Urs Schreiber on joint work with Hisham Sati:

invitation to our preprint: [arXiv:2507.00138]

Identifying Anyonic Topological Order in FQAH Systems

CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS

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جامعة نيويورك أبوظبي
NYU ABU DHABI

(September 2025) find these slides at: [ncatlab.org/schreiber/show/QMATH16]

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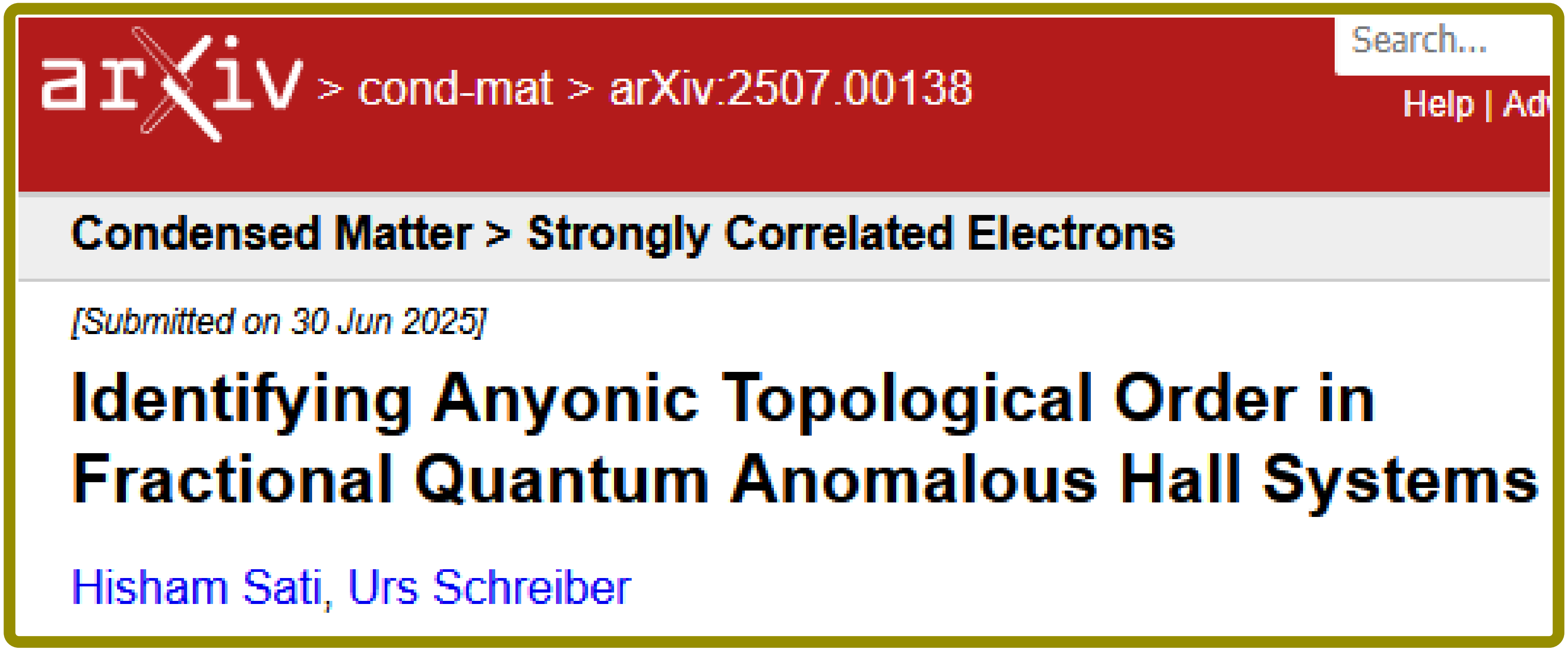
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[Submitted on 30 Jun 2025]

Identifying Anyonic Topological Order in Fractional Quantum Anomalous Hall Systems

Hisham Sati, Urs Schreiber

CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS @ جامعة نيويورك أبوظبي NYU ABU DHABI

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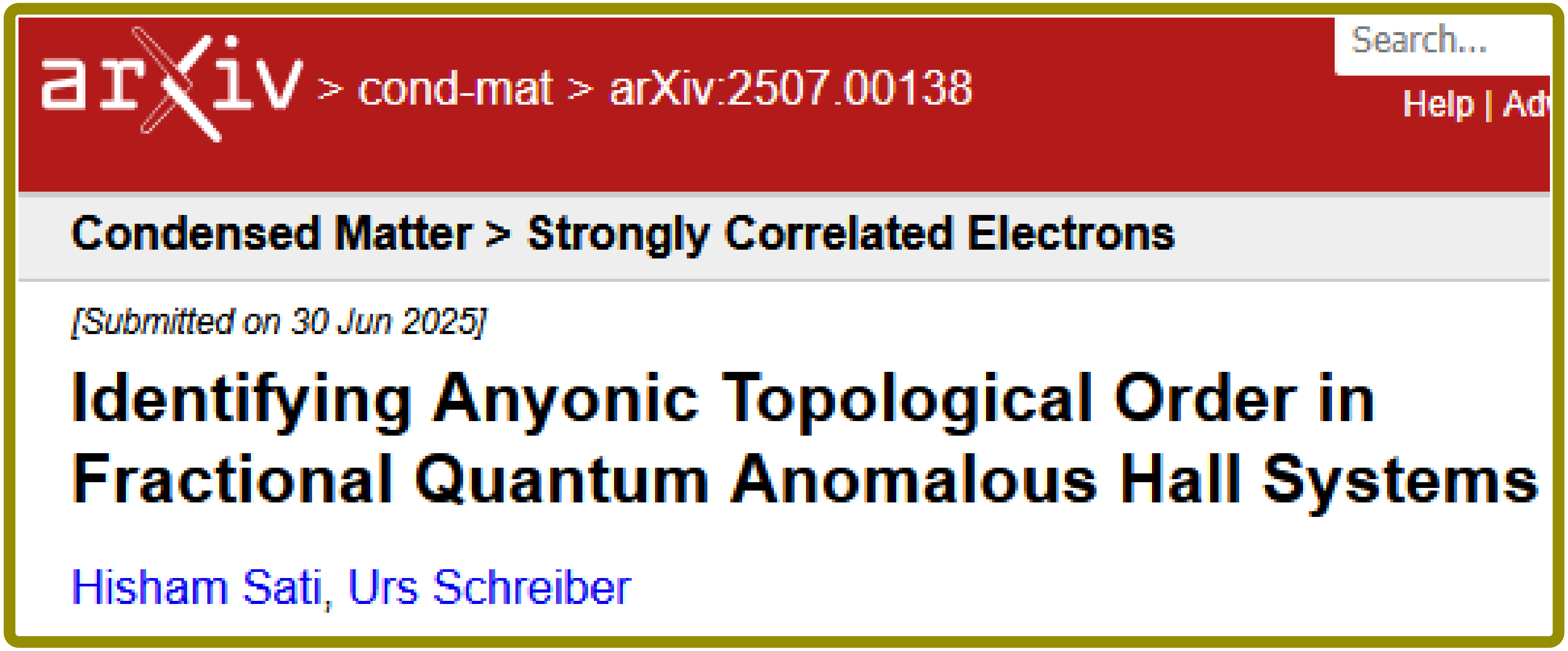

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quantumzeitgeist.com/topological-quantum-hardware-emerges-from-fractional-anomalous-hall-e

QUANTUM ZEITGEIST

QUANTUM COMPUTING TECHNOLOGY NEWS

QUANTUM TECHNOLOGY

Topological Quantum Hardware Emerges From Fractional Anomalous Hall Effect Physics.

July 2, 2025
BY QUANTUM NEWS

The pursuit of robust quantum computation necessitates the identification and control of exotic states of matter exhibiting topological order, where information is encoded not in local degrees of freedom but in the global properties of the system. Recent observations of fractional anomalous Hall (FQAH) states, characterised by fractionalised quantum Hall effects in the absence of an external magnetic field, present a promising avenue for realising such topological hardware. However, confirming the existence of the crucial anyonic excitations, quasiparticles obeying non-Abelian statistics essential for quantum computation, remains a significant challenge. Hisham Sati and Urs Schreiber, alongside colleagues at the Center for Quantum and Topological Systems at New York University Abu Dhabi, address this issue in their work, "Identifying Anyonic Topological Order in Fractional Quantum Anomalous Hall Systems". Their research establishes a link between the fragile topology of these systems and the identification of anyons within momentum space, utilising a theorem from algebraic topology dating back to 1980, and providing a framework for understanding symmetry-protected topological order in FQAH systems through computations in equivariant cohomotopy.

Fractional Quantum Hall systems.

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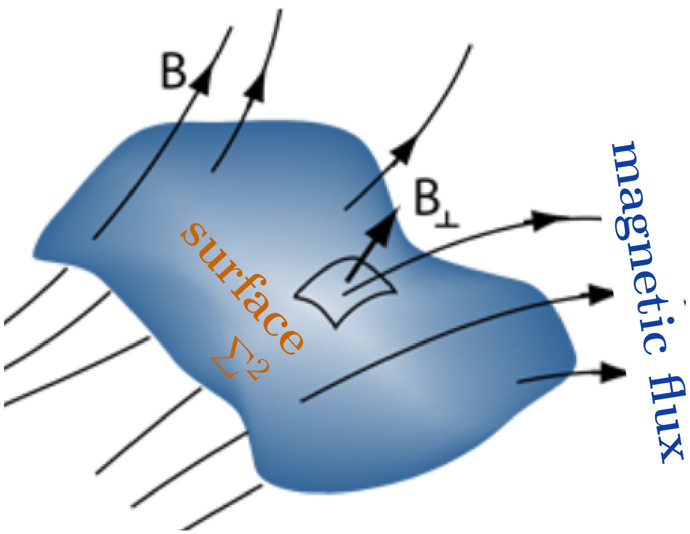
electron gas in 2D semiconductor Σ^2
subject to transverse magnetic field



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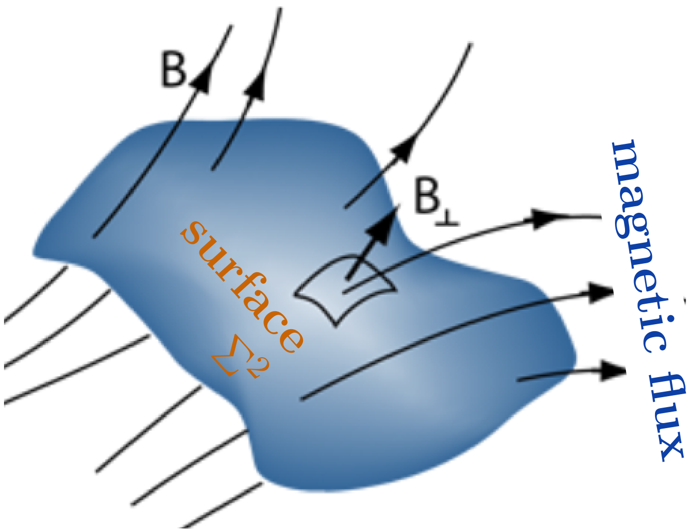

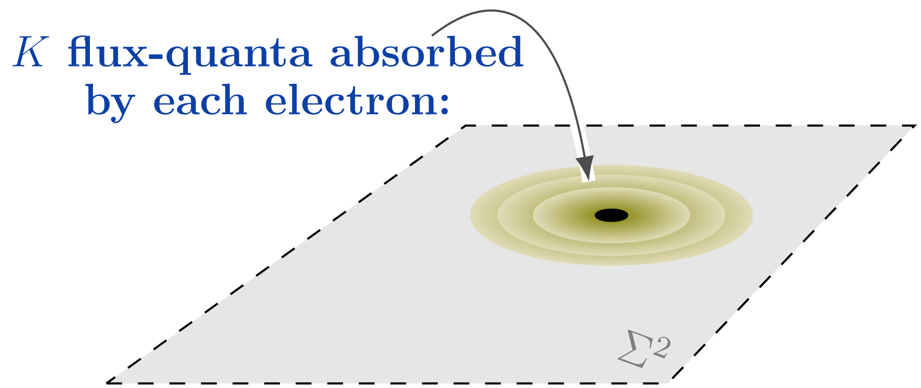

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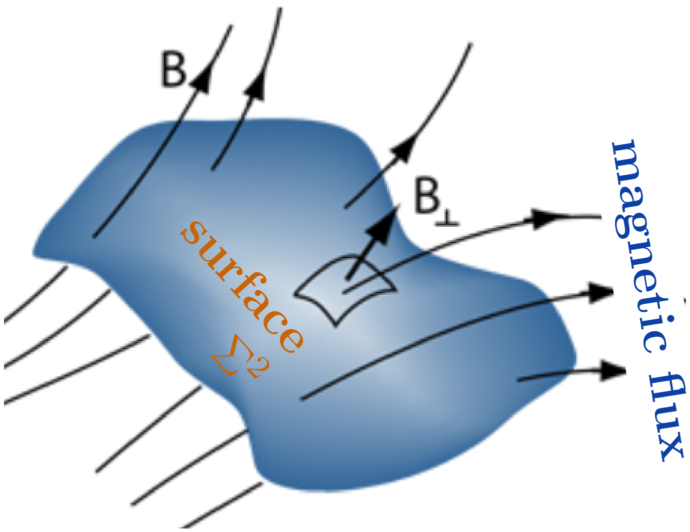

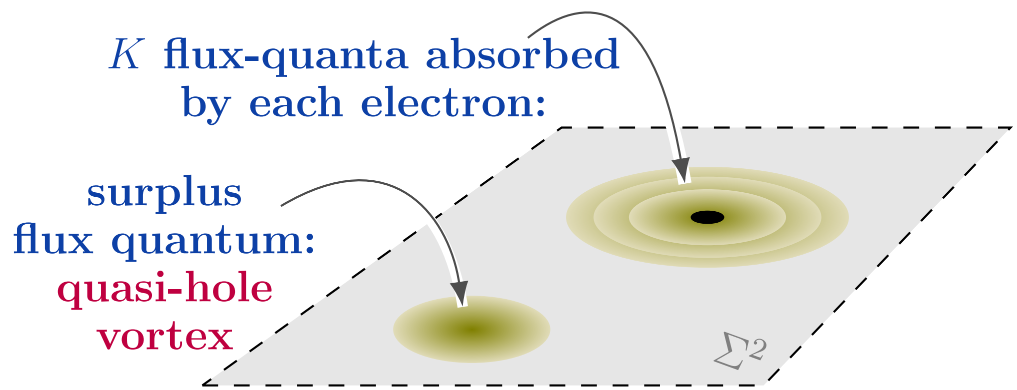

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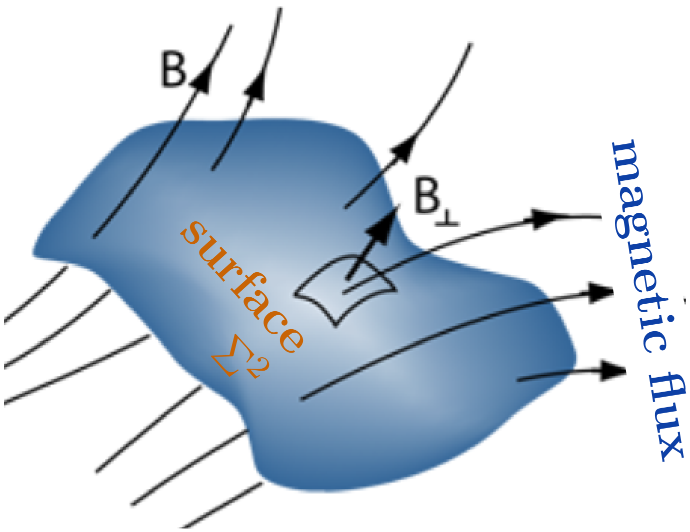

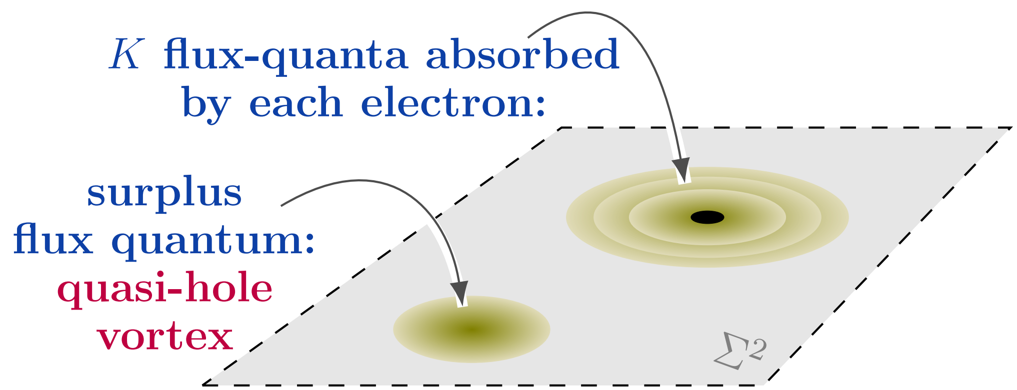

remarkably:
surplus magnetic flux quanta
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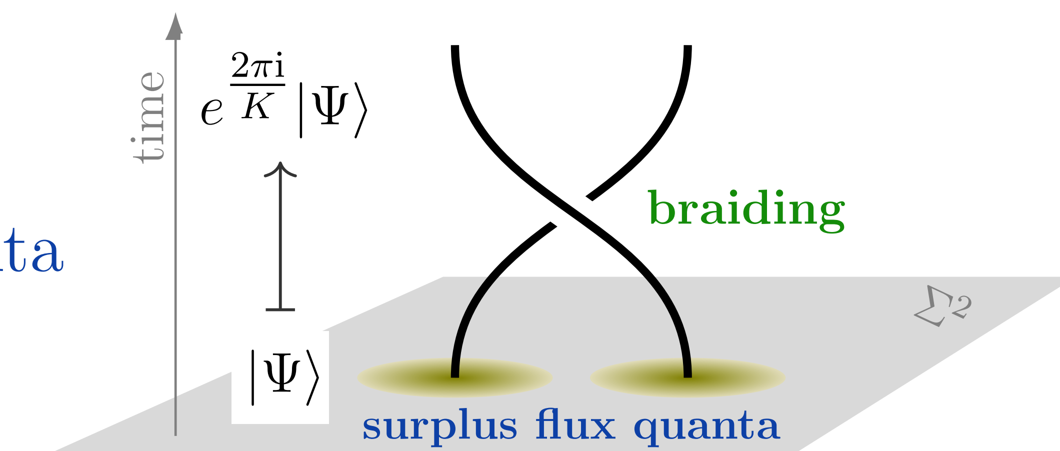

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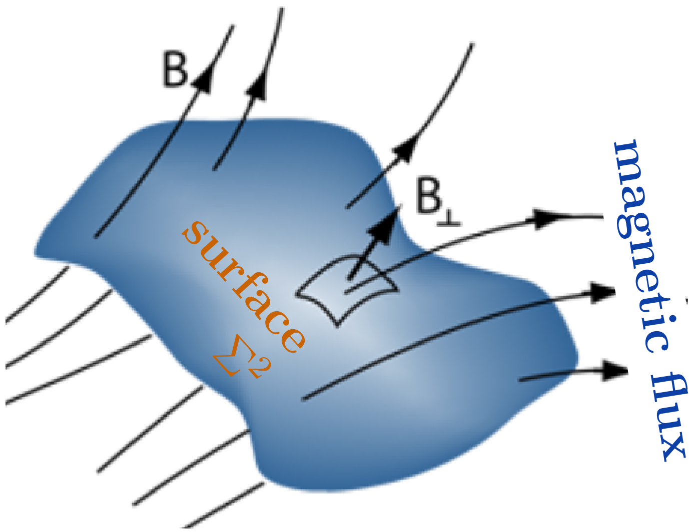

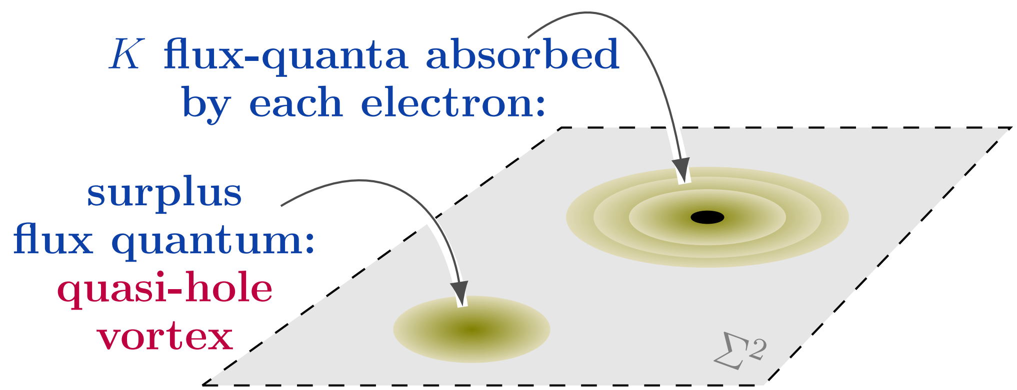

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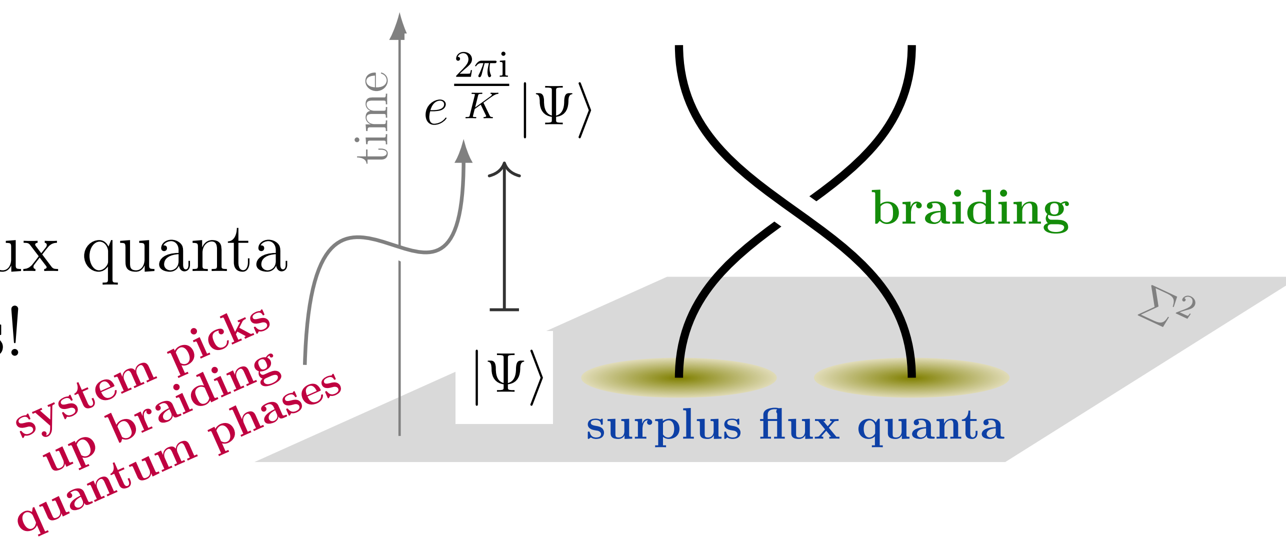

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Fractional Quantum Hall systems.

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[Nakamura et al. 2020]
[Nakamura et al. 2023]
[Ruelle et al. 2023]
[Glidic et al. 2023]
[Kundu et al. 2023]
[Veillon et al. 2024]
[Ghosh et al. 2025]

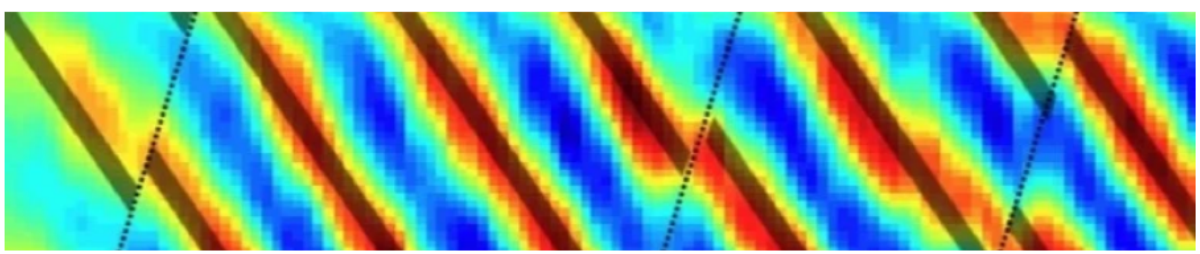
nature

NEWS | 03 July 2020

Welcome anyons! Physicists find best evidence yet for long-sought 2D structures

The 'quasiparticles' defy the categories of ordinary particles and herald a potential way to build quantum computers.

By Davide Castelvecchi



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Hall effect

Semiclassical electron equation of motion

$$\dot{\vec{k}} = -\partial\epsilon/\partial\vec{x} + \underbrace{\vec{x} \times e\vec{B}}_{\text{Lorentz force}}$$

$$\dot{\vec{x}} = \partial\epsilon/\partial\vec{k} - \underbrace{\vec{k} \times \vec{\Omega}}_{\text{anomalous velocity}}$$

position	magnetic flux density
$\vec{x}$	$e\vec{B}$
$\updownarrow$	$\updownarrow$
$\vec{k}$	$\vec{\Omega}$
momentum	Berry curvature

Anomalous Hall effect

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	Lorentz force	position	magnetic flux density
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Semiclassical electron equation of motion		$\updownarrow$	$\updownarrow$
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	anomalous velocity	momentum	Berry curvature

Anomalous Hall effect

nature

Article | Published: 17 August 2023

Observation of fractionally quantized anomalous Hall effect

Heonjoon Park, Jiaqi Cai, Eric Anderson, Yinong Zhang, Jiayi Zhu, Xiaoyu Liu, Chong Wang, William Holtzmann, Chaowei Hu, Zhaoyu Liu, Takashi Taniguchi, Kenji Watanabe, Jiun-Haw Chu, Ting Cao, Liang Fu, Wang Yao, Cui-Zu Chang, David Cobden, Di Xiao & Xiaodong Xu

Nature **622**, 74–79 (2023) | Cite this article

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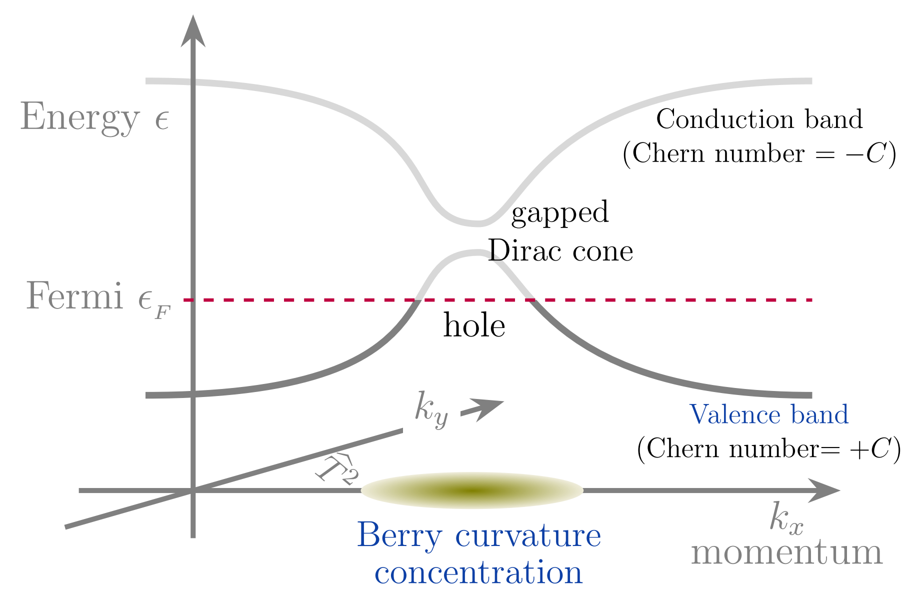
	Lorentz force	**position**	**magnetic flux density**
Semiclassical electron equation of motion	$\dot{\vec{k}} = -\partial\epsilon/\partial\vec{x} + \vec{x} \times e\vec{B}$	$\vec{x}$	$e\vec{B}$
	$\dot{\vec{x}} = \partial\epsilon/\partial\vec{k} - \vec{k} \times \vec{\Omega}$	$\vec{k}$	$\vec{\Omega}$
	anomalous velocity	**momentum**	**Berry curvature**

Anomalous Hall effect



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Lorentz force | **position** | **magnetic flux density**

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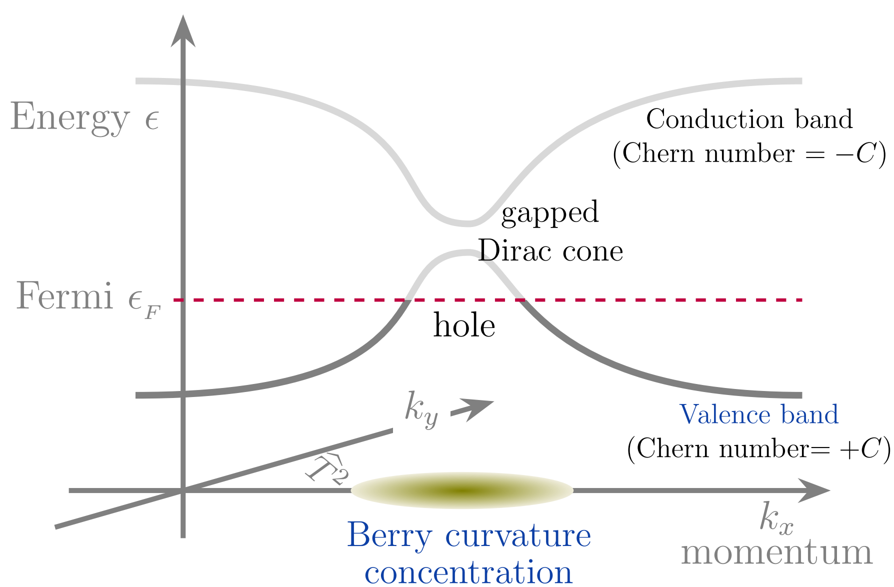
$\vec{x} \leftrightarrow \vec{k}$, $\quad e\vec{B} \leftrightarrow \vec{\Omega}$

anomalous velocity | **momentum** | **Berry curvature**

Anomalous Hall effect

suggests FQAH systems as viable topological quantum hardware!



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$$\begin{aligned} \dot{\vec{k}} &= -\partial\epsilon/\partial\vec{x} + \underbrace{\vec{x} \times e\vec{B}}_{\text{Lorentz force}} \\ \dot{\vec{x}} &= \partial\epsilon/\partial\vec{k} - \underbrace{\vec{k} \times \vec{\Omega}}_{\text{anomalous velocity}} \end{aligned}$$

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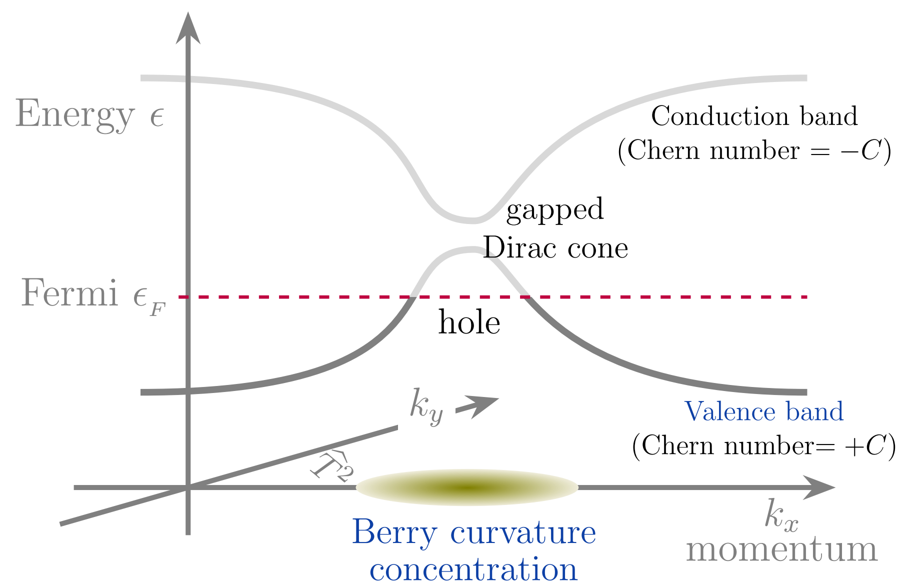
but **open question**:

Where & how are FQAH anyons?



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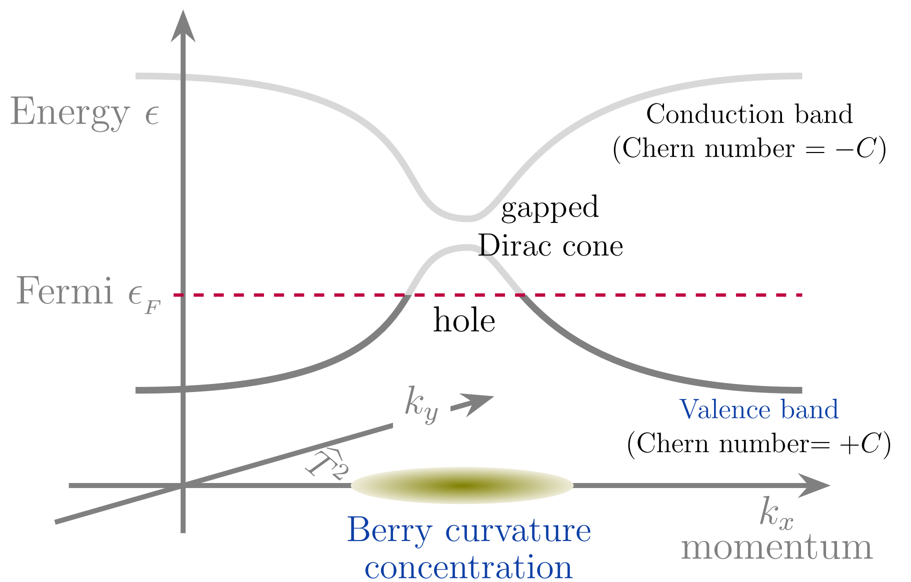

Answer – in general:

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Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

topological phase: $\ldots$

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Then:

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big)$

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the Bloch Hamiltonians are a map
from the Brillouin torus of crystal momenta
to the given space of Hamiltonians

$$
\begin{aligned}
\widehat{T}^2 &\longrightarrow \mathcal{A} \\
k &\longmapsto H_{\mathrm{Blch}}(k)
\end{aligned}
$$

Answer – in general:

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equivariance is the mathematical term
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$$\boxed{H_{\mathrm{Blch}}\big(g \cdot k\big) \;=\; U_g \circ H_{\mathrm{Blch}}(k) \circ U_g^{-1}}$$

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Answer – in general:

Given:

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Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

topological phase: $\color{blue}{C} \in \pi_{\color{purple}{0}}\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big)$

Answer – in general:

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connected components give the topological deformation classes of Bloch Hamiltonians

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Examples:

classifying space	$\mathcal{A}$	phases

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classifying space	$\mathcal{A}$	phases
$\mathrm{Gr}_v\big(\mathbb{C}^{v+c}\big) \;=$	$\frac{\mathrm{U}(c+v)}{\mathrm{U}(c)\times\mathrm{U}(v)}$	

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${\color{blue} B\mathrm{U}(1)}$	${\color{blue} =}$	${\color{blue} \bigcup_{c\in\mathbb{N}} \frac{\mathrm{U}(c+1)}{\mathrm{U}(c)\times\mathrm{U}(1)}}$	Chern

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$B\mathrm{U}$	$=$	$\bigcup\limits_{c,v\in\mathbb{N}} \frac{\mathrm{U}(c+v)}{\mathrm{U}(c)\times\mathrm{U}(v)}$	

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$\mathrm{Gr}_v\big(\mathbb{C}^{v+c}\big)$	$=$	$\frac{\mathrm{U}(c+v)}{\mathrm{U}(c)\times\mathrm{U}(v)}$	*fragile*
$B\mathrm{U}(1)$	$=$	$\bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(c+1)}{\mathrm{U}(c)\times\mathrm{U}(1)}$	Chern
$B\mathrm{U}$	$=$	$\bigcup\limits_{c,v\in\mathbb{N}} \frac{\mathrm{U}(c+v)}{\mathrm{U}(c)\times\mathrm{U}(v)}$	K-theoretic

Answer – in general:

Given:

crystal symmetry group G

Bloch Hamiltonian space $\mathcal{A}$

Then:

connected components

mapping space

equivariant maps

equivariant extraordinary nonabelian cohomology

topological phase: $C \in \pi_0\Big(\mathrm{Map}\big(\widehat{T}^2, \mathcal{A}\big)^G\Big) \equiv H^1_G\big(\widehat{T}^2; \Omega\mathcal{A}\big)$

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gapped ground state Hilbert spaces

form local system/flat bundle

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connected components mapping space equivariant maps equivariant extraordinary nonabelian cohomology

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fundamental group unitary irreps

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form local system/flat bundle $\quad \mathcal{H}_{p_1}$

over external parameter space $\quad p_1 \xrightarrow{\gamma_{12}} p_2$

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$$p_1 \overset{\gamma_{12}}{\rightsquigarrow} p_2$$

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Familiar example – FQH systems:

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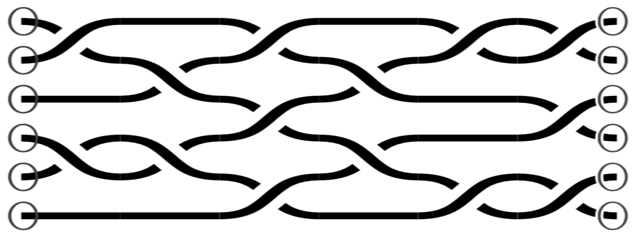
monodromy = braiding of their worldlines



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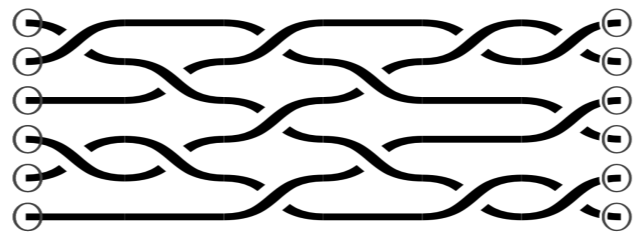

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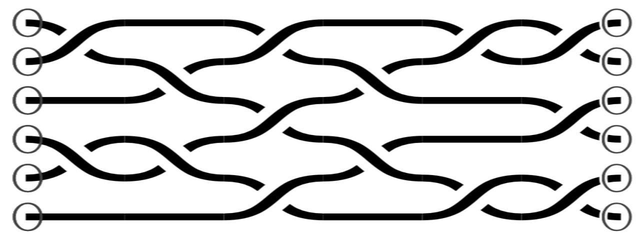

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Answer – in general:

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more precisely, the ground state TQFT is *generally covariant*

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$\Rightarrow$ band topologies differing by diffeos are equivalent to the TQFT:

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as seen by

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homotopy quotient

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homotopy quotient

Answer – for FQAH: 1.) Bloch Hamiltonian space

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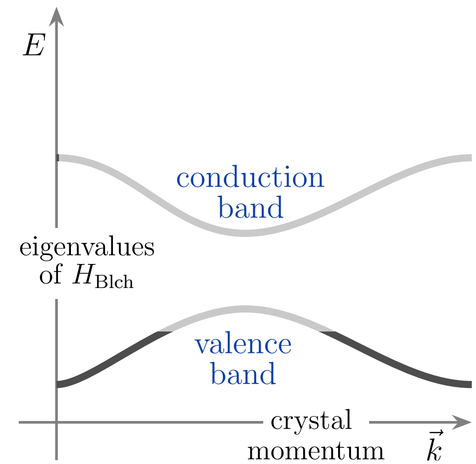
2-band systems



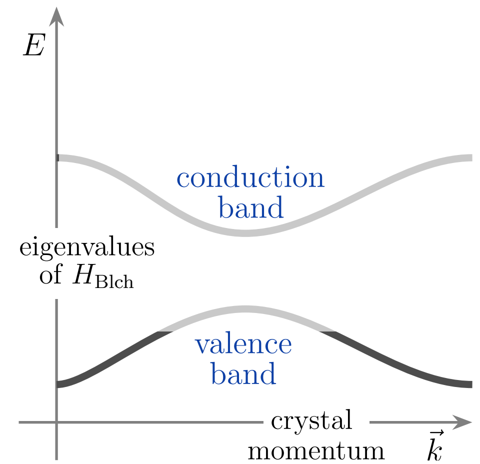
Answer – for FQAH: 1.) Bloch Hamiltonian space

2-band systems

$\Rightarrow$ Bloch Hamiltonian

$$\widehat{T}^2 \xrightarrow{H_{\mathrm{Blch}}} \mathrm{Mat}_{2\times 2}(\mathbb{C})$$

$$\vec{k} \longmapsto H_{\mathrm{Blch}}(\vec{k})$$



Answer – for FQAH: 1.) Bloch Hamiltonian space

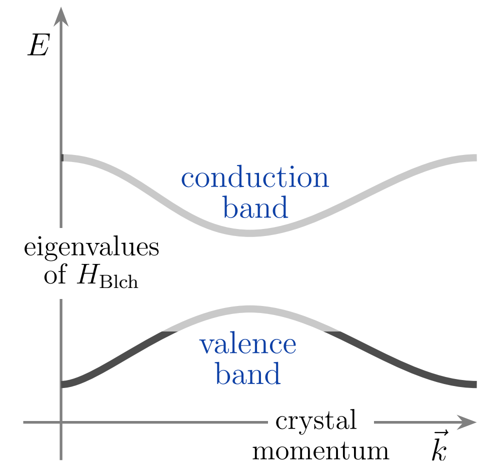
2-band systems

$\Rightarrow$ Bloch Hamiltonian expanded in Pauli σ

$$\widehat{T}^2 \xrightarrow{H_{\mathrm{Blch}}} \mathrm{Mat}_{2\times 2}(\mathbb{C})$$

$$\vec{k} \longmapsto H_{\mathrm{Blch}}(\vec{k})$$

$$\equiv h_0(\vec{k}) + \overbrace{\sum_{i=1}^{3} h_i(\vec{k})\,\sigma_i}^{\substack{\text{relative} \\ \text{Bloch Hamiltonian} \\ H(\vec{k})}}$$



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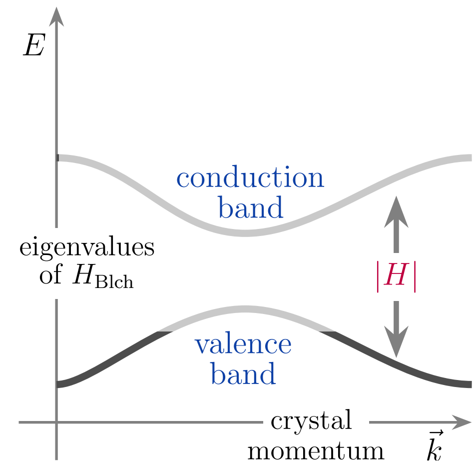

which are gapped

$$\Rightarrow \quad \big|H(\vec{k})\big| := \sqrt{\textstyle\sum_{i=1}^{3}\big(h_i(\vec{k})\big)^2} \;>\; 0\,,$$

Answer – for FQAH: 1.) Bloch Hamiltonian space

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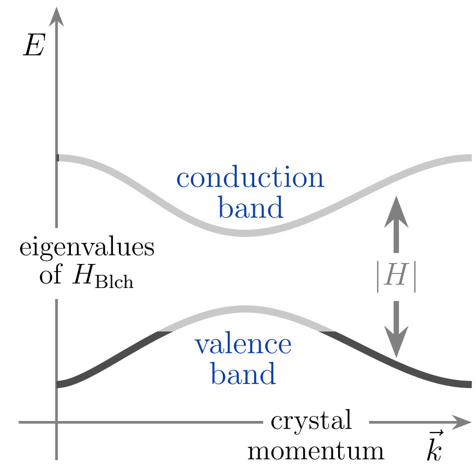
$$\begin{aligned} \widehat{T}^2 &\xrightarrow{H_{\mathrm{Blch}}} \mathrm{Mat}_{2\times 2}(\mathbb{C}) \\ \vec{k} &\longmapsto H_{\mathrm{Blch}}\big(\vec{k}\big) \\ &\equiv h_0\big(\vec{k}\big) + \overbrace{\textstyle\sum_{i=1}^{3} h_i\big(\vec{k}\big)\,\sigma_i}^{\substack{\text{relative}\\ \text{Bloch Hamiltonian}\\ H(\vec{k})}} \end{aligned}$$



which are gapped

$$\Rightarrow \quad \big|H(\vec{k}\,)\big| := \sqrt{\textstyle\sum_{i=1}^{3}\big(h_i(\vec{k}\,)\big)^2} \;>\; 0\,,$$

$\Rightarrow$ valence bundle is $\mathrm{Eig}_{-1}\big(H/|H|\big)$

Answer – for FQAH: 1.) Bloch Hamiltonian space

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$$
\begin{array}{ccl}
\widehat{T}^2 & \xrightarrow{H_{\mathrm{Blch}}} & \mathrm{Mat}_{2\times 2}(\mathbb{C}) \\
\vec{k} & \longmapsto & H_{\mathrm{Blch}}(\vec{k}) \\
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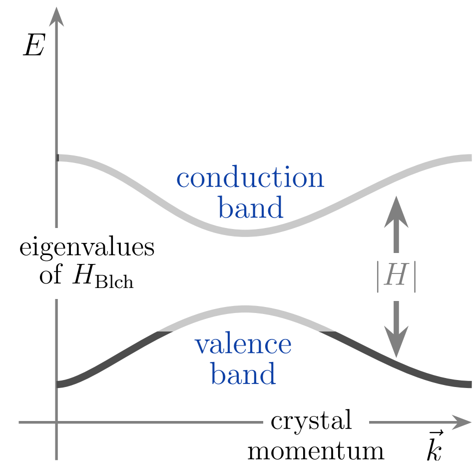
\end{array}
$$



which are gapped

$$\Rightarrow \quad \big|H(\vec{k}\,)\big| := \sqrt{\textstyle\sum_{i=1}^{3}\big(h_i(\vec{k}\,)\big)^2} \;>\; 0\,,$$

$\Rightarrow$ valence bundle is $\mathrm{Eig}_{-1}\big(H/|H|\big)$

$\Rightarrow$ Hamiltonian space is $\dfrac{\mathrm{U}(2)}{\mathrm{U}(1)\times\mathrm{U}(1)} \simeq S^2 \equiv \mathcal{A}$

$$
\begin{array}{ccl}
\widehat{T}^2 & \xrightarrow{\;\;H/|H|\;\;} & S^2 \subset \mathbb{R}^3 \\
\vec{k} & \longmapsto & \vec{h}(\vec{k})/|H|\,.
\end{array}
$$

Answer – for FQAH: 1.) Bloch Hamiltonian space

2-band systems

$\Rightarrow$ Bloch Hamiltonian expanded in Pauli σ

$$
\begin{array}{ccl}
\widehat{T}^2 & \xrightarrow{H_{\mathrm{Blch}}} & \mathrm{Mat}_{2\times 2}(\mathbb{C}) \\
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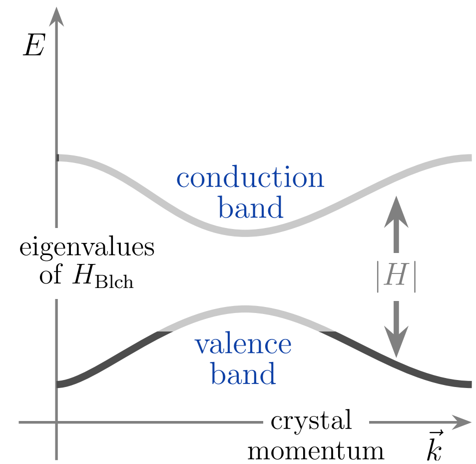
& & \equiv h_0(\vec{k}) + \underbrace{\textstyle\sum_{i=1}^{3} h_i(\vec{k})\,\sigma_i}_{\substack{\text{relative} \\ \text{Bloch Hamiltonian} \\ H(\vec{k})}}
\end{array}
$$



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\end{array}
$$

Answer – for FQAH: 2.) topological phases

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$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

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$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$\begin{array}{c}\text{valence}\\\text{bundle}\\[1em]\mathcal{V}\\\downarrow\\\widehat{T}^2\end{array}$$

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\underset{\text{bundle}}{\overset{\text{valence}}{\mathcal{V}}} \arrow[rr] \arrow[d] \arrow[d, phantom, "(\mathrm{pb})", shift left=4] & & \underset{\text{line bundle}}{\overset{\text{tautological complex}}{\mathcal{L}}} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = & \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}
\end{tikzcd}
$$

Answer – for FQAH: 2.) topological phases

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\begin{tikzcd}
\underset{\text{bundle}}{\overset{\text{valence}}{\mathcal{V}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & & \underset{\text{line bundle}}{\overset{\text{tautological complex}}{\mathcal{L}}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & & \underset{\text{line bundle}}{\overset{\text{universal complex}}{E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C}}} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = & \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[rr, "\sim"] & & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\underset{\text{valence bundle}}{\mathcal{V}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \underset{\text{tautological complex line bundle}}{\mathcal{L}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \underset{\text{universal complex line bundle}}{E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C}} \arrow[d] \\
\widehat{T}^2 \xrightarrow{H/|H|} S^2 = & \underbrace{\frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}}_{\text{fragile band topology}} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[r, "\sim"] & B\mathrm{U}(1)\,,
\end{tikzcd}
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\begin{tikzcd}
\underset{\text{bundle}}{\overset{\text{valence}}{\mathcal{V}}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \underset{\text{line bundle}}{\overset{\text{tautological complex}}{\mathcal{L}}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \underset{\text{line bundle}}{\overset{\text{universal complex}}{E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C}}} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = \underset{\substack{\text{fragile}\\\text{band topology}}}{\frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[rr, "\sim"', "\substack{\text{stable}\\\text{band topology}}"] & & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

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$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

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\begin{tikzcd}
\underset{\text{bundle}}{\overset{\text{valence}}{}} \mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \underset{\text{line bundle}}{\overset{\text{tautological complex}}{}} \mathcal{L} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \underset{\text{line bundle}}{\overset{\text{universal complex}}{}} E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \xrightarrow{H/|H|} S^2 = \underset{\substack{\text{fragile} \\ \text{band topology}}}{\frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[rr, "\sim"', "\substack{\text{stable} \\ \text{band topology}}"] & & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\underset{\text{bundle}}{\overset{\text{valence}}{\mathcal{V}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \overset{\text{tautological complex line bundle}}{\mathcal{L}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \overset{\text{universal complex line bundle}}{E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C}} \arrow[d] \\
\widehat{T}^2 \xrightarrow{H/|H|} S^2 = & \underset{\substack{\text{fragile}\\\text{band topology}}}{\dfrac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}} \arrow[r, hook] & \underset{\substack{\text{stable}\\\text{band topology}}}{\displaystyle\bigcup_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)}} \arrow[r, "\sim"] & B\mathrm{U}(1)\,,
\end{tikzcd}
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$\Rightarrow$ topological phases characterized by an integer

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\underset{\text{bundle}}{\overset{\text{valence}}{}} \mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \underset{\text{line bundle}}{\overset{\text{tautological complex}}{}} \mathcal{L} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \underset{\text{line bundle}}{\overset{\text{universal complex}}{}} E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \xrightarrow{H/|H|} S^2 = \underbrace{\frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}}_{\substack{\text{fragile} \\ \text{band topology}}} \arrow[r, hook] & \bigcup_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[rr, "\sim"] & & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

stable
band topology

$\Rightarrow$ topological phases characterized by an integer

$C := \big[H/|H|\big]$

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\text{valence bundle} & & \text{tautological complex line bundle} & & \text{universal complex line bundle} \\
\mathcal{V} \arrow[rr] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & & \mathcal{L} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 \arrow[r, equal] & \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[r, "\sim"] & B\mathrm{U}(1)\,, \\
& & \text{fragile band topology} & \text{stable band topology} &
\end{tikzcd}
$$

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$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

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\mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{L} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)} \arrow[r, hook] & \bigcup\limits_{c \in \mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[r, "\sim"] & B\mathrm{U}(1)\,, \\
& \text{fragile band topology} & \text{stable band topology} &
\end{tikzcd}
$$

$\Rightarrow$ topological phases characterized by an integer

$$C := \big[H/|H|\big] \;\in\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big)$$

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

valence bundle — tautological complex line bundle — universal complex line bundle

$$
\begin{tikzcd}
\mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{L} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[r, "\sim"] & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

fragile band topology — stable band topology

$\Rightarrow$ topological phases characterized by an integer

$$C := \big[H/|H|\big] \;\;\in\;\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big)$$

winding number

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

valence bundle — tautological complex line bundle — universal complex line bundle

$$
\begin{tikzcd}
\mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{L} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \xrightarrow{H/|H|} S^2 & = \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)} \arrow[r, hook] & \underset{c\in\mathbb{N}}{\bigcup} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[r, "\sim"] & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

fragile band topology — stable band topology

$\Rightarrow$ topological phases characterized by an integer

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C := \big[H/|H|\big] \;\in\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big)
$$

winding number

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\underset{\text{bundle}}{\overset{\text{valence}}{\mathcal{V}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \underset{\text{line bundle}}{\overset{\text{tautological complex}}{\mathcal{L}}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & \underset{\text{line bundle}}{\overset{\text{universal complex}}{E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C}}} \arrow[d] \\
\widehat{T}^2 \xrightarrow{H/|H|} S^2 = \underset{\substack{\text{fragile} \\ \text{band topology}}}{\dfrac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}} \arrow[r, hook] & \underset{\substack{\text{stable} \\ \text{band topology}}}{\bigcup_{c \in \mathbb{N}} \dfrac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)}} \arrow[rr, "\sim"] & & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

$\Rightarrow$ topological phases characterized by an integer

$$C := \big[H/|H|\big] \;\in\; \underset{\substack{\textbf{winding}\\\textbf{number}}}{\pi_0 \, \mathrm{Map}\big(\widehat{T}^2,\, S^2\big)} \;\simeq\; \pi_0 \, \mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)$$

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

$$
\begin{tikzcd}
\underset{\text{bundle}}{\overset{\text{valence}}{\mathcal{V}}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] && \underset{\text{line bundle}}{\overset{\text{tautological complex}}{\mathcal{L}}} \arrow[rrr] \arrow[d] \arrow[drrr, phantom, "\text{(pb)}"] &&& \underset{\text{line bundle}}{\overset{\text{universal complex}}{E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C}}} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = & \underset{\text{band topology}}{\overset{\text{fragile}}{\frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}}} \arrow[r, hook] & \underset{\text{band topology}}{\overset{\text{stable}}{\bigcup_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)}}} \arrow[rr, "\sim"] && B\mathrm{U}(1)\,,
\end{tikzcd}
$$

$\Rightarrow$ topological phases characterized by an integer

$$
C := \big[H/|H|\big] \;\in\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big) \;\simeq\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)
$$

winding number **Chern number**

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

valence bundle — tautological complex line bundle — universal complex line bundle

$$
\begin{tikzcd}
\mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{L} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \xrightarrow{H/|H|} S^2 & = \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[r, "\sim"] & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

fragile band topology — stable band topology

$\Rightarrow$ topological phases characterized by an integer

$$
C := \big[H/|H|\big] \;\in\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big) \;\simeq\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big)
$$

winding number — **Chern number**

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

valence bundle — tautological complex line bundle — universal complex line bundle

$$
\begin{tikzcd}
\mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{L} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)} \arrow[r, hook] & \underset{c\in\mathbb{N}}{\bigcup} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[r, "\sim"] & B\mathrm{U}(1)\,,
\end{tikzcd}
$$

fragile band topology — stable band topology

$\Rightarrow$ topological phases characterized by an integer

$$
C := \big[H/|H|\big] \;\in\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big) \;\simeq\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big) \;\simeq\; \mathbb{Z}
$$

winding number — **Chern number**

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

valence bundle | tautological complex line bundle | universal complex line bundle

$$
\begin{tikzcd}
\mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{L} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & EU(1) \underset{U(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = \frac{U(1)}{U(1)\times U(1)} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{U(1+c)}{U(1)\times U(c)} \arrow[r, "\sim"] & BU(1)\,,
\end{tikzcd}
$$

fragile band topology | stable band topology

$\Rightarrow$ topological phases characterized by an integer

$$
C := \big[H/|H|\big] \;\in\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big) \;\simeq\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, BU(1)\big) \;\simeq\; \mathbb{Z}
$$

winding number | **Chern number**

$\Rightarrow$ fragile = stable, for topological phases

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

valence bundle — tautological complex line bundle — universal complex line bundle

$$\begin{tikzcd} \mathcal{V} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathcal{L} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\ \widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[r, "\sim"] & B\mathrm{U}(1)\,, \end{tikzcd}$$

fragile band topology — stable band topology

$\Rightarrow$ topological phases characterized by an integer

$$C := \big[H/|H|\big] \;\in\; \pi_0 \,\mathrm{Map}\big(\widehat{T}^2,\, S^2\big) \;\simeq\; \pi_0\,\mathrm{Map}\big(\widehat{T}^2,\, B\mathrm{U}(1)\big) \;\simeq\; \mathbb{Z}$$

winding number — **Chern number**

$\Rightarrow$ fragile = stable, for topological phases

but not so for topological order...

Answer – for FQAH: 2.) topological phases

$\Rightarrow$ valence bundle is pullback of tautological/universal line bundle

valence bundle — tautological complex line bundle — universal complex line bundle

$$
\begin{tikzcd}
\mathcal{V} \arrow[rr] \arrow[d] \arrow[rrd, phantom, "\text{(pb)}"] & & \mathcal{L} \arrow[rr] \arrow[d] \arrow[rrd, phantom, "\text{(pb)}"] & & E\mathrm{U}(1) \underset{\mathrm{U}(1)}{\times} \mathbb{C} \arrow[d] \\
\widehat{T}^2 \arrow[r, "H/|H|"] & S^2 = \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)} \arrow[r, phantom] & \frac{\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)} \arrow[r, hook] & \bigcup\limits_{c\in\mathbb{N}} \frac{\mathrm{U}(1+c)}{\mathrm{U}(1)\times\mathrm{U}(c)} \arrow[r, "\sim"] & B\mathrm{U}(1)\,,
\end{tikzcd}
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fragile band topology — stable band topology

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fragile crystalline Chern phases

(apparently not computed before)

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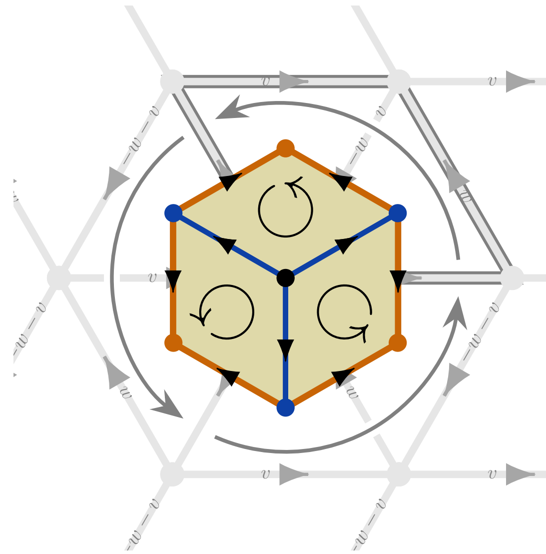

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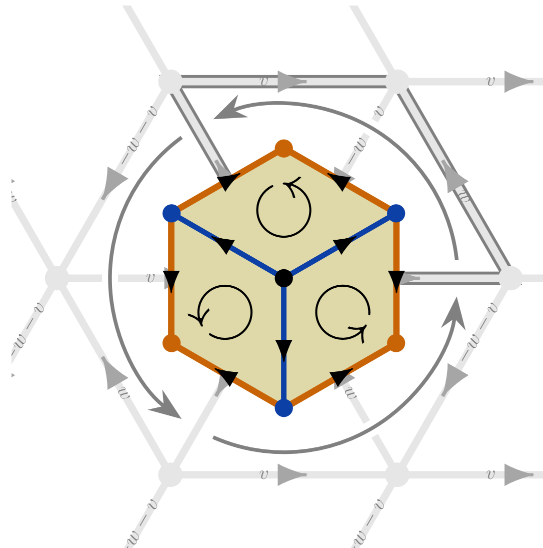

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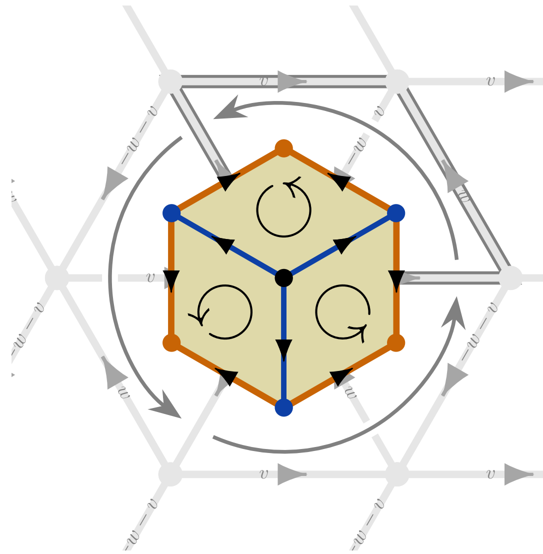

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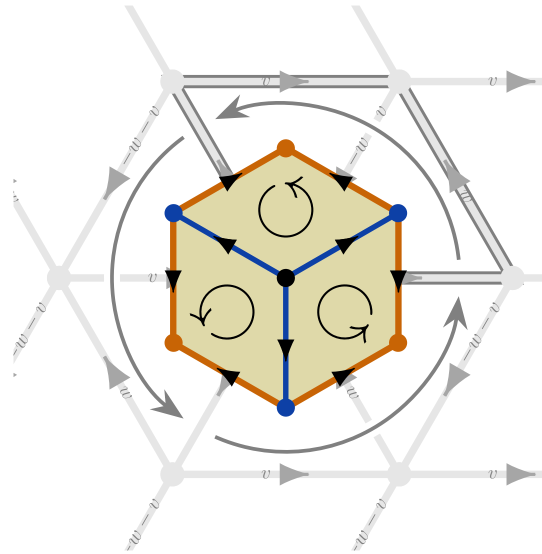

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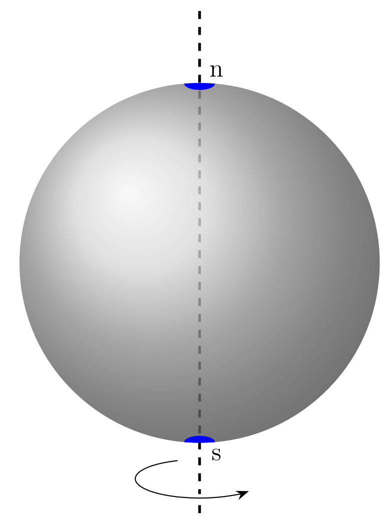

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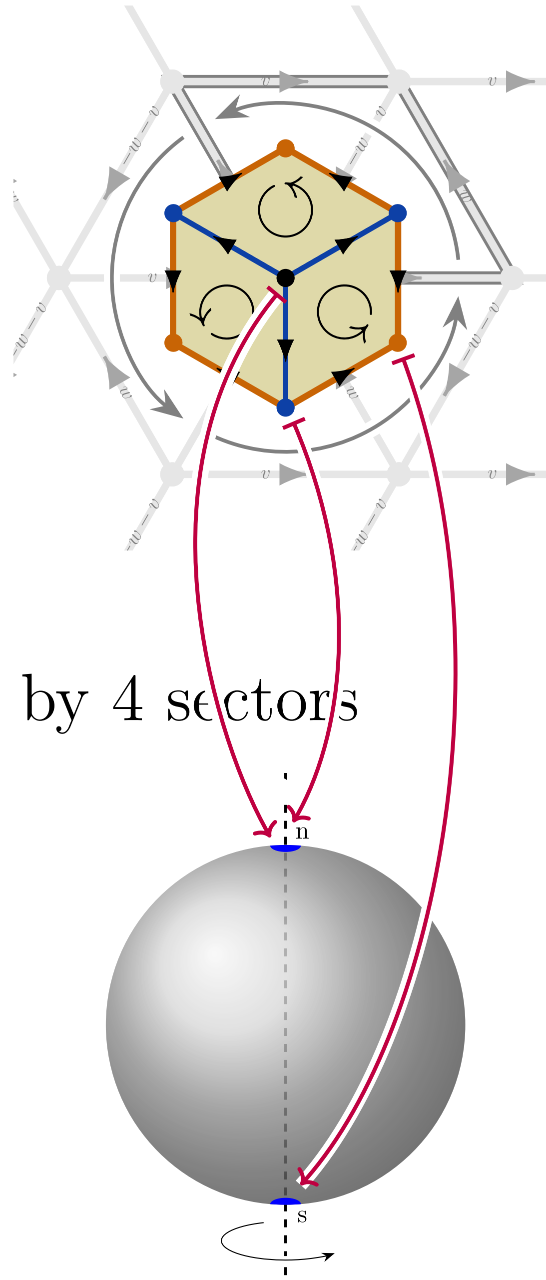

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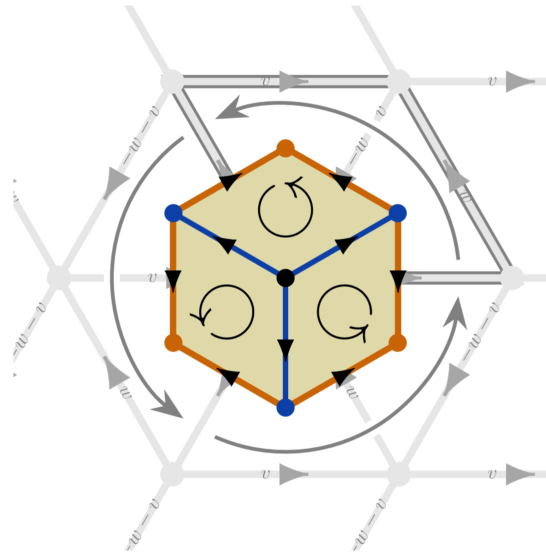

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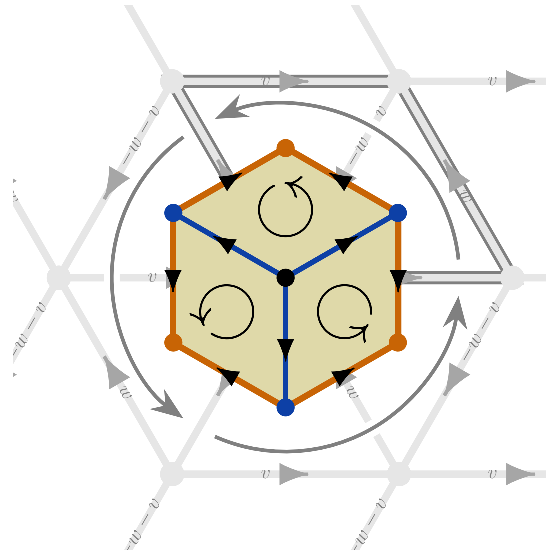

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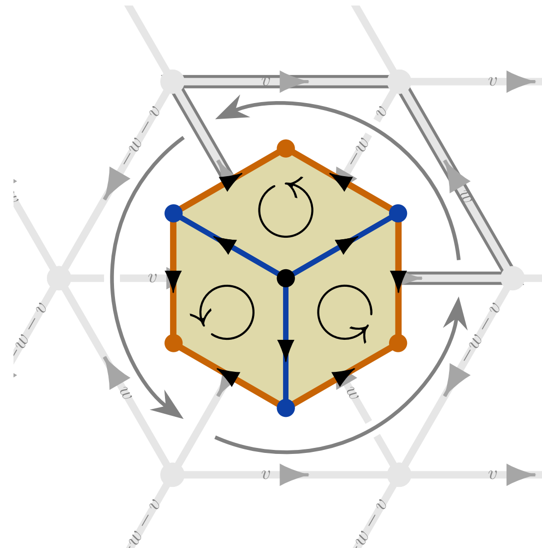

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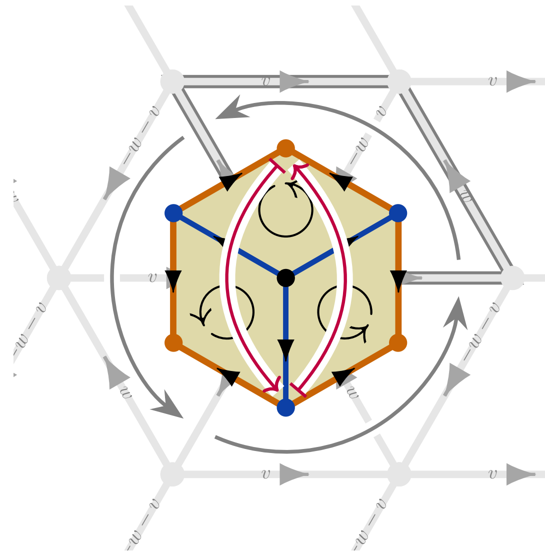

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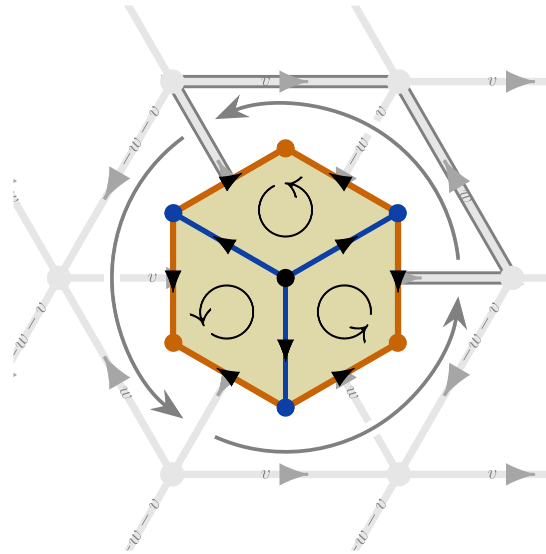

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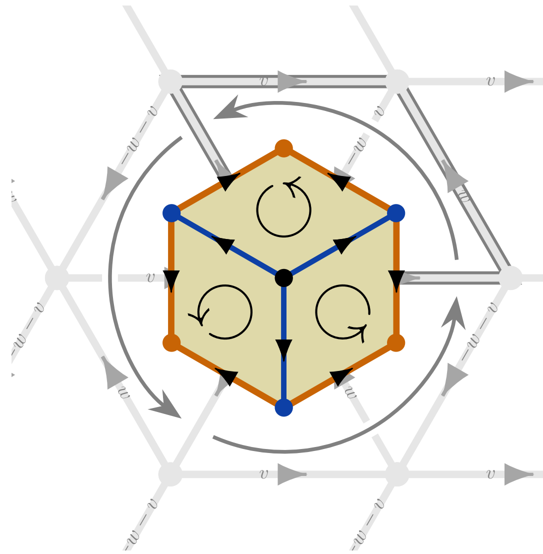

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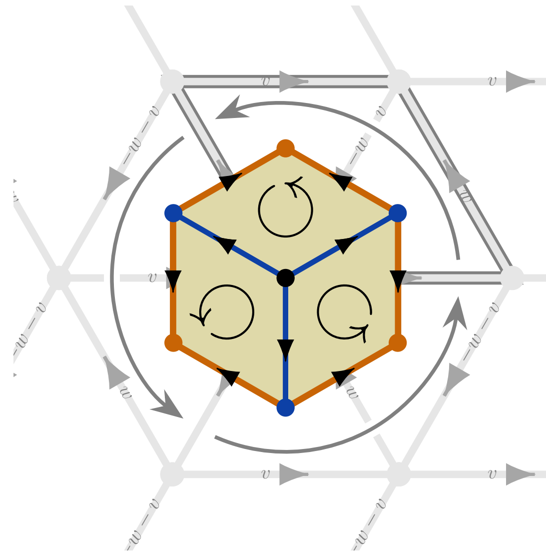

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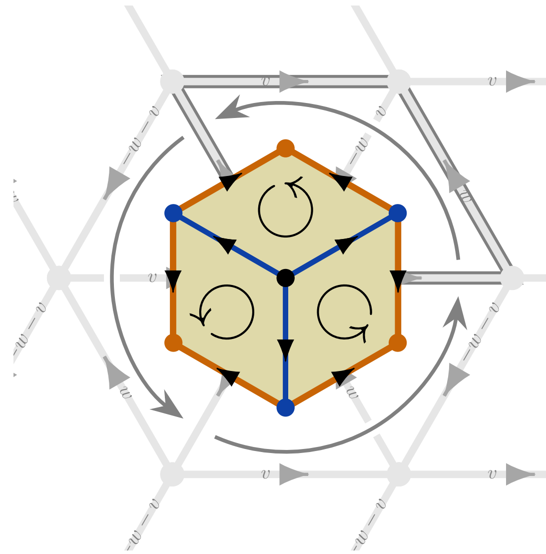

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standard Sym(3) irrep implements Z-and rotation-gate $R_y(2\pi/3)$

[Sati & S 2025, Prop. 3.61]

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Urs Schreiber on joint work with Hisham Sati:

surveying our preprint: [arXiv:2507.00138]

Identifying Anyonic Topological Order in FQAH Systems

Thanks!

CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS @ NYU ABU DHABI جامعة نيويورك أبوظبي

(September 2025) find these slides at: [ncatlab.org/schreiber/show/QMATH16]