# Abstracts

## Quantization via twisted generalized cohomology

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Recently there has been much progress in identifying mathematical axioms for quantum field theories, taking into account more of the *local* structure than had been considered in the past. In the Schrödinger picture of QFT this means refining the functor that assigns linear operators to spacetimes or worldvolumes by some kind of $n$-functor with values in some higher category of higher spaces of quantum states. In the Heisenberg picture it similarly means refining the functor that assigns algebras of observables to small regions of spacetimes or worldvolumes to some kind of homotopical functor with values in higher algebras of observables. See the various talks on these matters at this meeting.

All this concerns quantum, hence *quantized* field theories. What has received less attention is the development of the corresponding higher local geometric structures on the side of classical field theories, as well as the higher local refinement of the process of quantization that should take the latter to the former.

For more review of the problem, exposition of the following partial solution, and for references to the literature, see [3, 4].

For example there are competing proposals for the higher functorial formulation of standard Chern-Simons quantum field theory. What does not however exist to date is a systematic prescription that would produce any of these from the only datum that justifies calling any quantum field theory a Chern-Simons theory: the Chern-Simons Lagrangian density, defined on a certain family of jet bundles. This is true even if one considers just the linear spaces of quantum states and ignores the more subtle issue of the evolution operators between these: There does exist a systematic prescription for the construction, starting from the local Lagrangian density, of the spaces of quantum states that Chern-Simons theory assigns in codimension 1 – namely the *geometric quantization* of the Chern-Simons functional. But it is an open problem to refine the process of geometric quantization to higher codimension.

(Here I should amplify that I am talking about complete, hence non-perturbative, quantization, to which the prescription of geometric quantization is particularly adapted. Alternatively there are various concepts of formal deformation quantization, where one considers infinitesimal deformations of classical field theory.)

This problem is of more general relevance than it may superficially seem. By the famous relation of 3-dimensional Chern-Simons theory to 2-dimensional WZW-type field theory, these spaces of quantum states of the topological field theory in 3d are identified with the spaces of solutions of the Ward identities in 2-dimensional conformal field theory. In fact the only existing complete mathematical construction and classification of 2d conformal field theories, namely of rational 2d conformal field theories by Fuch-Runkel-Schweigert, proceeds "holographically" from the 3d Chern-Simons theory. It has been suggested by Witten that analogous relations hold in higher dimensions. Notably the geometric quantization of some 7-dimensional Chern-Simons theory is to similarly encode the correlation functions of conformal self-dual higher gauge field theory in 6 dimensions, a field theory that has been attracting a lot of attention in recent years due to the conjecture that it is effectively 4d (super-)Yang-Mills theory with its S-duality made geometrically manifest.

Moreover, there are various other field theories of interest which are secretly of higher Chern-Simons type. This includes all field theories of AKSZ type. It is reasonable to expect that, in analogy, the geometric quantization of non-pertrubative AKSZ theory in dimension $d+1$ encodes on its boundary the quantization of a non-topological quantum field theory in $d$ dimensions.

In fact, the simplest non-trivial example of an AKSZ field theory, the 2-dimensional Poisson sigma-model, has famously been shown by Kontsevich and Cattaneo-Felder, in perturbation theory, to holographically encode 1-dimensional field theory, namely quantum mechanics: The perturbative quantization of the 2d Poisson sigma-model was shown to induce the perturbative formal deformation qantization of the underlying Poisson manifold.

In summary, this suggests that it should be interesting to consider the general problem of finding the correct formalism for higher geometric quantization of higher topological field theories in positive codimension.

This here is to advertize some first progress in this direction, as worked out in two master theses that I had advised [1, 2], see [3, 5] for the wrap-up:

In certain good situations (which are satisfied in particular for Chern-Simons theory) ordinary geometric quantization is in fact equivalent to push-forward of the pre-quantum line bundle in $G$-equivariant K-theory, where $G$ is a given group of prequantized Hamiltonian symmetries acting leaf-wise on a given Poisson manifold. This is in refinement to an observation that goes back to Bott and is often known as $\mathrm{Spin}^c$-quantization.

This is noteworthy for the following reason: The traditional prescription for geometric quantization – in terms of polarized sections of a prequantum line bundle with symplectic curvature – is problematic when generalizing to higher dimensional local field theory, where the prequantum line bundle becomes a higher pre-quantum gerbe. On the other hand, the concept of push-forward in twisted generalized cohomology is intrinsically homotopy theoretic and as such naturally lends itself to such a generalization.

Second, the geometric quantization of any (compact) Poisson manifold this way may naturally be understood as the non-perturbative boundary quantization of the 2d-Poisson sigma model with that Poisson manifold as target space. This may be regarded as a non-perturbative refinement of the famous perturbative result by Kontsevich and Cattaneo-Felder.

Finally, this construction is such that it has an evident generalization to higher dimensional field theories, provided the correct higher pre-quantum data: on the moduli stack of boundary fields of some $d+1$-dimensional (topological) field theory we need a pre-quantum $(d-1)$-gerbe, and then we need a "superposition principle" embodied in a choice of some $E_\infty$-ring spectrum $E$ (playing the role of the ground field and replacing the complex numbers in traditional quantum mechanics) together with a homomorphism $\mathbf{B}^{d-1}U(1) \to \mathrm{GL}_1(E)$ from the "$\infty$-group of phases" to the $\infty$-group of units of $E$.

One finds that in this perspective all the analytic subtleties of quantum theory are packaged into the choice of ring spectrum $E$. For instance the reason that the K-theory spectrum $E = \mathrm{KU}$ knows about quantum mechanics is ultimately due to the fact that $C^*$-algebras and Hilbert bimodules present the category of KU-modules, via KK-theory. In [5] I discuss evidence that one plausible choice of ring spectrum for quantizing 2-dimensional conformal field theory as the boundary theory of 3-dimensional Chern-Simons theory is $E = \mathrm{tmf}$, the spectrum of topological modular forms. Again, as the name suggests, this captures just the kind of analytical data that controls 2d CFTs.

The main open questions remaining in this approach of higher geometric quantization via twisted generalized cohomology theory are 1) the identification of the ring spectra that quantize a given type of $d$-dimensional field theory, and 2) the refinement of the whole prescription from single boundaries to an $n$-functor on higher codimension singularities.

## References


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[3] U. Schreiber, *What, and for what is Higher geometric quantization?*, talks at *Workshop on Higher Gauge Theory and Higher Quantization*, Edinburgh, 25-26 June 2014 and at *Symmetries and correspondences in number theory, geometry, algebra, physics: intra-disciplinary trends*, Oxford, July 5-8 2014
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[4] U. Schreiber, *Higher Prequantum Geometry*, chapter in G. Catren, M. Anel (eds.) *New Spaces for Mathematics and Physics*
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