Urs Schreiber[a] on joint work with Hisham Sati[a]:

Introduction to *Hypothesis H*

lecture series at

45th Srní Winter School GEOMETRY AND PHYSICS

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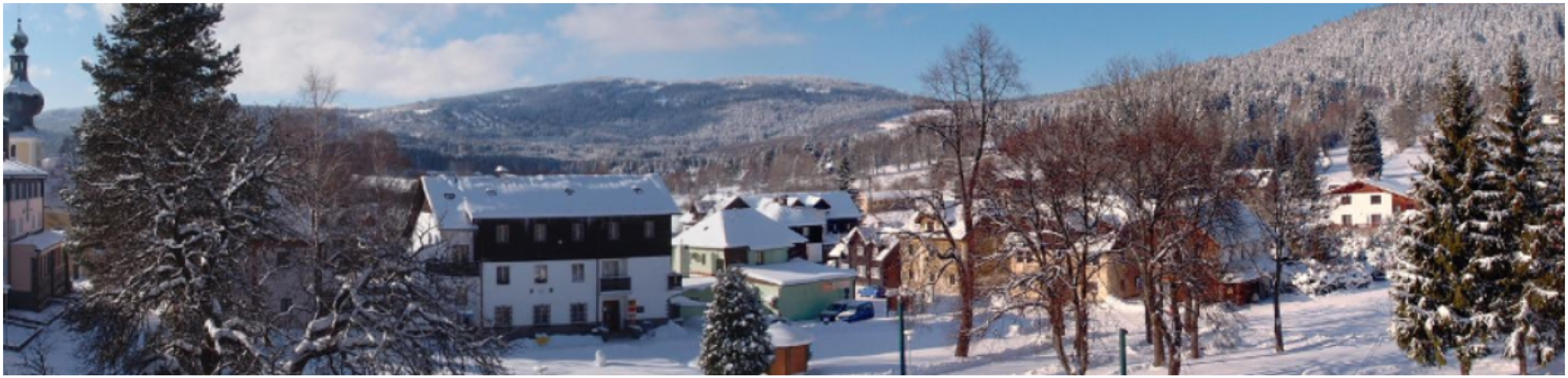


Session 1: **Non-Linear Flux Quantization in general** (Context)

[1]

Session 2: **Flux Quantization on probe M5-Branes** (Hypothesis H)

[2]

Session 3: **Topological Order on probe M5-Branes** (Consequences)

Materials:

[1] *Flux Quantization*, Enzyclopedia of Mathematical Physics 2nd ed. **4** (2025) 281-324
[ncatlab.org/schreiber/show/Flux+Quantization]

[2] *Engineering of Anyons on M5 probes via Flux Quantization*, lecture notes (2025)
[ncatlab.org/schreiber/show/Engineering+of+Anyons+on+M5-Probes]

Today, to explain Hypothesis H.

Given higher Maxwell-type equations of motion,
last time we saw the *rules by which to choose* flux-quantization laws.

But there is still a *choice* involved, namely of a cohomology theory.
That choice is a *choice of completion* of the higher gauge theory,
hence a *Hypothesis* about the non-perturbative physics it describes.

For the RR-field of 10D supergravity
a traditional choice is: $\boldsymbol{K}$-theoretic cohomology theory;
this choice could be called *Hypothesis* $\boldsymbol{K}$.

For the C-field of 11D supergravity,
and the B-field on its M5-probes
an admissible choice is a
twisted/twistorial form of
"Co-***H****omotopy* Theory", hence of "***H****omotopy* cohomology theory".

Hypothesis $\boldsymbol{H}$ states that: *This is the correct choice for M-theory.*

The Bianchi identities

are these:

A-field	$\mathrm{d}\,F_2^s = 0$	$\mathrm{d}\,G_4^s = 0$	**C-field**
self-dual B-field	$\mathrm{d}\,H_3^s = \phi_s^* G_4^s + \theta\, F_2^s\, F_2^s$	$\mathrm{d}\,G_7^s = \tfrac{1}{2} G_4^s\, G_4^s$	**dual C-field**

M5 probe $\quad \Sigma^{1,5\,|\,2\cdot\mathbf{8}_+} \xrightarrow[\text{1/2BPS immersion}]{\phi_s} X^{1,11\,|\,\mathbf{32}}$ $\quad$ **SuGra bulk**

The next two slides indicate the remarkable way in which these equations come about in 11D supergravity (technical, may be skipped).

Bianchi identities on M5-Probes of 11D SuGra via super-geometry. Consider the 11D super-tangent space

$$\underset{\text{super-Minkowski}}{\mathbb{R}^{1,10\,|\,\mathbf{32}}} \hookrightarrow \underset{\text{super-Poincaré}}{\mathfrak{isom}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big)} \twoheadrightarrow \underset{\text{Lorentz}}{\mathfrak{so}(1,10)}$$

with its super-invariant 1-forms (cf. [49, §2.1])

$$\mathrm{CE}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big) \simeq \underset{\text{super-transl. invar. forms}}{\Omega^{\bullet}_{\mathrm{dR}}\big(\mathbb{R}^{1,10|\mathbf{32}}\big)^{\mathrm{li}}} \simeq \mathbb{R}_{\mathrm{d}}\left[\begin{array}{l}(\Psi^\alpha)_{\alpha=1}^{32} \\ (E^a)_{a=0}^{10}\end{array}\right] \Big/ \left(\begin{array}{rcl}\mathrm{d}\,\Psi^\alpha &=& 0 \\ \mathrm{d}\,E^a &=& \big(\overline{\Psi}\,\Gamma^a\,\Psi\big)\end{array}\right).$$

Remarkably, the quartic Fierz identities entail that [19][84][49, Prop. 2.73]:

$$\left.\begin{array}{rcl} G_4^0 &:=& \tfrac{1}{2}\big(\overline{\Psi}\,\Gamma_{a_1 a_2}\,\Psi\big)\,E^{a_1}\,E^{a_2} \\ G_7^0 &:=& \tfrac{1}{5!}\big(\overline{\Psi}\,\Gamma_{a_1\cdots a_5}\,\Psi\big)\,E^{a_1}\cdots E^{a_5}\end{array}\right\} \in \underset{\text{fully super-invariant forms}}{\mathrm{CE}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big)^{\mathrm{Spin}(1,10)}} \quad \text{satisfy :} \quad \begin{array}{rcl}\mathrm{d}\,G_4^0 &=& 0 \\ \mathrm{d}\,G_7^0 &=& \tfrac{1}{2}G_4^0\,G_4^0\end{array}$$

To globalize this situation, say that an **11D super-spacetime** X is a super-manifold equipped with a super-Cartan connection, locally on an open cover $\widetilde{X} \twoheadrightarrow X$ given by

$$\left.\begin{array}{l}(\Psi^\alpha)_{\alpha=1}^{32} \\ (E^a)_{a=0}^{10} \\ \big(\Omega^{ab} = -\Omega^{ba}\big)_{a,b=0}^{10}\end{array}\right\} \in \Omega^1_{\mathrm{dR}}\big(\widetilde{X}\big) \quad \text{such that the super-torsion vanishes} \quad \mathrm{d}\,E^a - \Omega^a{}_b\,E^b = \big(\overline{\Psi}\,\Gamma^a\,\Psi\big),$$

and say that **C-field super-flux** on such a super-spacetime are super-forms with these co-frame components:

$$\begin{array}{rcl} G_4^s &:=& G_4 + G_4^0 &:=& \tfrac{1}{4!}(G_4)_{a_1\cdots a_4}E^{a_1}\cdots E^{a_4} + \tfrac{1}{2}\big(\overline{\Psi}\,\Gamma_{a_1 a_2}\,\Psi\big)E^{a_1}\,E^{a_2} \\ G_7^s &:=& G_7 + G_7^0 &:=& \tfrac{1}{7!}(G_4)_{a_1\cdots a_7}E^{a_1}\cdots E^{a_7} + \tfrac{1}{5!}\big(\overline{\Psi}\,\Gamma_{a_1\cdots a_5}\,\Psi\big)E^{a_1}\cdots E^{a_5}\end{array}$$

Theorem [49, Thm. 3.1]: On an 11D super-spacetime X with C-field super-flux (G_4^s, G_7^s):

The duality-symmetric super-Bianchi identity $\left\{\begin{array}{rcl}\mathrm{d}\,G_4^s &=& 0 \\ \mathrm{d}\,G_7^s &=& \tfrac{1}{2}\,G_4^s\,G_4^s\end{array}\right\}$ *is equivalent to* *the full 11D SuGra equations of motion!*

Next, on the super-subspace $\mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}_+} \xhookrightarrow{\phi_0} \mathbb{R}^{1,10\,|\,\mathbf{32}}$ fixed by the involution $\Gamma_{012345} \in \mathrm{Pin}^+(1,10)$ we have:

$$H_3^0 \;:=\; 0 \;\in\; \mathrm{CE}\big(\mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}_+}\big)^{\mathrm{Spin}(1,5)} \qquad \text{satisfies :} \qquad \mathrm{d}\,H_3^0 \;=\; \phi_0^\ast\, G_4^0$$

To globalize this situation, say that a super-immersion $\Sigma^{1,5\,|\,2\cdot\mathbf{8}_+} \xrightarrow{\phi_s} X^{1,10\,|\,\mathbf{32}}$ is ½**BPS M5** if it is "locally like" ϕ_0, and say that **B-field super-flux** on such an M5-probe is a super-form with these co-frame components:

$$\boxed{H_3^s \;:=\; H_3 \,+\, H_3^0 \;:=\; \tfrac{1}{3!}(H_3)_{a_1 a_2 a_3} e^{a_1}\, e^{a_2}\, e^{a_3} \;+\; 0} \qquad \big(e^{a<6} \;:=\; \phi_s^\ast E^a\big)$$

Theorem [50, §3.3]: On a super-immersion ϕ_s with B-field super-flux H_3^s:

The super-Bianchi identity	$\big\{\mathrm{d}\,H_3^s \;=\; \phi_s^\ast G_4^s\big\}$	*is equivalent to*	*the M5's B-field equations of motion.*

In particular, the (non-linear self-)duality conditions on the ordinary fluxes are *implied*: $G_4 \leftrightarrow G_7$ and $H_3 \leftrightarrow H_3$.

Seeing from this that also trivial tangent super-cochains may have non-trivial globalization, observe next that:

$$F_2^0 \;:=\; \big(\overline{\psi}\,\psi\big) \;=\; 0 \;\in\; \mathrm{CE}\big(\mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}_+}\big)^{\mathrm{Spin}(1,5)} \qquad \text{satisfies :} \qquad \mathrm{d}\,F_2^0 \;=\; 0$$

Globalizing this to $\Sigma^{1,5\,|\,2\cdot\mathbf{8}_+}$ via

$$\boxed{F_2^s \;:=\; F_2 \,+\, F_2^s \;:=\; \tfrac{1}{2}(F_2)_{a_1 a_2}\, e^{a_1} e^{a2} \;+\; 0}$$

we have on top of the above:

Theorem [108, p 7]:

The super-Bianchi identity	$\big\{\mathrm{d}\,F_2^s \;=\; 0\big\}$	*is equivalent to*	*the Chern-Simons E.O.M.:* $F_2 \;=\; 0$.

Flux quantization in Twistorial Cohomotopy. In summary, a remarkable kind of higher super-Cartan geometry locally modeled on the 11D super-Minkowski spacetime $\mathbb{R}^{1,10\,|\,\mathbf{32}}$ entails that on-shell 11D supergravity probed by magnetized 1/2BPS M5-branes implies and is entirely governed by these Bianchi identities on super-flux densities:

$$
\begin{array}{lllll}
\text{A-field} & \mathrm{d}\, F_2^s \;=\; 0 & & \mathrm{d}\, G_4^s \;=\; 0 & \text{C-field} \\
\text{self-dual B-field} & \mathrm{d}\, H_3^s \;=\; \phi_s^\ast G_4^s + \theta\, F_2^s\, F_2^s & & \mathrm{d}\, G_7^s \;=\; \tfrac{1}{2} G_4^s\, G_4^s & \text{dual C-field} \\
\text{M5 probe} & \Sigma^{1,5\,|\,2\cdot\mathbf{8}_+} & \xrightarrow[\text{1/2BPS immersion}]{\phi_s} & X^{1,11\,|\,\mathbf{32}} & \text{SuGra bulk}
\end{array}
\qquad (2)
$$

Here we have observed that the Green-Schwarz term $F_2^s F_2^s$ may equivalently be included for any theta-angle $\theta \in \mathbb{R}$ without affecting the equations of motion (since, recall, the CS e.o.m. $F_2^s = 0$ is already implied by $\mathrm{d}\, F_2^s = 0$).

But non-vanishing theta-angle does affect the admissible flux-quantization laws and hence the global solitonic and torsion charges of the fields. The choice of flux quantization according to *Hypothesis H* [32][33] is the following:

Admissible fibrations of classifying spaces for cohomology theories with the above character images (2). The homotopy quotient of S^7 is (i) for $\theta = 0$ by the trivial action and (ii) for $\theta \neq 0$ by the principal action of the complex Hopf fibration.

$$
\begin{array}{lccccc}
\boxed{\theta = 0} & S^7 /\!\!/_0\, \mathrm{U}(1) \simeq S^7 \times \mathbb{C}P^\infty & \twoheadrightarrow & S^7 & \xrightarrow[\mathbb{H}\text{-Hopf fibration}]{h_{\mathbb{H}}} & \mathbb{H}P^1 \\
& \downarrow & & \downarrow {\scriptstyle \mathbb{C}\text{-Hopf fibration}} & & \| \\
\boxed{\theta \neq 0} & S^7 /\!\!/ \mathrm{U}(1) & \xrightarrow{\sim} & \mathbb{C}P^3 & \xrightarrow[\text{Twistor fibration}]{t_{\mathbb{H}}} & \mathbb{H}P^1
\end{array}
$$

Proof. This may be seen as follows [33, Lem. 2.13]:

Since the real cohomology of projective space is a truncated polynomial algebra,

$$
\begin{array}{ll}
H^\bullet(\mathbb{C}P^n;\mathbb{R}) \simeq \mathbb{R}[\overbrace{c_1}^{\deg=2}]/(c_1^{n+1}) & H^\bullet(\overbrace{\mathbb{C}P^\infty}^{\simeq BU(1)};\mathbb{R}) \simeq \mathbb{R}[c_1] \\
H^\bullet(\mathbb{H}P^n;\mathbb{R}) \simeq \mathbb{R}[\underbrace{\tfrac{1}{2}p_1}_{\deg=4}]/(p_1^{n+1}) & H^\bullet(\underbrace{\mathbb{C}P^\infty}_{\simeq BSp(1)\simeq BSU(2)};\mathbb{R}) \simeq \mathbb{R}[\tfrac{1}{2}p_1]
\end{array}
$$

the minimal dgc-algebra model for $\mathbb{C}P^n$ needs a closed generator f_2 to span the cohomology and a generator h_{2n+1} in order to truncate it; analogously for $\mathbb{H}P^n$.

$$
\mathrm{CE}(\mathfrak{l}\mathbb{C}P^n) \simeq \mathbb{R}_{\mathrm{d}}\begin{bmatrix} f_2 \\ h_{2n+1}\end{bmatrix} \Big/ \left(\begin{array}{ll} \mathrm{d}\, f_2 & = 0 \\ \mathrm{d}\, h_{2n+1} & = (f_2)^{n+1}\end{array}\right)
$$

$$
\mathrm{CE}(\mathfrak{l}\mathbb{H}P^n) \simeq \mathbb{R}_{\mathrm{d}}\begin{bmatrix} g_4 \\ g_{4n+3}\end{bmatrix} \Big/ \left(\begin{array}{ll} \mathrm{d}\, g_4 & = 0 \\ \mathrm{d}\, g_{4n+3} & = (g_4)^{n+1}\end{array}\right)
$$

Furthermore, since the second Chern class of an $S(\mathrm{U}(1)^2)$-bundle is minus the cup square of the first Chern class (by the Whitney sum rule)

$$
\begin{array}{ccc}
BU(1) & \xrightarrow{B(c\mapsto \mathrm{diag}(c,c^*))} & B\mathrm{SU}(2) \\
-(c_1)^2 & \longleftarrow & \tfrac{1}{2}p_1 = c_2
\end{array}
$$

the minimal model of $\mathbb{C}P^3$ *relative* to that of $\mathbb{H}P_1$ needs to adjoin to the latter not only f_2 but also a generator h_3 imposing this relation in cohomology.

$$
\mathrm{CE}(\mathfrak{l}_{\mathbb{H}P^1}\mathbb{C}P^3) \simeq \mathbb{R}_{\mathrm{d}}\begin{bmatrix} f_2 \\ h_3 \\ g_4 \\ g_7\end{bmatrix} \Big/ \left(\begin{array}{ll} \mathrm{d}\, f_2 & = 0 \\ \mathrm{d}\, h_3 & = g_4 + f_2 f_2 \\ \mathrm{d}\, g_4 & = 0 \\ \mathrm{d}\, g_7 & = \tfrac{1}{2} g_4\, g_4\end{array}\right)
$$

The resulting fibration of L_∞-algebras is manifestly just that classifying the desired Bianchi identities (2) we are showing the case $\theta \neq 0$, which by isomorphic rescaling may be taken to be $\theta = 1$):

$$
\begin{array}{ccccccccc}
\Sigma^6 & \dashrightarrow & \Omega^1_{\mathrm{dR}}(-;\mathfrak{l}_{\mathbb{H}P^1}\mathbb{C}P^3)_{\mathrm{clsd}} & & \Omega^\bullet_{\mathrm{dR}}(\Sigma^6) & \dashleftarrow & \mathrm{CE}(\mathfrak{l}_{\mathbb{H}P^1}\mathbb{C}P^3) & & \begin{array}{l} F_2 \\ H_3 \end{array} \in \Omega^\bullet_{\mathrm{dR}}(\Sigma^6) \left| \begin{array}{ll} \mathrm{d}\, F_2 & = 0 \\ \mathrm{d}\, H_3 & = G_4 + F_2\, F_2 \end{array}\right. \\
{\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle (\mathfrak{l}\, t_{\mathbb{H}})_*} & \Leftrightarrow & {\scriptstyle \phi^*}\uparrow & & \downarrow{\scriptstyle (\mathfrak{l}\, t_{\mathbb{H}})^*} & \Leftrightarrow & \\
\Sigma^{11} & \dashrightarrow & \Omega^1_{\mathrm{dR}}(-;\mathfrak{l}\mathbb{H}P^1)_{\mathrm{clsd}} & & \Omega^\bullet_{\mathrm{dR}}(X^{11}) & \dashleftarrow & \mathrm{CE}(\mathfrak{l}\mathbb{H}P^1) & & \begin{array}{l} G_4 \\ G_7 \end{array} \in \Omega^\bullet_{\mathrm{dR}}(X^{11}) \left| \begin{array}{ll} \mathrm{d}\, G_4 & = 0 \\ \mathrm{d}\, G_7 & = \tfrac{1}{2} G_4\, G_4 \end{array}\right.
\end{array}
$$

Aside: Projective Spaces and their Fibrations – some classical facts.

Consider:

division algebras	$\mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{H}$ generically denoted $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$
groups of units	$\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$ understood with the multiplicative group structure
projective spaces	$\mathbb{K}P^n := (\mathbb{K}^{n+1} \setminus \{0\}) / \mathbb{K}^\times$
higher spheres	$S^n \simeq (\mathbb{R}^{n+1} \setminus \{0\}) / \mathbb{R}_{>0}$

$\mathbb{K}$-Hopf fibrations are the quotient co-projections induced by $\iota : \mathbb{R}_{>0} \hookrightarrow \mathbb{K}$

The classical Hopf fibrations $h_{\mathbb{K}}$ are:

$$
\begin{array}{ccc}
S^0 \simeq \mathbb{R}^\times / \mathbb{R}_{>0} & S^1 \simeq \mathbb{C}^\times / \mathbb{R}_{>0} & S^3 \simeq \mathbb{H}^\times / \mathbb{R}_{>0} \\
\big\downarrow{\scriptstyle \ker} & \big\downarrow{\scriptstyle \ker} & \big\downarrow{\scriptstyle \ker} \\
S^1 \simeq (\mathbb{R}^2 \setminus \{0\}) / \mathbb{R}_{>0} & S^3 \simeq (\mathbb{C}^2 \setminus \{0\}) / \mathbb{R}_{>0} & S^7 \simeq (\mathbb{H}^2 \setminus \{0\}) / \mathbb{R}_{>0} \\
\big\downarrow{\scriptstyle h_{\mathbb{R}}} \quad \big\downarrow{\scriptstyle \iota_*} & \big\downarrow{\scriptstyle h_{\mathbb{C}}} \quad \big\downarrow{\scriptstyle \iota_*} & \big\downarrow{\scriptstyle h_{\mathbb{H}}} \quad \big\downarrow{\scriptstyle \iota_*} \\
S^1 \simeq \underbrace{(\mathbb{R}^2 \setminus \{0\}) / \mathbb{R}^\times}_{\mathbb{R}P^1} & S^2 \simeq \underbrace{(\mathbb{C}^2 \setminus \{0\}) / \mathbb{C}^\times}_{\mathbb{C}P^1} & S^4 \simeq \underbrace{(\mathbb{H}^2 \setminus \{0\}) / \mathbb{H}^\times}_{\mathbb{H}P^1}
\end{array}
$$

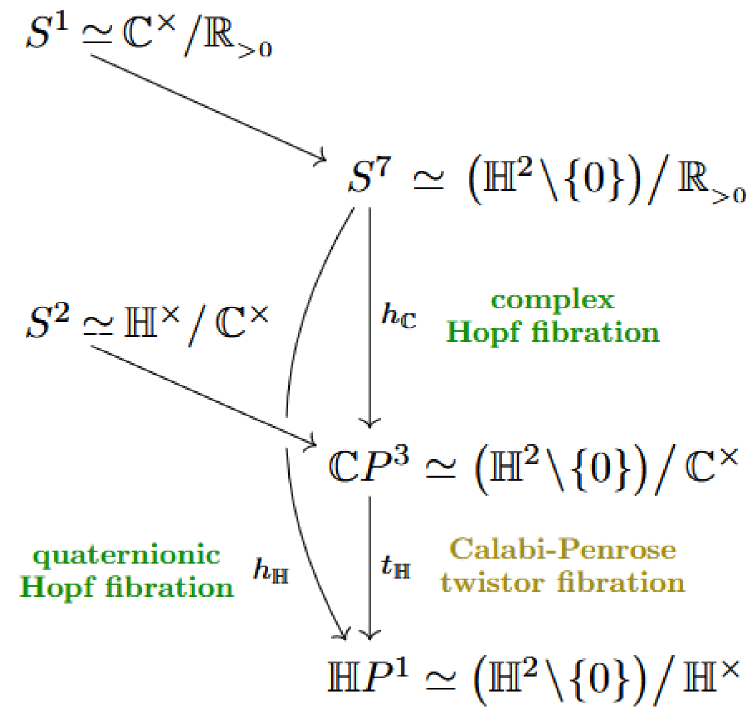
The Hopf fibrations in higher dimensions are the attaching maps exhibiting the topological cell-complex structure of projective spaces [88], from which the (cellular) cohomology follows readily.

$$
\begin{array}{ccc}
S(\mathbb{K}^{n+1}) & \longrightarrow & * \\
{\scriptstyle h_{\mathbb{K}}}\big\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle (\mathrm{po})} & \big\downarrow \\
\mathbb{K}P^n & \hookrightarrow & \mathbb{K}P^{n+1}
\end{array}
$$

Further factor-fibrations arise by factoring the Hopf fibrations via the stage-wise quotienting along

$$\mathbb{R}_{>0} \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{H}.$$

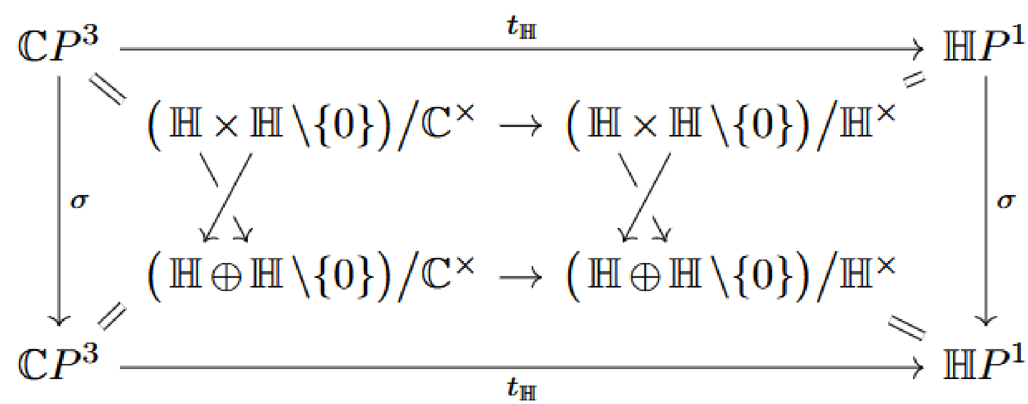
Notably, the classical quaternionic Hopf fibration $h_{\mathbb{H}}$ factors through a higher-dimensional complex Hopf fibration followed by the
Calabi-Penrose twistor fibration $t_{\mathbb{H}}$ [33, §2].



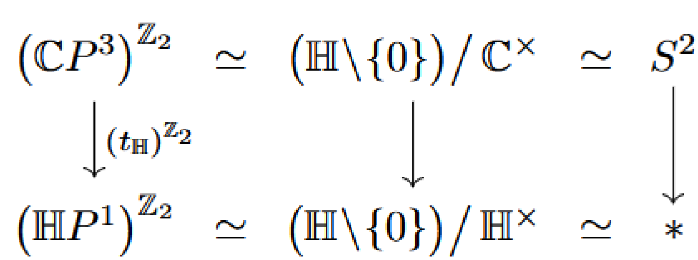
Equivariantization: Since the quotienting is by right actions, these fibrations are equivariant under the left action of

$$\mathrm{Spin}(5) \simeq \mathrm{Sp}(2) := \left\{ g \in \mathrm{GL}_2(\mathbb{H}) \,\middle|\, g^\dagger \cdot g = \mathrm{e} \right\}.$$

For example, the involution $\sigma := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{Sp}(2)$ swaps the two copies of $\mathbb{H}$:



The resulting $\mathbb{Z}_2$-fixed locus is the 2-sphere:



Aside: Implications of Hypothesis H, in view of traditional expectations for M-theory.

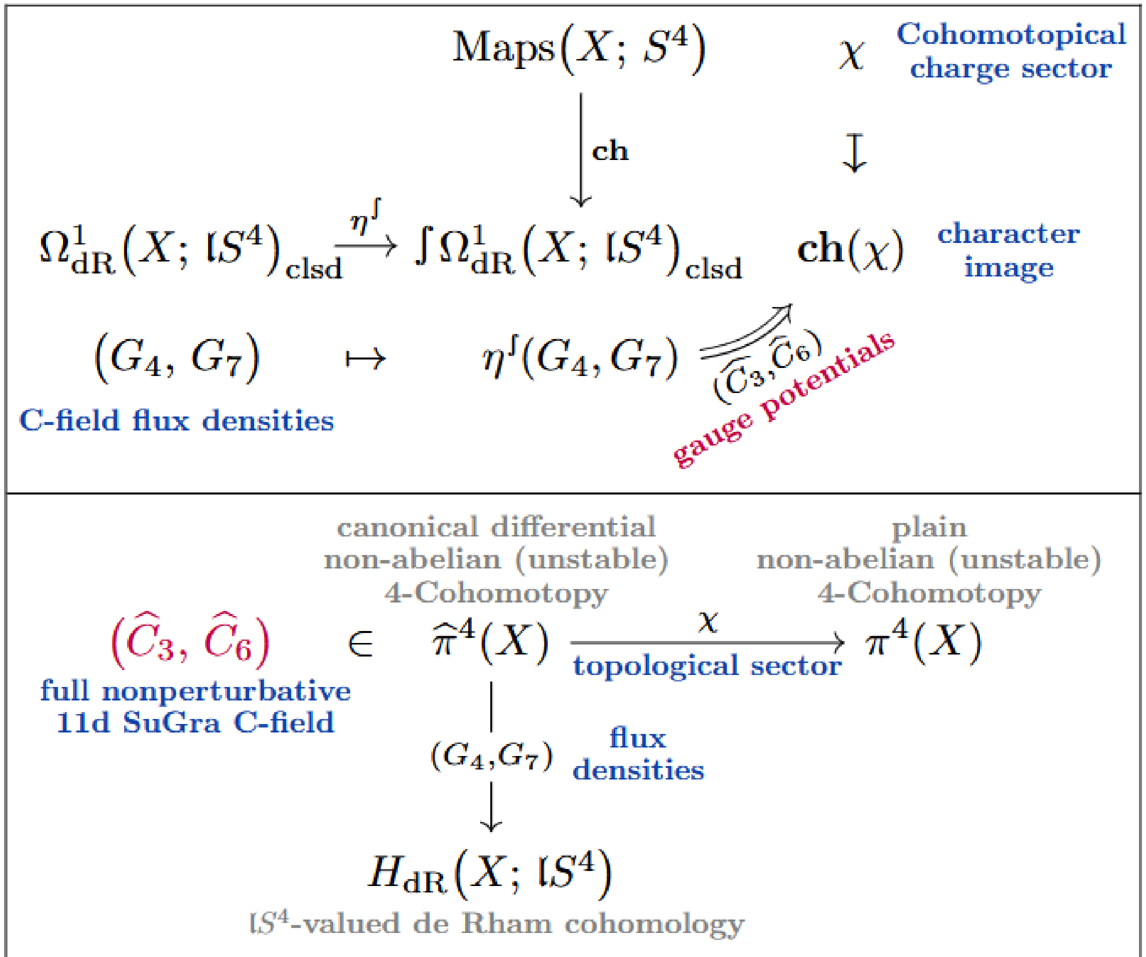
The plain Hypothesis H in the bulk says that the non-perturbative completion of the C-field in 11d supergravity involves a map χ from spacetime to the homotopy type of the 4-sphere, with the C-field gauge potentials $(\widehat{C}_3, \widehat{C}_6)$ exhibiting the flux densities (G_4, G_7) as $\mathbb{R}$-rational representatives of χ.

In other words, this is the postulate that the non-perturbative C-field is a cocycle in canonical, unstable *differential 4-Cohomotopy* $\widehat{\pi}^4$ [30, §4][54, §3.1][36, Ex. 9.3].

As an immediate plausibility check: This implies, from the well-known homotopy groups of spheres in low degrees, that:

integral quantization of charges carried by singular M5-brane branes *and*

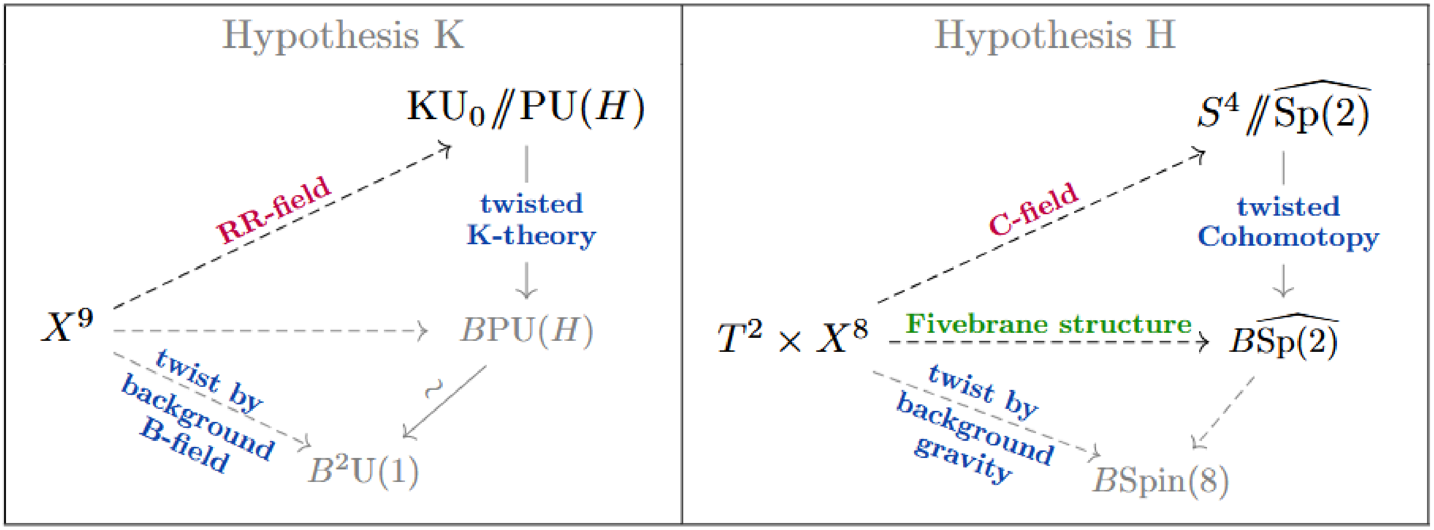
integral quantization of charges carried by singular M2-branes... plus a torsion-contribution (a first prediction of Hypothesis H).



$$\pi^4\big(\mathbb{R}^{10,1} \setminus \mathbb{R}^{5,1}\big) = \pi^4\big(\mathbb{R}^{5,1} \times \mathbb{R}_+ \times S^4\big)$$
$$= \pi^4(S^4) = \pi_4(S^4) = \mathbb{Z}$$

$$\pi^4\big(\mathbb{R}^{10,1} \setminus \mathbb{R}^{2,1}\big) = \pi^4\big(\mathbb{R}^{2,1} \times \mathbb{R}_+ \times S^7\big)$$
$$= \pi^4(S^7) = \pi_7(S^4) = \mathbb{Z} \oplus \mathbb{Z}_{12}$$

Hypothesis H with curvature corrections. More generally, the curvature corrections from the coupling to the background gravity are postulated to be reflected in *tangentially twisted* 4-Cohomotopy [33], analogous to the well-known twisting of the RR-field flux-quantization in K-theory by its background B-field:



To distinguish M2/M5-charge, the tangential twisting needs to preserve the $\mathbb{H}$-Hopf fibration $\Rightarrow$ tangential $Sp(2) \hookrightarrow Spin(8)$-structure [33, §2.3]. With this, integrality of M2's Page charge & anomaly-cancellation of the M5's Hopf-WZ term follows from trivialization of the Euler 8-class, which means lift to the *Fivebrane* 6-group $\widehat{Sp(2)} \to Sp(2)$ [32, §4].

This implies [33, Prop. 3.13][32, Thm. 4.8]:

(i) half-integrally shifted quantization of M5-brane charge in curved backgrounds, *and*

(ii) integral quantization of the Page charge of M2-branes.

$$[\widetilde{G}_4] := \underbrace{[G_4]}_{\text{C-field 4-flux}} + \tfrac{1}{2}\underbrace{\big(\tfrac{1}{2}p_1(TX^8)\big)}_{\text{integral Spin-Pontrjagin class}} \in H^4\big(X^8;\mathbb{Z}\big)$$

$$2[\widetilde{G}_7] := 2\big([G_7] + \tfrac{1}{2}[H_3 \wedge \widetilde{G}_4]\big) \in H^7(\widehat{X}^8;\mathbb{Z})$$

Both of these quantization conditions on M-brane charge are thought to be crucial for M-theory to make any sense. Previously, item (i) had remained enigmatic and item (ii) had remained wide open.

There is more:

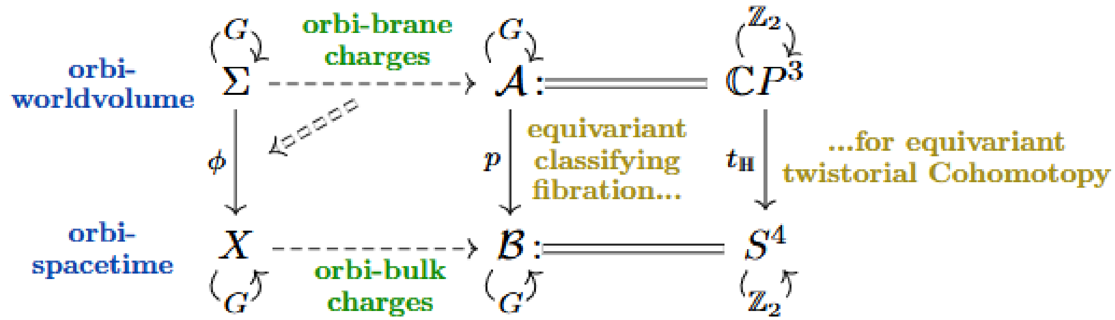
Provable implications from Hypothesis H of subtle effects expected in M-theory:

- half-integral shift of 4-flux [33, Prop. 3.13]
- DMW anomaly cancellation [33, Prop. 3.7]
- the C-field's "integral EoM" [33, §3.6]
- M2 Page charge quantization [32, Thm. 4.8]
- integrality of $\frac{1}{6}(G_4)^3$ [54, Rem. 2.9]
- M5-brane anomaly cancellation [106]
- non-abelian gerbe field on M5 [34]

It is these results which suggest that Hypothesis H goes towards the correct flux-quantization law for the C-field in M-theory.

Orbi-worldvolumes and Equivariant charges. Flux-quantization generalizes to *orbifolds* [4] by generalizing the cohomology of the charges to *equivariant cohomology* [102].

In terms of classifying spaces this simply means that all spaces are now equipped with the action of a finite group G and all maps are required to be G-equivariant.
We take $G := \mathbb{Z}_2$ and the classifying fibration to be the **twistor fibration** $p := t_{\mathbb{H}}$ equivariant under swapping the $\mathbb{H}$-summands,



and the brane/bulk orbifold we take to be as on p. 3:

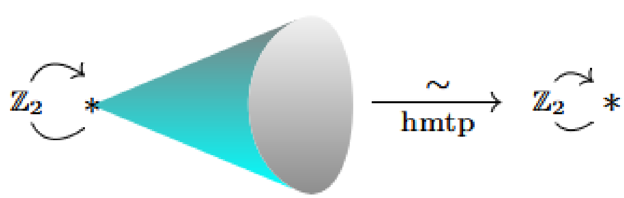
The orbi-brane diagram for a flat M5-brane wrapped on a trivial Seifert-fibered orbi-singularity. Shaded is the $\mathbb{Z}_2$-fixed locus/orbi-singularity.

We are adjoining the *point at infinity* to the space $\mathbb{R}^2_{\cup\{\infty\}} \underset{\text{homeo}}{\simeq} S^2$ which is thereby designated as transverse to any worldvolume solitons to be measured in reduced cohomology.

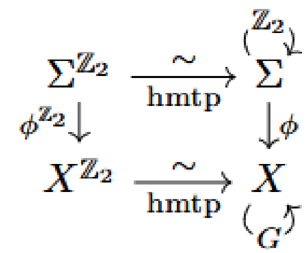
$$
\begin{array}{l}
\Sigma := \mathbb{R}^{1,0} \times \mathbb{R}^2_{\cup\{\infty\}} \times S^1 \times \mathbb{R}^2_{\mathrm{sgn}} \\
\downarrow \phi \\
X := \mathbb{R}^{1,0} \times \mathbb{R}^2_{\cup\{\infty\}} \times S^1 \times \mathbb{R}^2_{\mathrm{sgn}} \times \mathbb{R}^5
\end{array}
$$

time — trnsvrs space to solitons — M/IIA-circle — orbi-cone — trnsvrs space to M5-brane

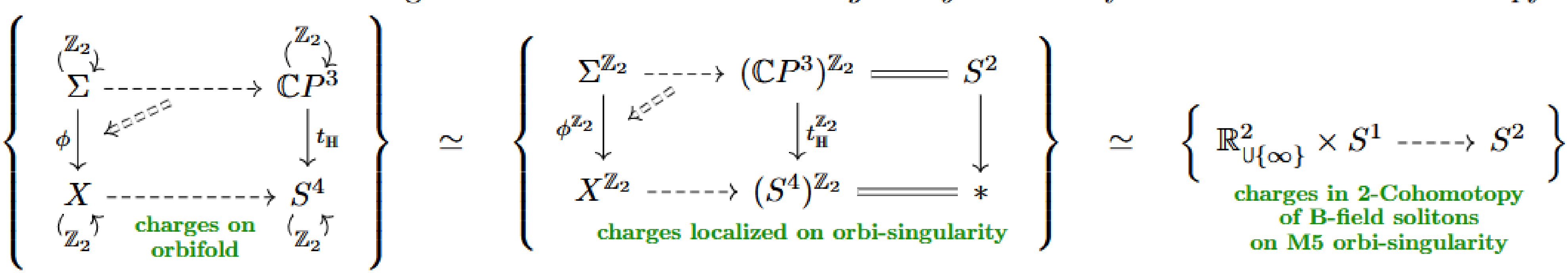
But since the cone $\mathbb{Z}_2 \circlearrowright \mathbb{R}^2_{\mathrm{sgn}}$ is equivariantly contractible,



the inclusion of the $\mathbb{Z}_2$-fixed loci is actually a homotopy equivalence

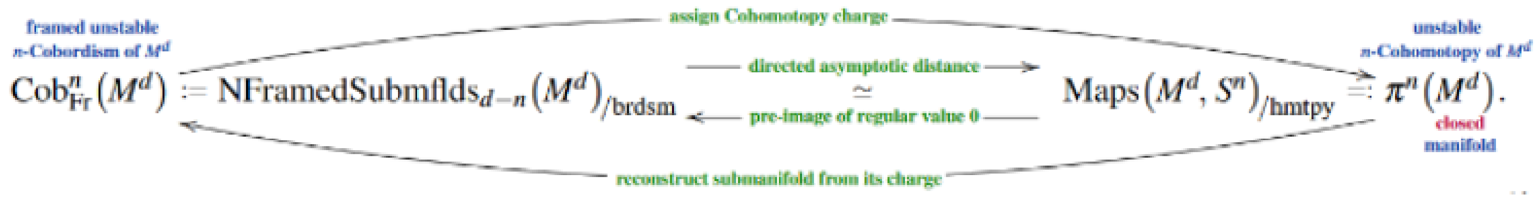


Therefore our equivariant classifying maps are determined up to equivariant homotopy by their restriction to the fixed-locus and hence the charges are *localized on the orbi-singularity* where they take values in 2-Cohomotopy:



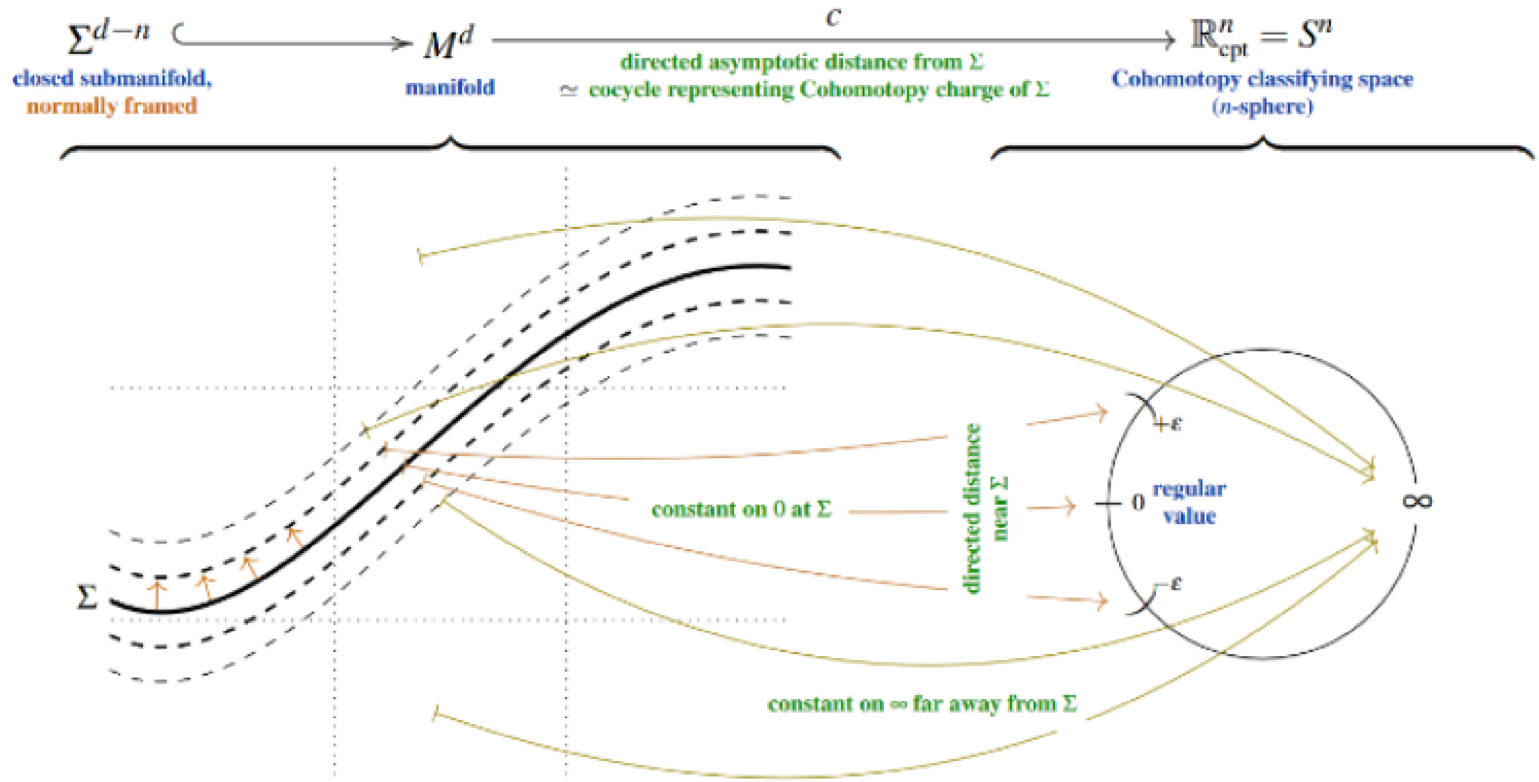
Remarkably, there is an equivalence between *Cohomotopy* of spacetime/worldvolumes and *Cobordism* classes of submanifolds behaving like solitonic branes carrying the corresponding Cohomotopy charge [103, §2.2] [101, §2.1]:

The **Pontrjagin theorem** [70][69, §IX] identifies the unstable n-Cohomotopy of a closed manifold with the cobordism classes of its normally framed submanifolds of co-dimension n.

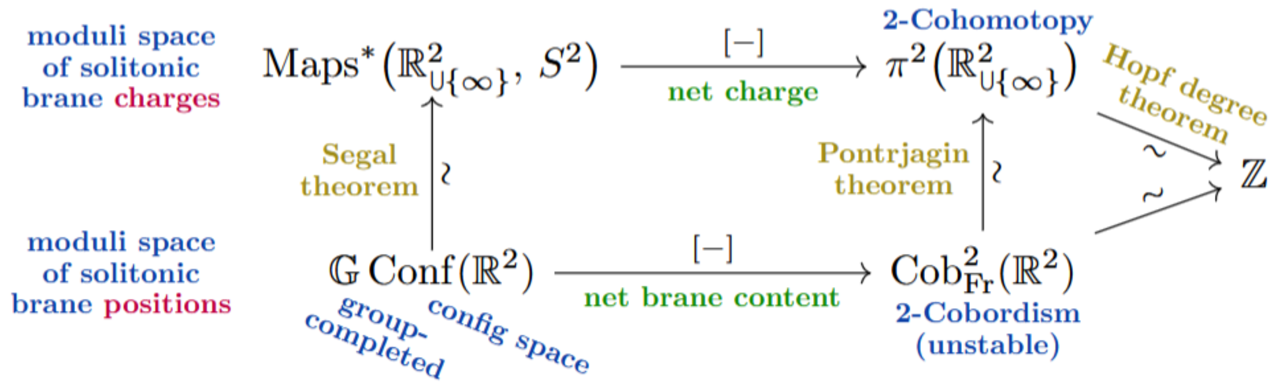


The **Cohomotopy charge** of a normally framed submanifold (aka *scanning map* or *Pontrjagin-Thom collapse*) is represented by mapping points of the ambient space to their directed distance if inside a tubular neighbourhood, else to ∞.

Conversely, every Cohomotopy class is representated by a smooth map with 0 a regular value, whose pre-image is a normally framed submanifold with that Cohomotopy charge.

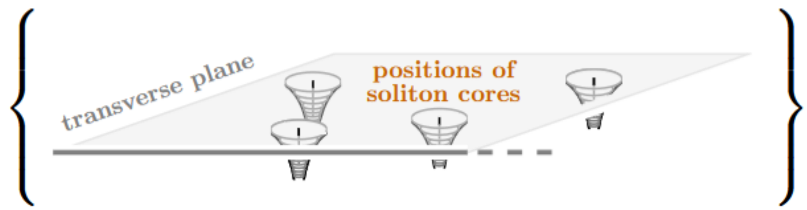


Moduli space of soliton configurations. But the Pontrjagin theorem concerns only the total cohomotopical charge, identifying it with the *net* (anti-)brane content. Beyond that we have the whole *moduli space* of charges (considered now specialized to our 2D transverse space), and **Segal's theorem** [111] says that the cohomotopy charge map (scanning map) identifies this with a moduli space of brane positions, namely with the *group-completed configuration space of points* [15][120][43]:

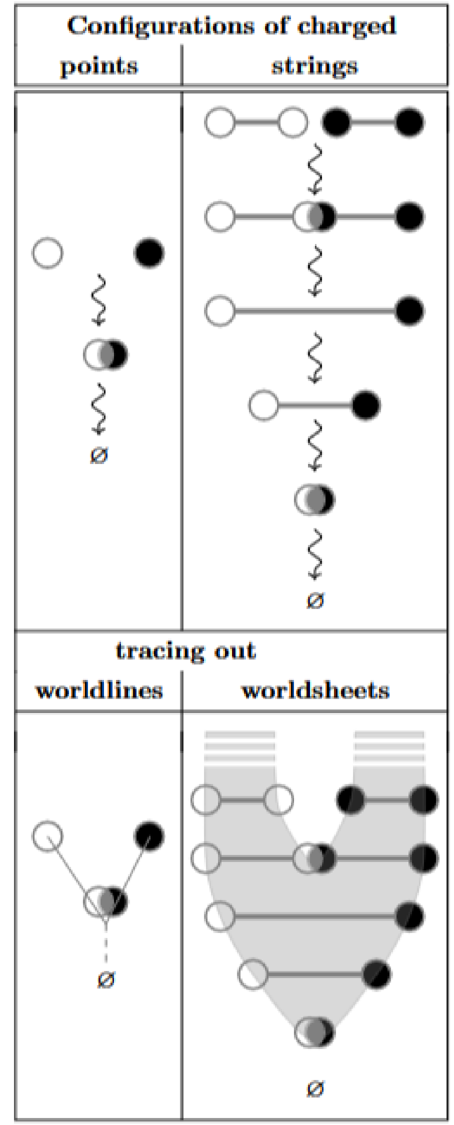


where the *configuration space of points* is the space of finite subsets of $\mathbb{R}^2$ – here understood as the space of positions of cores of solitons of unit charge +1,

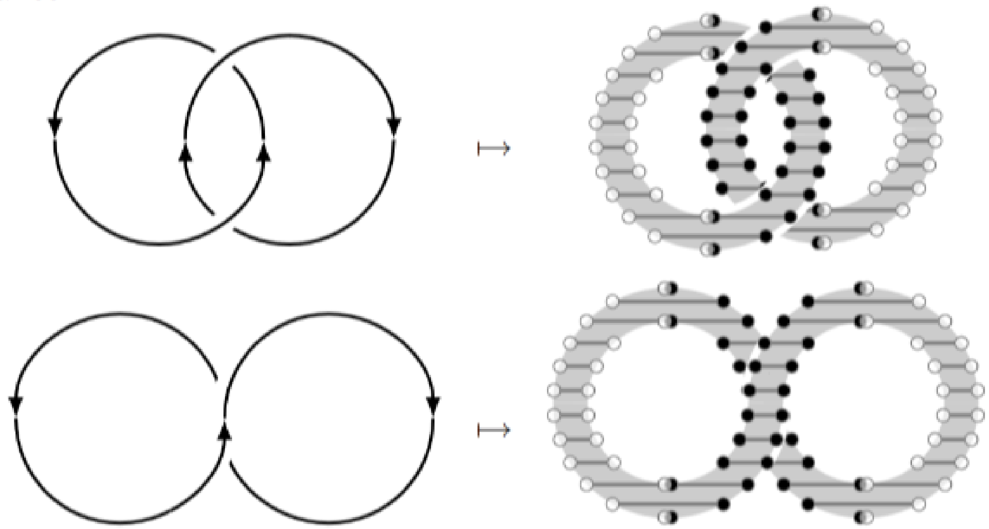
$\mathrm{Conf}(\mathbb{R}^2) \;=$



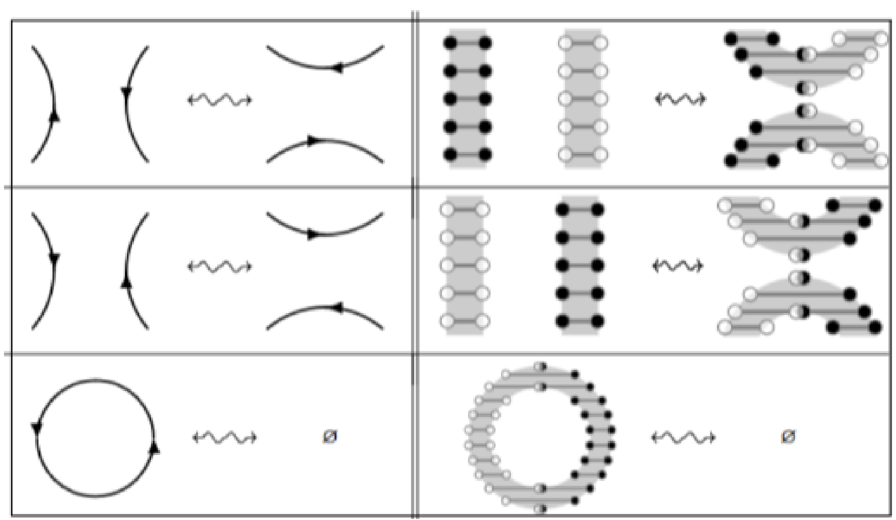
and its *group completion* $\mathbb{G}(-)$ is the topological completion of the topological partial monoid structure given by disjoint union of soliton configurations.
Naïvely this is given by including also **anti-solitons** in the form of configurations of *$\pm$-charged points*, topologized such as to allow for their pair annihilation/creation as shown in the left column on the right.
Remarkably, closer analysis reveals [89] that the group completion $\mathbb{G}(-)$ produces configurations of **strings** (extending parallel to one axis in $\mathbb{R}^3$) **with charged endpoints** whose pair annihilation/creation is smeared-out to string worldsheets as shown in the right column.



This means [105] that the **vacuum-to-vacuum soliton scattering processes**, forming the loop space $\Omega\,\mathbb{G}\,\mathrm{Conf}(\mathbb{R}^2)$, are identified with *framed links* ([90, pp 15]), for instance



subject to *link cobordism* (cf. [76]):



It follows [105, Thm 3.17] that the charge of a soliton scattering process L is the sum over crossings of the *crossing number* $\#(\text{positive crossing}) = +1$, $\#(\text{negative crossing}) = -1$, which equals the linking+framing number:

$$\Omega\mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \xrightarrow{\sim} \Omega\mathrm{Maps}^{*/}(\mathbb{R}^2_{\cup\{\infty\}}, S^2) \xrightarrow{[-]} \pi_3(S^2) \simeq \mathbb{Z}$$
$$L \longmapsto \#L \quad \text{(total crossing number = linking + framing number)}$$

But this is precisely the **Wilson loop observable** of L **in** (abelian) **Chern-Simons theory!** [105, §4]
As we explain next.

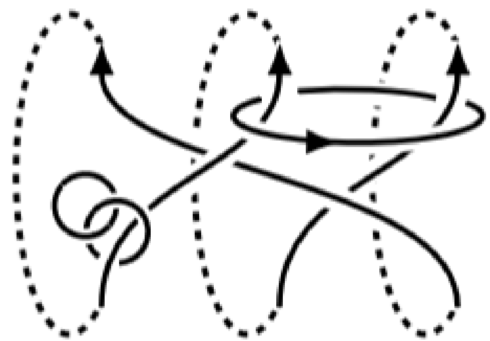
The k-Soliton sector. More generally, we may consider loops based in the kth connected component of the moduli space, corresponding to scattering process from k to k net number of solitons.

$$\begin{array}{ccc} \textbf{net charge } k & & \textbf{Hopf degree } k \\ \mathbb{G}\mathrm{Conf}_k(\mathbb{R}^2) & \xrightarrow{\;\sim\;} & \mathrm{Maps}^*_k\big(\mathbb{R}^2_{\cup\{\infty\}}, S^2\big) \\ \downarrow & & \downarrow \\ \mathbb{G}\mathrm{Conf}(\mathbb{R}^2) & \xrightarrow{\;\sim\;} & \mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}}, S^2\big) \end{array}$$

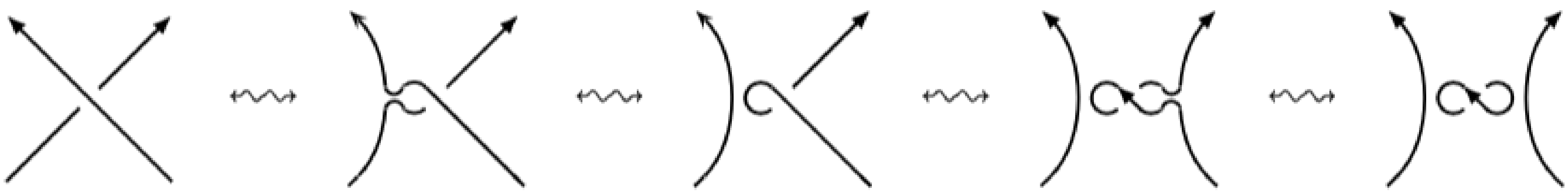
Since the double loop space $\mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}}, S^2\big)$ admits the structure of a topological group, all these connected components have the same homotopy type, and hence these scattering processes L are again classified by the integer total crossing number $\#L$ which is the abelian Chern-Simons Wilson-loop observable.

$$\begin{array}{ccc} \Omega_k\,\mathbb{G}\mathrm{Conf}(\mathbb{R}^2) & \xrightarrow{\;L \,\mapsto\, \#L\;} & \mathbb{Z} \\ \downarrow & & \\ \pi_0\Omega_k\,\mathbb{G}\mathrm{Conf}(\mathbb{R}^2) & \xrightarrow[\sim]{} & \mathbb{Z} \end{array}$$

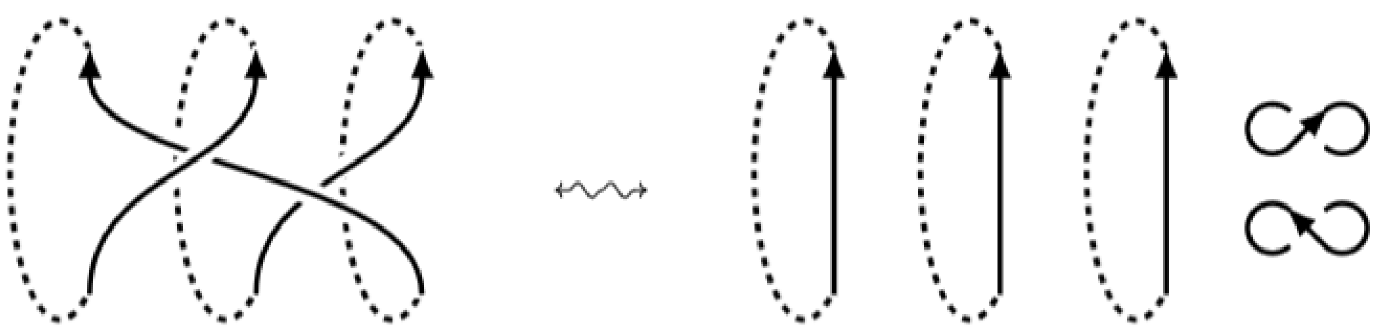
For instance, a generic $k=3$ process looks like this:



and via the framed cobordism moves



it computes to the trivial scattering process accompanied by $\#L$ vacuum pair braiding processes:



Chern-Simons level. We will see below further meanings of the number k:

This integer k is equivalently $\begin{cases} \text{the } \textit{number} \text{ of fractional quasi-hole vortices in a quantum Hall system,} \\ \text{the } \textit{level} \text{ of their effective abelian Chern-Simons theory,} \\ \text{the } \textit{maximal denominator} \text{ for filling fractions of their quantum states.} \end{cases}$

Generally, we will recover in a novel *non-Lagrangian* way the features of quantum Chern-Simons theory that are traditionally argued starting with the kth multiple of the local Lagrangian density $a \wedge da$ for a gauge potential 1-form a.

The situation on the 2-Sphere.
Furthermore consider k solitons on the actual 2-sphere S^2.
Here the 2-Cohomotopy moduli space satisfies (cf. [42]):

$$\pi_0 \Omega_k \, \mathrm{Maps}(S^2, S^2) \simeq \mathbb{Z}_{2|k|},$$

and the long homotopy fiber sequence induced by point evaluation shows that the generator of this cyclic group is again identified with the basic half-braiding operation:

$$\mathrm{Maps}^*(\mathbb{R}^2_{\cup\{\infty\}}, S^2) \xrightarrow{\text{fiber of...}} \mathrm{Maps}(S^2, S^2) \xrightarrow{\substack{\text{point-}\\ \text{evaluation}}} S^2$$

$$\underbrace{\pi_2(S^2)}_{\mathbb{Z}} \xrightarrow{2k} \underbrace{\pi_0 \Omega_k \mathrm{Maps}^*(\mathbb{R}^2_{\cup\{\infty\}}, S^2)}_{\mathbb{Z}} \longrightarrow \underbrace{\pi_0 \Omega_k \mathrm{Maps}(S^2, S^2)}_{\mathbb{Z}_{2|k|}} \longrightarrow \underbrace{\pi_1(S^2)}_{1}$$

With flux-quantized fields being equipped with a classifying space $\mathcal{A}$, there is a neat way to directly obtain the topological quantum observables – via the following observation:

Topological flux observables in Yang-Mills theory – Theorem [104]. For G-Yang-Mills theory on $\mathbb{R}^{1,1} \times \Sigma^2$, non-perturbative quantization of the algebra of flux observables through the closed surface Σ^2 is given, via a choice of Ad-invariant lattice $\Lambda \subset \mathfrak{g}$, by the group C^*-algebra $\mathbb{C}[-]$ of the Fréchet-Lie group of smooth maps $\Sigma^2 \to G \ltimes (\mathfrak{g}/\Lambda)$ — and the subalgebra of topological observables coincides with the Pontrjagin homology algebra of pointed maps $(\mathbb{R}^1 \times \Sigma^2)_{\cup\{\infty\}} \to B\big(G \ltimes (\mathfrak{g}/\Lambda)\big)$:

$$\underbrace{\mathbb{C}\Big[C^\infty(\Sigma^2, G) \ltimes C^\infty\big(\Sigma^2, (\mathfrak{g}/\Lambda)\big)\Big]}_{\text{non-perturbative quantum algebra of observables on flux through } \Sigma^2} \xleftarrow{(\pi^0)^*} \underbrace{\mathbb{C}\Big[H^0(\Sigma^2; G) \ltimes H^1(\Sigma^2; \Lambda)\Big]}_{\text{sub-algebra of topological observables}} \simeq \underbrace{H_0\Big(\mathrm{Maps}^*\big((\mathbb{R}^1 \times \Sigma^2)_{\cup\{\infty\}}, B(G \ltimes (\mathfrak{g}/\Lambda)\big); \mathbb{C}\Big)}_{\text{Pontrjagin homology algebra of moduli space of soliton charges}}$$

For example in electromagnetism, with $G = \mathrm{U}(1)$ and $\Lambda := \mathbb{Z} \hookrightarrow \mathbb{R}$:

$$\mathbb{C}\Big[\underbrace{H^1(\Sigma^2;\mathbb{Z})}_{\text{electric}} \times \underbrace{H^1(\Sigma^2;\mathbb{Z})}_{\text{magentic}}\Big] \simeq H_0\Big(\mathrm{Maps}^*\big((\mathbb{R}^1 \times \Sigma^2)_{\cup\{\infty\}}, \underbrace{B\mathrm{U}(1) \times B\mathrm{U}(1)}_{\text{classifying space for Dirac flux quantization}}\big); \mathbb{C}\Big)$$

This allows to generalize:

Topological flux observables of any higher gauge theory.

For a higher gauge theory flux-quantized in $\mathcal{A}$-cohomology the quantum algebra of topological flux observables on a spacetime of the form $\mathbb{R}^{1,1} \times \Sigma^{D-2}$ is the Pontrjagin homology algebra of the soliton moduli hence in $\deg = 0$ is the group algebra of vacuum soliton processes "**on the light-cone**":

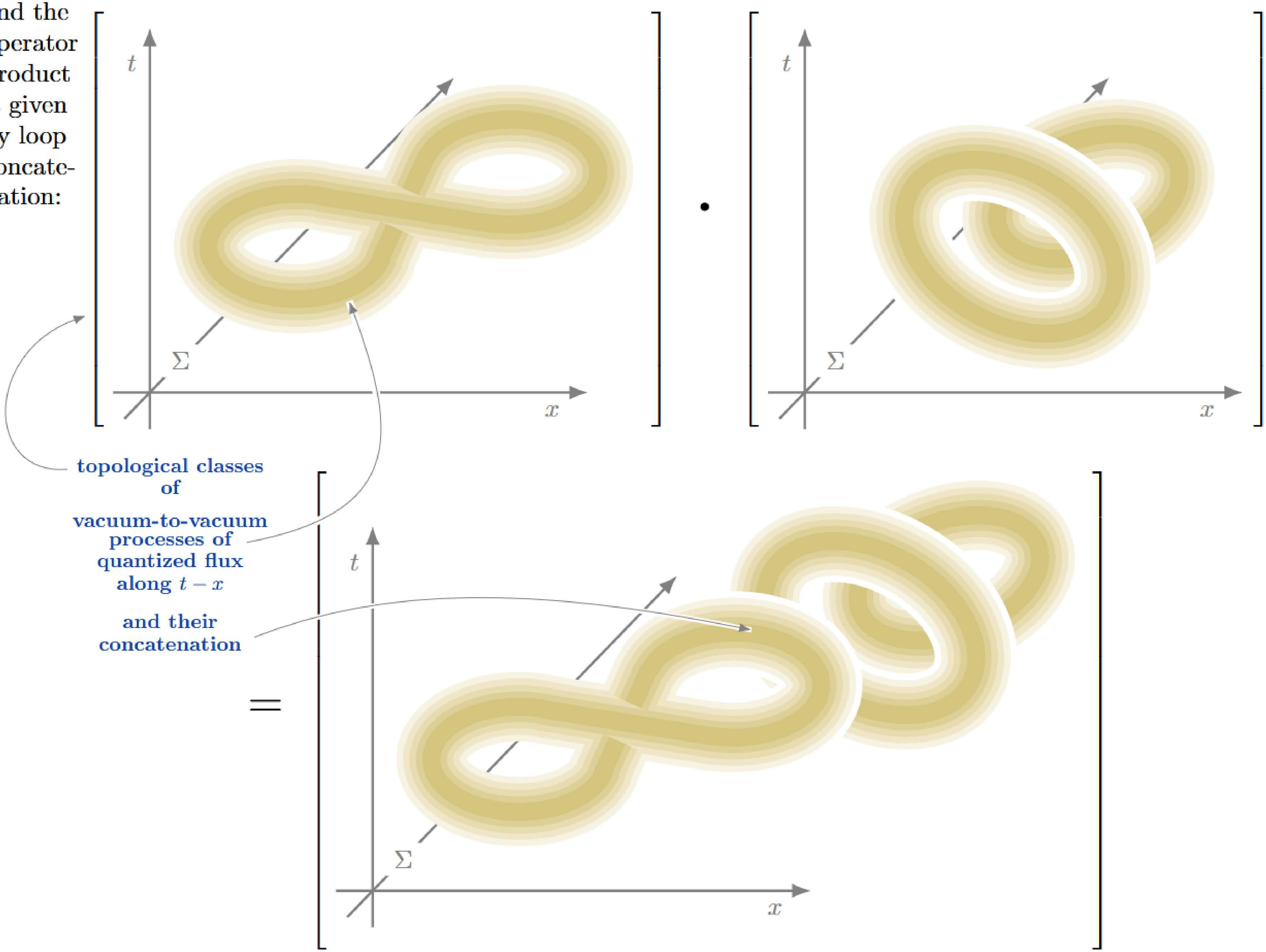
$$\begin{aligned}
\mathrm{Obs}_\bullet &:= H_\bullet\Big(\mathrm{Maps}^*\big((\mathbb{R}^1 \times \Sigma^{D-2})_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\
&\simeq H_\bullet\Big(\Omega\,\mathrm{Maps}(\Sigma^{D-2}, \mathcal{A}); \mathbb{C}\Big) \\
\mathrm{Obs}_0 &= \mathbb{C}\Big[\pi_0 \Omega\,\mathrm{Maps}(\Sigma^{D-2}, \mathcal{A})\Big]
\end{aligned}$$

For note that the star-involution is given by the *combination* of

- complex conjugation (time reversal)
- loop reversal (hence x-reversal)

where $\mathbb{R}^{1,1} \simeq \mathbb{R}\langle t, x \rangle$,

and the operator product is given by loop concatenation:



In the next lecture we discuss
concretely these quantum observables
on orbi-M5 probes
finding them exhibit
anyonic topological order
as in fractional quantum Hall systems.