Quantum Programming via Linear Homotopy Types

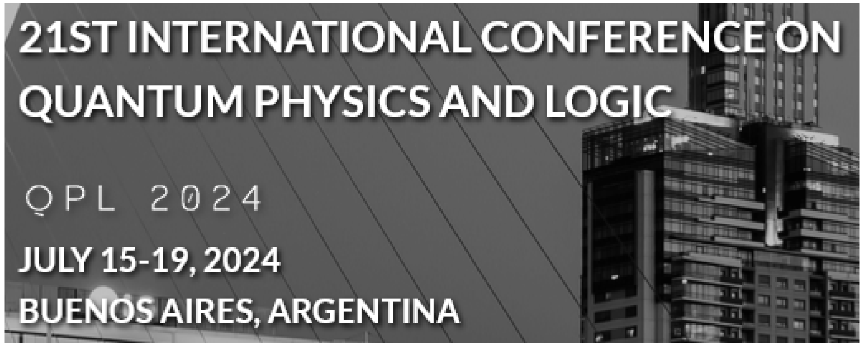
Urs Schreiber

CENTER FOR QUANTUM & TOPOLOGICAL SYSTEMS

جامعة نيويورك أبوظبي

NYU ABU DHABI

talk at:



21ST INTERNATIONAL CONFERENCE ON QUANTUM PHYSICS AND LOGIC

QPL 2024

JULY 15-19, 2024

BUENOS AIRES, ARGENTINA

these slides available at: `ncatlab.org/schreiber/show/QPL2024`

The Gist

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classical logic	**quantum logic**

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traditionally:

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lcc category	tensor category
dependent types	linear types

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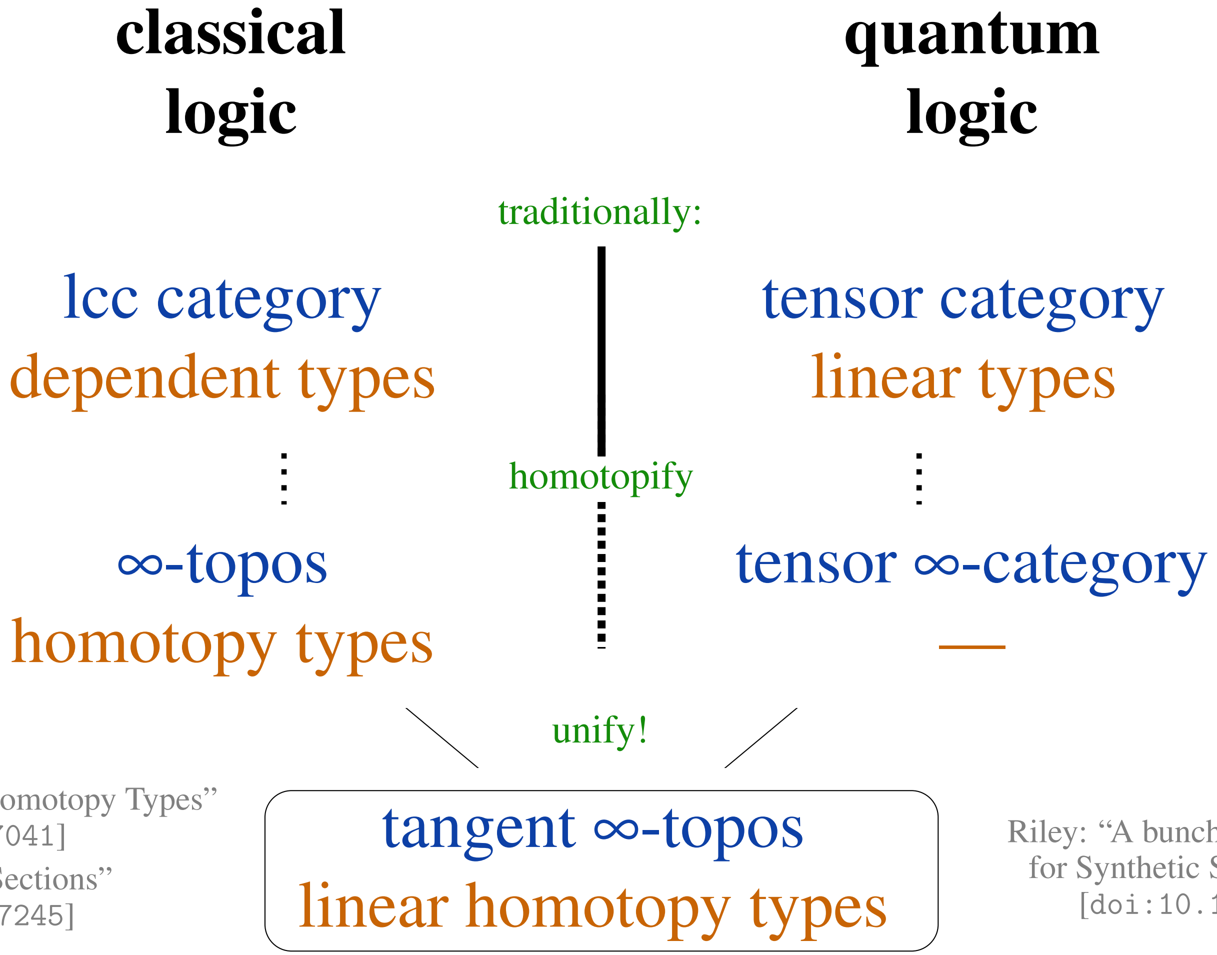
lcc category dependent types		tensor category linear types
	homotopify	
∞-topos homotopy types		tensor ∞-category —

The Gist



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unify!

tangent ∞-topos
linear homotopy types

“Quantization via Linear Homotopy Types”
`[arXiv:1402.7041]`
“Entanglement of Sections”
`[arXiv:2309.07245]`

Riley: “A bunched Homotopy Type Theory
for Synthetic Stable Homotopy Theory”
`[doi:10.14418/wes01.3.139]`

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expresses

“Topological Quantum Gates
in Homotopy Type Theory”
`[arXiv:2303.02382]`

topological gates
type transport

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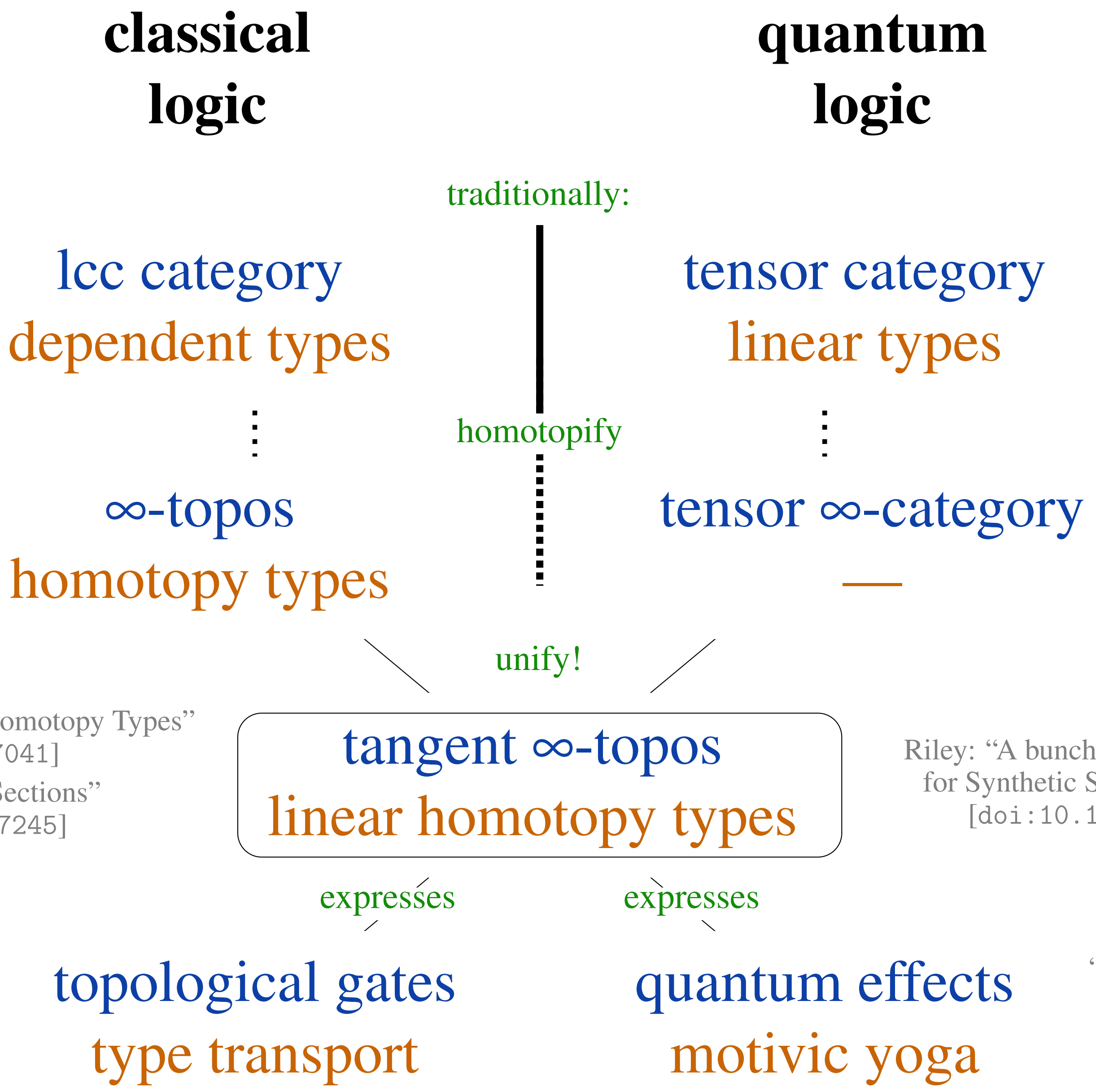

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certification
identity types

Riley: “Linear HoTT as a Quantum Certification Language”
[mvr.hosting.nyu.edu/pubs/translation.pdf]

Algebraic Topology — **Quantum Physics** — **Linear Homotopy Types**

Highlights

Algebraic Topology — Quantum Physics — Linear Homotopy Types

Highlights

Parameterized Quantum Systems

local system of state spaces
i.e., flat vector bundle over
classical parameter space

$$\begin{tikzcd}
\textbf{linear state space bundle} & \mathcal{H} \arrow[r] \arrow[d, "\text{quantum state fibration}"'] & \widehat{\mathrm{LinType}} \arrow[d, "\text{univalent linear fibration}"] & \\
\textbf{parameter space} & X \arrow[r, "\vdash \mathcal{H}"] & \mathrm{LinType} & \textbf{linear type universe}
\end{tikzcd}$$

linear types dependent
on homotopy type

Algebraic Topology — Quantum Physics — Linear Homotopy Types

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Topological Quantum Gates

parallel (adiabatic) **transport**
along path in parameter space

$$\begin{tikzcd}
\mathcal{H}_x \arrow[rr, bend left, "U_p", "\text{gate operator}"'{name=U}] & & \mathcal{H}_y \\
x \arrow[rr, squiggly, "p"{name=P}, "\text{path in parameter space}"'] & & y
\arrow[from=P, to=U, "\text{lift}"']
\end{tikzcd}$$

type transport along
identification in base type

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type transport along
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Quantum Measurement

counit of the
descent comonad
(“necessity” modality)

$$\begin{tikzcd}
\underset{\text{actual world}}{x'} : X \vdash \bigoplus_X \mathbb{C} \arrow[rr, "\sum_x q_x |x\rangle \mapsto q_{x'} |x'\rangle", "\text{state collapse}"'] & & \mathbb{C} \\
\Box \mathbb{1}_X \arrow[rr, "\eta^{\Box}", "\text{necessity counit}"'] \arrow[d] & & \mathbb{1}_X \arrow[d] \\
\underset{\text{possible worlds}}{X} \arrow[rr, equal] & & \underset{\text{measurement basis (finite)}}{X}
\end{tikzcd}$$

eliminator of
dependent product
(here: direct sum)

Default Categorical Semantics

Tech glimpse

Default Categorical Semantics

Naïve Categorical Semantics:

vector bundles over sets
i.e. over discrete spaces
(coproduct completion of Vect)
`Quipper` has semantics here

“Quantum Monadology”
§2.1 [`arXiv:2310.15735`]

Tech glimpse

$$\left\{ \begin{tikzcd} \mathcal{H} \arrow[r, "\text{fiberwise linear map}"] \arrow[d, dash pattern, "\text{quantum state fibration}"'] \arrow[dr, phantom, "\longrightarrow"] & \mathcal{H}' \arrow[d, "\text{quantum state fibration}"] \\ X \arrow[r, "\text{map of base spaces}"'] & X' \end{tikzcd} \right\}$$

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$\hookrightarrow$

∞-Categorical Semantics

$H\mathbb{C}$-module spectra parameterized over ∞-groupoids
(“tangent ∞-topos” $T^{\mathbb{C}}\mathrm{Grpd}_\infty$)
needs model ctgr presentation

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§3 [arXiv:2309.07245]

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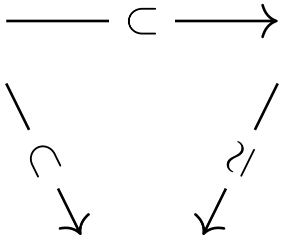

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Model Categorical Semantics:

simplicial chain complexes parameterized over (simplicial) groupoids
(“simplicial local systems”)

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$$\left\{\begin{tikzcd} \mathcal{H} \arrow[r, "\text{fiberwise linear map}"] \arrow[d, "\text{quantum state fibration}"'] \arrow[r, phantom, shift right=8ex, "\longrightarrow"] & \mathcal{H}' \arrow[d, "\text{quantum state fibration}"] \\ X \arrow[r, "\text{map of base spaces}"'] & X' \end{tikzcd}\right\}$$

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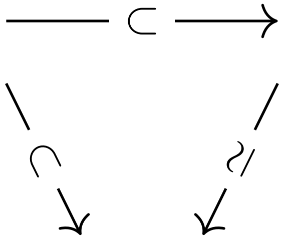

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Tech glimpse

In all cases: the corresponding Grothendieck construction of tensor categories of linear bundles $\mathcal{H}$ over base spaces X:

$$\overbrace{\int_{\underset{\text{...classical base types.}}{X \in \mathrm{ClaType}}}^{\text{...linear types over...}} \mathrm{QuType}_X}^{\substack{\text{all types are...}\\ \mathrm{Type}}} = \left\{ \begin{tikzcd} \mathcal{H} \arrow[rr, "\text{fiberwise linear map}"] \arrow[d, "\text{quantum state fibration}"', ""{name=L}] & & \mathcal{H}' \arrow[d, "\text{quantum state fibration}", ""'{name=R}] \\ X \arrow[rr, "\text{map of base spaces}"'] & & X' \arrow[from=L, to=R, shorten=10mm] \end{tikzcd} \right\}$$

these dependent linear types are “infinitesimally cohesive”

“Diff. cohomology in cohesive ∞-topos”
Prop. 4.1.9 [`arXiv:1310.7930`]

Tech glimpse

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Prop. 4.1.9 `[arXiv:1310.7930]`

modality:
“classically”

$$\begin{tikzcd}
\left[\begin{array}{c} 0_X \\ \downarrow \\ X \end{array}\right] & & X \arrow[ll, mapsto] \\
\text{Type} \arrow[loop left, "\natural"] \arrow[rr, shift right=2] \arrow[rr, phantom, "\bot\ \top"] & & \text{ClaType} \arrow[ll, hook', shift right=2] \\
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Tech glimpse

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Linear HoTT is foremost HoTT
with infntsml cohesive modality $\natural$

Finster et al: “Synthetic Spectra” `[arXiv:2102.04099]`

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Tech glimpse

this allows to *construct*
linear/quantum types

$$\mathrm{QuType} \;\equiv\; \big(\mathcal{H} : \mathrm{Type}\big) \times \big(\natural\mathcal{H} = *\big)$$

modality:
“quantumly”

$$\begin{tikzcd} \left[\begin{array}{c} \mathcal{H} \\ \downarrow \\ * \end{array}\right] & & \mathcal{H} \arrow[ll, mapsto] \\ \mathrm{Type} \arrow[loop left, "\triangleright"] \arrow[rr, shift right=2] & \top & \mathrm{QuType} \arrow[ll, hook', shift right=2] \\ \left[\begin{array}{c} \mathcal{H} \\ \downarrow \\ X \end{array}\right] \arrow[rr, mapsto] & & \bigoplus\limits_{x:X} \mathcal{H}_x \end{tikzcd}$$

these dependent linear types are “infinitesimally cohesive”

“Diff. cohomology in cohesive ∞-topos”
Prop. 4.1.9 `[arXiv:1310.7930]`

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\left[\begin{smallmatrix} \mathcal{H} \\ \downarrow \\ X \end{smallmatrix}\right] \arrow[rr, mapsto] & & \bigoplus\limits_{x:X} \mathcal{H}_x
\end{tikzcd}$$

and a Girard !-modality of linear logic via a Bierman-Benton adjunction

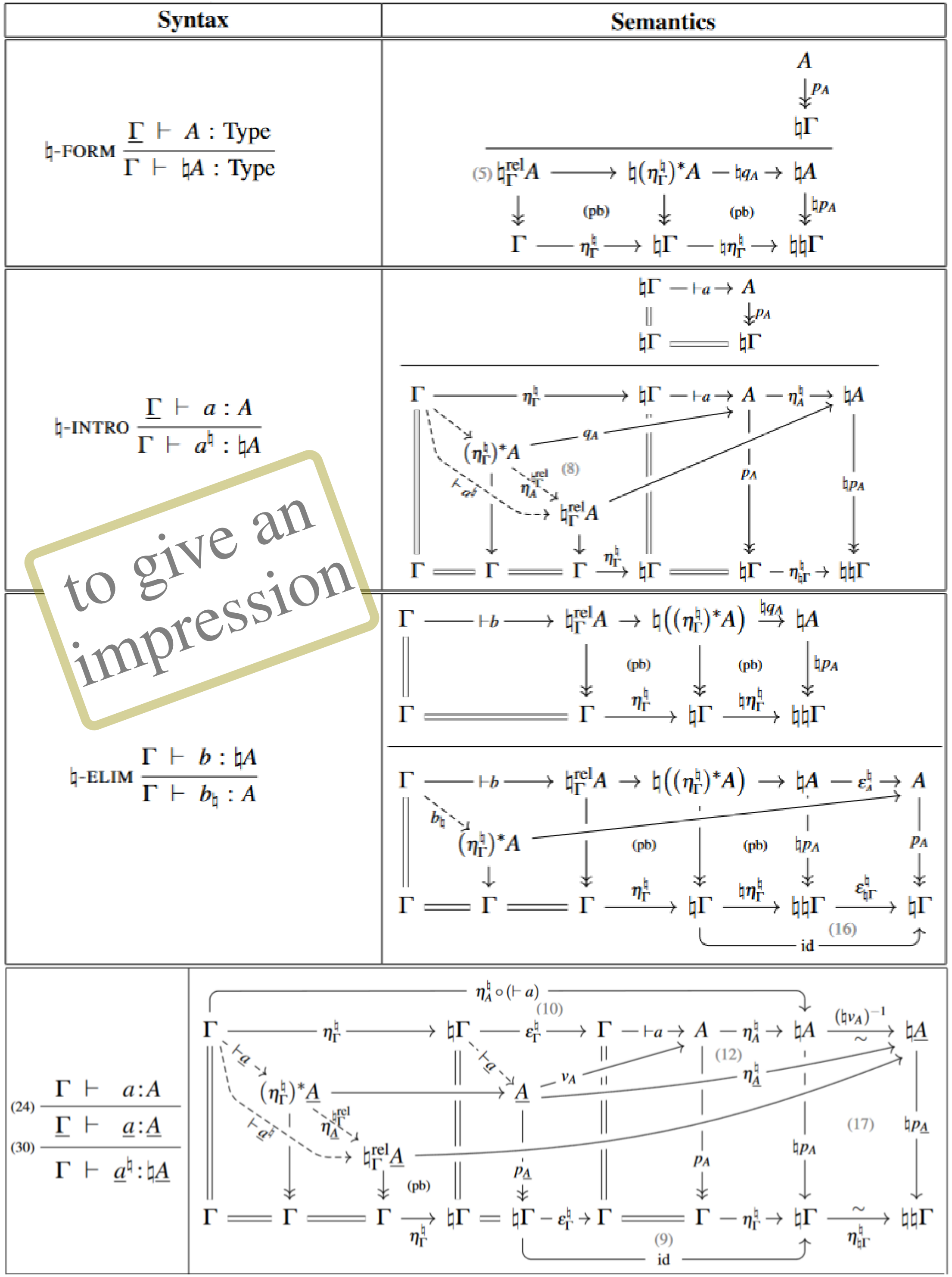
“The Quantum Monadology”
Prop. 2.7 `[arXiv:2310.15735]`

$$\begin{tikzcd}
\mathrm{QuType} \arrow[loop left, "!"] \arrow[rr, hook, shift right=2] \arrow[rr, phantom, "\bot"] \arrow[rrrr, bend right=30, "\text{“\textbf{classicized}”}\ C \,\equiv\, \Omega^\infty_+"'] & & \mathrm{Type} \arrow[ll, shift right=2, "\triangleright"'] \arrow[rr, shift right=2, "\natural(\mathbb{1}\to(\text{-}))"'] \arrow[rr, phantom, "\bot"] & & \mathrm{ClaType} \arrow[ll, shift right=2, "\mathbb{1}\times(\text{-})"'] \arrow[llll, bend right=30, "\text{“\textbf{quantized}”}\ Q \,\equiv\, \Sigma^\infty_+"']
\end{tikzcd}$$

some inference rules

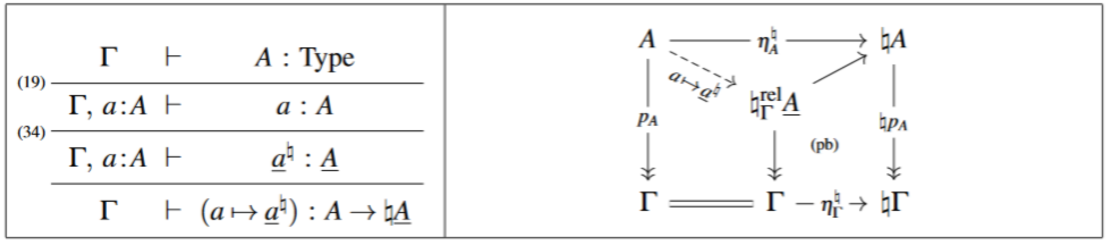
to give an impression

some inference rules for the *classically* modality



Syntax	Semantics
♮-FORM $\dfrac{\underline{\Gamma} \vdash A : \mathrm{Type}}{\Gamma \vdash \natural A : \mathrm{Type}}$	$$\frac{\begin{tikzcd} A \arrow[d, two heads, "p_A"] \\ \natural\Gamma \end{tikzcd}}{\text{(5)}\;\begin{tikzcd} \natural^{\mathrm{rel}}_{\Gamma}A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\text{(pb)}"] & \natural(\eta^{\natural}_{\Gamma})^{*}A \arrow[r, "\natural q_A"] \arrow[d, two heads] \arrow[dr, phantom, "\text{(pb)}"] & \natural A \arrow[d, two heads, "\natural p_A"] \\ \Gamma \arrow[r, "\eta^{\natural}_{\Gamma}"] & \natural\Gamma \arrow[r, "\natural\eta^{\natural}_{\Gamma}"] & \natural\natural\Gamma \end{tikzcd}}$$
♮-INTRO $\dfrac{\underline{\Gamma} \vdash a : A}{\Gamma \vdash a^{\natural} : \natural A}$	$$\frac{\begin{tikzcd} \natural\Gamma \arrow[r, "\vdash a"] \arrow[d, equal] & A \arrow[d, two heads, "p_A"] \\ \natural\Gamma \arrow[r, equal] & \natural\Gamma \end{tikzcd}}{\begin{tikzcd} \Gamma \arrow[rrr, "\eta^{\natural}_{\Gamma}"] \arrow[dddd, equal] \arrow[dr, dashed] \arrow[ddrr, dashed, bend right, "\vdash a^{\natural}"'] & & & \natural\Gamma \arrow[r, "\vdash a"] \arrow[dddd, equal] & A \arrow[r, "\eta^{\natural}_{A}"] \arrow[dddd, two heads, "p_A"] & \natural A \arrow[dddd, two heads, "\natural p_A"] \\ & (\eta^{\natural}_{\Gamma})^{*}A \arrow[urrr, "q_A"] \arrow[dr, dashed, "\eta^{\natural^{\mathrm{rel}}_{\Gamma}}_{A}"] \arrow[ddd] & \text{(8)} & & & \\ & & \natural^{\mathrm{rel}}_{\Gamma}A \arrow[uurrr] \arrow[dd] & & & \\ & & & & & \\ \Gamma \arrow[r, equal] & \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^{\natural}_{\Gamma}"] & \natural\Gamma \arrow[r, equal] & \natural\Gamma \arrow[r, "\eta^{\natural}_{\natural\Gamma}"] & \natural\natural\Gamma \end{tikzcd}}$$
♮-ELIM $\dfrac{\Gamma \vdash b : \natural A}{\Gamma \vdash b_{\natural} : A}$	$$\frac{\begin{tikzcd} \Gamma \arrow[r, "\vdash b"] \arrow[d, equal] & \natural^{\mathrm{rel}}_{\Gamma}A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\text{(pb)}"] & \natural\big((\eta^{\natural}_{\Gamma})^{*}A\big) \arrow[r, "\natural q_A"] \arrow[d, two heads] \arrow[dr, phantom, "\text{(pb)}"] & \natural A \arrow[d, two heads, "\natural p_A"] \\ \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^{\natural}_{\Gamma}"] & \natural\Gamma \arrow[r, "\natural\eta^{\natural}_{\Gamma}"] & \natural\natural\Gamma \end{tikzcd}}{\begin{tikzcd} \Gamma \arrow[rr, "\vdash b"] \arrow[dd, equal] \arrow[dr, dashed, "b_{\natural}"'] & & \natural^{\mathrm{rel}}_{\Gamma}A \arrow[r] \arrow[dd, two heads] \arrow[ddr, phantom, "\text{(pb)}"] & \natural\big((\eta^{\natural}_{\Gamma})^{*}A\big) \arrow[r] \arrow[dd, two heads] \arrow[ddr, phantom, "\text{(pb)}"] & \natural A \arrow[r, "\varepsilon^{\natural}_{A}"] \arrow[dd, two heads, "\natural p_A"] & A \arrow[dd, two heads, "p_A"] \\ & (\eta^{\natural}_{\Gamma})^{*}A \arrow[urrrr] \arrow[d] & & & & \\ \Gamma \arrow[r, equal] & \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^{\natural}_{\Gamma}"] & \natural\Gamma \arrow[r, "\natural\eta^{\natural}_{\Gamma}"] \arrow[rr, bend right, "\mathrm{id}"', "\text{(16)}"] & \natural\natural\Gamma \arrow[r, "\varepsilon^{\natural}_{\natural\Gamma}"] & \natural\Gamma \end{tikzcd}}$$
$\begin{array}{rc} & \Gamma \vdash a : A \\ \cline{2-2} \text{(24)} & \underline{\Gamma} \vdash \underline{a} : \underline{A} \\ \cline{2-2} \text{(30)} & \Gamma \vdash \underline{a}^{\natural} : \natural\underline{A} \end{array}$	$$\begin{tikzcd} \Gamma \arrow[rrr, "\eta^{\natural}_{\Gamma}"] \arrow[rrrrrrr, bend left, "\eta^{\natural}_{A} \circ (\vdash a)"] \arrow[ddd, equal] \arrow[dr, dashed, "\vdash \underline{a}"] \arrow[ddrr, dashed, bend right, "\vdash \underline{a}^{\natural}"'] & & & \natural\Gamma \arrow[rr, "\varepsilon^{\natural}_{\Gamma}", "\text{(10)}"'] \arrow[ddd, equal] \arrow[dr, dashed, "\vdash \underline{a}"] & & \Gamma \arrow[r, "\vdash a"] \arrow[ddd, equal] & A \arrow[r, "\eta^{\natural}_{A}"] \arrow[ddd, two heads, "p_A"] & \natural A \arrow[r, "(\natural\nu_A)^{-1}", "\sim"'] \arrow[ddd, "\natural p_A"] & \natural\underline{A} \arrow[ddd, "\natural p_{\underline{A}}"] \\ & (\eta^{\natural}_{\Gamma})^{*}\underline{A} \arrow[rrr] \arrow[dd, two heads] \arrow[dr, dashed, "\eta^{\natural^{\mathrm{rel}}_{\Gamma}}_{\underline{A}}"] & & & \underline{A} \arrow[urr, "\nu_A"] \arrow[urrrr, "\eta^{\natural}_{\underline{A}}", "\text{(12)}"'] \arrow[dd, two heads, "p_{\underline{A}}"] & & & \text{(17)} & \\ & & \natural^{\mathrm{rel}}_{\Gamma}\underline{A} \arrow[uurrrrrr] \arrow[d, two heads] \arrow[dr, phantom, "\text{(pb)}"] & & & & & & \\ \Gamma \arrow[r, equal] & \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^{\natural}_{\Gamma}"'] & \natural\Gamma \arrow[r, equal] & \natural\Gamma \arrow[r, "\varepsilon^{\natural}_{\Gamma}"] \arrow[rrr, bend right, "\mathrm{id}"', "\text{(9)}"] & \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^{\natural}_{\Gamma}"] & \natural\Gamma \arrow[r, "\sim", "\eta^{\natural}_{\natural\Gamma}"'] & \natural\natural\Gamma \end{tikzcd}$$

to give an impression



$\begin{array}{rc} & \Gamma \;\vdash\; A : \mathrm{Type} \\ \cline{2-2} \text{(19)} & \Gamma, a{:}A \;\vdash\; a : A \\ \cline{2-2} \text{(34)} & \Gamma, a{:}A \;\vdash\; \underline{a}^{\natural} : \underline{A} \\ \cline{2-2} & \Gamma \;\vdash\; (a \mapsto \underline{a}^{\natural}) : A \to \natural\underline{A} \end{array}$	$$\begin{tikzcd} A \arrow[rr, "\eta^{\natural}_{A}"] \arrow[dr, dashed, "a \mapsto \underline{a}^{\natural}"] \arrow[dd, two heads, "p_A"] & & \natural A \arrow[dd, two heads, "\natural p_A"] \\ & \natural^{\mathrm{rel}}_{\Gamma}\underline{A} \arrow[ur] \arrow[d, two heads] \arrow[dr, phantom, "\text{(pb)}"] & \\ \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^{\natural}_{\Gamma}"] & \natural\Gamma \end{tikzcd}$$

Schreiber@QPL2024: "Quantum Programming via Linear Homotopy types"

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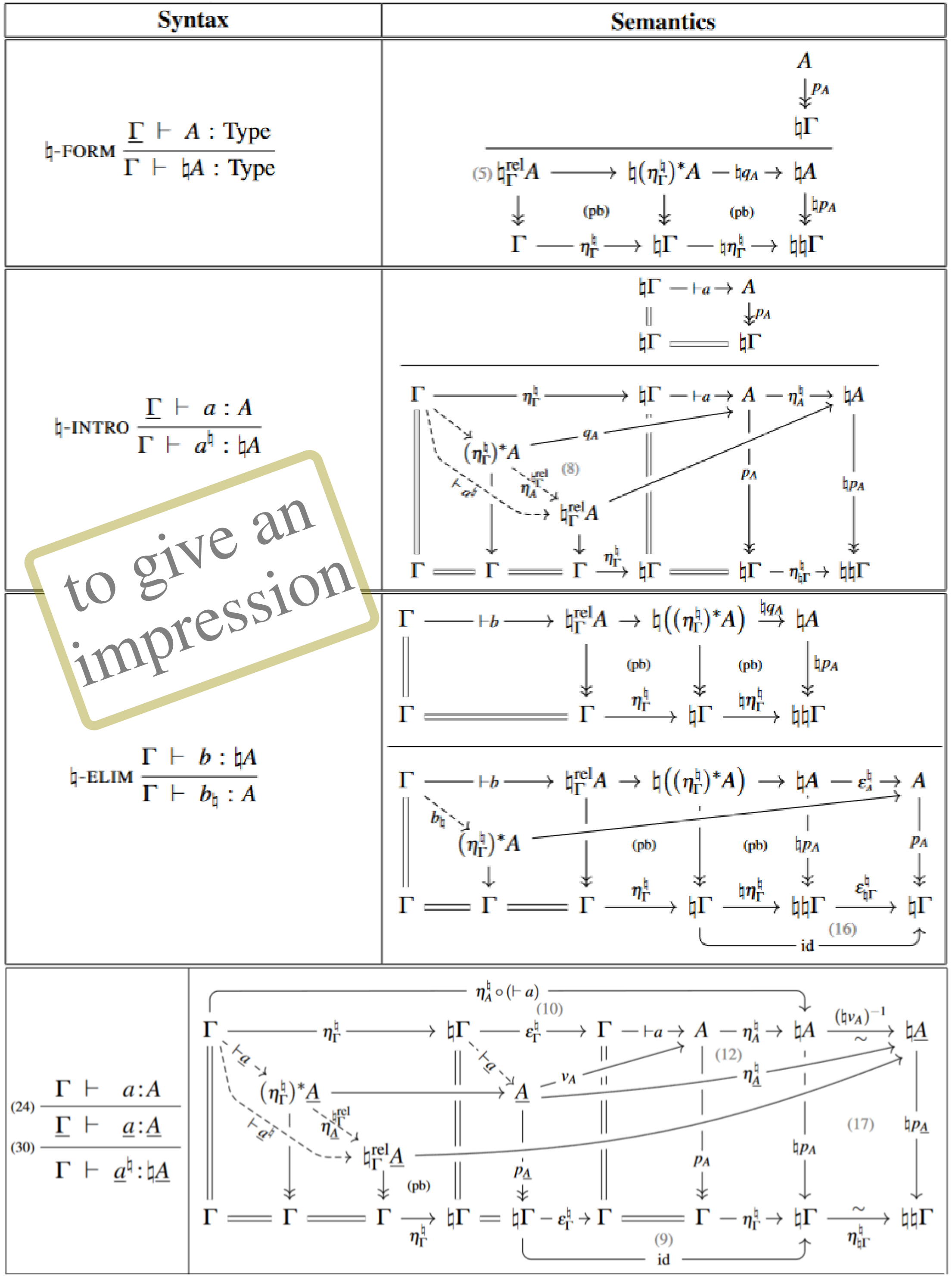
TOPOLOGICAL
SYSTEMS



Syntax | **Semantics**

$$\natural\text{-FORM}\ \frac{\Gamma \vdash A : \mathrm{Type}}{\Gamma \vdash \natural A : \mathrm{Type}}$$

$$\begin{tikzcd} A \arrow[d, two heads, "p_A"] \\ \natural\Gamma \end{tikzcd}$$

$$\begin{tikzcd} (5)\ \natural^{\mathrm{rel}}_{\Gamma}A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "(\mathrm{pb})"] & \natural(\eta^\natural_\Gamma)^*A \arrow[r, "\natural q_A"] \arrow[d, two heads] \arrow[dr, phantom, "(\mathrm{pb})"] & \natural A \arrow[d, two heads, "\natural p_A"] \\ \Gamma \arrow[r, "\eta^\natural_\Gamma"] & \natural\Gamma \arrow[r, "\natural\eta^\natural_\Gamma"] & \natural\natural\Gamma \end{tikzcd}$$

$$\natural\text{-INTRO}\ \frac{\Gamma \vdash a : A}{\Gamma \vdash a^\natural : \natural A}$$

$$\begin{tikzcd} \natural\Gamma \arrow[r, "\vdash a"] \arrow[d, equal] & A \arrow[d, two heads, "p_A"] \\ \natural\Gamma \arrow[r, equal] & \natural\Gamma \end{tikzcd}$$

$$\begin{tikzcd} \Gamma \arrow[rrr, "\eta^\natural_\Gamma"] \arrow[dddd, equal] \arrow[dr, dashed] \arrow[ddrr, dashed, bend right, "\vdash a^\natural"'] & & & \natural\Gamma \arrow[r, "\vdash a"] \arrow[dddd, equal] & A \arrow[r, "\eta^\natural_A"] \arrow[dddd, two heads, "p_A"] & \natural A \arrow[dddd, two heads, "\natural p_A"] \\ & (\eta^\natural_\Gamma)^*A \arrow[urrr, "q_A"] \arrow[dr, dashed, "\eta^{\natural^{\mathrm{rel}}_\Gamma}_A"] \arrow[ddd] & \text{(8)} & & & \\ & & \natural^{\mathrm{rel}}_\Gamma A \arrow[uurrr] \arrow[dd] & & & \\ & & & & & \\ \Gamma \arrow[r, equal] & \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^\natural_\Gamma"] & \natural\Gamma \arrow[r, equal] & \natural\Gamma \arrow[r, "\eta^\natural_{\natural\Gamma}"] & \natural\natural\Gamma \end{tikzcd}$$

to give an impression

$$\natural\text{-ELIM}\ \frac{\Gamma \vdash b : \natural A}{\Gamma \vdash b_\natural : A}$$

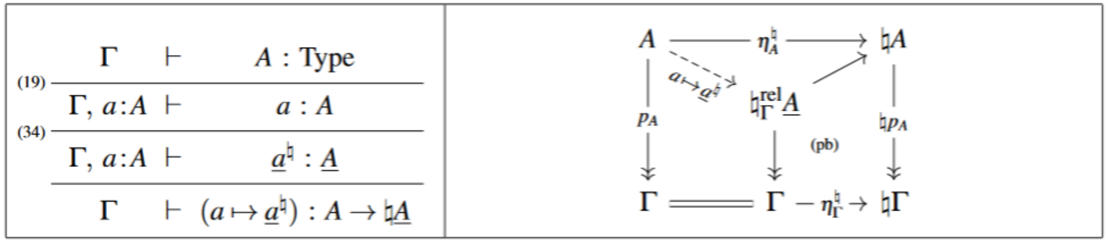
$$\begin{tikzcd} \Gamma \arrow[r, "\vdash b"] \arrow[d, equal] & \natural^{\mathrm{rel}}_\Gamma A \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "(\mathrm{pb})"] & \natural\big((\eta^\natural_\Gamma)^*A\big) \arrow[r, "\natural q_A"] \arrow[d, two heads] \arrow[dr, phantom, "(\mathrm{pb})"] & \natural A \arrow[d, two heads, "\natural p_A"] \\ \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^\natural_\Gamma"] & \natural\Gamma \arrow[r, "\natural\eta^\natural_\Gamma"] & \natural\natural\Gamma \end{tikzcd}$$

$$\begin{tikzcd} \Gamma \arrow[rr, "\vdash b"] \arrow[dd, equal] \arrow[dr, dashed, "b_\natural"'] & & \natural^{\mathrm{rel}}_\Gamma A \arrow[r] \arrow[dd, two heads] \arrow[ddr, phantom, "(\mathrm{pb})"] & \natural\big((\eta^\natural_\Gamma)^*A\big) \arrow[r] \arrow[dd, two heads] \arrow[ddr, phantom, "(\mathrm{pb})"] & \natural A \arrow[r, "\varepsilon^\natural_A"] \arrow[dd, two heads, "\natural p_A"] & A \arrow[dd, two heads, "p_A"] \\ & (\eta^\natural_\Gamma)^*A \arrow[urrrr] \arrow[d] & & & & \\ \Gamma \arrow[r, equal] & \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^\natural_\Gamma"] & \natural\Gamma \arrow[r, "\natural\eta^\natural_\Gamma"] \arrow[rr, bend right, "\mathrm{id}"', "\text{(16)}"] & \natural\natural\Gamma \arrow[r, "\varepsilon^\natural_{\natural\Gamma}"] & \natural\Gamma \end{tikzcd}$$

$$\begin{array}{l} \text{(24)}\ \dfrac{\Gamma \vdash a : A}{\Gamma \vdash \underline{a} : \underline{A}} \\ \text{(30)}\ \dfrac{}{\Gamma \vdash \underline{a}^\natural : \natural\underline{A}} \end{array}$$

$$\begin{tikzcd} \Gamma \arrow[rrrrrrr, bend left, "\eta^\natural_A \circ (\vdash a)", "\text{(10)}"'] \arrow[rrr, "\eta^\natural_\Gamma"] \arrow[dddd, equal] \arrow[dr, dashed, "\vdash a"] \arrow[ddrr, dashed, bend right, "\vdash \underline{a}^\natural"'] & & & \natural\Gamma \arrow[rr, "\varepsilon^\natural_\Gamma"] \arrow[dddd, equal] \arrow[dr, dashed, "\vdash a"] & & \Gamma \arrow[r, "\vdash a"] \arrow[dddd, equal] & A \arrow[r, "\eta^\natural_A"] \arrow[dddd, two heads, "p_A"] & \natural A \arrow[r, "(\natural\nu_A)^{-1}", "\sim"'] \arrow[dddd, "\natural p_A"] & \natural\underline{A} \arrow[dddd, "\natural p_{\underline{A}}"] \\ & (\eta^\natural_\Gamma)^*\underline{A} \arrow[rrr] \arrow[dddd, two heads] \arrow[dr, dashed, "\eta^{\natural^{\mathrm{rel}}_\Gamma}_{\underline{A}}"] & & & \underline{A} \arrow[urr, "\nu_A"] \arrow[urrrr, "\eta^\natural_{\underline{A}}", "\text{(12)}"'] \arrow[ddd, two heads, "p_{\underline{A}}"] & & & \text{(17)} & \\ & & \natural^{\mathrm{rel}}_\Gamma\underline{A} \arrow[uurrrrrr] \arrow[dd, two heads] \arrow[ddr, phantom, "(\mathrm{pb})"] & & & & & & \\ & & & & & & & & \\ \Gamma \arrow[r, equal] & \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^\natural_\Gamma"'] & \natural\Gamma \arrow[r, equal] & \natural\Gamma \arrow[r, "\varepsilon^\natural_\Gamma"] \arrow[rrr, bend right, "\mathrm{id}"', "\text{(9)}"] & \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^\natural_\Gamma"] & \natural\Gamma \arrow[r, "\sim", "\eta^\natural_{\natural\Gamma}"'] & \natural\natural\Gamma \end{tikzcd}$$

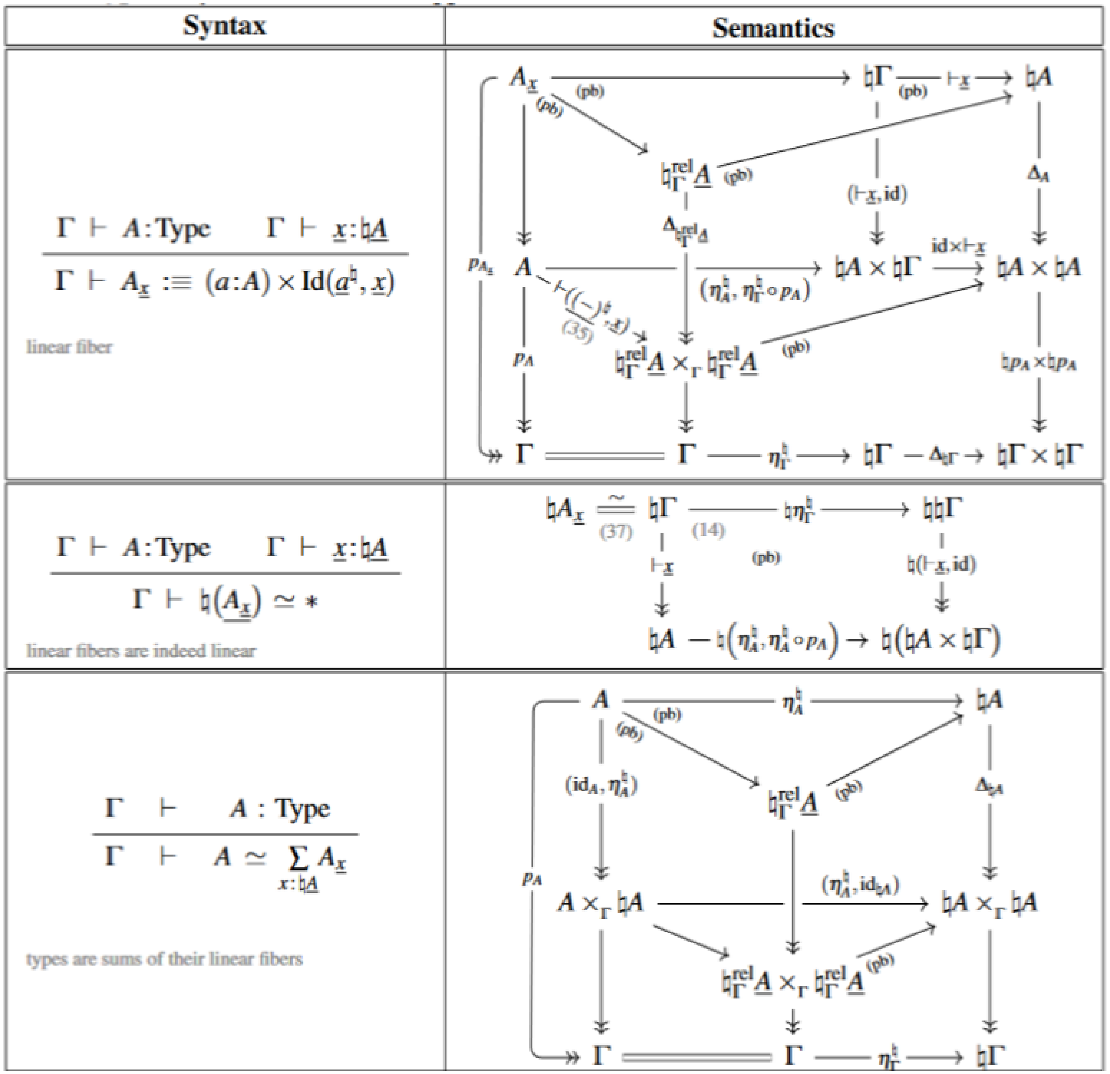
some inference rules for the *classically* modality



$$\begin{array}{rl} & \Gamma \vdash A : \mathrm{Type} \\ \text{(19)} & \overline{\Gamma,\ a{:}A \vdash a : A} \\ \text{(34)} & \overline{\Gamma,\ a{:}A \vdash \underline{a}^\natural : \underline{A}} \\ & \overline{\Gamma \vdash (a \mapsto \underline{a}^\natural) : A \to \natural\underline{A}} \end{array}$$

$$\begin{tikzcd} A \arrow[rr, "\eta^\natural_A"] \arrow[dr, dashed, "a \mapsto \underline{a}^\natural"] \arrow[dd, two heads, "p_A"] & & \natural A \arrow[dd, two heads, "\natural p_A"] \\ & \natural^{\mathrm{rel}}_\Gamma\underline{A} \arrow[ur] \arrow[d, two heads] \arrow[dr, phantom, "(\mathrm{pb})"] & \\ \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^\natural_\Gamma"] & \natural\Gamma \end{tikzcd}$$

some inference rules for linear types:



Syntax | **Semantics**

$$\frac{\Gamma \vdash A{:}\mathrm{Type} \qquad \Gamma \vdash \underline{x}{:}\natural A}{\Gamma \vdash A_{\underline{x}} :\equiv (a{:}A) \times \mathrm{Id}(\underline{a}^\natural, \underline{x})}$$

linear fiber

$$\begin{tikzcd} A_{\underline{x}} \arrow[rrr] \arrow[dr, "(\mathrm{pb})"] \arrow[dd, two heads] \arrow[dddd, bend right, two heads, "p_{A_{\underline{x}}}"'] & (\mathrm{pb}) & & \natural\Gamma \arrow[r, "\vdash \underline{x}", "(\mathrm{pb})"'] \arrow[dd, "(\vdash\underline{x},\mathrm{id})"] & \natural A \arrow[dd, two heads, "\Delta_A"] \\ & \natural^{\mathrm{rel}}_\Gamma\underline{A} \arrow[urrr, "(\mathrm{pb})"'] \arrow[dd, "\Delta_{\natural^{\mathrm{rel}}_\Gamma\underline{A}}"] & & & \\ A \arrow[rrr, "(\eta^\natural_A,\, \eta^\natural_\Gamma \circ p_A)"'] \arrow[dr, "\vdash((-)^\natural,\underline{x})", "\text{(35)}"'] \arrow[dd, two heads, "p_A"] & & & \natural A \times \natural\Gamma \arrow[r, "\mathrm{id}\times\vdash\underline{x}"] & \natural A \times \natural A \arrow[dd, two heads, "\natural p_A \times \natural p_A"] \\ & \natural^{\mathrm{rel}}_\Gamma\underline{A} \times_\Gamma \natural^{\mathrm{rel}}_\Gamma\underline{A} \arrow[urrr, "(\mathrm{pb})"] \arrow[d, two heads] & & & \\ \Gamma \arrow[r, equal] & \Gamma \arrow[rr, "\eta^\natural_\Gamma"] & & \natural\Gamma \arrow[r, "\Delta_{\natural\Gamma}"] & \natural\Gamma \times \natural\Gamma \end{tikzcd}$$

$$\frac{\Gamma \vdash A{:}\mathrm{Type} \qquad \Gamma \vdash \underline{x}{:}\natural A}{\Gamma \vdash \natural(A_{\underline{x}}) \simeq *}$$

linear fibers are indeed linear

$$\begin{tikzcd} \natural A_{\underline{x}} \arrow[r, equal, "\sim", "\text{(37)}"'] & \natural\Gamma \arrow[r, "\natural\eta^\natural_\Gamma", "\text{(14)}"'] \arrow[d, two heads, "\vdash\underline{x}"] \arrow[dr, phantom, "(\mathrm{pb})"] & \natural\natural\Gamma \arrow[d, two heads, "\natural(\vdash\underline{x},\mathrm{id})"] \\ & \natural A \arrow[r, "\natural(\eta^\natural_A,\, \eta^\natural_A \circ p_A)"] & \natural(\natural A \times \natural\Gamma) \end{tikzcd}$$

$$\frac{\Gamma \vdash A : \mathrm{Type}}{\Gamma \vdash A \simeq \sum\limits_{x:\natural A} A_{\underline{x}}}$$

types are sums of their linear fibers

$$\begin{tikzcd} A \arrow[rr, "\eta^\natural_A"] \arrow[dr, "(\mathrm{pb})"] \arrow[d, two heads, "(\mathrm{id}_A,\, \eta^\natural_A)"] \arrow[dd, bend right, two heads, "p_A"'] & (\mathrm{pb}) & \natural A \arrow[d, two heads, "\Delta_{\natural A}"] \\ A \times_\Gamma \natural A \arrow[rr, "(\eta^\natural_A,\, \mathrm{id}_{\natural A})"] \arrow[dr] \arrow[d, two heads] & \natural^{\mathrm{rel}}_\Gamma\underline{A} \arrow[ur, "(\mathrm{pb})"'] \arrow[d, two heads] & \natural A \times_\Gamma \natural A \arrow[d, two heads] \\ & \natural^{\mathrm{rel}}_\Gamma\underline{A} \times_\Gamma \natural^{\mathrm{rel}}_\Gamma\underline{A} \arrow[ur, "(\mathrm{pb})"'] \arrow[d, two heads] & \\ \Gamma \arrow[r, equal] & \Gamma \arrow[r, "\eta^\natural_\Gamma"] & \natural\Gamma \end{tikzcd}$$

key: **quantum measurement** effect handling captured by dependent linear type formation

miracle
occurs

“Quantum Monadology”
§2 [`arXiv:2310.15735`]

miracle
occurs

key: **quantum measurement** effect handling captured by dependent linear type formation

recall: given a context extension

classical type $X \twoheadrightarrow *$

then dependent sum and dependent product form the base change adjoint triple, where...

dependent sum

X-dependent... ...classical types

$$\begin{tikzcd}[column sep=huge]
\mathrm{ClaType}_X \arrow[r, shift left=4, "\Sigma_X"] \arrow[r, shift left=2, phantom, "\perp"] \arrow[r, shift right=2, phantom, "\perp"'] \arrow[r, shift right=4, "\Pi_X"'] & \mathrm{ClaType} \arrow[l, "*_X \times (\text{-})" description]
\end{tikzcd}$$

dependent product

...the induced (co-)modalities exhibit S5 modal logic

“Quantum Monadology”
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miracle occurs

key: **quantum measurement** effect handling captured by dependent linear type formation

recall: given a context extension

$$\begin{tikzcd} \text{classical type } X \arrow[rr, two heads] & & * \end{tikzcd}$$

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$$\begin{tikzcd}[column sep=huge] \overset{X\text{-dependent...}}{\text{ClaType}_X} \arrow[r, shift left=10, "\overset{\textbf{dependent sum}}{\Sigma_X}"] \arrow[r, phantom, shift left=5, "\perp"] \arrow[r, phantom, shift right=5, "\perp"] \arrow[r, shift right=10, "\underset{\textbf{dependent product}}{\Pi_X}"'] & \overset{\text{...classical types}}{\text{ClaType}} \arrow[l, "*_X \times (\text{-})"'] \end{tikzcd}$$

...the induced (co-)modalities exhibit S5 modal logic

---

now for dependent *linear* types over finite bases this is the direct sum...

$$\begin{tikzcd}[column sep=huge] \overset{X\text{-dependent...}}{\text{QuType}_X} \arrow[loop above left, "\Diamond_X"] \arrow[loop left, phantom, "\wr"] \arrow[loop below left, "\Box_X"'] \arrow[r, shift left=10, "\overset{\textbf{direct sum}}{\oplus_X}"] \arrow[r, phantom, shift left=5, "\perp"] \arrow[r, phantom, shift right=5, "\perp"] \arrow[r, shift right=10, "\underset{\textbf{direct sum}}{\oplus_X}"'] & \overset{\text{...linear types}}{\text{QuType}} \arrow[l, "\mathbb{1}_X \otimes (\text{-})"'] \arrow[loop above right, "\star_X"] \arrow[loop right, phantom, "\wr"] \arrow[loop below right, "\bigcirc_X"'] \end{tikzcd}$$

X finite:

... which yields pair of Frobenius monads

“Quantum Monadology”
§2 [`arXiv:2310.15735`]

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TOPOLOGICAL
SYSTEMS

miracle
occurs

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---

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$$\begin{tikzcd}[column sep=huge]
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\end{tikzcd}$$

then dependent sum and dependent product form the base change adjoint triple, where...

$$\begin{tikzcd}[column sep=huge]
\overset{X\text{-dependent...}}{\mathrm{ClaType}_X}
\arrow[rr, shift left=20pt, "\overset{\textbf{dependent sum}}{\Sigma_X}"]
\arrow[rr, phantom, shift left=10pt, "\perp"]
\arrow[rr, phantom, shift right=10pt, "\perp"]
\arrow[rr, shift right=20pt, "\underset{\textbf{dependent product}}{\Pi_X}"']
&& \overset{\text{...classical types}}{\mathrm{ClaType}}
\arrow[ll, "*_X \times (\text{-})" description]
\end{tikzcd}$$

...the induced (co-)modalities exhibit S5 modal logic

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X finite:

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\overset{X\text{-dependent...}}{\mathrm{QuType}_X}
\arrow[loop above, "\Diamond_X"]
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\arrow[loop right, phantom, "\wr"]
\end{tikzcd}$$

... which yields pair of Frobenius monads

⇒ **quantum necessity monad**
models “dynamic lifting”
of quantum measurement;
Kleisli equivalence proves
deferred msrmnt principle

“Quantum Monadology”
§2 [`arXiv:2310.15735`]

quantum indefiniteness monad
$\simeq$ Coecke’s measurement monad
as now used in `xz-calculus`

this way, quantum-logic & -effects
are incarnated in homotopy type theory
by a **system of modalities/monads**

1st punchline

this way, quantum-logic & -effects
are incarnated in homotopy type theory
by a **system of modalities/monads**

axiomatized:

Riley: “A bunched HoTT”
`[doi:10.14418/wes01.3.139]`

- $\natural$ classically
- $\otimes$ entangled

induced:

“Quantum Monadology”
§2 `[arXiv:2310.15735]`

- $\triangleright$ quantumly
- $!$ 2^{nd} quantized
- $\Box$ necessarily
- $\bigcirc$ indefinitely

quantum
measurement
effects

1^{st} punchline

“Quantum Monadology”
§3 `[arXiv:2310.15735]`

this way, quantum-logic & -effects are incarnated in homotopy type theory by a **system of modalities/monads**

via do-notation for monadic effects this permits a domain specific **quantum language** to be embedded into LHoTT

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- $\natural$ classically
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induced:

“Quantum Monadology” §2 [arXiv:2310.15735]

- $\triangleright$ quantumly
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- $\square$ necessarily
- $\bigcirc$ indefinitely

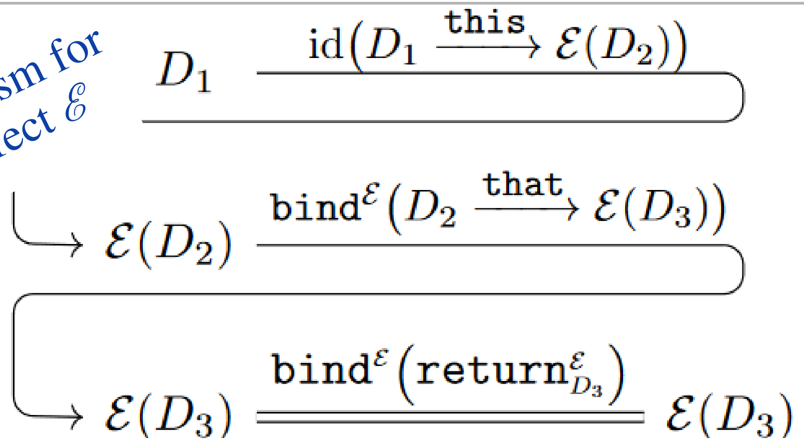
$\square$ necessarily, $\bigcirc$ indefinitely: quantum measurement effects

1^{st} punchline

“Quantum Monadology” §3 [arXiv:2310.15735]

“Quantum Monadology” Lit 1.19 [arXiv:2310.15735]

Kleisli morphism for monadic effect $\mathcal{E}$



D_1 $\quad \mathrm{id}\big(D_1 \xrightarrow{\texttt{this}} \mathcal{E}(D_2)\big)$

$\mathcal{E}(D_2)$ $\quad \texttt{bind}^{\mathcal{E}}\big(D_2 \xrightarrow{\texttt{that}} \mathcal{E}(D_3)\big)$

$\mathcal{E}(D_3)$ $\quad \texttt{bind}^{\mathcal{E}}\big(\texttt{return}^{\mathcal{E}}_{D_3}\big)$ $\quad \mathcal{E}(D_3)$

corresponding do-notation

prog : $D_1 \to \mathcal{E}(D_3)$

prog $\equiv$

```
for d2 in this d1
do
    for d3 in that d2
    do return d3
```

this way, quantum-logic & -effects are incarnated in homotopy type theory by a **system of modalities/monads**

axiomatized:

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- $\natural$ classically
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induced:

"Quantum Monadology" §2 [arXiv:2310.15735]

- $\triangleright$ quantumly
- ! 2^{nd} quantized
- $\square$ necessarily
- $\bigcirc$ indefinitely

necessarily, indefinitely: quantum measurement effects

via do-notation for monadic effects this permits a domain specific **quantum language** to be embedded into LHoTT

e.g. snippet from embedded error-correction peudocode:

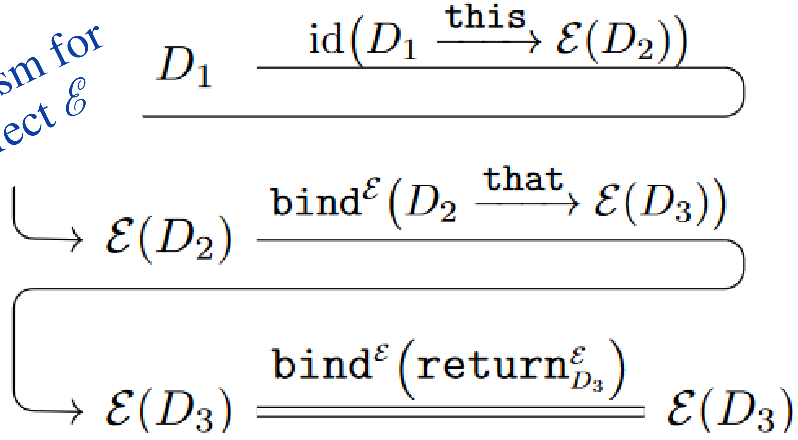
```
$\mathrm{correct\_error} \;⨾\; \mathrm{LgclQBit} \multimap \bigcirc_{\mathrm{Syndrome}} \mathrm{LgclQBit}$

$\mathrm{correct\_error} \equiv$
  for $|b_1, b_2, b_3\rangle$
  do
    for $|\psi\rangle$ in $\mathrm{measure\_syndrome}(|b_1, b_2, b_3\rangle)$
    do
      if measured $(s_1, s_2)$
      then $\mathrm{BitFlip}_{(s_1, s_2)}|\psi\rangle$
```

"Quantum Monadology" §3 [arXiv:2310.15735]

1^{st} punchline

"Quantum Monadology" Lit 1.19 [arXiv:2310.15735]

Kleisli morphism for monadic effect $\mathcal{E}$



$$D_1 \quad \mathrm{id}\big(D_1 \xrightarrow{\texttt{this}} \mathcal{E}(D_2)\big)$$

$$\hookrightarrow \mathcal{E}(D_2) \quad \mathtt{bind}^{\mathcal{E}}\big(D_2 \xrightarrow{\texttt{that}} \mathcal{E}(D_3)\big)$$

$$\hookrightarrow \mathcal{E}(D_3) \quad \xlongequal{\mathtt{bind}^{\mathcal{E}}\left(\mathtt{return}^{\mathcal{E}}_{D_3}\right)} \quad \mathcal{E}(D_3)$$

corresponding do-notation

```
$\mathrm{prog} \;:\; D_1 \to \mathcal{E}(D_3)$

$\mathrm{prog} \equiv$
  for d2 in this d1
  do
    for d3 in that d2
    do return d3
```

beyond offering certification for traditional quantum languages (e.g. `Quipper`)

the homotopy-typing reflects
topological quantum gates

2nd punchline

Riley: “Linear HoTT as a Quantum Certification Language”
[`mvr.hosting.nyu.edu/pubs/translation.pdf`]

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the homotopy-typing reflects
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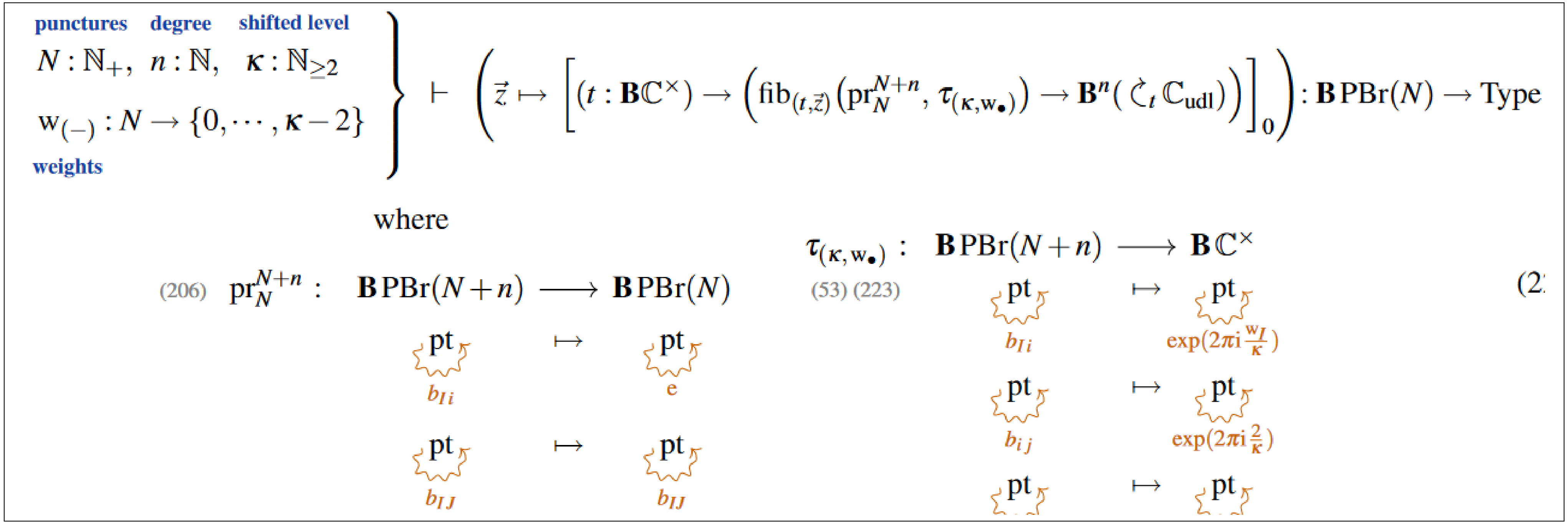
2^{nd} punchline

“Topological Quantum Gates
in Homotopy Type Theory”
Thm. 6.8 [`arXiv:2303.02382`]

Theorem: *Type transport in the following dependent type family expresses, in the above simplicial categorical semantics, topologial braid quantum gates of* $\mathfrak{su}(2)$*-anyons (Majorana, Fibonacci, ...)*



punctures **degree** **shifted level**

weights

$$\left.\begin{array}{l} N : \mathbb{N}_+,\;\; n : \mathbb{N},\;\; \kappa : \mathbb{N}_{\geq 2} \\ \mathrm{w}_{(-)} : N \to \{0, \cdots, \kappa - 2\} \end{array}\right\} \vdash \left(\vec{z} \mapsto \Big[(t : \mathbf{B}\mathbb{C}^\times) \to \Big(\mathrm{fib}_{(t,\vec{z})}\big(\mathrm{pr}^{N+n}_{N},\, \tau_{(\kappa,\mathrm{w}_\bullet)}\big) \to \mathbf{B}^n\big(\circlearrowleft_t \mathbb{C}_{\mathrm{udl}}\big)\Big)\Big]_0\right) : \mathbf{B}\,\mathrm{PBr}(N) \to \mathrm{Type}$$

where

(206) $$\mathrm{pr}^{N+n}_{N} : \begin{array}{ccc} \mathbf{B}\,\mathrm{PBr}(N+n) & \longrightarrow & \mathbf{B}\,\mathrm{PBr}(N) \\ \underset{b_{Ii}}{\mathrm{pt}} & \mapsto & \underset{\mathrm{e}}{\mathrm{pt}} \\ \underset{b_{IJ}}{\mathrm{pt}} & \mapsto & \underset{b_{IJ}}{\mathrm{pt}} \end{array}$$

(53) (223) $$\tau_{(\kappa,\mathrm{w}_\bullet)} : \begin{array}{ccc} \mathbf{B}\,\mathrm{PBr}(N+n) & \longrightarrow & \mathbf{B}\,\mathbb{C}^\times \\ \underset{b_{Ii}}{\mathrm{pt}} & \mapsto & \underset{\exp(2\pi\mathrm{i}\frac{\mathrm{w}_I}{\kappa})}{\mathrm{pt}} \\ \underset{b_{ij}}{\mathrm{pt}} & \mapsto & \underset{\exp(2\pi\mathrm{i}\frac{2}{\kappa})}{\mathrm{pt}} \\ \mathrm{pt} & \mapsto & \mathrm{pt} \end{array}$$

(2

beyond offering certification for traditional quantum languages (e.g. `Quipper`)

Riley: “Linear HoTT as a Quantum Certification Language”
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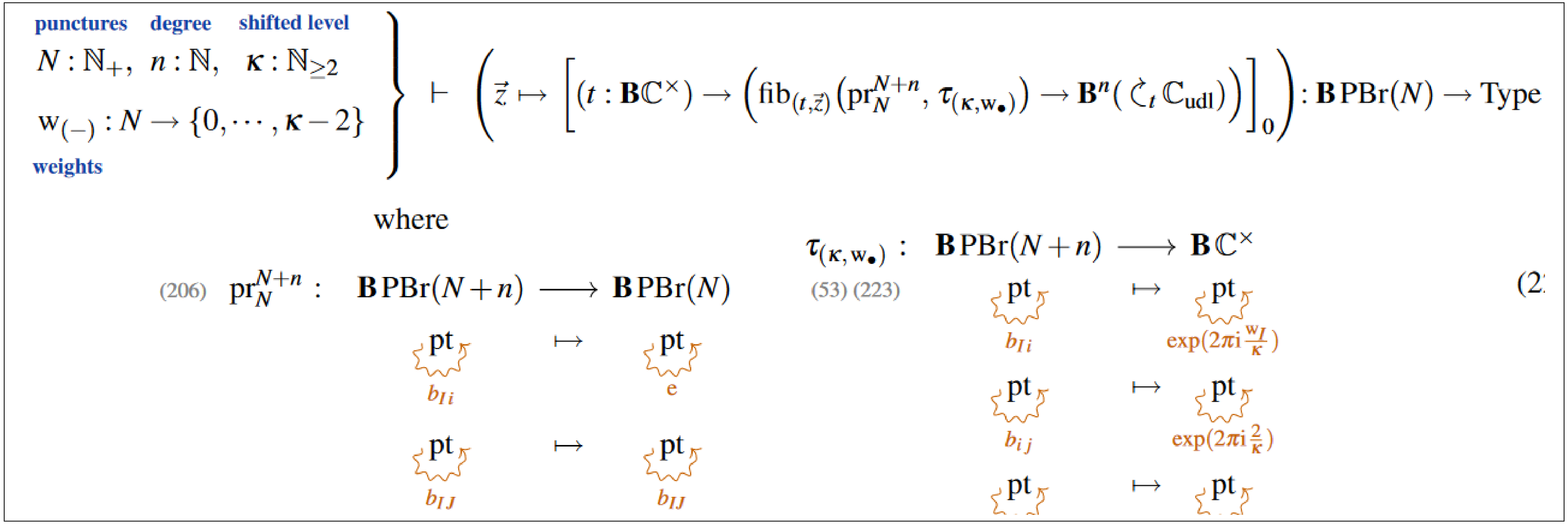
the homotopy-typing reflects
topological quantum gates

2nd punchline

Theorem: *Type transport in the following dependent type family expresses, in the above simplicial categorical semantics, topologial braid quantum gates of* $\mathfrak{su}(2)$*-anyons (Majorana, Fibonacci, ...)*

“Topological Quantum Gates in Homotopy Type Theory”
Thm. 6.8 [`arXiv:2303.02382`]



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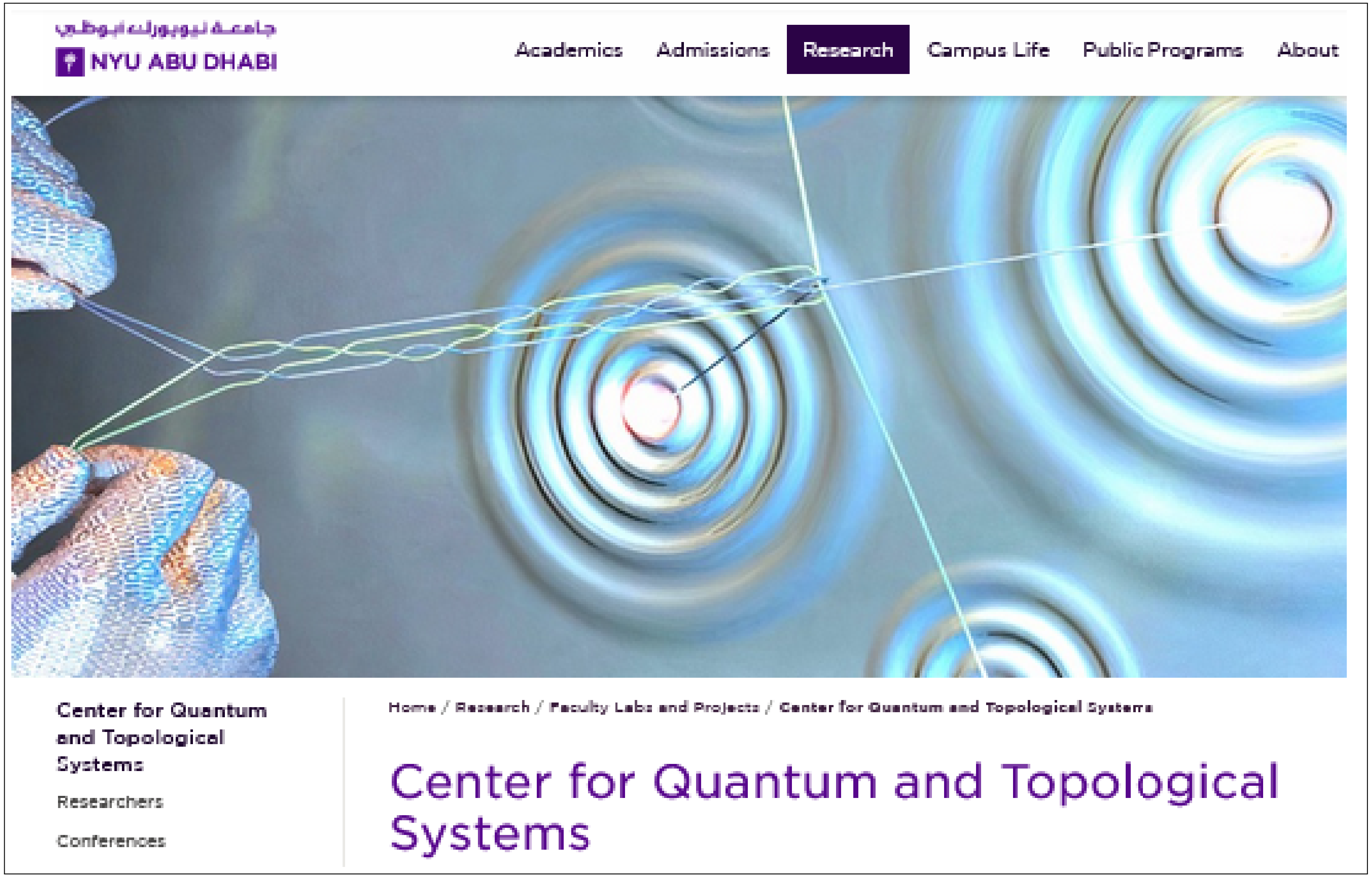
(206) $$\begin{array}{cccc} \mathrm{pr}^{N+n}_N : & \mathbf{B}\,\mathrm{PBr}(N+n) & \longrightarrow & \mathbf{B}\,\mathrm{PBr}(N) \\ & \underset{b_{Ii}}{\mathrm{pt}} & \mapsto & \underset{\mathrm{e}}{\mathrm{pt}} \\ & \underset{b_{IJ}}{\mathrm{pt}} & \mapsto & \underset{b_{IJ}}{\mathrm{pt}} \end{array}$$

(53) (223) $$\begin{array}{cccc} \tau_{(\kappa,\mathrm{w}_\bullet)} : & \mathbf{B}\,\mathrm{PBr}(N+n) & \longrightarrow & \mathbf{B}\mathbb{C}^\times \\ & \underset{b_{Ii}}{\mathrm{pt}} & \mapsto & \underset{\exp(2\pi\mathrm{i}\frac{\mathrm{w}_I}{\kappa})}{\mathrm{pt}} \\ & \underset{b_{ij}}{\mathrm{pt}} & \mapsto & \underset{\exp(2\pi\mathrm{i}\frac{2}{\kappa})}{\mathrm{pt}} \\ & \mathrm{pt} & \mapsto & \mathrm{pt} \end{array}$$ (2

in combination ⇒ major potential use-case:
certifying topological quantum compilation
(arguably inevitable for industry-scale TQC)

Thanks!



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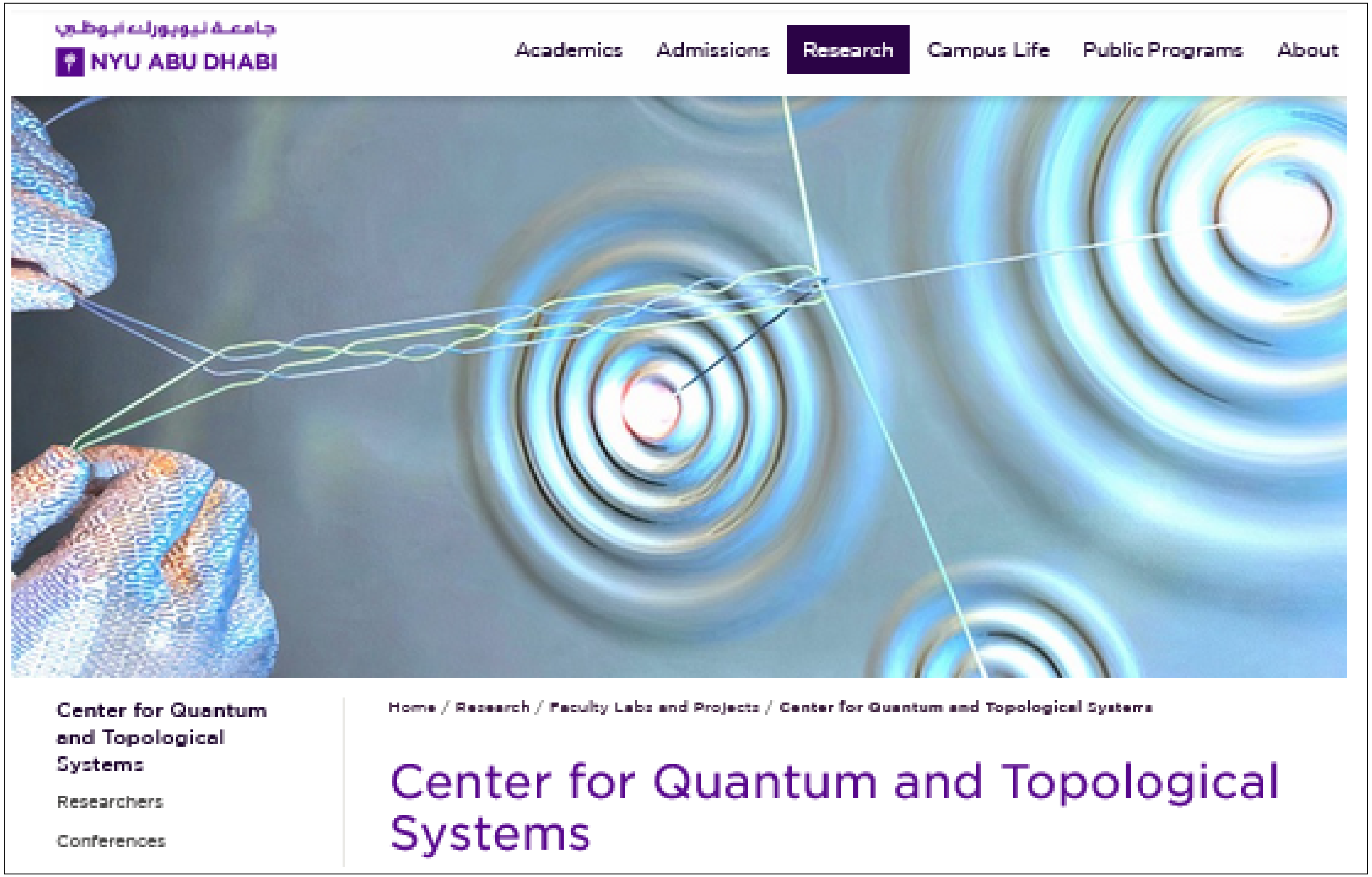
the reported results are theoretical (∃ on paper)

Thanks!



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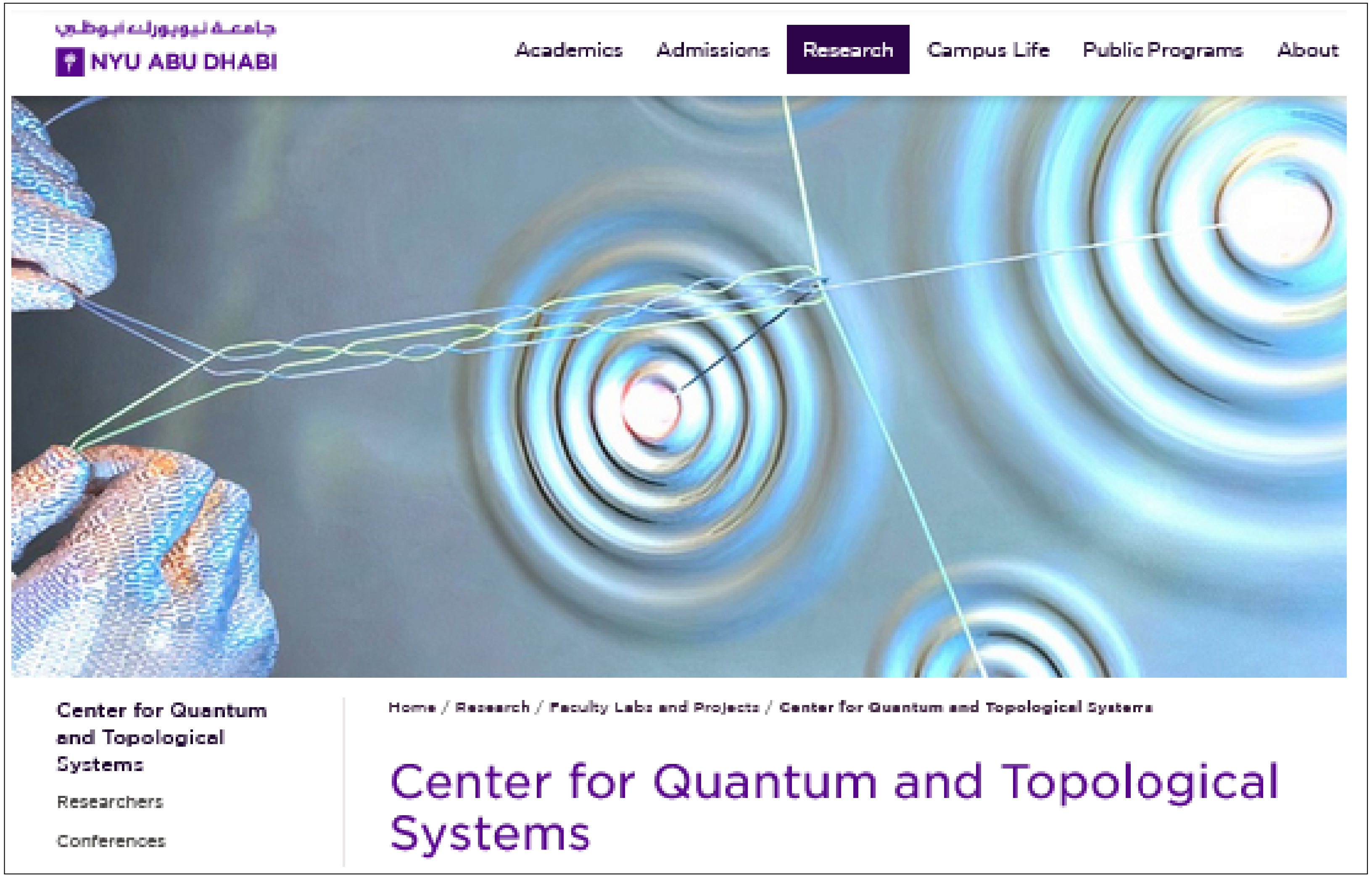
the reported results are theoretical ($\exists$ on paper)

If you are a young energetic type theorist
interested in coding implementations $\Rightarrow$
contact us: `nyuad.cqts.info@nyu.edu` !

Thanks!



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