Formalizing higher Cartan geometry in modal homotopy type theory

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but many other flavors of geometry play a role:

- symplectic geometry – phase spaces
- conformal geometry – e.g. 2d critical phenomena, RNS strings, gauge theories on solitonic branes
- complex geometry – complex polarized phase spaces, CY-compactifications

All flavors of geometry are unified by *Cartan geometry.*

Which we survey in a moment.

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U. Schreiber, *Differential cohomology in a cohesive topos*

ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos

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shows that Cartan geometry (and much more) has a useful synthetic axiomatization in differentially cohesive ∞-topoi.

U. Schreiber,
Some thoughts on the future of modal homotopy type theory,
talk at German Mathematical Society meeting 2015

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poses the problem of formalizing synthetic Cartan geometry in a homotopy type theory proof checker, such as HoTT-Agda.

Such a synthetic formalization of Cartan geometry
in HoTT with a modal operator
has now been obtained:

Felix Wellen,
Formalizing Cartan geometry in modal homotopy type theory
PhD thesis, in preparation
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Here I give motivation and introduction.

Felix Wellen in his talk will discuss details
of the implementation in modal HoTT.

First some informal survey.

Best to speak synthetic differential geometry right away.

Let X be a smooth space. For
every $k \in \mathbb{N} \cup \{\infty\}$
and every point $x \in X$
there is its *kth order infinitesimal neighbourhood*

$$\mathbb{D}_x^{(k)} \hookrightarrow X \,.$$

If $X = \mathbb{R}^d$,
then $\mathbb{D}_x^{(k)}$ is characterized by the fact that
smooth functions on $\mathbb{D}_x^{(k)}$ are equivalent to
Taylor expansions at x to order k of smooth functions on $\mathbb{R}^d$.

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If you care about the details: Let

$$\mathrm{SuperFormalSmoothCartSp} \hookrightarrow \mathrm{sCAlg}_{\mathbb{R}}^{\mathrm{op}}$$

$$\mathbb{R}^n \times \mathbb{D} \mapsto C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \left(\mathbb{R} \oplus \left\{ \begin{array}{c} \text{fin. dim.} \\ \text{nilpotent} \end{array} \right\}\right)$$

be the site of Cartesian spaces with infinitesimal thickening and smooth functions between them. (See Wellen's talk for details.)

Then

$$\mathbf{H} \simeq \left\{ \begin{array}{c} \text{simplicial presheaves} \\ \text{on } \mathit{SuperFormalSmoothCartSp} \end{array} \right\} \left[\begin{array}{c} \text{local weak} \\ \text{homotopy equivalences} \end{array} \right]^{-1}$$

Informally,
a *smooth manifold* X is
something that locally looks like $\mathbb{R}^d$,
glued by smooth functions.

The *tangent bundle* of a smooth manifold
is over each point x
the first order infinitesimal neighbourhood $\mathbb{D}_x^{(1)}$,
regarded as a vector space.

As one passes from one chart $U_i \simeq \mathbb{R}^d$ to the next via some gluing function f, then these infinitesimal neighbourhoods transform as

$$df : \mathbb{D}_x^{(1)} \xrightarrow{\simeq} \mathbb{D}_{f(x)}^{(1)}$$

hence under the group

$$\mathrm{Aut}(\mathbb{D}_x^{(1)}) \simeq \mathrm{GL}(d)\,.$$

One says that the tangent bundle is associated to a $\mathrm{GL}(d)$-principal bundle, the *frame bundle*.

We may require that
the infinitesimal neighbourhoods transform under only a subgroup

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Examples:

$G = O(d-1,1)$	pseudo-Riemannian metric (gravity)
$G = \mathrm{SO}(d-1,1)$	metric and orientation
$G = \mathrm{SO}(d,2)$	conformal structure
$\mathrm{GL}(d,\mathbb{C}) \hookrightarrow \mathrm{GL}(2d,\mathbb{R})$	almost complex structure
$\mathrm{U}(d) \hookrightarrow \mathrm{GL}(2d,\mathbb{R})$	almost Hermitian structure
$\mathrm{Sp}(d) \hookrightarrow \mathrm{GL}(2d,\mathbb{R})$	almost symplectic structure
$\vdots$	$\vdots$

Here is how it works in components:

Reduction of structure group $O(d) \hookrightarrow \mathrm{GL}(d)$ is locally exhibited by d differential 1-forms

$$E^a = \sum_{\mu=1}^{d} E^a_\mu dx^\mu \quad a \in \{1, \cdots, d\}$$

which identify the tangent space at any point with the model space $\mathbb{R}^d$.
("vielbein", "soldering form")

The model space $\mathbb{R}^d$ carries a canonical metric η, the Minkowski metric. The induced metric on X is

$$ds^2 = \sum_{a,b=1}^{d} \eta_{ab} E^a E^b \,.$$

Since the tangent bundle of $\mathbb{R}^d$ is trivialized by translation
the local model space $\mathbb{R}^d$ carries
a canonical G-structure for every choice of $G \hookrightarrow \mathrm{GL}(d)$.

Say that a G-structure on X is *flat of order k*
if restricted to every $\mathbb{D}_x^{(k)}$ $(x \in X)$
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Flatness to first order is equivalently *vanishing torsion*.

V. Guillemin, *The integrability problem for G-structures*,
Trans. Amer. Math. Soc. 116 (1965)

J. Lott, *The Geometry of Supergravity Torsion Constraints*,
Comm. Math. Phys. 133 (1990) arXiv:0108125

Examples of torsion free G-structures:

$G = \mathrm{GL}(d, \mathbb{C})$	complex structure
$G = \mathrm{U}(d)$	Hermitian structure
$G = \mathrm{Sp}(d)$	symplectic structure
$G = O(d-1, 1)$	pseudo-Riemannian metric (gravity)

---

every orthogonal structure has vanishing intrinsic torsion
$\Leftrightarrow$
for every metric there exists a torsion free metric connection
(the "Levi-Civita connection").

This is Einstein's **principle of equivalence**:
For X a spacetime with gravity,
then every $\mathbb{D}_x^{(1)}$ looks like Minkowski spacetime.

Here is how this works in components.

The model space $\mathbb{R}^d$ carries a canonical vielbein $e^a = dx^a$.
This has the special property that it is translation invariant

$$de^a = 0$$

But E^a on X is only defined up to Lorentz transformation in $O(d-1,1)$. Hence instead of asking for dE^a, we need to ask for the covariant derivative

$$\tau^a = dE^a + \sum_{b=1}^{d} \Omega^a{}_b \wedge E^b$$

for *some* 1-forms $\Omega^a{}_b$ that send tangent vectors to infinitesimal Lorentz transformations.
The *intrinsic* torsion is τ^a modulo terms of the form $\Delta\Omega^a{}_b \wedge E^b$.

generalization 1: Super-Cartan geometry

supergravity is Cartan geometry
for local model space a super-translation group $\mathbb{R}^{d-1,1|N}$
and reduction along $\mathrm{Spin}(d-1,1) \hookrightarrow \mathrm{GL}(d-1,1|N)$

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and reduction along $\mathrm{Spin}(d-1,1) \hookrightarrow \mathrm{GL}(d-1,1|N)$

A miracle happens:

the Einstein equations in 11-dimensions
are already equivalent to
torsion-free $\mathrm{Spin}(10,1) \hookrightarrow \mathrm{GL}(10,1|N)$-structure.

A. Candiello, K. Lechner, *Duality in Supergravity Theories*,
Nucl.Phys. B412 (1994) 479-501
arXiv:hep-th/9309143

P. Howe, *Weyl Superspace*,
Physics Letters B Volume 415, Issue 2 (1997)
arXiv:hep-th/9707184

generalization 2: Higher Cartan geometry

11d Supergravity with M-brane effects included ("M-theory"), is a hypothetical candidate for a "theory of everything" in physics.

physics jargon: "M5-brane moves in condensate of M2-branes"

mathematically: spacetime becomes a higher Cartan geometry modeled on a homotopy 3-type extension of $\mathbb{R}^{10,1|\mathbf{32}}$.
(stacky spacetime, higher Cartan geometry)

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D. Fiorenza, H. Sati, U. Schreiber, *Super Lie n-algebra extensions, higher WZW models and super p-branes with tensor multiplet fields* International Journal of Geometric Methods in Modern Physics Volume 12, Issue 02 (2015) 1550018 arXiv:1308.5264

H. Sati, U. Schreiber, *Lie n-algebras of BPS charges* ncatlab.org/schreiber/show/Lie+n-algebras+of+BPS+charges

Now some words

on the synthetic axiomatization of (higher, super) Cartan geometry,

that Felix Wellen has now implemented in Hott-Agda.

following

- U. Schreiber, *Differential cohomology in a cohesive topos*

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For every smooth space $X \in \mathbf{H}$
there is a smooth space $\Im^{(k)}X$ obtained from X
by contracting all $\mathbb{D}_x^{(k)}$ to a point.

Hence there is a canonical projection

$$\eta : X \longrightarrow \Im^{(k)}X$$

its fiber is $\mathbb{D}_x^{(k)}$

$$\begin{array}{ccc} \mathbb{D}_x^{(k)} & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \eta} \\ * & \underset{x}{\longrightarrow} & \Im^{(k)}X \end{array}$$

$\Im^{(\infty)}X$ is also called the *de Rham space* or *de Rham stack* of X.

Proposition:

A map between smooth manifolds $f : X \to Y$ is local diffeomorphism precisely if its $\Im$-naturality square

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \Im X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Im f} \\ Y & \xrightarrow[\eta_Y]{} & \Im Y \end{array}$$

is Cartesian (is a pullback square)

Generally, say that a map f with this property is *formally étale*.

Hence the atlas $U = \sqcup_i \mathbb{R}^d$ of a manifold X yields a diagram

$$\begin{array}{ccc} & U & \\ {}^{\mathrm{et}}\swarrow & & \searrow^{\mathrm{et}} \\ \mathbb{R}^d & & X \end{array}$$

Generally, the X in such a diagram
are *étale* ∞*-stacks*, e.g. *orbifolds*.

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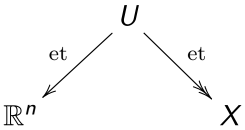

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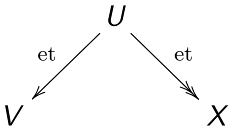
Definition:

For V a group object in $\mathbf{H}$, say that a *V-manifold* or *V-scheme* is an $X \in \mathbf{H}$ such that there exists a diagram



Idea of synthetic axiomatization:

Relevant properties of V-manifolds follow *formally* from the fact that $\Im$ is an idempotent monad.

Hence Cartan geometry makes sense in every ∞-topos equipped with an idempotent ∞-monad interpreted as $\Im$.

A V-manifold is whatever $\Im$ thinks it is.

	analytic	**synthetic**
axiomatize:	constitutents	properties

Regarding homotopy type theory
as the internal language of ∞-toposes
then assuming the existence of an ∞-monad $\Im$
corresponds to adding a *modal operator*.

Theorem:

1. The infinitesimal disk bundle on any ∞-group V is trivialized by left translation.
2. For every V-manifold X its infinitesimal disk bundle is a locally trivial $\mathbb{D}$-fiber bundle, for $\mathbb{D} = \mathbb{D}^V_e$, associated to an $\mathbf{Aut}(\mathbb{D})$-principal bundle (its *frame bundle*) classified by a map

$$\tau_X : X \longrightarrow \mathbf{BAut}(\mathbb{D})$$

Proof: Formalized in HoTT-Agda by

Felix Wellen,
Formalizing higher Cartan geometry in modal HoTT
PhD thesis, github.com/felixwellen/DCHoTT-Agda

Details are presented in Felix Wellen's talk at this meeting.

This theorem establishes the fundamental ingredient
of Cartan geometry: frame bundles of V-manifolds.

It is now straightforward to axiomatize the geometric concepts

- G-structure;
- torsion-freeness

etc.

We briefly state this now:

Definition:

Let

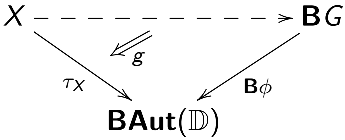
$$\phi : G \to \mathbf{Aut}(\mathbb{D})$$

be any group homomorphism, hence

$$\mathbf{B}\phi : \mathbf{B}G \to \mathbf{BAut}(\mathbb{D})$$

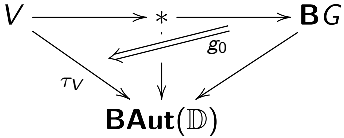
any map.

Then a *G-structure* on a V-manifold is a lift



Observation:

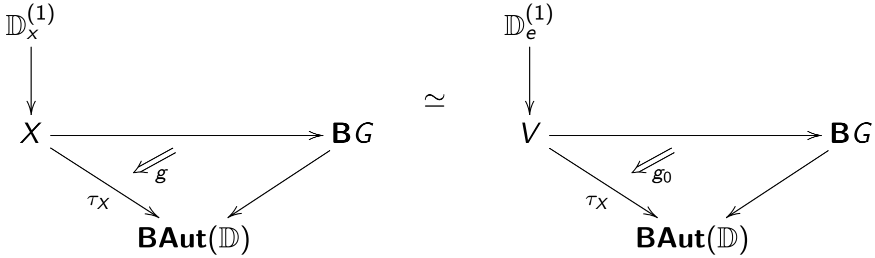
On V there is a canonical G-structure



where the left square exhibits
the left translation trivialization
of τ_V by the previous theorem

Definition:

A G-structure on a V-manifold X
is *torsion free*
if on every first order infinitesimal neighbourhood $\mathbb{D}^{(1)}$
it coincides with the canonical G-structure on V:



(remember: this is really Einstein's principle of equivalence)

One may similarly formalize much more, for instance

- fundamental theorem of calculus,
- Noether's theorem

by postulating a system of adjoint modalities

("differential cohesion")

$$
\begin{array}{ccccccccc}
\mathrm{id} & \dashv & \mathrm{id} & & & & & & \\
\vee & & \vee & & & & & & \\
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \mathrm{Rh} & & & & \\
 & & \vee & & \vee & & & & \\
 & & \Re & \dashv & \Im & \dashv & \text{Et} & & \\
 & & & & \vee & & \vee & & \\
 & & & & \int & \dashv & \flat & \dashv & \sharp \\
 & & & & & & \vee & & \vee \\
 & & & & & & \emptyset & \dashv & *
\end{array}
$$

So much for today.

For more exposition see

- U. Schreiber,
  Higher prequantum geometry
  in G. Catren and M. Anel (eds.)
  New Spaces in Mathematics and Physics
  arXiv:1601.05956

These slides with background material are kept online at

ncatlab.org/schreiber/show/Formalizing+Cartan+Geometry+in+Modal+HoTT