Super \( p \)-Brane Theory
emerging from
Super Homotopy Theory

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talk at String Math 2017

Based on arXiv:1611.06536 with D. Fiorenza and H. Sati
arXiv:1702.01774 with J. Huerta

des these slides are kept at ncatlab.org/schreiber/print/StringMath2017
Notorious Open Problem of String Theory:

What is the full non-perturbative Theory?
Notorious Open Problem of String Theory:

What is the full non-perturbative Theory?

We still have no fundamental formulation of “M-theory” - Work on formulating the fundamental principles underlying M-theory has noticeably waned. [...] If history is a good guide, then we should expect that anything as profound and far-reaching as a fully satisfactory formulation of M-theory is surely going to lead to new and novel mathematics. Regrettably, it is a problem the community seems to have put aside - temporarily. But, ultimately, Physical Mathematics must return to this grand issue.

G. Moore, Physical Mathematics and the Future, at Strings 2014
Notorious Open Problem of String Theory:

What is the full non-perturbative Theory?

What is even its Principle?
## Principles

<table>
<thead>
<tr>
<th>physics</th>
<th>mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>gauge principle</td>
<td>homotopy theory</td>
</tr>
<tr>
<td>&amp; Pauli exclusion</td>
<td>super-geometry</td>
</tr>
</tbody>
</table>

= super-homotopy theory
Homotopy Theory

plain

rational

super rational

stable → parameterized

stable

parameterized spectra

underlying parameter space →

spaces

rational cochain complexes
dg-modules
dgc-algebras

super rational cochain complexes

super dg-modules

super dgc-algebras

("FDA"s)
Homotopy Theory

plain | stable       | parameterized stable | plain

plain spectra parameterized over the point parameterized spectra underlying parameter space spaces

rational
dgc-algebras

super rational

Quillen 69, Sullivan 77: infinitesimal methods in homotopy theory
Homotopy Theory

- Stable spectra parameterized over the point
- Parameterized spectra underlying parameter space
- Spaces

Plain spectra parameterized over the point parameterized spectra underlying parameter space spaces

Rational super cochain complexes dgc-modules dgc-algebras

Super rational dgc-algebras ("FDA"s)

Nieuwenhuizen 82, D’Auria-Fré 82: FDAs efficiently construct SuGra-s
Homotopy Theory

\begin{center}
\begin{tikzpicture}[font=\small, auto, node distance=2cm, >=latex]

\node (plain) {plain};
\node[below of=plain] (rational) {rational};
\node[below of=rational] (super) {super rational};
\node[below right of=plain] (stable) {stable};
\node[below right of=rational] (cochain) {cochain complexes};
\node[below right of=super] (supercochain) {super cochain complexes};
\node[below right of=stable] (parameterized) {parameterized stable};
\node[below right of=cochain] (dgc) {dgc-algebras};
\node[below right of=supercochain] (superdgc) {super dgc-algebras ("FDA"s)};
\node[below right of=parameterized] (spaces) {spaces};

\draw[->] (plain) -- (stable); \node[below] at (plain.east) {spectra};
\draw[->] (parameterized) -- (spaces); \node[below] at (parameterized.east) {underlying parameter space};
\draw[->] (rational) -- (cochain); \node[below] at (rational.east) {cochain complexes};
\draw[->] (super) -- (supercochain); \node[below] at (super.east) {super cochain complexes};
\draw[->] (plain) -- (parameterized) node[midway, above] {parameterized over the point};

\node[blue] at (3.5,0) {Schwede-Shipley 03: stable homotopy theory subsumes homological algebra};
\end{tikzpicture}
\end{center}
Homotopy Theory

plain

stable

generation

parameterized

plain

stable

stable

parameterized

spectra

parameterized

spectra

parameterized

spectra

underlying

parameter space

spaces

rational

cochain

complexes

dg-modules

dgc-algebras

super

cochain

complexes

dg-modules

dgc-algebras

(“FDA”s)

Schlegel
Homotopy Theory

stable  parameterized  plain

parameterized over the point

underlying parameter space

plain spectra  parameterized spectra  spaces

rational cochain complexes  dg-modules  dgc-algebras

super rational super cochain complexes  super dg-modules super dgc-algebras (“FDA”s)
Cohomology

extension

homotopy fiber

space

cocycle

spectrum
Cohomology

extension

homotopy fiber

space
twisted cocycle
twist

parameter space

parameterized spectrum

Nikolaus-Schreiber-Stevenson 12, Ando-Blumberg-Gepner-Hopkins-Rezk 14
We now work out in *rational* super-homotopy theory a tower of extensions, each invariant wrt automorphisms modulo R-symmetries.
We now work out in *rational* super-homotopy theory a tower of extensions, each invariant wrt automorphisms modulo R-symmetries.

Beware:
Everything in the following holds in (super-) *rational* homotopy theory.
In the beginning
the atom of space:
the superpoint

0\,1
Its maximal torus extension is the super-line.
its type II version:
the $\mathcal{N} = 2$ superpoint
maximal torus extension:
\( d = 3, \ N = 1 \)

super-Minkowski spacetime
type II version:
\( d = 3, \ N = 2 \)
super-Minkowski spacetime

\[ T^{2,1|2+2} \leftrightarrow T^{2,1|2} \leftrightarrow T^{1|1} \]
\[ T^{0|1+1} \leftrightarrow T^{0|1} \]
maximal invariant torus extension:
\( d = 4, N = 1 \)
super-Minkowski spacetime
type II version:
\( \hat{d} = 4, \; N = 2 \)
super-Minkowski spacetime

\[
\begin{align*}
\mathbb{T}^{3,1|4+4} & \leftrightarrow \mathbb{T}^{3,1|4} \\
\mathbb{T}^{2,1|2+2} & \leftrightarrow \mathbb{T}^{2,1|2} \quad \mathbb{T}^{1|1} \\
\mathbb{T}^{0|1+1} & \leftrightarrow \mathbb{T}^{0|1}
\end{align*}
\]
maximal invariant torus extension:
\( d = 6, \ N = 1 \)
super-Minkowski spacetime
T^5,1|8+8 ←→ T^5,1|8

T^3,1|4+4 ←→ T^3,1|4

T^2,1|2+2 ←→ T^2,1|2

T^0|1+1 ←→ T^0|1

type IIB version:
\( d = 6, \, N = (2, 0) \)
super-Minkowski spacetime.
type IIA version:
\( d = 6, \ N = (1, 1) \)
super-Minkowski spacetime.
maximal invariant torus extension:
\( d = 10, \; N = 1 \)
super-Minkowski spacetime
$T^9,1|16 + 16 \leftrightarrow T^9,1|16$

$T^5,1|8 + 8 \leftrightarrow T^5,1|8 \leftrightarrow T^5,1|8 + \bar{8}$

$T^3,1|4 + 4 \leftrightarrow T^3,1|4$

$T^2,1|2 + 2 \leftrightarrow T^2,1|2 \leftrightarrow T^1|1$

$T^0|1 + 1 \leftrightarrow T^0|1$

Type IIB version:
$d = 10, N = (2, 0)$
Super-Minkowski spacetime
and its type IIA version:
$d = 10, \tilde{N} = (1, 1)$
super-Minkowski spacetime
\[ \mathbb{T}^{10,1|32} \]

\[ \mathbb{T}^{9,1|16+16} \leftarrow \mathbb{T}^{9,1|16} \rightarrow \mathbb{T}^{9,1|16+16} \]

\[ \mathbb{T}^{5,1|8+8} \leftarrow \mathbb{T}^{5,1|8} \rightarrow \mathbb{T}^{5,1|8+8} \]

\[ \mathbb{T}^{3,1|4+4} \leftarrow \mathbb{T}^{3,1|4} \]

\[ \mathbb{T}^{2,1|2+2} \leftarrow \mathbb{T}^{2,1|2} \rightarrow \mathbb{T}^{1|1} \]

\[ \mathbb{T}^{0|1+1} \leftarrow \mathbb{T}^{0|1} \]

maximal invariant torus extension:
\[ d = 11, \ N = 1 \]

super-Minkowski spacetime
In summary:

**Theorem (Huerta-Schreiber 17):** There exists a diagram as shown of maximal torus extensions at each stage invariant with respect to the semi-simple part of automorphisms modulo R-symmetry which happens to be the Lorentzian Spin-groups.
the higher invariant extensions: superstrings condense
these extensions are classified by WZW-term for the GS-Superstring
\[ \mu_{F1} = i\bar{\psi} \wedge \Gamma_a \psi \wedge e^a \]

\footnotesize{(Green-Schwarz 81, Henneaux-Mezincescu 85)
the next higher invariant extensions: D-branes condense
the next higher invariant extensions: D-branes condense
these extensions are classified by WZW-terms for the super D-branes
\[
\exp(F_2) \sum c_p \overline{\psi} \wedge \Gamma_{a_1 \cdots a_p} \psi \wedge e_{a_1} \wedge \cdots \wedge e_{a_p}
\]

Azcárraga et al. 99, Sakaguchi 00
Fiorenza-Sati-Schreiber 13
D0-brane condensate is 11-th dimension via homotopy pullback
m2brane

$\text{d5brane} \rightleftharpoons \text{d3brane} \rightleftharpoons \text{d1brane} \leftarrow \text{d7brane}$

$\text{d9brane} \rightarrow \text{string}_{\text{IIB}}$

$\text{string}_{\text{het}} \leftarrow \text{string}_{\text{IIA}} \leftarrow \text{d8brane}$

$\text{T}^{9,1|16+16} \iff \text{T}^{9,1|16} \iff \text{T}^{9,1|16+16}$

$\text{T}^{5,1|8+8} \iff \text{T}^{5,1|8} \iff \text{T}^{5,1|8+8}$

$\text{T}^{3,1|4+4} \iff \text{T}^{3,1|4}$

$\text{T}^{2,1|2+2} \iff \text{T}^{2,1|2}$

$\text{T}^{1|1}$

$\text{T}^{0|1+1} \iff \text{T}^{0|1}$

its own higher invariant extension: M2-branes condense
m5brane
↓
m2brane

\( \mathcal{D}_5\)brane \( \mathcal{D}_3\)brane \( \mathcal{D}_1\)brane
\( \mathcal{D}_7\)brane
\( \mathcal{D}_9\)brane → string_{IIB}

\( \mathcal{T}^{9,1|16+16} \leftarrow \mathcal{T}^{9,1|16} \rightarrow \mathcal{T}^{9,1|16+16} \)

\( \mathcal{T}^{5,1|8+8} \leftarrow \mathcal{T}^{5,1|8} \rightarrow \mathcal{T}^{5,1|8+8} \)

\( \mathcal{T}^{3,1|4+4} \leftarrow \mathcal{T}^{3,1|4} \)

\( \mathcal{T}^{2,1|2+2} \leftarrow \mathcal{T}^{2,1|2} \)

\( \mathcal{T}^{0|1+1} \leftarrow \mathcal{T}^{0|1} \)

\( \mathcal{T}^{10,1|32} \)

\( \mathcal{D}_0\)brane \( \mathcal{D}_2\)brane \( \mathcal{D}_4\)brane
\( \mathcal{D}_6\)brane

string_{het}

\( \mathcal{T}^{9,1|16} \)

\( \mathcal{D}_8\)brane

string_{IIA} ← \( \mathcal{D}_8\)brane

its next higher extension: M5-branes condense
spacetime and M-branes have emerged from the superpoint as iterated higher invariant extensions
m5brane

m2brane

$\mathbb{T}^{10,1|32}$

$\mathbb{T}^{9,1|16} \rightleftharpoons \mathbb{T}^{9,1|16+\overline{16}}$

$\mathbb{T}^{5,1|8} \rightleftharpoons \mathbb{T}^{5,1|8+\overline{8}}$

$\mathbb{T}^{3,1|4+4} \rightleftharpoons \mathbb{T}^{3,1|4}$

$\mathbb{T}^{2,1|2+2} \rightleftharpoons \mathbb{T}^{2,1|2}$

$\mathbb{T}^{0|1+1} \rightleftharpoons \mathbb{T}^{0|1}$

Perhaps we need to understand the nature of time itself better. [...] understand in what sense time itself is an emergent concept, [...] how pseudo-Riemannian geometry can emerge from more fundamental and abstract notions such as categories of branes.

(G. Moore, *Physical Mathematics and the Future*, at *Strings 2014*)

spacetime and M-branes have emerged from the superpoint as iterated higher invariant extensions
Spacetime and M-branes have emerged from the superpoint as iterated higher invariant extensions.
consider the M-brane sector
the M2-extension is classified by a 4-cocycle:
the GS-WZW-term of the M2-brane

\[ \mu_{M2} = \frac{i}{2} \bar{\psi} \wedge \Gamma_{a_1 a_2} \psi \wedge e^{a_1} \wedge e^{a_2} \]

D’Auria-Fré 82, Bergshoeff-Sezgin-Townsend 87, Fiorenza-Sati-Schreiber 13
the M5-extension is classified by a 7-cocycle:
the GS-WZW-terms of the M5-brane

\[
\mu_{M5} = \frac{1}{5!} \overline{\psi} \wedge \Gamma_{a_1 \ldots a_5} \psi \wedge e^{a_1} \wedge \cdots \wedge e^{a_5} + \frac{1}{2} c_3 \wedge \frac{1}{2} \overline{\psi} \wedge \Gamma_{a_1 a_2} \psi \wedge e^{a_1} \wedge e^{a_2}
\]

D’Auria-Fré 82, Pasti-Sorokin-Tonin 97, Fiorenza-Sati-Schreiber 13
to descend this means to ask for analogous fiber sequence on the coefficients

Nikolaus-Schreiber-Stevenson 12
this comes out to be:
quaternionic Hopf fibration
(rationally)

Fiorenza-Sati-Schreiber 15
M5-cocycle descends:
unified M2/M5-cocycle

Fiorenza-Sati-Schreiber 15
\begin{align*}
\text{dgc-model for } S^4: \\
&d\omega_4 = 0 \\
&d\omega_7 = -\frac{1}{2}\omega_4 \wedge \omega_4
\end{align*}

11d SuGra $C$-field equation of motion: 
\[dG_7 + \frac{1}{2}G_4 \wedge G_4 = 0\]
consider this
unified M-brane cocycle
\[ \text{Ext}(\mathbb{T}^{9,1|16+16}) \cong \mathbb{T}^{10,1|32} \xrightarrow{\mu_{M2/M5}} S^4 \]

\[ \text{remember that} \]

11d spacetime
is (maximal invariant) extension of type IIA spacetime
\[
\text{Ext}(\mathbb{T}^{9,1|16+\overline{16}}) = \mathbb{T}^{10,1|32} \xrightarrow{\mu_{M_2/M_5}} S^4 \xrightarrow{\mu_{M_2}} B^4\mathbb{Q} \xrightarrow{\mathbb{Q} \leftarrow S^4} \text{Ext}(S^4/S^1)
\]

Similarly, \(S^4\)

is homotopy extension of its \(S^1\) homotopy quotient via canonical \(SU(2)\)-action on \(S^4 \simeq S(\mathbb{R} \oplus \mathbb{H})\)
\[
\Ext(T^{9,1}\mathbf{16} + \overline{\mathbf{16}}) \longrightarrow T^{10,1}\mathbf{32} \xrightarrow{\mu_{M^2/M^5}} S^4 \longrightarrow \Ext(S^4/S^1)
\]

This orbifold \(S^4/C_n \rightarrow S^4/S^1\) happens to be the same as in the near-horizon geometry of the black M5-brane at an A-type singularity.

Medeiros, Figueroa-O’Farrill 10
hence the unified M2/M5-cocycle is really of this form
Theorem (Fiorenza-Sati-Schreiber 17): Ext has a derived right adjoint

\[
\begin{array}{c}
\text{Ext}(\mathbb{T}^{9,1|\overline{16}+\overline{16}}) \quad \mu_{M2/M5} \quad \text{Ext}(S^4/S^1) \\
\downarrow \quad \mu_{M2} \\
B^4\mathbb{Q} \\
\end{array}
\]

\text{SuperHomotopyTypes} \quad \text{Extension} \quad \perp \quad \text{Cyclification} \quad \text{SuperHomotopyTypes}_{/BS^1}

given by passing to twisted loop spaces / cyclic cohomology
apply the right adjoint
and compose with the adjunction unit
to obtain the Ext \dashv \text{Cyc-adjunct} of the unified M-brane cocycle
Theorem (Fiorenza-Sati-Schreiber 17): This is the Green-Schwarz WZW term of the double dimensional reduction of M2/M5 to $F_{IIA}^{IIA}/D0/D2/D4/NS5$:

\[
\begin{align*}
\text{dgc-algebra for CycExt}(S^4/S^1): \quad & \begin{cases} 
  dH_3 = 0, \ dH_7 = F_2 \wedge F_6 - \frac{1}{2} F_4 \wedge F_4 \\
  dF_2 = 0, \ dF_4 = H_3 \wedge F_2, \ dF_6 = H_3 \wedge F_4
\end{cases}
\end{align*}
\]
This gives rise to two questions:
1) Where are the $D(p \geq 6)$-branes (gauge enhancement)?
2) Is there a dashed lift as above?
let us first make some room...
consider the Goodwillie-linearized lifting problem:
form the fiberwise suspension spectrum over $S^3$
to obtain an $S^3$ parameterized spectrum
Theorem (Roig-Saralegi 00):
rationally, a direct summand of $\Omega_{S^3}^\infty \Sigma_{S^3}^\infty (S^4 / S^1)$
is twisted connective K-theory $\text{ku}/B^2\mathbb{Z}$
and now there is a lift:

the unified cocyle of all the type IIA D-branes

dgc-algebra for $B^3 \mathbb{Z} \simeq_\mathbb{Q} S^3$: $dH_3 = 0$
dg-module for $\text{ku}/B^2 \mathbb{Z}$: $dF_{2p+2} = H_3 \wedge F_{2p}$ $p \in \mathbb{N}$
Conclusion:

Double dimensional reduction of unified M-brane cocycle via cyclification is unified IIA-brane cocycle
Conclusion:

Double dimensional reduction of unified M-brane cocycle via cyclification is unified IIA-brane cocycle.
we repeat the process:

and consider the double dimensional reduction of the IIA-cocycle
to 9d super-spacetime $\mathbb{T}^{8,1|16+16}$
\[
\begin{align*}
\text{Cyc}(\mathbb{T}^{9,1|\overline{16}+\overline{16}}) \xrightarrow{\text{Cyc}(\mu_{F_1/D_p}^{\text{IIA}})} \text{Cyc}(\text{KU}/B^2\mathbb{Z}) \\
\downarrow \\
\text{CycExt}_{\text{IIA}}(\mathbb{T}^{8,1|\overline{16}+\overline{16}})
\end{align*}
\]

hence apply cyclification
and compose
with the adjunction unit
to obtain
the double dimensional reduction
but there was also
the type IIB extension
Cyc\(\mathbb{T}^{9,1|16+16}\) \quad \xrightarrow{\text{Cyc}(\mu_{F1/Dp})} \quad \text{Cyc}(KU/B^2\mathbb{Z})

\text{CycExt}_{IA}(\mathbb{T}^{8,1|16+16}) \quad \xrightarrow{\eta_{IA}} \quad \mathbb{T}^{8,1|16+16}

\text{Ext}_{IB}(\mathbb{T}^{8,1|16+16}) \quad \xrightarrow{\mu} \quad (\quad)

whatever cocycle it carries
has itself a double dimensional reduction
by adjunction
this defines $\mu$ in terms of $\mu_{\text{IIA}}$
Theorem A: (Fiorenza-Sati-Schreiber 17):
This is the cocycle in twisted $K^1$
for the F1/Dp-branes in type IIB
Theorem B: (Fiorenza-Sati-Schreiber 17):
The commutativity of this diagram is equivalently the Buscher rules for the RR-fields
(Hori 99)
Theorem C: (Fiorenza-Sati-Schreiber 17):
The commutativity of this diagram is equivalently the rules of “topological T-duality”
(Bouwknegt-Evslin-Mathai 04, Bunke-Rumpf-Schick 08)
rationally
Theorem D: (Fiorenza-Sati-Schreiber 17):
The homotopy fiber is the doubled generalized geometry 10d super-spacetime
Theorem E: (Fiorenza-Sati-Schreiber 17):
The homotopy pullback of type II doubled super-spacetime back to 11d super-spacetime is the local model for an F-theory fibration.
Conclusion:
A fair bit of
the expected structure of **M-theory**
emerges out of the superpoint
in rational super-homotopy theory.

Evident Conjecture:
The full theory emerges
once passing beyond the rational approximation
in **full super-geometric homotopy** theory.

Epilogue

In full super-geometric homotopy theory
the superpoint $\mathbb{R}^{0|1}$ itself
emerges from $\emptyset$

$id \vdash id$
$\forall \quad \forall$
$\Rightarrow \quad \vdash \Leftrightarrow \quad \dashv \mathbb{R}^{0|1}$
$\forall \quad \forall$
$\mathcal{R} \vdash \mathbf{D} \vdash \mathbf{E}$
$\forall \quad \forall$
$\mathbb{R} \vdash b \vdash \#$
$\forall \quad \forall$
$\emptyset \vdash *$

(Schreiber 16, FOMUS proceedings)