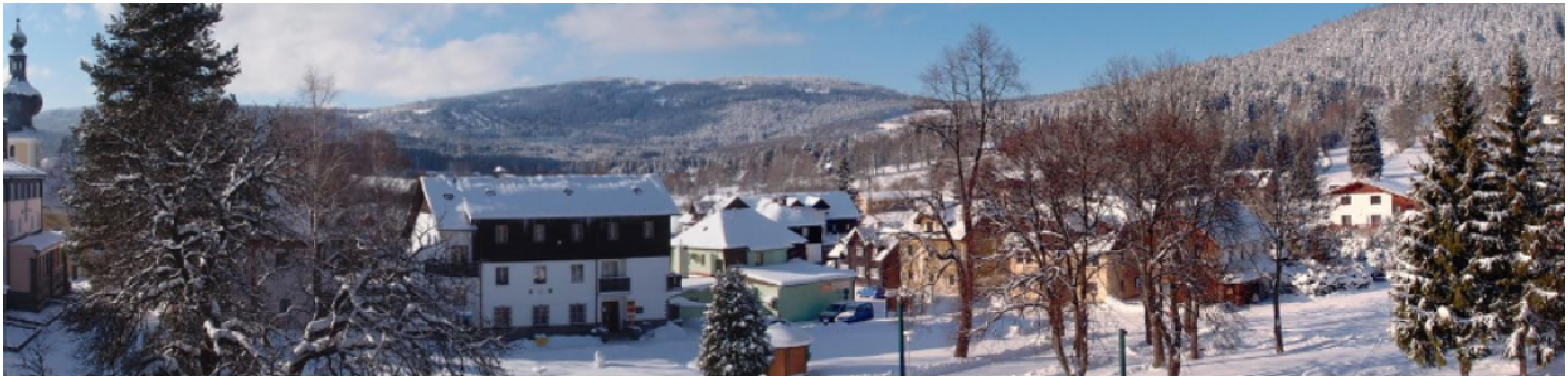
Urs Schreiber [a] on joint work with Hisham Sati [a]:

Introduction to *Hypothesis H*

lecture series at

45th Srní Winter School GEOMETRY AND PHYSICS
18-25 Jan 2025, Srní, Czechia



Session 1: **Non-Linear Flux Quantization in general**	[1]	(Context)
Session 2: **Flux Quantization on probe M5-Branes**		(Hypothesis H)
Session 3: **Topological Order on probe M5-Branes**	[2]	(Consequences)

Materials:

[1] *Flux Quantization*, Enzyclopedia of Mathematical Physics 2nd ed. 4 (2025) 281-324
[ncatlab.org/schreiber/show/Flux+Quantization]

[2] *Engineering of Anyons on M5 probes via Flux Quantization*, lecture notes (2025)
[ncatlab.org/schreiber/show/Engineering+of+Anyons+on+M5-Probes]

Today, to explain this phenomenon:

global definition of *higher gauge fields*	$\Leftrightarrow$	*flux-quantization* law	$\Leftrightarrow$	generalized cohomology

classical examples in electromagnetism:

Dirac charge-quantization of electromagnetic field in integral 2-cohomology
(makes quantum electron well-defined & stabilizes Abrikosov vortices in superconductors)

Dirac quantization of "statistical gauge field" in integral 2-cohomology
(quantizes electron number in effective field theory of quantum Hall effect)

famous proposals in 10D super-gravity:

flux-quantization of B-field in integral 3-cohomology
(makes quantum string well-defined & stabilizes NS5-branes)

flux-quantization of RR-field in K-theoretic cohomology
(stabilizes certain non-supersymmetric D-branes)

previous gap in 11D super-gravity: how to flux-quantize the C-field?

subtle because:

non-linear Maxwell equation	$\Rightarrow$	*non-abelian* cohomology

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If super-gravity does not motivate you:

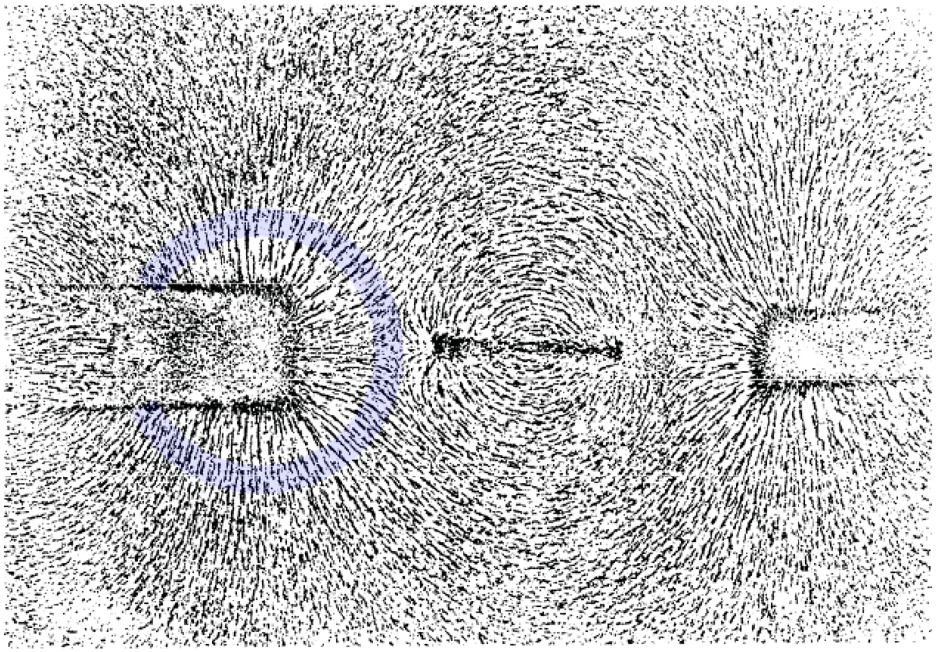
The third lecture will explain:

A non-standard flux-quantization
of the EM-field, in *2-Cohomotopy*,
which makes magnetic flux quanta behave
as seen in *fractional quantum Hall systems*.

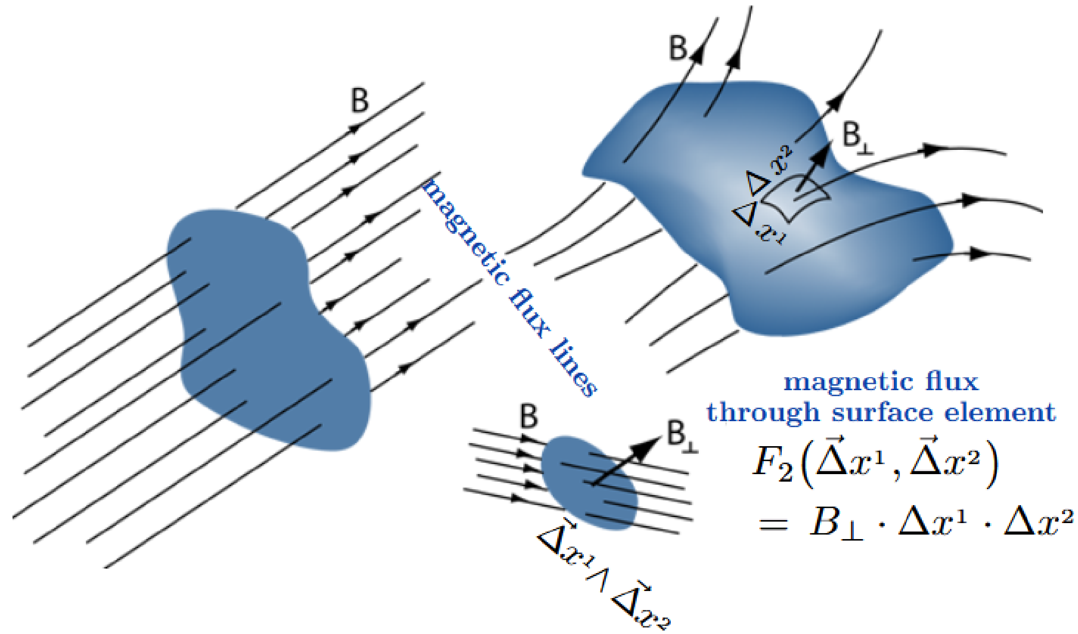
Question: But where would this exotic law come from?
Answer: Naturally from M5-probes of Seifert orbifolds in 11D supergravity.

Recall Electromagnetic Flux:

Faraday observed "lines of force" – now called flux of the magnetic field – concentrating towards the poles of rod magnets. In modern differential-geometric formulation, the density of these flux lines through any given surface-element is encoded in a differential 2-form F_2:



From Faraday's *Diary of experimental investigation*, vol VI, entry from 11th Dec. 1851, as reproduced in [Martin09]; the colored arc is our addition, for ease of comparison with the schematics on the right.



The density and orientation of magnetic field flux lines are encoded in a differential 2-form whose integral over a given surface is proportional to the total magnetic flux through that surface. (Graphics adapted from [Hyperphysics].)

More in detail, with respect to any foliation $X^4 \simeq \mathbb{R} \times X^3$ of a globally hyperbolic spacetime X^4 by spacelike Cauchy surfaces X^3, the spatial component of F_2 is the magnetic flux density B, while the Hodge dual (with respect to X^4) of the temporal component is the electric flux density E.

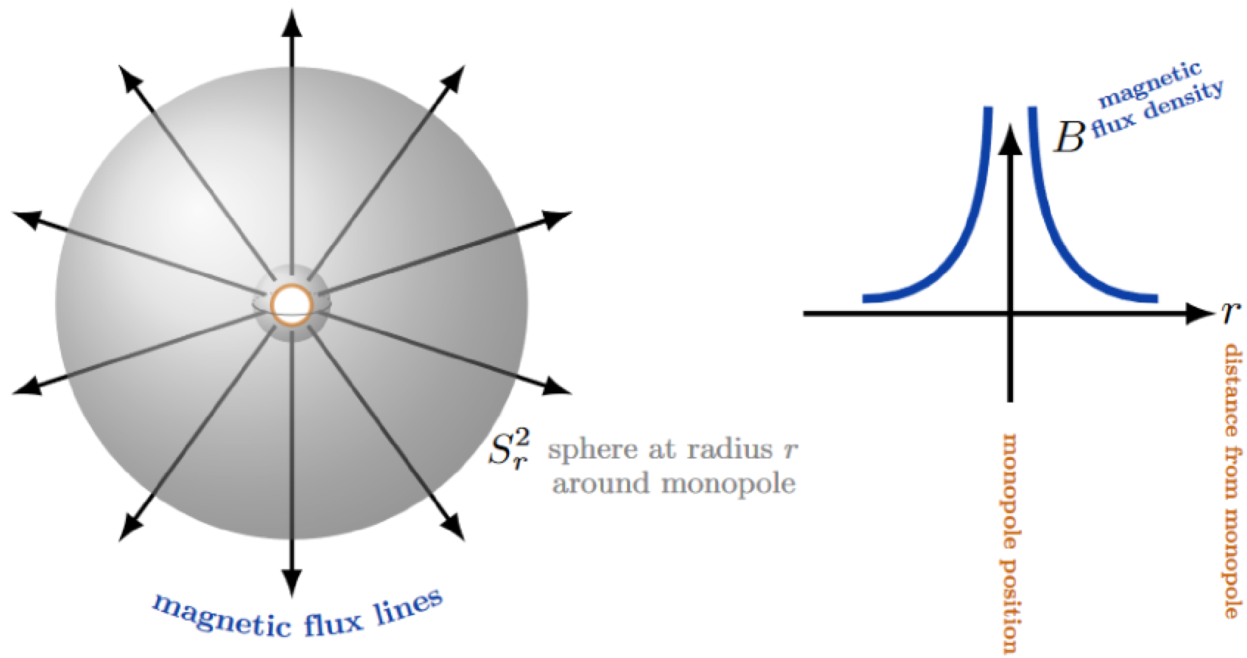
Electromagnetic flux density.	
X^4	spacetime 4-fold
$F_2 \in \Omega^2_{\mathrm{dR}}(X^4)$	Faraday tensor
$= \star\big(E_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j\big)$	electric flux density
$+ \; B_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j$	magnetic flux density

Example – Magnetic monopoles.

Imagining, with Dirac, that Faraday's rod magnet could be made *infinitely* long and thin, any one of its poles would look like an isolated mono-pole with flux concentrating towards it from all directions.

At the point of the idealized monopole itself, the flux density B per unit volume would diverge – a "singularity" much in the sense of black holes, which therefore is not to be regarded as part of space(-time): The spacetime domain on which to discuss the fluxes sourced by a magnetic monopole is (more on all this below in §2.2) not Minkowski spacetime $\mathbb{R}^{3,1}$ itself, but its complement around the worldline $\mathbb{R}^{0,1}$ of the would-be monopole.

$$\overset{\text{Faraday tensor}}{F_2} \;=\; \overset{\text{magnetic flux density}}{B(r)}\,\mathrm{dvol}_{S^2_r}\,, \qquad \overset{\substack{\text{Gauß law}\\ \text{away from singular locus}}}{\mathrm{d}\,F_2 \;=\; 0}\,, \qquad \int_{S^2_r} F_2 \;=\; \begin{array}{l}\text{integrated magnetic flux}\\ \text{through any sphere}\\ \text{around the monopole,}\\ \text{prop. to monopole charge}\end{array}$$



ambient space $\mathbb{R}^3 \setminus \{0\}$

magnetic **monopole**

$$\mathbb{R}^{3,1} \setminus \mathbb{R}^{0,1} \underset{\text{homeo}}{\simeq} \mathbb{R}^{0,1} \times \left(\mathbb{R}^3 \setminus \{0\}\right) \underset{\text{homeo}}{\simeq} \mathbb{R}^{0,1} \times R^1_{>0} \times S^2 \underset{\text{hmtp}}{\simeq} S^2 \,.$$

As such, magnetic monopoles are the **singular 0-branes** of electromagnetism (cf. §2.2) – in theory: Whether

Example – Abrikosov Vortices.

However, in the EM-field there are also **solitonic 1-branes** which are experimentally well-established as the *Abrikosov vortices* formed in type II superconductors within a transverse magnetic field [Abrikosov 1957] [Loudon et al. 2009] [Timm 2020, §6.5]. These may be regarded as *strings* approximated by a Nambu-Goto action [Nielsen et al. 1973] [Beekman et al. 2011].

$$\overset{\text{Faraday tensor}}{F_2} \;=\; \overset{\text{magnetic flux density}}{B(r)}\,\mathrm{d}x^1\wedge \mathrm{d}x^2\,, \qquad \overset{\text{Gauß law}}{\mathrm{d}\,F_2 \;=\; 0}\,, \qquad \int_{\mathbb{R}_1\times\mathbb{R}_2} F_2 \;=\; \int B\,\mathrm{d}x^1\,\mathrm{d}x^2 \;=\; \begin{array}{l}\text{total magnetic flux}\\ \text{prop. to \# vortices}\end{array}$$

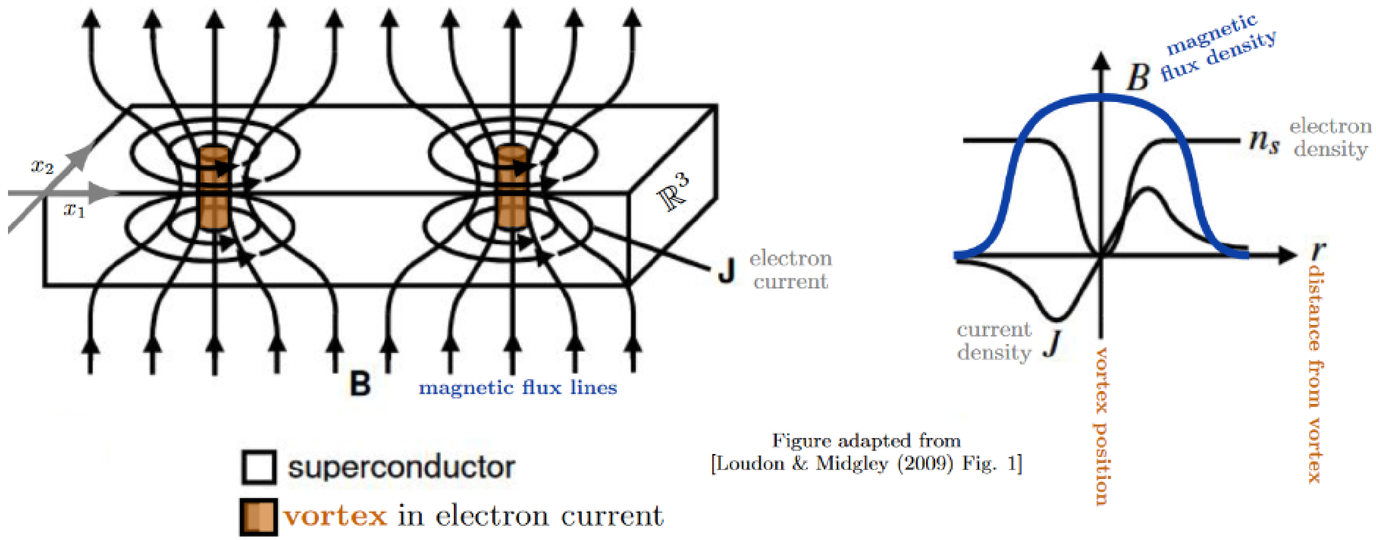


Figure adapted from [Loudon & Midgley (2009) Fig. 1]

Higher Flux Densities.

On this backdrop of ordinary electromagnetic flux (§2.1) and of the general rule for measuring flux sourced by singular branes or solitonic branes (§2.2) it clearly makes sense to consider physical theories of higher gauge fields whose precise nature remains to be discussed, but whose flux densities are reflected in higher-degree differential forms

$$F^{(i)} \in \Omega_{\mathrm{dR}}^{\deg_i}(X^D)\,, \tag{2}$$

these possibly being of different field species to be labeled by a finite index set $I \in \mathrm{FinSet}$ and jointly to be denoted as follows:

$$\vec{F} \equiv \Big\{ F^{(i)} \in \Omega_{\mathrm{dR}}^{\deg_i}(X^D) \Big\}\,. \tag{3}$$

Such higher flux densities appear in higher dimensional supergravity, namely as "super-partners" of the gravitino field that cannot be accounted for by the graviton itself. In particular, in $D = 10$ supergravity and $D = 11$ supergravity these higher flux densities are known under the (now) fairly standard symbols shown on the right, along with the standard name of the corresponding singular branes (the "higher-dimensional monopoles"), e.g. [Blumenhagen et al. 2013, §18.5].

	Field	**Flux**	**Singular source**
$D=4$ Maxwell theory	A-field	F_2	monopole 0-branes
$D=10$ supergravity	B-field	H_3	NS5-brane
		H_7	F1-branes
	RR-field	F_{8-p}	Dp-branes
$D=11$ supergravity	C-field	G_4	M5-branes
		G_7	M2-branes

(4)

Higher Maxwell-type Equations.

higher Maxwell-type equations of motion in duality-symmetric form

$$
\boxed{
\begin{array}{c}
\textbf{Bianchi identities} \\
\mathrm{d}\,\vec{F} \;=\; \vec{P}(\vec{F}) \\[6pt]
\star\, F \;=\; \vec{\mu}(\vec{F}) \\
\textbf{self-duality}
\end{array}
}
\quad
\boxed{
\begin{array}{l}
\overset{\text{flux species}}{I \in \mathrm{Set}}\,, \quad \overset{\text{flux degrees}}{\big(\mathrm{deg}_i \in \mathbb{N}_{\geq 1}\big)_{i \in I}}\,, \quad \vec{F} \equiv \overset{\text{flux densities}}{\Big(F^{(i)} \in \Omega^{\mathrm{deg}_i}_{\mathrm{dR}}\big(X^D\big)\Big)_{i \in I}} \\[10pt]
\underset{\text{flux self-sourcing}}{\vec{P} \text{ graded-symm. polynomial}}\;, \quad \underset{\text{vacuum permittivity}}{\vec{\mu} \text{ invertible matrix}}
\end{array}
}
\tag{6}
$$

Concretely:

- $\vec{P}$ is an I-tuple of graded-symmetric polynomials with rational coefficients in I variables of degrees $\vec{\deg}$,
- $\vec{\mu}$ is a linear endomorphism on the vector space spanned by these variables.

Example 2.9 (Motion of the ordinary electromagnetic fluxes).
The classical Maxwell equations expressed in terms of differential forms are as shown on the left (e.g. [Frankel 1997, §3.5 & §7.2b]), with their "premetric" form shown on the right.
Here the differential 3-form J_3 embodies the density of an electric current carrying an electric field and inducing a magnetic field.
This kind of *external* or *background* source term, where the source is not given by (a polynomial in) the flux densities themselves, does not fit into the Definition 2.6 and will be disregarded for the purpose of the present discussion, meaning that we focus on the special case of Maxwell's equations "in vacuum".

$$\boxed{\begin{array}{rcl} \mathrm{d}\,F_2 &=& 0 \\ \mathrm{d}\star F_2 &=& J_3 \end{array}} \quad \longleftarrow \left\{ \begin{array}{l} \boxed{\begin{array}{rcl} \mathrm{d}\,F_2 &=& 0 \\ \mathrm{d}\,G_2 &=& J_3 \end{array}} \\[1.5em] \boxed{G_2 \;=\; \star F_2} \end{array} \right. \tag{8}$$

$$\boxed{\begin{array}{rcl} \mathrm{d}\,F_2 &=& 0 \\ \mathrm{d}\star F_2 &=& 0 \end{array}} \quad \longleftarrow \left\{ \begin{array}{l} \boxed{\begin{array}{rcl} \mathrm{d}\,F_2 &=& 0 \\ \mathrm{d}\,G_2 &=& 0 \end{array}} \\[1.5em] \boxed{G_2 \;=\; \star F_2} \end{array} \right. \tag{9}$$

Example 2.10 (Motion of unbounded RR-field fluxes). The equations of motion of the RR-field fluxes in $D = 10$ supergravity in the case of vanishing B-field-fluxes are often taken to be as follows (e.g. [Mkrtchyan & Valach 2023])

$$
\boxed{
\begin{array}{rcl}
\mathrm{d}\,F_{2\bullet+\sigma} &=& 0 \\
\mathrm{d}\star F_{2\bullet+\sigma} &=& 0 \\
\multicolumn{3}{c}{\forall\, 2\bullet+\sigma \leq 5} \\
\star F_5 &=& F_5 \text{ if } \sigma = 1
\end{array}
}
\;\longleftarrow\;
\begin{array}{l}
\boxed{\mathrm{d}\,F_{2\bullet+\sigma} \;=\; 0 \quad \forall\, 2\bullet+\sigma}
\\[2em]
\boxed{F_{(10-2\bullet-\sigma)} \;=\; \star\, F_{2\bullet+\sigma}}
\end{array}
\qquad
\begin{array}{l}
\bullet \in \mathbb{N}
\\[1em]
\sigma = \left\{
\begin{array}{ll}
0 & \text{for type IIA} \\
1 & \text{for type IIB}
\end{array}
\right.
\end{array}
\tag{10}
$$

and, more generally, those with non-vanishing B-field as follows:

$$
\boxed{
\begin{array}{rclrcl}
\mathrm{d}\,F_{2\bullet+\sigma} &=& H_3 \wedge F_{2\bullet+\sigma-2} & \mathrm{d}\,H_3 &=& 0 \\
\mathrm{d}\star F_{2\bullet+\sigma} &=& H_3 \wedge \star F_{D-2\bullet-\sigma+2} & \mathrm{d}\,\star H_3 &=& \cdots
\end{array}
}
\;\longleftarrow\;
\begin{array}{l}
\boxed{
\begin{array}{rcl}
\mathrm{d}\,F_{2\bullet+\sigma} &=& H_3 \wedge F_{2\bullet+\sigma-2}
\end{array}
\;
\begin{array}{rcl}
\mathrm{d}\,H_3 &=& 0 \\
\mathrm{d}\,H_7 &=& \cdots
\end{array}
}
\\[2em]
\boxed{
F_{D=2\bullet-\sigma} \;=\; \star\, F_{2\bullet+\sigma} \qquad H_7 \;=\; \star\, H_3
}
\end{array}
\tag{11}
$$

Beware, while these equations are now often stated in this form, and while this is the form that motivates the traditional *Hypothesis K* (§4.1), it is at least subtle to see them in entirety as actually arising from ordinary $D = 10$ supergravity (namely from KK-compactification of $D = 11$ supergravity, in the case $\sigma = 0$), since in that context:

- The fluxes F_0 and F_{10} are not actually present: They are from *massive* type IIA, which has its own subtleties.
- The flux H_7 has a non-linear Bianchi ($\mathrm{d}H_7 = -F_4 \wedge F_4 + F_2 \wedge F_6$) which does not fit the pattern (cf. Ex. 2.13).

Example 2.11 (Motion of self-dual higher gauge field fluxes). Since Def. 2.6 regards *every* higher gauge theory (of Maxwell-type) as being "self-dual" in a sense, the equations of motion of flux densities of actual self-dual higher gauge fields — in the strict sense that one and the same flux density form is required to be Hodge dual to itself — are readily an example of Def. 2.6:

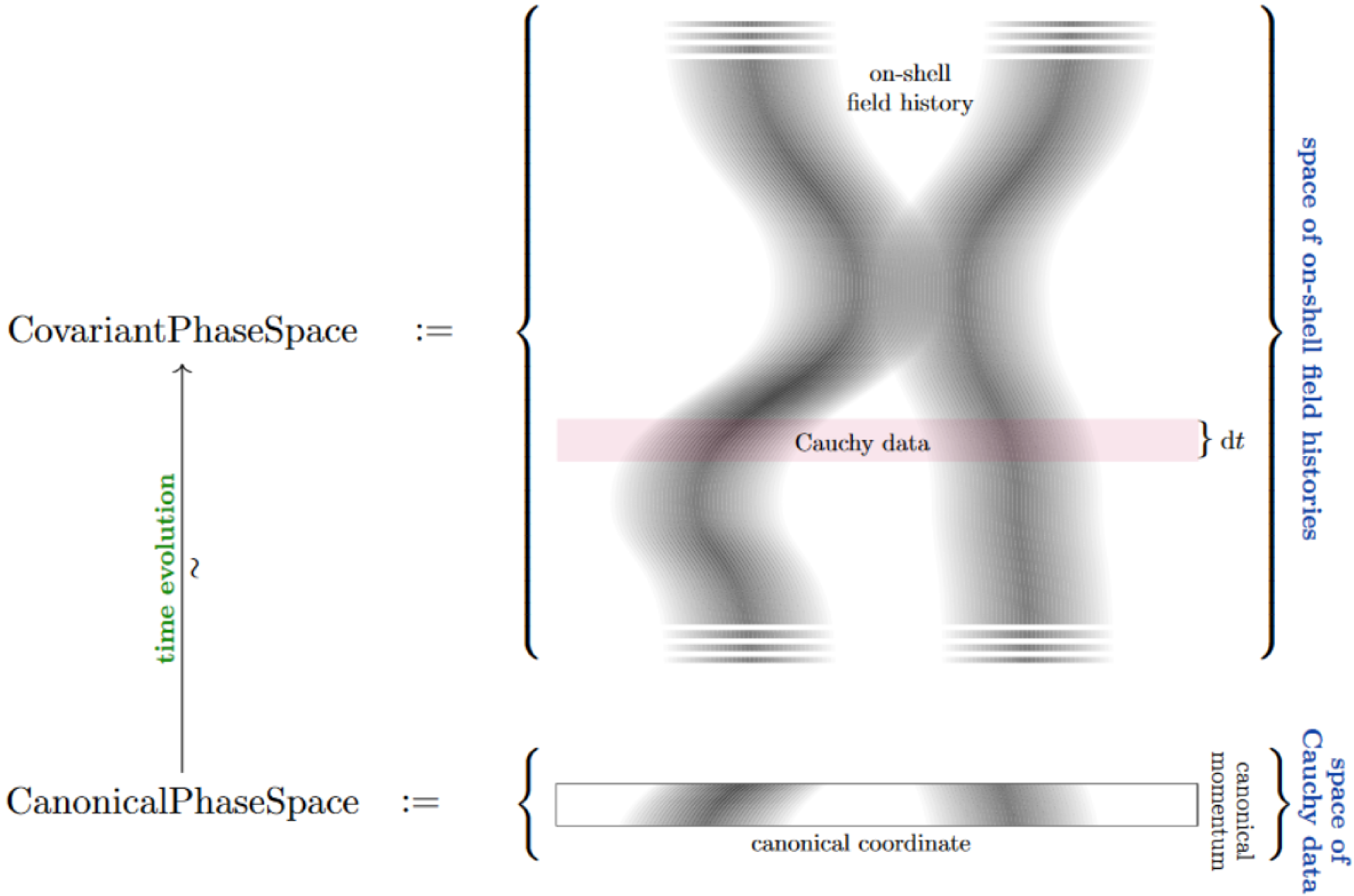
$$
\boxed{\begin{array}{c}\text{equations of motion of}\\ \text{self-dual higher gauge field}\\ \text{in } D = 4k+2\end{array}}
\;\longleftarrow\;
\left\{
\begin{array}{l}
\boxed{\mathrm{d}\, F_{D/2} \;=\; 0}\\[10pt]
\boxed{F_{D/2} \;=\; \star F_{D/2}}
\end{array}
\right.
\tag{12}
$$

Due to the properties of the square of the Hodge operator (7), this has non-trivial solutions iff the degree of the flux is odd, $\deg = 2k+1$, and hence iff spacetime dimension is $D = 4k+2$, $k \in \mathbb{N}$.

Example 2.12 (Motion of C-field fluxes). The equations of motion of the C-field in $D = 11$ supergravity (originally the "3-index A-field" due to [Cremmer et al. 1978] (cf. [Miemiec et al. 2006, p. 32]) are traditionally as shown on the left here, with their equivalent "duality-symmetric" reformulation [Bandos et al. 1998] shown on the right, cf. [Giotopoulos et al. 2024a].

$$
\boxed{\begin{aligned} \mathrm{d}\,G_4 &= 0 \\ \mathrm{d}\star G_4 &= -\tfrac{1}{2}G_4 \wedge G_4 \end{aligned}}
\;\longleftrightarrow\;
\left\{ \begin{array}{l} \boxed{\begin{aligned} \mathrm{d}\,G_4 &= 0 \\ \mathrm{d}\,G_7 &= -\tfrac{1}{2}G_4 \wedge G_4 \end{aligned}} \\[1em] \boxed{G_7 \;=\; \star G_4} \end{array} \right.
\tag{13}
$$

Recall Phase Space.



Higher Flux Solution Space.

Proposition 2.14 ([Sati & Schreiber2023b]). *On a globally hyperbolic spacetime $X^D \simeq \mathbb{R}^{0,1} \times X^d$, the solution space given higher Maxwell-equations of motion (Def. 2.6) is isomorphic to the solution of (just) the duality-symmetric Bianchi identities restricted (i.e.: pulled back to) to any Cauchy surface $\iota : X^d \hookrightarrow X^D$, there to be called the higher Gauss law:*

$$
\begin{aligned}
\underset{\substack{\text{Space of flux densities}\\ \text{on spacetime, solving}\\ \text{the equations of motion}}}{\mathrm{SolSpace}} \;&\equiv\; \left\{ \overset{\text{electromagnetic flux densities on spacetime}}{\vec{F} \equiv \Big(F^{(i)} \in \Omega^{\mathrm{deg}_i}_{\mathrm{dR}}\big(X^D\big)\Big)_{i \in I}} \;\middle|\; \begin{array}{c} \text{Bianchi identities} \\ \mathrm{d}\,\vec{F} = \vec{P}(\vec{F}) \\ \star\, F = \vec{\mu}(\vec{F}) \\ \text{self-duality} \end{array} \right\} \;\textbf{covariant form} \\
&\underset{\iota^*}{\simeq}\; \left\{ \overset{\text{magnetic flux densities on Cauchy surface}}{\vec{B} \equiv \Big(B^{(i)} \in \Omega^{\mathrm{deg}_i}_{\mathrm{dR}}\big(X^d\big)\Big)_{i \in I}} \;\middle|\; \overset{\text{Gauß law}}{\mathrm{d}\,\vec{B} = \vec{P}(\vec{B})} \right\} \;\textbf{canonical form}
\end{aligned}
\tag{14}
$$

The Idea of Flux-Quantization.

With the solution space (Prop. 2.14) of higher Maxwell-type equations of motion (Def. 2.6) in hand, the question of flux quantization is to further constrain the flux densities such that the total fluxes and their total source charges take values in some discrete space. The technical issue to be resolved here is that:

- this is a global condition on the flux densities: The local flux densities may take any value (compatible with the equations of motion) and yet the total accumulation of all these local contributions needs to be constrained;
- the evident idea of constraining the ordinary integrals of the flux densities (their "periods") makes sense only for closed differential forms and hence does not work for non-linear Bianchi identities (such as those of the C-field, Ex. 2.12, and the B&RR-field, Ex. 2.13).

To resolve this, one may first observe that:

- the integrals/periods of ordinary closed differential n-forms f_n over n-manifolds are in natural correspondence with their de Rham-classes, $[F_n] \in H^n_{\mathrm{dR}}(-)$, which in turn are equivalently their "deformation classes", namely their *concordance* classes: $H^n_{\mathrm{dR}}(-) \simeq \Omega^n_{\mathrm{dR}}(-)_{\mathrm{clsd}}/_{\mathrm{cncrdnc}}$;
- so that integrality of the closed flux density F_n is witnessed by an integral cohomology class $[\chi] \in H^n(X;\mathbb{Z})$ whose "de Rham character" image $\mathrm{ch}[\chi] \in H^n_{\mathrm{dR}}(X)$ coincides with the deformation class $[F_n]$;

$$
\begin{tikzcd}
 & & H^2(X;\mathbb{Z}) \arrow[d, "\mathrm{ch}"] & {[\chi]} \arrow[d, mapsto, "\text{character}"] & \text{integral charge} \\
 & \Omega^2_{\mathrm{dR}}(X)_{\mathrm{clsd}} \arrow[r, ->>] & H^2_{\mathrm{dR}}(X) & \mathrm{ch}[\chi] & \\
\text{flux density} & F_2 \arrow[r, mapsto] & {[F_2]} \arrow[ur, equal, bend right, "\text{integral total flux}"'] & &
\end{tikzcd}
$$

and, second, one may observe that this perspective generalizes [Fiorenza et al. 2023][Sati & Schreiber2023b]:

Higher Maxwell-type equations have a **characteristic L_∞-algebra $\mathfrak{a}$**: The flux densities are equivalently $\mathfrak{a}$-valued differential forms, and the Gauß law (14) is equivalently the condition that these be *closed* (i.e.: flat, aka "Maurer-Cartan elements"; in Italian SuGra literature: "satisfying an FDA").

$$\mathrm{SolSpace}(X^d) \simeq \left\{ \overset{\text{flux densities on Cauchy surface}}{\vec{B} \equiv \big(B^{(i)} \in \Omega^{\deg_i}_{\mathrm{dR}}(X^d)\big)_{i \in I}} \;\middle|\; \overset{\text{satisfying Gauß's law}}{\mathrm{d}\,\vec{B} = \vec{P}(\vec{B})} \right\}$$

$$\simeq \Omega_{\mathrm{dR}}(X^d;\, \mathfrak{a})_{\mathrm{clsd}} \quad \text{flat differential forms valued in characteristic } L_\infty\text{-algebra}$$

Also every topological space $\mathcal{A}$ (under mild conditions) has a characteristic L_∞-algebra: Its $\mathbb{R}$-rational **Whitehead bracket L_∞-algebra $\mathfrak{l}\mathcal{A}$.**

$$\begin{array}{c} \text{(homotopy type of)} \\ \text{a topological space} \end{array} \quad \mathcal{A} \underset{\mathbb{R}\text{-rationalization}}{\rightsquigarrow} \mathfrak{l}\mathcal{A} \quad \begin{array}{l} \text{Whitehead} \\ L_\infty\text{-algebra} \end{array}$$

The **nonabelian** Chern-Dold **character map** turns $\mathcal{A}$-valued maps into closed $\mathfrak{l}\mathcal{A}$-valued differential forms, generalizing the Chern character for $\mathcal{A} = \mathrm{KU}_0$.

$$\text{charge } \big(\chi : X^d \to \mathcal{A}\big) \underset{\text{character map in } \mathcal{A}\text{-cohomology}}{\longmapsto} \mathrm{ch}(\chi) \in \textstyle\int \Omega_{\mathrm{dR}}(X^d;\, \mathfrak{l}\mathcal{A})_{\mathrm{clsd}}$$

The **possible flux quantization laws** for a given higher gauge field are those spaces $\mathcal{A}$ whose Whitehead L_∞-algebra is the characteristic one.

$$\mathrm{FluxQuantLaws} = \left\{ \begin{array}{c} \mathcal{A} \\ \text{classifying} \\ \text{spaces} \end{array} \;\middle|\; \begin{array}{c} \mathfrak{l}\mathcal{A} \simeq \mathfrak{a} \\ \text{whose rational homotopy} \\ \text{encodes the Gauß law} \end{array} \right\}$$

Given a flux quantization law $\mathcal{A}$, the corresponding **higher gauge potentials** are deformations of the flux densities into characters of $\mathcal{A}$-valued maps, witnessing the flux densities as reflecting discrete charges quantized in $\mathcal{A}$-cohomology.
(It is not obvious that this reduces to the usual notion of gauge potentials, but it does.)

$$\begin{tikzcd} & & \chi \;\; \text{charge} \arrow[d, mapsto, "\text{character}"] \\ & & \mathrm{ch}(\chi) \arrow[dl, Rightarrow, bend left, "\widehat{A}\;\; \text{gauge potential}"] \\ \text{flux density} \;\; \vec{F} \arrow[r, mapsto, "\text{shape}"'] & \vec{F} & \end{tikzcd}$$

These non-perturbatively completed higher gauge fields form a *smooth higher groupoid*: the "canonical **differential $\mathcal{A}$-cohomology** moduli stack". Since these are now the flux-quantized on-shell fields, this is the **phase space** of the flux-quantized higher gauge theory (p. 11).

$$\begin{array}{c} \text{flux-quantized} \\ \text{phase space} \\ \text{stack is} \\ \widehat{\mathcal{A}}(X^d) := \\ \text{differential} \\ \mathcal{A}\text{-cohomology} \\ \text{moduli stack} \end{array} \left\{ \left(\begin{array}{lll} \vec{F} & \in & \Omega_{\mathrm{dR}}(X^d;\, \mathfrak{l}\mathcal{A})_{\mathrm{clsd}} \\ \chi & \in & \mathrm{Map}(X;\, \mathcal{A}) \\ \widehat{A} & : & \mathrm{ch}(\chi) \Rightarrow \vec{F} \end{array} \begin{array}{l} \text{flux} \\ \text{charge} \\ \text{gauge} \end{array} \right) \right\}$$

Key Observation:

Flux densities satisfying Gauß law are closed L_∞-valued differential forms. Remarkably, it follows that polynomials $\vec{P}$ defining Bianchi identities (6) and Gauss laws (14) are equivalently structure constants of L_∞-algebras $\mathfrak{a}$, such that the Bianchi/Gauß law is the closure/flatness condition on $\mathfrak{a}$-valued forms:

$$
\begin{array}{c}
\overbrace{\underset{\text{insert spacetime manifold here}}{\Omega^1_{\mathrm{dR}}\big(-;\,\mathfrak{a}\big)_{\mathrm{clsd}}} \;=\; \mathrm{Hom}_{\mathrm{dgAlg}}\big(\mathrm{CE}(\mathfrak{a}),\, \Omega^\bullet_{\mathrm{dR}}(-)\big)}^{\text{Sheaf of closed } L_\infty\text{-algebra-valued differential forms}} \;=\; \Big\{ \overset{\text{systems of flux densities}}{\vec{B} \equiv \big(B^{(i)} \in \Omega^{\deg_i}_{\mathrm{dR}}(-)\big)} \;\Big|\; \overset{\text{satisfying this Gauß law}}{\mathrm{d}\,\vec{B} \;=\; \vec{P}(\vec{B})} \Big\}
\\
\updownarrow
\\
\overset{\text{Chevalley-Eilenberg algebra of } L_\infty\text{-algebra}}{\mathrm{CE}(\,\mathfrak{a}\,)} \;=\; \overset{\text{free differential graded-commutative algebra on these graded generators}}{\mathbb{R}\Big[\big\{b^{(i)}_{\deg_i}\big\}_{i\in I}\Big]} \Big/ \overset{\text{satisfying these differential relations}}{\Big(\mathrm{d}\,\vec{b} \;=\; \vec{P}(\vec{b})\Big)}
\\
\updownarrow
\\
\overset{L_\infty\text{-algebra}}{\mathfrak{a}} \;=\; \overset{\text{graded vector space spanned by these graded generators}}{\mathbb{R}\Big\langle \big\{ v^{(i)}_{\deg_i - 1}\big\}_{i\in I}\Big\rangle} \quad \overset{\text{equipped with these higher Lie brackets}}{\big[v^{(i_1)}, \cdots, v^{(i_n)}\big] \;=\; \textstyle\sum_{i\in I} P^{(i)}_{i_1\cdots i_n}\, v^{(i)}}
\end{array}
\tag{19}
$$

Beware: Uncommon use of L_∞-valued forms.

Traditionally: gauge potentials — flatness is extra condition

Here: Flux densities — flatness = Bianchi identities !

Higher Flux Solution Space – Redux.

With Prop. 2.14, this means:

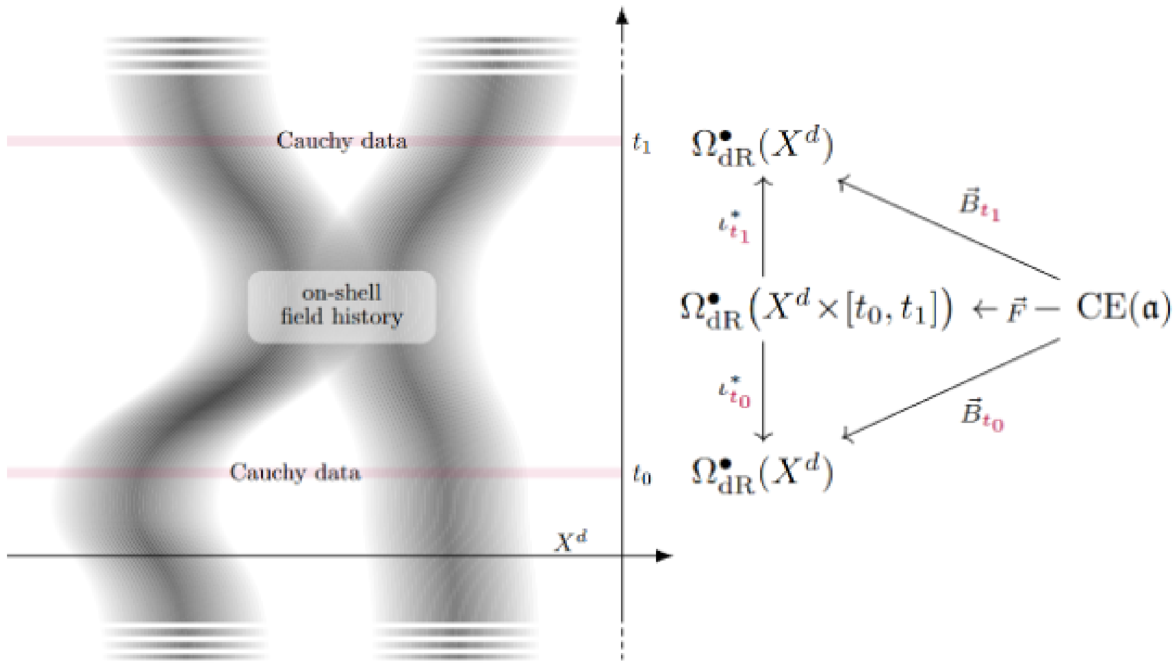
Proposition 3.1 (Flux solutions as closed L_∞-valued forms)**.** *Given a higher gauge theory of Maxwell-type (Def. 2.6) with Bianchi identities given by graded-symmetric polynomials $\vec{P}$ (6), its space of flux densities solving the higher Maxwell equations is identified with the space of closed differential forms with coefficients in the L_∞-algebra $\mathfrak{a}$ on $|I|$ $\vec{\deg}$-graded generators with structure constants $\vec{P}$:*

$$
\begin{aligned}
\underset{\substack{\text{Space of flux densities}\\ \text{on spacetime, solving}\\ \text{the equations of motion}}}{\mathrm{SolSpace}(X^D)} \;&\equiv\; \left\{ \overset{\text{electromagnetic flux densities on spacetime}}{\vec{F} \equiv \Big(F^{(i)} \in \Omega^{\deg_i}_{\mathrm{dR}}(X^D)\Big)_{i\in I}} \;\middle|\; \begin{array}{c} \text{Bianchi identities} \\ \mathrm{d}\,\vec{F} \;=\; \vec{P}(\vec{F}) \\ \star\,\vec{F} \;=\; \vec{\mu}(\vec{F}) \\ \text{self-duality} \end{array} \right\} \;\; \text{covariant form} \\
&\underset{\iota^*}{\simeq}\; \left\{ \overset{\text{magnetic flux densities on Cauchy surface}}{\vec{B} \equiv \Big(B^{(i)} \in \Omega^{\deg_i}_{\mathrm{dR}}(X^d)\Big)_{i\in I}} \;\middle|\; \begin{array}{c} \text{Gauß law} \\ \mathrm{d}\,\vec{B} \;=\; \vec{P}(\vec{B}) \end{array} \right\} \;\; \text{canonical form} \\
&\simeq\; \underset{\substack{\text{space of closed (flat)}\\ \mathfrak{a}\text{-valued differential forms}}}{\Omega^1_{\mathrm{dR}}\big(X^d;\,\mathfrak{a}\big)_{\mathrm{clsd}}}
\end{aligned}
\tag{20}
$$

Example 3.2. The characteristic L_∞-algebra of ordinary vacuum electromagnetism is the direct sum $b\mathfrak{u}(1) \oplus b\mathfrak{u}(1)$ of two copies of the line Lie 2-algebra, which by the previous example and Prop. 3.1 corresponds to:

$$\mathrm{SolSpace}_{\mathrm{EM}}(X^3) \;\simeq\; \Omega^1_{\mathrm{dR}}\big(X^3;\, b\mathfrak{u}(1) \times b\mathfrak{u}(1)\big)_{\mathrm{clsd}} \;\simeq\; \Omega^2_{\mathrm{dR}}(X^3) \times \Omega^2_{\mathrm{dR}}(X^3)\,.$$

Charges in Non-abelian de Rham cohomology.



Definition 3.3 (Non-abelian de Rham cohomology [Fiorenza et al. 2023, Def. 6.3]**).** Given an L_∞-algebra $\mathfrak{a}$ and a smooth manifold X^d, we say that a pair of flat *closed $\mathfrak{a}$-valued differential forms* $\vec B_0, \vec B_1 \in \Omega^1_{\mathrm{dR}}\big(X^d; \mathfrak{a}\big)_{\mathrm{clsd}}$ (16) are *cohomologous* iff they are concordant: iff there exists a closed $\mathfrak{a}$-valued differential form $\vec F$ on the cylinder over X^d whose pullback to the kth boundary component equals $\vec B_k$:

$$\vec B_0 \sim \vec B_1 \quad \Leftrightarrow \quad \exists\ \vec F \in \Omega^1_{\mathrm{dR}}\big(X^d \times [0,1];\, \mathfrak{a}\big)_{\mathrm{clsd}} \ \text{ with } \begin{cases} \vec B_1 = \iota_1^* \vec F\,, \\ \vec B_0 = \iota_0^* \vec F\,. \end{cases} \tag{22}$$

The quotient set by this equivalence relation is $\mathfrak{a}$-valued *nonabelian de Rham cohomology* of X^d:

$$H^1_{\mathrm{dR}}\big(X^d; \mathfrak{a}\big) \coloneqq \Omega^1_{\mathrm{dR}}\big(X^d;\, \mathfrak{a}\big)_{\mathrm{clsd}}/\!\sim\,. \tag{23}$$

Non-abelian cohomology.

Hence for flux-quantization
we need to understand
non-abelian de Rham cohomology
as an approximation to
non-abelian generalized cohomology.

Key Observation:

Reasonable cohomology theories have *classifying spaces*.

The archetypical examples are Eilenberg-MacLane spaces like $K(\mathbb{Z}, n)$ which classify ordinary cohomology such as integral cohomology, in any degree n. As n ranges, these EM-spaces happen to be loop spaces of each other, via weak homotopy equivalences: $K(\mathbb{Z}, n) \simeq \Omega K(\mathbb{Z}, n+1)$.

Generalizing from this classical example, one considers Whitehead-generalized cohomology theories which are classified by any sequence of pointed topological spaces $\{E_n\}_{n \in \mathbb{N}}$ equipped with weak homotopy equivalences $E_n \simeq \Omega E_{n+1}$, called a *spectrum of spaces* or just a *spectrum*.

This entails that each E_n is an infinite-loop space, which makes them be "abelian ∞-groups", reflecting the fact that the homotopy classes of maps into these spaces indeed have the structure of abelian groups.

Perhaps the most familiar example of such *abelian* generalized cohomology is topological K-theory, whose classifying space KU_0 may be identified with the space of Fredholm operators on an infinite-dimensional separable complex Hilbert space.

While Whitehead-generalized cohomology theory has received so much attention that it is now widely understood as the default or even the exclusive meaning of "generalized cohomology", historically long preceding it is the *non-abelian cohomology* of Chern-Weil theory, classified by the original classifying spaces BG of compact Lie groups G.

Unless G happens to be abelian itself, this nonabelian cohomology does not assign abelian cohomology groups, nor even any groups at all, but just pointed cohomology sets. Nevertheless, as the historical name "nonabelian cohomology" clearly indicates, these systems of cohomology sets may usefully be regarded as constituting a kind of cohomology theory, too.

ordinary cohomology — Eilenberg-MacLane space

$$H^n(X; \mathbb{Z}) \simeq \pi_0 \, \mathrm{Maps}\big(X,\, K(\mathbb{Z}, n)\big)$$

topological K-theory — space of Fredholm operators

$$K^0(X) \simeq \pi_0 \, \mathrm{Maps}\big(X,\, \mathrm{Fred}_{\mathbb{C}}\big)$$

Whitehead-generalized cohomology — stage in spectrum of spaces

$$E^n(X) \simeq \pi_0 \, \mathrm{Maps}\big(X,\, E_n\big)$$

nonabelian cohomology — classifing space of principal G-bundles

$$H^1(X; G) \simeq \pi_0 \, \mathrm{Maps}\big(X,\, BG\big)$$

coHomotopy — sphere

$$\pi^n(X) \simeq \pi_0 \, \mathrm{Maps}\big(X,\, S^n\big)$$

generalized nonabelian cohomology — any space

$$H^1(X, \Omega \mathcal{A}) := \pi_0 \, \mathrm{Maps}\big(X,\, \mathcal{A}\big)$$

$$= \pi_0 \left\{ \begin{tikzcd} X \arrow[rr, bend left=50, "\overset{\text{cocycle (map)}}{\mathcal{F}}"{name=U}] \arrow[rr, bend right=50, "\underset{\text{another cocycle}}{\mathcal{F}'}"'{name=D}] & & \mathcal{A} \arrow[from=U, to=D, Rightarrow, shorten=3pt, "\substack{\text{coboundary}\\ \text{(homotopy)}}"] \end{tikzcd} \right\}$$

Character maps on generalized cohomology. Moreover, it is classical that, over smooth manifolds, reasonable cohomology theories have their non-torsion content reflected in de Rham cohomology via *character maps*:

$$
\begin{array}{llll}
\text{Ordinary integral cohomology} & H^n(X;\mathbb{Z}) \xrightarrow{\text{de Rham map}} H^n_{\mathrm{dR}}(X) \simeq \mathrm{Hom}_{\mathrm{dgAlg}_{\mathbb{R}}}\big(\mathbb{R}[\omega_n],\, H^\bullet_{\mathrm{dR}}(X)\big) & \text{differential forms in degree } n \\
\text{Traditional nonabelian cohomology} & H^1(X;\,G) \xrightarrow{\text{Chern-Weil homomorphism}} \mathrm{Hom}_{\mathrm{dgAlg}_{\mathbb{R}}}\big(\mathrm{inv}^\bullet(\mathfrak{g}),\, H^\bullet_{\mathrm{dR}}(X)\big) & \text{differential forms for } \mathfrak{g}\text{-invariant polynomials} \\
\text{Topological K-theory} & K^0(X) \xrightarrow{\text{Chern character}} \mathrm{Hom}_{\mathrm{dgAlg}_{\mathbb{R}}}\big(\mathbb{R}[\omega_0,\omega_2,\omega_4,\cdots],\, H^\bullet_{\mathrm{dR}}(X)\big) & \text{differential forms in every even degree} \\
\text{abelian Whitehead-generalized cohomology} & E^n(X) \xrightarrow{\text{Chern-Dold character}} \mathrm{Hom}_{\mathrm{dgAlg}_{\mathbb{R}}}\big(\wedge^\bullet(\pi_\bullet(E)\otimes_{\mathbb{Z}}\mathbb{R})^\vee,\, H^{\bullet+n}_{\mathrm{dR}}(X)\big) & \text{differential forms for rational homotopy groups of the classifying space} \\
\text{Generalized non-abelian cohomology} & H^1(X;\,\Omega\mathcal{A}) \xrightarrow{\text{nonabelian character}} H^1_{\mathrm{dR}}(X;\,\mathfrak{l}A) := \mathrm{Hom}_{\mathrm{dgAlg}_{\mathbb{R}}}\big(\mathrm{CE}(\mathfrak{l}\mathcal{A}),\,\Omega^\bullet_{\mathrm{dR}}(X)\big)/_\sim & \text{differential forms with coefficients in Whitehead } L_\infty\text{-algebra}
\end{array}
\tag{25}
$$

L_∞-algebras approxima of spaces.

Proposition 3.7. Quillen-Sullivan-Whitehead L_∞-algebra cf. [Fiorenza et al. 2023, Prop. 4.23, 5.6 & 5.13]
For a topological space $\mathcal{A}$ which is

- *simply connected:* $\pi_0\mathcal{A} = *$, $\pi_1\mathcal{A} = 1$;
- *of rational finite type:* $\dim_{\mathbb{Q}}\big(H^n(\mathcal{A};\mathbb{Q})\big) < \infty$,

there is a polynomial dgc-algebra over $\mathbb{R}$, unique up to dga-isomorphism, whose

- *generators are the $\mathbb{R}$-rational homotopy groups of $\mathcal{A}$,*

$$\mathrm{CE}(\mathfrak{l}\mathcal{A}) \;=\; \Big(\wedge^\bullet\big(\pi_\bullet(\Omega\mathcal{A})\otimes_{\mathbb{Z}}\mathbb{R}\big)^\vee,\, \mathrm{d}_{\mathrm{CE}(\mathfrak{l}\mathcal{A})}\Big)$$

- *cochain cohomology is the ordinary real cohomology of $\mathcal{A}$*

$$H^\bullet\big(\mathrm{CE}(\mathfrak{l}\mathcal{A})\big) = H^\bullet(\mathcal{A};\mathbb{R})\,.$$

This dgc-algebra is known as the *minimal Sullivan model* of $\mathcal{A}$. By (15) it is the Chevalley-Eilenberg algebra of an L_∞-algebra which we denote $\mathfrak{l}\mathcal{A}$: The Whitehead bracket algebra structure on the $\mathbb{R}$-rational homotopy groups of the loop space (think of "$\mathfrak{l}(-)$" as standing for "Lie" or for "loops"):

$$\mathfrak{l}\mathcal{A} \;=\; \pi_\bullet(\Omega\mathcal{A})\otimes_{\mathbb{Z}}\mathbb{R}\,. \tag{26}$$

Circle: $\mathcal{A} \equiv S^1 \simeq B\mathbb{Z}$. $\big(\pi_\bullet(S^1) \otimes_{\mathbb{Z}} \mathbb{R}\big)^\vee \simeq \mathbb{R}\langle \omega_1 \rangle, \quad H^\bullet(S^1;\mathbb{R}) \simeq \mathbb{R}[\omega_1]$ Since $\mathbb{R}[\omega_1]$ is already the correct cohomology ring, it must be that $\mathrm{d}_{S^1} = 0$ and hence $\mathrm{CE}\big(\mathfrak{l}S^1\big) \;\simeq\; \mathbb{R}[\omega_1]/\big(\mathrm{d}\,\omega_1 = 0\big)$	While the circle is not simply connected, it is a "nilpotent space", and Sullivan's theorem actually applies in this generality. Nilpotent spaces have nilpotent fundamental group (e.g.: abelian) such that all higher homotopy groups are nilpotent modules (e.g.: trivial modules).
2-Sphere: $\mathcal{A} \equiv S^2$. $\big(\pi_\bullet(S^2) \otimes_{\mathbb{Z}} \mathbb{R}\big)^\vee \simeq \mathbb{R}\langle \omega_2, \omega_3 \rangle, \;\; H^\bullet(S^2;\mathbb{R}) \simeq \mathbb{R}[\omega_2]/(\omega_2^2)$ The differential on $\mathbb{R}[\omega_2, \omega_3]$ needs to remove ω_2^2 and ω_3 from cohomology, hence it must be that: $\mathrm{CE}\big(\mathfrak{l}S^2\big) \;\simeq\; \mathbb{R}\begin{bmatrix}\omega_3, \\ \omega_2\end{bmatrix} \Big/ \begin{pmatrix}\mathrm{d}\,\omega_3 = -\tfrac{1}{2}\omega_2 \wedge \omega_2 \\ \mathrm{d}\,\omega_2 = 0\end{pmatrix}$	The homotopy group coresponding to the generator ω_3 is that represented by the *complex Hopf fibration* $S^3 \xrightarrow{h_{\mathbb{C}}} S^2$.
3-Sphere: $\mathcal{A} \equiv S^3$. $\big(\pi_\bullet(S^3) \otimes_{\mathbb{Z}} \mathbb{R}\big)^\vee \simeq \mathbb{R}\langle \omega_3 \rangle, \quad H^\bullet(S^3;\mathbb{R}) \simeq \mathbb{R}[\omega_3]$ Since $\mathbb{R}[\omega_3]$ is already the correct cohomology ring, it must be that $\mathrm{d}_{S^3} = 0$ and hence $\mathrm{CE}\big(\mathfrak{l}S^3\big) \;\simeq\; \mathbb{R}[\omega_3]/\big(\mathrm{d}\,\omega_3 = 0\big)$	While $S^3 \simeq \mathrm{SU}(2)$, we see that $\mathfrak{l}\,\mathrm{SU}(2)$ is different from $\mathfrak{su}(2)$. But the former captures the cocycles of the latter: $\mathfrak{su}(2) \longrightarrow \mathfrak{l}\,\mathrm{SU}(2)$ $\mathrm{CE}\big(\mathfrak{su}(2)\big) \longleftarrow \mathrm{CE}\big(\mathfrak{l}\,\mathrm{SU}(2)\big)$ $\mathrm{tr}\big(-,[-,-]\big) \;\longleftarrow\!\shortmid\; \omega_3$
4-Sphere: $\mathcal{A} \equiv S^4$. $\big(\pi_\bullet(S^4) \otimes_{\mathbb{Z}} \mathbb{R}\big)^\vee \simeq \mathbb{R}\langle \omega_4, \omega_7 \rangle, \;\; H^\bullet(S^4;\mathbb{R}) \simeq \mathbb{R}[\omega_4]/(\omega_4^2)$ The differential on $\mathbb{R}[\omega_4, \omega_7]$ needs to remove ω_4^2 and ω_7 from cohomology, hence it must be that: $\mathrm{CE}\big(\mathfrak{l}S^4\big) \;\simeq\; \mathbb{R}\begin{bmatrix}\omega_7, \\ \omega_4\end{bmatrix} \Big/ \begin{pmatrix}\mathrm{d}\,\omega_7 = -\tfrac{1}{2}\omega_4 \wedge \omega_4 \\ \mathrm{d}\,\omega_4 = 0\end{pmatrix}$	The homotopy group corresponding to the generator ω_7 is that represented by the *quaternionic Hopf fibration* $S^7 \xrightarrow{h_{\mathbb{H}}} S^4$

Complex Projective space: $\mathcal{A} \equiv \mathbb{C}P^n$. $\big(\pi_\bullet(\mathbb{C}P^n)\otimes_{\mathbb{Z}}\mathbb{R}\big)^\vee \simeq \mathbb{R}\langle \omega_2,\, \omega_{2n+1}\rangle$, $H^\bullet(\mathbb{C}P^n;\mathbb{R}) \simeq \mathbb{R}[\omega_2]/(\omega_2^{n+1})$ The differential on $\mathbb{R}[\omega_2, \omega_{2n+1}]$ needs to remove ω_2^{n+1} from cohomology, hence it must be that: $\mathrm{CE}\big(\mathfrak{l}\mathbb{C}P^n\big) \;\simeq\; \mathbb{R}\left[\begin{array}{l}\omega_{2n+1},\\ \omega_2\end{array}\right] \Big/ \left(\begin{array}{ll}\mathrm{d}\,\omega_{2n+1} & = -\omega_2^{n+1}\\ \mathrm{d}\,\omega_2 & = 0\end{array}\right)$	This is related to the above sequence of examples by the fact that $\mathbb{C}P^n$ is an S^1-quotient of S^{2n+1}: $\begin{array}{ccc} S^1 & \hookrightarrow & S^{2n+1} \\ & & \big\downarrow \\ & & \mathbb{C}P^n \end{array}$
Infinite Projective space: $\mathcal{A} \equiv \mathbb{C}P^\infty \simeq B\mathrm{U}(1) \simeq B^2\mathbb{Z}$. $\big(\pi_\bullet(\mathbb{C}P^\infty)\otimes_{\mathbb{Z}}\mathbb{R}\big)^\vee \simeq \mathbb{R}\langle \omega_2\rangle$, $H^\bullet(\mathbb{C}P^n;\mathbb{R}) \simeq \mathbb{R}[\omega_2]$ Since $\mathbb{R}[\omega_2]$ is already the correct cohomology ring, it must be that $\mathrm{d}_{\mathbb{C}P^\infty} = 0$: $\mathrm{CE}\big(\mathfrak{l}\mathbb{C}P^\infty\big) \;\simeq\; \mathbb{R}\,[\omega_2]\,/\,\big(\mathrm{d}\,\omega_2 = 0\big)$	This is the Lie 2-algebra of the shifted circle group: $\mathfrak{l}B\mathrm{U}(1) \;\simeq\; b\,\mathfrak{u}(1)$
Eilenberg-MacLane space: $\mathcal{A} \equiv B^n\mathrm{U}(1) \simeq B^{n+1}\mathbb{Z}$. $\big(\pi_\bullet(B^{n+1}\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}\big)^\vee \simeq \mathbb{R}\langle \omega_{n+1}\rangle$, $H^\bullet(B^{n+1}\mathbb{Z}) \simeq \mathbb{R}[\omega_{n+1}]$ Since $\mathbb{R}[\omega_{n+1}]$ is already the correct cohomology ring, it must be that $\mathrm{d}_{B^{n+1}\mathbb{Z}} = 0$: $\mathrm{CE}\big(\mathfrak{l}B^{n+1}\mathbb{Z}\big) \;\simeq\; \mathbb{R}\,[\omega_{n+1}]\,/\,\big(\mathrm{d}\,\omega_{n+1} = 0\big)$	This is the Lie $(n+1)$-algebra of the circle $(n+1)$-group: $\mathfrak{l}B^n\mathrm{U}(1) \;\simeq\; b^n\,\mathfrak{u}(1)$
Classifying space: $\mathcal{A} \equiv BG$ of cpt. 1-conn. Lie group. $H^\bullet(BG;\mathbb{R}) \simeq \mathrm{inv}^\bullet(\mathfrak{g})$ the invar. polynomials on Lie alg. (Chern-Weil theory) Since $H^\bullet(BG;\mathbb{R})$ is already a free graded-symmetric ring it must be that $\mathrm{d}_{BG} = 0$ (cf. [Fiorenza et al. 2023, Lem. 8.2]): $\mathrm{CE}\big(\mathfrak{l}BG\big) \;\simeq\; \mathrm{inv}^\bullet(\mathfrak{g})/\big(\mathrm{d}_{BG} = 0\big)$	$\mathfrak{l}BG$ captures all the curvature invariants hence all the invariant flux densities of $\mathfrak{g}$-connections $A \in \Omega^1_{\mathrm{dR}}(X)\otimes\mathfrak{g}$, e.g. $\begin{array}{ccc} \mathrm{CE}\big(\mathfrak{l}B\mathrm{SU}(2)\big) & \longrightarrow & \Omega^\bullet_{\mathrm{dR}}(X) \\ \mathrm{tr}(-,-) & \mapsto & \delta_{ij}F_A^{(i)}\wedge F_A^{(j)} \end{array}$

Rational homotopy theory: Discarding torsion in nonabelian cohomology. From the perspective (above) that any topological space $\mathcal{A}$ serves as the classifying space of a generalized nonabelian cohomology theory, the idea of rational homotopy theory (survey in [Hess 2006]; [Fiorenza et al. 2023, §4]) becomes that of extracting the *non-torsion* content of such a cohomology theory, which we will see is, over smooth manifolds, that shadow of it that is reflected in the non-abelian de Rham cohomology (Def. 3.3) of $\mathfrak{l}\mathcal{A}$-valued differential forms.

regard spaces as classifying spaces

Homotopy theory	Rational	Sullivan model
Nonabelian cohomology	Non-torsion	de Rham cohomology

(28)

Hence to have a classifying space for the non-torsion part of $\mathcal{A}$-cohomology means to ask for:

The rationalization of $\mathcal{A}$:

A topological space	$L^{\mathbb{Q}}\mathcal{A}$
all whose homotopy groups have the structure of $\mathbb{Q}$-vector spaces	$\pi_n\big(L^{\mathbb{Q}}\mathcal{A}\big) \in \mathrm{Mod}_{\mathbb{Q}}$
equipped with a map from $\mathcal{A}$	$\mathcal{A} \xrightarrow{\;\eta^{\mathbb{Q}}_{\mathcal{A}}\;} L^{\mathbb{Q}}\mathcal{A}$
which induces isomorphisms on rationalized homotopy groups	$\pi_n(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow[\sim]{\;\eta^{\mathbb{Q}}_{\mathcal{A}} \otimes_{\mathbb{Z}} \mathbb{Q}\;} \pi_n\big(L^{\mathbb{Q}}\mathcal{A}\big) \otimes_{\mathbb{Z}} \mathbb{Q}$
and is universal with this property	

The Fundamental Theorem of dg-Algebraic Rational Homotopy Theory (review in [Fiorenza et al. 2023, Prop. 5.6]) says that the homotopy theory of rational spaces (simply-connected with fin-dim rational cohomology) is all encoded by their Whitehead L_∞-algebras (26) over the rational numbers. In particular, for X a CW-complex, the homotopy classes of maps into the rationalization $L^{\mathbb{Q}}\mathcal{A}$ (29) of a space $\mathcal{A}$ is identified with dg-homotopy classes of homomorphisms from the rational Sullivan model of $\mathcal{A}$ to the "piecewise $\mathbb{Q}$-polynomial de Rham complex" of the topological space X:

$$\mathrm{Map}\big(X,\, L^{\mathbb{Q}}\mathcal{A}\big)_{/\mathrm{homotopy}} \;\simeq\; \mathrm{Hom}_{\mathrm{dgAlg}}\Big(\mathrm{CE}\big(\mathfrak{l}^{\mathbb{Q}}\mathcal{A}\big),\, \Omega^\bullet_{\mathrm{P}\mathbb{Q}\mathrm{LdR}}(X)\Big)_{/\mathrm{concordance},} \tag{31}$$

The general non-abelian character map is now immediate [Fiorenza et al. 2023, Def. IV.2]: It is the cohomology operation induced by $\mathbb{R}$-rationalization of classifying spaces (32), seen under the non-abelian de Rham theorem (33):

$$\begin{tikzcd}
H^1(X;\Omega\mathcal{A}) \arrow[r, "\text{rationalization}"] \arrow[rrr, bend left, "\text{character map on } \mathcal{A}\text{-cohomology}"] \arrow[d, equal] & H^1\big(X;\, L^{\mathbb{Q}}\Omega\mathcal{A}\big) \arrow[r, "\text{extension}", "\text{of scalars}"'] \arrow[d, equal] & H^1\big(X;\, L^{\mathbb{R}}\Omega\mathcal{A}\big) \arrow[r, "\text{nonabelian}", "\text{de Rham theorem}"'] \arrow[d, equal] & H^1_{\mathrm{dR}}\big(X;\, \mathfrak{l}\mathcal{A}\big) \arrow[d, equal] \\
\pi_0\mathrm{Map}\big(X,\, \mathcal{A}\big) \arrow[r, "(\eta^{\mathbb{Q}}_{\mathcal{A}})_*"] & \pi_0\mathrm{Map}\big(X,\, L^{\mathbb{Q}}\mathcal{A}\big) \arrow[r, "(\eta^{\mathrm{ext}}_{L^{\mathbb{Q}}\mathcal{A}})_*"] & \pi_0\mathrm{Map}\big(X,\, L^{\mathbb{R}}\mathcal{A}\big) \arrow[r, "\sim", "\substack{\text{fundamental theorem}\\ \text{of dg-algebraic RHT}}"'] & \mathrm{Hom}_{\mathrm{dgAlg}}\big(\mathrm{CE}(\mathfrak{l}\mathcal{A}),\, \Omega^\bullet_{\mathrm{dR}}(X)\big)_{/\mathrm{cncrd}}
\end{tikzcd} \tag{35}$$

Conclusion.

Global flux quantization. Higher gauge fields on a spatial Cauchy surface satisfying their Gauß law constraint are equivalently closed L_∞-valued forms for some characteristic L_∞-algebra $\mathfrak{a}$; the global *total flux* is their class in nonabelian de Rham cohomology.
A compatible *flux quantization law* is a choice of classifying space $\mathcal{A}$ with Whitehead L_∞-algebra $\mathfrak{l}\mathcal{A} \simeq \mathfrak{a}$; and to quantize total flux is to lift it through the *character map* to nonabelian $\mathcal{A}$-cohomology.

choice of $\mathcal{A}$-cohomology
with $\mathfrak{l}\mathcal{A} \simeq \mathfrak{a}$

$$
\begin{tikzcd}
 & & H^1(X^d;\, \Omega\mathcal{A}) \arrow[d, "\mathrm{ch}_{\mathcal{A}}(X^d)", "\text{sourced flux}"'] \\
\ast \arrow[urr, dashed, "c", "\text{charge quantum in } \mathcal{A}\text{-cohomology}"'] \arrow[r, "\vec{B}"] & \Omega^1_{\mathrm{dR}}(X^d;\, \mathfrak{a})_{\mathrm{clsd}} \arrow[r, "\substack{\text{total}\\ \text{flux}}"] & H^1_{\mathrm{dR}}(X^d;\, \mathfrak{a})
\end{tikzcd}
$$

flux densities on Cauchy surface
satisfying their higher Gauß law

$\mathfrak{a}$-valued
de Rham cohomology

Example 3.9 (Flux quantization laws for ordinary electromagnetism). By Ex. 2.15, the characteristic L_∞-algebra of vacuum electromagnetism is two copies of the line Lie 2-algebra $b\mathfrak{u}(1)$. This is the Whitehead L_∞-algebra of the classifying space $B\mathrm{U}(1) \simeq B^2\mathbb{Z}$ and hence of its rationalization $B^2\mathbb{Q}$. Therefore — among many further variants — there are the following choices of flux quantization laws for ordinary electromagnetism:

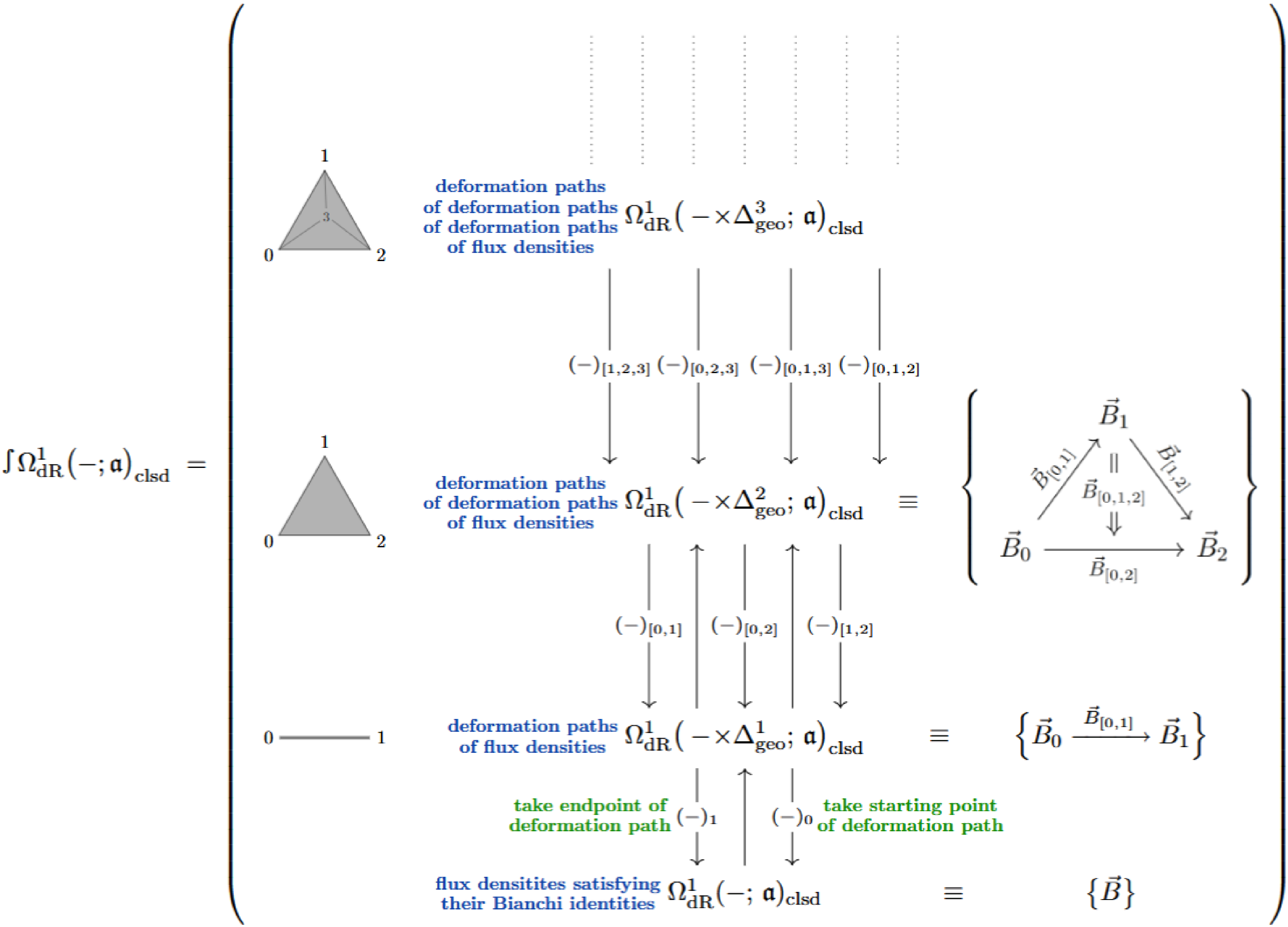
$\underbrace{B^2\mathbb{Q}}_{\mathrm{mag}} \times \underbrace{B^2\mathbb{Q}}_{\mathrm{el}}$	This choice imposes essentially *no* flux quantization (it does rule out irrational total fluxes) and as such was the tacit choice since [Maxwell 1865] until [Dirac 1931].
$\underbrace{B^2\mathbb{Z}}_{\mathrm{mag}} \times \underbrace{B^2\mathbb{Q}}_{\mathrm{el}}$	This choice imposes integrality of magnetic charge but no further condition on electric flux — common choice since [Dirac 1931], for instance in [Alvarez 1985, p. 299] [Brylinski 1993, §7.1][Freed 2000, Ex. 2.1.2].
$\underbrace{B^2\mathbb{Z}}_{\mathrm{mag}} \times \underbrace{B^2\mathbb{Z}}_{\mathrm{el}}$	This choice imposes integrality of both magnetic and electric charge — considered in [Freed et al. 2007b][Freed et al. 2007c][Becker et al. 2017, Rem. 2.3] [Lazaroiu & Shahbazi 2022][Lazaroiu & Shahbazi 2023]
$\underbrace{B^2\mathbb{Z}}_{\mathrm{mag}} \rtimes \underbrace{BK \ltimes B^2\mathbb{Z}}_{\mathrm{el}}$	For a finite group $K \to \mathrm{Aut}(\mathbb{Z})$ — this choice induces non-commutativity between EL/EL- and EL/M-fluxes, an example of a "non-evident" flux quantization condition considered in [Sati & Schreiber 2023c].

Outlook.

The full definition of
flux-quantized higher gauge fields
needs *geometric homotopy theory*
a.k.a. *higher topos theory*
where unification happens of:
differential forms & classifying spaces

$$
\begin{tikzcd}[column sep=huge]
\textbf{smooth sets} & \textbf{smooth } \infty\textbf{-groupoids} & \infty\textbf{-groupoids} \\
\mathrm{SmthSet} \arrow[r, hook, "\textbf{smooth structure}", "\text{(no gauge transf.)}"'] \arrow[d, equal] & \mathrm{SmthGrpd}_\infty \arrow[d, equal] & \mathrm{Grpd}_\infty \arrow[l, hook', "\textbf{higher gauge transf.}"', "\text{(discrete smooth struc.)}"] \arrow[d, equal] \\
L^{\mathrm{liso}}\,\mathrm{PSh}(\mathrm{CartSp}) \arrow[d, phantom, "\Cup"] & L^{\mathrm{lheq}}\,\mathrm{PSh}\big(\mathrm{CartSp},\, \mathrm{PSh}(\Delta)_{\mathrm{Kan}}\big) \arrow[d, phantom, "\Cup"] & \mathrm{PSh}(\Delta)_{\mathrm{Kan}} \arrow[d, phantom, "\Cup"] \\
\Omega^1_{\mathrm{dR}}\big(-;\mathfrak{a}\big)_{\mathrm{clsd}} \arrow[r, "\eta^{\int}", "\textbf{shape unit}"'] & \int \Omega^1_{\mathrm{dR}}\big(-;\mathfrak{a}\big)_{\mathrm{clsd}} & \mathcal{A} \arrow[l, "\mathbf{ch}_{\mathcal{A}}"', "\textbf{differential character}"] \\
\begin{array}{c}\textbf{moduli of}\\ \textbf{flux densities}\end{array} & \begin{array}{c}\textbf{deformations of}\\ \textbf{flux densities}\end{array} & \begin{array}{c}\textbf{moduli of}\\ \textbf{charges}\end{array}
\end{tikzcd}
$$

Here exists the moduli stack of flux densities:



The full definition of flux-quantized higher gauge fields.

this object that flux densities become comparable to their charges:

(i) There is an evident inclusion $\Omega^1_{\mathrm{dR}}(-;\mathfrak{a})_{\mathrm{clsd}} \xrightarrow{\text{shape unit}} \int \Omega^1_{\mathrm{dR}}(-;\mathfrak{a})_{\mathrm{clsd}}$ [Fiorenza et al. 2023, (9.3)], which we may identify as the *shape unit* of the moduli of flux densities;

(ii) given an identification $\mathfrak{a} \simeq \mathfrak{l}\mathcal{A}$ with a Whitehead L_∞-algebra (37), then the fundamental theorem of dg-algebraic rational homotopy theory (31) furthermore says [Fiorenza et al. 2023, Lem. 9.1] that we have a (homotopy-) equivalence to the $\mathbb{R}$-rationalization $L^{\mathbb{R}}\mathcal{A}$ of $\mathcal{A}$ (32), so that rationalization gives a *differential character map* [Fiorenza et al. 2023, Def. 9.2]:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\text{rationalization}"] \arrow[rrr, bend right, "\mathbf{ch}"', "\text{differential character map}"'] & L^{\mathbb{Q}}\mathcal{A} \arrow[r, "\substack{\text{extension}\\ \text{of scalars}}"] & L^{\mathbb{R}}\mathcal{A} \arrow[r, "\substack{\text{fundamental thm. of RHT}\\ \text{piecewise smooth version}}", "\sim"'] & \int \Omega^1_{\mathrm{dR}}(-;\mathfrak{l}\mathcal{A})_{\mathrm{clsd}}
\end{tikzcd}$$

Local flux quantization: Gauge potentials in differential cohomology. This way one may now *locally* implement flux quantization, by taking the higher gauge field fields on X^d to be *homotopies* deforming flux densities $\vec{B}$ into the differential character of local charges χ. On equivalence classes, this reproduces the quantization of total fluxes (37) and thereby lifts it to a local structure. Indeed, the higher gauge fields defined this way are the cocyles of the nonabelian *differential* $\mathcal{A}$-cohomology [Fiorenza et al. 2023, Def. 9.3].	$\begin{tikzcd} \widehat{X}^d \arrow[r, dashed, "\text{charges}", "\chi"'] \arrow[d, dashed, "\vec{B}", "\substack{\text{flux}\\ \text{densities}}"'] \arrow[dr, phantom, "\substack{\widehat{A}\ \Swarrow\ \text{gauge}\\ \text{potentials}}"] & \mathcal{A} \arrow[d, "\mathbf{ch}"', "\substack{\text{differential}\\ \text{character}}"] \\ \Omega^1_{\mathrm{dR}}(-;\mathfrak{a})_{\mathrm{clsd}} \arrow[r, "\eta^{\int}", "\substack{\text{shape}\\ \text{unit}}"'] & \int \Omega^1_{\mathrm{dR}}(-;\mathfrak{a})_{\mathrm{clsd}} \end{tikzcd}$ (39)