U}(v)}$	*fragile*

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

Examples:

classifying space		$\mathcal{A}$	phases
$\mathrm{Gr}_v\big(\mathbb{C}^{v+c}\big)$	$=$	$\frac{\mathrm{U}(c+v)}{\mathrm{U}(c)\times\mathrm{U}(v)}$	*fragile*
$B\mathrm{U}(1)$	$=$	$\bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(c+1)}{\mathrm{U}(c)\times\mathrm{U}(1)}$	Chern

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $\quad C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

Examples:

classifying space	$\mathcal{A}$	phases
$\mathrm{Gr}_v\big(\mathbb{C}^{v+c}\big) \;=$	$\frac{\mathrm{U}(c+v)}{\mathrm{U}(c)\times\mathrm{U}(v)}$	*fragile*
$B\mathrm{U}(1) \;=$	$\bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(c+1)}{\mathrm{U}(c)\times\mathrm{U}(1)}$	Chern

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $\quad C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

Examples:

classifying space		$\mathcal{A}$	phases
$\mathrm{Gr}_v\big(\mathbb{C}^{v+c}\big)$	$=$	$\frac{\mathrm{U}(c+v)}{\mathrm{U}(c)\times\mathrm{U}(v)}$	*fragile*
$B\mathrm{U}(1)$	$=$	$\bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(c+1)}{\mathrm{U}(c)\times\mathrm{U}(1)}$	Chern
$B\mathrm{U}$	$=$	$\bigcup\limits_{c,v\in\mathbb{N}} \frac{\mathrm{U}(c+v)}{\mathrm{U}(c)\times\mathrm{U}(v)}$	

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $\quad {\color{blue}C} \in \pi_{\color{purple}0}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

Examples:

classifying space		$\mathcal{A}$	phases
$\mathrm{Gr}_v\big(\mathbb{C}^{v+c}\big)$	$=$	$\frac{\mathrm{U}(c+v)}{\mathrm{U}(c)\times\mathrm{U}(v)}$	*fragile*
$B\mathrm{U}(1)$	$=$	$\bigcup_{c\in\mathbb{N}} \frac{\mathrm{U}(c+1)}{\mathrm{U}(c)\times\mathrm{U}(1)}$	Chern
$B\mathrm{U}$	$=$	$\bigcup_{c,v\in\mathbb{N}} \frac{\mathrm{U}(c+v)}{\mathrm{U}(c)\times\mathrm{U}(v)}$	K-theoretic

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

Examples:

classifying space		$\mathcal{A}$	phases
$\mathrm{Gr}_v\big(\mathbb{C}^{v+c}\big)$	$=$	$\frac{\mathrm{U}(c+v)}{\mathrm{U}(c)\times\mathrm{U}(v)}$	*fragile*
$B\mathrm{U}(1)$	$=$	$\bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(c+1)}{\mathrm{U}(c)\times\mathrm{U}(1)}$	Chern
$B\mathrm{U}$	$=$	$\bigcup\limits_{c,v\in\mathbb{N}} \frac{\mathrm{U}(c+v)}{\mathrm{U}(c)\times\mathrm{U}(v)}$	K-theoretic

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $\color{blue}{C} \in \pi_{\color{purple}{0}}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order:

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components mapping space equivariant maps equivariant extraordinary nonabelian cohomology

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\mathcal{H} \in \pi_1\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_C\Big)$-IrrRep

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $\color{blue}{C} \in \pi_{\color{purple}{0}}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\mathcal{H} \in \pi_{\color{purple}{1}}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_{\color{blue}{C}}\Big)$-IrrRep

fundamental group

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $\quad {\color{blue}C} \in \pi_{\color{purple}0}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\quad \mathcal{H} \in \pi_{\color{purple}1}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_{\color{blue}C}\Big)$-IrrRep

fundamental group

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $\quad C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\quad \mathcal{H} \in \pi_1\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_C\Big)$-IrrRep

fundamental group

unitary irreps

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

Then:

topological phase: $\quad {\color{blue}C} \in \pi_{\color{purple}0}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\quad \mathcal{H} \in \pi_{\color{purple}1}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_{\color{blue}C}\Big)\text{-IrrRep}$

fundamental group

unitary irreps

by topological adiabatic theorem:

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components — mapping space — equivariant maps — equivariant extraordinary nonabelian cohomology

topological phase: $\color{blue}{C} \in \pi_{\color{purple}{0}}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\mathcal{H} \in \pi_{\color{purple}{1}}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_{\color{blue}{C}}\Big)\text{-IrrRep}$

fundamental group — unitary irreps

by topological adiabatic theorem:

gapped ground state Hilbert spaces

form local system/flat bundle

over topological moduli space

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $\quad {\color{blue}C} \in \pi_{\color{purple}0}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\quad \mathcal{H} \in \pi_{\color{purple}1}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_{\color{blue}C}\Big)$-IrrRep

fundamental group

unitary irreps

by topological adiabatic theorem:

gapped ground state Hilbert spaces

form local system/flat bundle $\quad {\color{blue}\mathcal{H}_{p_1}}$

over topological moduli space $\quad {\color{blue}p_1}$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

Then:

topological phase: $\quad {\color{blue}C} \in \pi_{\color{purple}0}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\quad \mathcal{H} \in \pi_{\color{purple}1}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_{\color{blue}C}\Big)$-IrrRep

fundamental group

unitary irreps

by topological adiabatic theorem:

gapped ground state Hilbert spaces

form local system/flat bundle $\quad \mathcal{H}_{p_1}$

over topological moduli space $\quad p_1 \xrightarrow{\gamma_{12}} {\color{blue}p_2}$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components · mapping space · equivariant maps · equivariant extraordinary nonabelian cohomology

topological phase: $\quad C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\quad \mathcal{H} \in \pi_1\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_C\Big)$-IrrRep

fundamental group · unitary irreps

by topological adiabatic theorem:

gapped ground state Hilbert spaces

form local system/flat bundle

over topological moduli space

$$\mathcal{H}_{p_1} \xrightarrow{U_{\gamma_{12}}} \mathcal{H}_{p_2}$$

$$p_1 \overset{\gamma_{12}}{\rightsquigarrow} p_2$$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components | mapping space | equivariant maps | equivariant extraordinary nonabelian cohomology

topological phase: $\quad {\color{blue}C} \in \pi_{\color{purple}0}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\quad \mathcal{H} \in \pi_{\color{purple}1}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_{\color{blue}C}\Big)$-IrrRep

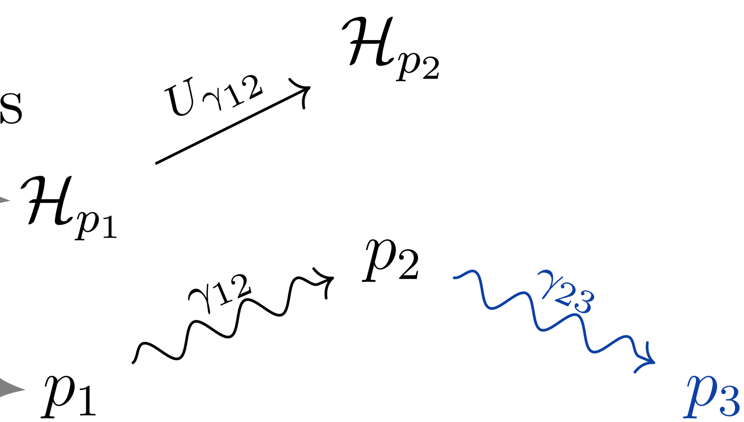
fundamental group | unitary irreps

by topological adiabatic theorem:

gapped ground state Hilbert spaces

form local system/flat bundle

over topological moduli space



Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\mathcal{H} \in \pi_1\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_C\Big)$-IrrRep

fundamental group

unitary irreps

by topological adiabatic theorem:

gapped ground state Hilbert spaces

form local system/flat bundle

over topological moduli space

$$\mathcal{H}_{p_1} \xrightarrow{U_{\gamma_{12}}} \mathcal{H}_{p_2} \xrightarrow{U_{\gamma_{23}}} \mathcal{H}_{p_3}$$

$$p_1 \overset{\gamma_{12}}{\rightsquigarrow} p_2 \overset{\gamma_{23}}{\rightsquigarrow} p_3$$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

Then:

topological phase: $\quad {\color{blue}C} \in \pi_{\color{purple}0}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\quad \mathcal{H} \in \pi_{\color{purple}1}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_{\color{blue}C}\Big)\text{-IrrRep}$

fundamental group

unitary irreps

by topological adiabatic theorem:

gapped ground state Hilbert spaces

form local system/flat bundle

over topological moduli space

$$
\begin{tikzcd}
 & \mathcal{H}_{p_2} \arrow[dr, "U_{\gamma_{23}}"] & \\
\mathcal{H}_{p_1} \arrow[ur, "U_{\gamma_{12}}"] & & \mathcal{H}_{p_3}
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & p_2 \arrow[dr, squiggly, "\gamma_{23}"] & \\
p_1 \arrow[ur, squiggly, "\gamma_{12}"] \arrow[rr, squiggly, blue, "\gamma_{13}"] & & p_3
\end{tikzcd}
$$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $\quad {\color{blue}C} \in \pi_{\color{purple}0}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\quad \mathcal{H} \in \pi_{\color{purple}1}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_{\color{blue}C}\Big)$-IrrRep

fundamental group

unitary irreps

by topological adiabatic theorem:

gapped ground state Hilbert spaces

form local system/flat bundle

over topological moduli space

$$\begin{tikzcd} & \mathcal{H}_{p_2} \arrow[dr, "U_{\gamma_{23}}"] & \\ \mathcal{H}_{p_1} \arrow[ur, "U_{\gamma_{12}}"] \arrow[rr, "{\color{blue}U_{\gamma_{13}}}"] & & \mathcal{H}_{p_3} \end{tikzcd}$$

$$\begin{tikzcd} & p_2 \arrow[dr, rightsquigarrow, "\gamma_{23}"] & \\ p_1 \arrow[ur, rightsquigarrow, "\gamma_{12}"] \arrow[rr, rightsquigarrow, "{\color{blue}\gamma_{13}}"] & & p_3 \end{tikzcd}$$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components | mapping space | equivariant maps | equivariant extraordinary nonabelian cohomology

topological phase: $\quad {\color{blue}C} \in \pi_{\color{purple}0}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\quad \mathcal{H} \in \pi_{\color{purple}1}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_{\color{blue}C}\Big)\text{-IrrRep}$

fundamental group | unitary irreps

by topological adiabatic theorem:

gapped ground state Hilbert spaces

form local system/flat bundle

over topological moduli space

${\color{blue}\Leftrightarrow \text{ rep of fundamental group}}$

$$\begin{tikzcd} & \mathcal{H}_{p_2} \arrow[dr, "U_{\gamma_{23}}"] & \\ \mathcal{H}_{p_1} \arrow[ur, "U_{\gamma_{12}}"] \arrow[rr, "U_{\gamma_{13}}"] & & \mathcal{H}_{p_3} \end{tikzcd}$$

$$\begin{tikzcd} & p_2 \arrow[dr, squiggly, "\gamma_{23}"] & \\ p_1 \arrow[ur, squiggly, "\gamma_{12}"] \arrow[rr, squiggly, "\gamma_{13}"] & & p_3 \end{tikzcd}$$

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\mathcal{H} \in \pi_1\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_C\Big)$-IrrRep

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\mathcal{H} \in \pi_1\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_C\Big)$-IrrRep

more precisely, the ground state TQFT is *generally covariant*

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\mathcal{H} \in \pi_1\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_C\Big)$-IrrRep

more precisely, the ground state TQFT is *generally covariant*

$\Rightarrow$ band topologies differing by diffeos are equivalent to the TQFT:

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\mathcal{H} \in \pi_1\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_C\Big)$-IrrRep

more precisely, the ground state TQFT is *generally covariant*

$\Rightarrow$ band topologies differing by diffeos are equivalent to the TQFT:

$$\text{topological phase: } C = \pi_0\Big(\underbrace{\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G}_{\text{band topology}} /\!\!/ \underbrace{\mathrm{Diff}^+\big(\widehat{T}^2\big)^G}_{\text{diffeos}} \Big)$$

as seen by

$$\text{topological order: } \mathcal{H} \in \pi_1\Big(\overbrace{\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G} /\!\!/ \overbrace{\mathrm{Diff}^+\big(\widehat{T}^2\big)^G}, C \Big)\text{-IrrRep}$$

homotopy quotient

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

topological order: $\mathcal{H} \in \pi_1\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G_C\Big)$-IrrRep

more precisely, the ground state TQFT is *generally covariant*

$\Rightarrow$ band topologies differing by diffeos are equivalent to the TQFT:

topological phase: $C = \pi_0\Big(\underbrace{\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G}_{\text{band topology}} /\!\!/ \underbrace{\mathrm{Diff}^+\big(\widehat{T}^2\big)^G}_{\text{diffeos}}\Big)$

as seen by

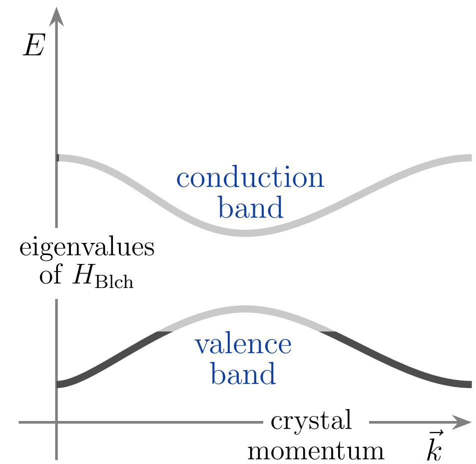
topological order: $\mathcal{H} \in \pi_1\Big(\overbrace{\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G}^{\text{band topology}} /\!\!/ \overbrace{\mathrm{Diff}^+\big(\widehat{T}^2\big)^G}^{\text{diffeos}}, C\Big)$-IrrRep

homotopy quotient

Answer – for FQAH: 1.) Bloch Hamiltonian space

Answer – for FQAH: 1.) Bloch Hamiltonian space

2-band systems

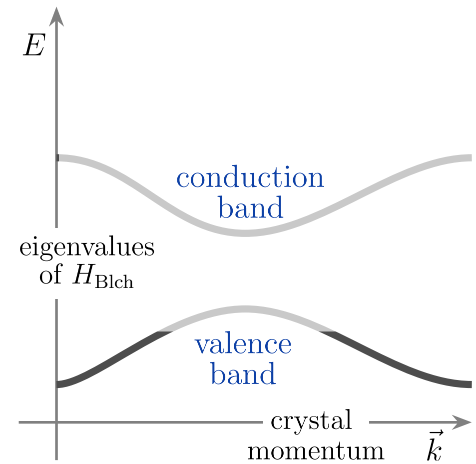


Answer – for FQAH: 1.) Bloch Hamiltonian space

2-band systems

$\Rightarrow$ Bloch Hamiltonian

$$\begin{array}{ccc} \widehat{T}^2 & \xrightarrow{H_{\mathrm{Blch}}} & \mathrm{Mat}_{2\times 2}(\mathbb{C}) \\ \vec{k} & \longmapsto & H_{\mathrm{Blch}}(\vec{k}) \end{array}$$



Answer – for FQAH: 1.) Bloch Hamiltonian space

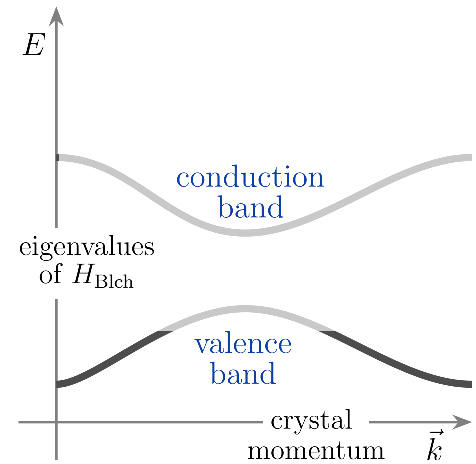
2-band systems

$\Rightarrow$ Bloch Hamiltonian expanded in Pauli σ

$$\widehat{T}^2 \xrightarrow{H_{\mathrm{Blch}}} \mathrm{Mat}_{2\times 2}(\mathbb{C})$$

$$\vec{k} \longmapsto H_{\mathrm{Blch}}(\vec{k})$$

$$\equiv h_0(\vec{k}) + \overbrace{\sum_{i=1}^{3} h_i(\vec{k})\,\sigma_i}^{\substack{\text{relative}\\ \text{Bloch Hamiltonian}\\ H(\vec{k})}}$$



Answer – for FQAH: 1.) Bloch Hamiltonian space

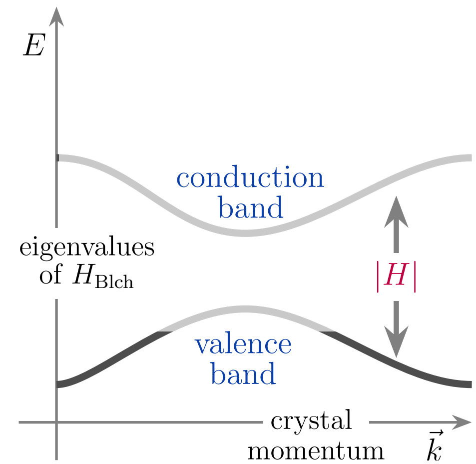
2-band systems

$\Rightarrow$ Bloch Hamiltonian expanded in Pauli σ

$$\widehat{T}^2 \xrightarrow{H_{\mathrm{Blch}}} \mathrm{Mat}_{2\times 2}(\mathbb{C})$$

$$\vec{k} \longmapsto H_{\mathrm{Blch}}(\vec{k})$$

$$\equiv h_0(\vec{k}) + \overbrace{\textstyle\sum_{i=1}^{3} h_i(\vec{k})\,\sigma_i}^{\substack{\text{relative} \\ \text{Bloch Hamiltonian} \\ H(\vec{k})}}$$



which are gapped

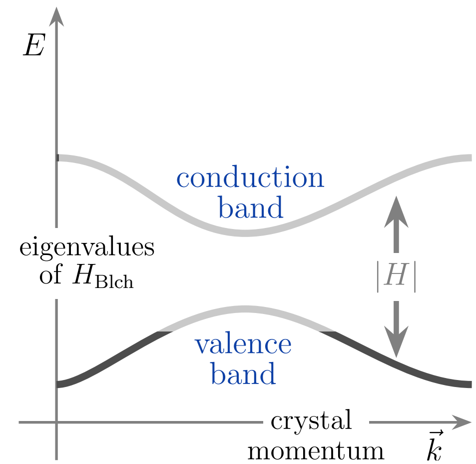
$$\Rightarrow \quad \left|H(\vec{k})\right| := \sqrt{\textstyle\sum_{i=1}^{3} \big(h_i(\vec{k})\big)^2} \; > \; 0\,,$$

Answer – for FQAH: 1.) Bloch Hamiltonian space

2-band systems

$\Rightarrow$ Bloch Hamiltonian expanded in Pauli σ

$$
\begin{aligned}
\widehat{T}^2 \;&\xrightarrow{H_{\text{Blch}}}\; \mathrm{Mat}_{2\times 2}(\mathbb{C}) \\
\vec{k} \;&\longmapsto\; H_{\text{Blch}}(\vec{k}) \\
&\equiv h_0(\vec{k}) + \overbrace{\textstyle\sum_{i=1}^{3} h_i(\vec{k})\,\sigma_i}^{\substack{\text{relative} \\ \text{Bloch Hamiltonian} \\ H(\vec{k})}}
\end{aligned}
$$



which are gapped

$\Rightarrow$ $$\big|H(\vec{k})\big| := \sqrt{\textstyle\sum_{i=1}^{3}\big(h_i(\vec{k})\big)^2} \;>\; 0\,,$$

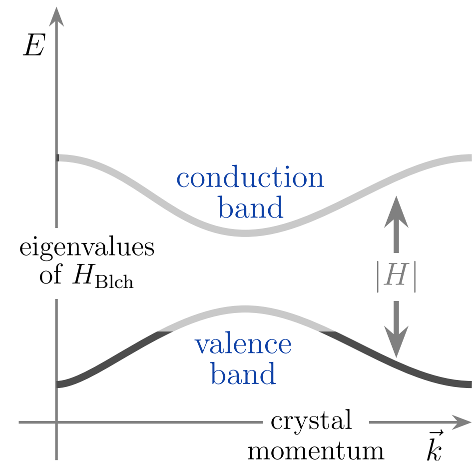
$\Rightarrow$ valence bundle is $\mathrm{Eig}_{-1}\big(H/|H|\big)$

Answer – for FQAH: 1.) Bloch Hamiltonian space

2-band systems

$\Rightarrow$ Bloch Hamiltonian expanded in Pauli σ

$$
\begin{array}{ccl}
\widehat{T}^2 & \xrightarrow{H_{\mathrm{Blch}}} & \mathrm{Mat}_{2\times 2}(\mathbb{C}) \\
\vec{k} & \longmapsto & H_{\mathrm{Blch}}(\vec{k}) \\
 & & \equiv h_0(\vec{k}) + \overbrace{\textstyle\sum_{i=1}^{3} h_i(\vec{k})\,\sigma_i}^{\substack{\text{relative}\\ \text{Bloch Hamiltonian}\\ H(\vec{k})}}
\end{array}
$$



which are gapped

$\Rightarrow$ $$\big|H(\vec{k}\,)\big| := \sqrt{\textstyle\sum_{i=1}^{3}\big(h_i(\vec{k}\,)\big)^2} \;>\; 0\,,$$

$\Rightarrow$ valence bundle is $\mathrm{Eig}_{-1}\big(H/|H|\big)$

$\Rightarrow$ Hamiltonian space is $\dfrac{\mathrm{U}(2)}{\mathrm{U}(1)\times\mathrm{U}(1)} \simeq S^2 \equiv \mathcal{A}$

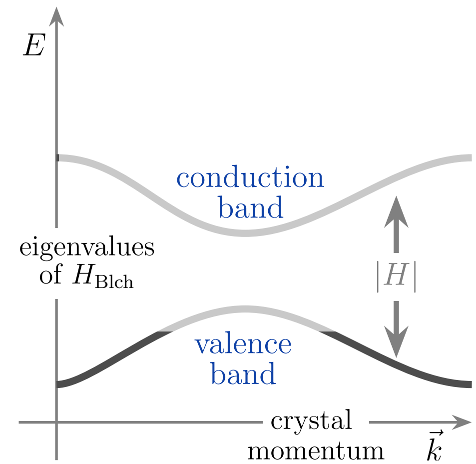
$$
\begin{array}{ccl}
\widehat{T}^2 & \xrightarrow{\quad H/|H| \quad} & S^2 \subset \mathbb{R}^3 \\
\vec{k} & \longmapsto & \vec{h}(\vec{k})/|H|\,.
\end{array}
$$

Answer – for FQAH: 1.) Bloch Hamiltonian space

2-band systems

$\Rightarrow$ Bloch Hamiltonian expanded in Pauli σ

$$\begin{aligned}
\widehat{T}^2 &\xrightarrow{H_{\mathrm{Blch}}} \mathrm{Mat}_{2\times 2}(\mathbb{C}) \\
\vec{k} &\longmapsto H_{\mathrm{Blch}}(\vec{k}) \\
&\equiv h_0(\vec{k}) + \overbrace{\textstyle\sum_{i=1}^{3} h_i(\vec{k})\,\sigma_i}^{\substack{\text{relative} \\ \text{Bloch Hamiltonian} \\ H(\vec{k})}}
\end{aligned}$$



which are gapped

$$\Rightarrow \quad \big|H(\vec{k})\big| := \sqrt{\textstyle\sum_{i=1}^{3}\big(h_i(\vec{k})\big)^2} \;>\; 0\,,$$

$\Rightarrow$ valence bundle is $\mathrm{Eig}_{-1}\big(H/|H|\big)$

$\Rightarrow$ Hamiltonian space is $\frac{\mathrm{U}(2)}{\mathrm{U}(1)\times\mathrm{U}(1)} \simeq S^2 \equiv \mathcal{A}$

$$\begin{aligned}
\widehat{T}^2 &\xrightarrow{\quad H/|H| \quad} S^2 \subset \mathbb{R}^3 \\
\vec{k} &\longmapsto \vec{h}(\vec{k})/|H|\,.
\end{aligned}$$

Answer – for FQAH: 2.) topological phases

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$\begin{array}{c}\text{valence}\\\text{bundle}\\[4pt]\mathcal{V}\\\downarrow\\\widehat{T}^2\end{array}$$

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

valence bundle

tautological complex line bundle

$$
\begin{tikzcd}
\mathcal{V} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \mathcal{L} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = & \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}
\end{tikzcd}
$$

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\substack{\text{valence}\\\text{bundle}} & & \substack{\text{tautological complex}\\\text{line bundle}} & & \substack{\text{universal complex}\\\text{line bundle}} \\
\mathcal{V} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \mathcal{L} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 \arrow[r, equal] & \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[r, "\sim"] & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

valence bundle

tautological complex line bundle

universal complex line bundle

$$
\begin{tikzcd}
\mathcal{V} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \mathcal{L} \arrow[rrr] \arrow[d] \arrow[drrr, phantom, "\text{(pb)}"] & & & EU(1) \underset{U(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = & \frac{U(1)}{U(1)\times U(1)} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{U(1+c)}{U(1)\times U(c)} \arrow[rr, "\sim"] & & BU(1)\,,
\end{tikzcd}
$$

fragile band topology

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

valence bundle — tautological complex line bundle — universal complex line bundle

$$
\begin{tikzcd}
\mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{L} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[r, "\sim"] & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

fragile band topology — stable band topology

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\text{valence bundle} & & \text{tautological complex line bundle} & & \text{universal complex line bundle} \\
\mathcal{V} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \mathcal{L} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = & \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[r, "\sim"] & B\mathrm{U}(1)\,, \\
& & \text{fragile band topology} & \text{stable band topology} &
\end{tikzcd}
$$

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\overset{\text{valence bundle}}{\mathcal{V}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\text{tautological complex line bundle}}{\mathcal{L}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \overset{\text{universal complex line bundle}}{E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C}} \arrow[d] \\
\widehat{T}^2 \xrightarrow{H/|H|} S^2 = \underset{\substack{\text{fragile}\\ \text{band topology}}}{\dfrac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \dfrac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[rr, "\sim"] & & B\mathrm{U}(1)\,, \\
& & \text{stable band topology} &
\end{tikzcd}
$$

$\Rightarrow$ topological phases characterized by an integer

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\underset{\text{bundle}}{\overset{\text{valence}}{\mathcal{V}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \underset{\text{line bundle}}{\overset{\text{tautological complex}}{\mathcal{L}}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \overset{\substack{\text{universal complex} \\ \text{line bundle}}}{E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C}} \arrow[d] \\
\widehat{T}^2 \xrightarrow{H/|H|} S^2 = \underset{\substack{\text{fragile} \\ \text{band topology}}}{\dfrac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}} \arrow[r, hook] & \underset{c\in\mathbb{N}}{\bigcup} \dfrac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[rr, "\sim", "\substack{\text{stable} \\ \text{band topology}}"'] & & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

$\Rightarrow$ topological phases characterized by an integer

$C := \big[H/|H|\big]$

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\underset{\text{bundle}}{\overset{\text{valence}}{\mathcal{V}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \underset{\text{line bundle}}{\overset{\text{tautological complex}}{\mathcal{L}}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \underset{\text{line bundle}}{\overset{\text{universal complex}}{E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C}}} \arrow[d] \\
\widehat{T}^2 \xrightarrow{H/|H|} S^2 = & \underset{\text{band topology}}{\overset{\text{fragile}}{\frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}}} \arrow[r, hook] & \displaystyle\bigcup_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[r, "\sim"] \arrow[r, phantom, shift right=8, "\text{stable band topology}"] & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

$\Rightarrow$ topological phases characterized by an integer

$$C := \big[H/|H|\big]$$

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\underset{\text{bundle}}{\overset{\text{valence}}{\mathcal{V}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \underset{\text{line bundle}}{\overset{\text{tautological complex}}{\mathcal{L}}} \arrow[rrr] \arrow[d] \arrow[drrr, phantom, "\text{(pb)}"] & & & \underset{\text{line bundle}}{\overset{\text{universal complex}}{E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C}}} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = \underset{\text{band topology}}{\underset{\text{fragile}}{\frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}}} \arrow[r, hook] & \underset{\text{band topology}}{\underset{\text{stable}}{\bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)}}} \arrow[rr, "\sim"] & & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

$\Rightarrow$ topological phases characterized by an integer

$$C := \big[H/|H|\big] \;\in\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big)$$

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\underset{\text{bundle}}{\overset{\text{valence}}{}} \mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\text{tautological complex line bundle}}{\mathcal{L}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\text{universal complex line bundle}}{E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C}} \arrow[d] \\
\widehat{T}^2 \xrightarrow{H/|H|} S^2 = \underset{\substack{\text{fragile} \\ \text{band topology}}}{\frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}} \arrow[r, hook] & \underset{\substack{\text{stable} \\ \text{band topology}}}{\bigcup_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)}} \arrow[r, "\sim"] & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

$\Rightarrow$ topological phases characterized by an integer

$$
C := \big[H/|H|\big] \;\in\; \underset{\textbf{winding number}}{\pi_0}\,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big)
$$

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\overset{\text{valence bundle}}{\mathcal{V}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\text{tautological complex line bundle}}{\mathcal{L}} \arrow[rrr] \arrow[d] \arrow[drrr, phantom, "\text{(pb)}"] & & & \overset{\text{universal complex line bundle}}{E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C}} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = \underset{\text{fragile band topology}}{\frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}} \arrow[r, hook] & \underset{\text{stable band topology}}{\bigcup_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)}} \arrow[rr, "\sim"] & & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

$\Rightarrow$ topological phases characterized by an integer

$$
C := \big[H/|H|\big] \;\in\; \underset{\textbf{winding number}}{\pi_0 \,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big)}
$$

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

valence bundle — tautological complex line bundle — universal complex line bundle

$$
\begin{tikzcd}
\mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{L} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \xrightarrow{H/|H|} S^2 = & \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)} \arrow[r, hook] & \underset{c\in\mathbb{N}}{\bigcup} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[r, "\sim"] & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

fragile band topology — stable band topology

$\Rightarrow$ topological phases characterized by an integer

$$C := \big[H/|H|\big] \;\in\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big) \;\simeq\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)$$

winding number

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

valence bundle — tautological complex line bundle — universal complex line bundle

$$
\begin{tikzcd}
\mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{L} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \xrightarrow{H/|H|} S^2 = \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[rr, "\sim"] & & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

fragile band topology — stable band topology

$\Rightarrow$ topological phases characterized by an integer

$$
C := \big[H/|H|\big] \;\in\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big) \;\simeq\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)
$$

winding number — **Chern number**

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

valence bundle — tautological complex line bundle — universal complex line bundle

$$
\begin{tikzcd}
\mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & \mathcal{L} \arrow[rr] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & & E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[r, "\sim"] & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

fragile band topology — stable band topology

$\Rightarrow$ topological phases characterized by an integer

$$
C := \big[H/|H|\big] \;\in\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big) \;\simeq\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)
$$

winding number — **Chern number**

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\underset{\text{bundle}}{\overset{\text{valence}}{\mathcal{V}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\text{tautological complex line bundle}}{\mathcal{L}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \overset{\text{universal complex line bundle}}{EU(1) \underset{U(1)}{\times} \mathbb{C}} \arrow[d] \\
\widehat{T}^2 \xrightarrow{H/|H|} S^2 = \underbrace{\frac{U(1)}{U(1)\times U(1)}}_{\text{fragile band topology}} & \hookrightarrow & \underbrace{\bigcup_{c \in \mathbb{N}} \frac{U(1+c)}{U(1)\times U(c)}}_{\text{stable band topology}} \xrightarrow{\sim} & BU(1)\,,
\end{tikzcd}
$$

$\Rightarrow$ topological phases characterized by an integer

$$
C := \big[H/|H|\big] \;\in\; \underbrace{\pi_0 \,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big)}_{\textbf{winding number}} \;\simeq\; \underbrace{\pi_0 \,\mathrm{Map}\big(\widehat{T}^2,\, BU(1)\big)}_{\textbf{Chern number}} \;\simeq\; \mathbb{Z}
$$

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\underset{\text{bundle}}{\overset{\text{valence}}{\mathcal{V}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \underset{\text{line bundle}}{\overset{\text{tautological complex}}{\mathcal{L}}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \underset{\text{line bundle}}{\overset{\text{universal complex}}{E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C}}} \arrow[d] \\
\widehat{T}^2 \xrightarrow{H/|H|} S^2 = \underbrace{\frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}}_{\substack{\text{fragile}\\ \text{band topology}}} \arrow[r, hook] & \underbrace{\bigcup_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)}}_{\substack{\text{stable}\\ \text{band topology}}} \arrow[rr, "\sim"] & & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

$\Rightarrow$ topological phases characterized by an integer

$$
C := \big[H/|H|\big] \;\in\; \underbrace{\pi_0 \,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big)}_{\substack{\textbf{winding}\\ \textbf{number}}} \;\simeq\; \underbrace{\pi_0 \,\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)}_{\substack{\textbf{Chern}\\ \textbf{number}}} \;\simeq\; \mathbb{Z}
$$

$\Rightarrow$ fragile = stable, for topological phases

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\underset{\text{bundle}}{\overset{\text{valence}}{\mathcal{V}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\text{tautological complex line bundle}}{\mathcal{L}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \overset{\text{universal complex line bundle}}{E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C}} \arrow[d] \\
\widehat{T}^2 \xrightarrow{H/|H|} S^2 = & \underset{\substack{\text{fragile} \\ \text{band topology}}}{\dfrac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}} \arrow[r, hook] & \underset{\substack{\text{stable} \\ \text{band topology}}}{\bigcup\limits_{c\in\mathbb{N}} \dfrac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)}} \arrow[r, "\sim"] & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

$\Rightarrow$ topological phases characterized by an integer

$$
C := \big[H/|H|\big] \;\in\; \underbrace{\pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big)}_{\substack{\textbf{winding}\\\textbf{number}}} \;\simeq\; \underbrace{\pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)}_{\substack{\textbf{Chern}\\\textbf{number}}} \;\simeq\; \mathbb{Z}
$$

$\Rightarrow$ fragile = stable, for topological phases

but not so for topological order...

Answer – for FQAH: 2.) topological phases

⇒ valence bundle is pullback of tautological/universal line bundle

valence bundle — tautological complex line bundle — universal complex line bundle

$$
\begin{tikzcd}
\mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & \mathcal{L} \arrow[rr] \arrow[d] \arrow[drr, phantom, "(\mathrm{pb})"] & & EU(1) \underset{U(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = \frac{U(1)}{U(1)\times U(1)} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{U(1+c)}{U(1)\times U(c)} \arrow[r, "\sim"] & BU(1)\,,
\end{tikzcd}
$$

fragile band topology — stable band topology

⇒ topological phases characterized by an integer

$$
C := \big[H/|H|\big] \;\in\; \pi_0 \,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big) \;\simeq\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, BU(1)\big) \;\simeq\; \mathbb{Z}
$$

winding number — **Chern number**

⇒ fragile = stable, for topological phases

but not so for topological order...

Answer – for FQAH: 3.) topological order

Answer – for FQAH: 3.) topological order

the stable band monodromy is

$$\pi_1\Big(\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)_C\Big) \;\simeq\; \Big\langle A, B \Big\rangle \Big/ \Big(AB = BA\Big) \;\simeq\; \mathbb{Z}^2$$

Answer – for FQAH: 3.) topological order

the stable band monodromy is

$$\pi_1\Big(\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)_C\Big) \;\simeq\; \Big\langle A, B \Big\rangle \Big/ \Big(AB = BA \Big) \;\simeq\; \mathbb{Z}^2$$

Maxwell Wilson loop observables

Answer – for FQAH: 3.) topological order

the stable band monodromy is

$$\pi_1\Big(\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)_C\Big) \;\simeq\; \Big\langle A, B \Big\rangle \Big/ \Big(AB = BA\Big) \;\simeq\; \mathbb{Z}^2$$

Maxwell Wilson loop observables

have 1-dim irreps $\Rightarrow$ no topological order

Answer – for FQAH: 3.) topological order

the stable band monodromy is

$$\pi_1\Big(\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)_C\Big) \;\simeq\; \Big\langle A, B \Big\rangle \Big/ \Big(AB = BA\Big) \;\simeq\; \mathbb{Z}^2$$

Maxwell Wilson loop observables

have 1-dim irreps $\Rightarrow$ no topological order

but

Answer – for FQAH: 3.) topological order

the stable band monodromy is

$$\pi_1\Big(\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)_C\Big) \;\simeq\; \Big\langle A, B \Big\rangle \Big/ \Big(AB = BA\Big) \;\simeq\; \mathbb{Z}^2$$

Maxwell Wilson loop observables

have 1-dim irreps $\Rightarrow$ no topological order

but

the *fragile* band monodromy is

[Sati & S. 2025, using Larmore & Thomas 1980, Kallel 2001, going back to Hansen 1974]

$$\pi_1\Big(\mathrm{Map}\big(\widehat{T}^2,\, S^2\big)_C\Big) \;\simeq\; \Big\langle A, B, \underset{\text{central}}{\zeta} \Big\rangle \Big/ \Big(AB = \zeta^2 BA\Big)$$

Answer – for FQAH: 3.) topological order

the stable band monodromy is

$$\pi_1\Big(\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)_C\Big) \;\simeq\; \Big\langle A, B \Big\rangle \Big/ \Big(AB = BA\Big) \;\simeq\; \mathbb{Z}^2$$

Maxwell Wilson loop observables

have 1-dim irreps $\Rightarrow$ no topological order

but

the *fragile* band monodromy is [Sati & S. 2025, using Larmore & Thomas 1980, Kallel 2001, going back to Hansen 1974]

$$\pi_1\Big(\mathrm{Map}\big(\widehat{T}^2,\, S^2\big)_C\Big) \;\simeq\; \Big\langle A, B, \underset{\text{central}}{\zeta} \Big\rangle \Big/ \Big(AB = \zeta^2 BA\Big)$$

Chern-Simons Wilson loop observables!

Answer – for FQAH: 3.) topological order

the stable band monodromy is

$$\pi_1\Big(\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)_C\Big) \;\simeq\; \Big\langle A, B \Big\rangle \Big/ \Big(AB = BA\Big) \;\simeq\; \mathbb{Z}^2$$

Maxwell Wilson loop observables

have 1-dim irreps $\Rightarrow$ no topological order

but

the *fragile* band monodromy is [Sati & S. 2025, using Larmore & Thomas 1980, Kallel 2001, going back to Hansen 1974]

$$\pi_1\Big(\mathrm{Map}\big(\widehat{T}^2,\, S^2\big)_C\Big) \;\simeq\; \Big\langle A, B, \underset{\text{central}}{\zeta} \Big\rangle \Big/ \Big(AB = \zeta^2 BA\Big)$$

Chern-Simons Wilson loop observables!

irreps are the anyonic

topological order on the torus!

Answer – for FQAH: 3.) topological order

the stable band monodromy is

$$\pi_1\Big(\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)_C\Big) \;\simeq\; \Big\langle A, B \Big\rangle \Big/ \Big(AB = BA\Big) \;\simeq\; \mathbb{Z}^2$$

Maxwell Wilson loop observables

have 1-dim irreps $\Rightarrow$ no topological order

but

the *fragile* band monodromy is [Sati & S. 2025, using Larmore & Thomas 1980, Kallel 2001, going back to Hansen 1974]

$$\pi_1\Big(\mathrm{Map}\big(\widehat{T}^2,\, S^2\big)_C\Big) \;\simeq\; \Big\langle A, B, \underset{\text{central}}{\zeta} \Big\rangle \Big/ \Big(AB = \zeta^2 BA\Big)$$

Chern-Simons Wilson loop observables!

irreps are the anyonic

topological order on the torus!

$\Rightarrow$ solitonic anyons localized in momentum space

Answer – for FQAH: 3.) topological order

the stable band monodromy is

$$\pi_1\Big(\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)_C\Big) \;\simeq\; \Big\langle A, B \Big\rangle \Big/ \Big(AB = BA\Big) \;\simeq\; \mathbb{Z}^2$$

Maxwell Wilson loop observables

have 1-dim irreps ⇒ no topological order

but

the *fragile* band monodromy is [Sati & S. 2025, using Larmore & Thomas 1980, Kallel 2001, going back to Hansen 1974]

$$\pi_1\Big(\mathrm{Map}\big(\widehat{T}^2,\, S^2\big)_C\Big) \;\simeq\; \Big\langle A, B, \underset{\text{central}}{\zeta} \Big\rangle \Big/ \Big(AB = \zeta^2 BA\Big)$$

Chern-Simons Wilson loop observables!

irreps are the anyonic

topological order on the torus!

⇒ solitonic anyons localized in momentum space

so far this assumed *all* symmetries broken

Answer – for FQAH: 3.) topological order

the stable band monodromy is

$$\pi_1\Big(\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)_C\Big) \;\simeq\; \Big\langle A, B \Big\rangle \Big/ \Big(AB = BA\Big) \;\simeq\; \mathbb{Z}^2$$

Maxwell Wilson loop observables

have 1-dim irreps $\Rightarrow$ no topological order

but

the *fragile* band monodromy is [Sati & S. 2025, using Larmore & Thomas 1980, Kallel 2001, going back to Hansen 1974]

$$\pi_1\Big(\mathrm{Map}\big(\widehat{T}^2,\, S^2\big)_C\Big) \;\simeq\; \Big\langle A, B, \underset{\text{central}}{\zeta} \Big\rangle \Big/ \Big(AB = \zeta^2 BA\Big)$$

Chern-Simons Wilson loop observables!

irreps are the anyonic

topological order on the torus!

$\Rightarrow$ solitonic anyons localized in momentum space

so far this assumed *all* symmetries broken

Answer – for FQAH: 4.) crystal symmetry

Answer – for FQAH: 4.) crystal symmetry

we compute

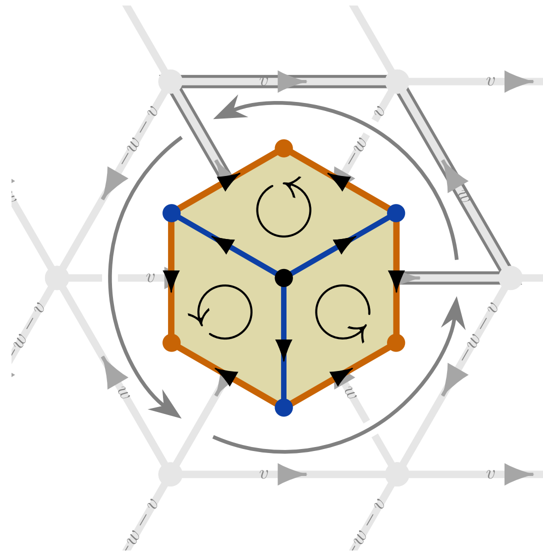
fragile crystalline Chern phases

(apparently not computed before)

Answer – for FQAH: 4.) crystal symmetry

fragile crystalline Chern phases

(apparently not computed before)

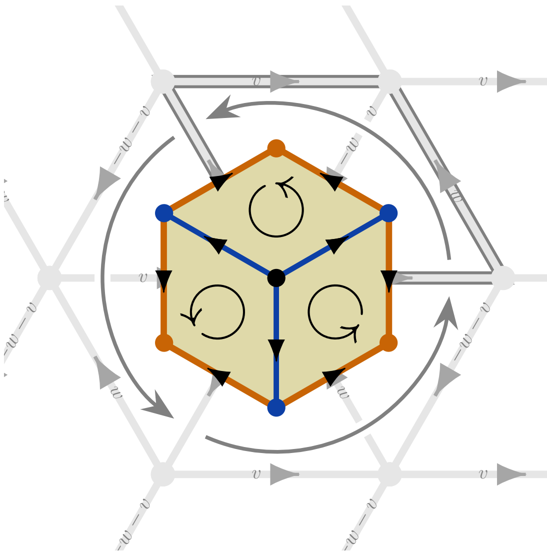


Example: crystalline p3 symmetry $\mathbb{Z}_3$

Answer – for FQAH: 4.) crystal symmetry

fragile crystalline Chern phases

(apparently not computed before)



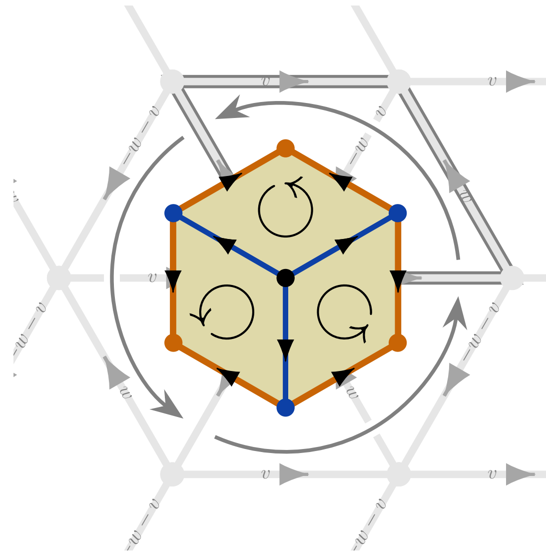
Example: crystalline p3 symmetry $\mathbb{Z}_3$

topological phase: $(C, c) \in \boxed{3\mathbb{Z} \times [4]}$

Answer – for FQAH: 4.) crystal symmetry

fragile crystalline Chern phases

(apparently not computed before)



Example: crystalline p3 symmetry $\mathbb{Z}_3$

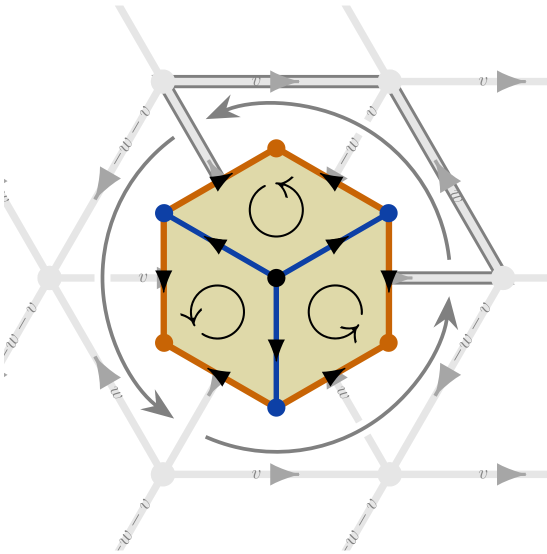
topological phase: $(C, c) \in \boxed{3\mathbb{Z} \times [4]}$

Chern number is multiple of 3 & adjoined by 4 sectors

Answer – for FQAH: 4.) crystal symmetry

fragile crystalline Chern phases

(apparently not computed before)

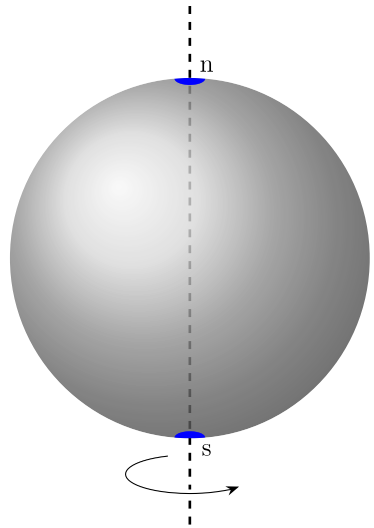


Example: crystalline p3 symmetry $\mathbb{Z}_3$

topological phase: $(C, c) \in \boxed{3\mathbb{Z} \times [4]}$

Chern number is multiple of 3 & adjoined by 4 sectors

labeled by how
the 3 fixed/high symmetry points
map to either fixed pole on the sphere



Answer – for FQAH: 4.) crystal symmetry

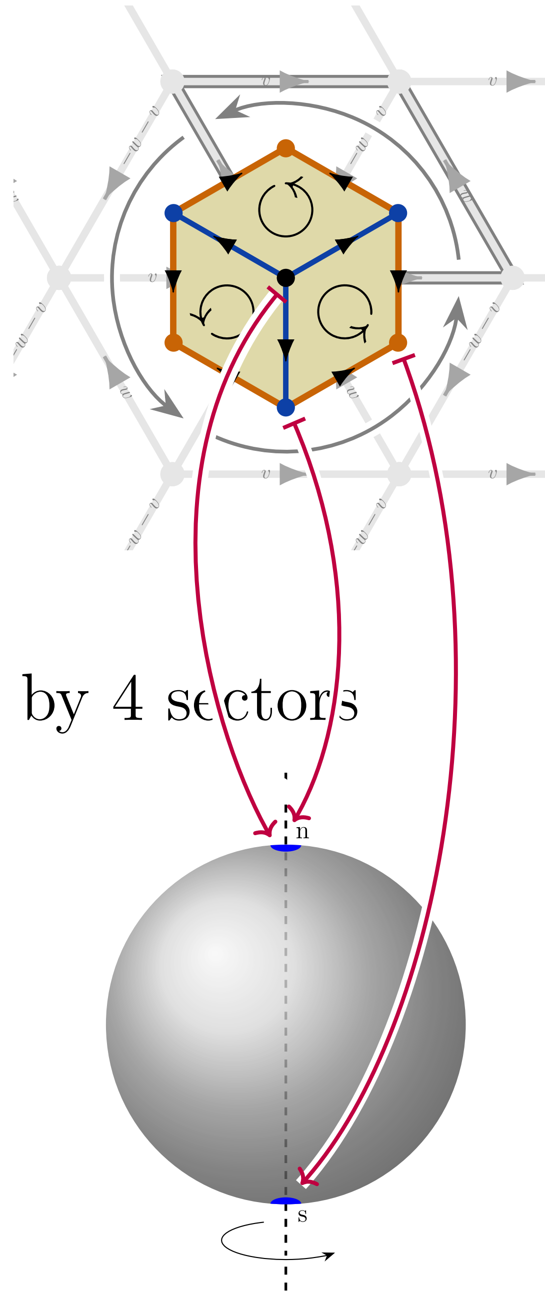
fragile crystalline Chern phases

(apparently not computed before)

Example: crystalline p3 symmetry $\mathbb{Z}_3$

topological phase: $(C, c) \in \boxed{3\mathbb{Z} \times [4]}$

Chern number is multiple of 3 & adjoined by 4 sectors

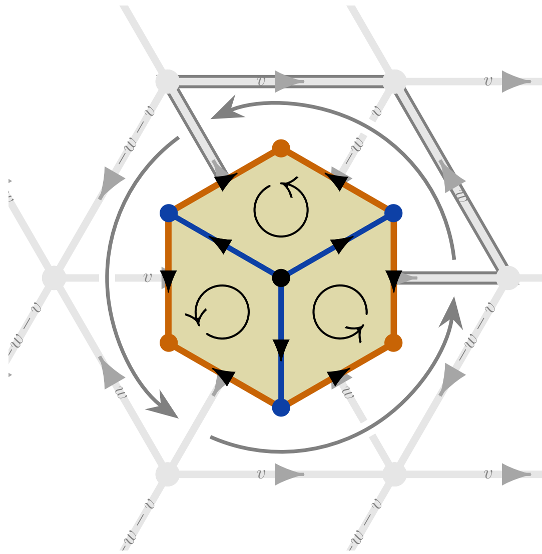


labeled by how
the 3 fixed/high symmetry points
map to either fixed pole on the sphere

Answer – for FQAH: 4.) crystal symmetry

fragile crystalline Chern phases

(apparently not computed before)

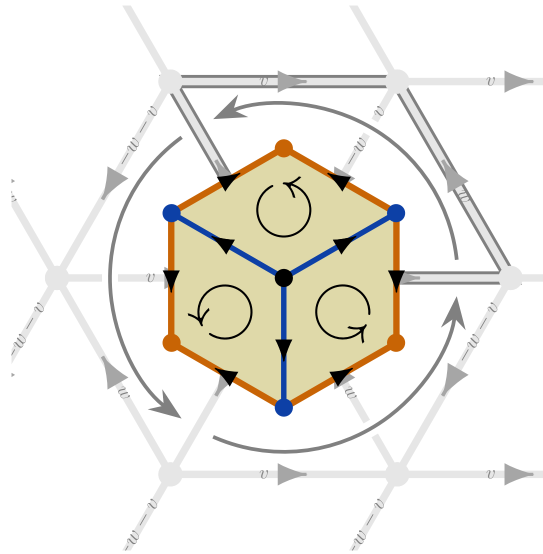


Example: crystalline p3 symmetry $\mathbb{Z}_3$

topological phase: $(C, c) \in \boxed{3\mathbb{Z} \times [4]}$

Chern number is multiple of 3 & adjoined by 4 sectors

Answer – for FQAH: 4.) crystal symmetry



fragile crystalline Chern phases

(apparently not computed before)

Example: crystalline p3 symmetry $\mathbb{Z}_3$

topological phase: $(C, c) \in \boxed{3\mathbb{Z} \times [4]}$

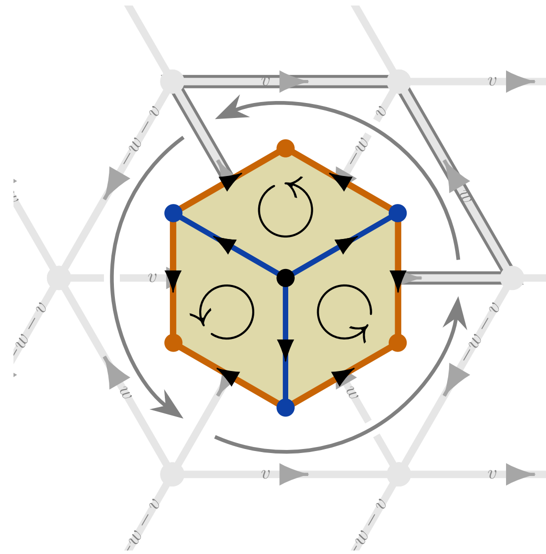
Chern number is multiple of 3 & adjoined by 4 sectors

topological order: $\mathcal{H} \in \left(\boxed{\mathbb{Z}^3 \rtimes \mathrm{Sym}(3)} \right)$-IrrRep

Answer – for FQAH: 4.) crystal symmetry

fragile crystalline Chern phases

(apparently not computed before)



Example: crystalline p3 symmetry $\mathbb{Z}_3$

topological phase: $(C, c) \in \boxed{3\mathbb{Z} \times [4]}$

Chern number is multiple of 3 & adjoined by 4 sectors

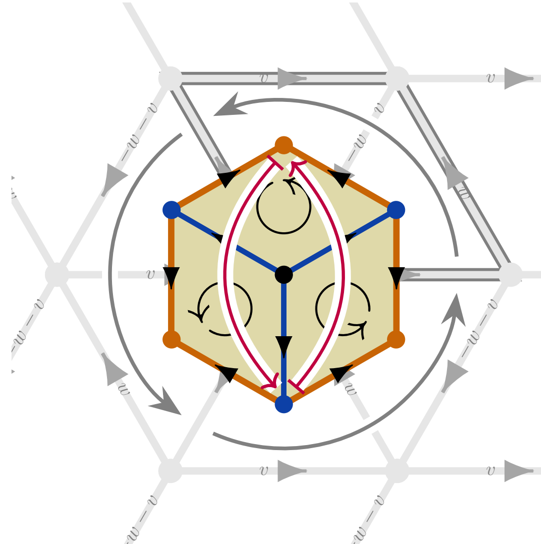
topological order: $\mathcal{H} \in \left(\boxed{\mathbb{Z}^3 \rtimes \mathrm{Sym}(3)} \right)$-IrrRep

braid the three high-symmetry points

Answer – for FQAH: 4.) crystal symmetry

fragile crystalline Chern phases

(apparently not computed before)



Example: crystalline p3 symmetry $\mathbb{Z}_3$

topological phase: $(C, c) \in \boxed{3\mathbb{Z} \times [4]}$

Chern number is multiple of 3 & adjoined by 4 sectors

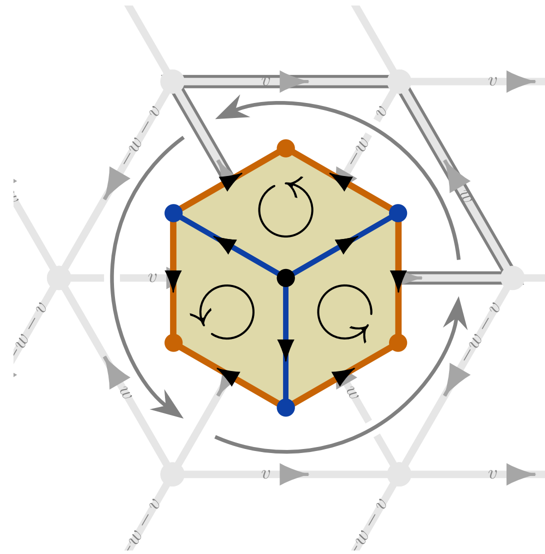
topological order: $\mathcal{H} \in \left(\boxed{\mathbb{Z}^3 \rtimes \mathrm{Sym}(3)} \right)$-IrrRep

braid the three
high-symmetry points

Answer – for FQAH: 4.) crystal symmetry

fragile crystalline Chern phases

(apparently not computed before)



Example: crystalline p3 symmetry $\mathbb{Z}_3$

topological phase: $(C, c) \in \boxed{3\mathbb{Z} \times [4]}$

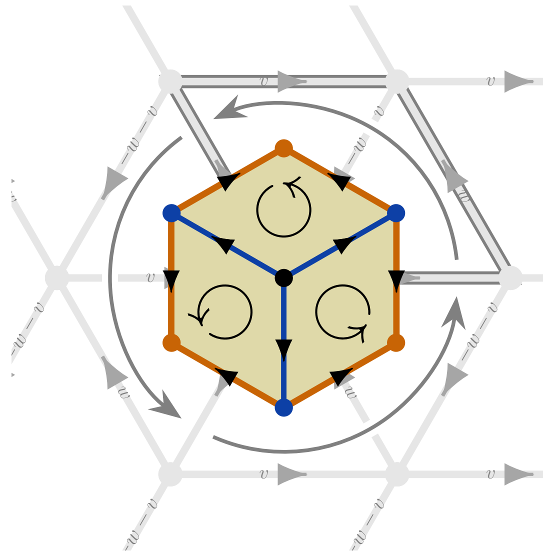
Chern number is multiple of 3 & adjoined by 4 sectors

topological order: $\mathcal{H} \in \Big(\boxed{\mathbb{Z}^3 \rtimes \mathrm{Sym}(3)} \Big)$-IrrRep

braid the three high-symmetry points

nonabelian!

Answer – for FQAH: 4.) crystal symmetry



fragile crystalline Chern phases

(apparently not computed before)

Example: crystalline p3 symmetry $\mathbb{Z}_3$

topological phase: $(C, c) \in \boxed{3\mathbb{Z} \times [4]}$

Chern number is multiple of 3 & adjoined by 4 sectors

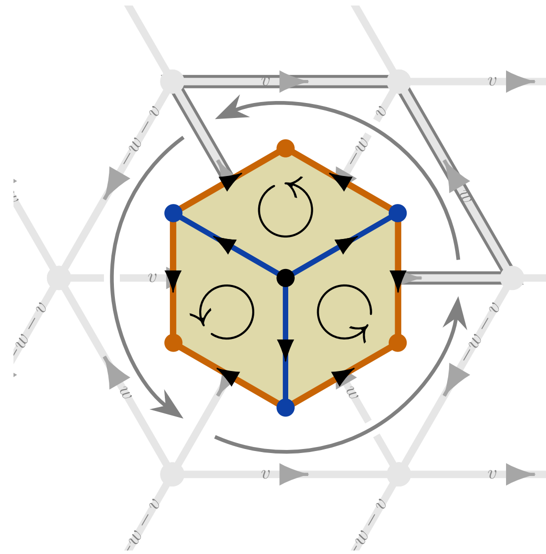
topological order: $\mathcal{H} \in \left(\boxed{\mathbb{Z}^3 \rtimes \mathrm{Sym}(3)} \right)$-IrrRep

braid the three high-symmetry points

nonabelian!

high symmetry points as defect para-anyons!

Answer – for FQAH: 4.) crystal symmetry



fragile crystalline Chern phases

(apparently not computed before)

Example: crystalline p3 symmetry $\mathbb{Z}_3$

topological phase: $(C, c) \in \boxed{3\mathbb{Z} \times [4]}$

Chern number is multiple of 3 & adjoined by 4 sectors

topological order: $\mathcal{H} \in \Big(\boxed{\mathbb{Z}^3 \rtimes \mathrm{Sym}(3)} \Big)$-IrrRep

braid the three high-symmetry points

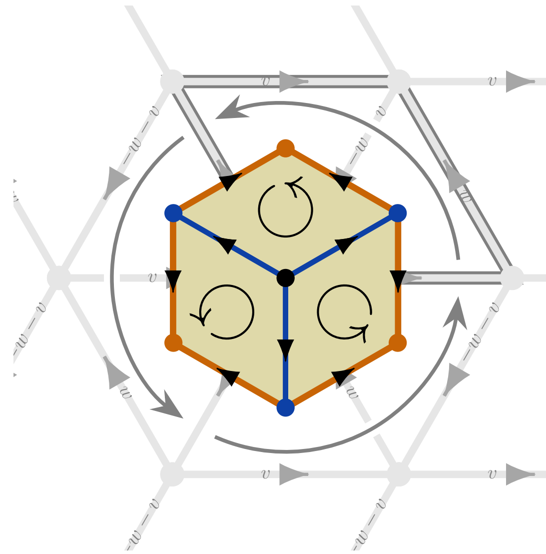
nonabelian!

high symmetry points as defect para-anyons!

Answer – for FQAH: 4.) crystal symmetry

fragile crystalline Chern phases

(apparently not computed before)



Example: crystalline p3 symmetry $\mathbb{Z}_3$

topological phase: $(C, c) \in \boxed{3\mathbb{Z} \times [4]}$

Chern number is multiple of 3 & adjoined by 4 sectors

topological order: $\mathcal{H} \in \Big(\boxed{\mathbb{Z}^3 \rtimes \mathrm{Sym}(3)} \Big)$-IrrRep

braid the three high-symmetry points

nonabelian!

high symmetry points as defect para-anyons!

standard Sym(3) irrep implements Z-and rotation-gate $R_y(2\pi/3)$

[Sati & S 2025, Prop. 3.61]

Conclusion.

Conclusion.

FQAH topological order theoretically seen in

monodromy of fragile band topology

$\Rightarrow$ clear predictions via equivariant (co)homotopy theory

Conclusion.

FQAH topological order theoretically seen in

monodromy of fragile band topology

$\Rightarrow$ clear predictions via equivariant (co)homotopy theory

band topology is maps to Bloch Hamiltonian space $\mathcal{A}$

Conclusion.

FQAH topological order theoretically seen in

monodromy of fragile band topology

$\Rightarrow$ clear predictions via equivariant (co)homotopy theory

band topology is maps to Bloch Hamiltonian space $\mathcal{A}$

topological phases is components π_0

Conclusion.

FQAH topological order theoretically seen in

monodromy of fragile band topology

$\Rightarrow$ clear predictions via equivariant (co)homotopy theory

band topology is maps to Bloch Hamiltonian space $\mathcal{A}$

topological phases is components π_0

topological order is monodromy π_1

Conclusion.

FQAH topological order theoretically seen in

monodromy of fragile band topology

$\Rightarrow$ clear predictions via equivariant (co)homotopy theory

band topology is maps to Bloch Hamiltonian space $\mathcal{A}$

topological phases is components π_0

topological order is monodromy π_1

For FQAH Chern insulators: $\mathcal{A} = S^2$ (2-Cohomotopy)

Conclusion.

FQAH topological order theoretically seen in

monodromy of fragile band topology

$\Rightarrow$ clear predictions via equivariant (co)homotopy theory

band topology is maps to Bloch Hamiltonian space $\mathcal{A}$

topological phases is components π_0

topological order is monodromy π_1

For FQAH Chern insulators: $\mathcal{A} = S^2$ (2-Cohomotopy)

all symmetry broken $\Rightarrow$ solitonic anyons in momentum space!

Conclusion.

FQAH topological order theoretically seen in

monodromy of fragile band topology

$\Rightarrow$ clear predictions via equivariant (co)homotopy theory

band topology is maps to Bloch Hamiltonian space $\mathcal{A}$

topological phases is components π_0

topological order is monodromy π_1

For FQAH Chern insulators: $\mathcal{A} = S^2$ (2-Cohomotopy)

all symmetry broken $\Rightarrow$ solitonic anyons in momentum space!

p3 symmetry $\mathbb{Z}_3$ $\Rightarrow$ defect para-anyons in momentum space!

Conclusion.

FQAH topological order theoretically seen in

monodromy of fragile band topology

$\Rightarrow$ clear predictions via equivariant (co)homotopy theory

band topology is maps to Bloch Hamiltonian space $\mathcal{A}$

topological phases is components π_0

topological order is monodromy π_1

For FQAH Chern insulators: $\mathcal{A} = S^2$ (2-Cohomotopy)

all symmetry broken $\Rightarrow$ solitonic anyons in momentum space!

p3 symmetry $\mathbb{Z}_3$ $\Rightarrow$ defect para-anyons in momentum space!

$\Rightarrow$ tantalizing targets for experiment

Conclusion.

FQAH topological order theoretically seen in

monodromy of fragile band topology

$\Rightarrow$ clear predictions via equivariant (co)homotopy theory

band topology is maps to Bloch Hamiltonian space $\mathcal{A}$

topological phases is components π_0

topological order is monodromy π_1

For FQAH Chern insulators: $\mathcal{A} = S^2$ (2-Cohomotopy)

all symmetry broken $\Rightarrow$ solitonic anyons in momentum space!

p3 symmetry $\mathbb{Z}_3$ $\Rightarrow$ defect para-anyons in momentum space!

$\Rightarrow$ tantalizing targets for experiment

Urs Schreiber on joint work with Hisham Sati:

surveying our preprint: [arXiv:2507.00138]

Identifying Anyonic Topological Order in FQAH Systems

Thanks!

CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS @ جامعة نيويورك أبوظبي NYU ABU DHABI

(September 2025) find these slides at: [ncatlab.org/schreiber/show/QMATH16]