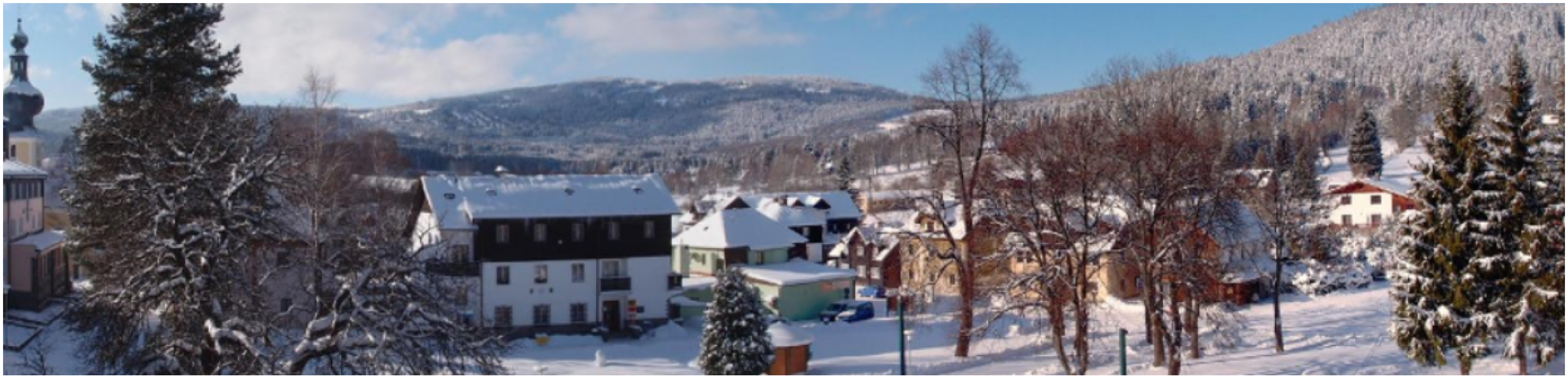
Urs Schreiber[a] on joint work with Hisham Sati[a]:

Introduction to *Hypothesis H*

lecture series at

45th Srní Winter School GEOMETRY AND PHYSICS

18-25 Jan 2025, Srní, Czechia



Session 1: **Non-Linear Flux Quantization in general**	[1]	(Context)
Session 2: **Flux Quantization on probe M5-Branes**		(Hypothesis H)
Session 3: **Topological Order on probe M5-Branes**	[2]	(Consequences)

Materials:

[1] *Flux Quantization*, Enzyclopedia of Mathematical Physics 2nd ed. **4** (2025) 281-324
[ncatlab.org/schreiber/show/Flux+Quantization]

[2] *Engineering of Anyons on M5 probes via Flux Quantization*, lecture notes (2025)
[ncatlab.org/schreiber/show/Engineering+of+Anyons+on+M5-Probes]

Today, to explain Hypothesis H.

Given higher Maxwell-type equations of motion,
last time we saw the *rules by which to* *choose* flux-quantization laws.

But there is still a *choice* involved, namely of a cohomology theory.
That choice is a *choice of completion* of the higher gauge theory,
hence a *Hypothesis* about the non-perturbative physics it describes.

For the RR-field of 10D supergravity
a traditional choice is: $\boldsymbol{K}$-theoretic cohomology theory;
this choice could be called *Hypothesis* $\boldsymbol{K}$.

For the C-field of 11D supergravity,
and the B-field on its M5-probes
an admissible choice is a
twisted/twistorial form of
"Co-***H****omotopy* Theory", hence "***H****omotopy* cohomology theory".

Hypothesis $\boldsymbol{H}$ states that: *This is the correct choice for M-theory.*

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The Bianchi identities

for fluxes on M5-probes of 11D supergravity
are these:

A-field	$\mathrm{d}\, F_2^s \;=\; 0$	$\mathrm{d}\, G_4^s \;=\; 0$	**C-field**
self-dual B-field	$\mathrm{d}\, H_3^s \;=\; \phi_s^* G_4^s \;+\; \theta\, F_2^s\, F_2^s$	$\mathrm{d}\, G_7^s \;=\; \tfrac{1}{2} G_4^s\, G_4^s$	**dual C-field**

M5 probe $\quad \Sigma^{1,5\,\vert\,2\cdot\mathbf{8}_+} \xrightarrow[\text{1/2BPS immersion}]{\phi_s} X^{1,10\,\vert\,\mathbf{32}}$ $\quad$ **SuGra bulk**

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Flux-quantizing this situation means to find/specify a fibration of spaces which has these same formulas as its relative minimal Sullivan model.

But first:
The next two slides indicate the remarkable way in which these Bianchi identites come about in 11D supergravity (technical, may be skipped).

Bianchi identities on M5-Probes of 11D SuGra via super-geometry. Consider the 11D super-tangent space

$$\underset{\text{super-Minkowski}}{\mathbb{R}^{1,10\,|\,\mathbf{32}}} \hookrightarrow \underset{\text{super-Poincaré}}{\mathfrak{isom}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big)} \twoheadrightarrow \underset{\text{Lorentz}}{\mathfrak{so}(1,10)}$$

with its super-invariant 1-forms (cf. [49, §2.1])

$$\mathrm{CE}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big) \simeq \underset{\text{super-transl. invar. forms}}{\Omega^{\bullet}_{\mathrm{dR}}\big(\mathbb{R}^{1,10|\mathbf{32}}\big)^{\mathrm{li}}} \simeq \mathbb{R}_{\mathrm{d}}\left[\begin{array}{l}(\Psi^\alpha)_{\alpha=1}^{32} \\ (E^a)_{a=0}^{10}\end{array}\right] \Big/ \left(\begin{array}{rcl}\mathrm{d}\,\Psi^\alpha &=& 0 \\ \mathrm{d}\,E^a &=& \big(\overline{\Psi}\,\Gamma^a\,\Psi\big)\end{array}\right).$$

Remarkably, the quartic Fierz identities entail that [19][84][49, Prop. 2.73]:

$$\left.\begin{array}{rcl} G_4^0 &:=& \tfrac{1}{2}\big(\overline{\Psi}\,\Gamma_{a_1 a_2}\,\Psi\big)\,E^{a_1}\,E^{a_2} \\ G_7^0 &:=& \tfrac{1}{5!}\big(\overline{\Psi}\,\Gamma_{a_1\cdots a_5}\,\Psi\big)\,E^{a_1}\cdots E^{a_5}\end{array}\right\} \in \underset{\text{fully super-invariant forms}}{\mathrm{CE}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big)^{\mathrm{Spin}(1,10)}} \qquad \text{satisfy :} \qquad \begin{array}{rcl}\mathrm{d}\,G_4^0 &=& 0 \\ \mathrm{d}\,G_7^0 &=& \tfrac{1}{2}G_4^0\,G_4^0\end{array}$$

To globalize this situation, say that an **11D super-spacetime** X is a super-manifold equipped with a super-Cartan connection, locally on an open cover $\widetilde{X} \twoheadrightarrow X$ given by

$$\left.\begin{array}{l}(\Psi^\alpha)_{\alpha=1}^{32} \\ (E^a)_{a=0}^{10} \\ \big(\Omega^{ab} = -\Omega^{ba}\big)_{a,b=0}^{10}\end{array}\right\} \in \Omega^1_{\mathrm{dR}}\big(\widetilde{X}\big) \qquad \text{such that the super-torsion vanishes} \qquad \mathrm{d}\,E^a - \Omega^a{}_b\,E^b = \big(\overline{\Psi}\,\Gamma^a\,\Psi\big),$$

and say that **C-field super-flux** on such a super-spacetime are super-forms with these co-frame components:

$$\boxed{\begin{array}{rcccl} G_4^s &:=& G_4 + G_4^0 &:=& \tfrac{1}{4!}(G_4)_{a_1\cdots a_4}E^{a_1}\cdots E^{a_4} + \tfrac{1}{2}\big(\overline{\Psi}\,\Gamma_{a_1 a_2}\,\Psi\big)E^{a_1}\,E^{a_2} \\ G_7^s &:=& G_7 + G_7^0 &:=& \tfrac{1}{7!}(G_4)_{a_1\cdots a_7}E^{a_1}\cdots E^{a_7} + \tfrac{1}{5!}\big(\overline{\Psi}\,\Gamma_{a_1\cdots a_5}\,\Psi\big)E^{a_1}\cdots E^{a_5}\end{array}}$$

Theorem [49, Thm. 3.1]: On an 11D super-spacetime X with C-field super-flux (G_4^s, G_7^s):

The duality-symmetric super-Bianchi identity $\left\{\begin{array}{rcl}\mathrm{d}\,G_4^s &=& 0 \\ \mathrm{d}\,G_7^s &=& \tfrac{1}{2}\,G_4^s\,G_4^s\end{array}\right\}$ *is equivalent to* *the full 11D SuGra equations of motion!*

Next, on the super-subspace $\mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}_+} \xhookrightarrow{\phi_0} \mathbb{R}^{1,10\,|\,\mathbf{32}}$ fixed by the involution $\Gamma_{012345} \in \mathrm{Pin}^+(1,10)$ we have:

$$H_3^0 \;:=\; 0 \;\in\; \mathrm{CE}\big(\mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}_+}\big)^{\mathrm{Spin}(1,5)} \qquad \text{satisfies :} \qquad \mathrm{d}\,H_3^0 \;=\; \phi_0^\ast\, G_4^0$$

To globalize this situation, say that a super-immersion $\Sigma^{1,5\,|\,2\cdot\mathbf{8}_+} \xrightarrow{\phi_s} X^{1,10\,|\,\mathbf{32}}$ is **1/2BPS M5** if it is "locally like" ϕ_0, and say that **B-field super-flux** on such an M5-probe is a super-form with these co-frame components:

$$\boxed{H_3^s \;:=\; H_3 \,+\, H_3^0 \;:=\; \tfrac{1}{3!}(H_3)_{a_1a_2a_3} e^{a_1}\, e^{a_2}\, e^{a_3} \,+\, 0} \qquad \big(e^{a<6} \;:=\; \phi_s^\ast E^a\big)$$

Theorem [50, §3.3]: On a super-immersion ϕ_s with B-field super-flux H_3^s:

The super-Bianchi identity $\big\{\mathrm{d}\,H_3^s \;=\; \phi_s^\ast G_4^s\big\}$ *is equivalent to* *the M5's B-field equations of motion.*

In particular, the (non-linear self-)duality conditions on the ordinary fluxes are *implied*: $G_4 \leftrightarrow G_7$ and $H_3 \leftrightarrow H_3$.

Seeing from this that also trivial tangent super-cochains may have non-trivial globalization, observe next that:

$$F_2^0 \;:=\; \big(\overline{\psi}\,\psi\big) \;=\; 0 \;\in\; \mathrm{CE}\big(\mathbb{R}^{1,5\,|\,2\cdot\mathbf{8}_+}\big)^{\mathrm{Spin}(1,5)} \qquad \text{satisfies :} \qquad \mathrm{d}\,F_2^0 \;=\; 0$$

Globalizing this to $\Sigma^{1,5\,|\,2\cdot\mathbf{8}_+}$ via

$$\boxed{F_2^s \;:=\; F_2 \,+\, F_2^s \;:=\; \tfrac{1}{2}(F_2)_{a_1a_2}\, e^{a_1} e^{a2} \,+\, 0}$$

we have on top of the above:

Theorem [108, p 7]:

The super-Bianchi identity $\big\{\mathrm{d}\,F_2^s \;=\; 0\big\}$ *is equivalent to* *the Chern-Simons E.O.M.:* $F_2 \;=\; 0.$

Flux quantization in Twistorial Cohomotopy. In summary, a remarkable kind of higher super-Cartan geometry locally modeled on the 11D super-Minkowski spacetime $\mathbb{R}^{1,10\,|\,\mathbf{32}}$ entails that on-shell 11D supergravity probed by magnetized 1/2BPS M5-branes implies and is entirely governed by these Bianchi identities on super-flux densities:

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\begin{array}{cccc}
\textbf{A-field} & \mathrm{d}\,F_2^s \;=\; 0 & \mathrm{d}\,G_4^s \;=\; 0 & \textbf{C-field} \\
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\end{array}
\tag{2}
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Here we have observed that the Green-Schwarz term $F_2^s F_2^s$ may equivalently be included for any theta-angle $\theta \in \mathbb{R}$ without affecting the equations of motion (since, recall, the CS e.o.m. $F_2^s = 0$ is already implied by $\mathrm{d}\,F_2^s = 0$).

But non-vanishing theta-angle does affect the admissible flux-quantization laws and hence the global solitonic and torsion charges of the fields. The choice of flux quantization according to *Hypothesis H* [33][35] is the following:

Admissible fibrations of classifying spaces for cohomology theories with the above character images (2). The homotopy quotient of S^7 is (i) for $\theta = 0$ by the trivial action and (ii) for $\theta \neq 0$ by the principal action of the complex Hopf fibration.

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Choosing the flux-quantization law represented by this fibration means that the charges on a spacetime domain with M5-probe are maps as on the right:

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\begin{array}{c}\textbf{M5-probe}\\ \textbf{worldvolume}\end{array} & \Sigma^6 \arrow[r, dashed, "\text{B-field charge}", "b_2"'] \arrow[d, "\phi"'] & \mathbb{C}P^3 \arrow[d, "t_{\mathbb{H}}"] \arrow[dl, Rightarrow, dashed] \\
\begin{array}{c}\textbf{bulk}\\ \textbf{spacetime}\end{array} & X^{11} \arrow[r, dashed, "c_3", "\text{C-field charge}"'] & S^4
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That this choice is the "right" one is Hypothesis H.

Proof. This may be seen as follows [33, Lem. 2.13]:

Since the real cohomology of projective space is a truncated polynomial algebra,

$$H^\bullet(\mathbb{C}P^n;\mathbb{R}) \simeq \mathbb{R}[\overbrace{c_1}^{\deg=2}]/(c_1^{n+1}) \qquad H^\bullet(\overbrace{\mathbb{C}P^\infty}^{\simeq BU(1)};\mathbb{R}) \simeq \mathbb{R}[c_1]$$

$$H^\bullet(\mathbb{H}P^n;\mathbb{R}) \simeq \mathbb{R}[\underbrace{\tfrac{1}{2}p_1}_{\deg=4}]/(p_1^{n+1}) \qquad H^\bullet(\underbrace{\mathbb{C}P^\infty}_{\simeq BSp(1)\simeq BSU(2)};\mathbb{R}) \simeq \mathbb{R}[\tfrac{1}{2}p_1]$$

the minimal dgc-algebra model for $\mathbb{C}P^n$ needs a closed generator f_2 to span the cohomology and a generator h_{2n+1} in order to truncate it; analogously for $\mathbb{H}P^n$.

$$\mathrm{CE}(\mathfrak{l}\mathbb{C}P^n) \simeq \mathbb{R}_{\mathrm{d}}\begin{bmatrix} f_2 \\ h_{2n+1}\end{bmatrix} \Big/ \left(\begin{aligned} \mathrm{d}\,f_2 &= 0 \\ \mathrm{d}\,h_{2n+1} &= (f_2)^{n+1}\end{aligned}\right)$$

$$\mathrm{CE}(\mathfrak{l}\mathbb{H}P^n) \simeq \mathbb{R}_{\mathrm{d}}\begin{bmatrix} g_4 \\ g_{4n+3}\end{bmatrix} \Big/ \left(\begin{aligned} \mathrm{d}\,g_4 &= 0 \\ \mathrm{d}\,g_{4n+3} &= (g_4)^{n+1}\end{aligned}\right)$$

Furthermore, since the second Chern class of an $S(\mathrm{U}(1)^2)$-bundle is minus the cup square of the first Chern class (by the Whitney sum rule)

$$\begin{array}{ccc} BU(1) & \xrightarrow{B(c\mapsto \mathrm{diag}(c,c^*))} & B\mathrm{SU}(2) \\ -(c_1)^2 & \longleftarrow\!\shortmid & \tfrac{1}{2}p_1 = c_2 \end{array}$$

the minimal model of $\mathbb{C}P^3$ *relative* to that of $\mathbb{H}P_1$ needs to adjoin to the latter not only f_2 but also a generator h_3 imposing this relation in cohomology.

$$\mathrm{CE}(\mathfrak{l}_{\mathbb{H}P^1}\mathbb{C}P^3) \simeq \mathbb{R}_{\mathrm{d}}\begin{bmatrix} f_2 \\ h_3 \\ g_4 \\ g_7\end{bmatrix} \Big/ \left(\begin{aligned} \mathrm{d}\,f_2 &= 0 \\ \mathrm{d}\,h_3 &= g_4 + f_2 f_2 \\ \mathrm{d}\,g_4 &= 0 \\ \mathrm{d}\,g_7 &= \tfrac{1}{2}g_4\,g_4\end{aligned}\right)$$

The resulting fibration of L_∞-algebras is manifestly just that classifying the desired Bianchi identities (2) we are showing the case $\theta \neq 0$, which by isomorphic rescaling may be taken to be $\theta = 1$):

$$\begin{tikzcd} \Sigma^6 \arrow[r, dashed] \arrow[d, "\phi"] & \Omega^1_{\mathrm{dR}}(-;\mathfrak{l}_{\mathbb{H}P^1}\mathbb{C}P^3)_{\mathrm{clsd}} \arrow[d, "(\mathfrak{l}t_{\mathbb{H}})_*"] \\ \Sigma^{11} \arrow[r, dashed] & \Omega^1_{\mathrm{dR}}(-;\mathfrak{l}\mathbb{H}P^1)_{\mathrm{clsd}} \end{tikzcd} \quad\Leftrightarrow\quad \begin{tikzcd} \Omega^\bullet_{\mathrm{dR}}(\Sigma^6) & \mathrm{CE}(\mathfrak{l}_{\mathbb{H}P^1}\mathbb{C}P^3) \arrow[l, dashed] \arrow[d, "(\mathfrak{l}t_{\mathbb{H}})^*"] \\ \Omega^\bullet_{\mathrm{dR}}(X^{11}) \arrow[u, "\phi^*"] & \mathrm{CE}(\mathfrak{l}\mathbb{H}P^1) \arrow[l, dashed] \end{tikzcd} \quad\Leftrightarrow\quad \begin{array}{l} \begin{matrix} F_2 \\ H_3 \end{matrix} \in \Omega^\bullet_{\mathrm{dR}}(\Sigma^6) \left| \begin{aligned} \mathrm{d}\,F_2 &= 0 \\ \mathrm{d}\,H_3 &= G_4 + F_2\,F_2 \end{aligned}\right. \\[3em] \begin{matrix} G_4 \\ G_7 \end{matrix} \in \Omega^\bullet_{\mathrm{dR}}(X^{11}) \left| \begin{aligned} \mathrm{d}\,G_4 &= 0 \\ \mathrm{d}\,G_7 &= \tfrac{1}{2}G_4\,G_4 \end{aligned}\right. \end{array}$$

Aside: Projective Spaces and their Fibrations – some classical facts.

Consider:

division algebras	$\mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{H}$ generically denoted $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$
groups of units	$\mathbb{K}^\times := \mathbb{K} \setminus \{0\}$ understood with the multiplicative group structure
projective spaces	$\mathbb{K}P^n := (\mathbb{K}^{n+1} \setminus \{0\})/\mathbb{K}^\times$
higher spheres	$S^n \simeq (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}_{>0}$
$\mathbb{K}$-Hopf fibrations are the quotient co-projections induced by $\iota : \mathbb{R}_{>0} \hookrightarrow \mathbb{K}$	

The classical Hopf fibrations $h_{\mathbb{K}}$ are:

$$
\begin{tikzcd}
S^0 \simeq \mathbb{R}^\times/\mathbb{R}_{>0} \arrow[d, hook, "\ker"] \\
S^1 \simeq (\mathbb{R}^2\setminus\{0\})/\mathbb{R}_{>0} \arrow[d, two heads, "h_{\mathbb{R}}"] \arrow[d, shift left=20, two heads, "\iota_*"] \\
S^1 \simeq \underbrace{(\mathbb{R}^2\setminus\{0\})/\mathbb{R}^\times}_{\mathbb{R}P^1}
\end{tikzcd}
\qquad
\begin{tikzcd}
S^1 \simeq \mathbb{C}^\times/\mathbb{R}_{>0} \arrow[d, hook, "\ker"] \\
S^3 \simeq (\mathbb{C}^2\setminus\{0\})/\mathbb{R}_{>0} \arrow[d, two heads, "h_{\mathbb{C}}"] \arrow[d, shift left=20, two heads, "\iota_*"] \\
S^2 \simeq \underbrace{(\mathbb{C}^2\setminus\{0\})/\mathbb{C}^\times}_{\mathbb{C}P^1}
\end{tikzcd}
\qquad
\begin{tikzcd}
S^3 \simeq \mathbb{H}^\times/\mathbb{R}_{>0} \arrow[d, hook, "\ker"] \\
S^7 \simeq (\mathbb{H}^2\setminus\{0\})/\mathbb{R}_{>0} \arrow[d, two heads, "h_{\mathbb{H}}"] \arrow[d, shift left=20, two heads, "\iota_*"] \\
S^4 \simeq \underbrace{(\mathbb{H}^2\setminus\{0\})/\mathbb{H}^\times}_{\mathbb{H}P^1}
\end{tikzcd}
$$

The Hopf fibrations in higher dimensions are the attaching maps exhibiting the topological cell-complex structure of projective spaces [88], from which the (cellular) cohomology follows readily.

$$
\begin{tikzcd}
S(\mathbb{K}^{n+1}) \arrow[r] \arrow[d, "h_{\mathbb{K}}"'] \arrow[dr, phantom, "(\mathrm{po})"] & * \arrow[d] \\
\mathbb{K}P^n \arrow[r, hook] & \mathbb{K}P^{n+1}
\end{tikzcd}
$$

Further factor-fibrations arise by factoring the Hopf fibrations via the stage-wise quotienting along

$$\mathbb{R}_{>0} \hookrightarrow \mathbb{R} \hookrightarrow \mathbb{C} \hookrightarrow \mathbb{H}.$$

Notably, the classical quaternionic Hopf fibration $h_{\mathbb{H}}$ factors through a higher-dimensional complex Hopf fibration followed by the

Calabi-Penrose twistor fibration $t_{\mathbb{H}}$ [33, §2].

$$\begin{tikzcd}
S^1 \simeq \mathbb{C}^\times/\mathbb{R}_{>0} \arrow[dr] & \\
& S^7 \simeq (\mathbb{H}^2\backslash\{0\})/\mathbb{R}_{>0} \arrow[d, "h_{\mathbb{C}}"', "\text{complex Hopf fibration}"] \arrow[dd, bend right, "h_{\mathbb{H}}"', "\text{quaternionic Hopf fibration}"'] \\
S^2 \simeq \mathbb{H}^\times/\mathbb{C}^\times \arrow[r] & \mathbb{C}P^3 \simeq (\mathbb{H}^2\backslash\{0\})/\mathbb{C}^\times \arrow[d, "t_{\mathbb{H}}"', "\text{Calabi-Penrose twistor fibration}"] \\
& \mathbb{H}P^1 \simeq (\mathbb{H}^2\backslash\{0\})/\mathbb{H}^\times
\end{tikzcd}$$

Equivariantization: Since the quotienting is by right actions, these fibrations are equivariant under the left action of

$$\mathrm{Spin}(5) \simeq \mathrm{Sp}(2) := \left\{ g \in \mathrm{GL}_2(\mathbb{H}) \,\middle|\, g^\dagger \cdot g = \mathrm{e} \right\}.$$

For example, the involution $\sigma := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in \mathrm{Sp}(2)$ swaps the two copies of $\mathbb{H}$:

$$\begin{tikzcd}
\mathbb{C}P^3 \arrow[rrr, "t_{\mathbb{H}}"] \arrow[ddd, "\sigma"] \arrow[dr, equal] & & & \mathbb{H}P^1 \arrow[ddd, "\sigma"] \arrow[dl, equal] \\
& (\mathbb{H}\times\mathbb{H}\backslash\{0\})/\mathbb{C}^\times \arrow[r] \arrow[d] & (\mathbb{H}\times\mathbb{H}\backslash\{0\})/\mathbb{H}^\times \arrow[d] & \\
& (\mathbb{H}\oplus\mathbb{H}\backslash\{0\})/\mathbb{C}^\times \arrow[r] & (\mathbb{H}\oplus\mathbb{H}\backslash\{0\})/\mathbb{H}^\times & \\
\mathbb{C}P^3 \arrow[rrr, "t_{\mathbb{H}}"'] \arrow[ur, equal] & & & \mathbb{H}P^1 \arrow[ul, equal]
\end{tikzcd}$$

The resulting $\mathbb{Z}_2$-fixed locus is the 2-sphere:

$$\begin{tikzcd}
(\mathbb{C}P^3)^{\mathbb{Z}_2} \arrow[d, "(t_{\mathbb{H}})^{\mathbb{Z}_2}"] & \simeq & (\mathbb{H}\backslash\{0\})/\mathbb{C}^\times \arrow[d] & \simeq & S^2 \arrow[d] \\
(\mathbb{H}P^1)^{\mathbb{Z}_2} & \simeq & (\mathbb{H}\backslash\{0\})/\mathbb{H}^\times & \simeq & *
\end{tikzcd}$$

Aside: Implications of Hypothesis H, in view of traditional expectations for M-theory.

The plain Hypothesis H in the bulk says that the non-perturbative completion of the C-field in 11d supergravity involves a map χ from spacetime to the homotopy type of the 4-sphere, with the C-field gauge potentials $(\widehat{C}_3, \widehat{C}_6)$ exhibiting the flux densities (G_4, G_7) as $\mathbb{R}$-rational representatives of χ.

In other words, this is the postulate that the non-perturbative C-field is a cocycle in canonical, unstable *differential 4-Cohomotopy* $\widehat{\pi}^4$ [30, §4][54, §3.1][36, Ex. 9.3].

As an immediate plausibility check: This implies, from the well-known homotopy groups of spheres in low degrees, that:

integral quantization of charges carried by singular M5-brane branes *and*

integral quantization of charges carried by singular M2-branes... plus a torsion-contribution (a first prediction of Hypothesis H).

$$
\begin{tikzcd}
 & \mathrm{Maps}(X;\, S^4) \arrow[d, "\mathbf{ch}"] & \chi \arrow[d, mapsto] & \text{Cohomotopical charge sector} \\
\Omega^1_{\mathrm{dR}}(X;\, \mathfrak{l}S^4)_{\mathrm{clsd}} \arrow[r, "\eta^{\int}"] & \int \Omega^1_{\mathrm{dR}}(X;\, \mathfrak{l}S^4)_{\mathrm{clsd}} & \mathbf{ch}(\chi) & \text{character image} \\
(G_4,\, G_7) \arrow[r, mapsto] & \eta^{\int}(G_4, G_7) \arrow[ur, Rightarrow, "{(\widehat{C}_3, \widehat{C}_6)}", "\text{gauge potentials}"'] & &
\end{tikzcd}
$$

C-field flux densities

$$
\begin{tikzcd}
\underset{\substack{\text{full nonperturbative} \\ \text{11d SuGra C-field}}}{(\widehat{C}_3,\, \widehat{C}_6)} \in & \overset{\substack{\text{canonical differential} \\ \text{non-abelian (unstable)} \\ \text{4-Cohomotopy}}}{\widehat{\pi}^4(X)} \arrow[r, "\chi", "\text{topological sector}"'] \arrow[d, "{(G_4,G_7)}"', "\substack{\text{flux} \\ \text{densities}}"] & \overset{\substack{\text{plain} \\ \text{non-abelian (unstable)} \\ \text{4-Cohomotopy}}}{\pi^4(X)} \\
 & \underset{\mathfrak{l}S^4\text{-valued de Rham cohomology}}{H_{\mathrm{dR}}(X;\, \mathfrak{l}S^4)} &
\end{tikzcd}
$$

$$
\begin{aligned}
\pi^4\big(\mathbb{R}^{10,1} \setminus \mathbb{R}^{5,1}\big) \;&=\; \pi^4\big(\mathbb{R}^{5,1} \times \mathbb{R}_+ \times S^4\big) \\
=\; \pi^4(S^4) \;&=\; \pi_4(S^4) \;=\; \mathbb{Z}
\end{aligned}
$$

$$
\begin{aligned}
\pi^4\big(\mathbb{R}^{10,1} \setminus \mathbb{R}^{2,1}\big) \;&=\; \pi^4\big(\mathbb{R}^{2,1} \times \mathbb{R}_+ \times S^7\big) \\
=\; \pi^4(S^7) \;&=\; \pi_7(S^4) \;=\; \mathbb{Z} \oplus \mathbb{Z}_{12}
\end{aligned}
$$

Hypothesis H with curvature corrections. More generally, the curvature corrections from the coupling to the background gravity are postulated to be reflected in *tangentially twisted* 4-Cohomotopy [33], analogous to the well-known twisting of the RR-field flux-quantization in K-theory by its background B-field:

Hypothesis K	Hypothesis H
$\begin{tikzcd} & & \mathrm{KU}_0 /\!\!/ \mathrm{PU}(H) \arrow[d, "\text{twisted K-theory}"] \\ X^9 \arrow[urr, dashed, "\text{RR-field}"] \arrow[rr, dashed] \arrow[dr, dashed, "\text{twist by background B-field}"'] & & B\mathrm{PU}(H) \arrow[dl, "\sim"] \\ & B^2\mathrm{U}(1) & \end{tikzcd}$	$\begin{tikzcd} & & S^4 /\!\!/ \widehat{\mathrm{Sp}(2)} \arrow[d, "\text{twisted Cohomotopy}"] \\ T^2 \times X^8 \arrow[urr, dashed, "\text{C-field}"] \arrow[rr, dashed, "\text{Fivebrane structure}"] \arrow[dr, dashed, "\text{twist by background gravity}"'] & & B\widehat{\mathrm{Sp}(2)} \arrow[dl, dashed] \\ & B\mathrm{Spin}(8) & \end{tikzcd}$

To distinguish M2/M5-charge, the tangential twisting needs to preserve the $\mathbb{H}$-Hopf fibration $\Rightarrow$ tangential $\mathrm{Sp}(2) \hookrightarrow \mathrm{Spin}(8)$-structure [33, §2.3]. With this, integrality of M2's Page charge & anomaly-cancellation of the M5's Hopf-WZ term follows from trivialization of the Euler 8-class, which means lift to the *Fivebrane* 6-group $\widehat{\mathrm{Sp}(2)} \to \mathrm{Sp}(2)$ [32, §4].

This implies [33, Prop. 3.13][32, Thm. 4.8]:

(i) half-integrally shifted quantization of M5-brane charge in curved backgrounds, *and*

(ii) integral quantization of the Page charge of M2-branes.

$$[\widetilde{G}_4] \;:=\; \underbrace{[G_4]}_{\substack{\text{C-field}\\ \text{4-flux}}} + \tfrac{1}{2}\underbrace{\left(\tfrac{1}{2}p_1(TX^8)\right)}_{\substack{\text{integral Spin-}\\ \text{Pontrjagin class}}} \in H^4\left(X^8; \mathbb{Z}\right)$$

$$2[\widetilde{G}_7] \;:=\; 2\left([G_7] + \tfrac{1}{2}[H_3 \wedge \widetilde{G}_4]\right) \;\in\; H^7(\widehat{X}^8; \mathbb{Z})$$

Both of these quantization conditions on M-brane charge are thought to be crucial for M-theory to make any sense. Previously, item (i) had remained enigmatic and item (ii) had remained wide open.

There is more:

Provable implications from Hypothesis H of subtle effects expected in M-theory:

- half-integral shift of 4-flux [33, Prop. 3.13]
- DMW anomaly cancellation [33, Prop. 3.7]
- the C-field's "integral EoM" [33, §3.6]
- M2 Page charge quantization [32, Thm. 4.8]
- integrality of $\frac{1}{6}(G_4)^3$ [54, Rem. 2.9]
- M5-brane anomaly cancellation [106]
- non-abelian gerbe field on M5 [34]

It is these results which suggest that Hypothesis H goes towards the correct flux-quantization law for the C-field in M-theory.

Orbi-worldvolumes and Equivariant charges. Flux-quantization generalizes to *orbifolds* [4] by generalizing the cohomology of the charges to *equivariant cohomology* [102].

In terms of classifying spaces this simply means that all spaces are now equipped with the action of a finite group G and all maps are required to be G-equivariant.
We take $G := \mathbb{Z}_2$ and the classifying fibration to be the **twistor fibration** $p := t_{\mathbb{H}}$ equivariant under swapping the $\mathbb{H}$-summands,

$$
\begin{tikzcd}
\text{orbi-worldvolume} & \overset{G \curvearrowright}{\Sigma} \arrow[rr, dashed, "\text{orbi-brane charges}"] \arrow[d, "\phi"'] && \overset{G \curvearrowright}{\mathcal{A}} \arrow[r, equal] \arrow[d, "p"', "\substack{\text{equivariant}\\ \text{classifying}\\ \text{fibration...}}"] \arrow[dll, phantom, "\Leftarrow"] & \overset{\mathbb{Z}_2 \curvearrowright}{\mathbb{C}P^3} \arrow[d, "t_{\mathbb{H}}"', "\substack{\text{...for equivariant}\\ \text{twistorial Cohomotopy}}"] \\
\text{orbi-spacetime} & \underset{G \curvearrowleft}{X} \arrow[rr, dashed, "\text{orbi-bulk charges}"'] && \underset{G \curvearrowleft}{\mathcal{B}} \arrow[r, equal] & \underset{\mathbb{Z}_2 \curvearrowleft}{S^4}
\end{tikzcd}
$$

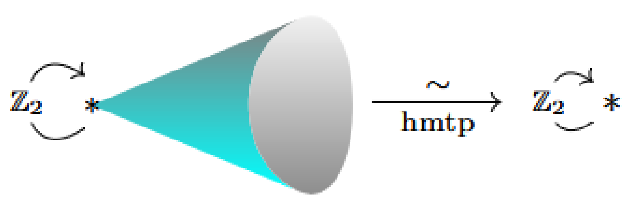
and the brane/bulk orbifold we take to be as on p. 3:

The orbi-brane diagram for a flat M5-brane wrapped on a trivial Seifert-fibered orbi-singularity. Shaded is the $\mathbb{Z}_2$-fixed locus/orbi-singularity.

We are adjoining the *point at infinity* to the space $\mathbb{R}^2_{\cup\{\infty\}} \underset{\text{homeo}}{\simeq} S^2$ which is thereby designated as transverse to any worldvolume solitons to be measured in reduced cohomology.

$$
\begin{array}{ccccccccccc}
\overset{\mathbb{Z}_2 \curvearrowright}{\Sigma} & := & \mathbb{R}^{1,0} & \times & \mathbb{R}^2_{\cup\{\infty\}} & \times & S^1 & \times & \overset{\mathbb{Z}_2 \curvearrowright}{\mathbb{R}^2_{\mathrm{sgn}}} & & \\
\downarrow \phi & & & & & & & & & & \\
\underset{\mathbb{Z}_2 \curvearrowleft}{X} & := & \mathbb{R}^{1,0} & \times & \mathbb{R}^2_{\cup\{\infty\}} & \times & S^1 & \times & \overset{\mathbb{Z}_2 \curvearrowright}{\mathbb{R}^2_{\mathrm{sgn}}} & \times & \mathbb{R}^5 \\
 & & \text{time} & & \substack{\text{trnsvrs space}\\ \text{to solitons}} & & \substack{\text{M/IIA-}\\ \text{circle}} & & \substack{\text{orbi-}\\ \text{cone}} & & \substack{\text{trnsvrs space}\\ \text{to M5-brane}}
\end{array}
$$

But since the cone $\mathbb{Z}_2 \circlearrowright \mathbb{R}^2_{\mathrm{sgn}}$ is equivariantly contractible,



the inclusion of the $\mathbb{Z}_2$-fixed loci is actually a homotopy equivalence

$$
\begin{tikzcd}
\Sigma^{\mathbb{Z}_2} \arrow[r, "\sim", "\text{hmtp}"'] \arrow[d, "\phi^{\mathbb{Z}_2}"'] & \overset{\mathbb{Z}_2 \curvearrowright}{\Sigma} \arrow[d, "\phi"] \\
X^{\mathbb{Z}_2} \arrow[r, "\sim", "\text{hmtp}"'] & \underset{G \curvearrowleft}{X}
\end{tikzcd}
$$

Therefore our equivariant classifying maps are determined up to equivariant homotopy by their restriction to the fixed-locus and hence the charges are *localized on the orbi-singularity* where they take values in 2-Cohomotopy:

$$
\left\{
\begin{tikzcd}
\overset{\mathbb{Z}_2 \curvearrowright}{\Sigma} \arrow[r, dashed] \arrow[d, "\phi"'] & \overset{\mathbb{Z}_2 \curvearrowright}{\mathbb{C}P^3} \arrow[d, "t_{\mathbb{H}}"] \arrow[dl, phantom, "\Leftarrow"] \\
\underset{\mathbb{Z}_2 \curvearrowleft}{X} \arrow[r, dashed, "\substack{\text{charges on}\\ \text{orbifold}}"'] & \underset{\mathbb{Z}_2 \curvearrowleft}{S^4}
\end{tikzcd}
\right\}
\simeq
\left\{
\begin{tikzcd}
\Sigma^{\mathbb{Z}_2} \arrow[r, dashed] \arrow[d, "\phi^{\mathbb{Z}_2}"'] & (\mathbb{C}P^3)^{\mathbb{Z}_2} \arrow[r, equal] \arrow[d, "t_{\mathbb{H}}^{\mathbb{Z}_2}"] \arrow[dl, phantom, "\Leftarrow"] & S^2 \arrow[d] \\
X^{\mathbb{Z}_2} \arrow[r, dashed] & (S^4)^{\mathbb{Z}_2} \arrow[r, equal] & \ast
\end{tikzcd}
\right\}
\simeq
\left\{
\mathbb{R}^2_{\cup\{\infty\}} \times S^1 \dashrightarrow S^2
\right\}
$$

charges localized on orbi-singularity

charges in 2-Cohomotopy of B-field solitons on M5 orbi-singularity

Orbi-worldvolumes and Equivariant charges. Flux-quantization generalizes to *orbifolds* [4] by generalizing the cohomology of the charges to *equivariant cohomology* [102].

In terms of classifying spaces this simply means that all spaces are now equipped with the action of a finite group G and all maps are required to be G-equivariant.
We take $G := \mathbb{Z}_2$ and the classifying fibration to be the **twistor fibration** $p := t_{\mathbb{H}}$ equivariant under swapping the $\mathbb{H}$-summands,

$$
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\text{orbi-spacetime} & \underset{G \circlearrowleft}{X} \arrow[r, dashed, "\text{orbi-bulk charges}"'] & \underset{G \circlearrowleft}{\mathcal{B}} \arrow[r, equal] & \underset{\mathbb{Z}_2 \circlearrowleft}{S^4}
\end{tikzcd}
$$

equivariant classifying fibration... ...for equivariant twistorial Cohomotopy

and the brane/bulk orbifold we take to be as on p. 3:

This situation to be

analyzed in the third lecture,

using the following general tools:

- Pontrjagin theorem: Cohomotopy sets $\leftrightarrow$ Cobordism classes of solitons
- Segal theorem: Cohomotopy moduli $\leftrightarrow$ configuration spaces of solitons
- Pontrjagin algebra of Cohomotopy moduli: quantum observables on solitons

Therefore our equivariant classifying maps are determined up to equivariant homotopy by their restriction to the fixed-locus and hence the charges are *localized on the orbi-singularity* where they take values in 2-Cohomotopy:

$$
\left\{
\begin{tikzcd}
\overset{\mathbb{Z}_2 \circlearrowright}{\Sigma} \arrow[r, dashed] \arrow[d, "\phi"'] & \overset{\mathbb{Z}_2 \circlearrowright}{\mathbb{C}P^3} \arrow[d, "t_{\mathbb{H}}"] \arrow[dl, phantom, "\Leftarrow"] \\
\underset{\mathbb{Z}_2 \circlearrowleft}{X} \arrow[r, dashed, "\text{charges on orbifold}"'] & \underset{\mathbb{Z}_2 \circlearrowleft}{S^4}
\end{tikzcd}
\right\}
\simeq
\left\{
\begin{tikzcd}
\Sigma^{\mathbb{Z}_2} \arrow[r, dashed] \arrow[d, "\phi^{\mathbb{Z}_2}"'] & (\mathbb{C}P^3)^{\mathbb{Z}_2} \arrow[r, equal] \arrow[d, "t_{\mathbb{H}}^{\mathbb{Z}_2}"] \arrow[dl, phantom, "\Leftarrow"] & S^2 \arrow[d] \\
X^{\mathbb{Z}_2} \arrow[r, dashed] & (S^4)^{\mathbb{Z}_2} \arrow[r, equal] & *
\end{tikzcd}
\right\}
\simeq
\left\{
\mathbb{R}^2_{\cup\{\infty\}} \times S^1 \dashrightarrow S^2
\right\}
$$

charges localized on orbi-singularity

charges in 2-Cohomotopy of B-field solitons on M5 orbi-singularity

Remarkably, there is an equivalence between *Cohomotopy* of spacetime/worldvolumes and *Cobordism* classes of submanifolds behaving like solitonic branes carrying the corresponding Cohomotopy charge [103, §2.2] [101, §2.1]:

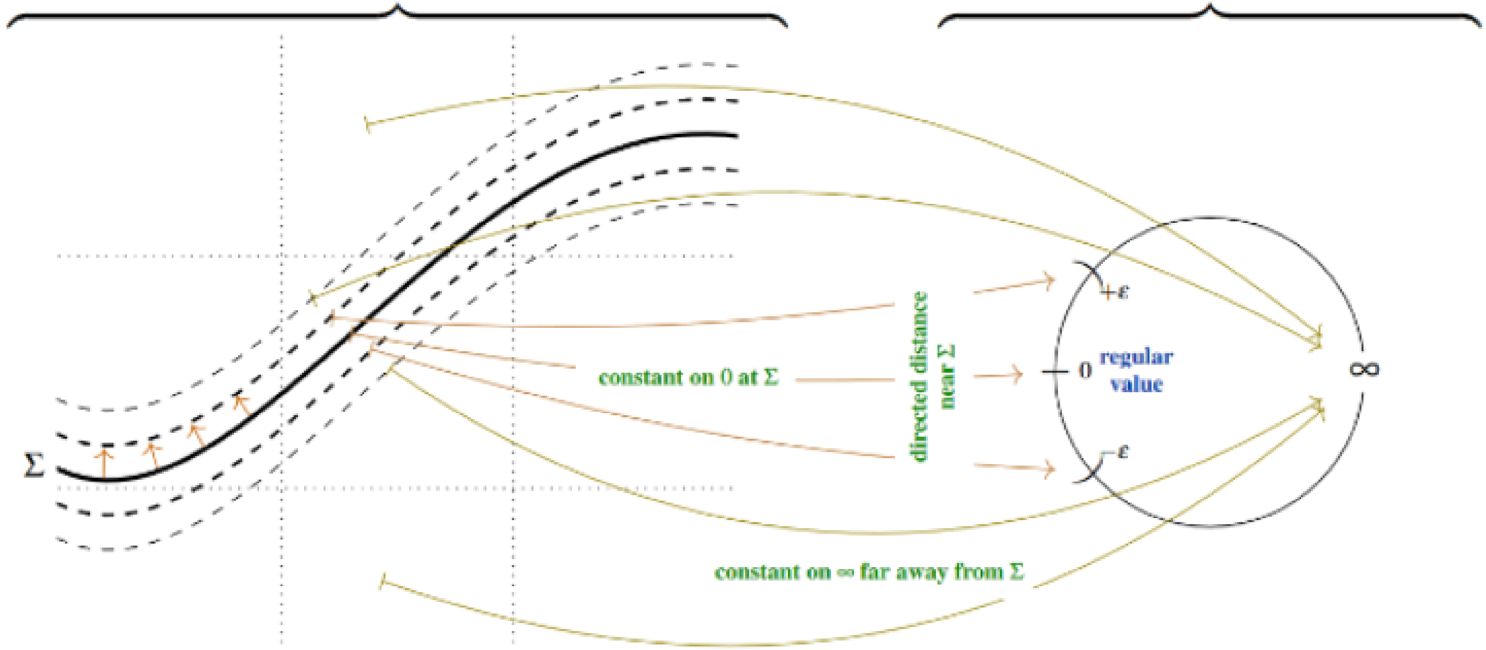
The **Pontrjagin theorem** [70][69, §IX] identifies the unstable n-Cohomotopy of a closed manifold with the cobordism classes of its normally framed submanifolds of co-dimension n.

$$
\begin{tikzcd}[column sep=huge]
\overset{\substack{\text{framed unstable}\\ n\text{-Cobordism of } M^d}}{\mathrm{Cob}^n_{\mathrm{Fr}}(M^d)} := \mathrm{NFramedSubmflds}_{d-n}(M^d)_{/\mathrm{brdsm}}
\arrow[r, shift left=2, "\text{directed asymptotic distance}"]
\arrow[r, phantom, "\simeq"]
\arrow[r, bend left=30, "\text{assign Cohomotopy charge}"]
&
\mathrm{Maps}(M^d, S^n)_{/\mathrm{hmtpy}} =: \overset{\substack{\text{unstable}\\ n\text{-Cohomotopy of } M^d}}{\underset{\substack{\text{closed}\\ \text{manifold}}}{\pi^n(M^d)}}.
\arrow[l, shift left=2, "\text{pre-image of regular value 0}"]
\arrow[l, bend left=30, "\text{reconstruct submanifold from its charge}"]
\end{tikzcd}
$$

The **Cohomotopy charge** of a normally framed submanifold (aka *scanning map* or *Pontrjagin-Thom collapse*) is represented by mapping points of the ambient space to their directed distance if inside a tubular neighbourhood, else to ∞.

Conversely, every Cohomotopy class is representated by a smooth map with 0 a regular value, whose preimage is a normally framed submanifold with that Cohomotopy charge.

$$
\underset{\substack{\text{closed submanifold,}\\ \text{normally framed}}}{\Sigma^{d-n}} \hookrightarrow \underset{\text{manifold}}{M^d} \xrightarrow[\substack{\text{directed asymptotic distance from }\Sigma\\ \simeq\text{ cocycle representing Cohomotopy charge of }\Sigma}]{c} \underset{\substack{\text{Cohomotopy classifying space}\\ (n\text{-sphere})}}{\mathbb{R}^n_{\mathrm{cpt}} = S^n}
$$



Moduli space of soliton configurations. But the Pontrjagin theorem concerns only the total cohomotopical charge, identifying it with the *net* (anti-)brane content. Beyond that we have the whole *moduli space* of charges (considered now specialized to our 2D transverse space), and **Segal's theorem** [111] says that the cohomotopy charge map (scanning map) identifies this with a moduli space of brane positions, namely with the *group-completed configuration space of points* [15][120][43]:

$$\begin{tikzcd}[column sep=huge] \mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big) \arrow[r, "{[-]}", "\text{net charge}"'] & \pi^2\big(\mathbb{R}^2_{\cup\{\infty\}}\big) \arrow[dr, "\sim"', "\text{Hopf degree theorem}"] & \\ \mathbb{G}\,\mathrm{Conf}(\mathbb{R}^2) \arrow[u, "\wr"', "\text{Segal theorem}"] \arrow[r, "{[-]}", "\text{net brane content}"'] & \mathrm{Cob}^2_{\mathrm{Fr}}(\mathbb{R}^2) \arrow[u, "\wr", "\text{Pontrjagin theorem}"'] \arrow[r, "\sim"] & \mathbb{Z} \end{tikzcd}$$

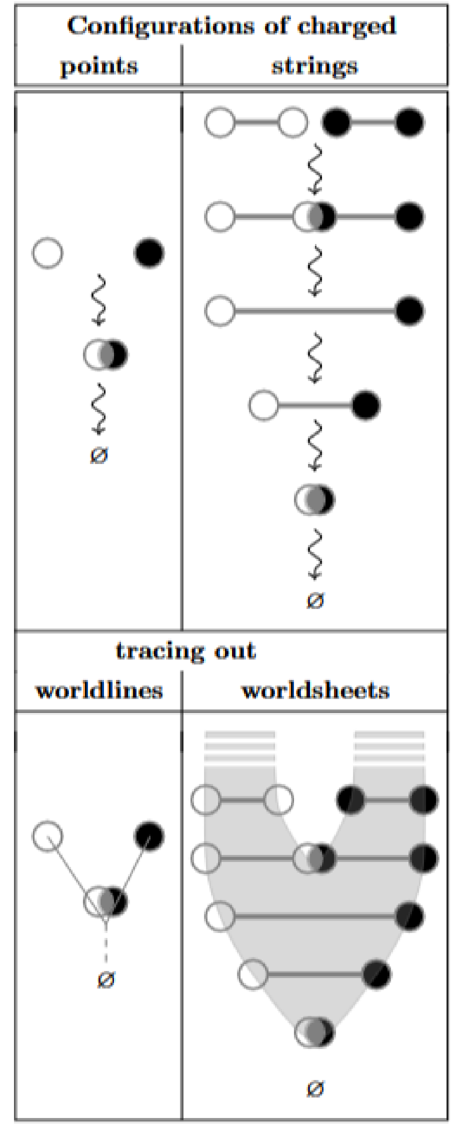
moduli space of solitonic brane charges | 2-Cohomotopy

moduli space of solitonic brane positions | group-completed config space | 2-Cobordism (unstable)

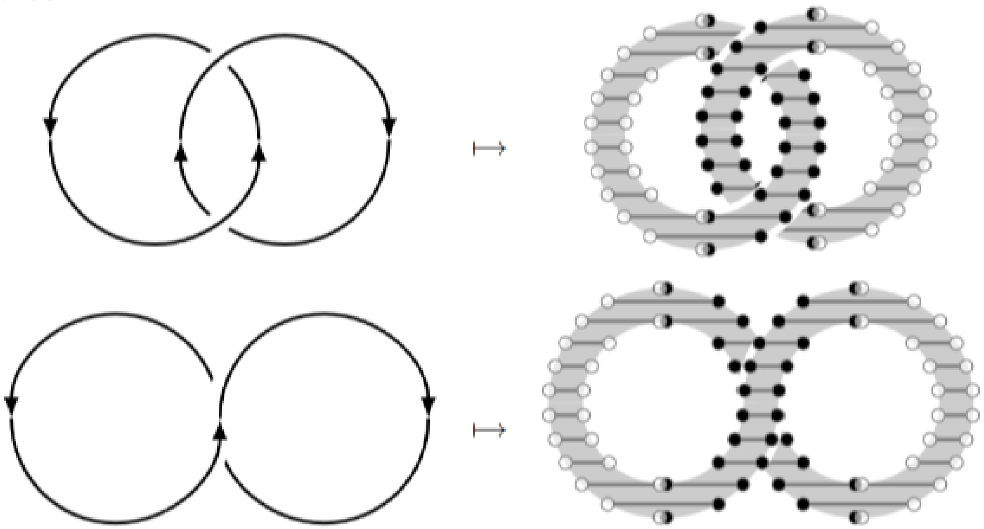
where the *configuration space of points* is the space of finite subsets of $\mathbb{R}^2$ – here understood as the space of positions of cores of solitons of unit charge $+1$,

$$\mathrm{Conf}(\mathbb{R}^2) \;=\; \Big\{$$  $$\Big\}$$

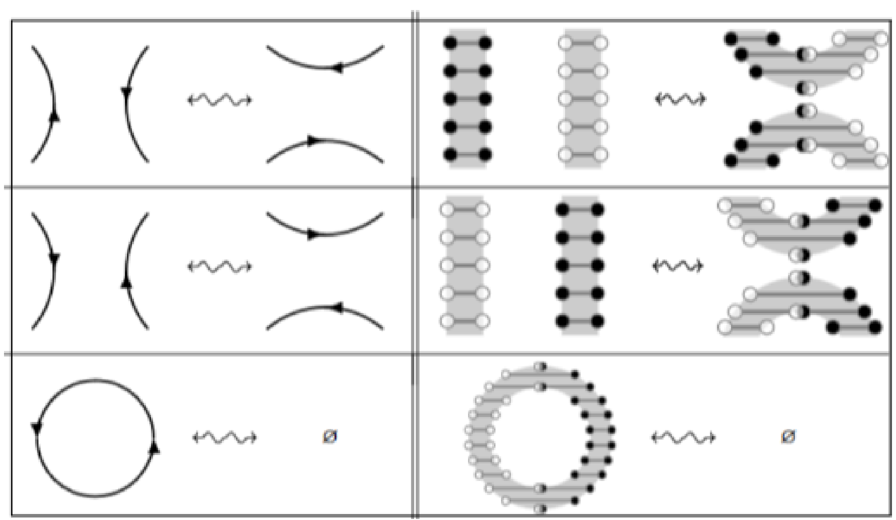
and its *group completion* $\mathbb{G}(-)$ is the topological completion of the topological partial monoid structure given by disjoint union of soliton configurations.
Naïvely this is given by including also **anti-solitons** in the form of configurations of $\pm$-*charged points*, topologized such as to allow for their pair annihilation/creation as shown in the left column on the right.
Remarkably, closer analysis reveals [89] that the group completion $\mathbb{G}(-)$ produces configurations of **strings** (extending parallel to one axis in $\mathbb{R}^3$) **with charged endpoints** whose pair annihilation/creation is smeared-out to string worldsheets as shown in the right column.

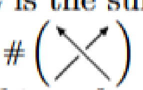
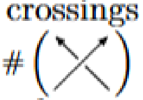


This means [105] that the **vacuum-to-vacuum soliton scattering processes**, forming the loop space $\Omega\,\mathbb{G}\,\mathrm{Conf}(\mathbb{R}^2)$, are identified with *framed links* ([90, pp 15]), for instance



subject to *link cobordism* (cf. [76]):



It follows [105, Thm 3.17] that the charge of a soliton scattering process L is the sum over crossings of the *crossing number* #  $= +1$, #  $= -1$, which equals the linking+framing number:

$$\Omega\mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \overset{\sim}{\to} \Omega\mathrm{Maps}^{*/}(\mathbb{R}^2_{\cup\{\infty\}}, S^2) \overset{[-]}{\to} \pi_3(S^2) \simeq \mathbb{Z}$$

$$L \xmapsto[\text{linking + framing number}]{\text{total crossing number =}} \#L$$

But this is precisely the **Wilson loop observable** of L **in** (abelian) **Chern-Simons theory!** [105, §4]
As we explain next.

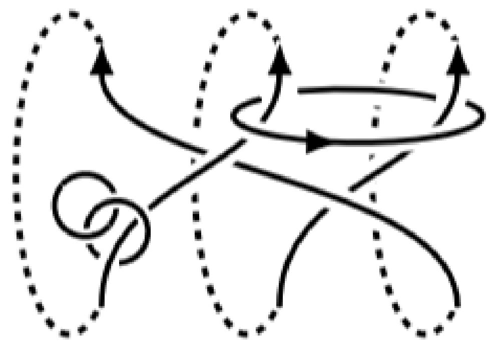
The k-Soliton sector. More generally, we may consider loops based in the kth connected component of the moduli space, corresponding to scattering process from k to k net number of solitons.

$$
\begin{tikzcd}
\text{net charge } k & \text{Hopf degree } k \\
\mathbb{G}\mathrm{Conf}_k(\mathbb{R}^2) \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{Maps}^*_k\big(\mathbb{R}^2_{\cup\{\infty\}}, S^2\big) \arrow[d, hook] \\
\mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \arrow[r, "\sim"] & \mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}}, S^2\big)
\end{tikzcd}
$$

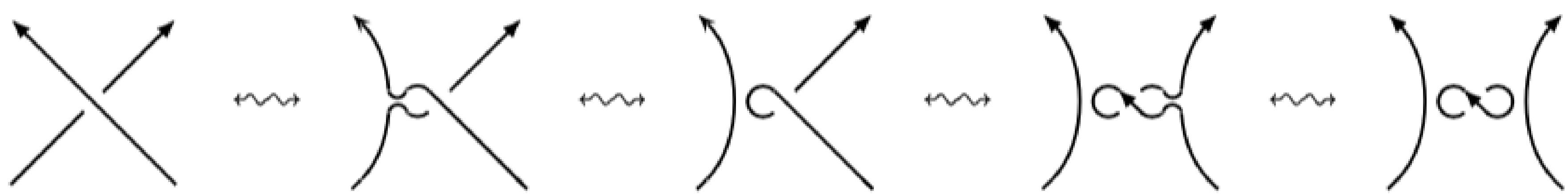
Since the double loop space $\mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}}, S^2\big)$ admits the structure of a topological group, all these connected components have the same homotopy type, and hence these scattering processes L are again classified by the integer total crossing number $\#L$ which is the abelian Chern-Simons Wilson-loop observable.

$$
\begin{tikzcd}
\Omega_k\, \mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \arrow[d, two heads] \arrow[dr, "L \,\mapsto\, \#L"] & \\
\pi_0\Omega_k\, \mathbb{G}\mathrm{Conf}(\mathbb{R}^2) \arrow[r, "\sim"'] & \mathbb{Z}
\end{tikzcd}
$$

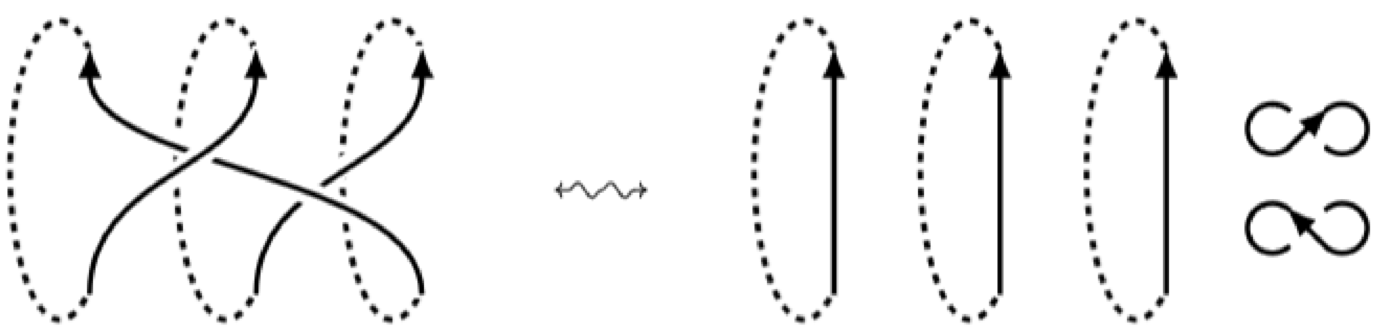
For instance, a generic $k = 3$ process looks like this:



and via the framed cobordism moves



it computes to the trivial scattering process accompanied by $\#L$ vacuum pair braiding processes:



Chern-Simons level. We will see below further meanings of the number k:

This integer k is equivalently $\left\{ \begin{array}{l} \text{the } \textit{number} \text{ of fractional quasi-hole vortices in a quantum Hall system,} \\ \text{the } \textit{level} \text{ of their effective abelian Chern-Simons theory,} \\ \text{the } \textit{maximal denominator} \text{ for filling fractions of their quantum states.} \end{array} \right.$

Generally, we will recover in a novel *non-Lagrangian* way the features of quantum Chern-Simons theory that are traditionally argued starting with the kth multiple of the local Lagrangian density $a \wedge da$ for a gauge potential 1-form a.

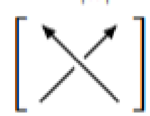
The situation on the 2-Sphere.
Furthermore consider k solitons on the actual 2-sphere S^2.
Here the 2-Cohomotopy moduli space satisfies (cf. [42]):

$$\pi_0 \Omega_k \, \mathrm{Maps}\big(S^2,\, S^2\big) \;\simeq\; \mathbb{Z}_{2|k|}\,,$$

and the long homotopy fiber sequence induced by point evaluation shows that the generator of this cyclic group is again identified with the basic half-braiding operation:

$$\mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}}, S^2\big) \xrightarrow{\text{fiber of...}} \mathrm{Maps}\big(S^2, S^2\big) \xrightarrow{\substack{\text{point-}\\ \text{evaluation}}} S^2$$

$$\underbrace{\pi_2\big(S^2\big)}_{\mathbb{Z}} \xrightarrow{2k} \underbrace{\pi_0 \Omega_k \mathrm{Maps}^*\big(\mathbb{R}^2_{\cup\{\infty\}}, S^2\big)}_{\mathbb{Z}} \longrightarrow \underbrace{\pi_0 \Omega_k \mathrm{Maps}\big(S^2, S^2\big)}_{\mathbb{Z}_{2|k|}} \longrightarrow \underbrace{\pi_1\big(S^2\big)}_{1}$$

 $\longmapsto$ 

With flux-quantized fields being equipped with a classifying space $\mathcal{A}$, there is a neat way to directly obtain the topological quantum observables – via the following observation:

Topological flux observables in Yang-Mills theory – Theorem [108]. For G-Yang-Mills theory on $\mathbb{R}^{1,1} \times \Sigma^2$, with a choice of Ad-invariant lattice $\Lambda \subset \mathfrak{g}$:
(i) Non-perturbative quantization of the algebra of flux observables through the closed surface Σ^2 is given by the group C^*-algebra $\mathbb{C}[-]$ of the Fréchet-Lie group of smooth maps $\Sigma^2 \to G \ltimes (\mathfrak{g}/\Lambda)$
(ii) the subalgebra of topological observables coincides with the Pontrjagin homology algebra of pointed maps $(\mathbb{R}^1 \times \Sigma^2)_{\cup\{\infty\}} \to B\big(G \ltimes (\mathfrak{g}/\Lambda)\big)$:

$$\underbrace{\mathbb{C}\Big[C^\infty\big(\Sigma^2, G\big) \ltimes C^\infty\big(\Sigma^2, (\mathfrak{g}/\Lambda)\big)\Big]}_{\substack{\text{non-perturbative quantum algebra of} \\ \text{observables on flux through } \Sigma^2}} \xleftarrow{(\pi^0)^*} \underbrace{\mathbb{C}\Big[H^0\big(\Sigma^2; G\big) \ltimes H^1\big(\Sigma^2; \Lambda\big)\Big]}_{\substack{\text{sub-algebra of} \\ \text{topological observables}}} \simeq \underbrace{H_0\Big(\mathrm{Maps}^*\big((\mathbb{R}^1 \times \Sigma^2)_{\cup\{\infty\}}, B(G \ltimes (\mathfrak{g}/\Lambda)\big); \mathbb{C}\Big)}_{\substack{\text{Pontrjagin homology algebra of} \\ \text{moduli space of soliton charges}}}$$

For example in electromagnetism, with $G = \mathrm{U}(1)$ and $\Lambda := \mathbb{Z} \hookrightarrow \mathbb{R}$:

$$\mathbb{C}\Big[\underbrace{H^1(\Sigma^2;\mathbb{Z})}_{\text{electric}} \times \underbrace{H^1(\Sigma^2;\mathbb{Z})}_{\text{magentic}}\Big] \simeq H_0\Big(\mathrm{Maps}^*\big((\mathbb{R}^1 \times \Sigma^2)_{\cup\{\infty\}}, \underbrace{B\mathrm{U}(1) \times B\mathrm{U}(1)}_{\substack{\text{classifying space for} \\ \text{Dirac flux quantization}}}\big); \mathbb{C}\Big)$$

This allows to generalize:

Topological flux observables of any higher gauge theory.

For a higher gauge theory flux-quantized in $\mathcal{A}$-cohomology the quantum algebra of topological flux observables on a spacetime of the form $\mathbb{R}^{1,1} \times \Sigma^{D-2}$ is the Pontrjagin homology algebra of the soliton moduli hence in deg $= 0$ is the group algebra of vacuum soliton processes "**on the light-cone**":

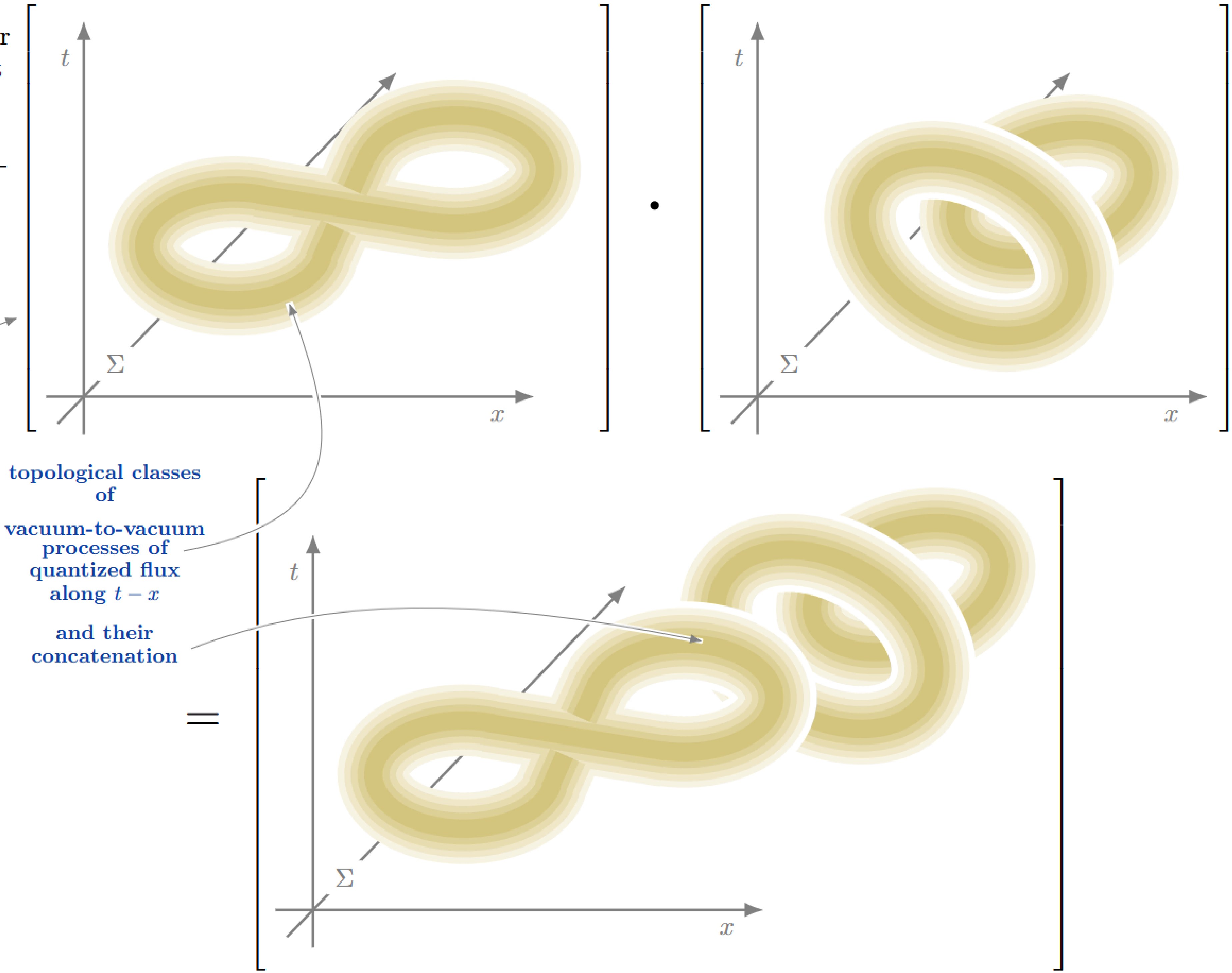
$$\begin{aligned} \mathrm{Obs}_\bullet &:= H_\bullet\Big(\mathrm{Maps}^*\big((\mathbb{R}^1 \times \Sigma^{D-2})_{\cup\{\infty\}}, \mathcal{A}\big); \mathbb{C}\Big) \\ &\simeq H_\bullet\Big(\Omega\,\mathrm{Maps}\big(\Sigma^{D-2}, \mathcal{A}\big); \mathbb{C}\Big) \\ \mathrm{Obs}_0 &= \mathbb{C}\Big[\pi_0 \Omega\,\mathrm{Maps}\big(\Sigma^{D-2}, \mathcal{A}\big)\Big] \end{aligned}$$

For note that the star-involution is given by the *combination* of

- complex conjugation (time reversal)
- loop reversal (hence x-reversal)

where $\mathbb{R}^{1,1} \simeq \mathbb{R}\langle t, x \rangle$,

and the operator product is given by loop concatenation:



In the next lecture we discuss
concretely these quantum observables
on orbi-M5 probes
finding them exhibit
anyonic topological order
as in fractional quantum Hall systems.