Effective Quantum Certification via Linear Homotopy Types

Urs Schreiber (NYU Abu Dhabi)
on joint work at CQTS with
D. J. Myers, M. Riley,
and Hisham Sati

CENTER FOR
QUANTUM &
TOPOLOGICAL
SYSTEMS

presentation at:

The Topos Institute Colloquium, 13 Apr 2023

slides and further pointers at: ncatlab.org/Quantum+Certification+via+Linear+Homotopy+Types#TI2023

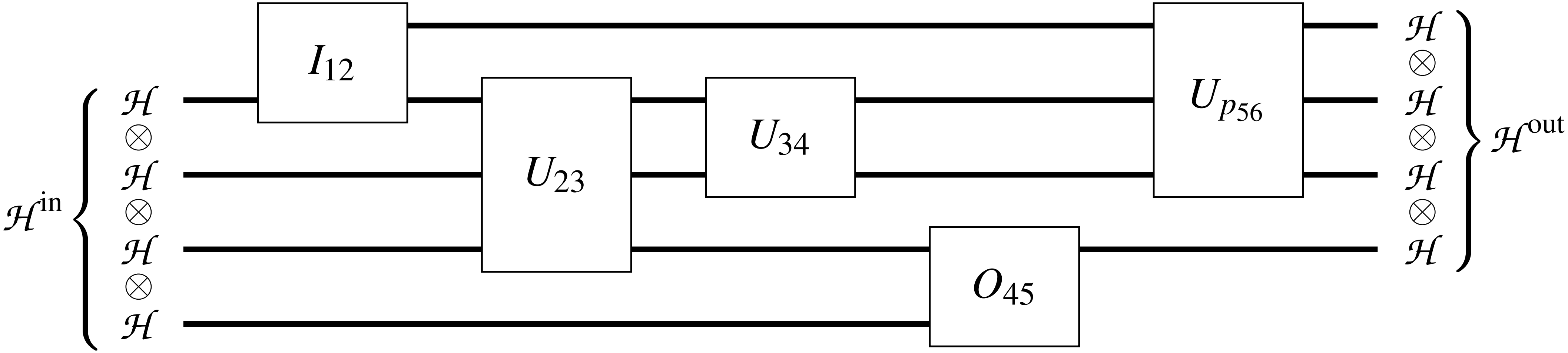
The Problem

in

Quantum Computing

Pure quantum circuits are easy...

Linear operator composed & tensored from given *quantum logic gates*


I_{12}
U_{23}
U_{34}
U_{p56}
O_{45}
$\mathcal{H}^{\text{in}}$
$\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$
$\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$
$\mathcal{H}^{\text{out}}$

Hilbert space of possible input quantum states

linear transformation
upon execution

Hilbert space of possible output quantum states

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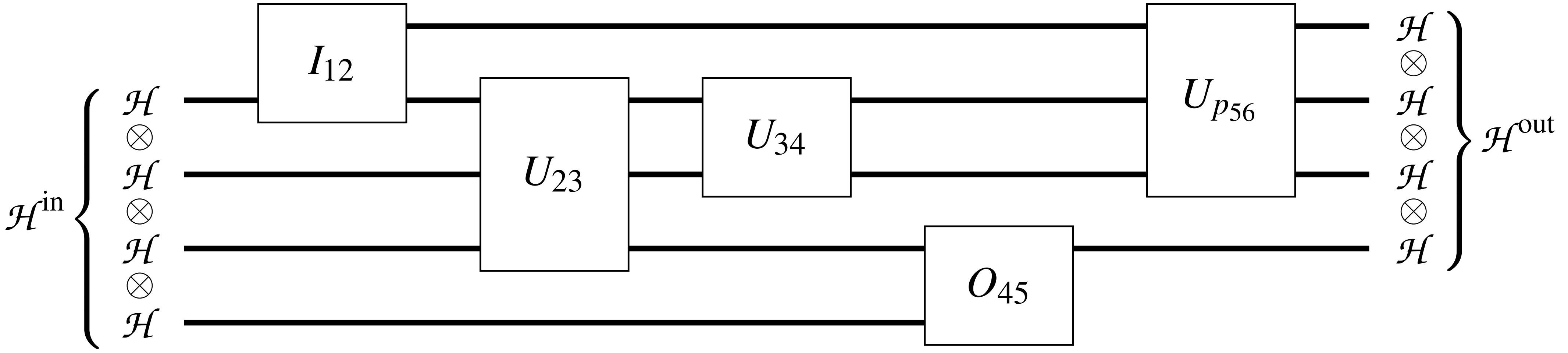
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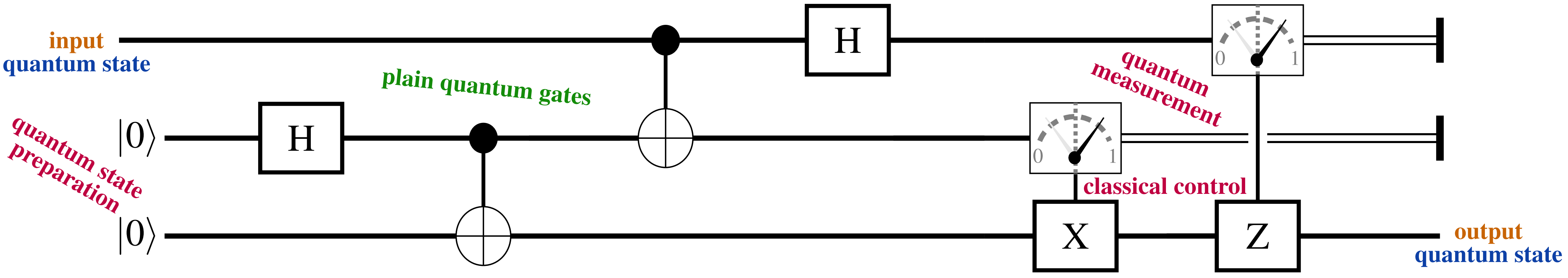
Hilbert space of possible input quantum states

linear transformation upon execution

Hilbert space of possible output quantum states

but real quantum circuits have classical control & effects

(Example: QBit Teleportation protocol)



input quantum state

plain quantum gates

quantum measurement

quantum state preparation

classical control

output quantum state

full reality is a loop: Classical $\underset{\text{prepare}}{\overset{\text{measure}}{\leftrightarrows}}$ Quantum

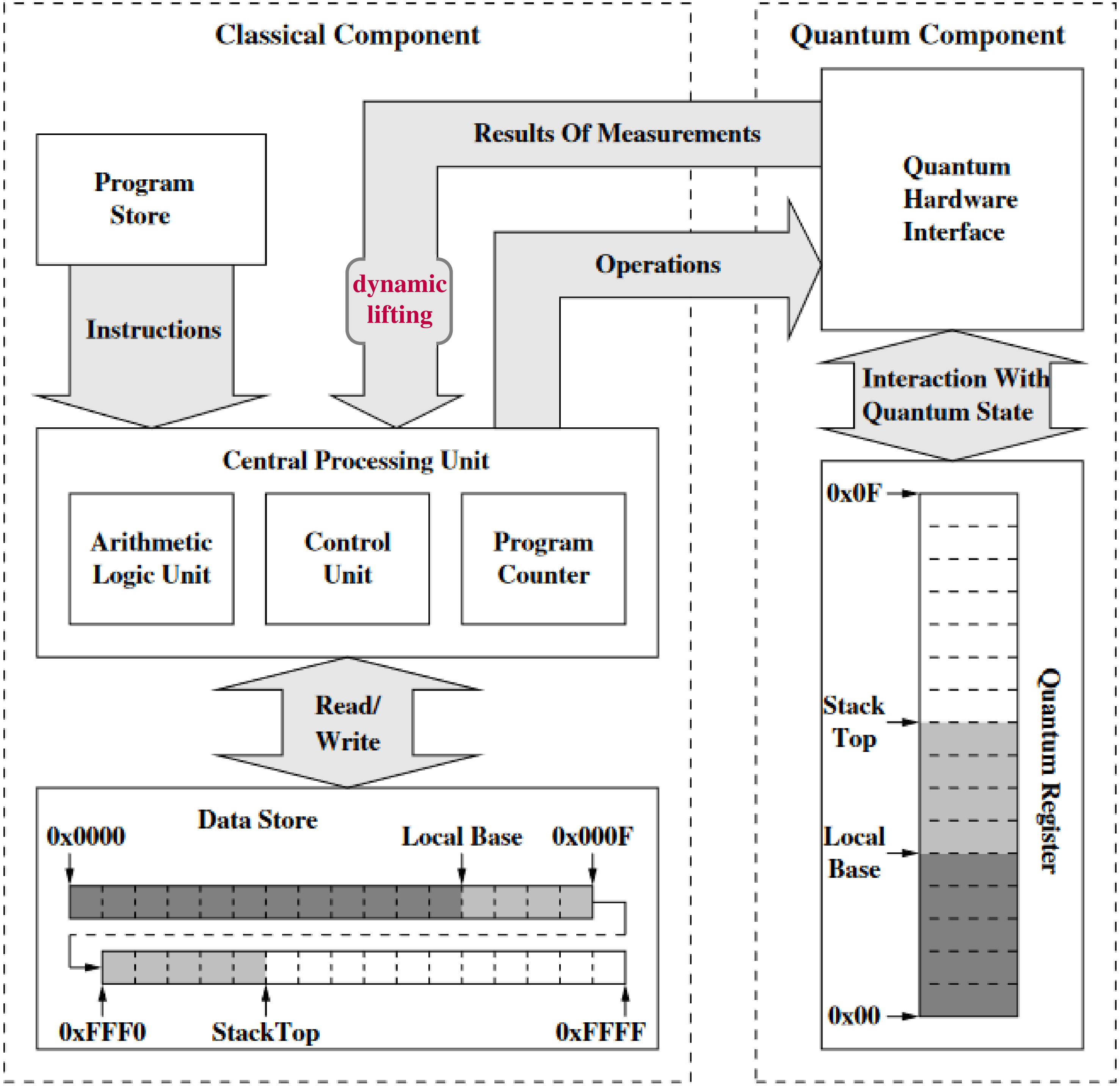


diagram adapted from Nagarajan et al. (2007)

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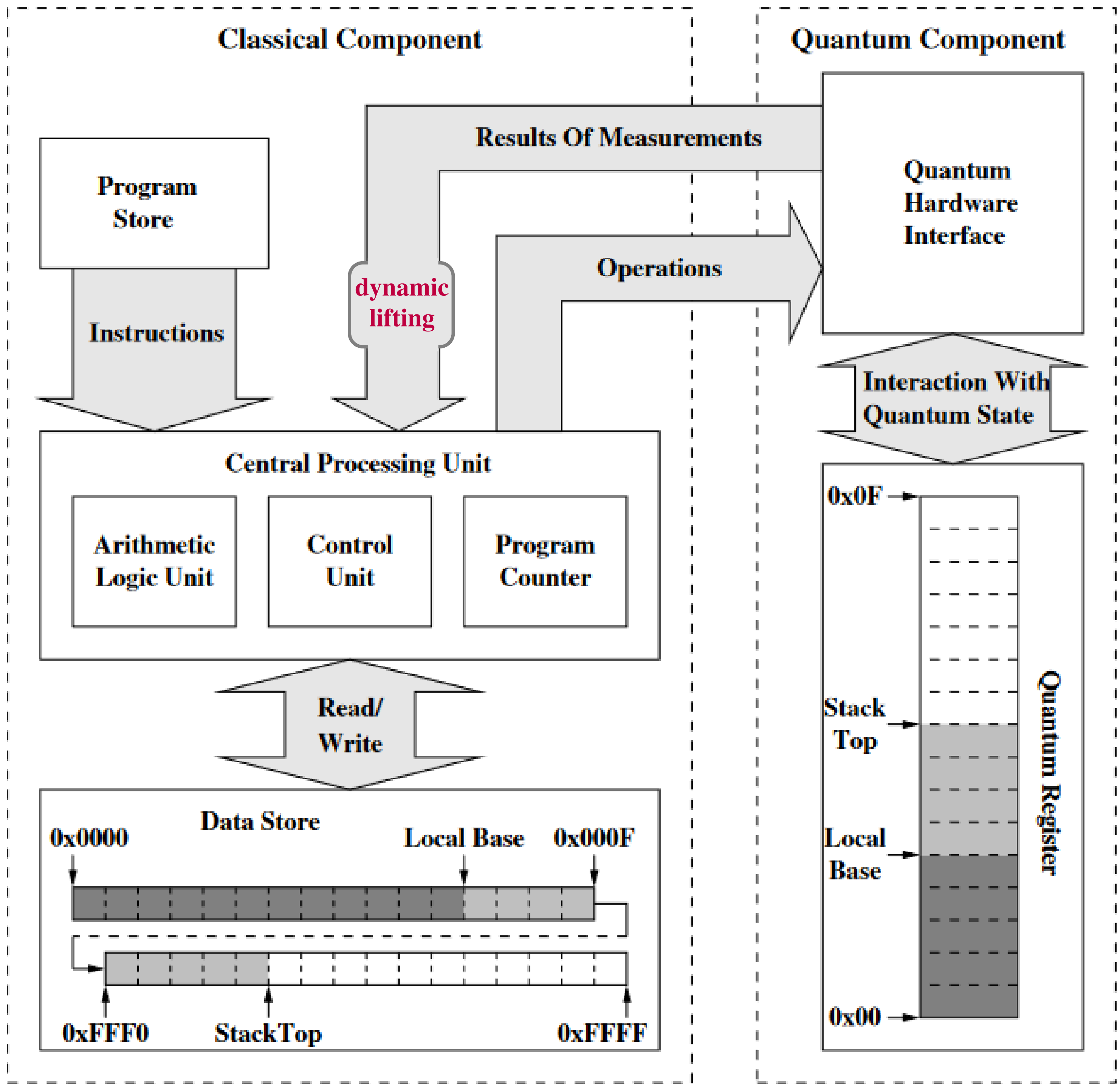

full loop needed e.g. for quantum error correction

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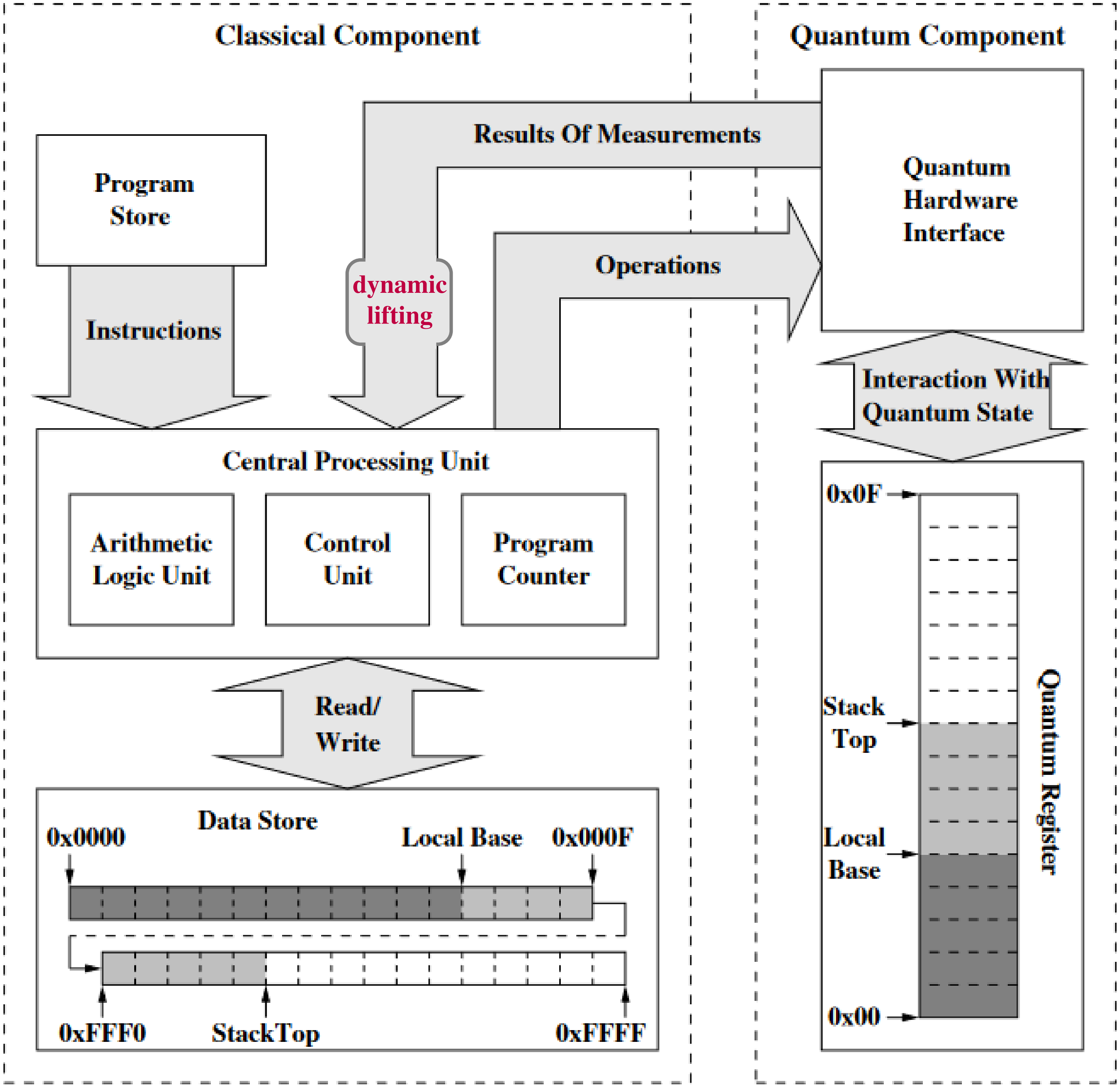

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full loop needed e.g. for quantum error correction

but its formal language theory had remained thin

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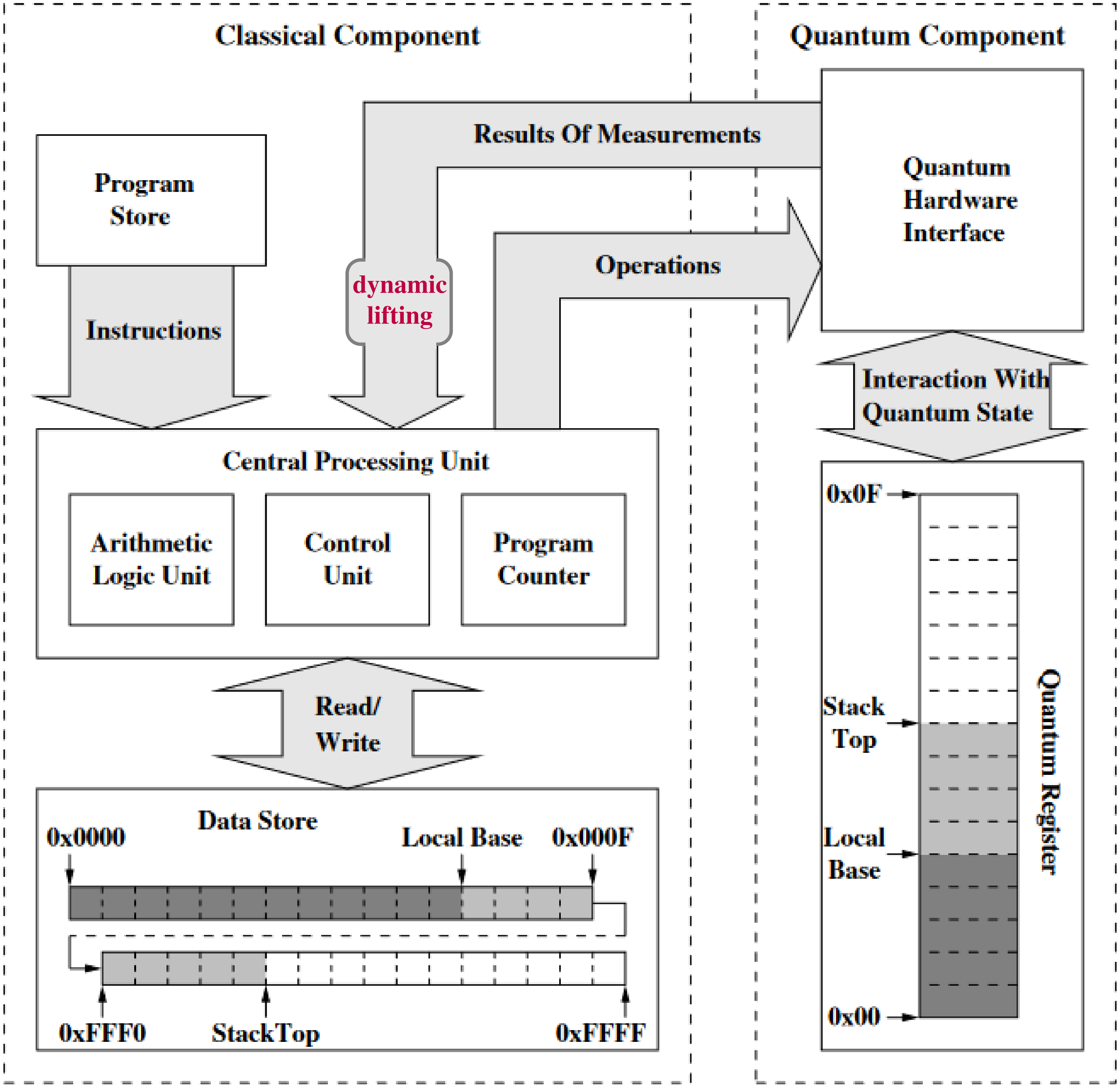

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full loop needed e.g. for quantum error correction

but its formal language theory had remained thin

existing models for dynamic lifting are ad hoc & unverified

Existing quantum typed circuit languages

are embedded inside *classical* type theories:

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Classical Type Theory (e.g. `Haskell`, `Coq`, ...)

for universal classical computation

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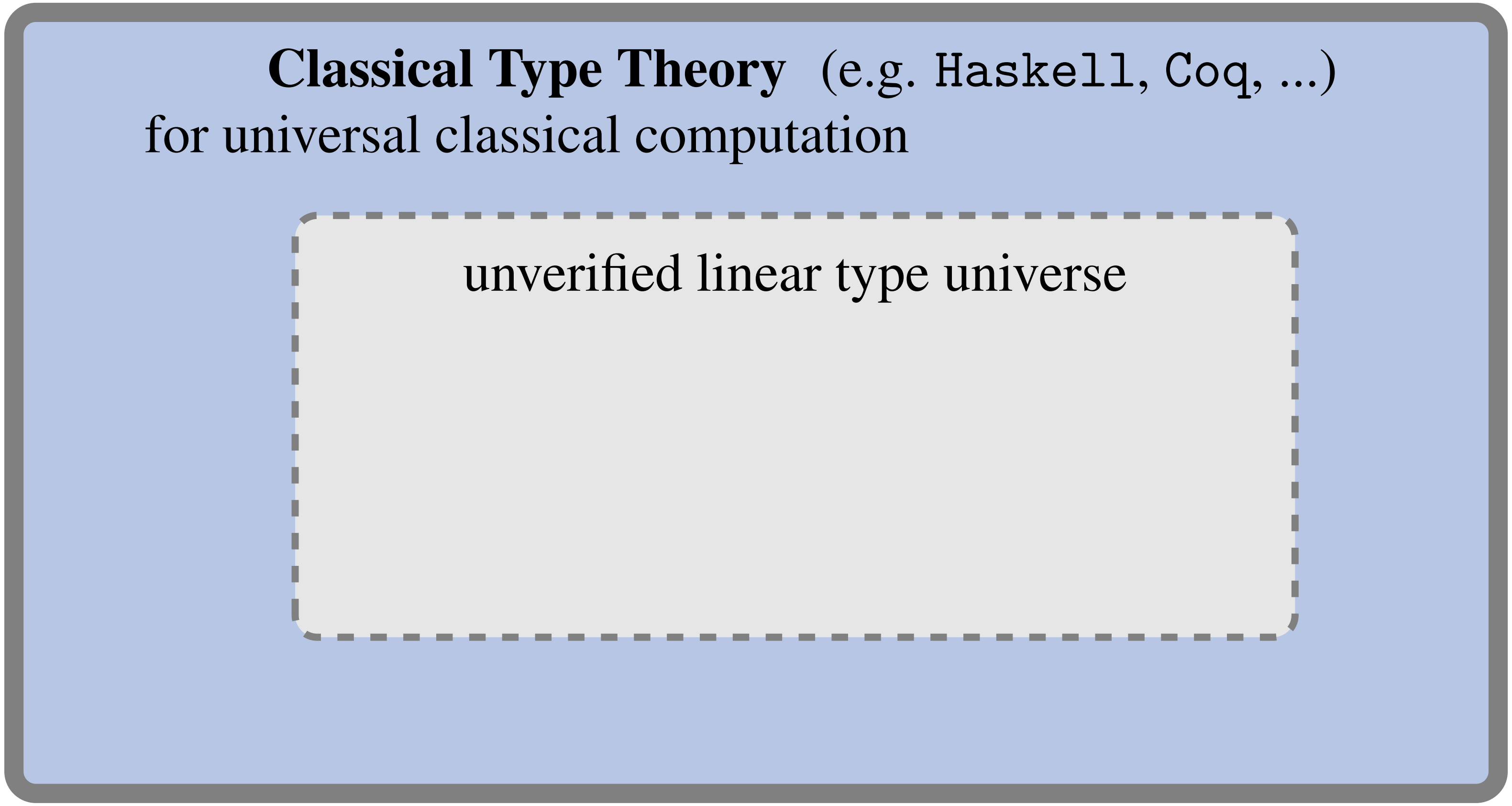

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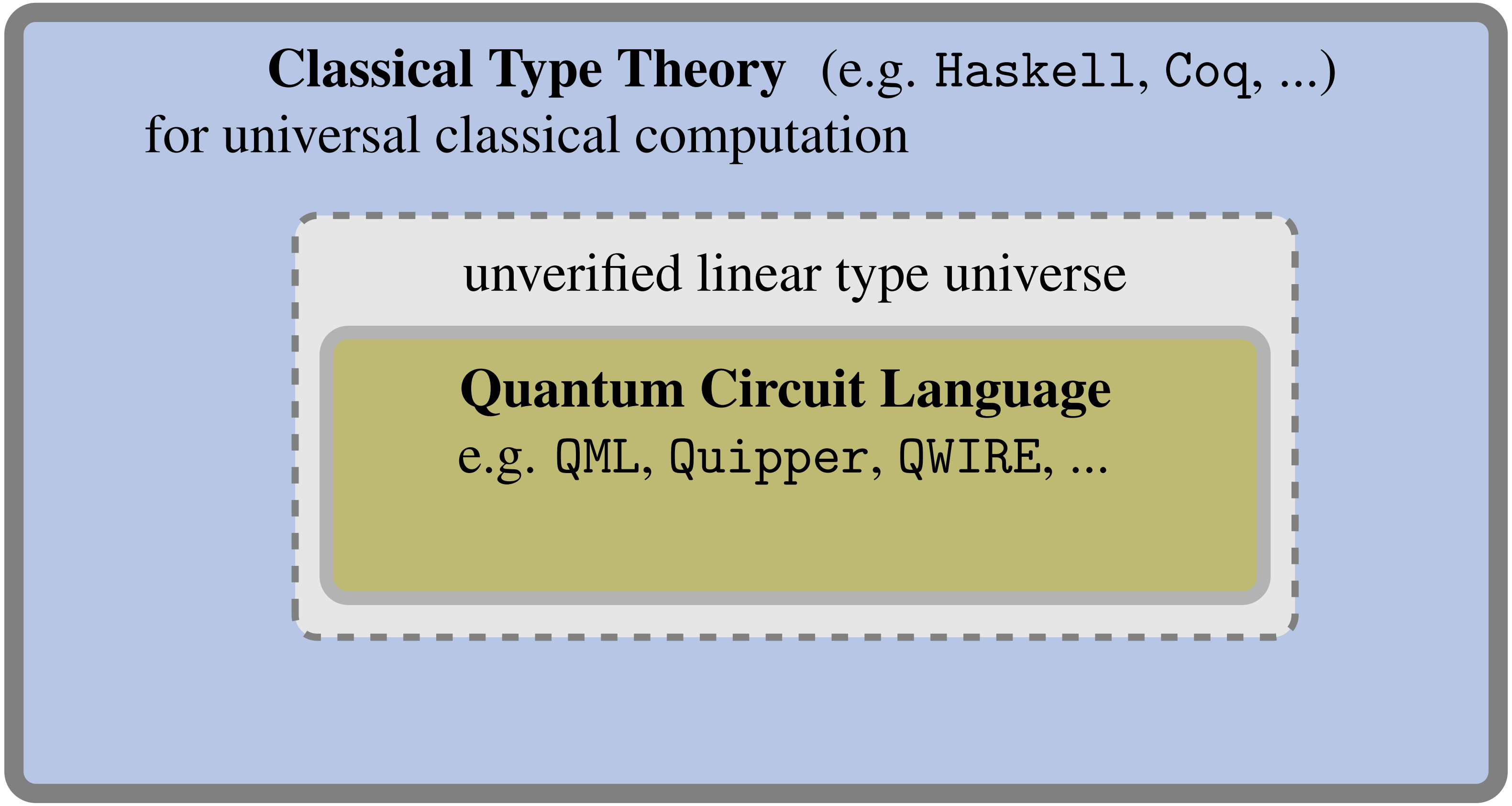

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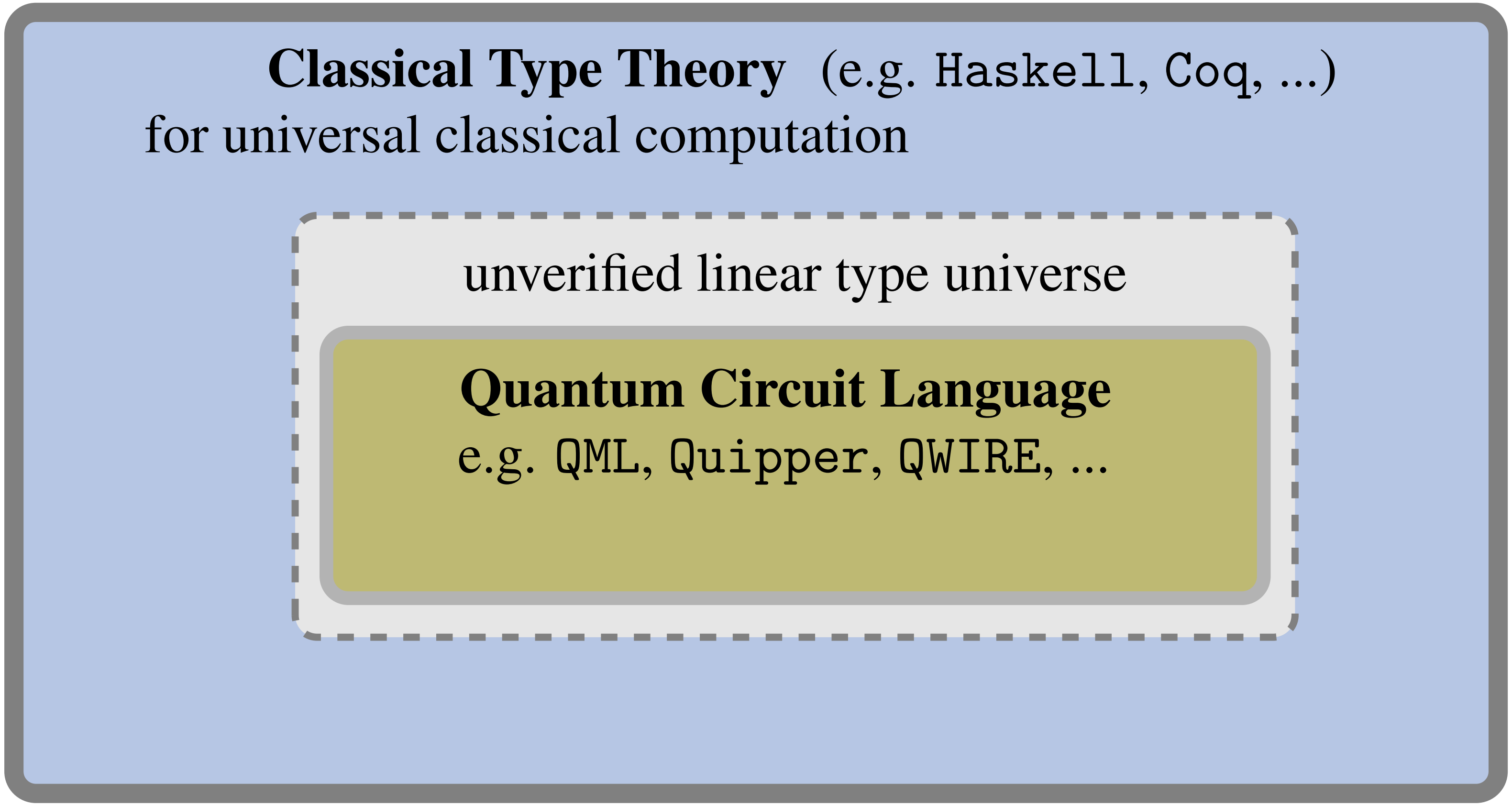

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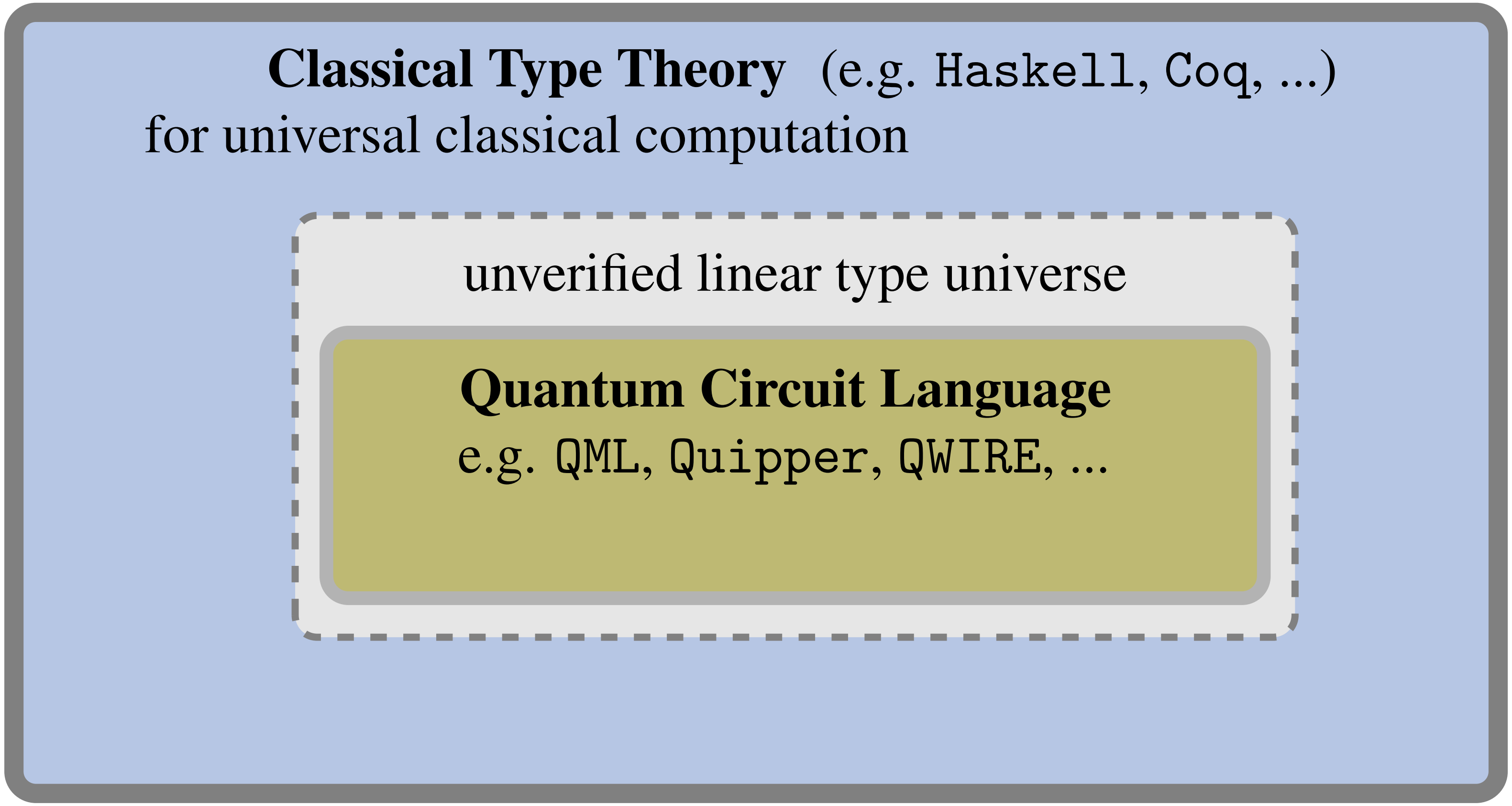

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Why did that not exist?

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Type Theory

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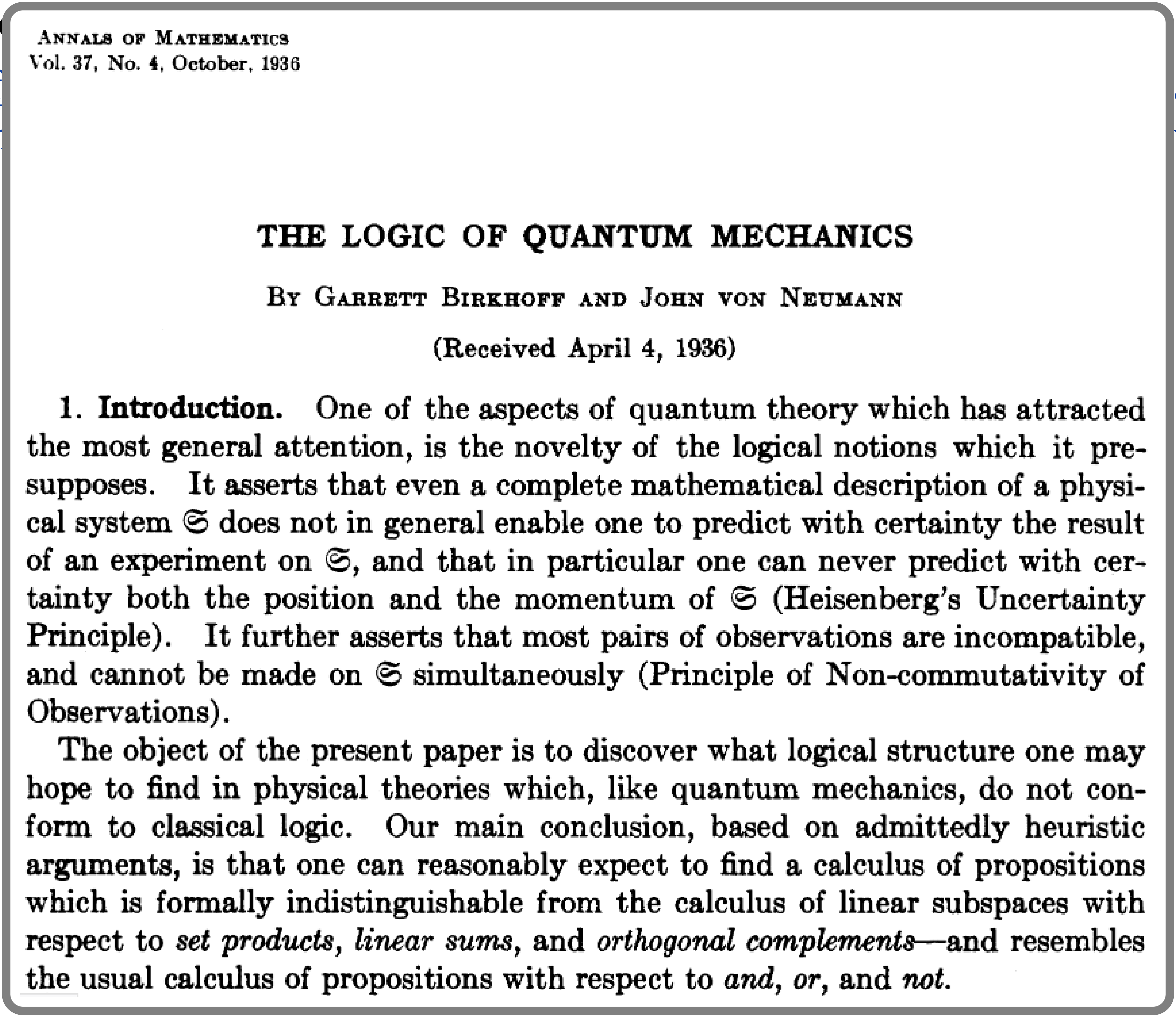

ANNALS OF MATHEMATICS
Vol. 37, No. 4, October, 1936

THE LOGIC OF QUANTUM MECHANICS

BY GARRETT BIRKHOFF AND JOHN VON NEUMANN

(Received April 4, 1936)

1. Introduction. One of the aspects of quantum theory which has attracted the most general attention, is the novelty of the logical notions which it presupposes. It asserts that even a complete mathematical description of a physical system $\mathfrak{S}$ does not in general enable one to predict with certainty the result of an experiment on $\mathfrak{S}$, and that in particular one can never predict with certainty both the position and the momentum of $\mathfrak{S}$ (Heisenberg's Uncertainty Principle). It further asserts that most pairs of observations are incompatible, and cannot be made on $\mathfrak{S}$ simultaneously (Principle of Non-commutativity of Observations).

The object of the present paper is to discover what logical structure one may hope to find in physical theories which, like quantum mechanics, do not conform to classical logic. Our main conclusion, based on admittedly heuristic arguments, is that one can reasonably expect to find a calculus of propositions which is formally indistinguishable from the calculus of linear subspaces with respect to *set products*, *linear sums*, and *orthogonal complements*—and resembles the usual calculus of propositions with respect to *and*, *or*, and *not*.

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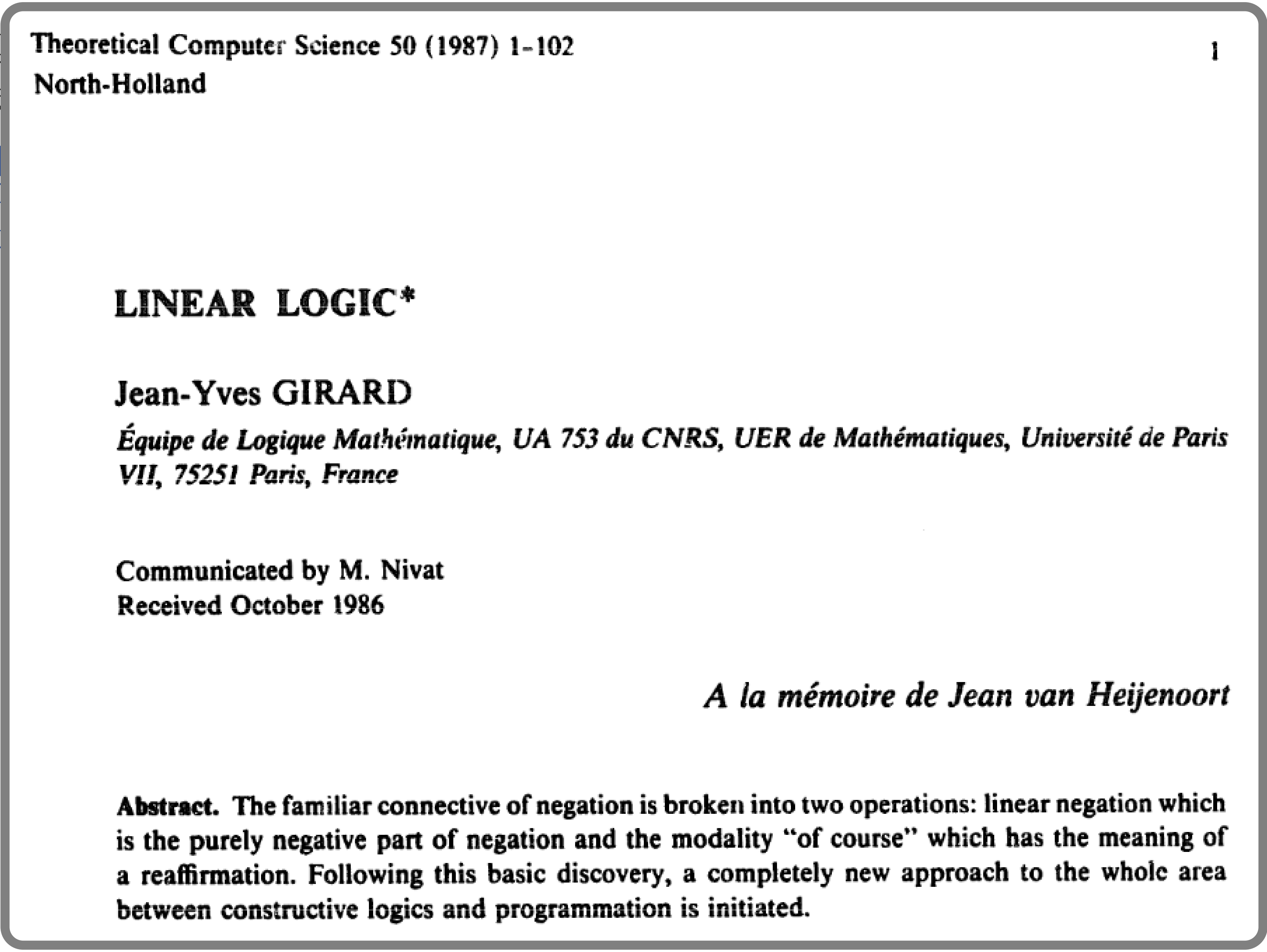

Theoretical Computer Science 50 (1987) 1-102
North-Holland

1

LINEAR LOGIC*

Jean-Yves GIRARD

Équipe de Logique Mathématique, UA 753 du CNRS, UER de Mathématiques, Université de Paris VII, 75251 Paris, France

Communicated by M. Nivat
Received October 1986

A la mémoire de Jean van Heijenoort

Abstract. The familiar connective of negation is broken into two operations: linear negation which is the purely negative part of negation and the modality “of course” which has the meaning of a reaffirmation. Following this basic discovery, a completely new approach to the whole area between constructive logics and programmation is initiated.

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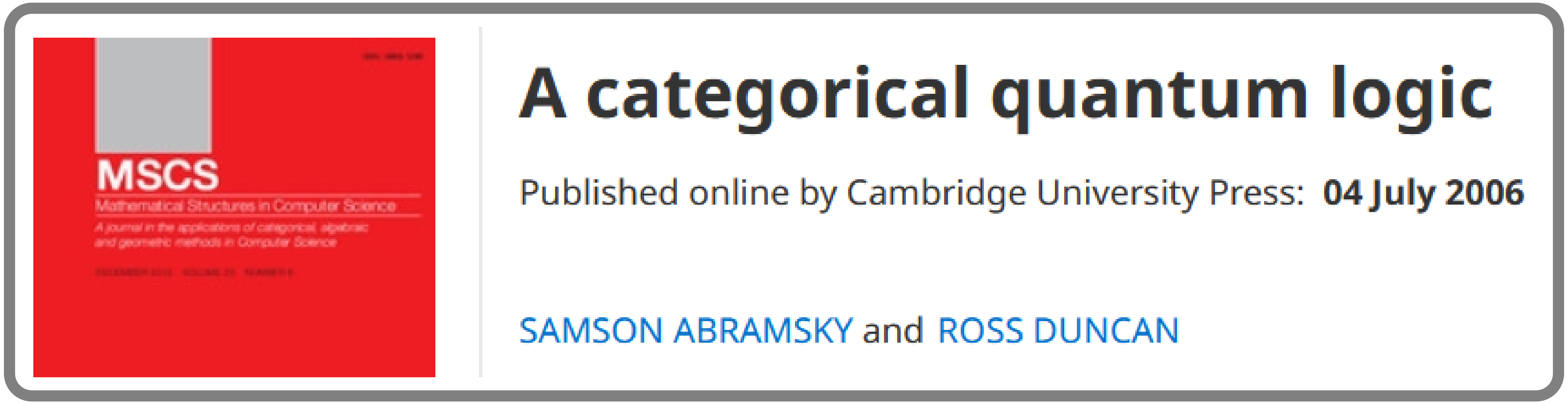
MSCS
Mathematical Structures in Computer Science
A journal in the applications of categorical, algebraic and geometric methods in Computer Science

A categorical quantum logic

Published online by Cambridge University Press: **04 July 2006**

SAMSON ABRAMSKY and ROSS DUNCAN

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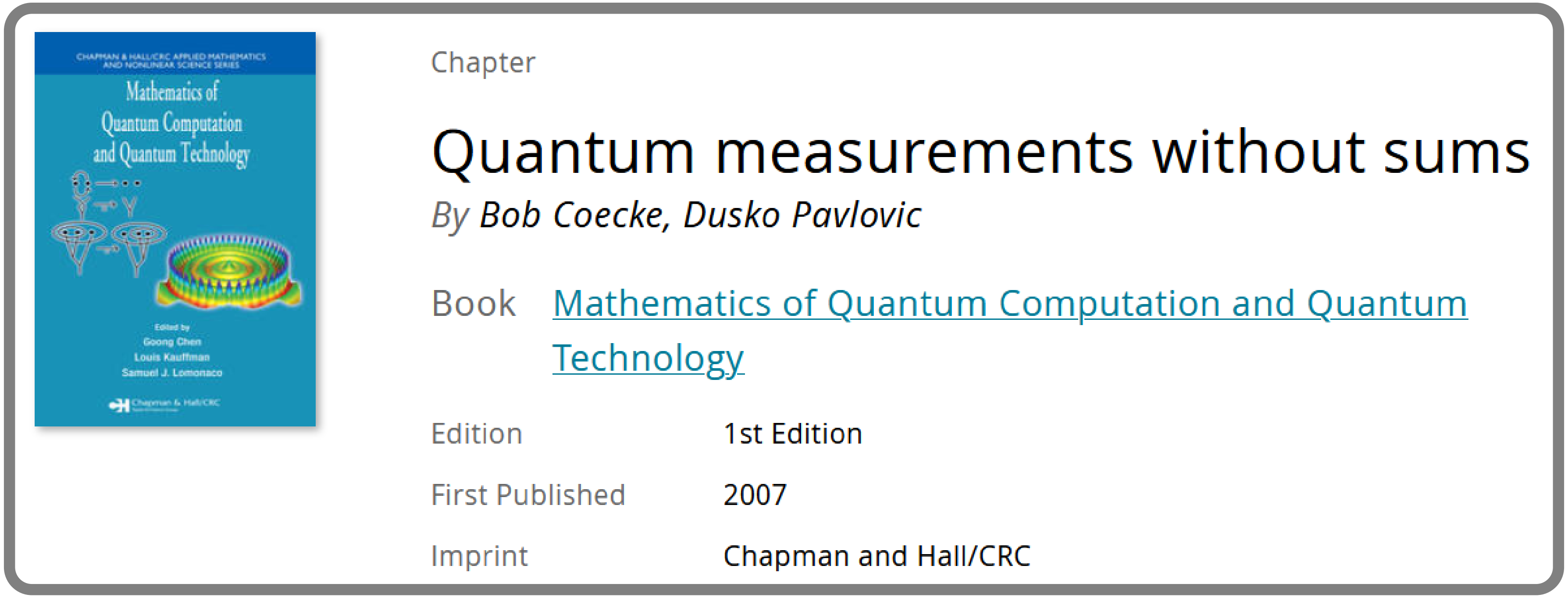
CHAPMAN & HALL/CRC APPLIED MATHEMATICS AND NONLINEAR SCIENCE SERIES
Mathematics of Quantum Computation and Quantum Technology
Edited by Goong Chen Louis Kauffman Samuel J. Lomonaco
Chapman & Hall/CRC

Chapter

Quantum measurements without sums

By *Bob Coecke, Dusko Pavlovic*

Book Mathematics of Quantum Computation and Quantum Technology

Edition	1st Edition
First Published	2007
Imprint	Chapman and Hall/CRC

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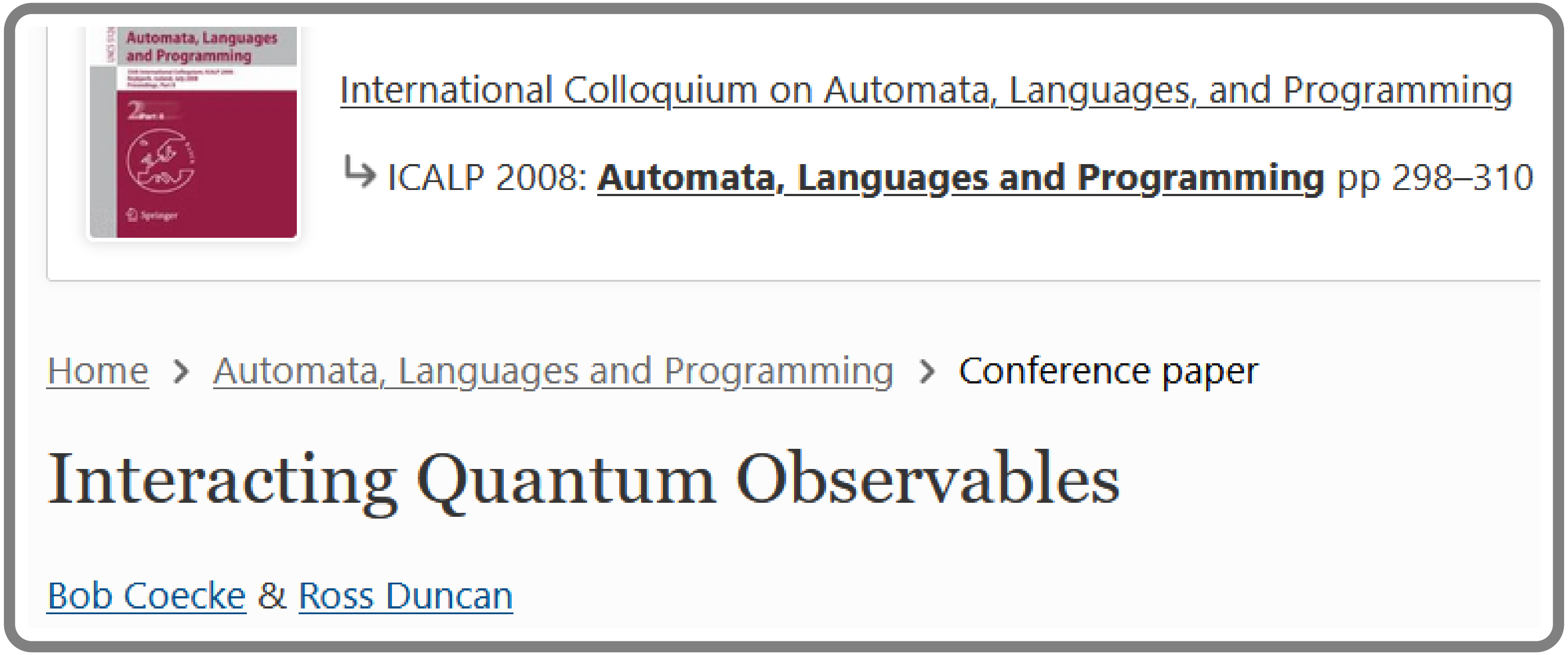

International Colloquium on Automata, Languages, and Programming

↳ ICALP 2008: **Automata, Languages and Programming** pp 298–310

Home › Automata, Languages and Programming › Conference paper

Interacting Quantum Observables

Bob Coecke & Ross Duncan

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Internal logic in $\mathrm{FDVect}_{/\mathcal{H}}$		
proposition	**conjunction**	**disjunction**
$\mathcal{P}$	$\begin{tikzcd} \mathcal{P}_1 & \mathcal{P}_1 \cap \mathcal{P}_2 \arrow[l] \arrow[r] & \mathcal{P}_2 \end{tikzcd}$	$\begin{tikzcd} & \mathcal{P}_1 \oplus \mathcal{P}_2 \arrow[d, two heads] & \\ \mathcal{P}_1 \arrow[ur] & \mathrm{Span}(\mathcal{P}_1, \mathcal{P}_2) & \mathcal{P}_2 \arrow[ul] \end{tikzcd}$

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proposition	**conjunction**	**disjunction**
$$\begin{tikzcd} \mathcal{P} \\ {} \arrow[u, hook, "p"] \end{tikzcd}$$	$$\begin{tikzcd} \mathcal{P}_1 & \mathcal{P}_1 \cap \mathcal{P}_2 \arrow[l] \arrow[r] & \mathcal{P}_2 \\ & {} \arrow[ul, hook, "p_1"] \arrow[u, hook, "p_1 \wedge p_2"] \arrow[ur, hook, "p_2"'] & \end{tikzcd}$$	$$\begin{tikzcd} & \mathcal{P}_1 \oplus \mathcal{P}_2 \arrow[d, two heads] & \\ \mathcal{P}_1 \arrow[ur] & \mathrm{Span}(\mathcal{P}_1, \mathcal{P}_2) & \mathcal{P}_2 \arrow[ul] \\ & {} \arrow[ul, hook, "p_1"] \arrow[u, hook, "p_1 \vee p_2"] \arrow[ur, hook, "p_2"'] & \end{tikzcd}$$

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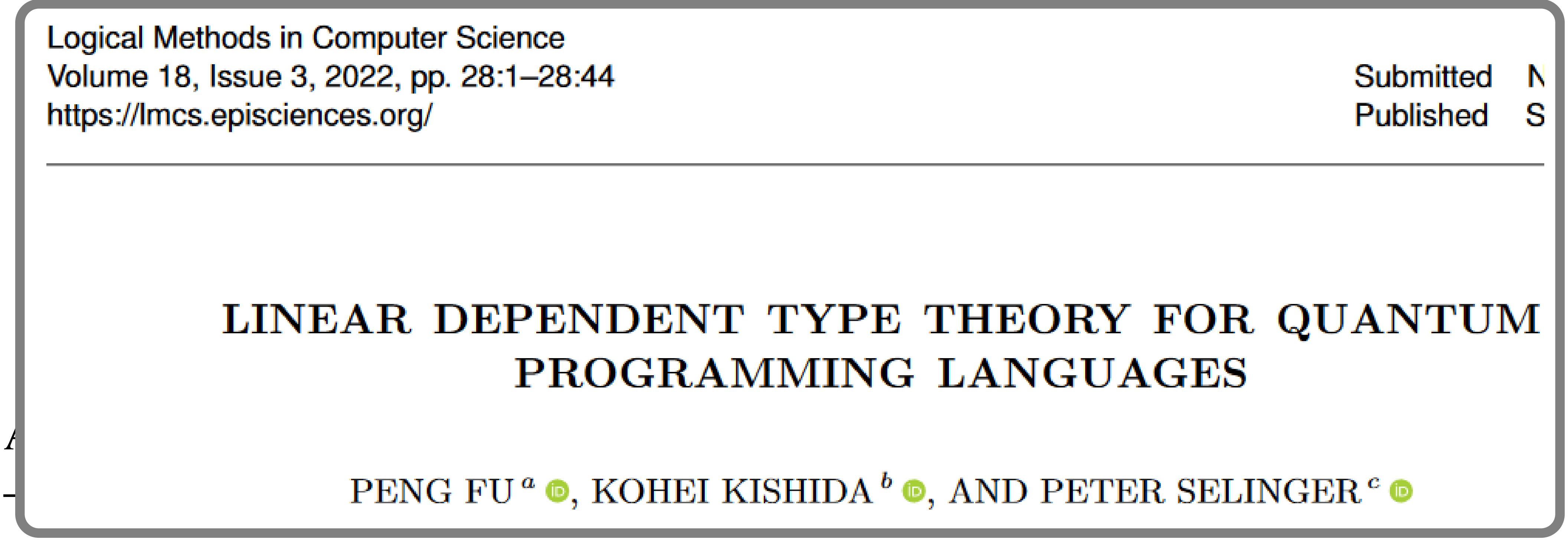

Logical Methods in Computer Science
Volume 18, Issue 3, 2022, pp. 28:1–28:44
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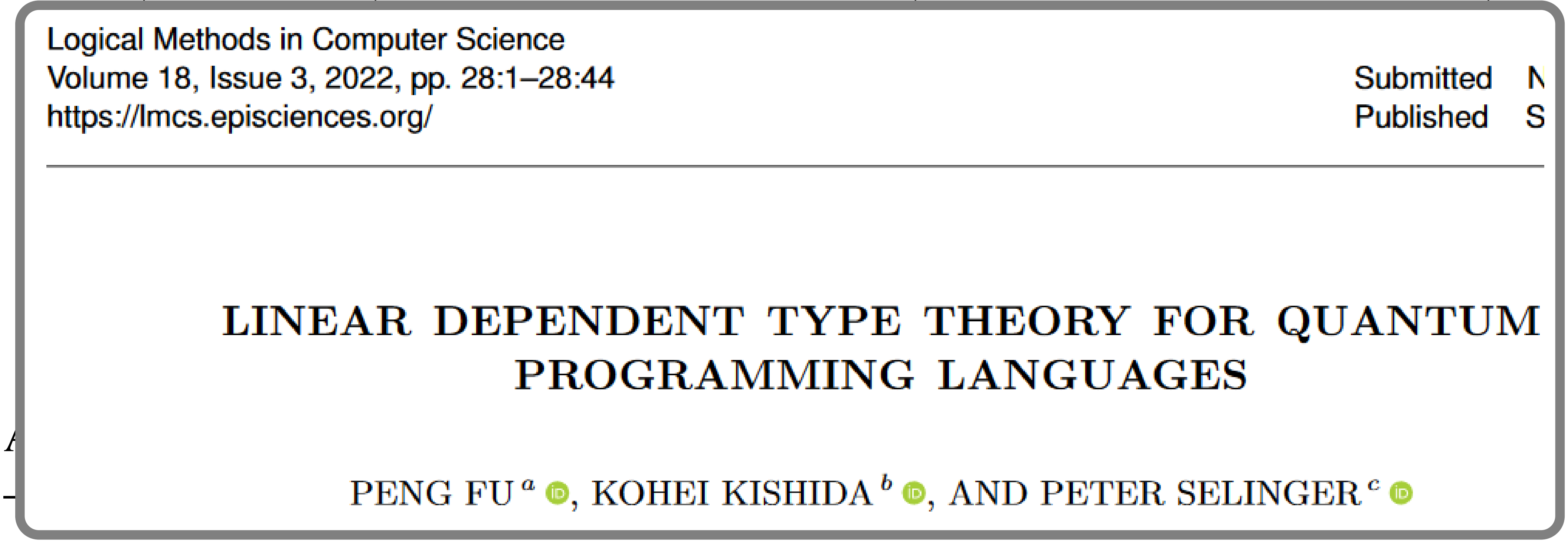

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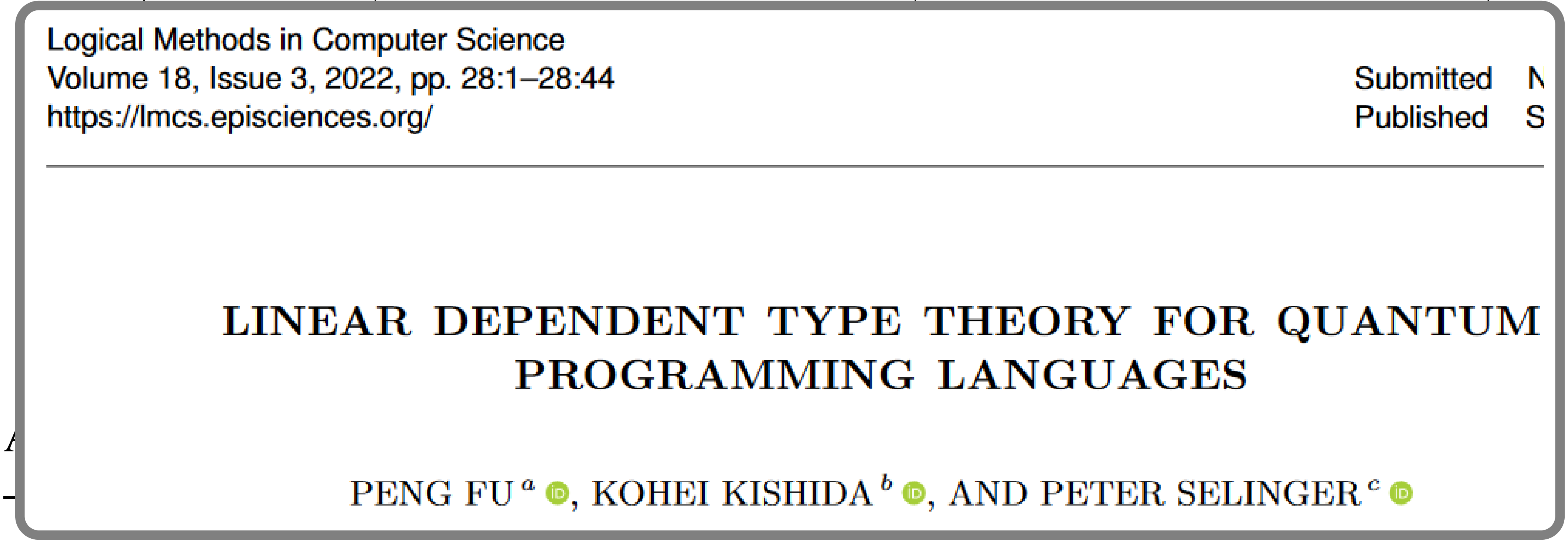

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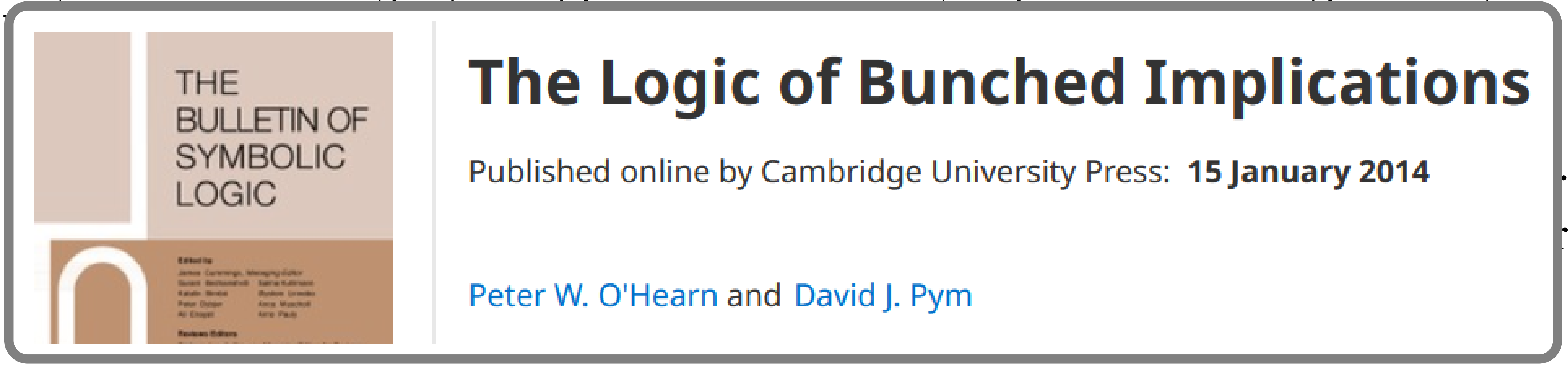

THE BULLETIN OF SYMBOLIC LOGIC

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Published online by Cambridge University Press: **15 January 2014**

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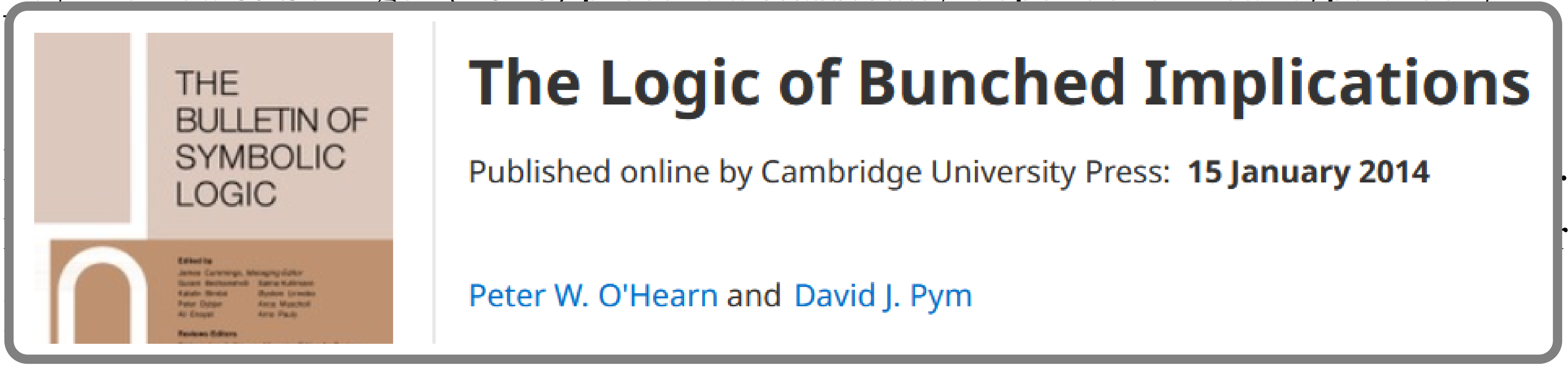

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Schöpp & Stark (2004) make progress towards bunched dependent type theory

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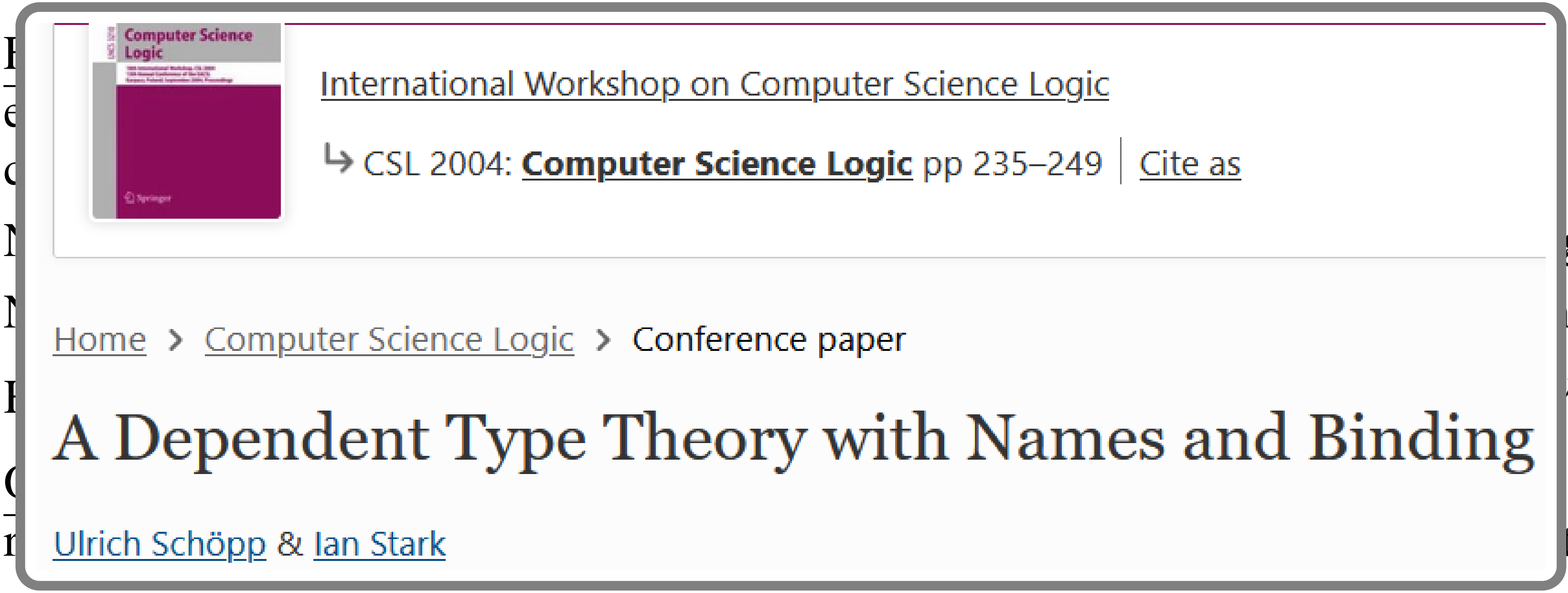

International Workshop on Computer Science Logic

↳ CSL 2004: **Computer Science Logic** pp 235–249 | Cite as

Home › Computer Science Logic › Conference paper

A Dependent Type Theory with Names and Binding

Ulrich Schöpp & Ian Stark

Schöpp & Stark (2004) make progress towards bunched dependent type theory

$$\begin{tikzcd}
\mathcal{P} \arrow[d, hook, "p"] \\
\mathcal{H}
\end{tikzcd}$$

$$\begin{tikzcd}
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& {} \arrow[d, two heads] & \\
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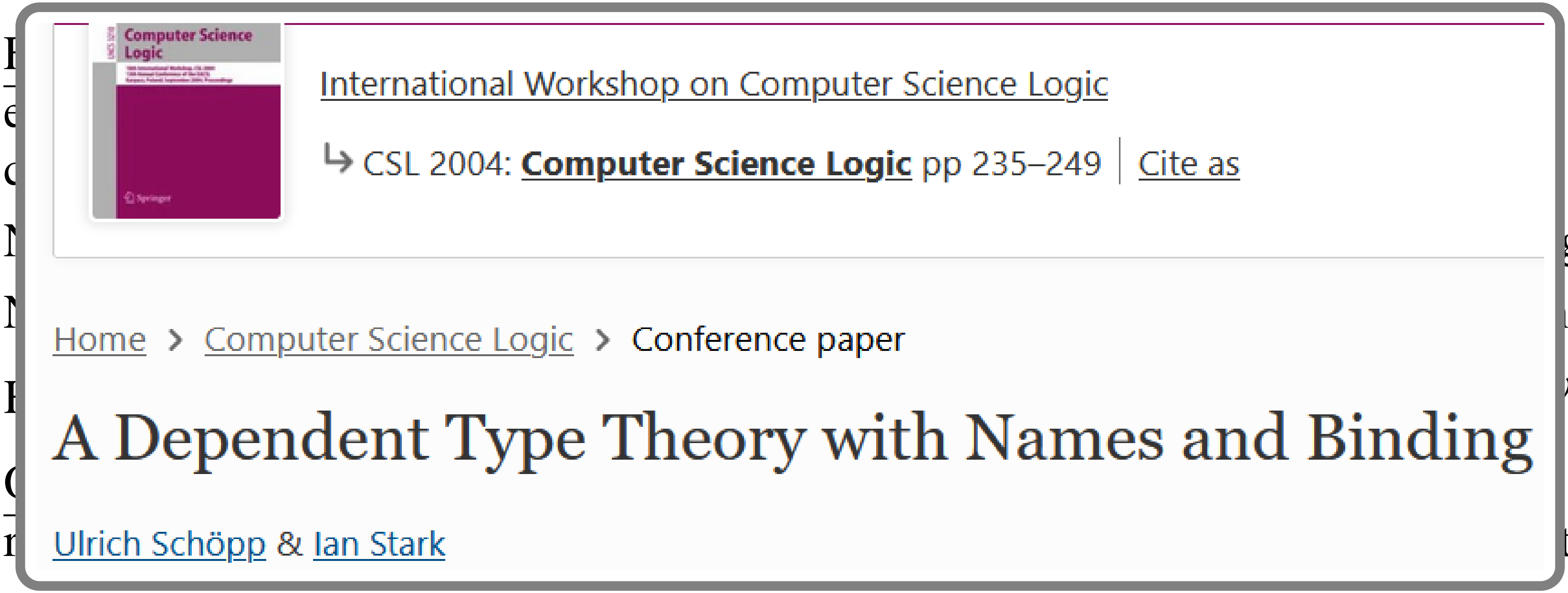

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$$\begin{tikzcd}
\mathcal{P}_1 \arrow[dr, hook, "p_1"] & \mathcal{P}_1 \cap \mathcal{P}_2 \arrow[l] \arrow[r] \arrow[d, hook, "p_1 \wedge p_2"] & \mathcal{P}_2 \arrow[dl, hook', "p_2"'] \\
& \mathcal{H} &
\end{tikzcd}$$

$$\begin{tikzcd}
& {} \arrow[d, two heads] & \\
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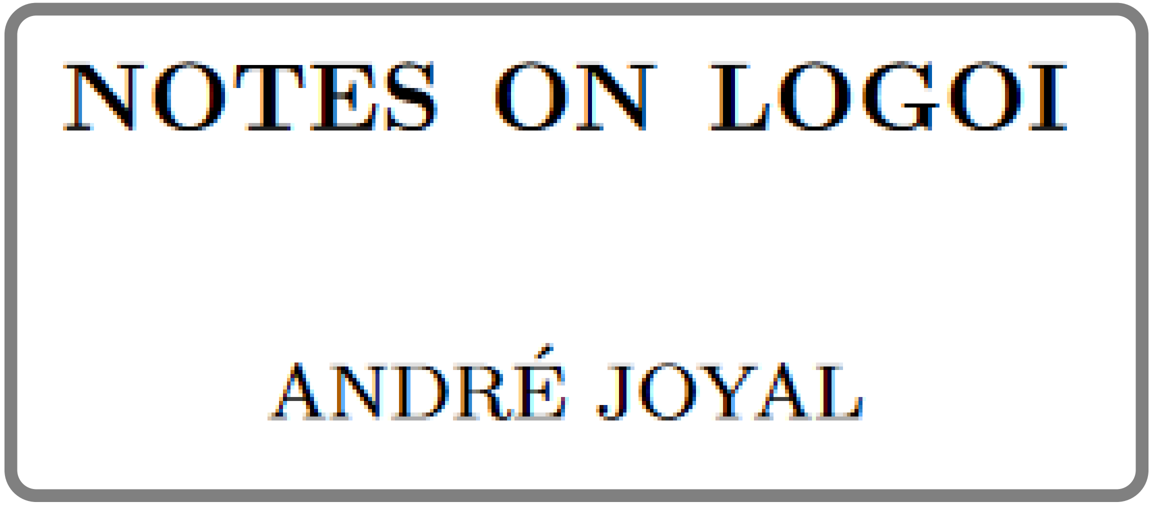

NOTES ON LOGOI

ANDRÉ JOYAL

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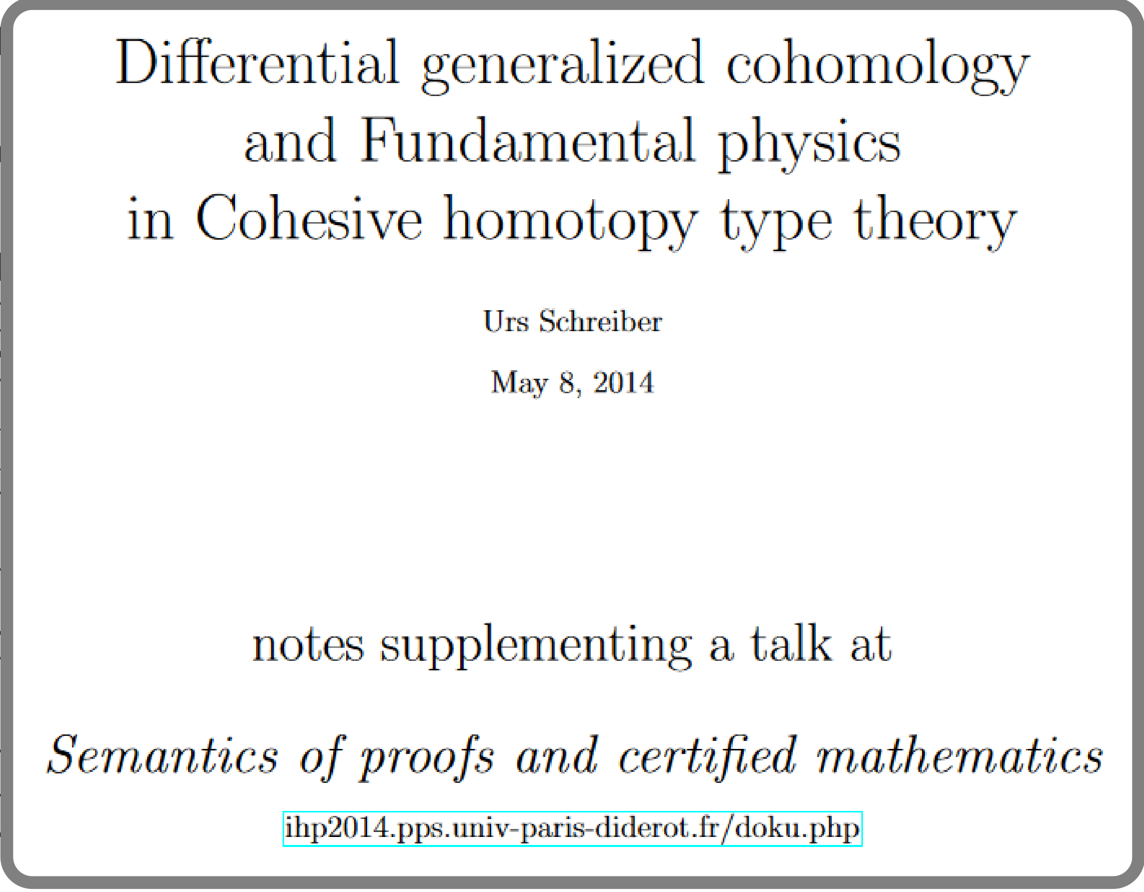

Differential generalized cohomology
and Fundamental physics
in Cohesive homotopy type theory

Urs Schreiber

May 8, 2014

notes supplementing a talk at

Semantics of proofs and certified mathematics

ihp2014.pps.univ-paris-diderot.fr/doku.php

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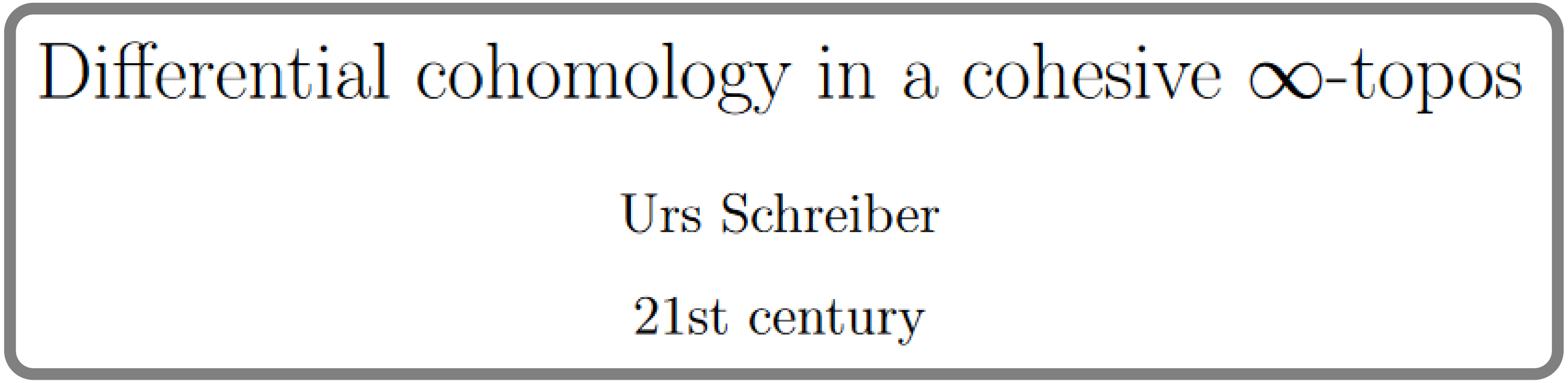

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21st century

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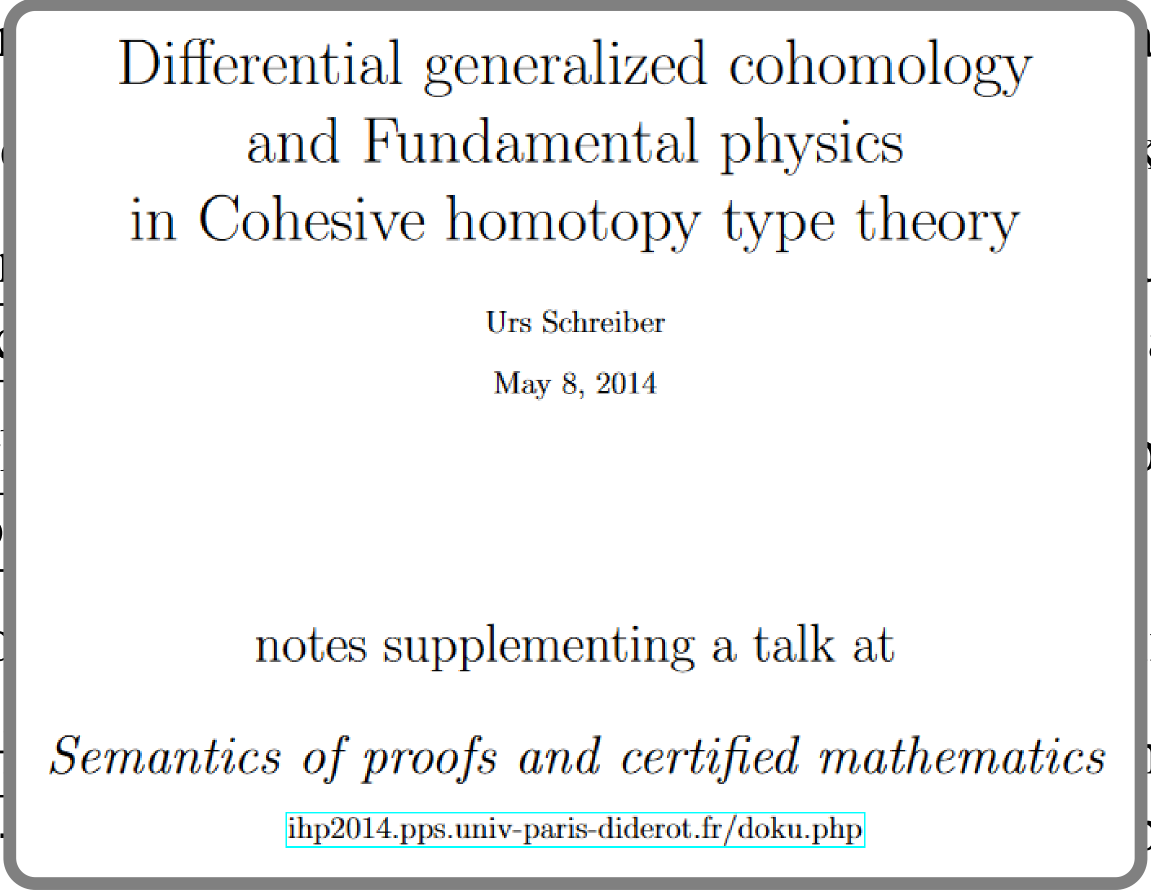

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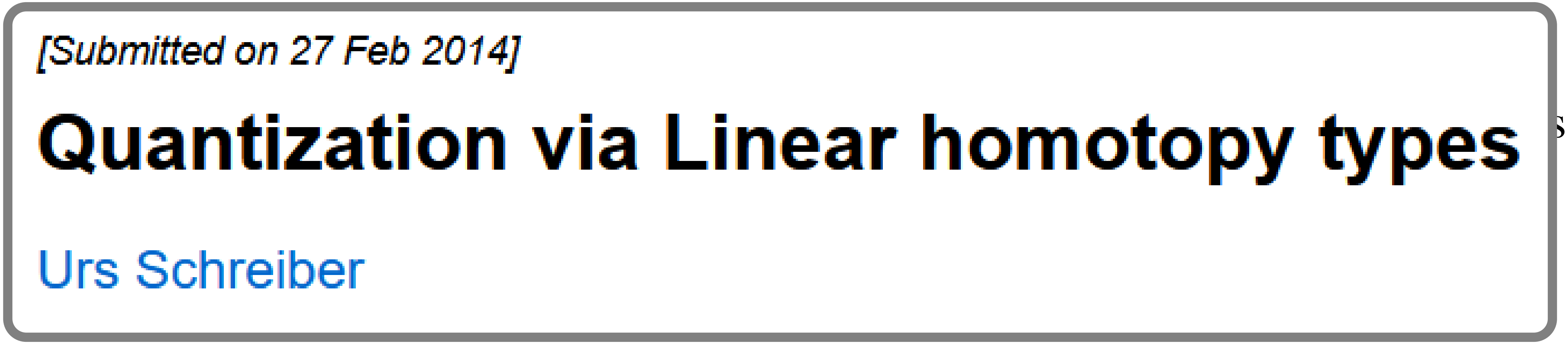

[Submitted on 27 Feb 2014]

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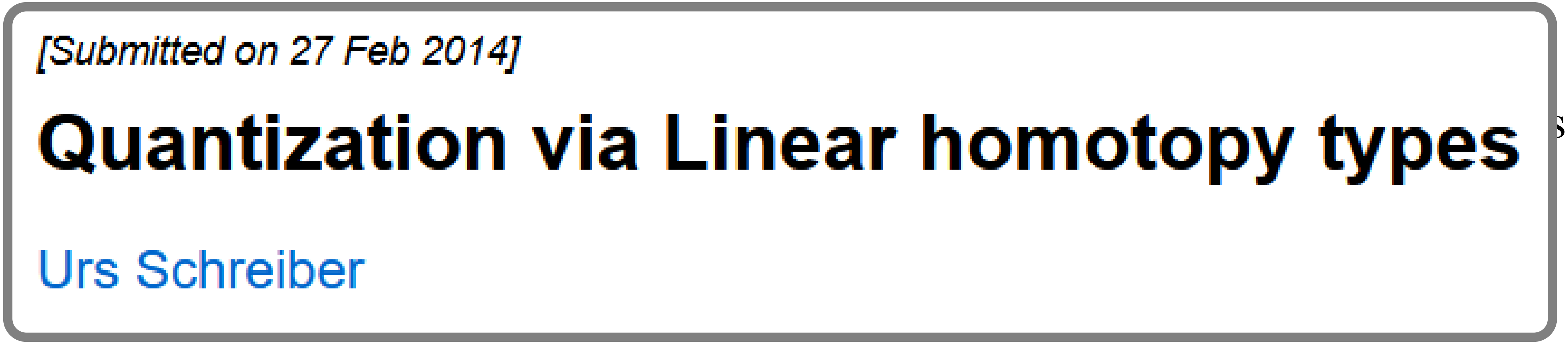

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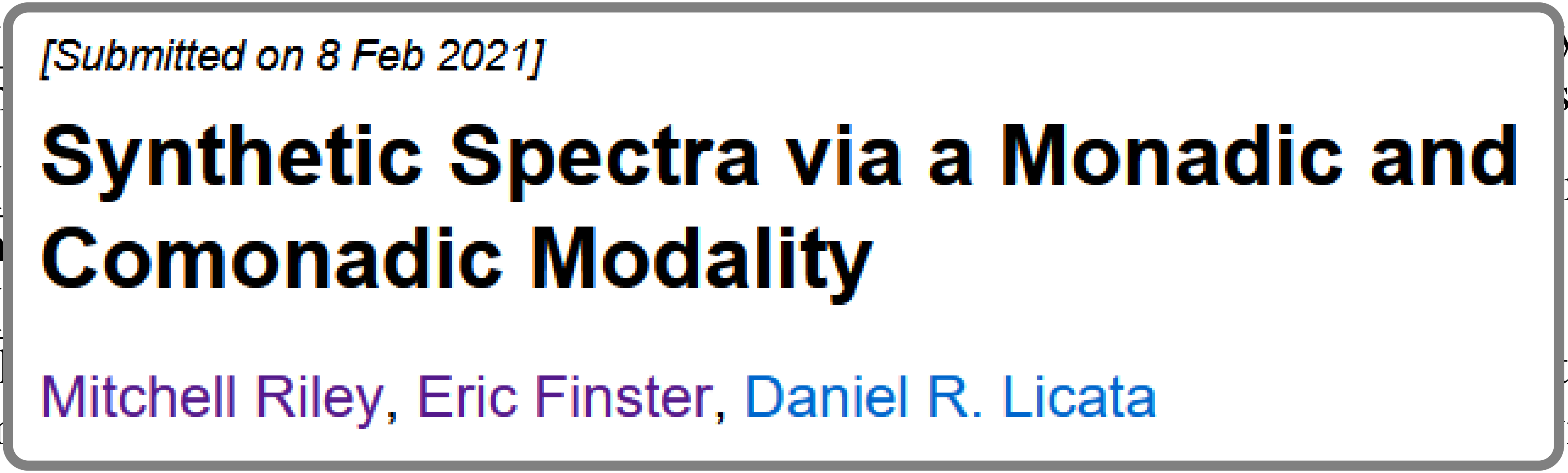

[Submitted on 8 Feb 2021]

Synthetic Spectra via a Monadic and Comonadic Modality

Mitchell Riley, Eric Finster, Daniel R. Licata

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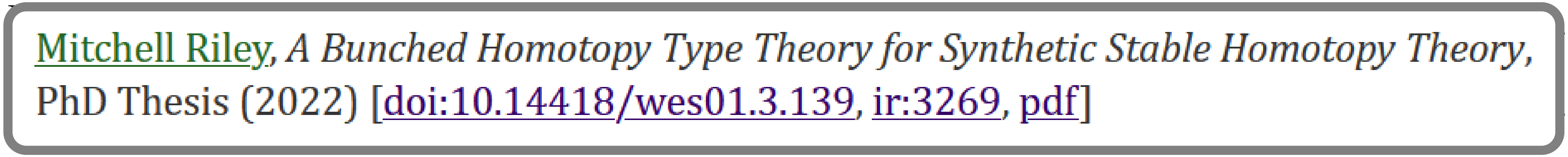

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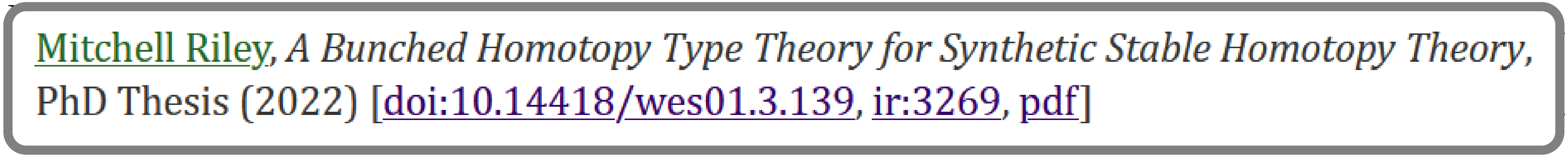

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Effective Quantum Certification via Linear Homotopy Types

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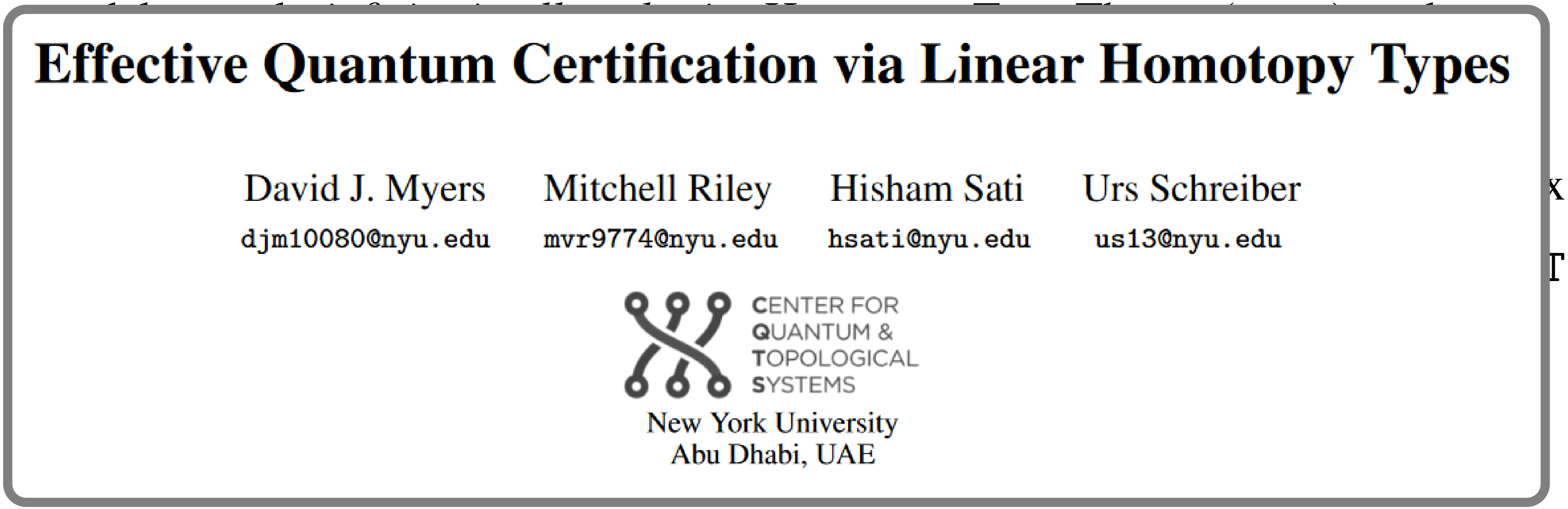

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Our Solution

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`LHoTT` is like a quantum microscope for Classical Data Types B

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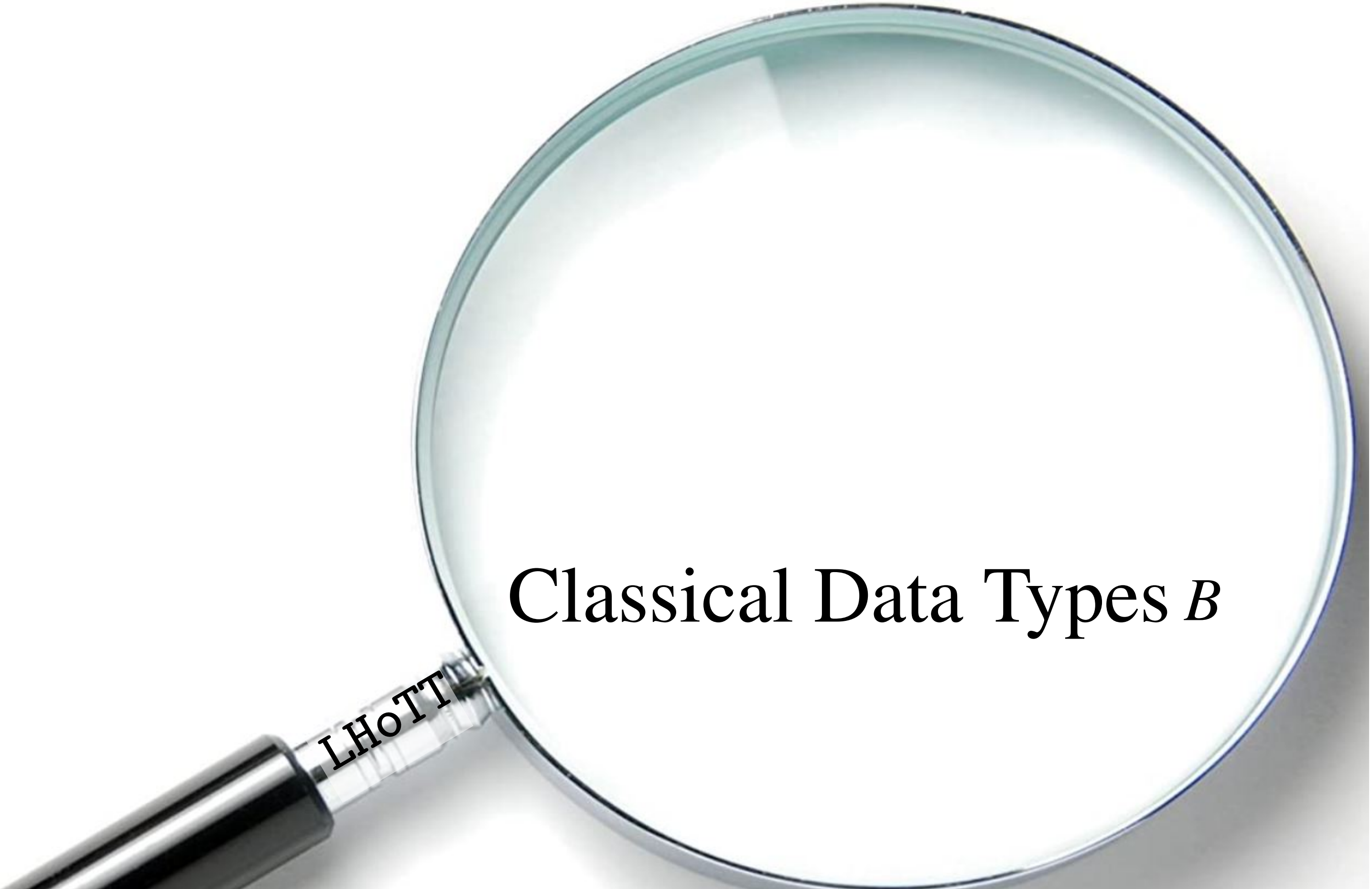

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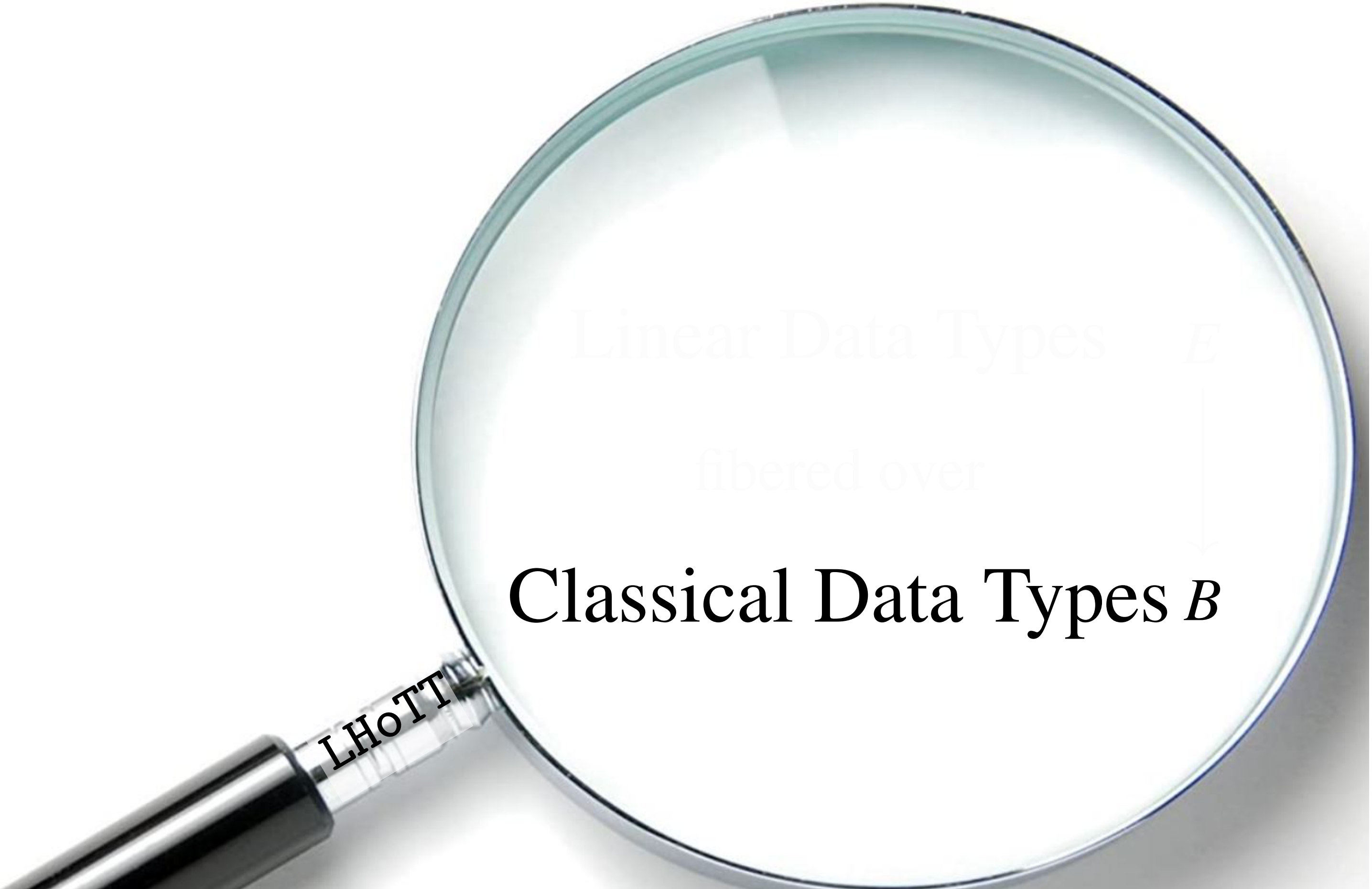

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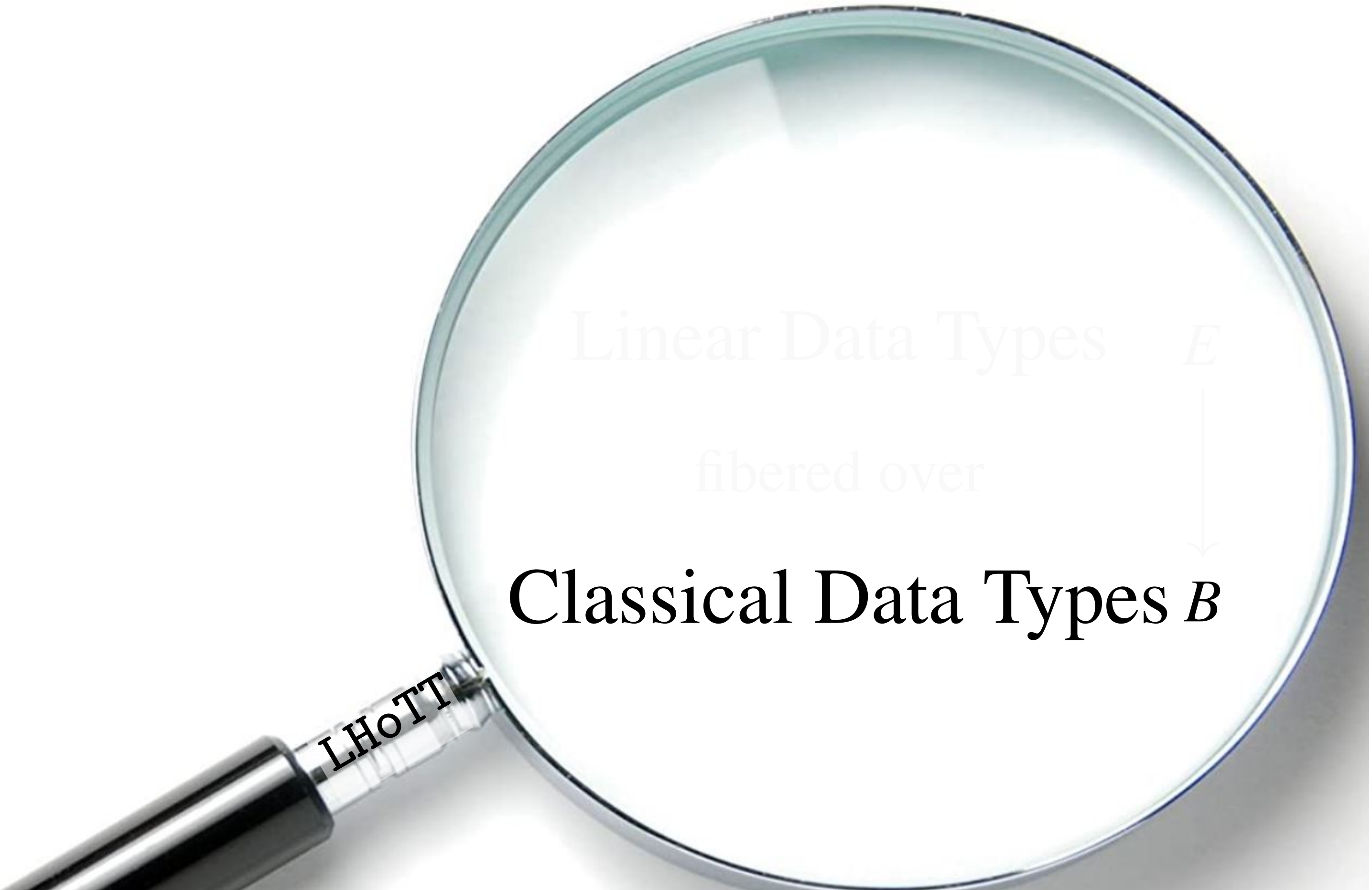

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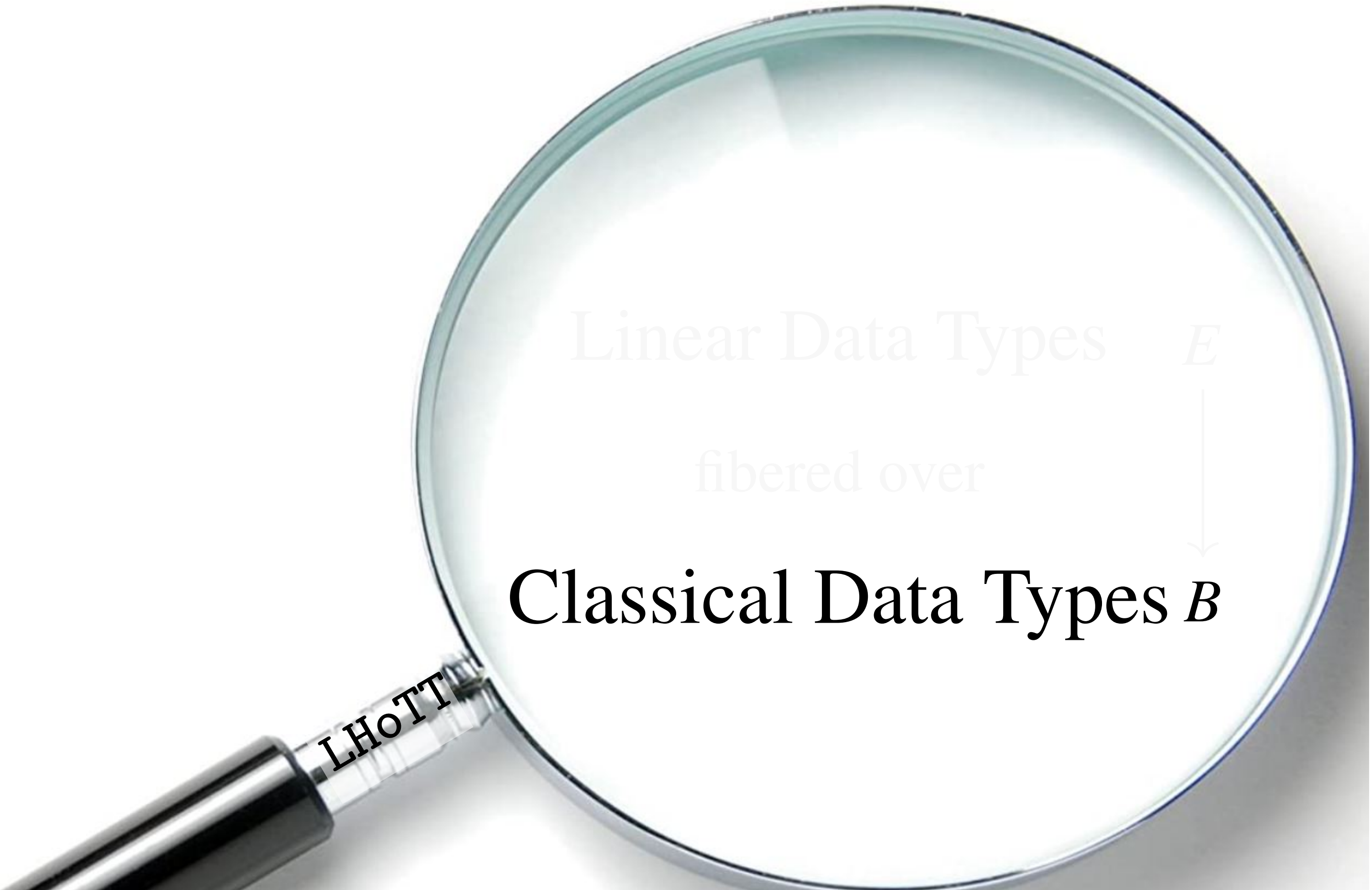

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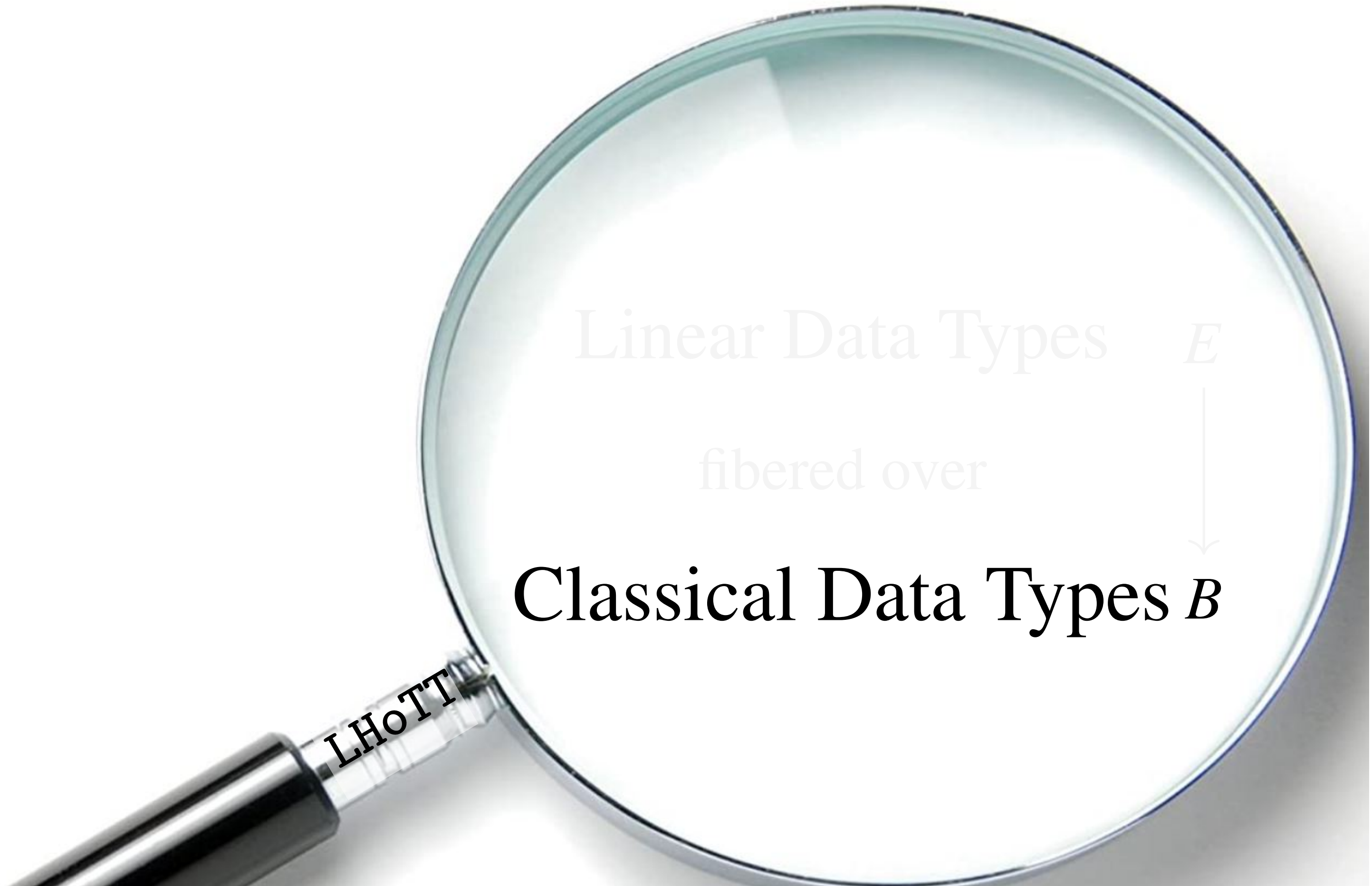

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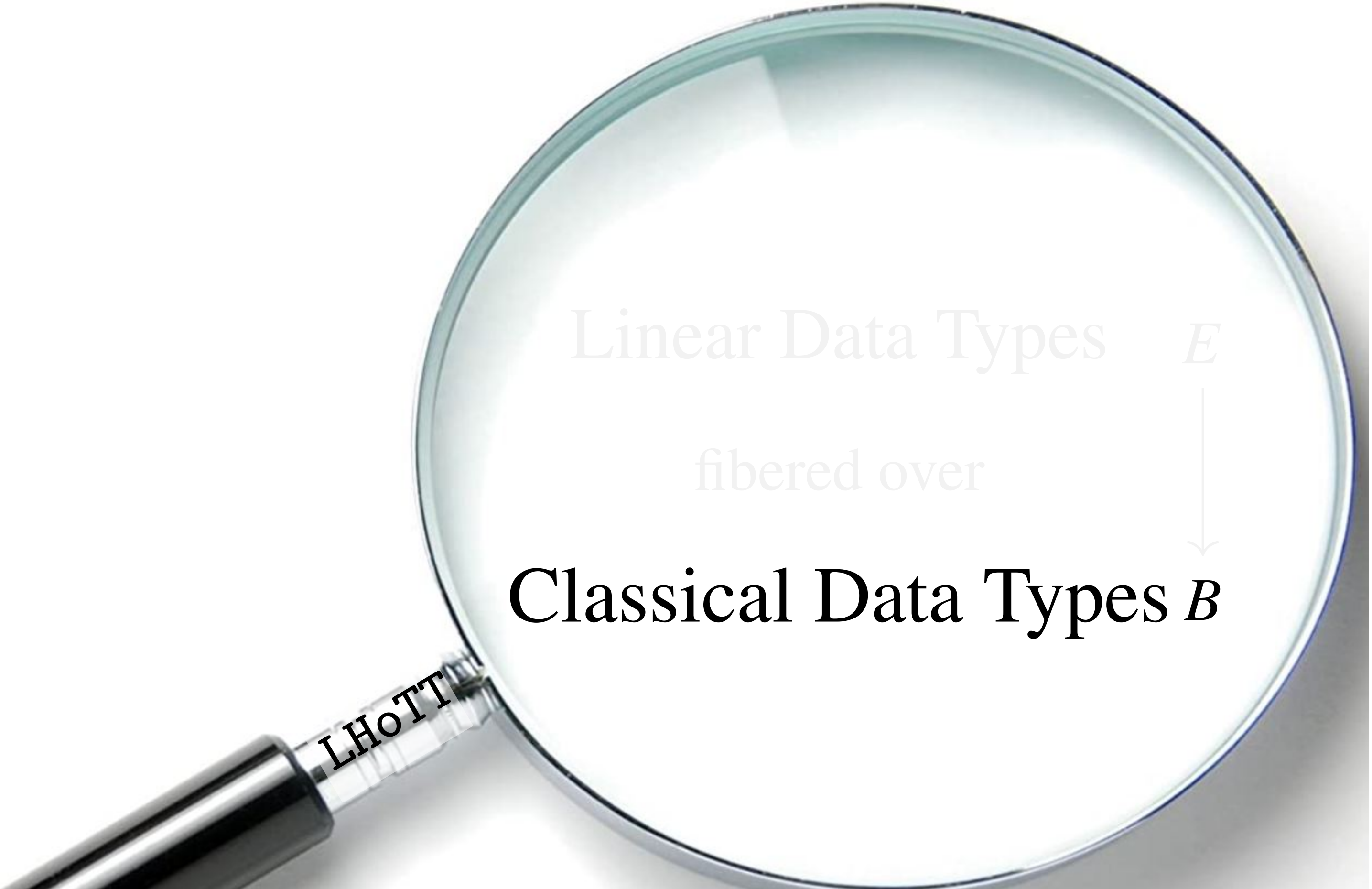

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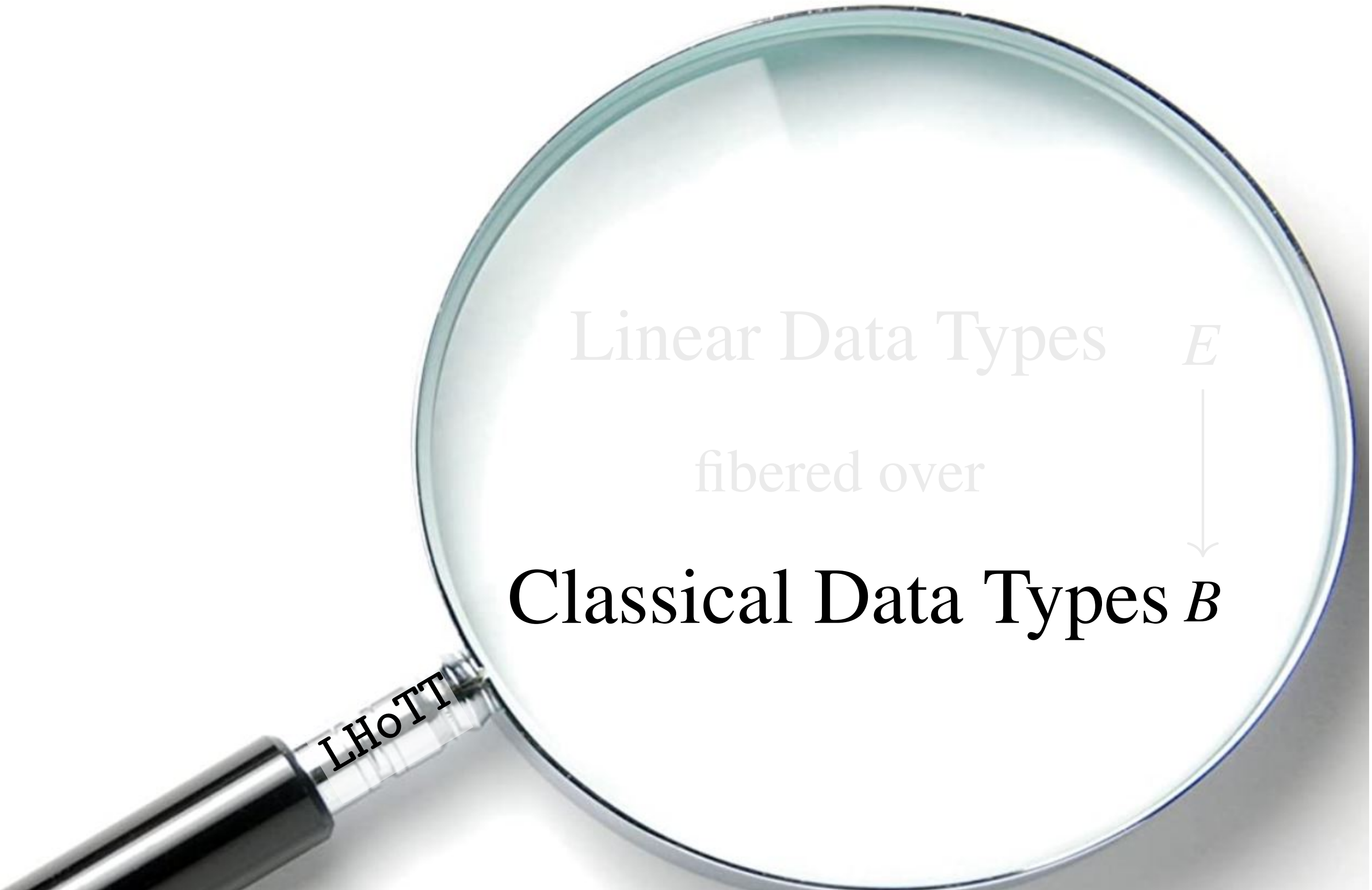

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Classical Data Types B

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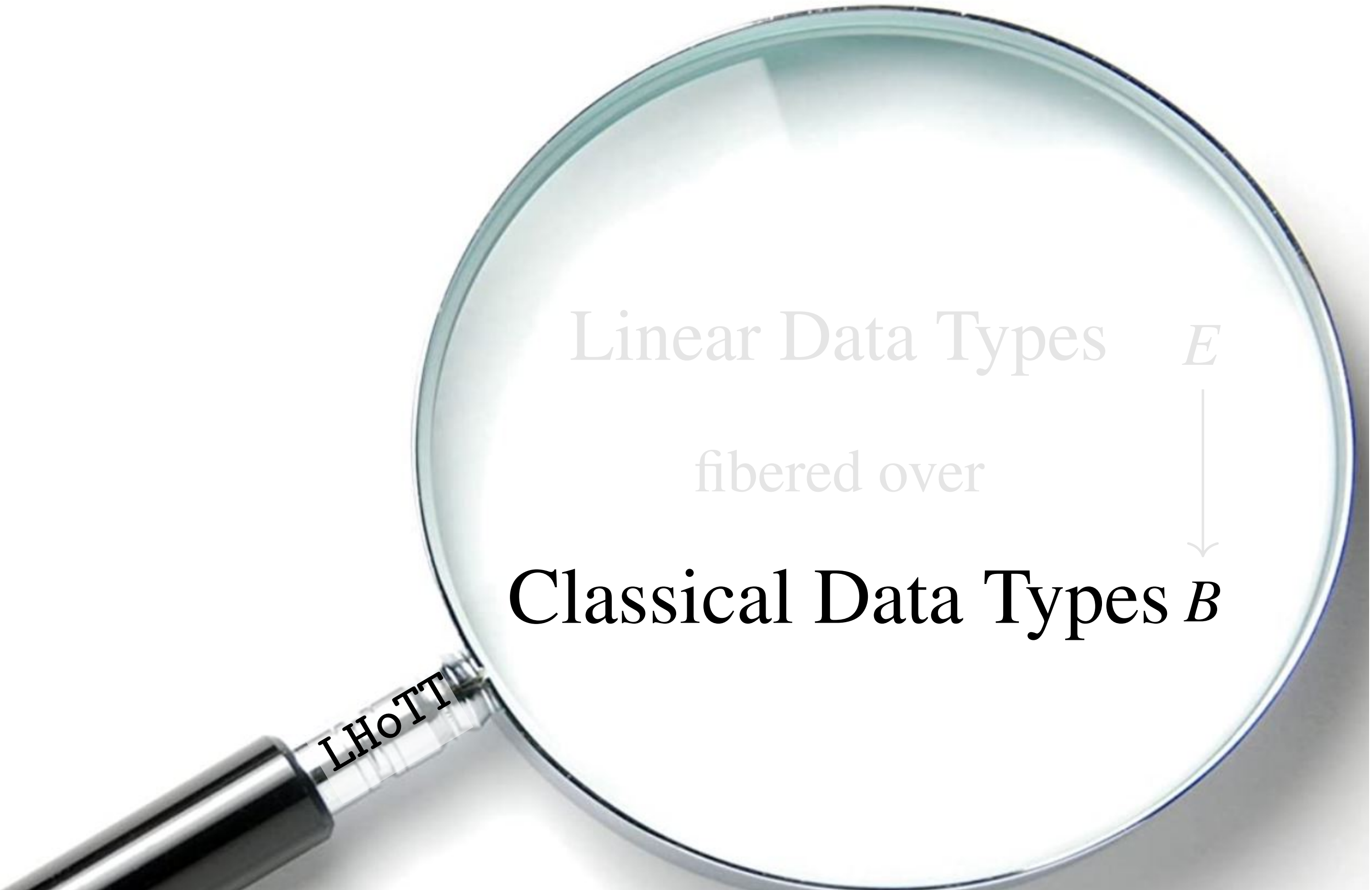

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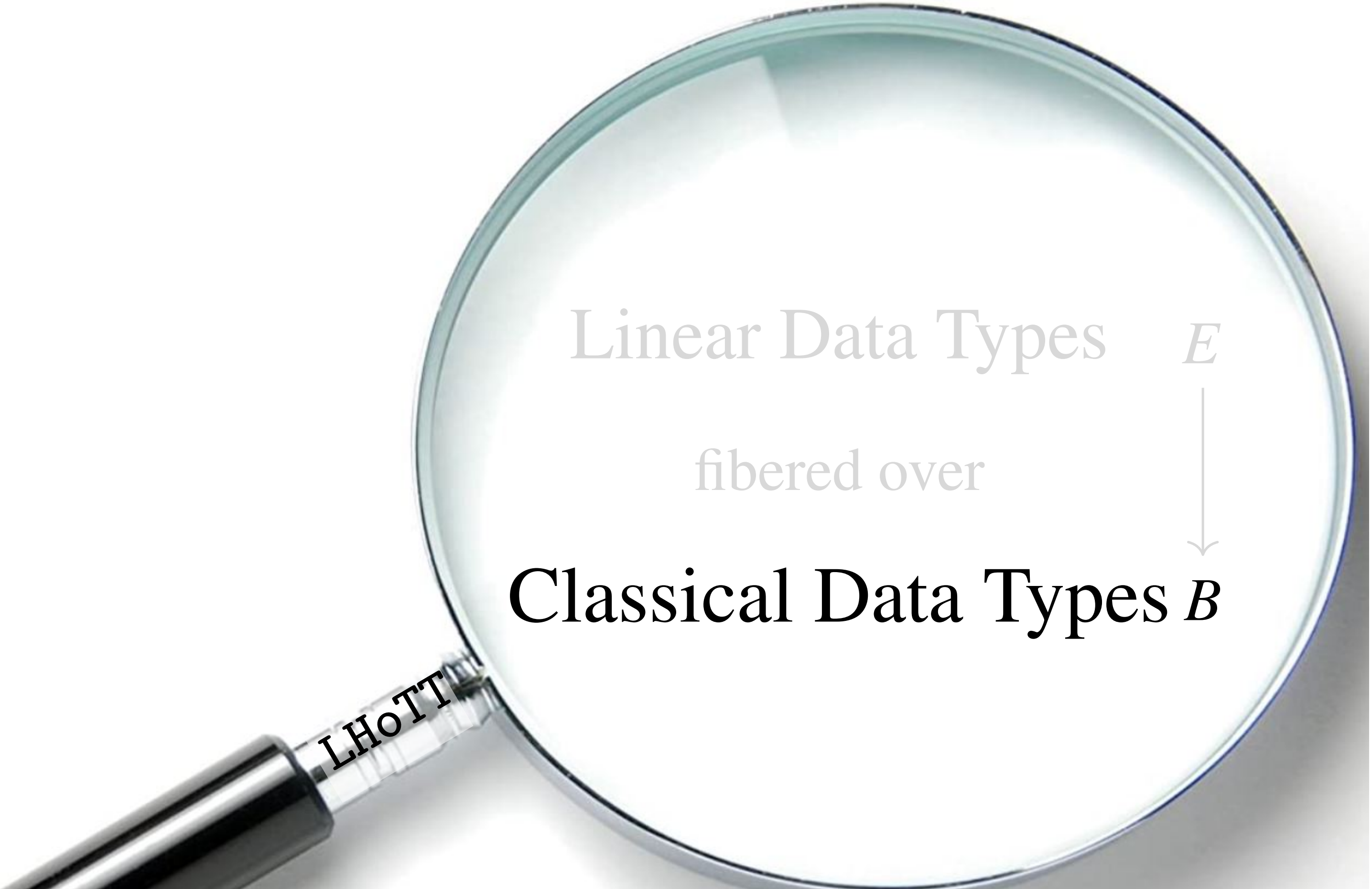

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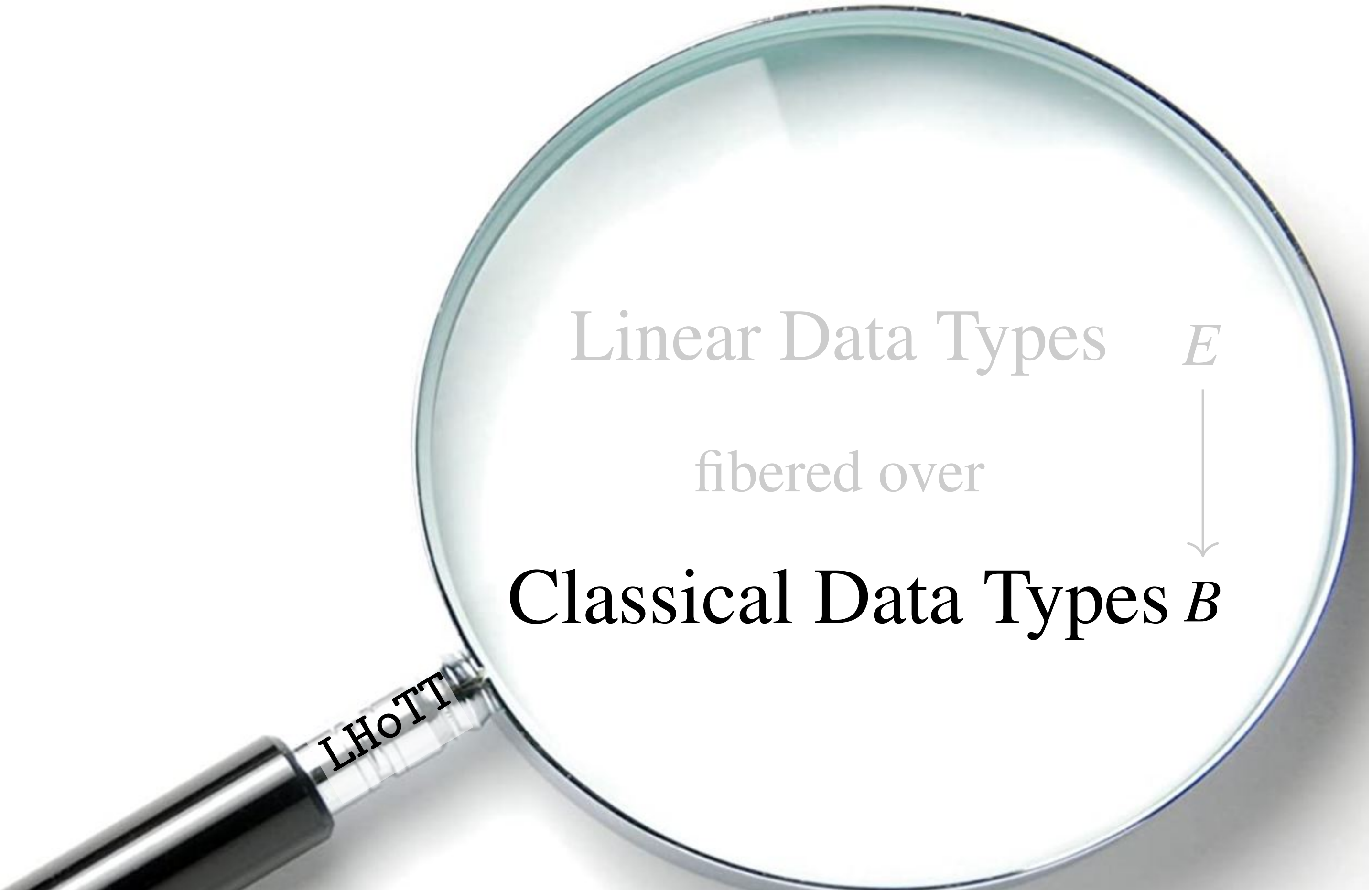

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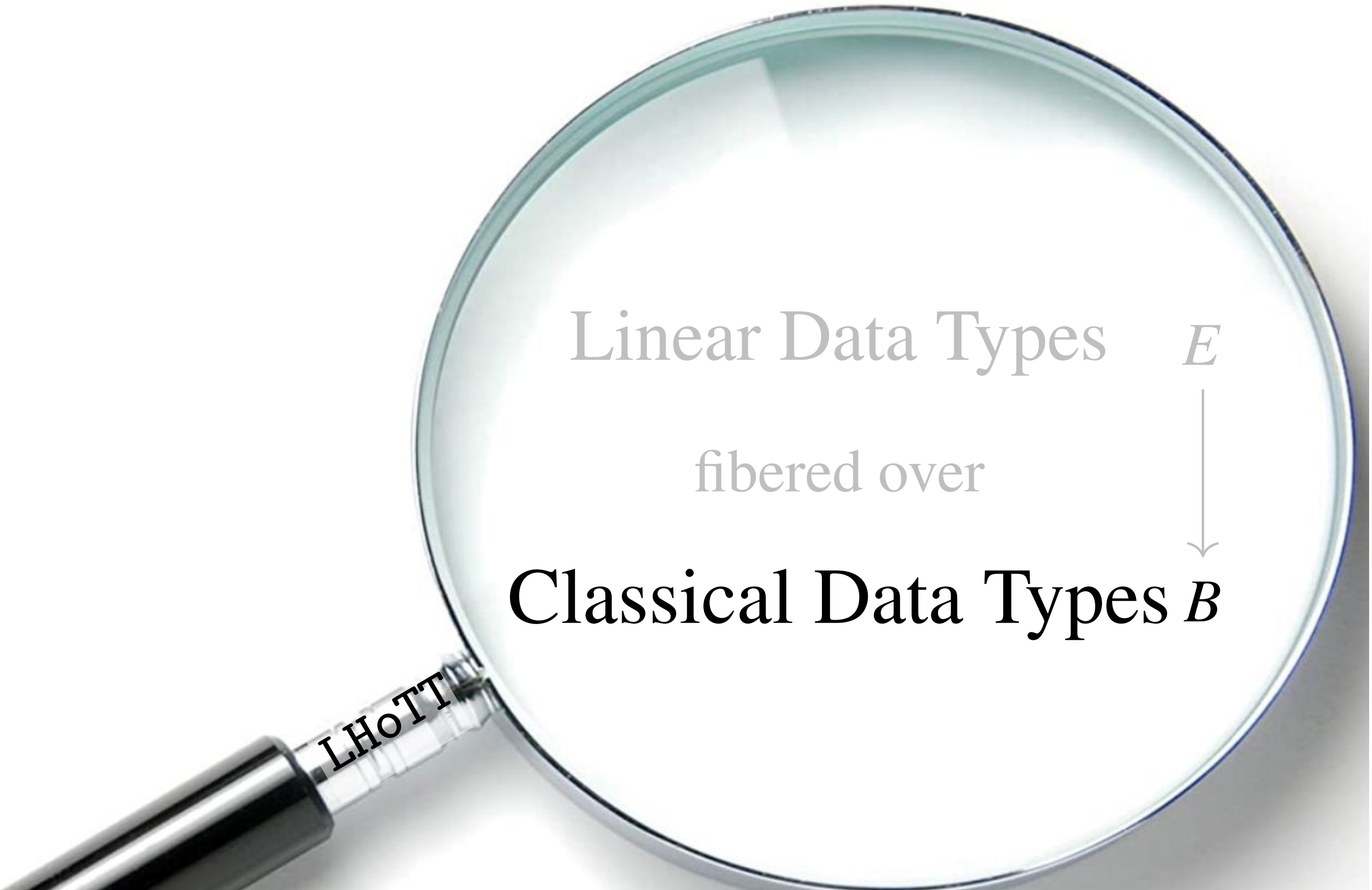

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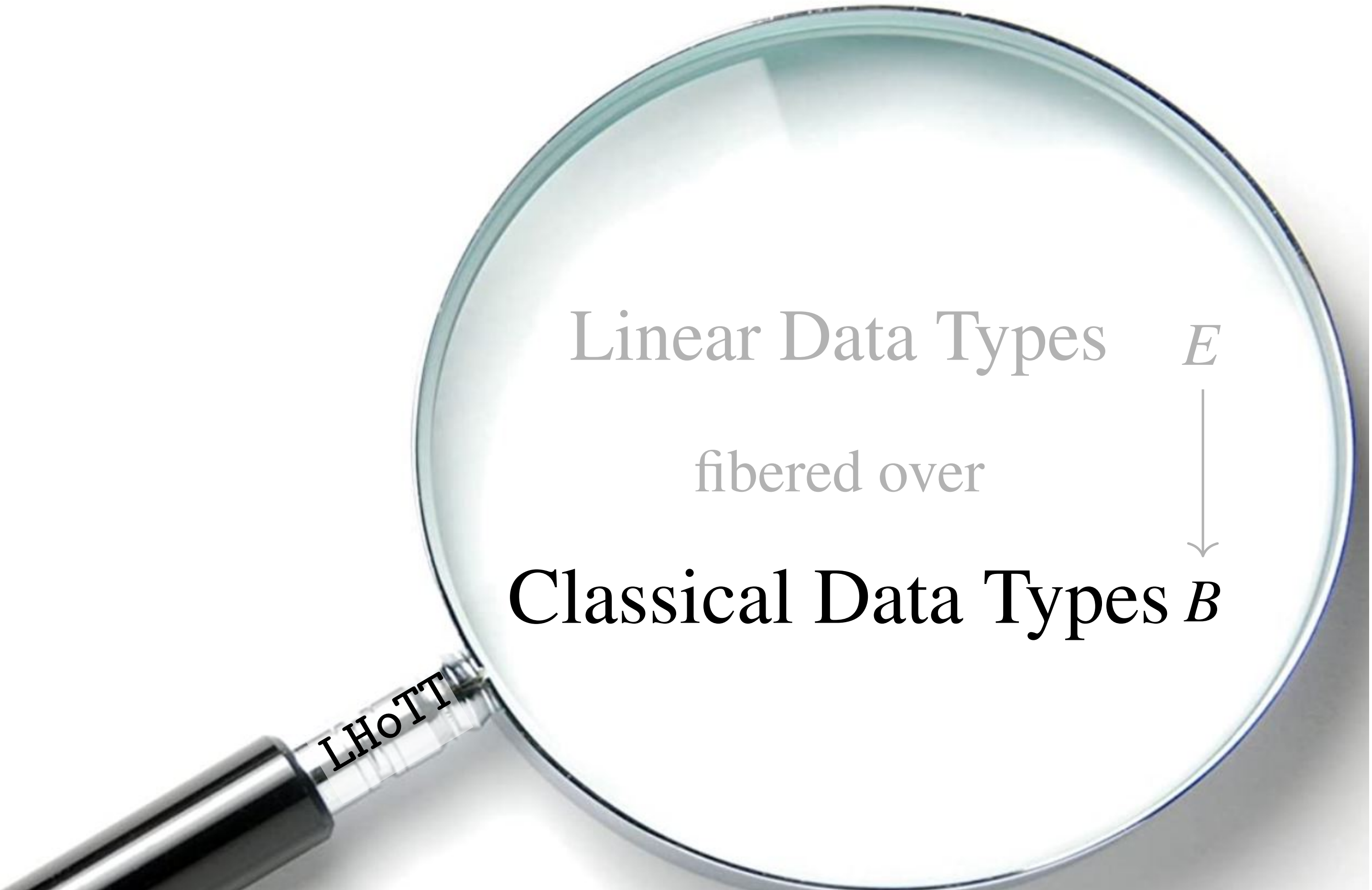

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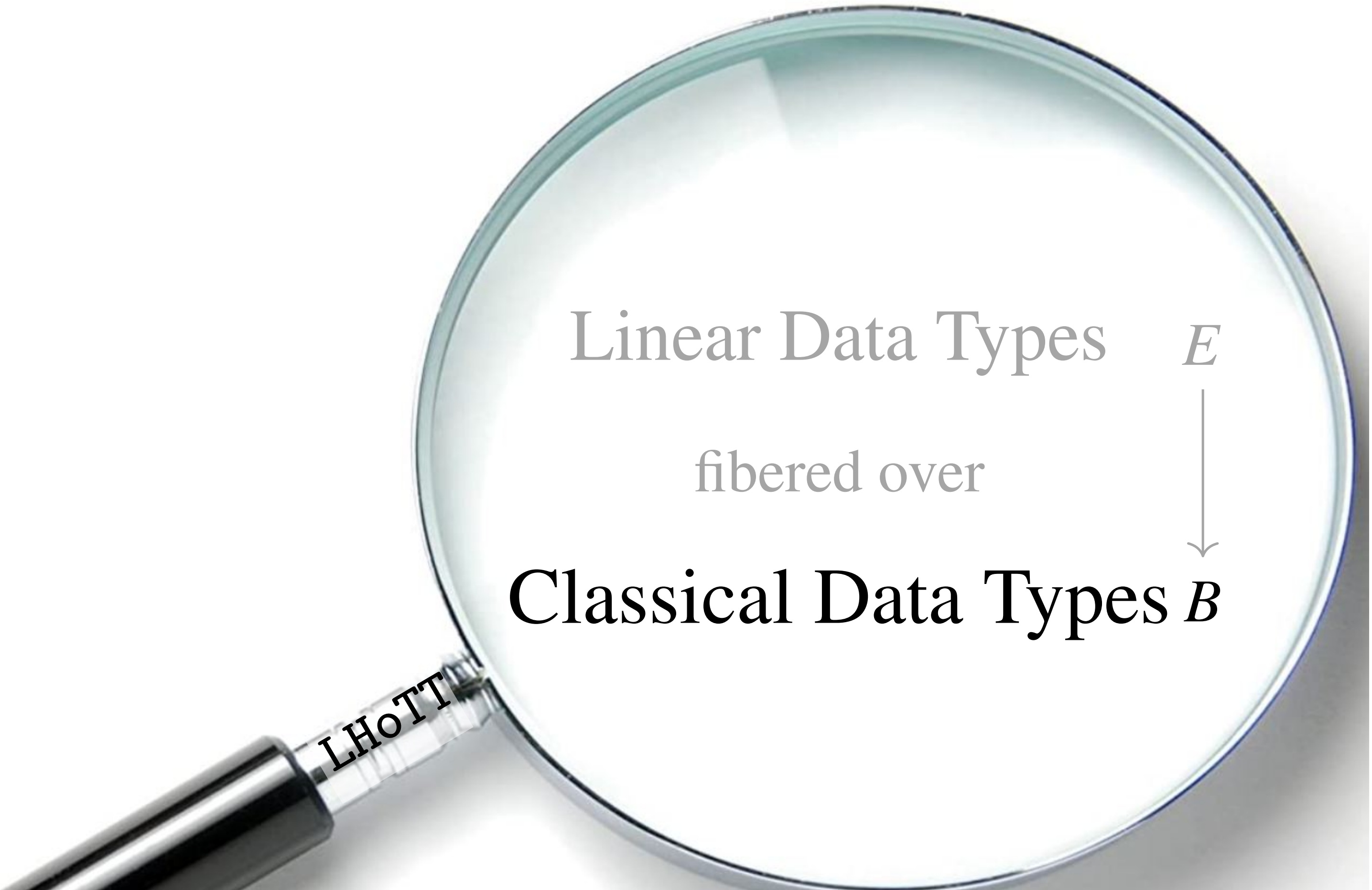

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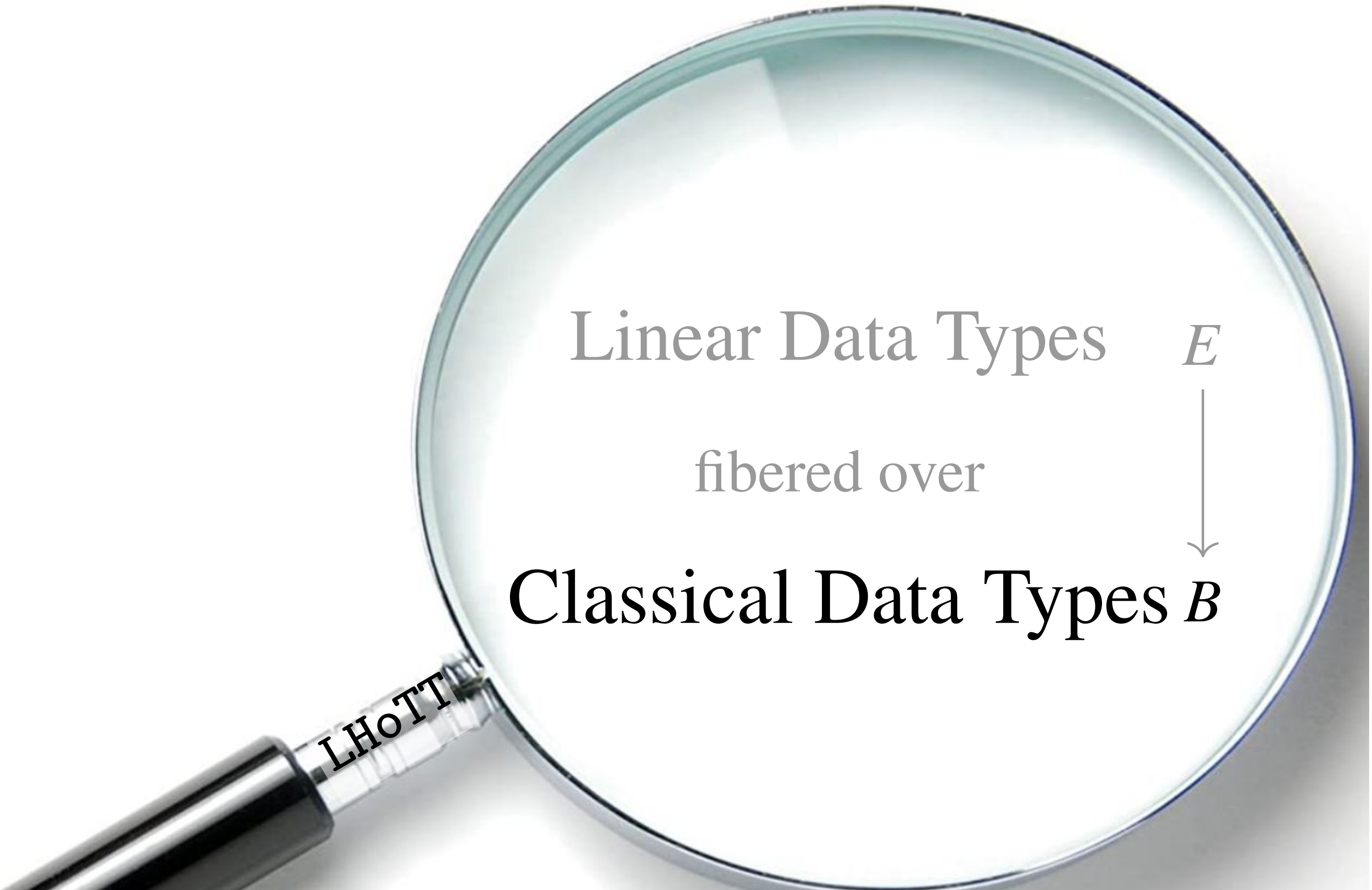

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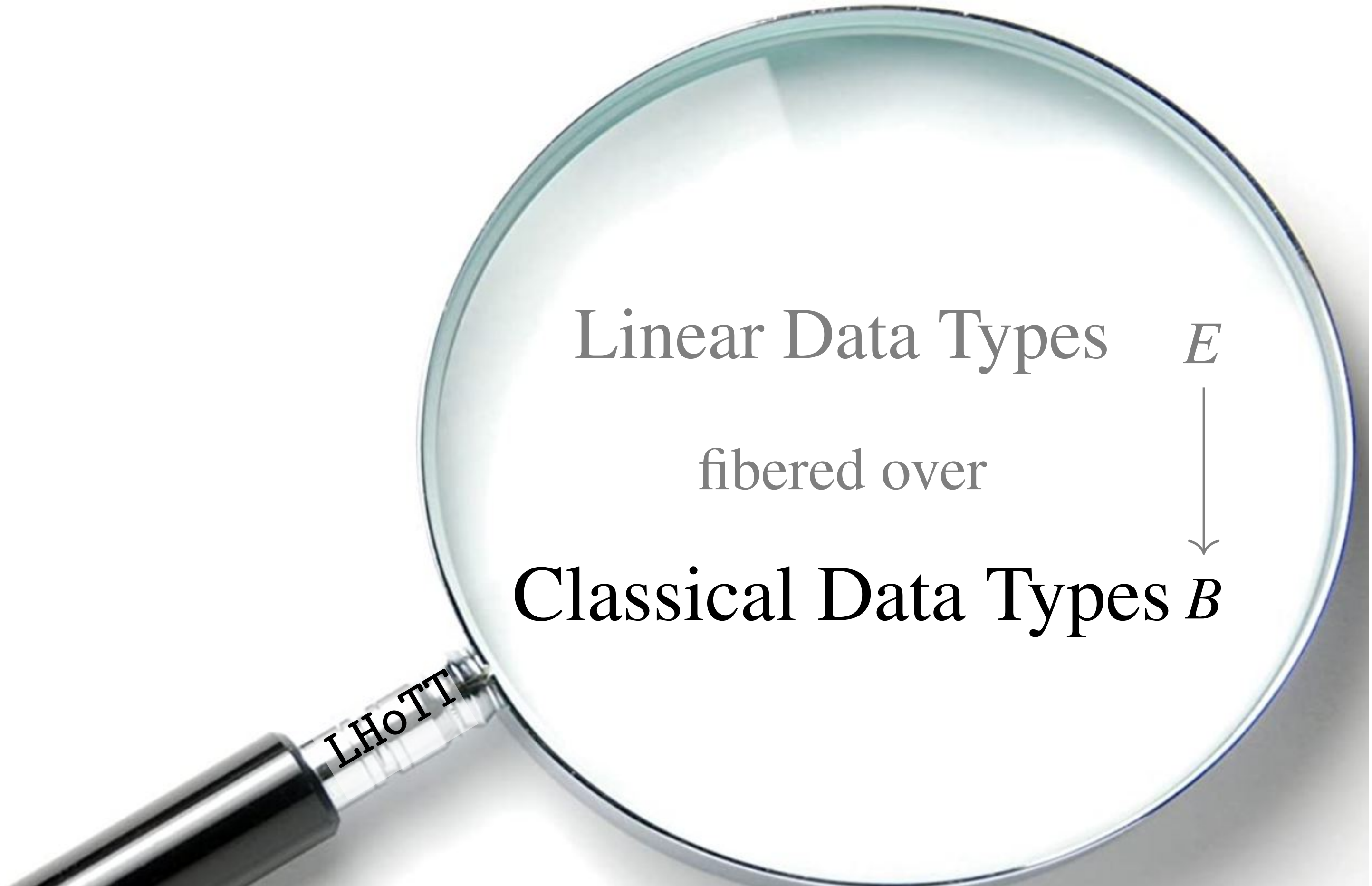

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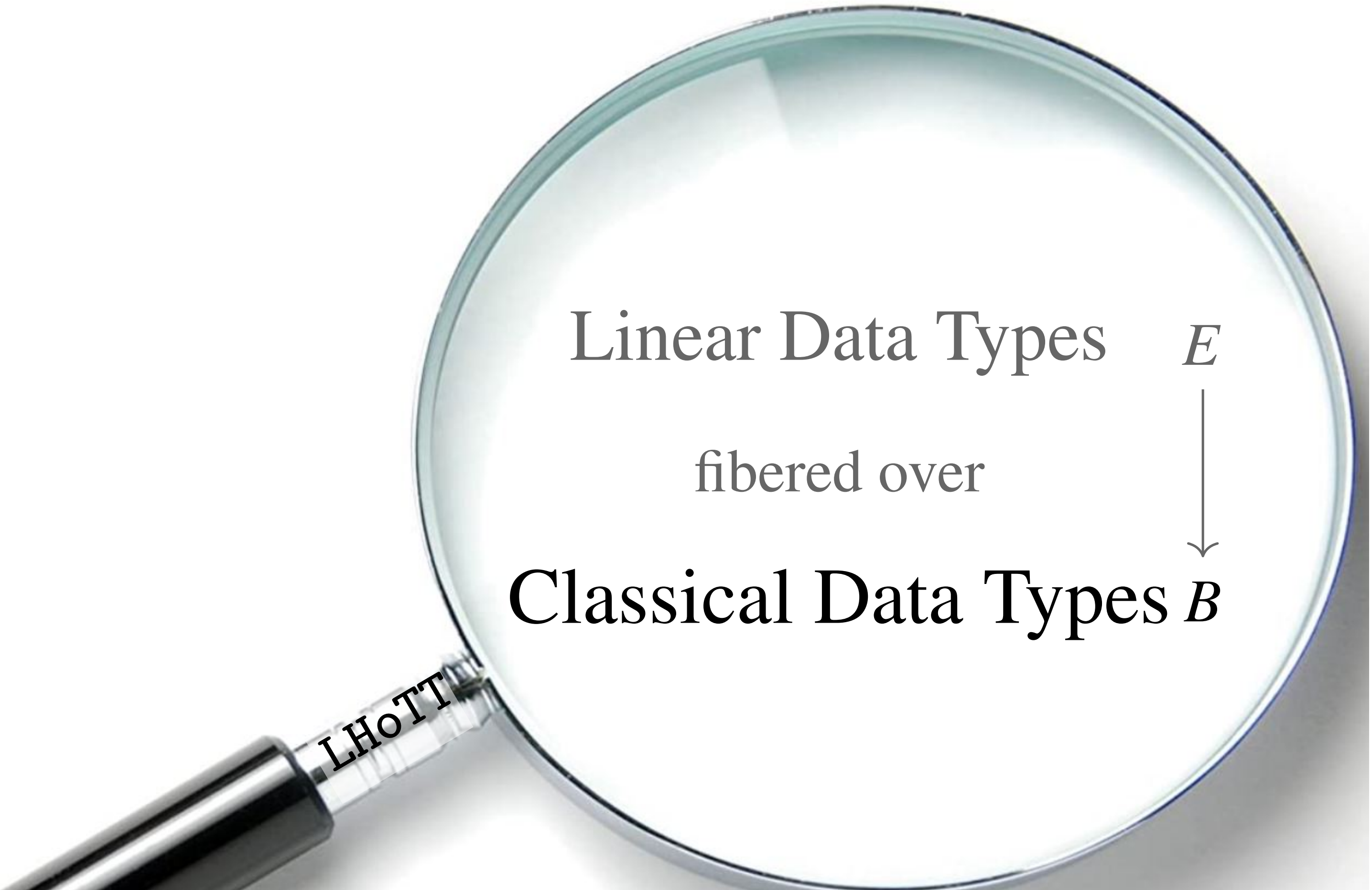

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LHoTT
module
bundle
E
B
parameter
base

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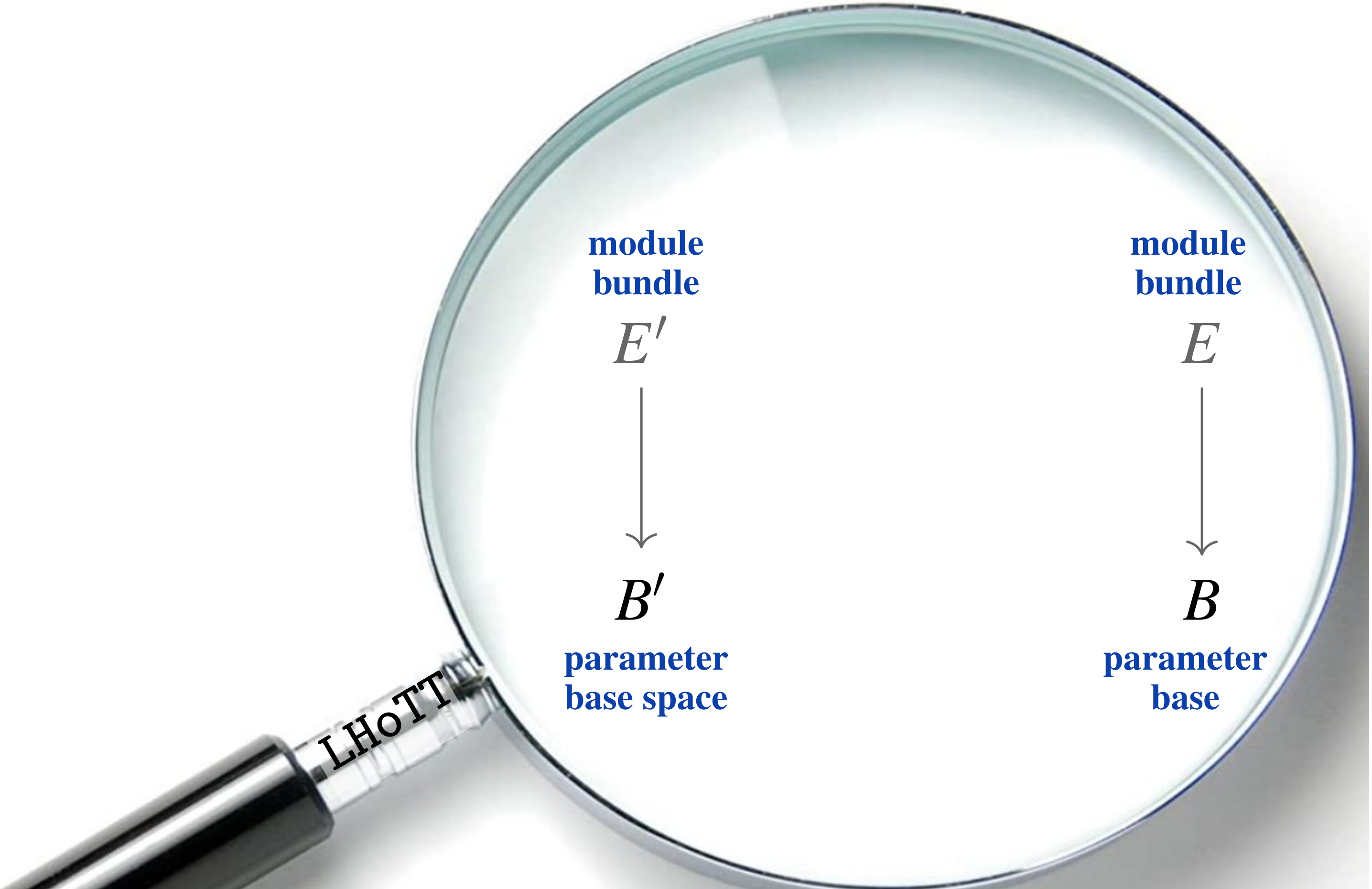

$$\begin{tikzcd}
\text{module bundle} & & \text{module bundle} \\
E' \arrow[d] & & E \arrow[d] \\
B' & & B \\
\text{parameter base space} & & \text{parameter base}
\end{tikzcd}$$

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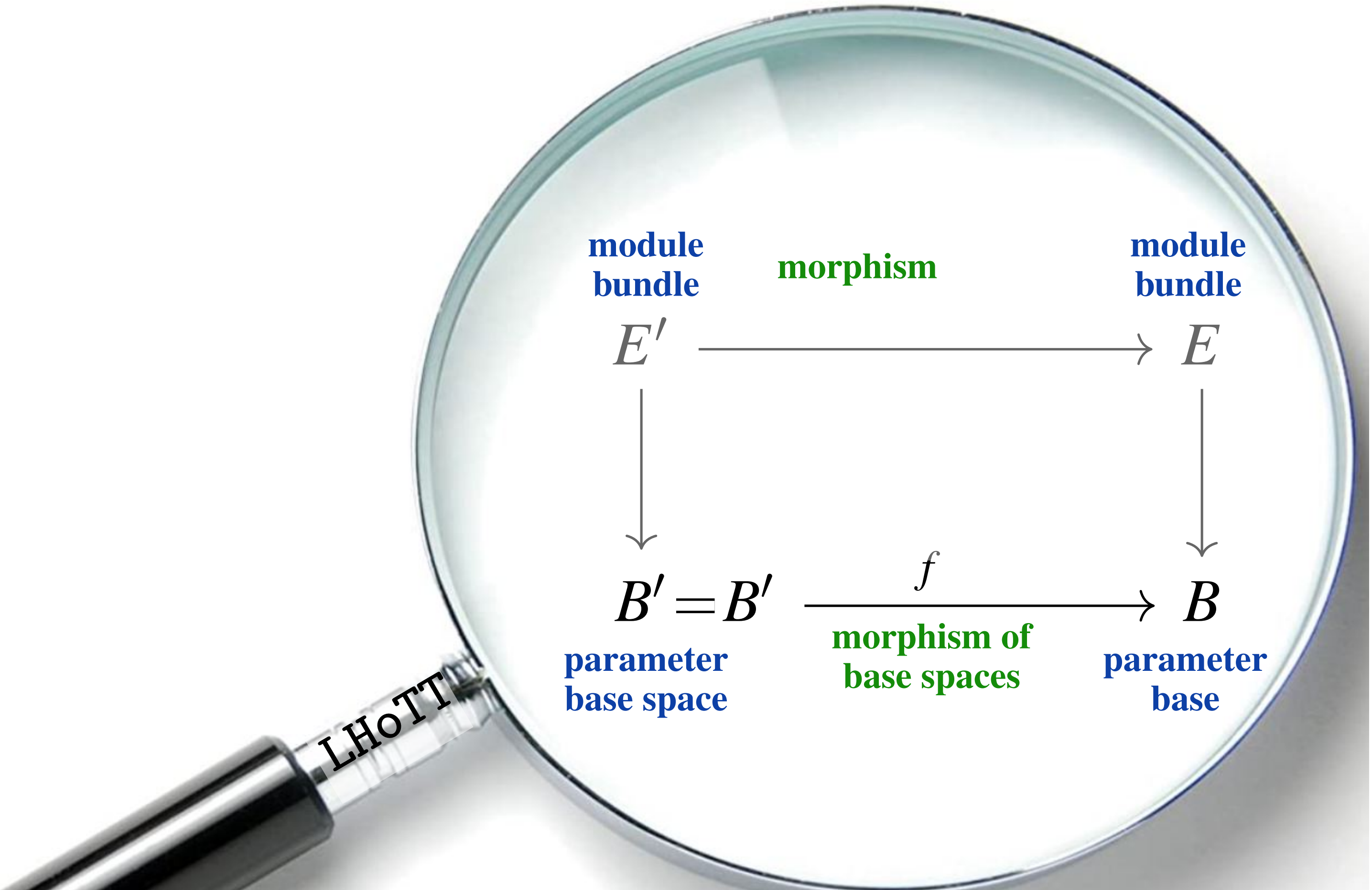

$$\begin{tikzcd}
E' \arrow[r] \arrow[d] & E \arrow[d] \\
B' = B' \arrow[r, "f"] & B
\end{tikzcd}$$

module bundle

morphism

module bundle

parameter base space

morphism of base spaces

parameter base

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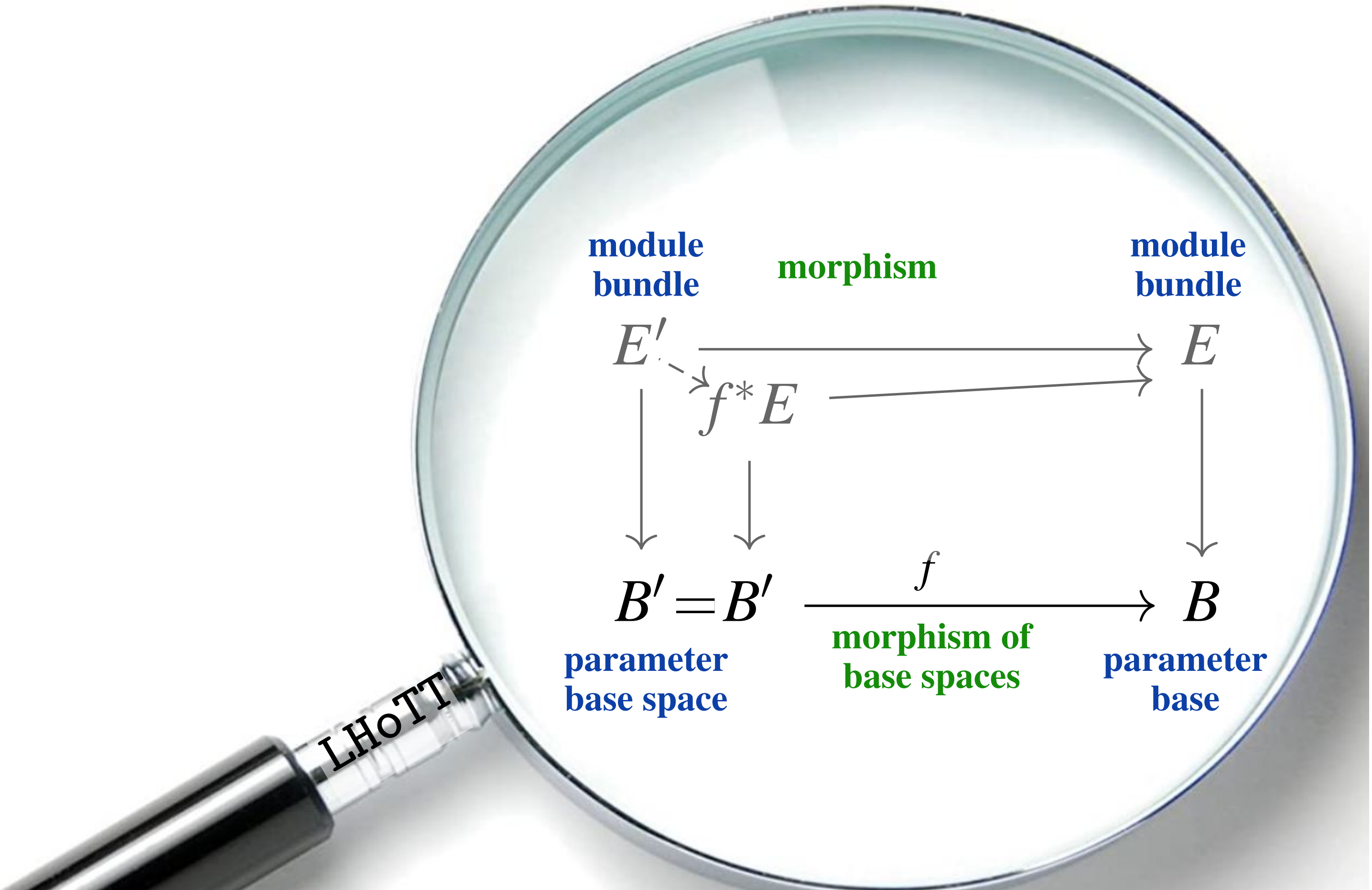

$$\begin{tikzcd}
E' \arrow[rrr] \arrow[dr, dashed] \arrow[dd] & & & E \arrow[dd] \\
& f^*E \arrow[urr] \arrow[d] & & \\
B' \arrow[r, phantom, "="] & B' \arrow[rr, "f"] & & B
\end{tikzcd}$$

module bundle

morphism

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morphism of base spaces

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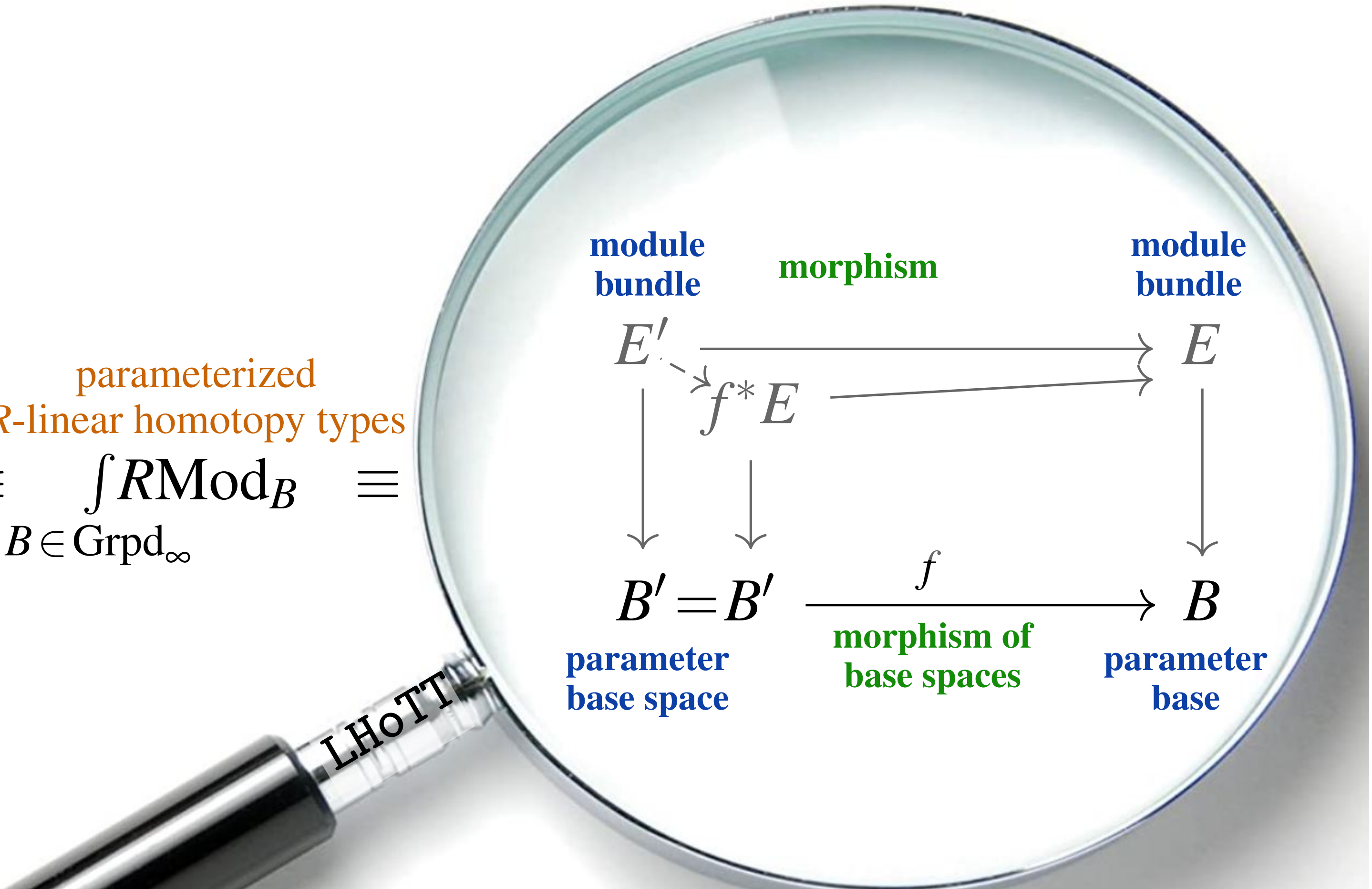
tangent ∞-topos of

parameterized
R-linear homotopy types

$$T\mathrm{Grpd}_\infty \;:\equiv\; \int_{B\in\mathrm{Grpd}_\infty} R\mathrm{Mod}_B \;\equiv$$



module bundle

morphism

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parameter base space

morphism of base spaces

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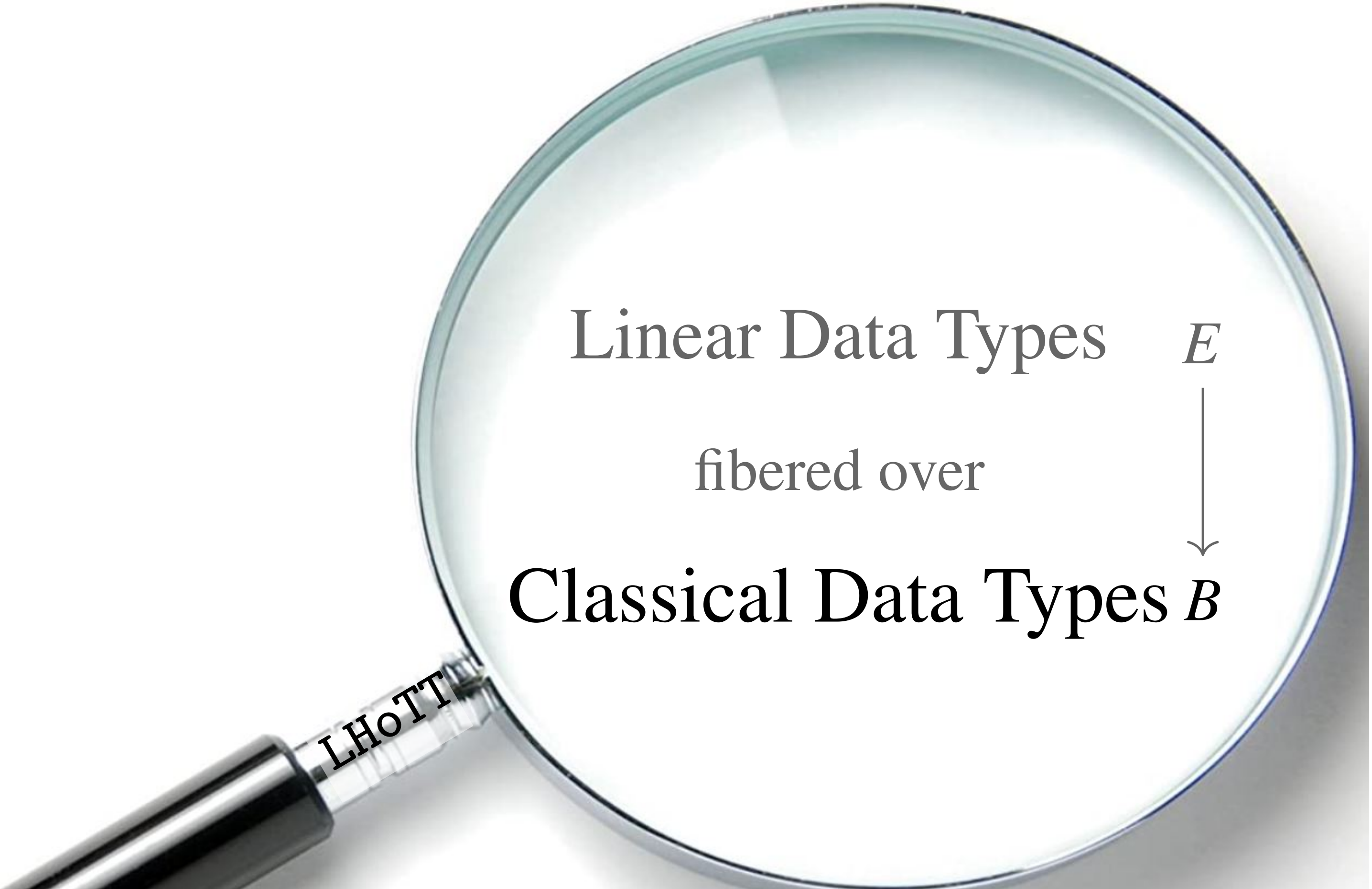

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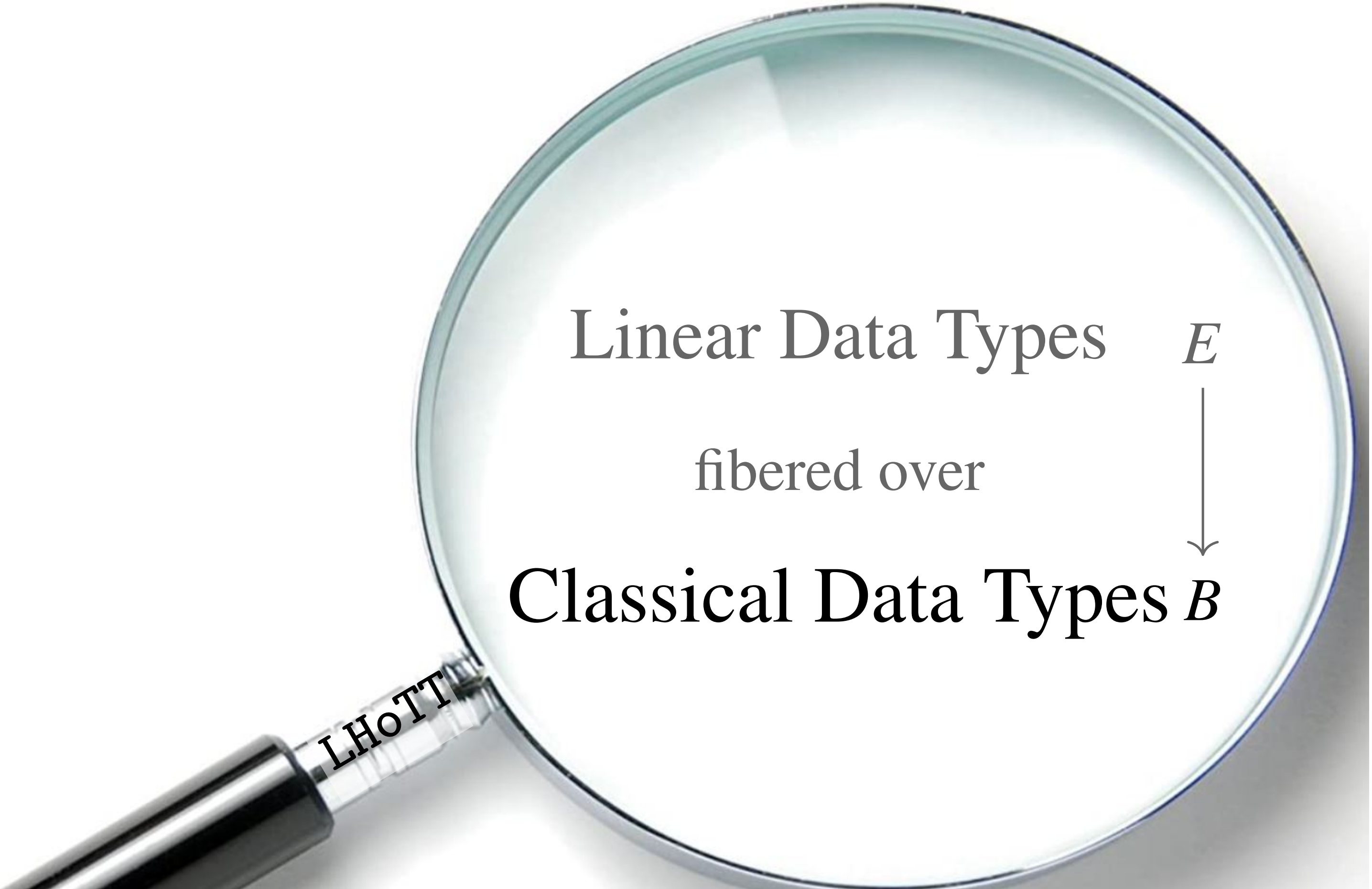

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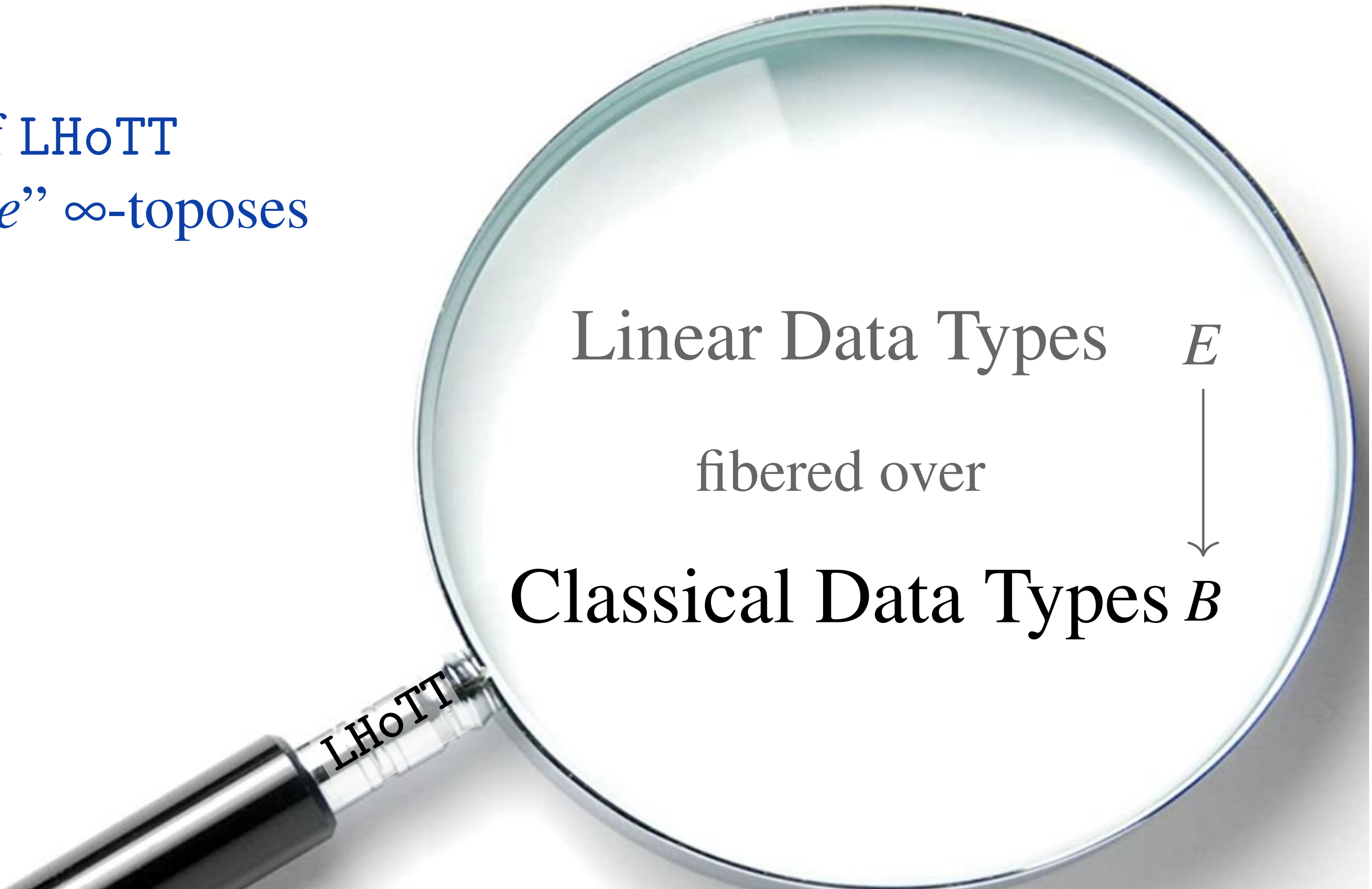
The categorical semantics of LHoTT
is in "*infinitesimally cohesive*" ∞-toposes



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∃ classical & linear dependent type theory
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The categorical semantics of `LHoTT`
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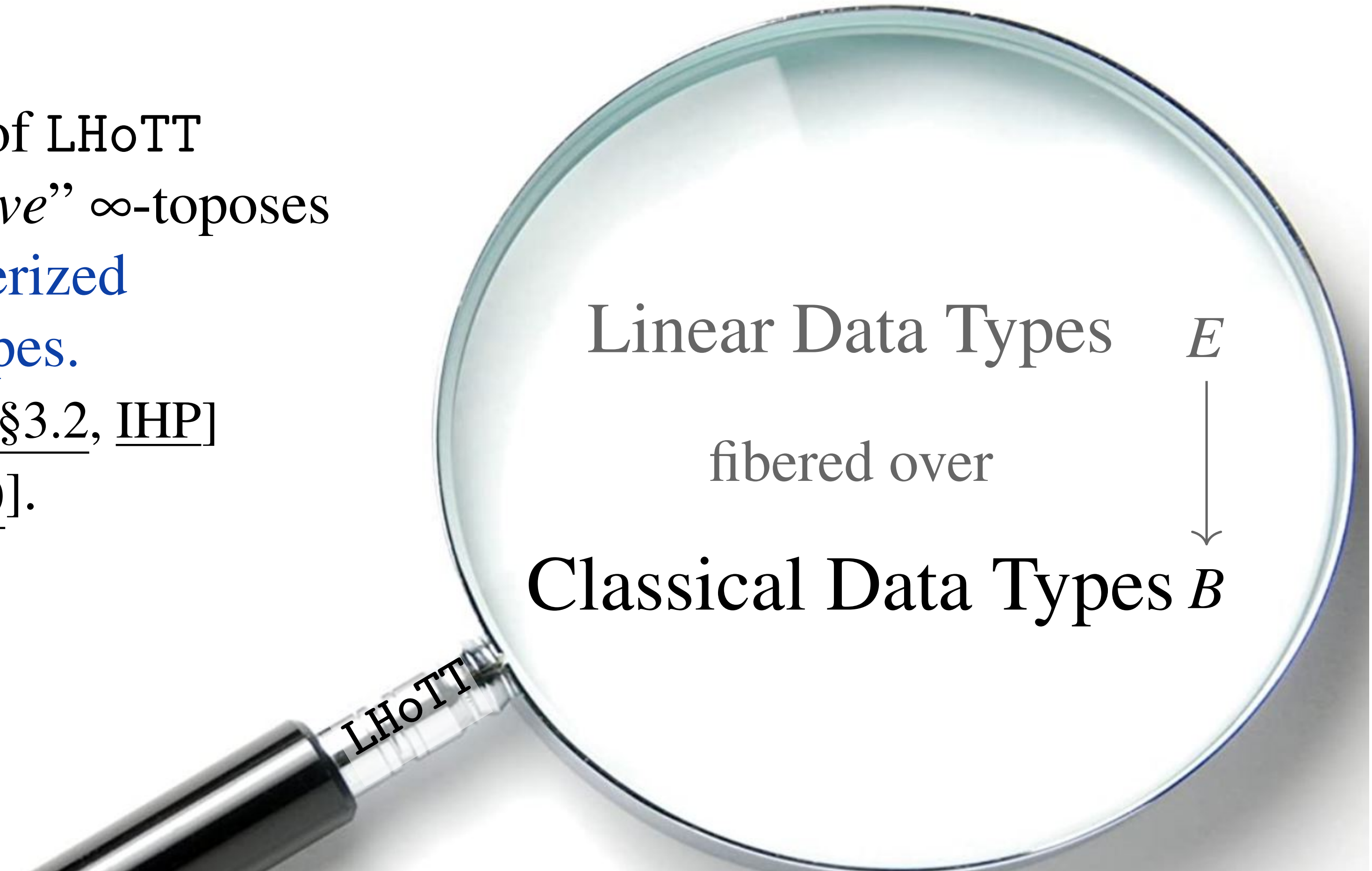

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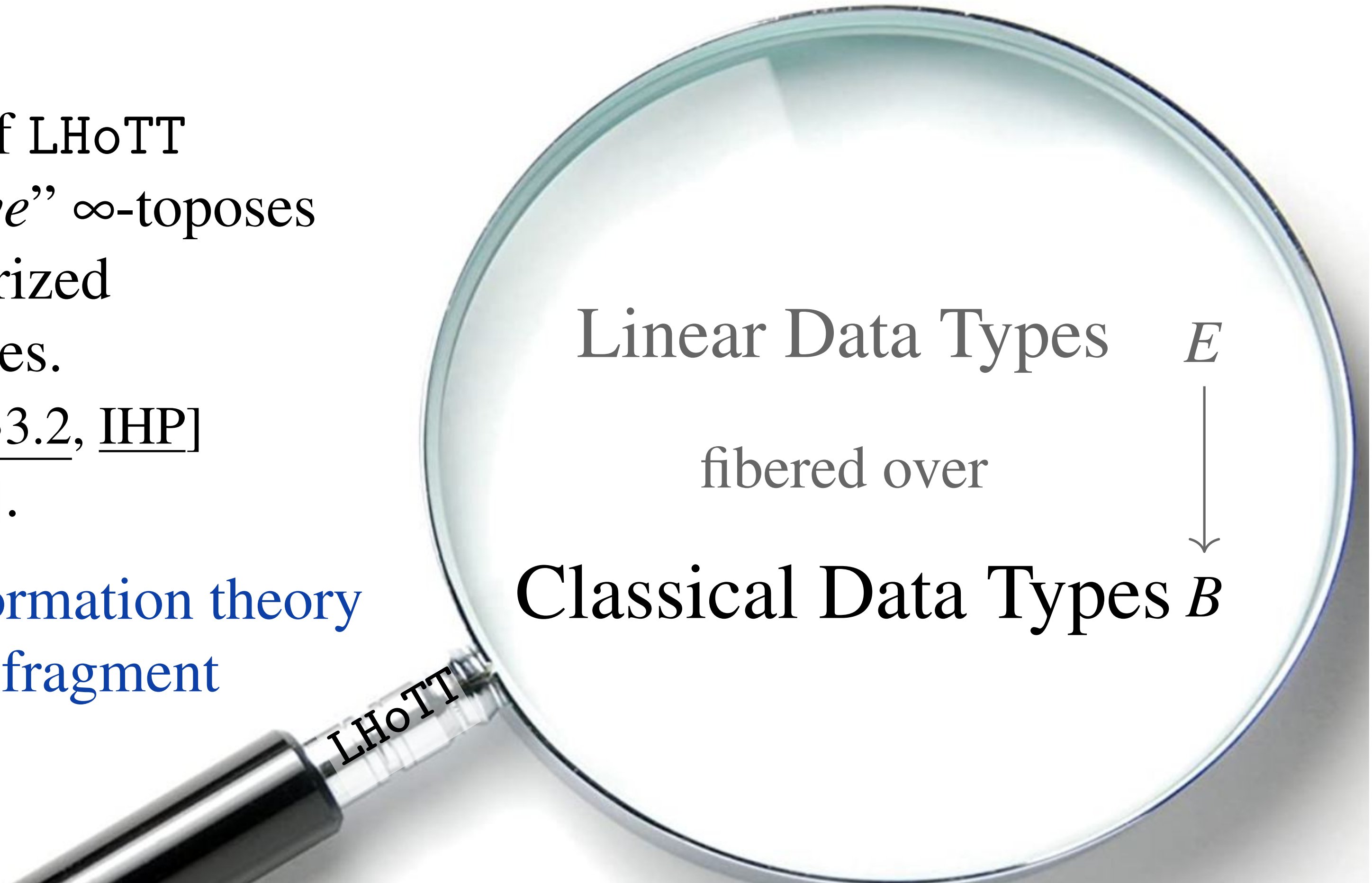

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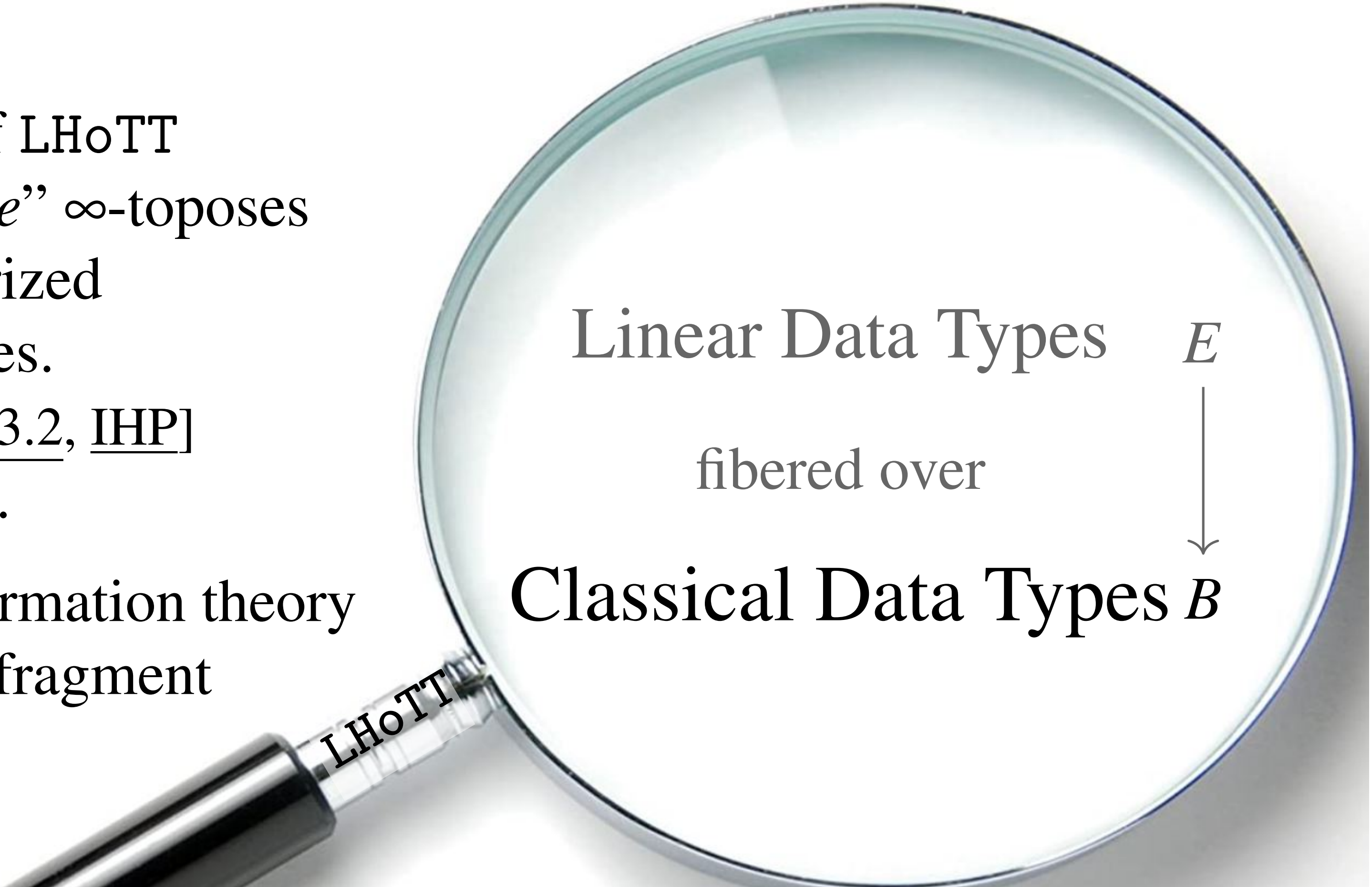

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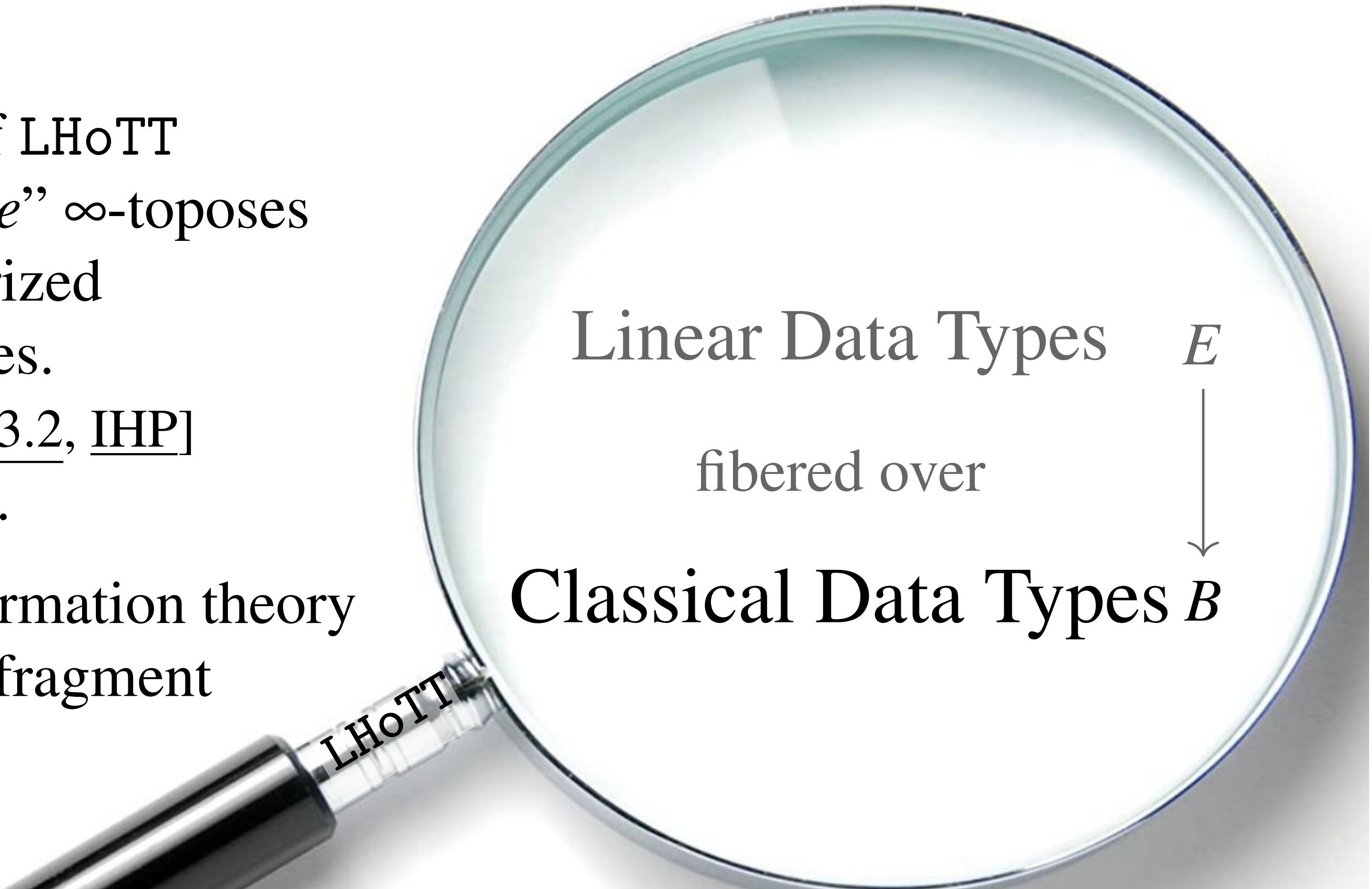

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complex EM-spectrum **EM-module spectra** **chain complexes**

$$R :\equiv H\mathbb{C} \quad \Rightarrow \quad (H\mathbb{C})\mathrm{Mod} \;\simeq\; \mathrm{Ch}_\bullet(\mathbb{C})$$

connective spectra

$$\begin{tikzcd} (H\mathbb{C})\mathrm{Mod} \arrow[loop left] \arrow[r, shift right=2, "\tau_{\geq 0}"'] \arrow[r, phantom, "\perp"] & (H\mathbb{C})\mathrm{Mod}_{\geq 0} \arrow[l, hook', shift right=2] \end{tikzcd}$$

connective cover

t-structure

$$(H\mathbb{C})\mathrm{Mod}^{\heartsuit} :\equiv (H\mathbb{C})\mathrm{Mod}_{\geq 0} \cap (H\mathbb{C})\mathrm{Mod}_{\leq 0} \;\simeq\; \mathbb{C}\mathrm{Vect}$$

the heart **complex vector spaces**

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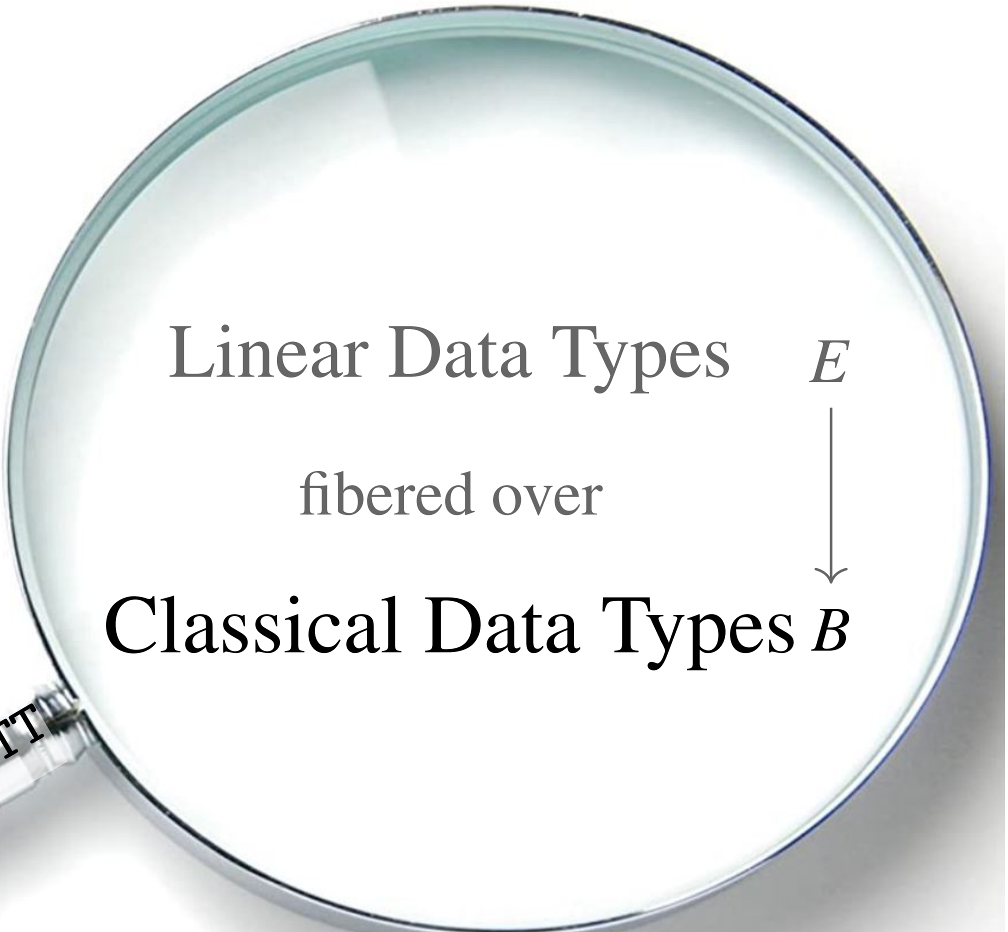

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$\rightsquigarrow$ full-blown Quantum Systems language emerges embedded in LHoTT

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⇝ full-blown Quantum Systems language emerges embedded in LHoTT

Linear Homotopy Type Theory (LHoTT)

for universal algorithmic quantum computation

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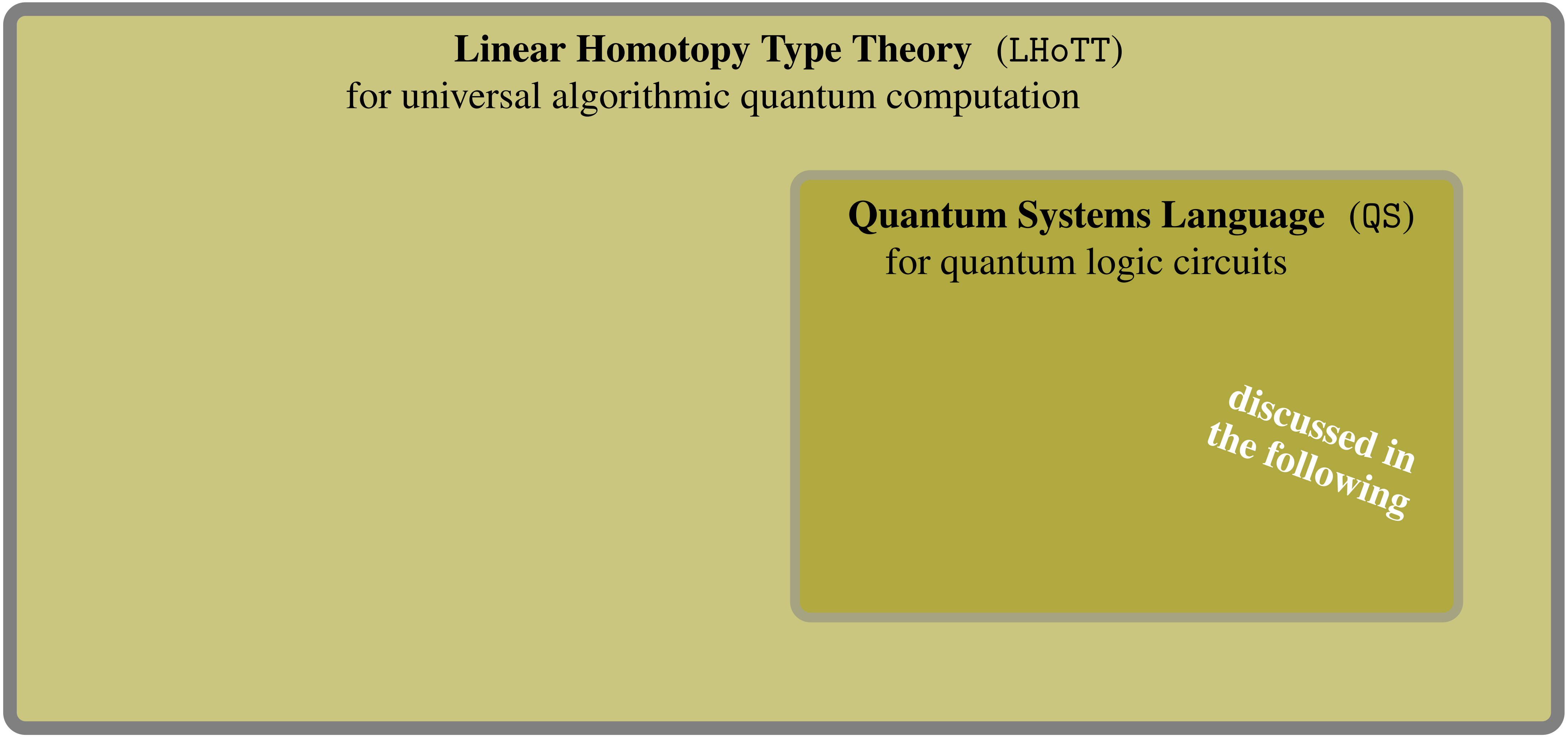


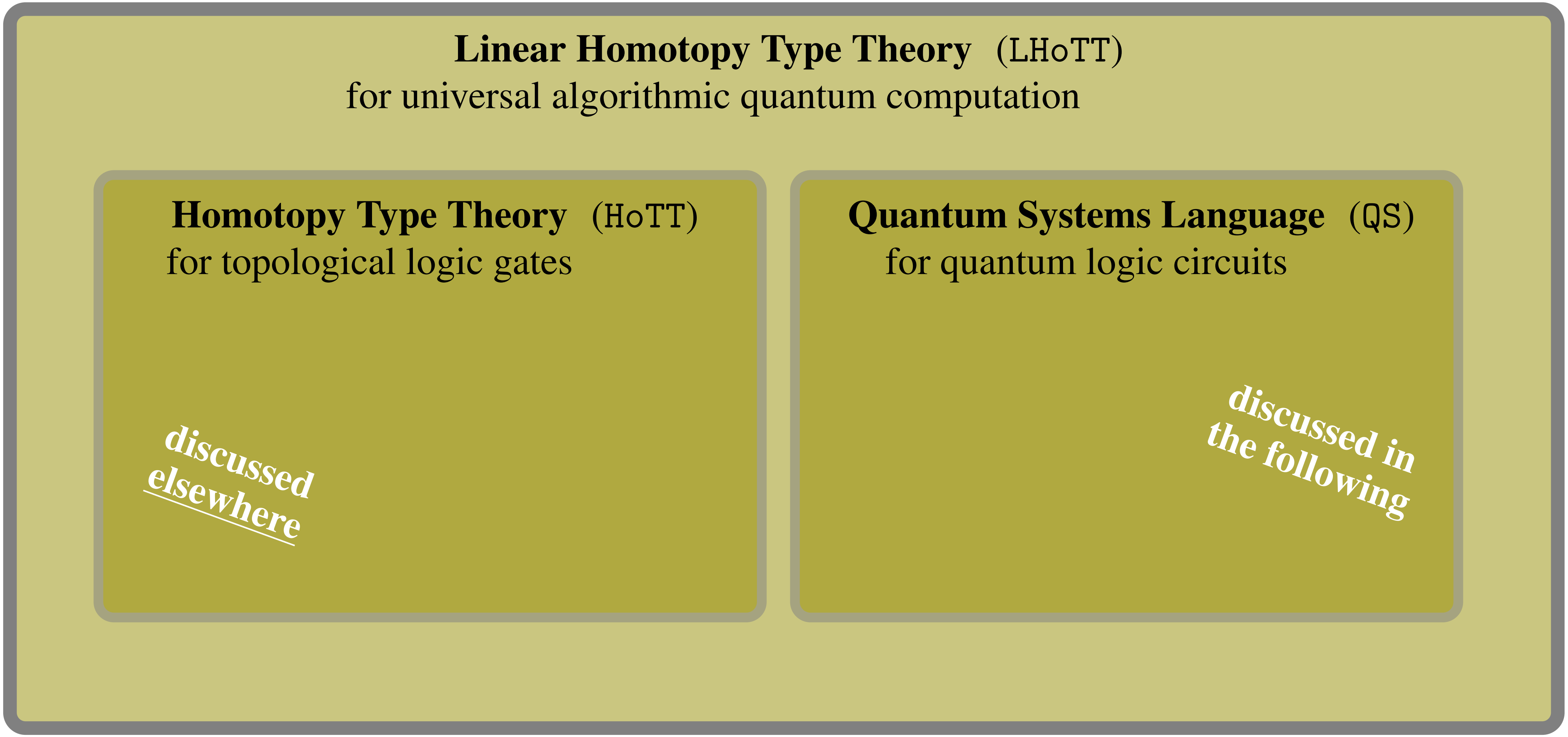
Linear Homotopy Type Theory (LHoTT)
for universal algorithmic quantum computation

Quantum Systems Language (QS)
for quantum logic circuits

discussed in the following

QS – Quantum Systems language @ CQTS

⇝ full-blown Quantum Systems language emerges embedded in LHoTT



Linear Homotopy Type Theory (LHoTT)
for universal algorithmic quantum computation

Homotopy Type Theory (HoTT)
for topological logic gates

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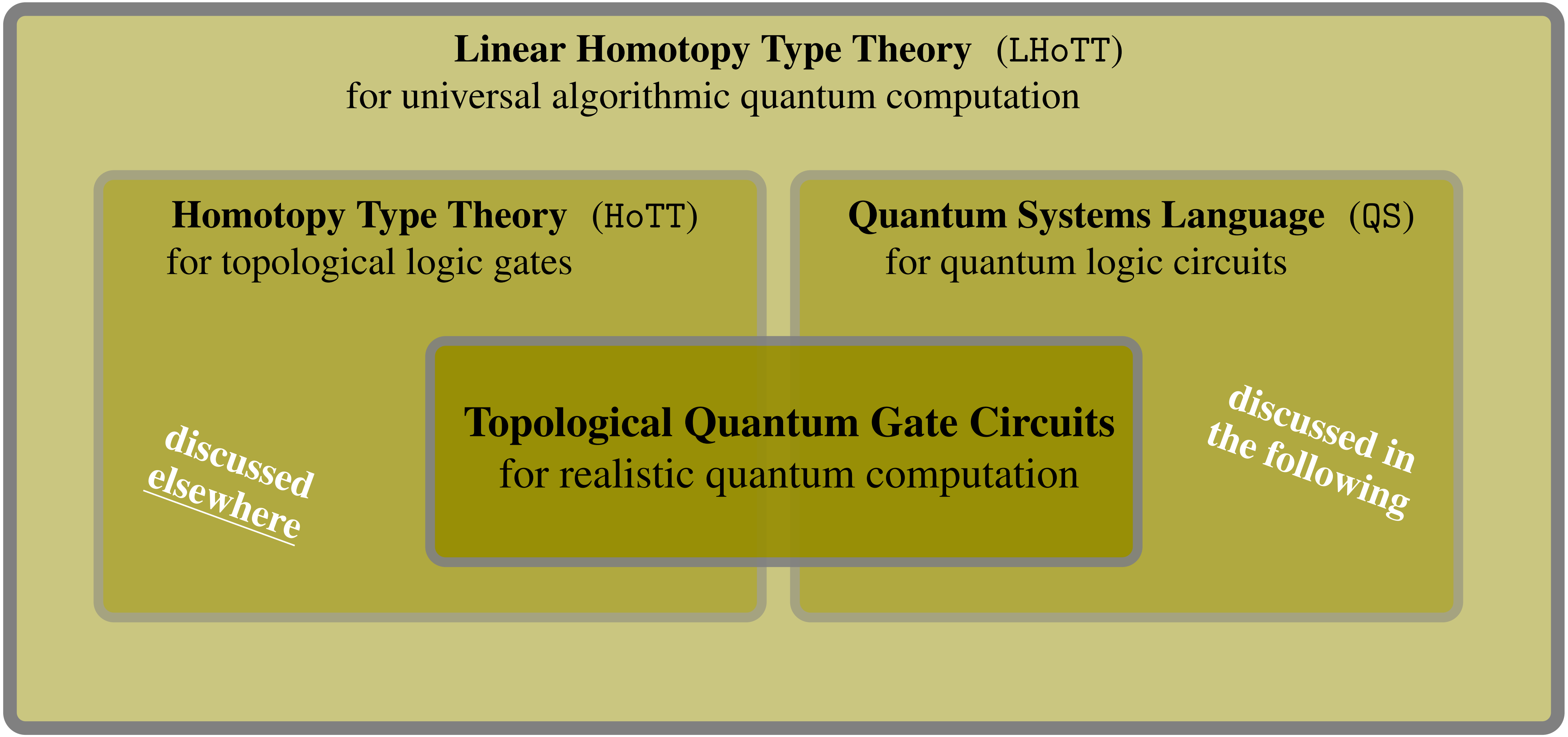

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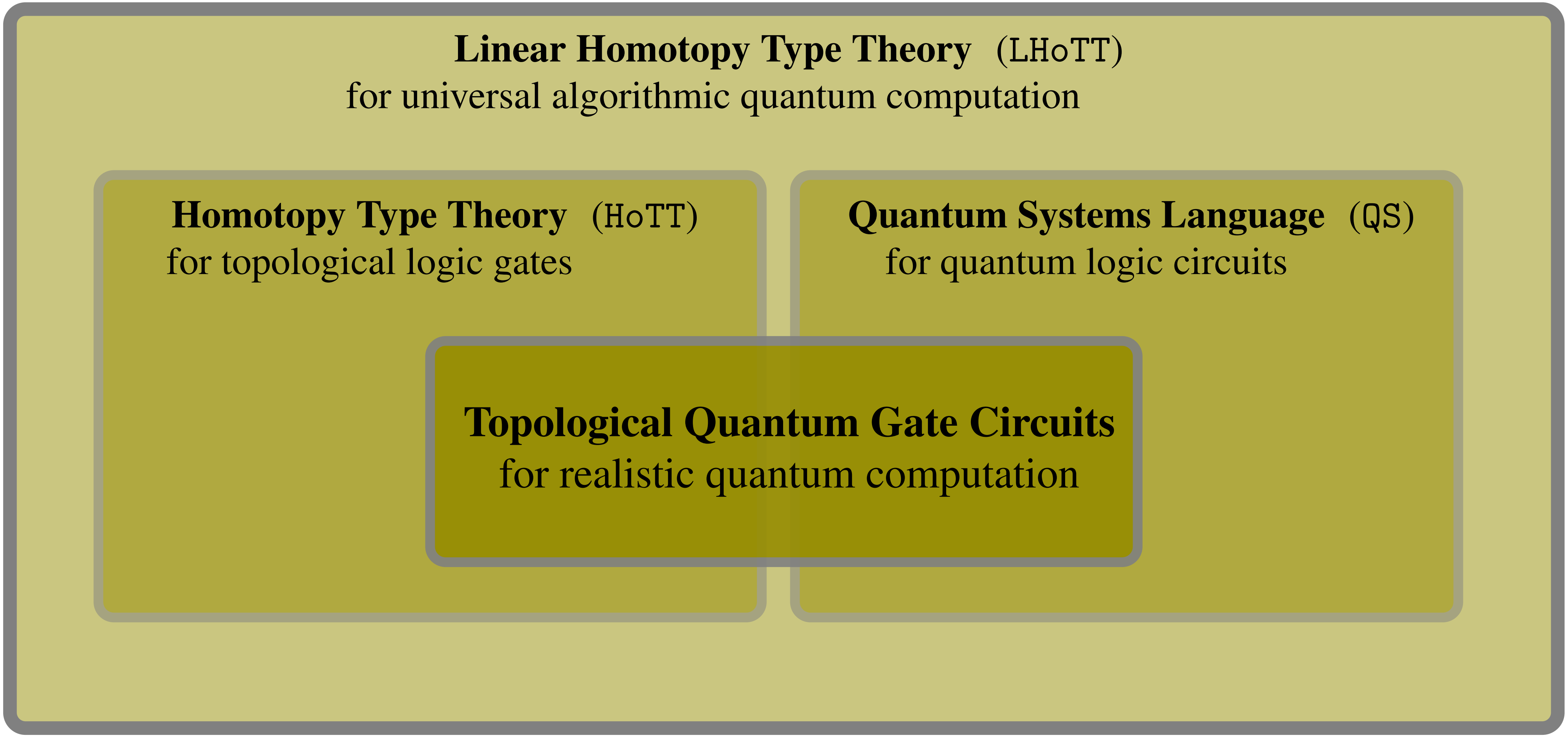

Linear Homotopy Type Theory (LHoTT)
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Homotopy Type Theory (HoTT)
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Topological Quantum Gate Circuits
for realistic quantum computation

ambient LHoTT	verifies	classically dependent quantum linear types
ambient HoTT	provides	specification of topological quantum gates
ambient dTT	provides	full verified classical control

Quantum Data Types

Linear/Quantum Data Types

Characteristic Property			
Symbol			
Formula (for B : FinType)			
AlgTop Jargon			
Linear Logic			
Physics Meaning			

Linear/Quantum Data Types

<table>
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Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:		
Symbol	$\oplus$ direct sum		
Formula (for B : FinType)			
AlgTop Jargon			
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Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	
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Linear/Quantum Data Types

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Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd}
B \arrow[r, "p_B"] & *
\end{tikzcd}$$

classical type system dependent on context

$$\mathrm{BType}_B \qquad\qquad \mathrm{BType}$$

classical type system

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Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
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Formula (for B : FinType)	cart. product co-product $\prod_B \mathcal{H}_b \simeq \bigoplus_B \mathcal{H}_b \simeq \coprod_B \mathcal{H}_b$ direct sum	$\mathcal{V} \otimes \Big(\bigoplus\limits_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus\limits_{b:B} (\mathcal{V} \otimes \mathcal{H}_b)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd} B \arrow[r, "p_B"] & * \end{tikzcd}$$

classical context extension

classical type system dependent on context

$$\begin{tikzcd} \mathrm{BType}_B & \mathrm{BType} \arrow[l, "*_B \times"'] \end{tikzcd}$$

classical type system

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\bigoplus$ direct sum	$\bigotimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)	$\overset{\text{cart. product}}{\prod_B \mathcal{H}_b} \simeq \underset{\text{direct sum}}{\bigoplus_B \mathcal{H}_b} \simeq \overset{\text{co-product}}{\coprod_B \mathcal{H}_b}$	$\mathcal{V} \otimes \Big(\bigoplus_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus_{b:B} (\mathcal{V} \otimes \mathcal{H}_b)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd} B \arrow[r, "p_B"] & * \end{tikzcd}$$

co-product

classical type system dependent on context

$$\begin{tikzcd} \mathrm{BType}_B \arrow[r, shift left=2, "\coprod_{b:B}"] \arrow[r, phantom, "\bot"] & \mathrm{BType} \arrow[l, shift left=2, "*_B \times"] \end{tikzcd}$$

classical type system

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\oplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)	$\overset{\textbf{cart. product}}{\prod_B \mathcal{H}_b} \simeq \underset{\textbf{direct sum}}{\oplus_B \mathcal{H}_b} \simeq \overset{\textbf{co-product}}{\coprod_B \mathcal{H}_b}$	$\mathcal{V} \otimes \Big(\bigoplus_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus_{b:B} (\mathcal{V} \otimes \mathcal{H}_b)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd}
B \arrow[r, "p_B"] & *
\end{tikzcd}$$

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge]
\mathrm{BType}_B \arrow[r, shift left=6, "\coprod_{b:B}", "\text{co-product}"'{yshift=20pt}] \arrow[r, phantom, shift left=3, "\perp"] & \mathrm{BType} \arrow[l, "*_B \times"'] \\
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
\mathrm{BType}_B \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, shift right=3, "\prod_{b:B}"', "\text{product}"{yshift=-20pt}] & \mathrm{BType}
\end{tikzcd}$$

classical type system

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\oplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)	$\overset{\text{cart. product}}{\prod_B \mathcal{H}_b} \simeq \underset{\text{direct sum}}{\bigoplus_B \mathcal{H}_b} \simeq \overset{\text{co-product}}{\coprod_B \mathcal{H}_b}$	$\mathcal{V} \otimes \Big(\bigoplus_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus_{b:B} (\mathcal{V} \otimes \mathcal{H}_b)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \; \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd}
B \arrow[r, "p_B"] & *
\end{tikzcd}$$

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge]
\mathrm{BType}_B \arrow[r, shift left=5, "\overset{\text{co-product}}{\coprod_{b:B}}"] \arrow[r, phantom, shift left=2.5, "\perp"] \arrow[r, phantom, shift right=2.5, "\perp"] \arrow[r, shift right=5, "\underset{\text{product}}{\prod_{b:B}}"'] & \mathrm{BType} \arrow[l, "*_B \times" description]
\end{tikzcd}$$

classical type system

classical base change / classical quantification

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\oplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)	**cart. product** **co-product** $\prod_B \mathcal{H}_b \simeq \bigoplus_B \mathcal{H}_b \simeq \coprod_B \mathcal{H}_b$ **direct sum**	$\mathcal{V} \otimes \Big(\bigoplus\limits_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus\limits_{b:B} (\mathcal{V} \otimes \mathcal{H}_b)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd} B \arrow[r, "p_B"] & * \end{tikzcd}$$

co-product

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge] \mathrm{BType}_B \arrow[r, shift left=5, "\coprod_{b:B}"] \arrow[r, phantom, shift left=2.5, "\perp"] \arrow[r, phantom, shift right=2.5, "\perp"] \arrow[r, shift right=5, "\prod_{b:B}"'] & \mathrm{BType} \arrow[l, "*_B \times" description] \end{tikzcd}$$

classical type system

product

classical base change / classical quantification

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\bigoplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)	$\overset{\text{cart. product}}{\prod_B \mathcal{H}_b} \simeq \underset{\text{direct sum}}{\bigoplus_B \mathcal{H}_b} \simeq \overset{\text{co-product}}{\coprod_B \mathcal{H}_b}$	$\mathcal{V} \otimes \Big(\bigoplus_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus_{b:B} (\mathcal{V} \otimes \mathcal{H}_b)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd}
B \arrow[rr, "p_B"] & & *
\end{tikzcd}$$

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge]
\mathrm{BType}_B \arrow[r, bend left=50, "\overset{\text{co-product}}{\coprod_{b:B}}"] \arrow[r, bend right=50, "\underset{\text{product}}{\prod_{b:B}}"'] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] & \mathrm{BType} \arrow[l, "*_B \times"']
\end{tikzcd}$$

classical type system

classical base change / classical quantification

linear type system in classical context

$$\big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\big) \qquad\qquad \big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\big)$$

linear type system

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\bigoplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)	$\overset{\text{cart. product}}{\prod_B \mathcal{H}_b} \simeq \underset{\text{direct sum}}{\bigoplus_B \mathcal{H}_b} \simeq \overset{\text{co-product}}{\coprod_B \mathcal{H}_b}$	$\mathcal{V} \otimes \Big(\bigoplus_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus_{b:B} \big(\mathcal{V} \otimes \mathcal{H}_b\big)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

Dependent linear Type Formers

$$\begin{tikzcd}
\overset{\text{finite classical context (variables, parameters, ...)}}{B} \arrow[rr, "p_B"] & & \overset{\text{reference context}}{*}
\end{tikzcd}$$

classical type system dependent on context

$$\begin{tikzcd}
\mathrm{BType}_B \arrow[rr, shift left=12, "\overset{\text{co-product}}{\coprod_{b:B}}"] \arrow[rr, phantom, shift left=6, "\perp"] \arrow[rr, phantom, shift right=6, "\perp"] \arrow[rr, shift right=12, "\prod_{b:B}"', "\text{product}"'] & & \mathrm{BType} \arrow[ll, "*_B \times" description]
\end{tikzcd}$$

classical type system

linear type system in classical context

$$\begin{tikzcd}
\Big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\Big) & & \Big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\Big) \arrow[ll, "\overset{\text{linear context extension}}{\mathbb{1}_B \otimes}"']
\end{tikzcd}$$

linear type system

classical base change / classical quantification

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\bigoplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)	$\overset{\text{cart. product}}{\prod_B \mathcal{H}_b} \simeq \underset{\text{direct sum}}{\bigoplus_B \mathcal{H}_b} \simeq \overset{\text{co-product}}{\coprod_B \mathcal{H}_b}$	$\mathcal{V} \otimes \Big(\bigoplus_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus_{b:B} \big(\mathcal{V} \otimes \mathcal{H}_b\big)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd} B \arrow[rr, "p_B"] && * \end{tikzcd}$$

classical type system dependent on context

co-product

$$\begin{tikzcd}[column sep=huge] \mathrm{BType}_B \arrow[rr, shift left=8, "\coprod_{b:B}"] \arrow[rr, phantom, shift left=4, "\perp"] \arrow[rr, phantom, shift right=4, "\perp"] \arrow[rr, shift right=8, "\prod_{b:B}"'] && \mathrm{BType} \arrow[ll, "*_B \times"'] \end{tikzcd}$$

product

classical type system

classical base change / classical quantification

linear type system in classical context

direct sum

$$\begin{tikzcd}[column sep=huge] \big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\big) \arrow[rr, shift left=8, "\bigoplus_{b:B}"] \arrow[rr, phantom, shift left=4, "\perp"] \arrow[rr, phantom, shift right=4, "\perp"] \arrow[rr, shift right=8, "\bigoplus_{b:B}"'] && \big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\big) \arrow[ll, "\mathbb{1}_B \otimes"'] \end{tikzcd}$$

linear type system

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\bigoplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)	$\overset{\text{cart. product}}{\prod_B \mathcal{H}_b} \simeq \underset{\text{direct sum}}{\bigoplus_B \mathcal{H}_b} \simeq \overset{\text{co-product}}{\coprod_B \mathcal{H}_b}$	$\mathcal{V} \otimes \Big(\bigoplus_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus_{b:B} (\mathcal{V} \otimes \mathcal{H}_b)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd}
B \arrow[r, "p_B"] & *
\end{tikzcd}$$

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge]
\mathrm{BType}_B \arrow[r, shift left=6, "\overset{\text{co-product}}{\coprod_{b:B}}"{description}] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, shift right=6, "\underset{\text{product}}{\prod_{b:B}}"{description}] & \mathrm{BType} \arrow[l, "*_B \times"{description}]
\end{tikzcd}$$

classical type system

classical base change / classical quantification

linear type system in classical context

$$\begin{tikzcd}[column sep=huge]
\big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\big) \arrow[r, shift left=6, "\overset{\text{direct sum}}{\bigoplus_{b:B}}"{description}] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, shift right=6, "\bigoplus_{b:B}"{description}] & \big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\big) \arrow[l, "\mathbb{1}_B \otimes"{description}]
\end{tikzcd}$$

linear type system

quantum base change / Motivic Yoga

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\bigoplus$ direct sum	$\bigotimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)	$\overset{\text{cart. product}}{\prod_B \mathcal{H}_b} \simeq \underset{\text{direct sum}}{\bigoplus_B \mathcal{H}_b} \simeq \overset{\text{co-product}}{\coprod_B \mathcal{H}_b}$	$\mathcal{V} \otimes \Big(\bigoplus_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus_{b:B} \big(\mathcal{V} \otimes \mathcal{H}_b\big)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

Dependent linear Type Formers

finite classical context (variables, parameters, ...)

reference context

$$\begin{tikzcd} B \arrow[r, "p_B"] & * \end{tikzcd}$$

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge] \mathrm{BType}_B \arrow[r, shift left=5, "\overset{\text{co-product}}{\coprod_{b:B}}"] \arrow[r, phantom, shift left=2.5, "\perp"] \arrow[r, leftarrow, "*_B \times" description] \arrow[r, phantom, shift right=2.5, "\perp"] \arrow[r, shift right=5, "\underset{\text{product}}{\prod_{b:B}}"'] & \mathrm{BType} \end{tikzcd}$$

classical type system

classical base change / classical quantification

linear type system in classical context

$$\begin{tikzcd}[column sep=huge] \big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\big) \arrow[r, shift left=5, "\overset{\text{direct sum}}{\bigoplus_{b:B}}"] \arrow[r, phantom, shift left=2.5, "\perp"] \arrow[r, leftarrow, "\mathbb{1}_B \otimes" description] \arrow[r, phantom, shift right=2.5, "\perp"] \arrow[r, shift right=5, "\bigoplus_{b:B}"'] & \big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\big) \end{tikzcd}$$

linear type system

quantum base change / Motivic Yoga

Linear/Quantum Data Types

Characteristic Property	**1.** their cartesian product blends into the co-product:	**2.** a tensor product appears & distributes over direct sum	**3.** a linear function type appears adjoint to tensor
Symbol	$\bigoplus$ direct sum	$\otimes$ tensor product	$\multimap$ linear function type
Formula (for B : FinType)	$\overset{\textbf{cart. product}}{\prod_B \mathcal{H}_b} \simeq \underset{\textbf{direct sum}}{\bigoplus_B \mathcal{H}_b} \simeq \overset{\textbf{co-product}}{\coprod_B \mathcal{H}_b}$	$\mathcal{V} \otimes \Big(\bigoplus_{b:B} \mathcal{H}_b \Big) \simeq \bigoplus_{b:B} (\mathcal{V} \otimes \mathcal{H}_b)$	$(\mathcal{V} \otimes \mathcal{H}) \multimap \mathcal{K}$ $\simeq \mathcal{V} \multimap (\mathcal{H} \multimap \mathcal{K})$

all available in LHoTT

Dependent linear Type Formers

$$\begin{tikzcd}
\underset{B}{\overset{\substack{\textbf{finite classical context}\\ \textbf{(variables, parameters, ...)}}}{}} B \arrow[rr, "p_B"] & & \overset{\textbf{reference context}}{*}
\end{tikzcd}$$

classical type system dependent on context

$$\begin{tikzcd}[column sep=huge]
\mathrm{BType}_B \arrow[r, shift left=7, "\text{co-product}", "\coprod_{b:B}"'] \arrow[r, phantom, shift left=3.5, "\perp"] \arrow[r, leftarrow, "*_B \times"{description}] \arrow[r, phantom, shift right=3.5, "\perp"] \arrow[r, shift right=7, "\prod_{b:B}", "\text{product}"'] & \mathrm{BType}
\end{tikzcd}$$

classical type system

classical base change / classical quantification

linear type system in classical context

$$\begin{tikzcd}[column sep=huge]
\Big(\mathrm{LType}_B, \overset{\text{tensor}}{\otimes_B}\Big) \arrow[r, shift left=7, "\text{direct sum}", "\bigoplus_{b:B}"'] \arrow[r, phantom, shift left=3.5, "\perp"] \arrow[r, leftarrow, "\mathbb{1}_B \otimes"{description}] \arrow[r, phantom, shift right=3.5, "\perp"] \arrow[r, shift right=7, "\bigoplus_{b:B}"'] & \Big(\mathrm{LType}, \overset{\text{tensor}}{\otimes}\Big)
\end{tikzcd}$$

linear type system

quantum base change / Motivic Yoga

Quantum Effects

Recall: **Monadic computational effects.**

A *monad* $\mathscr{E}(-)$ on a data type system encodes *computational effects*:

effectful program

$$D_1 \xrightarrow{\;\mathrm{prog}_{12}\;} \mathscr{E}(D_2)$$

output data of nominal type D_2
causing effects of type $\mathscr{E}(-)$

Recall: **Monadic computational effects.**

A *monad* $\mathscr{E}(-)$ on a data type system encodes *computational effects*:

first program

$$D_1 \xrightarrow{\mathrm{prog}_{12}} \mathscr{E}(D_2)$$

output data of nominal type D_2
causing effects of type $\mathscr{E}(-)$

second program

$$D_2 \xrightarrow{\mathrm{prog}_{23}} \mathscr{E}(D_3)$$

input data of type D_2
causing effects of type $\mathscr{E}(-)$

Recall: **Monadic computational effects.**

A *monad* $\mathscr{E}(-)$ on a data type system encodes *computational effects*:

first program **second program**

$$\begin{tikzcd} D_1 \arrow[r, "\mathrm{prog}_{12}"] & \mathscr{E}(D_2) & D_2 \arrow[r, "\mathrm{prog}_{23}"] & \mathscr{E}(D_3) \arrow[d, mapsto, bend left, "\text{bind previous effects into second program}"] \\ D_1 \arrow[r, "\mathrm{prog}_{12}"] & \mathscr{E}(D_2) & \mathscr{E}(D_2) \arrow[r, "\mathrm{bind}^{\mathscr{E}}\,\mathrm{prog}_{23}"] & \mathscr{E}(D_3) \end{tikzcd}$$

output data of nominal type D_2
causing effects of type $\mathscr{E}(-)$

input data of type D_2
causing effects of type $\mathscr{E}(-)$

bind previous effects
into second program

carry any previous $\mathscr{E}(-)$-effects along

Recall: **Monadic computational effects.**

A *monad* $\mathscr{E}(-)$ on a data type system encodes *computational effects*:

first program

second program

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}"] & \mathscr{E}(D_2) & D_2 \arrow[r, "\mathrm{prog}_{23}"] & \mathscr{E}(D_3) & {} \arrow[d, mapsto, bend left, "\text{bind previous effects into second program}"] \\
D_1 \arrow[r, "\mathrm{prog}_{12}"] & \mathscr{E}(D_2) & \mathscr{E}(D_2) \arrow[r, "\mathrm{bind}^{\mathscr{E}}\,\mathrm{prog}_{23}"] & \mathscr{E}(D_3) & {} \arrow[d, mapsto, bend left, "\text{compose}"] \\
D_1 \arrow[rrr, "\mathrm{bind}^{\mathscr{E}}\,\mathrm{prog}_{23} \circ \mathrm{prog}_{12}"] & & & \mathscr{E}(D_3) & {}
\end{tikzcd}$$

output data of nominal type D_2
causing effects of type $\mathscr{E}(-)$

input data of type D_2
causing effects of type $\mathscr{E}(-)$

bind previous effects
into second program

carry any previous $\mathscr{E}(-)$-effects along

compose

$\mathscr{E}$-composite program

causing cumulative $\mathscr{E}(-)$-effects

Recall: **Monadic effect handlers.**

$$D_1 \xrightarrow[\textbf{in-effectful program}]{\mathrm{prog}_{12}} D_2 \qquad \textbf{data type to absorb } \mathscr{E}\textbf{-effects}$$

Recall: **Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}", "\textbf{in-effectful program}"'] & D_2 & {} \arrow[d, mapsto, bend left=60, "\begin{array}{c}\textbf{incorporate handling}\\ \textbf{of } \mathscr{E}(-)\textbf{-effects}\end{array}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\,\mathrm{prog}_{12}", "\begin{array}{c}\textbf{in-effectful program}\\ \textbf{handling effects of type } \mathscr{E}(-)\end{array}"'] & D_2 & {}
\end{tikzcd}$$

Recall: **Monadic effect handlers.**

$$\begin{tikzcd}[column sep=huge, row sep=huge]
D_1 \arrow[r, "\mathrm{prog}_{12}", "\text{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left, "\begin{array}{c}\text{incorporate handling}\\ \text{of } \mathscr{E}(-)\text{-effects}\end{array}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\begin{array}{c}\text{in-effectful program}\\ \text{handling effects of type } \mathscr{E}(-)\end{array}"'] & D_2
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
D_1 \arrow[r, "\mathrm{ret}^{\mathscr{E}}_{D_1}", "\begin{array}{c}\text{produce}\\ \text{trivial effect}\end{array}"'] \arrow[rr, bend right, "\begin{array}{c}\mathrm{prog}_{12}\\ \text{no effect}\end{array}"'] & \mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\begin{array}{c}\text{handle effects}\\ \text{running program}\end{array}"'] & D_2
\end{tikzcd}$$

consistency conditions

Recall: **Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}", "\text{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left, "\text{incorporate handling of } \mathscr{E}(-)\text{-effects}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\,\mathrm{prog}_{12}", "\text{in-effectful program handling effects of type } \mathscr{E}(-)"'] & D_2
\end{tikzcd}$$

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{ret}^{\mathscr{E}}_{D_1}", "\text{produce trivial effect}"'] \arrow[rr, bend right, "\mathrm{prog}_{12}\ \text{no effect}"'] & \mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\,\mathrm{prog}_{12}", "\text{handle effects running program}"'] & D_2
\end{tikzcd}$$

consistency conditions

$$\begin{tikzcd}
\mathscr{E}(D_0) \arrow[r, "\mathrm{bind}^{\mathscr{E}}\,\mathrm{prog}_{01}", "\text{carry effects along}"'] \arrow[rr, bend right] & \mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\,\mathrm{prog}_{12}", "\text{handle cumulative effects}"'] & D_2
\end{tikzcd}$$

$$\mathrm{hndl}^{\mathscr{E}}_{D_2}\Big(D_0 \xrightarrow{\mathrm{prog}_{01}} \mathscr{E}(D_1) \xrightarrow{\mathrm{hndl}^{\mathscr{E}}_{D_2}\,\mathrm{prog}_{12}} D_2\Big)$$

handle effects... consecutively

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}", "\textbf{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left, "\textbf{incorporate handling of } \mathscr{E}(-)\textbf{-effects}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\,\mathrm{prog}_{12}", "\textbf{in-effectful program handling effects of type } \mathscr{E}(-)"'] & D_2
\end{tikzcd}$$

“$\mathscr{E}$-modal data type”

Monadicity:

$\mathscr{E}$**-modales in** Type
(“EM-category”) $\mathrm{Type}^{\mathscr{E}}$

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}", "\textbf{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left, "\textbf{incorporate handling of } \mathscr{E}(-)\textbf{-effects}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\textbf{in-effectful program handling effects of type } \mathscr{E}(-)"'] & D_2 \arrow[r, phantom, "\text{“}\mathscr{E}\textbf{-modal data type”}"] & {}
\end{tikzcd}$$

Monadicity:

$$\begin{tikzcd}
\mathrm{Type} \arrow[loop right, "\mathscr{E}"]
\end{tikzcd}$$

monad

$\mathscr{E}$-modales in Type
(“EM-category”)

$$\mathrm{Type}^{\mathscr{E}}$$

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}", "\textbf{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left, "\substack{\textbf{incorporate handling}\\ \textbf{of } \mathscr{E}(-)\textbf{-effects}}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\substack{\textbf{in-effectful program}\\ \textbf{handling effects of type } \mathscr{E}(-)}"'] & D_2 & \text{“$\mathscr{E}$\textbf{-modal data type}”}
\end{tikzcd}$$

Monadicity:

$$\begin{tikzcd}
 & & \mathrm{Type} \arrow[loop right, "\mathscr{E}"] & \textbf{induced monad} \\
\substack{\mathscr{E}\textbf{-modales in } \mathrm{Type} \\ \text{(“EM-category”)}} & \mathrm{Type}^{\mathscr{E}} \arrow[ur, shift right=2, "U^{\mathscr{E}}"', "\textbf{monadic adjunction}"'] \arrow[ur, phantom, "\perp"] & & \\
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Type} \arrow[r, "F^{\mathscr{E}}"] & \mathrm{Type}^{\mathscr{E}}
\end{tikzcd}$$

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}", "\text{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left, "\text{incorporate handling of } \mathscr{E}(-)\text{-effects}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\text{in-effectful program handling effects of type } \mathscr{E}(-)"'] & D_2
\end{tikzcd}$$

“$\mathscr{E}$-modal data type”

Monadicity:

any adjunction on type system

$$\begin{tikzcd}
\mathrm{Type}' \arrow[d, "K^{UF}"] \arrow[rr, shift right=1.5, "U"'] \arrow[rr, phantom, "\perp"] & & \mathrm{Type} \arrow[ll, shift right=1.5, "F"'] \arrow[loop right, "\mathscr{E}"] \arrow[dll, shift right=1.5, "F^{\mathscr{E}}"'] \\
\mathrm{Type}^{\mathscr{E}} \arrow[urr, shift right=1.5, "U^{\mathscr{E}}"'] \arrow[urr, phantom, "\perp"] & &
\end{tikzcd}$$

induced monad

terminal comparison functor

monadic adjunction

$\mathscr{E}$**-modales in** Type

(“EM-category”)

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd}
D_1 \arrow[r, "\mathrm{prog}_{12}", "\text{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left, "\begin{array}{c}\text{incorporate handling}\\ \text{of } \mathscr{E}(-)\text{-effects}\end{array}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\begin{array}{c}\text{in-effectful program}\\ \text{handling effects of type } \mathscr{E}(-)\end{array}"'] & D_2 & \text{“$\mathscr{E}$-modal data type”}
\end{tikzcd}$$

Monadicity:

$$\begin{tikzcd}
\begin{array}{r}\text{free $\mathscr{E}$-modales in Type}\\ \text{(“Kleisli category”)}\end{array} & \mathrm{Type}_{\mathscr{E}} \arrow[d, hook, "K_{UF}", "\begin{array}{c}\text{initial}\\ \text{comparison}\\ \text{functor}\end{array}"'] \arrow[dr, shift right, "U_{\mathscr{E}}"'] \arrow[dr, phantom, "\perp"] \\
\begin{array}{r}\text{any adjunction}\\ \text{on type system}\end{array} & \mathrm{Type}' \arrow[r, shift right, "U"'] \arrow[r, phantom, "\perp"] \arrow[d, "K^{UF}", "\begin{array}{c}\text{terminal}\\ \text{comparison}\\ \text{functor}\end{array}"'] & \mathrm{Type} \arrow[l, shift right, "F"'] \arrow[ul, two heads, shift right, "F_{\mathscr{E}}"'] \arrow[dl, shift right, "F^{\mathscr{E}}"'] \arrow[dl, phantom, "\perp"] \arrow[loop right, "\mathscr{E}"] & \text{induced monad} \\
\begin{array}{r}\text{$\mathscr{E}$-modales in Type}\\ \text{(“EM-category”)}\end{array} & \mathrm{Type}^{\mathscr{E}} \arrow[ur, shift right, "U^{\mathscr{E}}"', "\text{monadic adjunction}"]
\end{tikzcd}$$

Recall: **Data type system of Monadic effect handlers.**

$$\begin{tikzcd}[column sep=huge, row sep=huge]
D_1 \arrow[r, "\mathrm{prog}_{12}", "\textbf{in-effectful program}"'] & D_2 \arrow[d, mapsto, bend left, "\textbf{incorporate handling of } \mathscr{E}(-)\textbf{-effects}"] \\
\mathscr{E}(D_1) \arrow[r, "\mathrm{hndl}^{\mathscr{E}}_{D_2}\mathrm{prog}_{12}", "\textbf{in-effectful program handling effects of type } \mathscr{E}(-)"'] & D_2
\end{tikzcd}$$

“$\mathscr{E}$-modal data type”

Monadicity:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{Type}_{\mathscr{E}} \arrow[d, hook, "K_{UF}", "\textbf{initial comparison functor}"'] \arrow[dr, bend left=10, "U_{\mathscr{E}}"'] & \\
\mathrm{Type}' \arrow[d, "K^{UF}", "\textbf{terminal comparison functor}"'] \arrow[r, bend right=10, "U"'] & \mathrm{Type} \arrow[ul, two heads, bend left=10, "F_{\mathscr{E}}"'] \arrow[ul, phantom, "\perp"] \arrow[l, bend right=10, "F"'] \arrow[l, phantom, "\perp"] \arrow[dl, bend right=10, "F^{\mathscr{E}}"'] \arrow[dl, phantom, "\perp"] \arrow[loop right, "\mathscr{E}"] \\
\mathrm{Type}^{\mathscr{E}} \arrow[ur, bend right=10, "U^{\mathscr{E}}"', "\textbf{monadic adjunction}"] &
\end{tikzcd}$$

free $\mathscr{E}$-modales in Type
(“Kleisli category”)

any adjunction
on type system

$\mathscr{E}$-modales in Type
(“EM-category”)

induced monad

Now just to work this out
for the **effects induced by**
dependent data type formers
in LHoTT
⟶

Given B : BType of possible measurement outcomes (“possible worlds”) **the monadic effects of *B*-dependent data type formers** constitute modalities of **actual and potential *B*-measurements**:

classical data types

$$\begin{tikzcd}[column sep=large]
\text{actual data} & \perp & \mathrm{Type}_B \arrow[loop above, "\overset{\text{possibly}}{\Diamond_B}"] \arrow[loop below, "\underset{\text{necessarily}}{\Box_B}"'] \arrow[rr, shift left=4, "\coprod_B"] \arrow[rr, phantom, shift left=2, "\perp"] \arrow[rr, phantom, shift right=2, "\perp"] \arrow[rr, shift right=4, "\prod_B"'] & & \mathrm{Type} \arrow[ll, "\times B" description] \arrow[loop above, "\overset{\text{randomly}}{\star_B}"] \arrow[loop below, "\underset{\text{indefinitely}}{\bigcirc_B}"'] & \perp & \text{potential data}
\end{tikzcd}$$

Given B : BType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

$$\begin{tikzcd}
 & \textbf{possibly} & \textbf{randomly} & \\
\textbf{actual data} \quad \perp & \mathrm{Type}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[r, shift left=3, "\coprod_B"] \arrow[r, phantom, shift left=1.5, "\perp"] \arrow[r, phantom, shift right=1.5, "\perp"] \arrow[r, shift right=3, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"'] & \perp \quad \textbf{potential data} \\
 & \textbf{necessarily} & \textbf{indefinitely} &
\end{tikzcd}$$

necessarily $P_\bullet$

$$\Box_B P_\bullet$$

$$b : B \;\vdash\; \prod_{b' : B} P_{b'}$$

Given B : BType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

actual data $\perp$ **potential data** $\perp$

$$\begin{tikzcd}
\mathrm{Type}_B \arrow[loop above, "\lozenge_B"] \arrow[loop below, "\square_B"'] \arrow[r, bend left, "\coprod_B"] \arrow[r, phantom, bend left=20, "\perp"] \arrow[r, phantom, bend right=20, "\perp"] \arrow[r, bend right, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"']
\end{tikzcd}$$

necessarily **indefinitely**

---

necessarily $P_\bullet$ **entails** **actually $P_\bullet$**

$$\square_B P_\bullet \xrightarrow{\;\epsilon^{\square_B}_{P_\bullet}\;} P_\bullet$$

$$b : B \;\vdash\; \prod_{b' : B} P_{b'} \xrightarrow{(p_{b'})_{b':B} \;\mapsto\; p_b} P_b$$

Given B : BType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

$$\begin{tikzcd}
\mathrm{Type}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[r, shift left=8, "\coprod_B"] \arrow[r, shift left=4, phantom, "\perp"] \arrow[r, shift right=4, phantom, "\perp"] \arrow[r, shift right=8, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B"'] \arrow[loop above, "\star_B"'] \arrow[loop below, "\bigcirc_B"]
\end{tikzcd}$$

actual data $\perp$ $\perp$ **potential data**

necessarily **indefinitely**

$$\begin{tikzcd}
\text{necessarily } P_\bullet & \text{entails} & \text{actually } P_\bullet & \text{entails} & \text{possibly } P_\bullet \\
\Box_B P_\bullet \arrow[rr, "\epsilon^{\Box_B}_{P_\bullet}"] & & P_\bullet \arrow[rr, "\eta^{\Diamond_B}_{P_\bullet}"] & & \Diamond_B P_\bullet
\end{tikzcd}$$

$$b : B \;\vdash\; \begin{tikzcd} \prod\limits_{b':B} P_{b'} \arrow[r, "(p_{b'})_{b':B} \,\mapsto\, p_b"] & P_b \arrow[r, "p_b \,\mapsto\, (p_b)_b"] & \coprod\limits_{b':B} P_{b'} \end{tikzcd}$$

Given B : BType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

$$\begin{tikzcd} \mathrm{Type}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[r, shift left=10pt, "\coprod_B"] \arrow[r, phantom, shift left=5pt, "\perp"] \arrow[r, phantom, shift right=5pt, "\perp"] \arrow[r, shift right=10pt, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B"] \arrow[loop above, "☆_B"] \arrow[loop below, "\bigcirc_B"'] \end{tikzcd}$$

actual data $\perp$ $\perp$ **potential data**

necessarily **indefinitely**

---

necessarily $P_\bullet$ **entails** **actually $P_\bullet$** **entails** **possibly $P_\bullet$**

$$\begin{tikzcd} \Box_B P_\bullet \arrow[r, "\epsilon^{\Box_B}_{P_\bullet}"] & P_\bullet \arrow[r, "\eta^{\Diamond_B}_{P_\bullet}"] & \Diamond_B P_\bullet \end{tikzcd}$$

$$b : B \;\vdash\; \begin{tikzcd} \prod\limits_{b':B} P_{b'} \arrow[r, "(p_{b'})_{b':B} \,\mapsto\, p_b"] & P_b \arrow[r, "p_b \,\mapsto\, (p_b)_b"] & \coprod\limits_{b':B} P_{b'} \end{tikzcd}$$

- -

randomly P

$☆_B P$

$\coprod\limits_{b:B} P$

Given B : BType of possible measurement outcomes ("possible worlds") **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

actual data **potential data**

necessarily **indefinitely**

$$\begin{tikzcd}
\arrow[r, phantom, "\perp"] & \mathrm{Type}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[r, shift left=8, "\coprod_B"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, phantom, shift right=4, "\perp"] \arrow[r, shift right=8, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B"'] \arrow[loop above, "\star_B"'] \arrow[loop below, "\bigcirc_B"] \arrow[r, phantom, "\perp"] & {}
\end{tikzcd}$$

necessarily $P_\bullet$ **entails** **actually $P_\bullet$** **entails** **possibly $P_\bullet$**

$$\begin{tikzcd}
\Box_B P_\bullet \arrow[r, "\epsilon^{\Box_B}_{P_\bullet}"] & P_\bullet \arrow[r, "\eta^{\Diamond_B}_{P_\bullet}"] & \Diamond_B P_\bullet
\end{tikzcd}$$

$$b : B \;\vdash\; \prod_{b':B} P_{b'} \xrightarrow{(p_{b'})_{b':B} \mapsto p_b} P_b \xrightarrow{p_b \mapsto (p_b)_b} \coprod_{b':B} P_{b'}$$

randomly P **entails** **potentially P**

$$\begin{tikzcd}
\star_B P \arrow[r, "\epsilon^{\star_B}_{P}"] & P
\end{tikzcd}$$

$$\coprod_{b:B} P \xrightarrow{(p)_b \mapsto p} P$$

Given B : BType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent data type formers** constitute modalities of **actual and potential B-measurements**:

classical data types

possibly **randomly**

actual data $\perp$ $\perp$ **potential data**

$$\begin{tikzcd}
\mathrm{Type}_B \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"] \arrow[r, shift left=4, "\coprod_B"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=4, "\prod_B"'] & \mathrm{Type} \arrow[l, "\times B"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"]
\end{tikzcd}$$

necessarily **indefinitely**

---

necessarily $P_\bullet$ **entails** **actually $P_\bullet$** **entails** **possibly $P_\bullet$**

$$\begin{tikzcd}
\Box_B P_\bullet \arrow[r, "\epsilon^{\Box_B}_{P_\bullet}"] & P_\bullet \arrow[r, "\eta^{\Diamond_B}_{P_\bullet}"] & \Diamond_B P_\bullet
\end{tikzcd}$$

$$\begin{tikzcd}
b : B \ \vdash \ \prod\limits_{b':B} P_{b'} \arrow[r, "(p_{b'})_{b':B} \mapsto p_b"] & P_b \arrow[r, "p_b \mapsto (p_b)_b"] & \coprod\limits_{b':B} P_{b'}
\end{tikzcd}$$

---

randomly P **entails** **potentially P** **entails** **indefinitely P**

$$\begin{tikzcd}
\star_B P \arrow[r, "\epsilon^{\star_B}_{P}"] & P \arrow[r, "\eta^{\bigcirc_B}_{P}"] & \bigcirc_B P
\end{tikzcd}$$

$$\begin{tikzcd}
\coprod\limits_{b:B} P \arrow[r, "(p)_b \mapsto p"] & P \arrow[r, "p \mapsto (p)_{b:B}"] & \prod\limits_{b:B} P
\end{tikzcd}$$

Given B : BType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

$$\begin{tikzcd}
 & (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\Diamond_B} \arrow[dr, phantom, "\simeq" sloped] & \\
\textbf{actual data} \perp \; \mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[loop above, "\overset{\textbf{possibly}}{\Diamond_B}"] \arrow[loop below, "\underset{\textbf{necessarily}}{\Box_B}"'] \arrow[ur, two heads, "\oplus_B"] \arrow[dr, two heads, "\oplus_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \; \perp \textbf{potential data} \arrow[ll, two heads, "\mathbb{1}_B\otimes"'] \arrow[loop above, "\overset{\textbf{randomly}}{\star_B}"] \arrow[loop below, "\underset{\textbf{indefinitely}}{\bigcirc_B}"'] \\
 & (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, phantom, "\simeq" sloped] &
\end{tikzcd}$$

Given B : BType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

possibly

randomly

actual data $\perp$

$\perp$ potential data

$$\begin{tikzcd}
& (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\Diamond_B} \arrow[dr, phantom, "\sim" sloped] & \\
\mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[ur, two heads, "\oplus_B"] \arrow[dr, two heads, "\oplus_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"'] \\
& (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, phantom, "\sim" sloped] &
\end{tikzcd}$$

necessarily

indefinitely

necessarily $\mathcal{H}_\bullet$

$\Box_B \mathcal{H}_\bullet$

Given... obtain...

$b : B \vdash \mathcal{H}$

measurement result

where $\mathcal{H} := \bigoplus_{b':B} \mathcal{H}_{b'}$

Given B : BType of possible measurement outcomes ("possible worlds") **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

$$\begin{tikzcd}
 & \textbf{possibly} & & & & \textbf{randomly} & \\
 & & & (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\Diamond_B} \arrow[dr, phantom, "\sim"] & & \\
\textbf{actual data} \; \perp & \mathrm{LType}_B \arrow[urr, two heads, "\bigoplus_B"] \arrow[drr, two heads, "\bigoplus_B"'] \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"] \arrow[r, phantom, "\simeq"] & \mathrm{LType}^{\bigcirc_B} & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"] & \perp \; \textbf{potential data} \\
 & & & (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, phantom, "\sim"] & & \\
 & \textbf{necessarily} & & & & \textbf{indefinitely} &
\end{tikzcd}$$

necessarily $\mathcal{H}_\bullet$ **entails** **actually** $\mathcal{H}_\bullet$

$$\begin{tikzcd}
\Box_B \mathcal{H}_\bullet \arrow[r, "\epsilon^{\Box_B}_{\mathcal{H}_\bullet}"] & \mathcal{H}_\bullet
\end{tikzcd}$$

Given... **obtain...**

$$\underbrace{b : B}_{\text{measurement result}} \;\vdash\; \mathcal{H} \xrightarrow[\text{measurement collapse}]{\sum_{b'} |\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle} \mathcal{H}_b \qquad \text{where } \mathcal{H} := \bigoplus_{b' : B} \mathcal{H}_{b'}$$

Given B : BType of possible measurement outcomes ("possible worlds") **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

possibly

actual data $\perp$

necessarily

$$\begin{tikzcd}
 & & (\mathrm{LType}_B)_{\lozenge_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\lozenge_B} \arrow[dr, phantom, "\wr"] & \\
\mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[loop above, "\lozenge_B"] \arrow[loop below, "\Box_B"] \arrow[urr, two heads, "\oplus_B"] \arrow[drr, two heads, "\oplus_B"'] & & \mathrm{LType}_{\bigcirc_B} \arrow[ll, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"] \\
 & & (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, phantom, "\wr"] &
\end{tikzcd}$$

randomly

$\perp$ potential data

indefinitely

---

necessarily $\mathcal{H}_\bullet$ **entails** **actually $\mathcal{H}_\bullet$** **entails** **possibly $\mathcal{H}_\bullet$**

$$\Box_B \mathcal{H}_\bullet \xrightarrow{\;\epsilon^{\Box_B}_{\mathcal{H}_\bullet}\;} \mathcal{H}_\bullet \xrightarrow{\;\eta^{\lozenge_B}_{\mathcal{H}_\bullet}\;} \lozenge_B \mathcal{H}_\bullet$$

Given... **obtain...**

$b : B \;\vdash$

measurement result

$$\begin{tikzcd}
\mathcal{H} \arrow[r, two heads, "\sum_{b'} |\psi_{b'}\rangle \,\mapsto\, |\psi_b\rangle", "\text{measurement collapse}"'] \arrow[rr, bend right, "\text{linear projector onto sub-Hilbert space } \mathcal{H}_b"'] & \mathcal{H}_b \arrow[r, hook, "|\psi_b\rangle \mapsto \oplus_{b'} \begin{cases} |\psi_b\rangle & \text{if } b'=b \\ 0 & \text{else} \end{cases}", "\text{state preparation}"'] & \mathcal{H},
\end{tikzcd}$$

where $\mathcal{H} := \bigoplus_{b':B} \mathcal{H}_{b'}$

Given B : BType of possible measurement outcomes (“possible worlds”)
the monadic effects of B-dependent linear data type formers constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

$$\begin{tikzcd}
 & (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] & (\mathrm{LType}_B)^{\Diamond_B} \arrow[dr, phantom, "\simeq"] & \\
\mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[ur, two heads, "\bigoplus_B"] \arrow[dr, two heads, "\bigoplus_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[u, phantom, "\perp"] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"'] \\
 & (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, phantom, "\simeq"] &
\end{tikzcd}$$

possibly

actual data $\perp$

necessarily

randomly

$\perp$ potential data

indefinitely

---

principle of quantum compulsion:

necessarily $\mathcal{H}_\bullet$ entails actually $\mathcal{H}_\bullet$ entails possibly $\mathcal{H}_\bullet$ is necessarily $\mathcal{H}_\bullet$

$$\begin{tikzcd}
\Box_B \mathcal{H}_\bullet \arrow[r, "\epsilon^{\Box_B}_{\mathcal{H}_\bullet}"] & \mathcal{H}_\bullet \arrow[r, "\eta^{\Diamond_B}_{\mathcal{H}_\bullet}"] & \Diamond_B \mathcal{H}_\bullet \arrow[r, phantom, "\simeq"] & \Box_B \mathcal{H}_\bullet
\end{tikzcd}$$

ambidexterity

Given... obtain...

$$\begin{tikzcd}
b : B \vdash \mathcal{H} \arrow[r, two heads, "\sum_{b'} \vert\psi_{b'}\rangle \;\mapsto\; \vert\psi_b\rangle", "\text{measurement collapse}"'] \arrow[rr, bend right, "\text{linear projector onto sub-Hilbert space } \mathcal{H}_b"'] & \mathcal{H}_b \arrow[r, hook, "\vert\psi_b\rangle \mapsto \oplus_{b'} \begin{cases} \vert\psi_b\rangle \text{ if } b'=b \\ 0 \quad \text{else} \end{cases}", "\text{state preparation}"'] & \mathcal{H},
\end{tikzcd}$$

where $\mathcal{H} := \bigoplus\limits_{b':B} \mathcal{H}_{b'}$

measurement result

Given $B : \mathrm{BType}$ of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

possibly

randomly

$$\begin{tikzcd}
 & (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\Diamond_B} \arrow[dr, phantom, "\simeq"] & \\
\mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] \arrow[ur, two heads, "\bigoplus_B"] \arrow[dr, two heads, "\bigoplus_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"'] \\
 & (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, phantom, "\simeq"] &
\end{tikzcd}$$

actual data $\perp$ $\perp$ **potential data**

necessarily

indefinitely

---

principle of quantum compulsion:

necessarily $\mathcal{H}_\bullet$ **entails**   **actually $\mathcal{H}_\bullet$**   **entails**   **possibly $\mathcal{H}_\bullet$**   **is**   **necessarily $\mathcal{H}_\bullet$**

$$\Box_B \mathcal{H}_\bullet \xrightarrow{\;\epsilon^{\Box_B}_{\mathcal{H}_\bullet}\;} \mathcal{H}_\bullet \xrightarrow{\;\eta^{\Diamond_B}_{\mathcal{H}_\bullet}\;} \Diamond_B \mathcal{H}_\bullet \simeq \Box_B \mathcal{H}_\bullet$$

ambidexterity

Given...   **obtain...**

$$b : B \;\vdash\; \begin{tikzcd}
\mathcal{H} \arrow[r, two heads, "\sum_{b'} |\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle", "\text{measurement collapse}"'] \arrow[rr, bend right, "\text{linear projector onto sub-Hilbert space } \mathcal{H}_b"'] & \mathcal{H}_b \arrow[r, hook, "|\psi_b\rangle \mapsto \oplus_{b'} \begin{cases} |\psi_b\rangle & \text{if } b'{=}b \\ 0 & \text{else} \end{cases}", "\text{state preparation}"'] & \mathcal{H},
\end{tikzcd} \qquad \text{where } \mathcal{H} := \bigoplus_{b':B} \mathcal{H}_{b'}$$

measurement result

---

randomly $\mathcal{H}$

$$\star_B \mathcal{H}$$

$$\bigoplus_{b:B} \mathcal{H}$$

Given B : BType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements**.

quantum data types

possibly

randomly

$$\begin{tikzcd}
& (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\Diamond_B} \arrow[dr, phantom, "\sim" sloped] & \\
\mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[ur, two heads, "\oplus_B"] \arrow[dr, two heads, "\oplus_B"'] \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, two heads, "\mathbb{1}_B \otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"'] \\
& (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, phantom, "\sim" sloped] &
\end{tikzcd}$$

actual data $\perp$

$\perp$ **potential data**

necessarily

indefinitely

---

principle of quantum compulsion:

necessarily $\mathcal{H}_\bullet$ **entails** **actually $\mathcal{H}_\bullet$** **entails** **possibly $\mathcal{H}_\bullet$** **is** **necessarily $\mathcal{H}_\bullet$**

$$\begin{tikzcd}
\Box_B \mathcal{H}_\bullet \arrow[r, "\epsilon^{\Box_B}_{\mathcal{H}_\bullet}"] & \mathcal{H}_\bullet \arrow[r, "\eta^{\Diamond_B}_{\mathcal{H}_\bullet}"] & \Diamond_B \mathcal{H}_\bullet \arrow[r, phantom, "\simeq"] & \Box_B \mathcal{H}_\bullet
\end{tikzcd}$$

ambidexterity

Given... **obtain...**

$$\begin{tikzcd}
b : B \arrow[r, phantom, "\vdash"] & \mathcal{H} \arrow[r, two heads, "\Sigma_{b'} |\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle", "\text{measurement collapse}"'] \arrow[rr, bend right, "\text{linear projector onto sub-Hilbert space } \mathcal{H}_b"'] & \mathcal{H}_b \arrow[r, hook, "|\psi_b\rangle \mapsto \oplus_{b'} \begin{cases} |\psi_b\rangle & \text{if } b'=b \\ 0 & \text{else} \end{cases}", "\text{state preparation}"'] & \mathcal{H},
\end{tikzcd}$$

measurement result

where $\mathcal{H} := \bigoplus\limits_{b':B} \mathcal{H}_{b'}$

randomly $\mathcal{H}$ **entails** **potentially $\mathcal{H}$**

$$\begin{tikzcd}
\star_B \mathcal{H} \arrow[r, "\epsilon^{\star_B}_{\mathcal{H}}"] & \mathcal{H}
\end{tikzcd}$$

$$\begin{tikzcd}
\bigoplus\limits_{b:B} \mathcal{H} \arrow[r, "\oplus_b |\psi_b\rangle \mapsto \Sigma_b |\psi_b\rangle", "\text{quantum superposition}"'] & \mathcal{H}
\end{tikzcd}$$

Given B : BType of possible measurement outcomes (“possible worlds”) **the monadic effects of B-dependent linear data type formers** constitute modalities of **actual and potential quantum B-measurements.**

quantum data types

possibly

actual data $\perp$

necessarily

randomly

$\perp$ **potential data**

indefinitely

$$\begin{tikzcd}
 & (\mathrm{LType}_B)_{\Diamond_B} \arrow[r, hook, "\sim"] \arrow[d, phantom, "\perp"] & (\mathrm{LType}_B)^{\Diamond_B} \arrow[dr, phantom, "\simeq"] & \\
\mathrm{LType}_B \simeq \mathrm{LType}^{\bigcirc_B} \arrow[ur, ->>, "\oplus_B"] \arrow[dr, ->>, "\oplus_B"'] \arrow[loop above, "\Diamond_B"] \arrow[loop below, "\Box_B"'] & \mathrm{LType}_{\bigcirc_B} \arrow[l, hook'] \arrow[d, phantom, "\perp"] & & \mathrm{LType} \arrow[ll, ->>, "\mathbb{1}_B\otimes"'] \arrow[loop above, "\star_B"] \arrow[loop below, "\bigcirc_B"'] \\
 & (\mathrm{LType}_B)_{\Box_B} \arrow[r, hook, "\sim"'] & (\mathrm{LType}_B)^{\Box_B} \arrow[ur, phantom, "\simeq"] &
\end{tikzcd}$$

---

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$$\begin{tikzcd}
\Box_B \mathcal{H}_\bullet \arrow[r, "\epsilon^{\Box_B}_{\mathcal{H}_\bullet}"] & \mathcal{H}_\bullet \arrow[r, "\eta^{\Diamond_B}_{\mathcal{H}_\bullet}"] & \Diamond_B \mathcal{H}_\bullet \arrow[r, phantom, "\simeq"] & \Box_B \mathcal{H}_\bullet
\end{tikzcd}$$

ambidexterity

Given... **obtain...**

$b : B \quad \vdash$

measurement result

$$\begin{tikzcd}
\mathcal{H} \arrow[r, ->>, "\Sigma_{b'} |\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle", "\text{measurement collapse}"'] \arrow[rr, bend right, "\text{linear projector onto sub-Hilbert space } \mathcal{H}_b"'] & \mathcal{H}_b \arrow[r, hook, "|\psi_b\rangle \mapsto \oplus_{b'} \begin{cases} |\psi_b\rangle \text{ if } b'=b \\ 0 \quad \text{else} \end{cases}", "\text{state preparation}"'] & \mathcal{H},
\end{tikzcd}$$

where $\mathcal{H} := \bigoplus\limits_{b':B} \mathcal{H}_{b'}$

---

randomly $\mathcal{H}$ **entails** **potentially $\mathcal{H}$** **entails** **indefinitely $\mathcal{H}$**

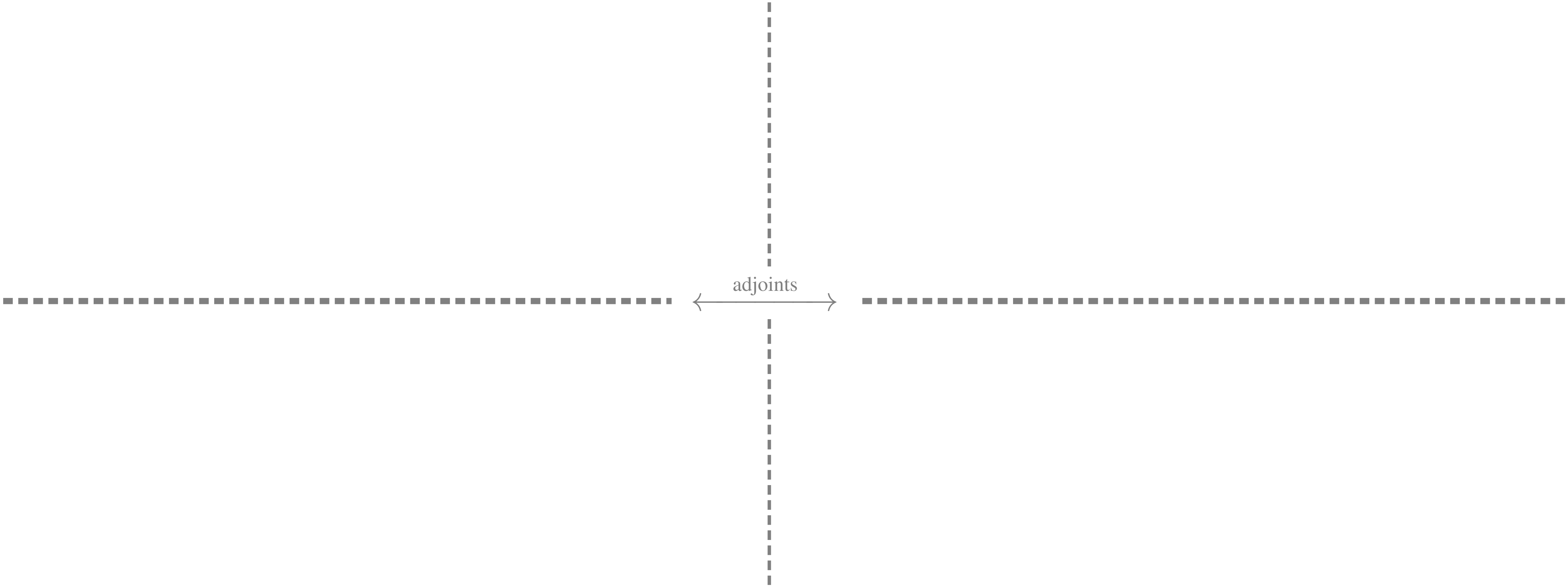
$$\begin{tikzcd}
\star_B \mathcal{H} \arrow[r, "\epsilon^{\star_B}_{\mathcal{H}}"] & \mathcal{H} \arrow[r, "\eta^{\bigcirc_B}_{\mathcal{H}}"] & \bigcirc_B \mathcal{H}
\end{tikzcd}$$

$$\begin{tikzcd}
\bigoplus\limits_{b:B} \mathcal{H} \arrow[r, "\oplus_b |\psi_b\rangle \mapsto \Sigma_b |\psi_b\rangle", "\text{quantum superposition}"'] & \mathcal{H} \arrow[r, "|\psi\rangle \mapsto \oplus_b |\psi\rangle_b", "\text{quantum parallelization}"'] & \bigoplus\limits_{b:B} \mathcal{H}
\end{tikzcd}$$

The pure effects of these modalities of dependent linear data type formation

are remarkable in their sheer quantum information-theoretic content.

To repeat:



The pure effects of these modalities of dependent linear data type formation

are remarkable in their sheer quantum information-theoretic content.

To repeat:

$$\overbrace{(p_B)^*(p_B)_*}^{\Box_B}\mathcal{H}_\bullet \xrightarrow[\textbf{necessity counit}]{\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

$$b : B \vdash \bigoplus_{b':B}\mathcal{H}_{b'} \xrightarrow[\text{quantum } \textbf{measurement}]{\oplus_{b'}|\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle} \mathcal{H}_b$$

“ the necessary becomes actual ”

adjoints

$$\longleftrightarrow$$

The pure effects of these modalities of dependent linear data type formation

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To repeat:

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“ the necessary becomes actual ”

adjoints ⟷

$$\mathcal{H}_\bullet \xrightarrow[\textbf{possibility unit}]{\eta^{\Diamond_B}_{\mathcal{H}_\bullet}} \overbrace{(p_B)^*(p_B)_!}^{\Diamond_B}\mathcal{H}_\bullet$$

$$b : B \vdash \mathcal{H}_b \xrightarrow[\text{quantum } \textbf{state preparation}]{|\psi_b\rangle \;\mapsto\; \oplus_{b'}\begin{cases}|\psi_b\rangle & \text{if } b'{=}b\\ 0 & \text{else}\end{cases}} \bigoplus_{b':B}\mathcal{H}_{b'}$$

“ the actual is possible ”

The pure effects of these modalities of dependent linear data type formation

are remarkable in their sheer quantum information-theoretic content.

To repeat:

$$\overbrace{(p_B)^*(p_B)_*}^{\Box_B}\mathcal{H}_\bullet \xrightarrow[\textbf{necessity counit}]{\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

$$b : B \vdash \bigoplus_{b':B}\mathcal{H}_{b'} \xrightarrow[\text{quantum }\textbf{measurement}]{\oplus_{b'}|\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle} \mathcal{H}_b$$

“ the necessary becomes actual ”

$$\mathcal{H}_\bullet \xrightarrow[\textbf{possibility unit}]{\eta^{\Diamond_B}_{\mathcal{H}_\bullet}} \overbrace{(p_B)^*(p_B)_!}^{\Diamond_B}\mathcal{H}_\bullet$$

$$b : B \vdash \mathcal{H}_b \xrightarrow[\text{quantum }\textbf{state preparation}]{|\psi_b\rangle \;\mapsto\; \oplus_{b'}\begin{cases}|\psi_b\rangle & \text{if } b'=b\\ 0 & \text{else}\end{cases}} \bigoplus_{b':B}\mathcal{H}_{b'}$$

“ the actual is possible ”

adjoints ⟷

“ the random becomes potential ”

$$\overbrace{(p_B)_!(p_B)^*}^{\star_B}\mathcal{H} \xhookrightarrow[\textbf{randomness counit}]{\varepsilon^{\star_B}_{\mathcal{H}}} \mathcal{H}$$

$$\bigoplus_{b:B}\mathcal{H} \xrightarrow[\text{quantum }\textbf{superposition}]{\oplus_b|\psi_b\rangle \;\mapsto\; \sum_b|\psi_b\rangle} \mathcal{H}$$

The pure effects of these modalities of dependent linear data type formation

are remarkable in their sheer quantum information-theoretic content.

To repeat:

$$\overbrace{(p_B)^*(p_B)_*}^{\Box_B}\mathcal{H}_\bullet \xrightarrow[\text{necessity counit}]{\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

$$b : B \vdash \bigoplus_{b':B}\mathcal{H}_{b'} \xrightarrow[\text{quantum measurement}]{\oplus_{b'}|\psi_{b'}\rangle \;\mapsto\; |\psi_b\rangle} \mathcal{H}_b$$

“ the necessary becomes actual ”

$$\mathcal{H}_\bullet \xrightarrow[\text{possibility unit}]{\eta^{\Diamond_B}_{\mathcal{H}_\bullet}} \overbrace{(p_B)^*(p_B)_!}^{\Diamond_B}\mathcal{H}_\bullet$$

$$b : B \vdash \mathcal{H}_b \xrightarrow[\text{quantum state preparation}]{|\psi_b\rangle \;\mapsto\; \oplus_{b'}\begin{cases}|\psi_b\rangle & \text{if } b'=b\\ 0 & \text{else}\end{cases}} \bigoplus_{b':B}\mathcal{H}_{b'}$$

“ the actual is possible ”

adjoints ↔

“ the random becomes potential ”

$$\overbrace{(p_B)_!(p_B)^*}^{\star_B}\mathcal{H} \xhookrightarrow[\text{randomness counit}]{\varepsilon^{\star_B}_{\mathcal{H}}} \mathcal{H}$$

$$\bigoplus_{b:B}\mathcal{H} \xrightarrow[\text{quantum superposition}]{\oplus_b|\psi_b\rangle \;\mapsto\; \Sigma_b|\psi_b\rangle} \mathcal{H}$$

“ the potential is inderminate ”

$$\mathcal{H} \xrightarrow[\text{indeterminacy unit}]{\eta^{\bigcirc_B}_{\mathcal{H}}} \overbrace{(p_B)_*(p_B)^*}^{\bigcirc_B}\mathcal{H}$$

$$\mathcal{H} \xrightarrow[\text{quantum parallelism}]{|\psi\rangle \;\mapsto\; \oplus_b|\psi\rangle_b} \bigoplus_{b:B}\mathcal{H}$$

Q-bits are the free linear indeterminacy-effect handlers over $\mathrm{Bit} = \{0,1\}$

Coherent q-bits:

$$\begin{tikzcd}
\mathrm{QBit} \arrow[d, phantom, "\rotatebox{90}{$:=$}"] : \mathrm{LType} \arrow[r, hook, "\mathbb{1}_{\mathrm{Bit}}\otimes"] & \mathrm{LType}_{\mathrm{Bit}} \arrow[r, "\bigoplus_{\mathrm{Bit}}", "\sim"'] & \mathrm{LType}^{\bigcirc_B} \\
\bigcirc_{\mathrm{Bit}} \mathbb{1} & &
\end{tikzcd}$$

Quantum gate with q-bit output:

De-cohered (measured) q-bits:

Q-bits are the free linear indeterminacy-effect handlers over $\mathrm{Bit} = \{0,1\}$

Coherent q-bits:

━━

$$\begin{tikzcd}
\mathrm{QBit} \arrow[d, phantom, "\overset{\cdot\cdot}{\parallel}"] & : & \mathrm{LType} \arrow[r, hook, "\mathbb{1}_{\mathrm{Bit}}\otimes"] & \mathrm{LType}_{\mathrm{Bit}} \arrow[r, "\oplus_{\mathrm{Bit}}", "\sim"'] & \mathrm{LType}^{\bigcirc_B} \\
\bigcirc_{\mathrm{Bit}}\mathbb{1} & = \bigoplus_{\{0,1\}}\mathbb{C} = \mathbb{C}\cdot|0\rangle \oplus \mathbb{C}\cdot|1\rangle
\end{tikzcd}$$

De-cohered (measured) q-bits:

Quantum gate with q-bit output:

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Coherent q-bits:

$$\begin{tikzcd}
\mathrm{QBit} \arrow[d, equal, ":"] & : & \mathrm{LType} \arrow[r, hook, "\mathbb{1}_{\mathrm{Bit}}\otimes"] & \mathrm{LType}_{\mathrm{Bit}} \arrow[r, "\bigoplus_{\mathrm{Bit}}", "\sim"'] & \mathrm{LType}^{\bigcirc_B} \\
\bigcirc_{\mathrm{Bit}} \mathbb{1} \;=\; \bigoplus_{\{0,1\}} \mathbb{C} \;=\; \mathbb{C}\cdot|0\rangle \oplus \mathbb{C}\cdot|1\rangle
\end{tikzcd}$$

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$$\mathrm{QBit} \;:\; \mathrm{LType} \xhookrightarrow{\;\mathbb{1}_{\mathrm{Bit}}\otimes\;} \mathrm{LType}_{\mathrm{Bit}} \underset{\sim}{\xrightarrow{\;\bigoplus_{\mathrm{Bit}}\;}} \mathrm{LType}^{\bigcirc_B}$$

$$\overset{\cdot\cdot}{\Vert}$$

$$\bigcirc_{\mathrm{Bit}} \mathbb{1} \;=\; \bigoplus_{\{0,1\}} \mathbb{C} \;=\; \mathbb{C}\cdot|0\rangle \oplus \mathbb{C}\cdot|1\rangle$$

$$\mathrm{QBit}$$

$$\otimes$$

$$\mathcal{H}$$

$$\Vert$$

$$\bigcirc_{\mathrm{Bit}} \mathcal{H} \;=\; \bigoplus_{\{0,1\}} \mathcal{H} \;=\; \mathcal{H}\otimes|0\rangle \;\oplus\; \mathcal{H}\otimes|0\rangle$$

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$$\rotatebox{90}{$=:$}$$

$$\bigcirc_{\mathrm{Bit}} \mathbb{1} \;=\; \bigoplus_{\{0,1\}} \mathbb{C} \;=\; \mathbb{C} \cdot |0\rangle \oplus \mathbb{C} \cdot |1\rangle$$

$$\mathrm{QBit}$$

$$\otimes$$

$$\mathcal{H}$$

$$\Vert$$

$$\bigcirc_{\mathrm{Bit}} \mathcal{H} \;=\; \bigoplus_{\{0,1\}} \mathcal{H} \;=\; \mathcal{H} \otimes |0\rangle \;\oplus\; \mathcal{H} \otimes |0\rangle$$

De-cohered (measured) q-bits:

$$\mathbb{1}_{\mathrm{Bit}} \;:\; \mathrm{LType}_{\mathrm{Bit}} \xrightarrow[\sim]{\bigoplus_{\mathrm{Bit}}} \mathrm{LType}^{\bigcirc_{\mathrm{Bit}}}$$

$$b : \mathrm{Bit} \quad \vdash \quad \mathbb{C} \cdot |b\rangle \;:\; \mathrm{LType}$$

Quantum gate with q-bit output:

Q-bits are the free linear indeterminacy-effect handlers over $\mathrm{Bit} = \{0,1\}$

Coherent q-bits:

$$\mathrm{QBit} \;:\; \mathrm{LType} \xhookrightarrow{\mathbb{1}_{\mathrm{Bit}} \otimes} \mathrm{LType}_{\mathrm{Bit}} \xrightarrow[\sim]{\oplus_{\mathrm{Bit}}} \mathrm{LType}^{\bigcirc_B}$$

$$\overset{\cdot\cdot}{\|}$$

$$\bigcirc_{\mathrm{Bit}} \mathbb{1} \;=\; \bigoplus_{\{0,1\}} \mathbb{C} \;=\; \mathbb{C} \cdot |0\rangle \oplus \mathbb{C} \cdot |1\rangle$$

$$\mathrm{QBit}$$

$$\otimes$$

$$\mathcal{H}$$

$$\|$$

$$\bigcirc_{\mathrm{Bit}} \mathcal{H} \;=\; \bigoplus_{\{0,1\}} \mathcal{H} \;=\; \mathcal{H} \otimes |0\rangle \;\oplus\; \mathcal{H} \otimes |0\rangle$$

Quantum gate with q-bit output:

De-cohered (measured) q-bits:

$$\mathbb{1}_{\mathrm{Bit}} \;:\; \mathrm{LType}_{\mathrm{Bit}} \xrightarrow[\sim]{\oplus_{\mathrm{Bit}}} \mathrm{LType}^{\bigcirc_{\mathrm{Bit}}}$$

$$b : \mathrm{Bit} \quad \vdash \quad \mathbb{C} \cdot |b\rangle \;:\; \mathrm{LType}$$

$$\mathbb{1}_{\mathrm{Bit}}$$

$$\otimes \qquad b : \mathrm{Bit} \quad \vdash \quad \mathcal{H} \otimes |b\rangle \;:\; \mathrm{LType}$$

$$\mathcal{H}$$

Q-bits are the free linear indeterminacy-effect handlers over $\mathrm{Bit} = \{0,1\}$

Coherent q-bits:

━━━ $\mathrm{QBit} \;:\; \mathrm{LType} \xhookrightarrow{\mathbb{1}_{\mathrm{Bit}}\otimes} \mathrm{LType}_{\mathrm{Bit}} \xrightarrow[\sim]{\bigoplus_{\mathrm{Bit}}} \mathrm{LType}^{\bigcirc_B}$

$$\overset{\cdot\cdot}{\|}$$

$$\bigcirc_{\mathrm{Bit}}\mathbb{1} \;=\; \bigoplus_{\{0,1\}}\mathbb{C} \;=\; \mathbb{C}\cdot|0\rangle \oplus \mathbb{C}\cdot|1\rangle$$

━━━ QBit

$\otimes$

━━━ $\mathcal{H}$

$$\|$$

$$\bigcirc_{\mathrm{Bit}}\mathcal{H} \;=\; \bigoplus_{\{0,1\}}\mathcal{H} \;=\; \mathcal{H}\otimes|0\rangle \;\oplus\; \mathcal{H}\otimes|0\rangle$$

De-cohered (measured) q-bits:

═══ $\mathbb{1}_{\mathrm{Bit}} \;:\; \mathrm{LType}_{\mathrm{Bit}} \xrightarrow[\sim]{\bigoplus_{\mathrm{Bit}}} \mathrm{LType}^{\bigcirc_{\mathrm{Bit}}}$

$$b : \mathrm{Bit} \quad \vdash \quad \mathbb{C}\cdot|b\rangle \;:\; \mathrm{LType}$$

═══ $\mathbb{1}_{\mathrm{Bit}}$

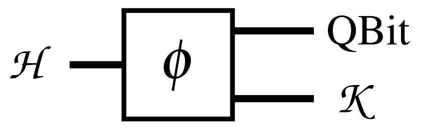
$\otimes$

━━━ $\mathcal{H}$

$$b : \mathrm{Bit} \quad \vdash \quad \mathcal{H}\otimes|b\rangle \;:\; \mathrm{LType}$$

Quantum gate with q-bit output:

A quantum gate which may handle $\bigcirc_{\mathrm{Bit}}$-effects is one with a QBit-output:



$$\mathcal{H} \xrightarrow{\phi} \mathrm{QBit}\otimes\mathcal{K} \simeq \bigcirc_{\mathrm{Bit}}\mathcal{K}$$

Q-bits are the free linear indeterminacy-effect handlers over $\mathrm{Bit} = \{0,1\}$

Coherent q-bits:

$$\mathrm{QBit} \;:\; \mathrm{LType} \overset{\mathbb{1}_{\mathrm{Bit}}\otimes}{\hookrightarrow} \mathrm{LType}_{\mathrm{Bit}} \underset{\sim}{\xrightarrow{\oplus_{\mathrm{Bit}}}} \mathrm{LType}^{\bigcirc_B}$$

$$\ddot{\|}$$

$$\bigcirc_{\mathrm{Bit}}\mathbb{1} \;=\; \bigoplus_{\{0,1\}}\mathbb{C} \;=\; \mathbb{C}\cdot|0\rangle \oplus \mathbb{C}\cdot|1\rangle$$

$$\begin{array}{c} \mathrm{QBit} \\ \otimes \\ \mathcal{H} \\ \| \end{array}$$

$$\bigcirc_{\mathrm{Bit}}\mathcal{H} \;=\; \bigoplus_{\{0,1\}}\mathcal{H} \;=\; \mathcal{H}\otimes|0\rangle \;\oplus\; \mathcal{H}\otimes|0\rangle$$

De-cohered (measured) q-bits:

$$\mathbb{1}_{\mathrm{Bit}} \;:\; \mathrm{LType}_{\mathrm{Bit}} \underset{\sim}{\xrightarrow{\oplus_{\mathrm{Bit}}}} \mathrm{LType}^{\bigcirc_{\mathrm{Bit}}}$$

$$b : \mathrm{Bit} \quad \vdash \quad \mathbb{C}\cdot|b\rangle \;:\; \mathrm{LType}$$

$$\begin{array}{c} \mathbb{1}_{\mathrm{Bit}} \\ \otimes \\ \mathcal{H} \end{array} \qquad b : \mathrm{Bit} \quad \vdash \quad \mathcal{H}\otimes|b\rangle \;:\; \mathrm{LType}$$

Quantum gate with q-bit output:

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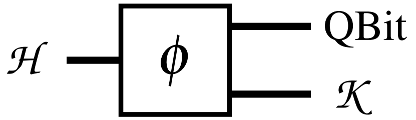
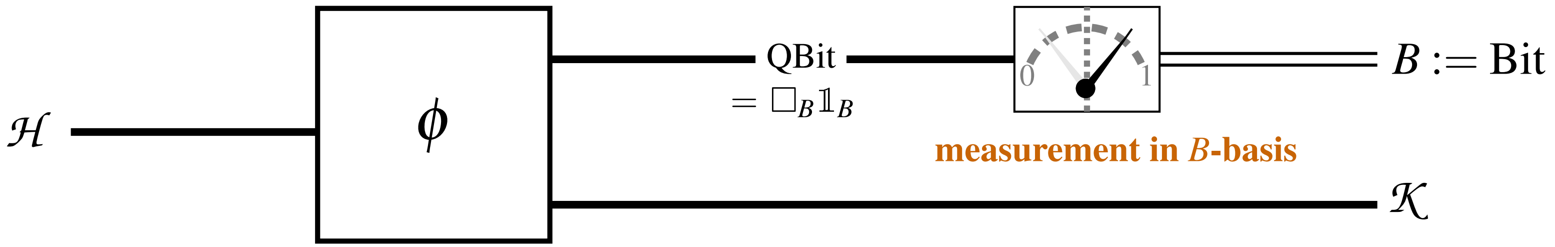


$$\mathcal{H} \xrightarrow{\;\phi\;} \mathrm{QBit}\otimes\mathcal{K} \simeq \bigcirc_{\mathrm{Bit}}\mathcal{K}$$

Quantum measurement is Linear indefiniteness-effect handling.

quantum circuit



quantum gate

$$\mathcal{H} \xrightarrow{\phi} \mathrm{QBit} \otimes \mathcal{K} \simeq \bigcirc_B \mathcal{K}$$

$\bigcirc_B$-effect handling

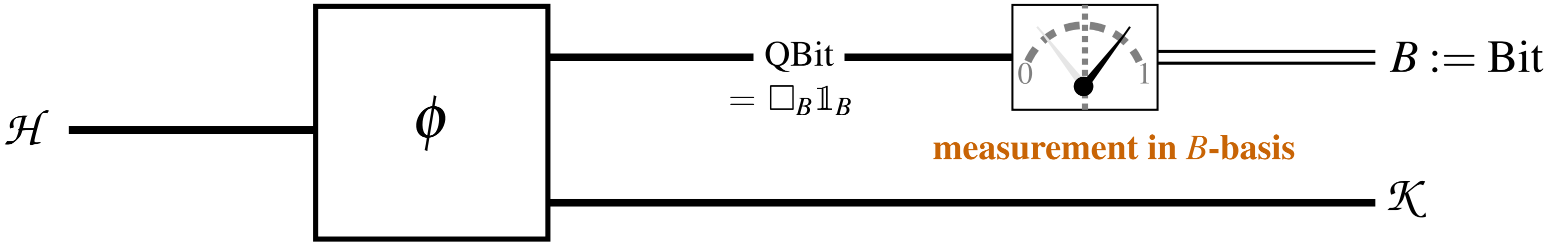
Quantum measurement is Linear indefiniteness-effect handling.

quantum circuit

formalization

$\bigcirc_B$-modal linear types

$\mathrm{LType}_{\bigcirc_B}$



quantum gate

$$\mathcal{H} \xrightarrow{\phi} \mathrm{QBit} \otimes \mathcal{K} \simeq \bigcirc_B \mathcal{K}$$

$\bigcirc_B$-effect handling

Quantum measurement is Linear indefiniteness-effect handling.

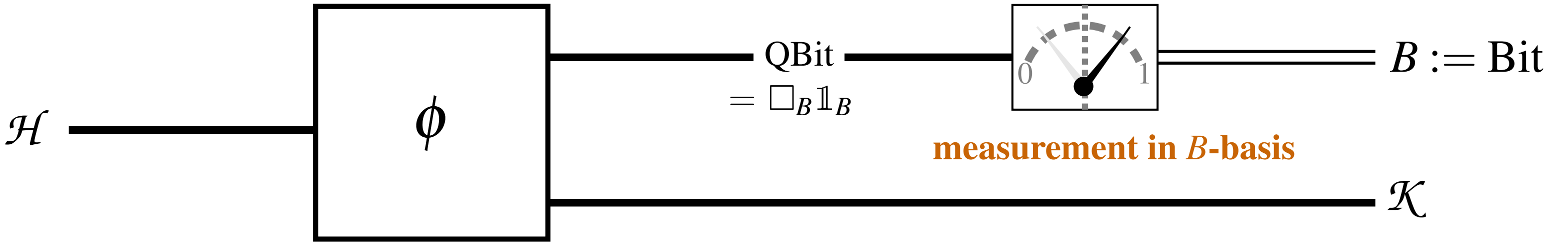
quantum circuit

$$\begin{tikzcd}
\text{quantum circuit} \arrow[d, squiggly, "\text{formalization}"] \\
\bigcirc_B\text{-modal linear types}
\end{tikzcd}$$

$\bigcirc_B$-modal linear types

$$\begin{tikzcd}
\mathrm{LType}_{\bigcirc_B} \arrow[d, hook, "\text{comparison functor}", "K_{(p_B)^*(p_B)_*}"'] \\
\mathrm{LType}_B
\end{tikzcd}$$

B-dependent linear types



$$\begin{tikzcd}
\bigcirc_B \mathcal{H} \arrow[loop above, "\bigcirc_B"] \arrow[rr, "\bigcirc_B\left(\mathcal{H} \xrightarrow{\phi} \mathrm{QBit} \otimes \mathcal{K} \simeq \bigcirc_B \mathcal{K}\right)"] && \bigcirc_B \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"] \arrow[rr, "\mathrm{hndl}^{\bigcirc_B}_{\bigcirc_B \mathcal{K}}", "\bigcirc_B\text{-effect handling}"'] && \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"]
\end{tikzcd}$$

subjection to $\bigcirc_B$-effects

quantum gate

$\bigcirc_B$-effect handling

Quantum measurement is Linear indefiniteness-effect handling.

quantum circuit

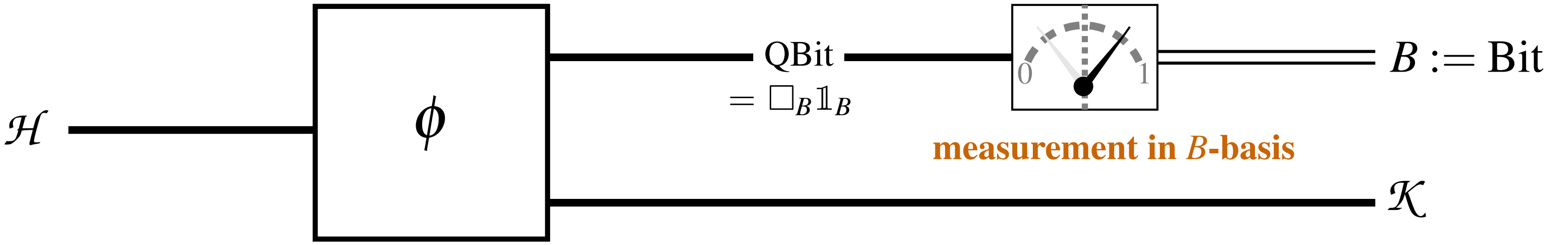
$$\begin{tikzcd}
\text{quantum circuit} \arrow[d, squiggly, "\text{formalization}"] \\
\mathrm{LType}_{\bigcirc_B} \arrow[d, "K_{(p_B)_*(p_B)^*}"', "\text{comparison functor}"] \\
\mathrm{LType}_B
\end{tikzcd}$$

$\bigcirc_B$-modal linear types

$\mathrm{LType}_{\bigcirc_B}$

LType_B

***B*-dependent linear types**



$$\begin{tikzcd}
\bigcirc_B \mathcal{H} \arrow[loop above, "\bigcirc_B"] \arrow[rr, "\bigcirc_B\left(\mathcal{H} \xrightarrow{\phi} \mathrm{QBit} \otimes \mathcal{K} \simeq \bigcirc_B \mathcal{K}\right)"] & & \bigcirc_B \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"] \arrow[rr, "\mathrm{hndl}^{\bigcirc_B}_{\bigcirc_B \mathcal{K}}", "\bigcirc_B\text{-effect handling}"'] & & \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"]
\end{tikzcd}$$

subjection to $\bigcirc_B$-effects

quantum gate

$\bigcirc_B$-effect handling

$$\begin{tikzcd}
\mathbb{1}_B \otimes \mathcal{H} \arrow[rr, "\mathbb{1}_B \otimes \phi"] & & \Box_B \mathbb{1}_B \otimes \mathcal{K} \arrow[rr, "\epsilon^{\Box_B}_{\mathbb{1}_B} \otimes \mathcal{K}", "\Box_B\text{-counit}"'] & & \mathbb{1}_B \otimes \mathcal{K}
\end{tikzcd}$$

$\Box_B$-counit

$$\begin{tikzcd}
b : B \ \vdash \ \mathcal{H} \arrow[rr, "\phi"] & & \mathrm{QBit} \otimes \mathcal{K} \arrow[rr, "\text{quantum state collapse}"] & & \mathcal{K}
\end{tikzcd}$$

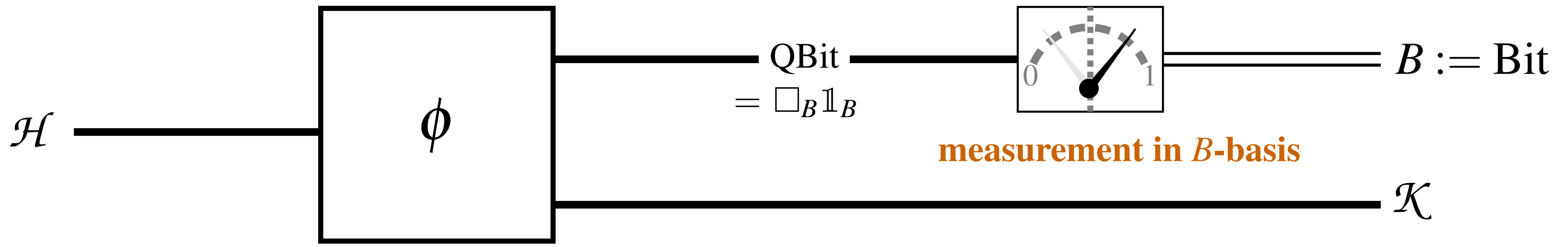
quantum state collapse

$$\begin{array}{c} |0\rangle \otimes |\psi_0\rangle \\ + |1\rangle \otimes |\psi_1\rangle \end{array} \mapsto |\psi_b\rangle$$

measurement result

Quantum measurement is Linear indefiniteness-effect handling.

$$\begin{tikzcd}
\text{quantum circuit} \arrow[d, squiggly, "\text{formalization}"] \\
\bigcirc_B\text{-modal linear types} \\
\mathrm{LType}_{\bigcirc_B} \arrow[d, "K_{(p_B)^*(p_B)_*}"', "\text{comparison functor}"] \\
\mathrm{LType}_B \\
B\text{-dependent linear types}
\end{tikzcd}$$



$$\begin{tikzcd}[column sep=huge]
\bigcirc_B \mathcal{H} \arrow[loop above, "\bigcirc_B"] \arrow[r, "{\substack{\text{subjection to } \bigcirc_B\text{-effects} \quad \text{quantum gate} \\ \bigcirc_B\left(\mathcal{H} \xrightarrow{\phi} \mathrm{QBit} \otimes \mathcal{K} \simeq \bigcirc_B \mathcal{K}\right)}}"] & \bigcirc_B \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"] \arrow[r, "\mathrm{hndl}^{\bigcirc_B}_{\bigcirc_B \mathcal{K}}", "\bigcirc_B\text{-effect handling}"'] & \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"]
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
\mathbb{1}_B \otimes \mathcal{H} \arrow[r, "\mathbb{1}_B \otimes \phi"] & \Box_B \mathbb{1}_B \otimes \mathcal{K} \arrow[r, "\varepsilon^{\Box_B}_{\mathbb{1}_B} \otimes \mathcal{K}", "\Box_B\text{-counit}"'] & \mathbb{1}_B \otimes \mathcal{K}
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
\underset{\text{measurement result}}{b} : B \vdash \mathcal{H} \arrow[r, "\phi"] & \mathrm{QBit} \otimes \mathcal{K} \arrow[r, "\text{quantum state collapse}", "{\substack{|0\rangle \otimes |\psi_0\rangle \\ + |1\rangle \otimes |\psi_1\rangle} \mapsto |\psi_b\rangle}"'] & \mathcal{K}
\end{tikzcd}$$

full linearly-typed detail of quantum measurement logic is emergent effect in LHoTT

Quantum measurement is Linear indefiniteness-effect handling.

quantum circuit

formalization

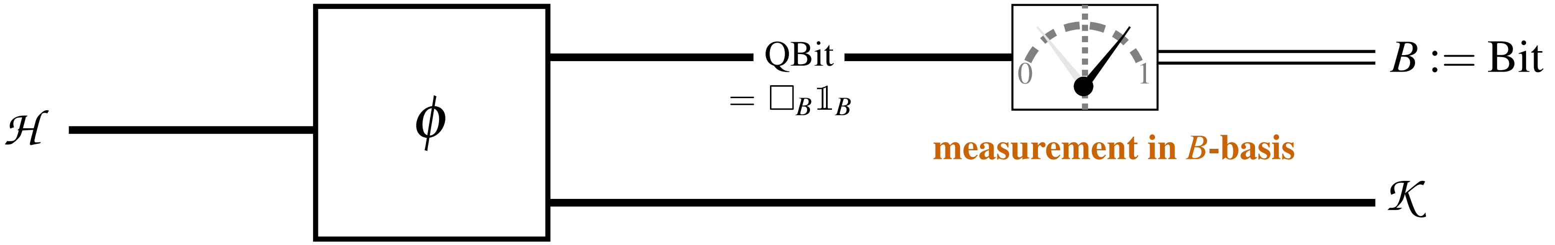
$\bigcirc_B$-modal linear types

$\mathrm{LType}_{\bigcirc_B}$

$$\begin{tikzcd} \mathrm{LType}_{\bigcirc_B} \arrow[d, "K_{(p_B)^*(p_B)_*}"', "\text{comparison functor}"] \\ \mathrm{LType}_B \end{tikzcd}$$

LType_B

B-dependent linear types



$$\begin{tikzcd}[column sep=huge] \bigcirc_B \mathcal{H} \arrow[loop above, "\bigcirc_B"] \arrow[r, "\bigcirc_B\left(\mathcal{H} \xrightarrow{\phi} \mathrm{QBit} \otimes \mathcal{K} \simeq \bigcirc_B \mathcal{K}\right)"] & \bigcirc_B \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"] \arrow[r, "\mathrm{hndl}^{\bigcirc_B}_{\bigcirc_B \mathcal{K}}", "\bigcirc_B\text{-effect handling}"'] & \bigcirc_B \mathcal{K} \arrow[loop above, "\bigcirc_B"] \end{tikzcd}$$

subjection to $\bigcirc_B$-effects

quantum gate

$$\begin{tikzcd}[column sep=huge] \mathbb{1}_B \otimes \mathcal{H} \arrow[r, "\mathbb{1}_B \otimes \phi"] & \Box_B \mathbb{1}_B \otimes \mathcal{K} \arrow[r, "\epsilon^{\Box_B}_{\mathbb{1}_B} \otimes \mathcal{K}", "\Box_B\text{-counit}"'] & \mathbb{1}_B \otimes \mathcal{K} \end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge] b : B \vdash \mathcal{H} \arrow[r, "\phi"] & \mathrm{QBit} \otimes \mathcal{K} \arrow[r, "\text{quantum state collapse}"] & \mathcal{K} \end{tikzcd}$$

measurement result

$$\begin{array}{l} |0\rangle \otimes |\psi_0\rangle \\ + |1\rangle \otimes |\psi_1\rangle \end{array} \mapsto |\psi_b\rangle$$

full linearly-typed detail of quantum measurement logic is emergent effect in LHoTT

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$\begin{tikzcd} \bigcirc_B \\ \mathrm{Monad}\big(\mathrm{LType}\big) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \end{tikzcd}$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$\begin{tikzcd}
\bigcirc_B \arrow[d, phantom, "\rotatebox{90}{$:=$}"] \\
B\text{-Reader} \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] \\
\mathrm{Monad}\big(\mathrm{LType}\big)
\end{tikzcd}$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$B : \mathrm{FinType} \;\vdash\; \begin{array}{c} \bigcirc_B \\ \rotatebox{90}{$=:$} \\ B\text{-Reader} \\ \rotatebox{90}{$\simeq$} \\ \mathbb{1}^B\text{-Writer} \\ \\ \rotatebox{90}{$\in$} \\ \mathrm{Monad}\big(\mathrm{LType}\big) \end{array}$$

$$\mathbb{1}^B\text{-Writer}(D) :\equiv \mathbb{1}^B \oplus D$$

$$\mathrm{bind}^{\mathbb{1}^B\mathrm{Writer}}\Big(D_1 \xrightarrow{\mathrm{prog}} \mathbb{1}^B \oplus D_2\Big) :\equiv \mathbb{1}^B \oplus D_1 \xrightarrow{\mathbb{1}^B \oplus \mathrm{prog}} \mathbb{1}^B \oplus \mathbb{1}^B \oplus D_2 \xrightarrow{\mu \oplus \mathrm{id}_{D_2}} \mathbb{1}^B \oplus D_2$$

Where $\mathbb{1}^B = \bigoplus\limits_{b:B} \mathbb{C} \cdot P_b \in \mathrm{CMon}(\mathrm{LType})$ is algebra of B-projection operators :

unit

$$\mathbb{1} \xrightarrow[1 \mapsto \sum\limits_{b:B} P_b]{\eta} \mathbb{1}^B$$

product

$$\mathbb{1}^B \otimes \mathbb{1}^B \xrightarrow[P_b \otimes P_{b'} \mapsto \begin{cases} P_b \text{ if } b = b' \\ 0 \text{ else} \end{cases}]{\mu} \mathbb{1}^B$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$B : \mathrm{FinType} \;\vdash\; \begin{tikzcd} & & \bigcirc_B \arrow[d, phantom, "=:"] \\ & & B\text{-Reader} \arrow[d, phantom, "\simeq"] \\ & & \mathbb{1}^B\text{-Writer} \arrow[dl, phantom, "\simeq"] \\ & \mathbb{1}^B\text{-DualWriter} \arrow[dr, phantom, "\subset"] & \\ & & \mathrm{Monad}(\mathrm{LType}) \\ & \mathrm{FrobMonad}(\mathrm{LType}) \arrow[ur] & \end{tikzcd}$$

$$\mathbb{1}^B\text{-Writer}(D) :\equiv \mathbb{1}^B \oplus D$$

$$\mathrm{bind}^{\mathbb{1}^B\mathrm{Writer}}\Big(D_1 \xrightarrow{\mathrm{prog}} \mathbb{1}^B \oplus D_2\Big) :\equiv \mathbb{1}^B \oplus D_1 \xrightarrow{\mathbb{1}^B \oplus \mathrm{prog}} \mathbb{1}^B \oplus \mathbb{1}^B \oplus D_2 \xrightarrow{\mu \oplus \mathrm{id}_{D_2}} \mathbb{1}^B \oplus D_2$$

Where $\mathbb{1}^B = \bigoplus_{b:B} \mathbb{C} \cdot P_b \in \mathrm{CMon}(\mathrm{LType})$ is Frobenius algebra of B-projection operators :

$$\begin{tikzcd}[column sep=huge] \mathbb{1} \arrow[r, "\overset{\textbf{unit}}{\eta}", "1 \mapsto \sum_{b:B} P_b"'] & \mathbb{1}^B \arrow[r, "\overset{\textbf{co-product}}{\delta}", "P_b \mapsto P_b \otimes P_b"'] & \mathbb{1}^B \otimes \mathbb{1}^B \arrow[r, "\overset{\textbf{product}}{\mu}", "P_b \otimes P_{b'} \mapsto \begin{cases} P_b & \text{if } b = b' \\ 0 & \text{else} \end{cases}"'] & \mathbb{1}^B \arrow[r, "\overset{\textbf{co-unit}}{\varepsilon}", "P_b \mapsto 1"'] & \mathbb{1} \end{tikzcd}$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$\begin{tikzcd}
 & & & \bigcirc_B \arrow[d, phantom, "\rotatebox{90}{$=:$}"] \\
 & & & B\text{-Reader} \arrow[d, phantom, "\rotatebox{90}{$\simeq$}"] \\
 & \mathbb{1}^B\text{-CoWriter} \arrow[dr, phantom, "\simeq"] & & \mathbb{1}^B\text{-Writer} \arrow[dl, phantom, "\simeq"] \\
B : \mathrm{FinType} \;\; \vdash & & \mathbb{1}^B\text{-DualWriter} \arrow[d, phantom, "\rotatebox{90}{$\in$}"] & \\
 & \mathrm{CoMonad}\big(\mathrm{LType}\big) & & \mathrm{Monad}\big(\mathrm{LType}\big) \\
 & & \mathrm{FrobMonad}\big(\mathrm{LType}\big) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$\mathbb{1}^B\text{-Writer}(D) := \mathbb{1}^B \oplus D$

$\mathrm{bind}^{\mathbb{1}^B\mathrm{Writer}}\Big(D_1 \xrightarrow{\mathrm{prog}} \mathbb{1}^B \oplus D_2\Big) := \mathbb{1}^B \oplus D_1 \xrightarrow{\mathbb{1}^B \oplus \mathrm{prog}} \mathbb{1}^B \oplus \mathbb{1}^B \oplus D_2 \xrightarrow{\mu \oplus \mathrm{id}_{D_2}} \mathbb{1}^B \oplus D_2$

Where $\mathbb{1}^B = \bigoplus_{b:B} \mathbb{C} \cdot P_b \in \mathrm{CMon}(\mathrm{LType})$ is Frobenius algebra of B-projection operators :

$$\begin{tikzcd}[column sep=huge]
\mathbb{1} \arrow[r, "\overset{\textbf{unit}}{\eta}", "1 \mapsto \sum_{b:B} P_b"'] & \mathbb{1}^B \arrow[r, "\overset{\textbf{co-product}}{\delta}", "P_b \mapsto P_b \otimes P_b"'] & \mathbb{1}^B \otimes \mathbb{1}^B \arrow[r, "\overset{\textbf{product}}{\mu}", "P_b \otimes P_{b'} \mapsto \begin{cases} P_b \text{ if } b = b' \\ 0 \text{ else} \end{cases}"'] & \mathbb{1}^B \arrow[r, "\overset{\textbf{co-unit}}{\varepsilon}", "P_b \mapsto 1"'] & \mathbb{1}
\end{tikzcd}$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$\begin{tikzcd}
 & & & \bigcirc_B \arrow[d, phantom, sloped, "=:"] \\
 & B\text{-CoReader} \arrow[d, phantom, sloped, "\simeq"] & & B\text{-Reader} \arrow[d, phantom, sloped, "\simeq"] \\
 & \mathbb{1}^B\text{-CoWriter} \arrow[dr, phantom, sloped, "\simeq"] & & \mathbb{1}^B\text{-Writer} \arrow[dl, phantom, sloped, "\simeq"] \\
B : \mathrm{FinType} \;\; \vdash & & \mathbb{1}^B\text{-DualWriter} \arrow[d, phantom, "\pitchfork"] & \\
 & \mathrm{CoMonad}\big(\mathrm{LType}\big) & & \mathrm{Monad}\big(\mathrm{LType}\big) \\
 & & \mathrm{FrobMonad}\big(\mathrm{LType}\big) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\mathbb{1}^B\text{-Writer}(D) := \mathbb{1}^B \oplus D$$

$$\mathrm{bind}^{\mathbb{1}^B\mathrm{Writer}}\Big(D_1 \xrightarrow{\mathrm{prog}} \mathbb{1}^B \oplus D_2\Big) := \mathbb{1}^B \oplus D_1 \xrightarrow{\mathbb{1}^B \oplus \mathrm{prog}} \mathbb{1}^B \oplus \mathbb{1}^B \oplus D_2 \xrightarrow{\mu \oplus \mathrm{id}_{D_2}} \mathbb{1}^B \oplus D_2$$

Where $\mathbb{1}^B = \bigoplus_{b:B} \mathbb{C} \cdot P_b \in \mathrm{CMon}(\mathrm{LType})$ is Frobenius algebra of B-projection operators :

$$\begin{tikzcd}[column sep=huge]
\mathbb{1} \arrow[r, "\overset{\textbf{unit}}{\eta}", "1 \mapsto \sum\limits_{b:B} P_b"'] & \mathbb{1}^B \arrow[r, "\overset{\textbf{co-product}}{\delta}", "P_b \mapsto P_b \otimes P_b"'] & \mathbb{1}^B \otimes \mathbb{1}^B \arrow[r, "\overset{\textbf{product}}{\mu}", "{P_b \otimes P_{b'} \mapsto \begin{cases} P_b \text{ if } b=b' \\ 0 \text{ else} \end{cases}}"'] & \mathbb{1}^B \arrow[r, "\overset{\textbf{co-unit}}{\varepsilon}", "P_b \mapsto 1"'] & \mathbb{1}
\end{tikzcd}$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

(see nLab:quantum+reader+monad)

$$\begin{tikzcd}
 & \textcolor{blue}{☆_B} \arrow[d, phantom, "\rotatebox{90}{$=:$}" description] & & \bigcirc_B \arrow[d, phantom, "\rotatebox{90}{$=:$}" description] \\
 & B\text{-CoReader} \arrow[d, phantom, "\rotatebox{90}{$\simeq$}" description] & & B\text{-Reader} \arrow[d, phantom, "\rotatebox{90}{$\simeq$}" description] \\
 & \mathbb{1}^B\text{-CoWriter} \arrow[dr, phantom, "\simeq" description] & & \mathbb{1}^B\text{-Writer} \arrow[dl, phantom, "\simeq" description] \\
B : \mathrm{FinType} \;\; \vdash & & \mathbb{1}^B\text{-DualWriter} & \\
 & \mathrm{CoMonad}\big(\mathrm{LType}\big) & \Cap & \mathrm{Monad}\big(\mathrm{LType}\big) \\
 & & \mathrm{FrobMonad}\big(\mathrm{LType}\big) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

$$\mathbb{1}^B\text{-Writer}\big(D\big) :\equiv \mathbb{1}^B \otimes D$$

$$\mathrm{bind}^{\mathbb{1}^B\mathrm{Writer}}\Big(D_1 \xrightarrow{\mathrm{prog}} \mathbb{1}^B \otimes D_2\Big) :\equiv \mathbb{1}^B \otimes D_1 \xrightarrow{\mathbb{1}^B \otimes \mathrm{prog}} \mathbb{1}^B \otimes \mathbb{1}^B \otimes D_2 \xrightarrow{\mu \otimes \mathrm{id}_{D_2}} \mathbb{1}^B \otimes D_2$$

Where $\mathbb{1}^B = \bigoplus\limits_{b:B} \mathbb{C} \cdot P_b \; \in \; \mathrm{CMon}\big(\mathrm{LType}\big)$ is Frobenius algebra of B-projection operators :

$$\mathbb{1} \xrightarrow[\;1 \,\mapsto\, \sum\limits_{b:B} P_b\;]{\overset{\textbf{unit}}{\eta}} \mathbb{1}^B \xrightarrow[\;P_b \,\mapsto\, P_b \otimes P_b\;]{\overset{\textbf{co-product}}{\delta}} \mathbb{1}^B \otimes \mathbb{1}^B \xrightarrow[\;P_b \otimes P_{b'} \,\mapsto\, \begin{cases} P_b & \text{if } b=b' \\ 0 & \text{else} \end{cases}\;]{\overset{\textbf{product}}{\mu}} \mathbb{1}^B \xrightarrow[\;P_b \,\mapsto\, 1\;]{\overset{\textbf{co-unit}}{\varepsilon}} \mathbb{1}$$

Aside: **Linear indefiniteness monad recovers Coecke's "classical structures".**

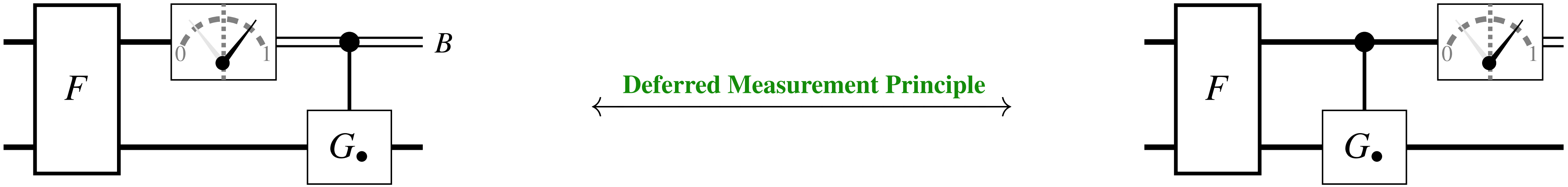
(see nLab:quantum+reader+monad)

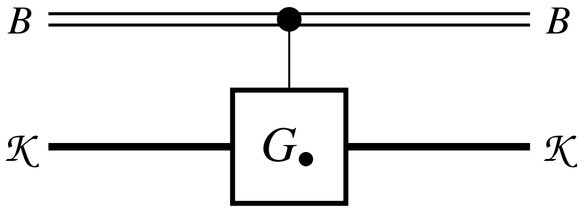
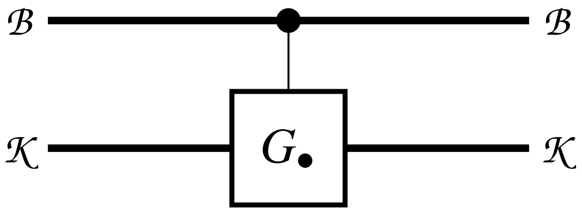
$$\begin{tikzcd}
 & & \star_B \arrow[rr, equal] \arrow[dr, equal] & & \bigcirc_B \arrow[dl, equal] & \\
B : \mathrm{FinType} \;\; \vdash & & & \mathbb{1}^B\text{-DualWriter} & & \\
 & & & \rotatebox{90}{$\in$} & & \\
 & & \mathrm{CoMonad}\big(\mathrm{LType}\big) & & \mathrm{Monad}\big(\mathrm{LType}\big) & \\
 & & & \mathrm{FrobMonad}\big(\mathrm{LType}\big) \arrow[ul] \arrow[ur] & &
\end{tikzcd}$$

Where $\mathbb{1}^B = \bigoplus_{b:B} \mathbb{C} \cdot P_b \in \mathrm{CMon}\big(\mathrm{LType}\big)$ is Frobenius algebra of B-projection operators :

$$\begin{tikzcd}[column sep=huge]
\mathbb{1} \arrow[r, "\overset{\textbf{unit}}{\eta}", "1 \mapsto \sum\limits_{b:B} P_b"'] & \mathbb{1}^B \arrow[r, "\overset{\textbf{co-product}}{\delta}", "P_b \mapsto P_b \otimes P_b"'] & \mathbb{1}^B \otimes \mathbb{1}^B \arrow[r, "\overset{\textbf{product}}{\mu}", "P_b \otimes P_{b'} \mapsto \begin{cases} P_b \text{ if } b{=}b' \\ 0 \text{ else} \end{cases}"'] & \mathbb{1}^B \arrow[r, "\overset{\textbf{co-unit}}{\varepsilon}", "P_b \mapsto 1"'] & \mathbb{1}
\end{tikzcd}$$

Exmp: Deferred measurement principle – Proven by monadic effect logic.

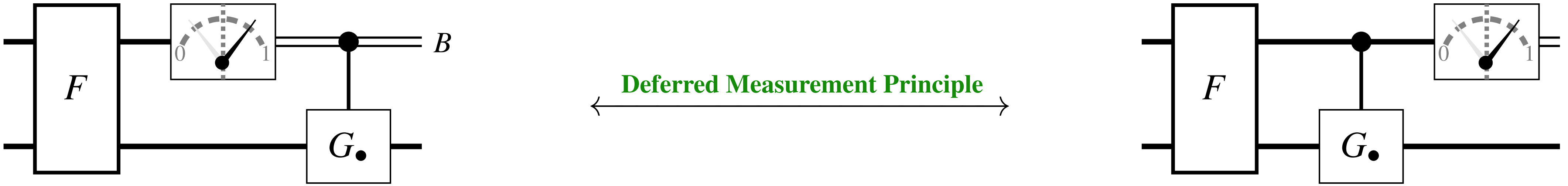


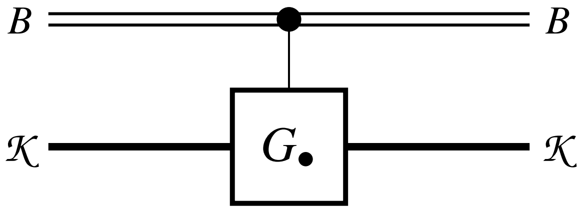
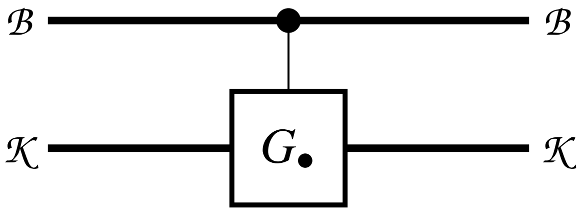
classically controlled gate	**quantumly controlled gate**
 $\mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{G_\bullet} \mathcal{B}_\bullet \boxtimes \mathcal{K}$ $b : B \;\vdash\; \mathcal{K} \xrightarrow{G_b} \mathcal{K}$	 $\Box_B \mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{\Box_B G_\bullet} \Box_B \mathcal{B}_\bullet \boxtimes \mathcal{K}$ $b : B \;\vdash\; \bigoplus_{b' B} \mathcal{K} \xrightarrow{\bigoplus_{b' : B} G_{b'}} \bigoplus_{b' B} \mathcal{K}$

Exmp: Deferred measurement principle – Proven by monadic effect logic.

$$\Box_B \mathcal{H}_\bullet \xrightarrow{F} \Box_B \mathcal{H}_\bullet \xrightarrow{\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet \xrightarrow{G_\bullet} \mathcal{H}_\bullet \quad\mapsto\quad \Box_B \mathcal{H}_\bullet \xrightarrow{\mathrm{diag}_B(G_\bullet)\circ F} \Box_B \mathcal{H}_\bullet \quad\mapsto\quad \Box_B \mathcal{H}_\bullet \xrightarrow{\mathrm{diag}_B(G_\bullet)\circ F} \Box_B \mathcal{H}_\bullet \xrightarrow{\varepsilon^{\Box_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

measurement-controlled quantum gate quantum-controlled quantum gate... ...followed by measurement



classically controlled gate	quantumly controlled gate
 $\mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{G_\bullet} \mathcal{B}_\bullet \boxtimes \mathcal{K}$ $b : B \;\vdash\; \mathcal{K} \xrightarrow{G_b} \mathcal{K}$	 $\mathcal{B}$ $\mathcal{B}$ $\mathcal{K}$ G• $\mathcal{K}$ $\Box_B \mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{\Box_B G_\bullet} \Box_B \mathcal{B}_\bullet \boxtimes \mathcal{K}$ $b : B \;\vdash\; \bigoplus_{b'B} \mathcal{K} \xrightarrow{\bigoplus_{b':B} G_{b'}} \bigoplus_{b'B} \mathcal{K}$

Exmp: Deferred measurement principle – Proven by monadic effect logic.

$$\begin{tikzcd} \mathrm{Kl}(\square_B) \arrow[r, "\sim", "\delta^B \circ \square_B(-)"'] \arrow[rr, bend left, "\mathrm{id}"] & \mathrm{LType}_{B\square_B} \arrow[r, "\sim", "\varepsilon^{\square_B} \circ (-)"'] & \mathrm{Kl}(\square_B) \end{tikzcd}$$

Kleisli equivalence

$\square_B$-Kleisli morphisms

$\square_B$-coalgebra homomorphisms

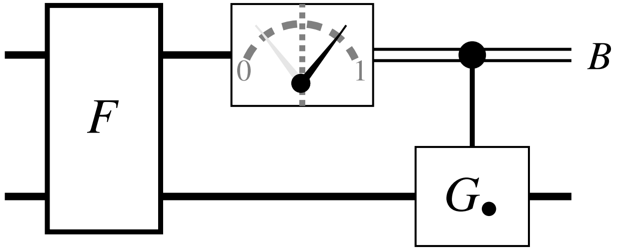
$\square_B$-Kleisli morphisms

$$\square_B\mathcal{H}_\bullet \xrightarrow{F} \square_B\mathcal{H}_\bullet \xrightarrow{\varepsilon^{\square_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet \xrightarrow{G_\bullet} \mathcal{H}_\bullet \quad \mapsto \quad \square_B\mathcal{H}_\bullet \xrightarrow{\mathrm{diag}_B(G_\bullet)\circ F} \square_B\mathcal{H}_\bullet \quad \mapsto \quad \square_B\mathcal{H}_\bullet \xrightarrow{\mathrm{diag}_B(G_\bullet)\circ F} \square_B\mathcal{H}_\bullet \xrightarrow{\varepsilon^{\square_B}_{\mathcal{H}_\bullet}} \mathcal{H}_\bullet$$

measurement-controlled quantum gate

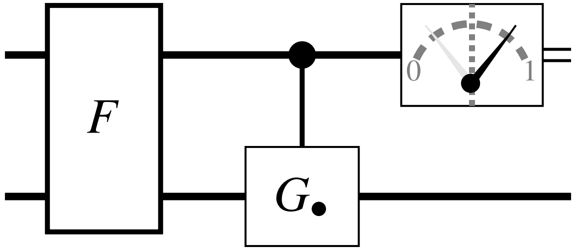
quantum-controlled quantum gate...

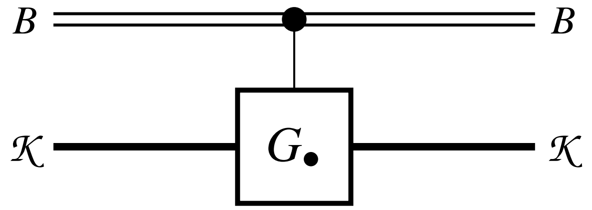
...followed by measurement

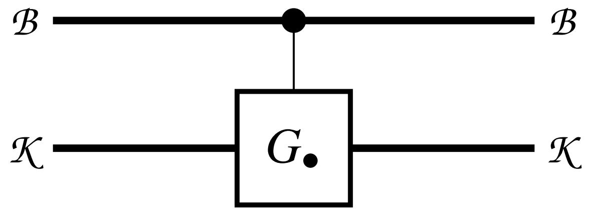


Deferred Measurement Principle

$\longleftrightarrow$



<table>
<tr><th>classically controlled gate</th><th>quantumly controlled gate</th></tr>
<tr><td>


$$\mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{G_\bullet} \mathcal{B}_\bullet \boxtimes \mathcal{K}$$
$$b : B \ \vdash \quad \mathcal{K} \xrightarrow{G_b} \mathcal{K}$$
</td><td>


$$\square_B\mathcal{B}_\bullet \boxtimes \mathcal{K} \xrightarrow{\square_B G_\bullet} \square_B\mathcal{B}_\bullet \boxtimes \mathcal{K}$$
$$b : B \ \vdash \quad \bigoplus_{b'B}\mathcal{K} \xrightarrow{\bigoplus_{b':B} G_{b'}} \bigoplus_{b'B}\mathcal{K}$$
</td></tr>
</table>

The Quantum modality.

Also the *exponential modality* traditionally postulated in linear logic is an emergent effect in `LHoTT`,

linear randomization

aka: stabilization/motivization

$$\begin{tikzcd}
\mathrm{Type} \arrow[r, shift left=2, "\Sigma^\infty_+ \,:\, B \,\mapsto\, \star_B \mathbb{1}"] \arrow[r, phantom, "\perp"] & \mathrm{LType} \arrow[l, shift left=2, "\Omega^\infty"] \arrow[loop right, "!"]
\end{tikzcd}$$

exponential modality

classical types

linear types

The Quantum modality.

Also the *exponential modality* traditionally postulated in linear logic is an emergent effect in LHoTT, as is the crucial *Quantum Modality*, not considered before:

linear randomization

aka: stabilization/motivization

quantum modality

$$\begin{tikzcd}
\mathrm{Type} \arrow[loop left, "Q"] \arrow[r, shift left=2, "\Sigma^\infty_+ \,:\, B \,\mapsto\, \star_B \mathbb{1}"] \arrow[r, phantom, "\perp"] & \mathrm{LType} \arrow[loop right, "!"] \arrow[l, shift left=2, "\Omega^\infty"]
\end{tikzcd}$$

exponential modality

classical types

linear types

The Quantum modality.

Also the *exponential modality* traditionally postulated in linear logic
is an emergent effect in LHoTT,
as is the crucial *Quantum Modality*, not considered before:

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$$\begin{tikzcd} \mathrm{Type} \arrow[loop left, "\mathrm{Q}"] \arrow[r, bend left, "\Sigma^\infty_+ \,:\, B \,\mapsto\, \star_B \mathbb{1}"] \arrow[r, phantom, "\perp"] & \mathrm{LType} \arrow[l, bend left, "\Omega^\infty"] \arrow[loop right, "!"] \end{tikzcd}$$

exponential modality

classical types

linear types

The Q-monad plays a crucial role in the full formulation of the QS-language.
It is the secret actor behind QBit $=$ Q(Bit)...

The Quantum modality.

Also the *exponential modality* traditionally postulated in linear logic
is an emergent effect in `LHoTT`,
as is the crucial *Quantum Modality*, not considered before:

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\end{tikzcd}$$

exponential modality

classical types

linear types

The Q-monad plays a crucial role in the full formulation of the `QS`-language.
It is the secret actor behind QBit $=$ Q(Bit)...

Quantum Circuits

Quantum effects are compatible with tensor product.

Linear Randomness and Indefiniteness are “very strong” effects, in that:

$$\bigcirc_B\big(D\otimes D'\big)\;\simeq\;\big(\bigcirc_B D\big)\otimes D'\,,\qquad \star_B\big(D\otimes D'\big)\;\simeq\;\big(\star_B D\big)\otimes D'$$

There is a whole system of them:

$$\bigcirc_B\bigcirc_{B'}\;\simeq\;\bigcirc_{B\times B'}\,,\quad \text{NB: }\bigcirc_B\bigcirc'_B\;\simeq\;\bigcirc_B\mathbb{1}\otimes\bigcirc'_B$$

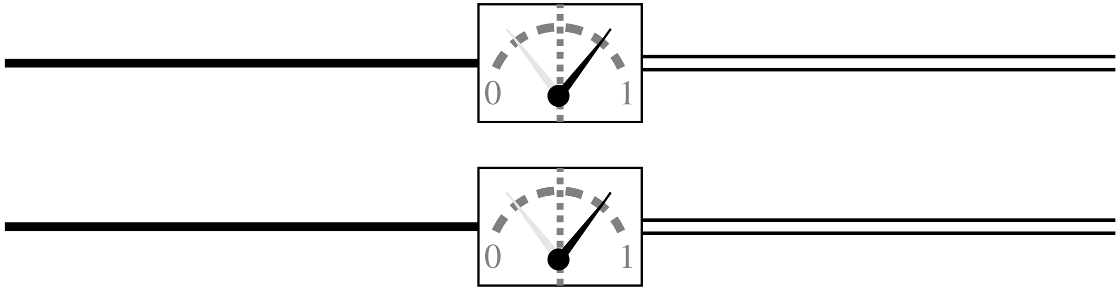
which under dynamic lifting (monadicity comparison functor)
gives the external tensor product of dependent linear types:

$$\begin{tikzcd}
\textbf{free } \bigcirc_B\textbf{-effect handlers in linear data types} & \mathrm{LType}_{\bigcirc_B} \arrow[d, hook, "K_{\oplus_B\mathbb{1}_B\otimes}", "\textbf{comparison functor}"'] & \bigcirc_{B\times B'}\bigcirc_{B\times B'}\mathcal{H} \arrow[loop above, "\bigcirc_{B\times B'}"] \arrow[d, mapsto] \\
B\textbf{-dependent linear data types} & \mathrm{LType}_B & \big(\Box_B\mathbb{1}_B\big)\boxtimes\big(\Box_{B'}\mathbb{1}_{B'}\big)\otimes\mathcal{H} \\
 & & \textbf{external tensor product}
\end{tikzcd}$$

Quantum circuits with classical control & effects

are the *effectful* string diagrams in the linear type system

E.g.
The dependent linear type of a measurement on a pair of qbits:



type of a pair of coherent qbits

pair of measurements

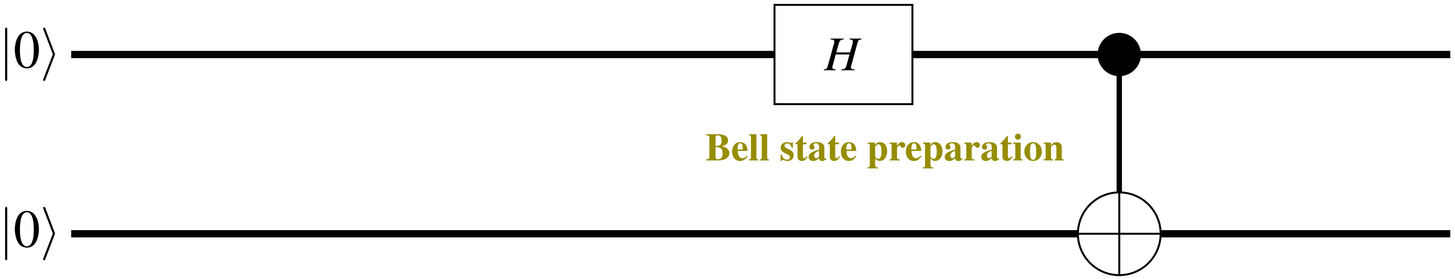
type of collapsed qbits dependent on measured bits b, b'

$$\square_{\mathrm{Bit}^2}\big(\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet\big) \xrightarrow{\;\varepsilon_{\mathrm{Bit}^2}(\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet)\;} \mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet$$

measured bits

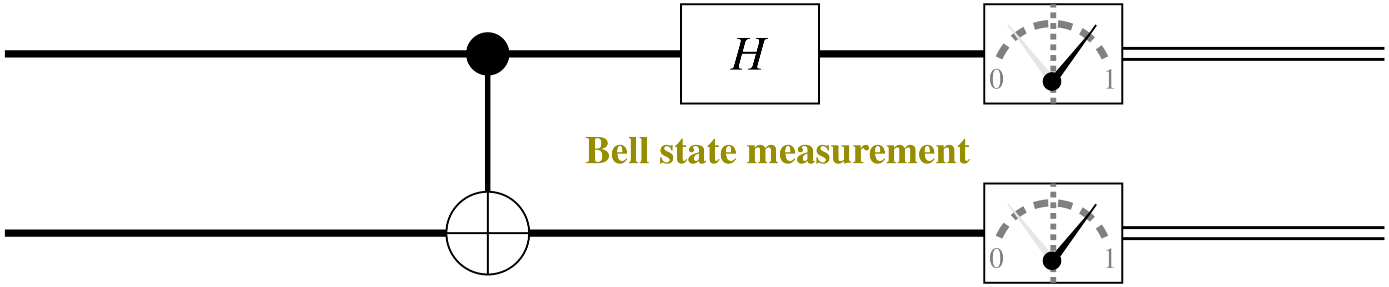
$$(b,b') : \mathrm{Bit}^2 \;\vdash\; \square_{\mathrm{Bit}^2}\big(\mathrm{QBit}_\bullet \boxtimes \mathrm{QBit}_\bullet\big)_{(b,b')} \simeq \mathbb{C}^2 \otimes \mathbb{C}^2 \xrightarrow[\text{collapse of the quantum state}]{\sum_{d,d'} q_{dd'}\, |d\rangle \otimes |d'\rangle \;\mapsto\; q_{bb'}\, |b\rangle \otimes |b'\rangle} \mathbb{C}\,.$$

Example: Bell states of q-bits are typed as follows (regarded in $\mathrm{LType}_{\mathrm{Bit}\times\mathrm{Bit}}$):



$$\mathrm{QBit}_{\bullet} \boxtimes \mathrm{QBit}_{\bullet} \longrightarrow \big(\lozenge_{\mathrm{Bit}}\mathrm{QBit}_{\bullet}\big) \boxtimes \big(\lozenge_{\mathrm{Bit}}\mathrm{QBit}_{\bullet}\big) \simeq \square_{\mathrm{Bit}^2}\big(\mathrm{QBit}_{\bullet} \boxtimes \mathrm{QBit}_{\bullet}\big) \longrightarrow \square_{\mathrm{Bit}^2}\big(\mathrm{QBit}_{\bullet} \boxtimes \mathrm{QBit}_{\bullet}\big)$$

$$b, b' : \mathrm{Bit} \vdash \; \mathbb{C} \xrightarrow{\;1 \;\mapsto\; |0\rangle\otimes|0\rangle \;\mapsto\; \frac{1}{\sqrt{2}}\big(|0\rangle+|1\rangle\big)\otimes|0\rangle \;\mapsto\; \frac{1}{\sqrt{2}}\big(|0\rangle\otimes|0\rangle+|1\rangle\otimes|1\rangle\big)\;} \mathbb{C}^2 \otimes \mathbb{C}^2$$

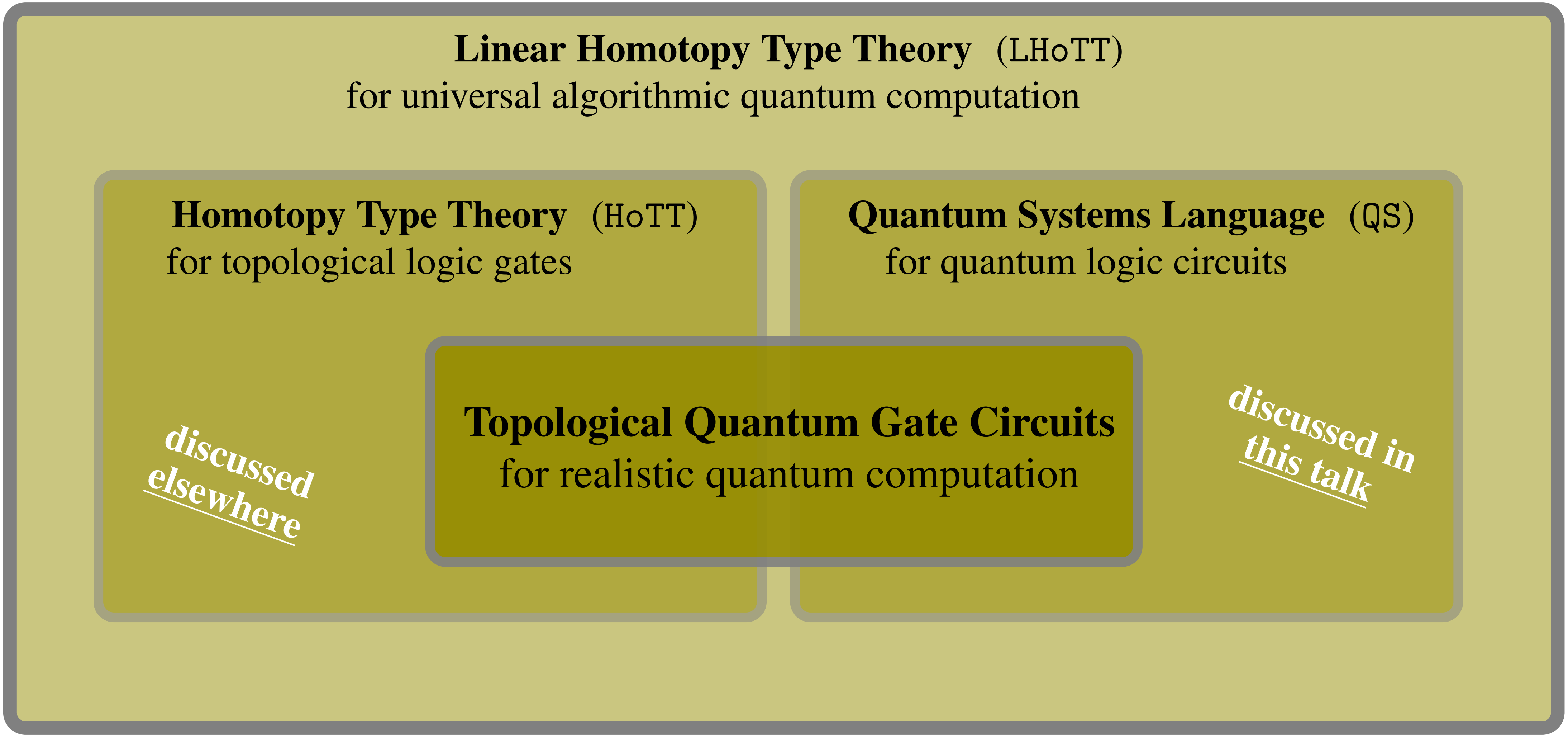


$$\square_{\mathrm{Bit}^2}\big(\mathrm{QBit}_{\bullet} \boxtimes \mathrm{QBit}_{\bullet}\big) \longrightarrow \mathrm{QBit}_{\bullet} \boxtimes \mathrm{QBit}_{\bullet}$$

$$b_1, b_2 : \mathrm{Bit} \quad \vdash \quad \mathbb{C}^2 \otimes \mathbb{C}^2 \xrightarrow{\;\Sigma_{b'_1 b'_2}\, q_{b'_1 b'_2} \cdot |b'_1\rangle \otimes |b'_2\rangle \;\mapsto\; \big(q_{0,b_2} + (-1)^{b_1} \cdot q_{1,(1-b_2)}\big) \cdot |b_1\rangle \otimes |b_2\rangle\;} \mathbb{C}$$

QS – Quantum Systems language @ CQTS

⇝ full-blown Quantum Systems language emerges embedded in LHoTT



Linear Homotopy Type Theory (LHoTT)
for universal algorithmic quantum computation

Homotopy Type Theory (HoTT)
for topological logic gates

Quantum Systems Language (QS)
for quantum logic circuits

Topological Quantum Gate Circuits
for realistic quantum computation

discussed elsewhere

discussed in this talk

Effective Quantum Certification via Linear Homotopy Types

Urs Schreiber (NYU Abu Dhabi)
on joint work at CQTS with
D. J. Myers, M. Riley,
and Hisham Sati

Thanks!

CENTER FOR
QUANTUM &
TOPOLOGICAL
SYSTEMS

presentation at:

The Topos Institute Colloquium, 13 Apr 2023

slides and further pointers at: ncatlab.org/Quantum+Certification+via+Linear+Homotopy+Types#TI2023