Obstruction theory for higher parameterized WZW terms

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joint with Domenico Fiorenza and Igor Khavkine

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www.math.uni-hamburg.de/DMV2015

A survey of results.
For method of proof see my talk on Thursday.

Contents

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1 Obstruction sequence 1

Consider the derived category over smooth manifolds... inside the homotopy theory of smooth stacks:

$$\mathrm{Sh}(\mathrm{SmoothMfd}, \mathrm{ChainCplx}) \xrightarrow{\text{Dold-Kan correspondence}} \mathrm{Sh}(\mathrm{SmoothMfd}, \mathrm{KanCplx}) \;.$$

Write

$\mathbf{\Omega}^k$:	sheaf of k-forms;
$\mathbf{B}^k\mathbb{Z} := \mathbb{Z}[k]$, $\mathbf{B}^k\flat\mathbb{R} := \mathbb{R}[k]$	locally constant coefficients

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The degree filtration on the de Rham complex induces a map

$$\mathbf{\Omega}^{p+2}_{\mathrm{cl}} \longrightarrow \mathbf{B}^{p+2}\flat\mathbb{R}$$

This induces the homotopy pullback

$$\begin{array}{ccccc} & & \mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}} & \longrightarrow & \mathbf{B}^{p+2}\mathbb{Z} \\ & \overset{\nabla}{\nearrow} & \downarrow^{\mathrm{curv}} & (\mathrm{pb}) & \downarrow^{2\pi\hbar} \\ V & \underset{F_\nabla}{\dashrightarrow} & \mathbf{\Omega}^{p+2}_{\mathrm{cl}} & \longrightarrow & \mathbf{B}^{p+2}\flat\mathbb{R} \end{array}$$

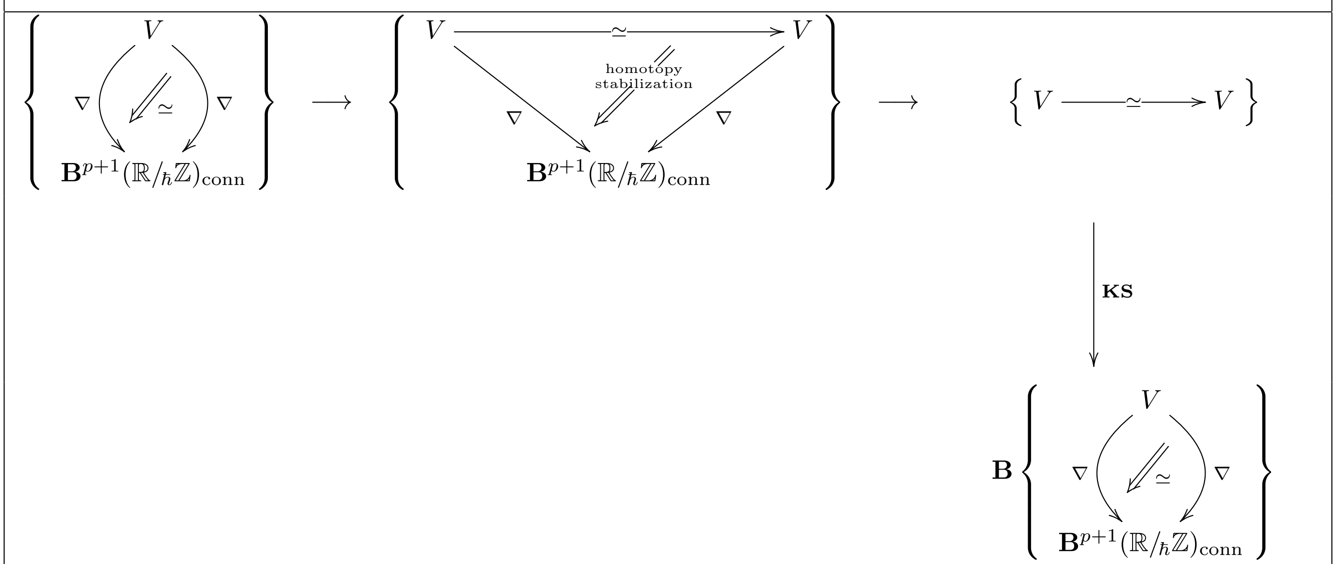
∇ is equivalently

- p-gerbe connection with band $(\mathbb{R}/_{\hbar}\mathbb{Z})$;
- principal $\mathbf{B}^p(\mathbb{R}/_{\hbar}\mathbb{Z})$-connection;
- Deligne cocycle of degree $(p+2)$.

on the stack V.

Theorem. [Fiorenza-Rogers-S 13a] For any ∇ there is a long homotopy fiber sequence of group stacks:

$$\mathbf{Ch}_\bullet(V, (\mathbb{R}/_{\hbar}\mathbb{Z})) \longrightarrow \mathbf{Stab}_{\mathbf{Aut}(V)}(\nabla) \longrightarrow \mathrm{im}(\mathbf{Stab}_{\mathbf{Aut}(V)}(\nabla) \to \mathbf{Aut}(V))$$



Theorem. [S 15]
Given a V-fiber bundle E, then the class $\mathbf{KS}(E)$ is the obstruction to parameterizing ∇ over E.

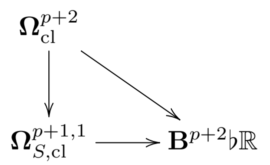
2 Obstruction sequence 2

For Σ a smooth manifold of dimension $(p+1)$,
consider now the derived category over PDEs over Σ.

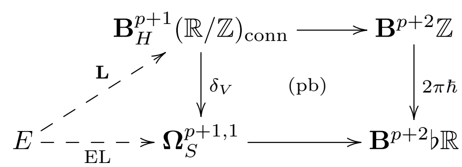
$$\mathrm{Sh}(\mathrm{SmoothMfd}_{/\Sigma}) \underset{\mathrm{Free}_!}{\overset{U\mathrm{nderlying}_!}{\rightleftarrows}} \mathrm{Sh}(\mathrm{PDE}_\Sigma) \;;\quad U(F(E)) \simeq J^\infty_\Sigma E \;\text{ jet bundle}$$

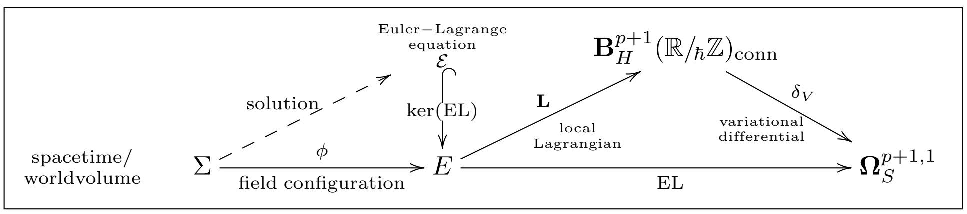
$\Omega^{p+1,1}_S(E) \subset \Omega^{p+2}(J^\infty E)$: forms of degree $(p+1)$-on Σ, vertical degree-1 on E

Now the bidegree filtration
of the de Rham complex
on $J^\infty_\Sigma E$
gives factorization

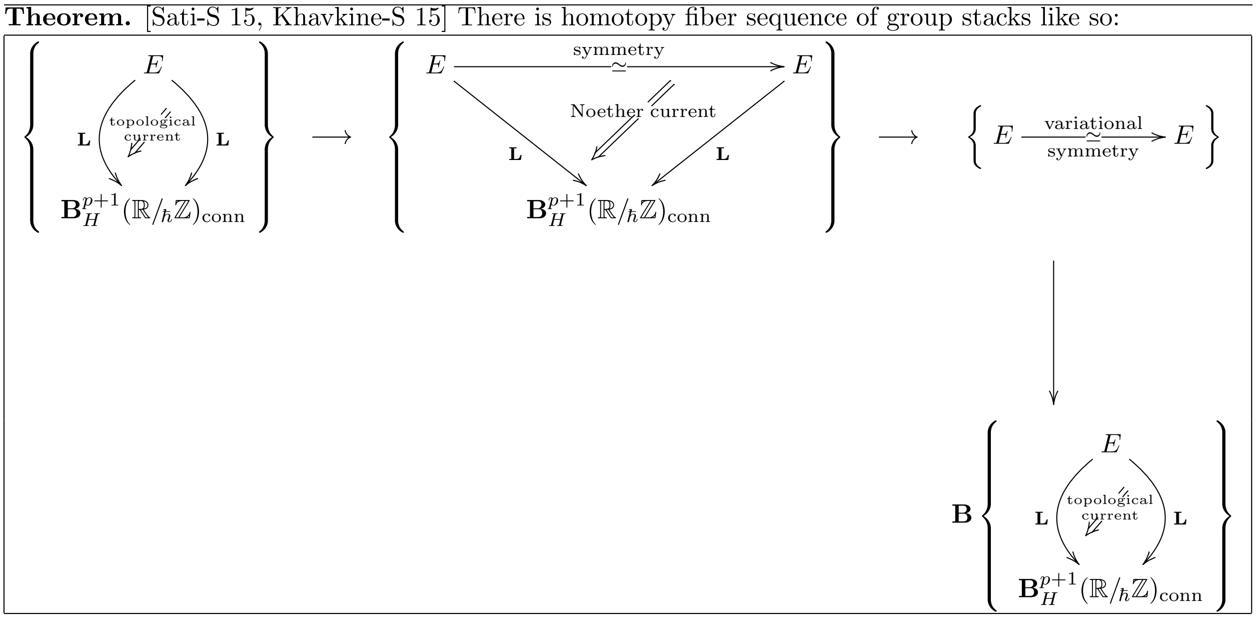


This induces
the homotopy pullback





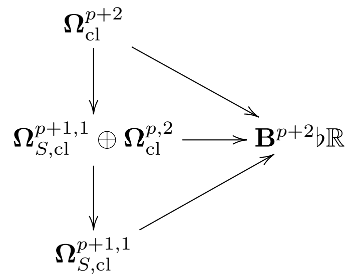
Theorem. [Sati-S 15, Khavkine-S 15] There is homotopy fiber sequence of group stacks like so:



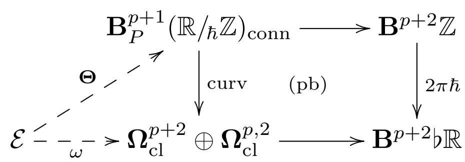
Theorem. [Khavkine-S 15] On shell, the Lie algebra is the Dickey bracket on conserved currents and for suitably regular non-gauge theories the Lie extension is the sharp version of Noether's theorem [Vinogradov 84, theorem 11.2].

3 Obstruction sequence 3

Yet one more factorization from bidegree filtration:



This induces the homotopy pullback [Khavkine-S 15]



ω:	canonical pre-symplectic current
$(\mathcal{E}, \omega)$;	covariant phase space
$(\mathcal{E}, \mathbf{\Theta})$:	its Kostant-Souriau prequantization

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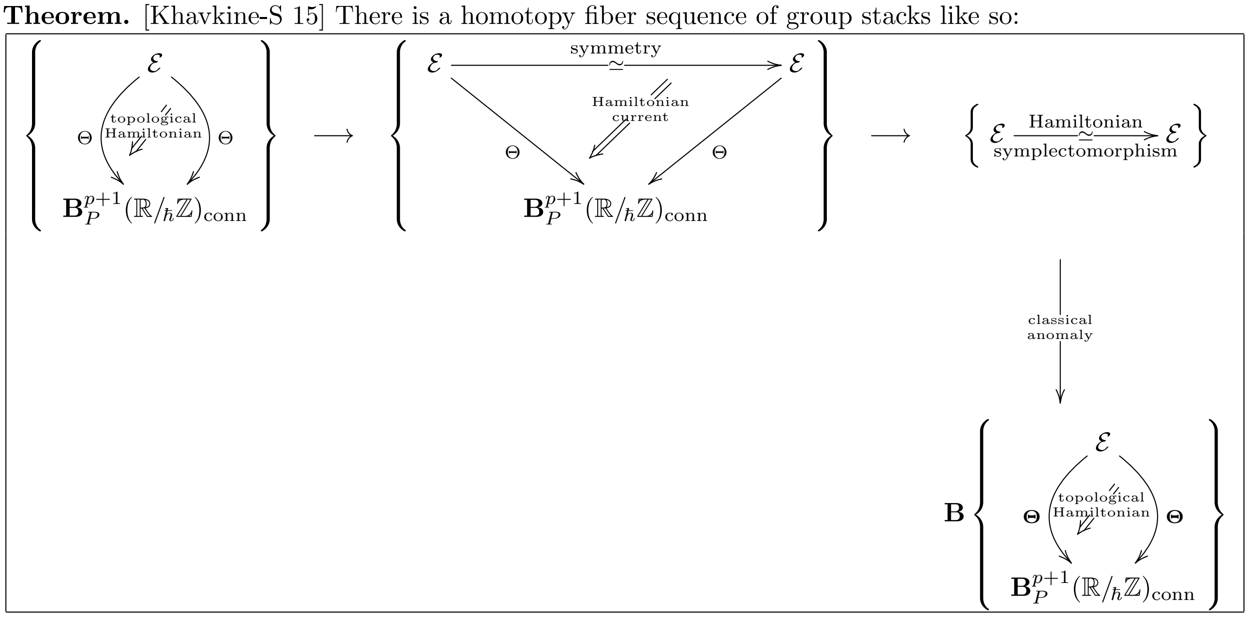
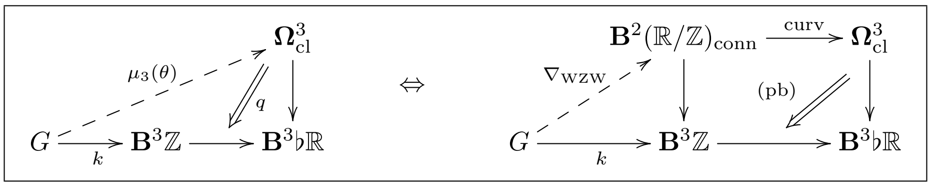
Theorem. [Khavkine-S 15] There is a homotopy fiber sequence of group stacks like so:



Theorem. [Fiorenza-Rogers-S 13b] The Lie algebra is the Poisson bracket.

4 The original WZW term

$\mathfrak{g}$	semisimple Lie algebra
G	its simply-connected Lie group
$\theta \in \Omega^1(G, \mathfrak{g})$	Maurer-Cartan form
$\langle -, - \rangle$	Killing metric
$\mu_3 = \langle -, [-, -] \rangle$	Lie algebra 3-cocycle
$k \in H^3(G, \mathbb{Z})$	level
$\mu_3(\theta \wedge \theta \wedge \theta) \xrightarrow[\simeq]{q} k_{\mathbb{R}}$	prequantization condition



Definition [Khavkine-S 15] Full WZW model is the $(p = 2)$-dimensional field theory with

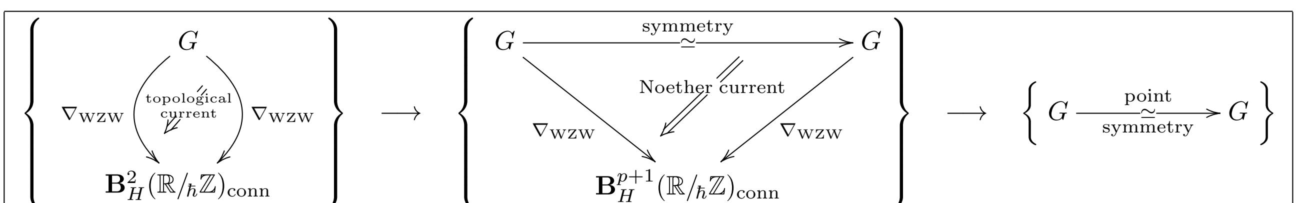
$$\mathbf{L} := \underbrace{\langle \theta_H \wedge \star\theta_H \rangle}_{\mathbf{L}_{\mathrm{kin}}} + \underbrace{(\nabla_{\mathrm{WZW}})_H}_{\mathbf{L}_{\mathrm{WZW}}} \ : \ \Sigma \times G \longrightarrow \mathbf{B}^{p+1}_H(\mathbb{R}/_\hbar\mathbb{Z})_{\mathrm{conn}} \,.$$

Theorem. [Baez-Crans-S-Stevenson 07] Write $\hat{\Omega}_k G$ for level-k Kac-Moody loop group extension of G. This has an adjoint action by the based path group $P_e G$. Write

$$\mathrm{String}(G) := P_e G // \hat{\Omega}_k G$$

for the homotopy quotient. This is a differentiable group stack, called the *string 2-group*.

Theorem. [Fiorenza-Rogers-S 13a] For the WZW model the above homotopy fiber sequence becomes:



$$\mathbf{B}U(1) \longrightarrow \mathrm{String}(G) \longrightarrow G$$

Theorem. [Fiorenza-Rogers-S 13a]
Obstruction to parameterizing ∇_{WZW} over G-principal bundle is canonical 4-class.

Example.
For $G = \mathrm{Spin} \times \mathrm{SU}$ this is the sum of fractional Pontryagin and second Chern class:

$$\tfrac{1}{2}p_1 - c_2 \,.$$

The vanishing of this class is the Green-Schwarz anomaly cancellation condition for the heterotic string. (This persepctive on the Green-Schwarz anomaly via paramterized WZW models had been suggested in [Distler-Sharpe 07].)

5 The GS-WZW cocycles

There are plenty of higher cocycles on higher Lie algebras.
Observation. [Fiorenza-Sati-S 13b] A 2-cocycle on a Lie algebra is equivalently an L_∞-homomorphism

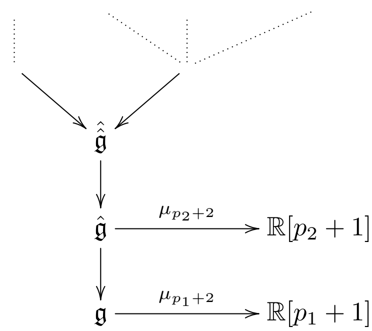
$$\mu_2 : \mathfrak{g} \longrightarrow \mathbb{R}[1]\,.$$

The central extension $\hat{\mathfrak{g}}$ that it classifies is equivalently homotopy fiber of this morphism

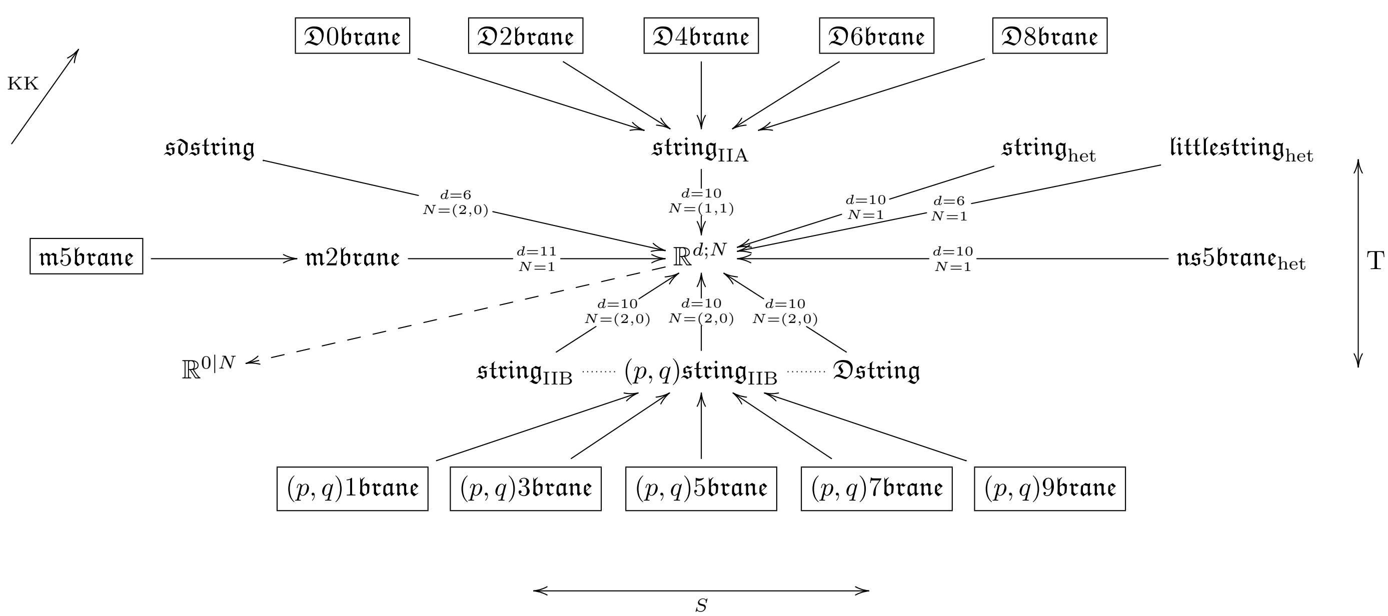
$$\begin{array}{ccc} \hat{\mathfrak{g}} & & \\ \downarrow & & \\ \mathfrak{g} & \xrightarrow{\mu_2} & \mathbb{R}[1] \end{array}$$

Hence generally, a $(p+2)$-cocycle on an L_∞-algebra $\mathfrak{g}$ is an L_∞-homomorphism to $\mathbb{R}[p+1]$, and the L_∞-extension that it classifies it the homotopy fiber of that.

This yields bouquets of higher extensions:



Theorem. [Fiorenza-Sati-S 13b] The Green-Schwarz WZW models for super p-branes come from the bouquet of cocycles on super-translation Lie algebras:



6 Higher WZW terms

For

$$\mu_{p+2} : \mathfrak{g} \longrightarrow \mathbb{R}[p+1]$$

an L_∞-cocycles, define the Lie integration of $\mathfrak{g}$ to be the looping of the stack

$$\mathbf{B}G : (U, k) \mapsto \mathrm{cosk}_{p+2}(\Omega_{\substack{\mathrm{vert} \\ \mathrm{flat}}}(U \times \Delta^\bullet, \mathfrak{g}))\,.$$

Theorem. [Fiorenza-S-Stasheff 10] μ_{p+2} integrates to homomorphism of group stacks

$$\exp(\mu_{p+2}) : G \longrightarrow \mathbf{B}^{p+1}(\mathbb{R}/\Gamma)$$

Theorem. [S] there are canonical Maurer-Cartan forms on this

$$\theta_G : G \longrightarrow \mathbf{\Omega}^{1 \leq \bullet \leq p+2}_{\mathrm{flat}}(-, \mathfrak{g})$$

Define $\tilde{G}$ as the homotopy pullback that universally turns the hyper-cocycle θ_G into a globally defined differential form $\theta_{\tilde{G}}$

$$\begin{array}{ccc} \tilde{G} & \longrightarrow & G \\ \downarrow{\scriptstyle \theta_{\tilde{G}}} & & \downarrow{\scriptstyle \theta_G} \\ \mathbf{\Omega}_{\mathrm{flat}}(-, \mathfrak{g}) & \longrightarrow & \mathbf{\Omega}^\bullet_{\mathrm{flat}}(-, \mathfrak{g}) \end{array}$$

Example:

1. for ordinary Lie algebra then $\tilde{G} \simeq G$;
2. for $\mathfrak{g} = \mathbb{R}[p+1]$ then $\tilde{G} = \mathbf{B}^{p+1}(\mathbb{R}/\Gamma)_{\mathrm{conn}}$

Hence the general case is a twisted product of these two cases. This means that sigma-models with target space $\tilde{G}$ contain gauge fields on their worldvolume (unification of sigma-model with gauge field theory in higher geometry)

Theorem. [Fiorenza-Sati-S 13b] [S 15] There is induced a unique, up to equivalence, WZW term

$$\tilde{G} \longrightarrow \mathbf{B}^{p+1}(\mathbb{R}/\Gamma)_{\mathrm{conn}}$$

with curvature $\mu(\theta_{\tilde{G}})$ and underlying bundle being the extension $\hat{G}$ classified by $\exp(\mu_{p+2})$

Theorem. [S 15] Given consecutive L_∞-cocycles

$$\begin{array}{ccc} \hat{\mathfrak{g}} & \xrightarrow{\mu_{p_2+2}} & \mathbb{R}[p_2+1] \\ \downarrow & & \\ \mathfrak{g} & \xrightarrow[\mu_{p_1+2}]{} & \mathbb{R}[p_1+1] \end{array}$$

then Lie integration yields two consecutive WZW terms, the second defined on a $\mathbf{B}^{p_1}(\mathbb{R}_{\hbar}\mathbb{Z})_{\mathrm{conn}}$-extension of G:

$$\begin{array}{ccccc} \mathbf{B}^p(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}} & \longrightarrow & \widetilde{\hat{G}} & \xrightarrow{\mathbf{L}^{p_2+1}_{\mathrm{WZW}}} & \mathbf{B}^{p_2+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}} \\ & & \downarrow & & \\ & & G & \xrightarrow[\mathbf{L}^{p_1+1}_{\mathrm{WZW}}]{} & \mathbf{B}^{p_1+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}} \end{array}$$

Application. Solves open problems in global definition of M2/M5-brane charge [Fiorenza-Sati-S 15, Sati-S 15].

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