Super p-Brane Theory

emerging from

Super Homotopy Theory

Urs Schreiber
(CAS Prague and HCM Bonn)

talk at String Math 2017

Based on	arXiv:1611.06536	with D. Fiorenza and H. Sati
	arXiv:1702.01774	with J. Huerta

these slides are kept at ncatlab.org/schreiber/print/StringMath2017

Notorious Open Problem of String Theory:

What is the full non-perturbative Theory?

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What is the full non-perturbative Theory?

We still have no fundamental formulation of "M-theory" - Work on formulating the fundamental principles underlying M-theory has noticeably waned. [. . .]. If history is a good guide, then we should expect that anything as profound and far-reaching as a fully satisfactory formulation of M-theory is surely going to lead to new and novel mathematics. Regrettably, it is a problem the community seems to have put aside - temporarily. But, ultimately, Physical Mathematics must return to this grand issue.

G. Moore, *Physical Mathematics and the Future*, at Strings 2014

Notorious Open Problem of String Theory:

What is the full non-perturbative Theory?

What is even its **Principle**?

Principles

	physics	mathematics
	gauge principle	homotopy theory
&	Pauli exclusion	super-geometry
=		super-homotopy theory

Homotopy Theory

	stable	parameterized stable	plain
plain	spectra $\xhookrightarrow{\text{parameterized over the point}}$	parameterized spectra $\xtwoheadrightarrow{\text{underlying parameter space}}$	spaces
rational			
super rational			

Homotopy Theory

	stable	parameterized stable	plain
plain	spectra $\xhookrightarrow{\text{parameterized over the point}}$	parameterized spectra $\xrightarrow{\text{underlying parameter space}}$	spaces
rational			dgc-algebras
super rational			

Quillen 69, Sullivan 77 : infinitesimal methods in homotopy theory

Homotopy Theory

	stable	parameterized stable	plain
plain	spectra $\xhookrightarrow{\text{parameterized over the point}}$	parameterized spectra $\twoheadrightarrow$ (underlying parameter space)	spaces
rational			dgc-algebras
super rational			super dgc-algebras (“FDA”s)

Nieuwenhuizen 82, D’Auria-Fré 82: FDAs efficiently construct SuGra-s

Homotopy Theory

	stable		parameterized stable		plain
plain	spectra	$\xhookrightarrow{\text{parameterized over the point}}$	parameterized spectra	$\xrightarrow{\text{underlying parameter space}}$	spaces
rational	cochain complexes				dgc-algebras
super rational	super cochain complexes				super dgc-algebras (“FDA”s)

Schwede-Shipley 03 : stable homotopy theory subsumes homological algebra

Homotopy Theory

	stable	parameterized stable	plain
plain	spectra $\xhookrightarrow{\text{parameterized over the point}}$	parameterized spectra $\xrightarrow{\text{underlying parameter space}}$	spaces
rational	cochain complexes	dg-modules	dgc-algebras
super rational	super cochain complexes	super dg-modules	super dgc-algebras (“FDA”s)

Schlegel

Homotopy Theory

	stable		parameterized stable		plain
plain	spectra	$\xhookrightarrow{\text{parameterized over the point}}$	parameterized spectra	$\xrightarrow{\text{underlying parameter space}}$	spaces
rational	cochain complexes		dg-modules		dgc-algebras
super rational	super cochain complexes		super dg-modules		super dgc-algebras (“FDA”s)

Cohomology

$$\begin{tikzcd} \text{extension} \arrow[d, "\text{homotopy fiber}"'] & \\ \text{space} \arrow[r, "\text{cocycle}"'] & \text{spectrum} \end{tikzcd}$$

Cohomology

$$
\begin{tikzcd}
\text{extension} \arrow[d, "\text{homotopy fiber}"'] & & \\
\text{space} \arrow[rr, "\text{twisted}", "\text{cocycle}"'] \arrow[dr, "\text{twist}"'] & & \text{parameterized spectrum} \arrow[dl] \\
& \text{parameter space} &
\end{tikzcd}
$$

Nikolaus-Schreiber-Stevenson 12, Ando-Blumberg-Gepner-Hopkins-Rezk 14

We now work out
in *rational* super-homotopy theory
a tower of extensions,
each invariant wrt
automorphisms modulo R-symmetries.

We now work out
in *rational* super-homotopy theory
a tower of extensions,
each invariant wrt
automorphisms modulo R-symmetries.

Beware:
Everything in the following holds
in (super-) *rational* homotopy theory.

In the beginning
the atom of space:
the superpoint

$\mathbb{T}^{0|\mathbf{1}}$

Its maximal torus extension
is the super-line

$$\begin{tikzcd} & \mathbb{T}^{1|1} \arrow[dl] \\ \mathbb{T}^{0|\mathbf{1}} & \end{tikzcd}$$

its type II version:
the $N = 2$ superpoint

$$
\begin{tikzcd}
 & & \mathbb{T}^{1|1} \arrow[dl] \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

maximal torus extension:
$d = 3$, $N = 1$
super-Minkowski spacetime

$$\begin{tikzcd}
 & \mathbb{T}^{2,1|\mathbf{2}} \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}$$

type II version:
$d = 3$, $N = 2$
super-Minkowski spacetime

$$
\begin{tikzcd}
{\color{blue}\mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

maximal invariant torus extension:
$d = 4$, $N = 1$
super-Minkowski spacetime

$$\begin{tikzcd}
 & {\color{blue}\mathbb{T}^{3,1|\mathbf{4}}} \arrow[dl] & \\
\mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}$$

type II version:
$d = 4$, $N = 2$
super-Minkowski spacetime

$$
\begin{tikzcd}
\mathbb{T}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathbb{T}^{5,1|\mathbf{8}} \arrow[dl] & \\
\mathbb{T}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

maximal invariant torus extension:
$d = 6$, $N = 1$
super-Minkowski spacetime

$$
\begin{tikzcd}
\mathbb{T}^{5,1\vert\mathbf{8}+\mathbf{8}} & \mathbb{T}^{5,1\vert\mathbf{8}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{3,1\vert\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1\vert\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{2,1\vert\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1\vert\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \mathbb{T}^{1\vert 1} \arrow[dl] \\
\mathbb{T}^{0\vert\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0\vert\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

type IIB version:
$d = 6$, $N = (2,0)$
super-Minkowski spacetime.

$$
\begin{tikzcd}
\mathbb{T}^{5,1|\mathbf{8}+\mathbf{8}} & \mathbb{T}^{5,1|\mathbf{8}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{T}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{T}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

type IIA version:
$d = 6$, $N = (1, 1)$
super-Minkowski spacetime.

$$
\begin{tikzcd}
 & \color{blue}{\mathbb{T}^{9,1|\mathbf{16}}} \arrow[dr] & \\
\mathbb{T}^{5,1|\mathbf{8}+\mathbf{8}} & \mathbb{T}^{5,1|\mathbf{8}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{T}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{T}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

maximal invariant torus extension:
$d = 10,\ N = 1$
super-Minkowski spacetime

$$
\begin{tikzcd}
{\color{blue}\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}}} & \mathbb{T}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dr] & \\
\mathbb{T}^{5,1|\mathbf{8}+\mathbf{8}} & \mathbb{T}^{5,1|\mathbf{8}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{T}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{T}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

type IIB version:
$d = 10$, $N = (2, 0)$
super-Minkowski spacetime

$$
\begin{tikzcd}
\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{T}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
\mathbb{T}^{5,1|\mathbf{8}+\mathbf{8}} & \mathbb{T}^{5,1|\mathbf{8}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{T}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{T}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

and its type IIA version:
$d = 10$, $N = (1, 1)$
super-Minkowski spacetime

$$
\begin{tikzcd}
 & \color{blue}{\mathbb{T}^{10,1|\mathbf{32}}} \arrow[dr] & \\
\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{T}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
\mathbb{T}^{5,1|\mathbf{8}+\mathbf{8}} & \mathbb{T}^{5,1|\mathbf{8}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{T}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{T}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

maximal invariant torus extension:
$d = 11$, $N = 1$
super-Minkowski spacetime

$$
\begin{tikzcd}
 & \mathbb{T}^{10,1|\mathbf{32}} \arrow[dr] & \\
\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{T}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
\mathbb{T}^{5,1|\mathbf{8}+\mathbf{8}} & \mathbb{T}^{5,1|\mathbf{8}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{T}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{T}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

In summary:

Theorem (Huerta-Schreiber 17):
There exists a diagram as shown
of maximal torus extensions
at each stage invariant
with respect to the semi-simple part
of automorphisms modulo R-symmetry
which happens to be
the Lorentzian Spin-groups.

$$
\begin{tikzcd}
 & \mathbb{T}^{10,1|\mathbf{32}} \arrow[dr] & \\
\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{T}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
\mathbb{T}^{5,1|\mathbf{8}+\mathbf{8}} & \mathbb{T}^{5,1|\mathbf{8}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{T}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{T}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathbb{T}^{10,1|\mathbf{32}} \arrow[ddr] & \\
\color{blue}{\mathfrak{string}_{\mathrm{IIB}}} \arrow[d] & \color{blue}{\mathfrak{string}_{\mathrm{het}}} \arrow[d] & \color{blue}{\mathfrak{string}_{\mathrm{IIA}}} \arrow[d] \\
\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{T}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
\mathbb{T}^{5,1|\mathbf{8}+\mathbf{8}} & \mathbb{T}^{5,1|\mathbf{8}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{T}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{T}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

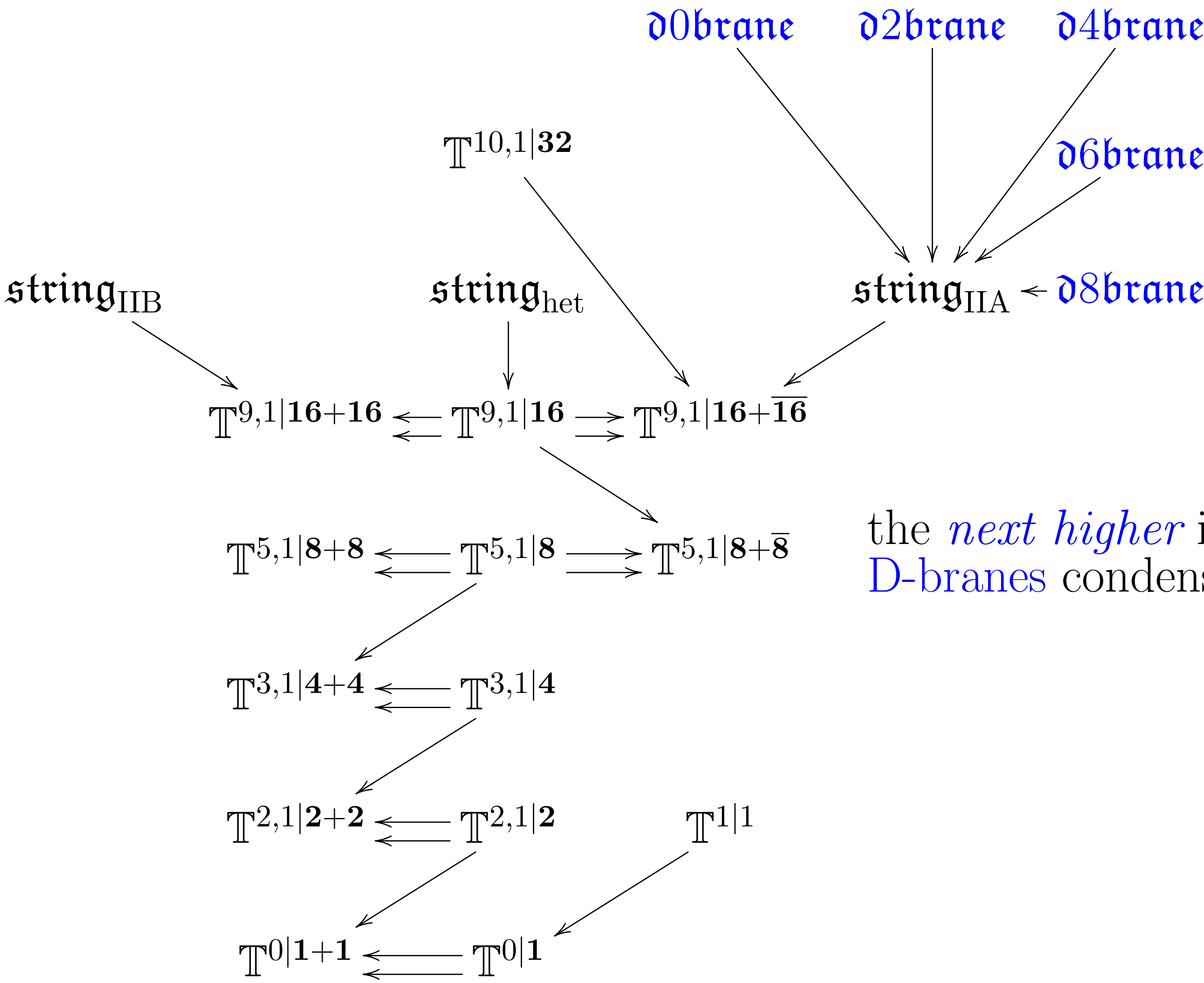
the *higher* invariant extensions:
superstrings condense

$$
\begin{tikzcd}
 & \mathbb{T}^{10,1|\mathbf{32}} \arrow[ddr] & & \\
\mathfrak{string}_{\mathrm{IIB}} \arrow[d] & \mathfrak{string}_{\mathrm{het}} \arrow[d] & \mathfrak{string}_{\mathrm{IIA}} \arrow[d, "\mathrm{hofib}(\mu_{F1})"'] & \\
\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{T}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \arrow[dr, "\mu_{F1}"] & \\
\mathbb{T}^{5,1|\mathbf{8}+\mathbf{8}} & \mathbb{T}^{5,1|\mathbf{8}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{T}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & B^3\mathbb{Q} \\
\mathbb{T}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & \\
\mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] & \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & &
\end{tikzcd}
$$

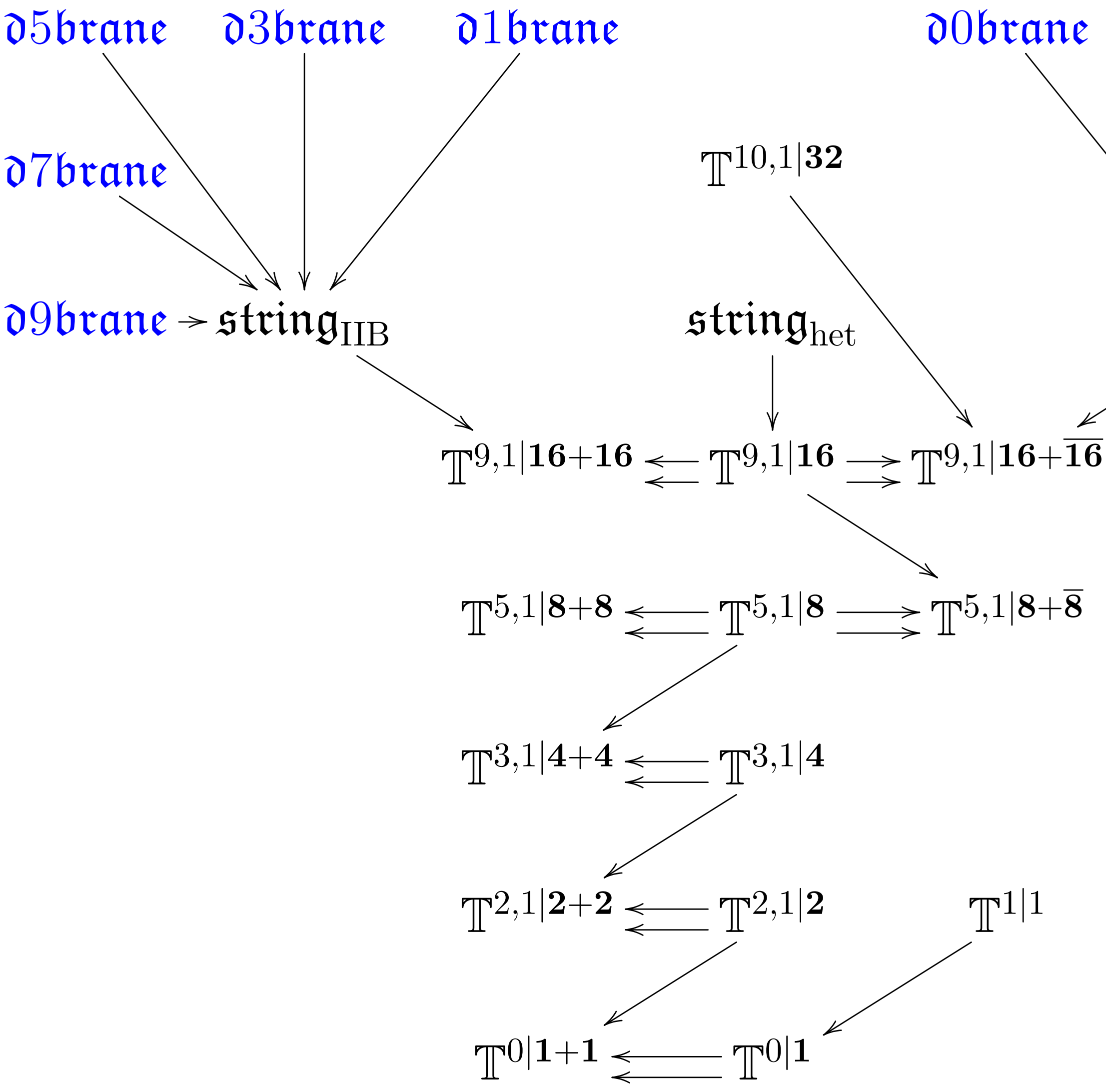
these extensions are classified by
WZW-term for the GS-Superstring
$\mu_{F1} = i\overline{\psi} \wedge \Gamma_a \psi \wedge e^a$

(Green-Schwarz 81, Henneaux-Mezincescu 85)

$$
\begin{tikzcd}
 & & \mathbb{T}^{10,1|\mathbf{32}} \arrow[ddr] & & \\
\mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & & \mathfrak{string}_{\mathrm{het}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] \\
 & \mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{T}^{9,1|\mathbf{16}} \arrow[l, shift right] \arrow[l, shift left] \arrow[r, shift right] \arrow[r, shift left] \arrow[dr] & \mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & \\
 & \mathbb{T}^{5,1|\mathbf{8}+\mathbf{8}} & \mathbb{T}^{5,1|\mathbf{8}} \arrow[l, shift right] \arrow[l, shift left] \arrow[r, shift right] \arrow[r, shift left] \arrow[dl] & \mathbb{T}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & \\
 & \mathbb{T}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1|\mathbf{4}} \arrow[l, shift right] \arrow[l, shift left] \arrow[dl] & & \\
 & \mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift right] \arrow[l, shift left] \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] & \\
 & \mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift right] \arrow[l, shift left] & &
\end{tikzcd}
$$



the *next higher* invariant extensions:
D-branes condense



the *next higher* invariant extensions:
D-branes condense

$$
\begin{tikzcd}
\mathfrak{d5brane} \arrow[ddr] & \mathfrak{d3brane} \arrow[dd] & \mathfrak{d1brane} \arrow[ddl] & & \mathfrak{d0brane} \arrow[ddr] & \mathfrak{d2brane} \arrow[dd] & \mathfrak{d4brane} \arrow[ddl] \\
\mathfrak{d7brane} \arrow[dr] & & & \mathbb{T}^{10,1|\mathbf{32}} \arrow[ddr] & & & \mathfrak{d6brane} \arrow[dl] \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & & \mathfrak{string}_{\mathrm{het}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] \\
& & \mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{T}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & \\
& & \mathbb{T}^{5,1|\mathbf{8}+\mathbf{8}} & \mathbb{T}^{5,1|\mathbf{8}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{T}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & \\
& & \mathbb{T}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & \\
& & \mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] & & \\
& & \mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & &
\end{tikzcd}
$$

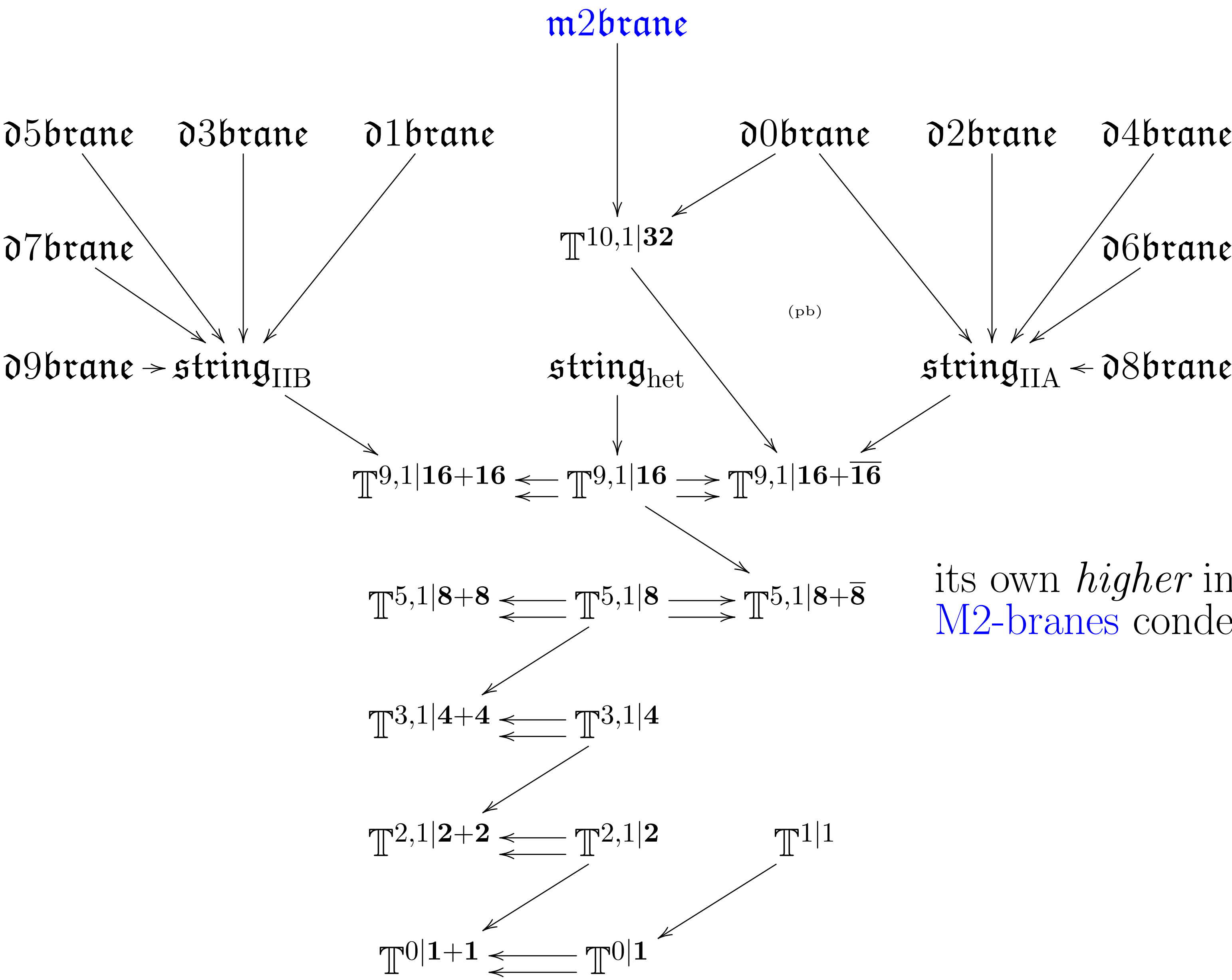
these extensions are classified by
WZW-terms for the super D-branes

$$\exp(F_2) \sum c_p \overline{\psi} \wedge \Gamma_{a_1 \cdots a_p} \psi \wedge e_{a_1} \wedge \cdots \wedge e_{a_p}$$

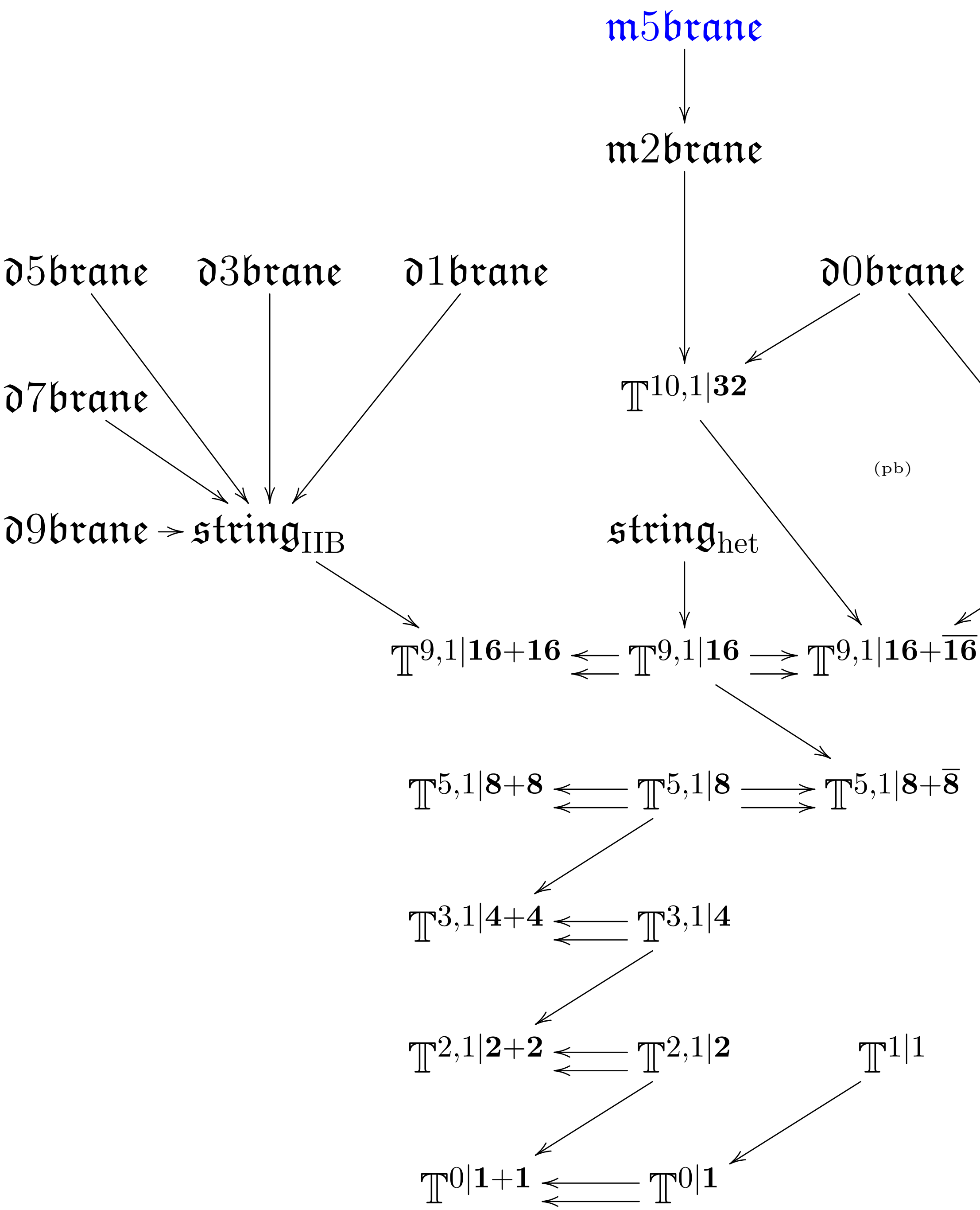
Azcárraga et al. 99, Sakaguchi 00
Fiorenza-Sati-Schreiber 13

$$
\begin{tikzcd}
\mathfrak{d5brane} \arrow[dddr] & \mathfrak{d3brane} \arrow[ddd] & \mathfrak{d1brane} \arrow[dddl] & & \mathfrak{d0brane} \arrow[dl] \arrow[dddr] & \mathfrak{d2brane} \arrow[ddd] & \mathfrak{d4brane} \arrow[dddl] \\
\mathfrak{d7brane} \arrow[ddr] & & & \mathbb{T}^{10,1|\mathbf{32}} \arrow[dddr] & & & \mathfrak{d6brane} \arrow[ddl] \\
& & & & \textcolor{blue}{\text{(pb)}} & & \\
\mathfrak{d9brane} \arrow[r] & \mathfrak{string}_{\mathrm{IIB}} \arrow[dr] & & \mathfrak{string}_{\mathrm{het}} \arrow[d] & & \mathfrak{string}_{\mathrm{IIA}} \arrow[dl] & \mathfrak{d8brane} \arrow[l] \\
& & \mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{T}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & \\
& & \mathbb{T}^{5,1|\mathbf{8}+\mathbf{8}} & \mathbb{T}^{5,1|\mathbf{8}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{T}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} & & \\
& & \mathbb{T}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & & & \\
& & \mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \mathbb{T}^{1|1} \arrow[dl] & & \\
& & \mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & & &
\end{tikzcd}
$$

D0-brane condensate
is 11-th dimension
via homotopy pullback



its own *higher* invariant extension:
M2-branes condense



its *next higher* extension:
M5-branes condense

$$
\begin{tikzcd}
 & \mathfrak{m5brane} \arrow[d] & \\
 & \mathfrak{m2brane} \arrow[d] & \\
 & \mathbb{T}^{10,1|\mathbf{32}} \arrow[dr] & \\
 & \mathbb{T}^{9,1|\mathbf{16}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
 & \mathbb{T}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{T}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{T}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

spacetime and M-branes
have emerged
from the superpoint
as iterated
higher invariant extensions

$$
\begin{tikzcd}
 & \color{blue}{\mathfrak{m5brane}} \arrow[d] & \\
 & \color{blue}{\mathfrak{m2brane}} \arrow[d] & \\
 & \mathbb{T}^{10,1|\mathbf{32}} \arrow[dr] & \\
 & \mathbb{T}^{9,1|\mathbf{16}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
 & \mathbb{T}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{T}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{T}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{0|\mathbf{1}+\mathbf{1}} & \color{blue}{\mathbb{T}^{0|\mathbf{1}}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

Perhaps we need to understand the nature of time itself better. [...]
understand in what sense time itself is an emergent concept, [...]
how pseudo-Riemannian geometry can emerge from
more fundamental and abstract notions such as categories of branes.

(G. Moore, *Physical Mathematics and the Future*, at Strings 2014)

spacetime and M-branes
have emerged
from the superpoint
as iterated
higher invariant extensions

$$
\begin{tikzcd}
 & \mathfrak{m5brane} \arrow[d] & \\
 & \mathfrak{m2brane} \arrow[d] & \\
 & \mathbb{T}^{10,1\vert\mathbf{32}} \arrow[dr] & \\
 & \mathbb{T}^{9,1\vert\mathbf{16}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{T}^{9,1\vert\mathbf{16}+\overline{\mathbf{16}}} \\
 & \mathbb{T}^{5,1\vert\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{T}^{5,1\vert\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{T}^{3,1\vert\mathbf{4}+\mathbf{4}} & \mathbb{T}^{3,1\vert\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{2,1\vert\mathbf{2}+\mathbf{2}} & \mathbb{T}^{2,1\vert\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{T}^{0\vert\mathbf{1}+\mathbf{1}} & \mathbb{T}^{0\vert\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] &
\end{tikzcd}
$$

spacetime and M-branes
have emerged
from the superpoint
as iterated
higher invariant extensions

$$
\begin{tikzcd}
\mathfrak{m5brane} \arrow[d] \\
\mathfrak{m2brane} \arrow[d] \\
\mathbb{T}^{10,1|\mathbf{32}}
\end{tikzcd}
$$

consider
the M-brane sector

$$\begin{tikzcd}
\mathfrak{m5brane} \arrow[d] & \\
\mathfrak{m2brane} \arrow[d, "\mathrm{hofib}(\mu_{\mathrm{M2}})"'] & \\
\mathbb{T}^{10,1|\mathbf{32}} \arrow[dr, "\mu_{M2}"'] & \\
& B^4\mathbb{Q}
\end{tikzcd}$$

the M2-extension is
classified by a 4-cocycle:
the GS-WZW-term of the M2-brane

$$\mu_{M2} = \frac{i}{2}\overline{\psi} \wedge \Gamma_{a_1 a_2}\psi \wedge e^{a_1} \wedge e^{a_2}$$

D'Auria-Fré 82 , Bergshoeff-Sezgin-Townsend 87,
Fiorenza-Sati-Schreiber 13

$$
\begin{tikzcd}
\mathfrak{m5brane} \arrow[d, "\mathrm{hofib}(\mu_{M5})"'] & & \\
\mathfrak{m2brane} \arrow[rr, "\mu_{M5}"] \arrow[d, "\mathrm{hofib}(\mu_{M2})"'] & & B^7\mathbb{Q} \\
\mathbb{T}^{10,1|\mathbf{32}} \arrow[dr, "\mu_{M2}"'] & & \\
& B^4\mathbb{Q} &
\end{tikzcd}
$$

the M5-extension is
classified by a 7-cocycle:

the GS-WZW-terms of the M5-brane

$$
\begin{aligned}
\mu_{M5} = & \tfrac{1}{5!}\overline{\psi} \wedge \Gamma_{a_1 \cdots a_5}\psi \wedge e^{a_1} \wedge \cdots \wedge e^{a_5} \\
& + \tfrac{1}{2} c_3 \wedge \tfrac{1}{2}\overline{\psi} \wedge \Gamma_{a_1 a_2}\psi \wedge e^{a_1} \wedge e^{a_2}
\end{aligned}
$$

D'Auria-Fré 82, Pasti-Sorokin-Tonin 97,
Fiorenza-Sati-Schreiber 13

$$
\begin{tikzcd}
\mathfrak{m5brane} \arrow[d, "\mathrm{hofib}(\mu_{M5})"'] & & \\
\mathfrak{m2brane} \arrow[rr, "\mu_{M5}"] \arrow[d, "\mathrm{hofib}(\mu_{M2})"'] & & B^7\mathbb{Q} \arrow[d, dashed, "\mathrm{hofib}(\rho)"] \\
\mathbb{T}^{10,1|\mathbf{32}} \arrow[dr, "\mu_{M2}"'] & & (B^7\mathbb{Q})/B^3\mathbb{Q} \arrow[dl, dashed, "\rho"] \\
& B^4\mathbb{Q} &
\end{tikzcd}
$$

to descend this means
to ask for analogous fiber sequence
on the coefficients

Nikolaus-Schreiber-Stevenson 12

$$
\begin{tikzcd}
\mathfrak{m5brane} \arrow[d, "\mathrm{hofib}(\mu_{M5})"'] & \\
\mathfrak{m2brane} \arrow[r, "\mu_{M5}"] \arrow[d, "\mathrm{hofib}(\mu_{M2})"'] & S^7 \arrow[d] \\
\mathbb{T}^{10,1|\mathbf{32}} \arrow[dr, "\mu_{M2}"'] & {\color{blue}S^4} \arrow[d] \\
& B^4\mathbb{Q}
\end{tikzcd}
$$

this comes out to be:
quaternionic Hopf fibration
(rationally)

Fiorenza-Sati-Schreiber 15

$$
\begin{tikzcd}
\mathfrak{m5brane} \arrow[d, "\mathrm{hofib}(\mu_{M5})"'] & \\
\mathfrak{m2brane} \arrow[r, "\mu_{M5}"] \arrow[d, "\mathrm{hofib}(\mu_{M2})"'] & S^7 \arrow[d] \\
\mathbb{T}^{10,1|\mathbf{32}} \arrow[r, dashed, "\color{blue}\mu_{M2/M5}"] \arrow[dr, "\mu_{M2}"'] & S^4 \arrow[d] \\
& B^4\mathbb{Q}
\end{tikzcd}
$$

M5-cocycle descends:
unified M2/M5-cocycle

Fiorenza-Sati-Schreiber 15

$$
\begin{tikzcd}
\mathfrak{m5brane} \arrow[d, "\mathrm{hofib}(\mu_{M5})"'] & & \\
\mathfrak{m2brane} \arrow[rr, "\mu_{M5}"] \arrow[d, "\mathrm{hofib}(\mu_{M2})"'] & & S^7 \arrow[d] \\
\mathbb{T}^{10,1|\mathbf{32}} \arrow[rr, dashed, "\color{blue}\mu_{M2/M5}"] \arrow[dr, "\mu_{M2}"'] & & S^4 \arrow[dl] \\
& B^4\mathbb{Q} &
\end{tikzcd}
$$

dgc-model for S^4:

$$
\begin{aligned}
d\omega_4 &= 0 \\
d\omega_7 &= -\tfrac{1}{2}\omega_4 \wedge \omega_4
\end{aligned}
$$

11d SuGra C-field equation of motion: $\quad dG_7 + \frac{1}{2}G_4 \wedge G_4 = 0$

$$
\begin{tikzcd}
\mathbb{T}^{10,1|\mathbf{32}} \arrow[rr, "\mu_{M2/M5}"] \arrow[dr, "\mu_{M2}"'] & & S^4 \arrow[dl] \\
& B^4\mathbb{Q} &
\end{tikzcd}
$$

consider this

unified M-brane cocycle

$$
\begin{tikzcd}
\color{blue}{\mathrm{Ext}\big(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}\big)} = \mathbb{T}^{10,1|\mathbf{32}} \arrow[rr, "\mu_{M2/M5}"] \arrow[dr, "\mu_{M2}"'] & & S^4 \arrow[dl] \\
& B^4\mathbb{Q} &
\end{tikzcd}
$$

remember that

11d spacetime
is (maximal invariant) extension of
type IIA spacetime

$$
\begin{tikzcd}
\mathrm{Ext}(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) = \mathbb{T}^{10,1|\mathbf{32}} \arrow[rr, "\mu_{M2/M5}"] \arrow[dr, "\mu_{M2}"'] & & S^4 = \mathrm{Ext}(S^4/S^1) \arrow[dl] \\
& B^4\mathbb{Q} &
\end{tikzcd}
$$

similarly S^4

is homotopy extension
of its S^1 homotopy quotient
via canonical $SU(2)$-action on
$S^4 \simeq S(\mathbb{R} \oplus \mathbb{H})$

$$
\begin{tikzcd}
\mathrm{Ext}\left(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}\right) = \mathbb{T}^{10,1|\mathbf{32}} \arrow[rr, "\mu_{M2/M5}"] \arrow[dr, "\mu_{M2}"'] & & S^4 \arrow[dl] \arrow[r, equal] & \color{blue}{\mathrm{Ext}(S^4/S^1)} \\
& B^4\mathbb{Q} & &
\end{tikzcd}
$$

This orbifold $S^4/C_n \to S^4/S^1$
happens to be the same as
in the near-horizon geometry
of the black M5-brane
at an A-type singularity
Medeiros, Figueroa-O'Farrill 10

$$
\begin{tikzcd}[column sep=large]
\mathrm{Ext}(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \arrow[rr, "\color{blue}\mu_{M2/M5}"] \arrow[dr, "\mu_{M2}"'] & & \mathrm{Ext}(S^4/S^1) \arrow[dl] \\
& B^4\mathbb{Q} &
\end{tikzcd}
$$

hence the

unified M2/M5-cocycle
is really of this form

$$
\begin{tikzcd}
\mathrm{Ext}(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \arrow[rr, "\mu_{M2/M5}"] \arrow[dr, "\mu_{M2}"'] & & \mathrm{Ext}(S^4/S^1) \arrow[dl] \\
& B^4\mathbb{Q} &
\end{tikzcd}
$$

Theorem (Fiorenza-Sati-Schreiber 17): Ext has a derived right adjoint

$$
\begin{tikzcd}
\mathrm{SuperHomotopyTypes} \arrow[r, phantom, "\perp"] & \mathrm{SuperHomotopyTypes}_{/BS^1} \arrow[l, shift right=2ex, "\mathrm{Extension}"'] \arrow[l, leftarrow, shift left=2ex, "\mathrm{Cyclification}"']
\end{tikzcd}
$$

given by passing to twisted loop spaces / cyclic cohomology

$$
\begin{tikzcd}
\textcolor{blue}{\mathrm{Cyc}}\mathrm{Ext}(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \arrow[rr, "\textcolor{blue}{\mathrm{Cyc}}(\mu_{M2/M5})"] \arrow[dr, "\mu_{F1}"'] & & \textcolor{blue}{\mathrm{Cyc}}\mathrm{Ext}(S^4/S^1) \arrow[dl] \\
& S^3 &
\end{tikzcd}
$$

apply the right adjoint

$$
\begin{tikzcd}
\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \arrow[d, "\eta_{\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}}"'] & & S^4/S^1 \arrow[d, "\eta_{S^4/S^1}"] \\
\mathrm{CycExt}(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \arrow[rr, "\mathrm{Cyc}(\mu_{M2/M5})"] \arrow[dr] & & \mathrm{CycExt}(S^4/S^1) \arrow[dl] \\
& S^3 &
\end{tikzcd}
$$

and compose
with the adjunction unit

$$
\begin{tikzcd}
\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \arrow[d, "\eta_{\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}}"'] \arrow[drr, "\widetilde{\mu_{M2/M5}}"] & & S^4/S^1 \arrow[d, "\eta_{S^4/S^1}"] \\
\mathrm{CycExt}(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \arrow[rr, "\mathrm{Cyc}(\mu_{M2/M5})"] \arrow[dr] & & \mathrm{CycExt}(S^4/S^1) \arrow[dl] \\
& S^3 &
\end{tikzcd}
$$

to obtain the
Ext ⊣ Cyc-adjunct
of the unified M-brane cocycle

$$
\begin{tikzcd}
\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \arrow[d, "\eta_{\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}}"'] \arrow[drr, "\mu_{F_1^{\mathrm{IIA}}/D0/D2/D4/NS5}"] & & S^4/S^1 \arrow[d, "\eta_{S^4/S^1}"] \\
\mathrm{CycExt}(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \arrow[rr, "\mathrm{Cyc}(\mu_{M2/M5})"] \arrow[dr] & & \mathrm{CycExt}(S^4/S^1) \arrow[dl] \\
& S^3 &
\end{tikzcd}
$$

Theorem (Fiorenza-Sati-Schreiber 17) : This is the Green-Schwarz WZW term of the double dimensional reduction of M2/M5 to $F_1^{\mathrm{IIA}}/D0/D2/D4/\mathrm{NS5}$:

dgc-algebra for $\mathrm{CycExt}(S^4/S^1)$: $\left\{ \begin{array}{c} dH_3 = 0 \; , \; dH_7 = F_2 \wedge F_6 - \tfrac{1}{2} F_4 \wedge F_4 \\ dF_2 = 0 \; , \; dF_4 = H_3 \wedge F_2 \; , \; dF_6 = H_3 \wedge F_4 \end{array} \right.$

$$
\begin{tikzcd}
\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \arrow[rr, dotted, bend left, "??"] \arrow[d, "\eta_{\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}}"'] \arrow[drr, "\mu_{F_1^{\mathrm{IIA}}/D0/D2/D4/NS5}"] & & S^4/S^1 \arrow[d, "\eta_{S^4/S^1}"] \\
\mathrm{CycExt}(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \arrow[rr, "\mathrm{Cyc}(\mu_{M2/M5})"] \arrow[dr] & & \mathrm{CycExt}(S^4/S^1) \arrow[dl] \\
& S^3 &
\end{tikzcd}
$$

This gives rise to two questions:
1) Where are the $D(p \geq 6)$-branes (gauge enhancement)?

2) Is there a dashed lift as above?

$$
\begin{tikzcd}
\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \arrow[d, "\eta_{\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}}"'] \arrow[drr, "\mu_{F1/D2/D4/D6/\mathrm{NS5}}"] & & S^4/S^1 \arrow[d, "\eta_{S^4/S^1}"] \\
\mathrm{CycExt}(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \arrow[rr, "\mathrm{Cyc}(\mu_{M2/M5})"] \arrow[dr] & & \mathrm{CycExt}(S^4/S^1) \arrow[dl] \\
& S^3 &
\end{tikzcd}
$$

let us first make some room...

$$\begin{tikzcd}
\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \arrow[d, "\eta_{\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}}"'] \arrow[dr, "\mu_{F1/D2/D4/D6/\mathrm{NS5}}"] & S^4/S^1 \arrow[r] \arrow[d, "\eta_{S^4/S^1}"] & \color{blue}{\Omega^\infty_{S^3}\Sigma^\infty_{S^3}}(S^4/S^1) \arrow[d, "\color{blue}{\Omega^\infty_{S^3}\Sigma^\infty_{S^3}}(\eta_{S^4/S^1})"] \\
\mathrm{CycExt}(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \arrow[r, "\mathrm{Cyc}(\mu_{M2/M5})"] \arrow[dr] & \mathrm{CycExt}(S^4/S^1) \arrow[r] \arrow[d] & \color{blue}{\Omega^\infty_{S^3}\Sigma^\infty_{S^3}}\mathrm{CycExt}(S^4/S^1) \arrow[dl] \\
& S^3 &
\end{tikzcd}$$

consider the Goodwillie-linearized lifting problem:
form the fiberwise suspension spectrum over S^3
to obtain an S^3 parameterized spectrum

$$
\begin{tikzcd}
\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \arrow[d, "\eta_{\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}}"'] & & \Omega^\infty_{S^3}\Sigma^\infty_{S^3}(S^4/S^1) \arrow[d, "\Omega^\infty_{S^3}\Sigma^\infty_{S^3}(\eta_{S^4/S^1})"] \\
\mathrm{CycExt}(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \arrow[rr, dashed, "\mathrm{Cyc}(\mu_{M2/M5})"] \arrow[dr] & & \Omega^\infty_{S^3}\Sigma^\infty_{S^3}\mathrm{CycExt}(S^4/S^1) \arrow[dl] \\
& S^3 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \arrow[d, "\eta_{\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}}"'] & & \mathrm{ku}/_{B^2\mathbb{Z}} \arrow[r, hook] & \Omega^\infty_{S^3}\Sigma^\infty_{S^3}(S^4/S^1) \arrow[d, "\Omega^\infty_{S^3}\Sigma^\infty_{S^3}(\eta_{S^4/S^1})"] \\
\mathrm{CycExt}(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \arrow[rrr, dashed, "\mathrm{Cyc}(\mu_{M2/M5})"] \arrow[dr] & & & \Omega^\infty_{S^3}\Sigma^\infty_{S^3}\mathrm{CycExt}(S^4/S^1) \arrow[dll] \\
& S^3 & &
\end{tikzcd}
$$

$\boxed{\begin{array}{l} \text{Theorem (Roig-Saralegi 00):} \\ \text{rationally, a direct summand of } \Omega^\infty_{S^3}\Sigma^\infty_{S^3}(S^4/S^1) \\ \text{is twisted connective K-theory } \mathrm{ku}/_{B^2\mathbb{Z}} \end{array}}$

$$
\begin{tikzcd}
\mathbb{T}^{9,1\vert \mathbf{16}+\overline{\mathbf{16}}} \arrow[rr, dashed, "\color{blue}{\mu^{\mathrm{IIA}}_{F1/Dp}}"] \arrow[d, "\eta_{\mathbb{T}^{9,1\vert \mathbf{16}+\overline{\mathbf{16}}}}"'] & & \mathrm{ku}/_{B^2\mathbb{Z}} \arrow[r, hook] & \Omega^\infty_{S^3}\Sigma^\infty_{S^3}(S^4/S^1) \arrow[d, "\Omega^\infty_{S^3}\Sigma^\infty_{S^3}(\eta_{S^4/S^1})"] \\
\mathrm{CycExt}(\mathbb{T}^{9,1\vert \mathbf{16}+\overline{\mathbf{16}}}) \arrow[rrr, dashed, "\mathrm{Cyc}(\mu_{M2/M5})"] \arrow[dr] & & & \Omega^\infty_{S^3}\Sigma^\infty_{S^3}\mathrm{CycExt}(S^4/S^1) \arrow[dll] \\
& S^3 & &
\end{tikzcd}
$$

and now there is a lift:
the unified cocyle of all the type IIA D-branes

dgc-algebra for $B^3\mathbb{Z} \simeq_{\mathbb{Q}} S^3$: $dH_3 = 0$

dg-module for $\mathrm{ku}/B^2\mathbb{Z}$: $dF_{2p+2} = H_3 \wedge F_{2p}$ $\quad p \in \mathbb{N}$

$$\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \xrightarrow{\mu^{\mathrm{IIA}}_{F1/Dp}} \mathrm{ku}/_{B^2\mathbb{Z}}$$

Conclusion:

Double dimensional reduction
of unified M-brane cocycle
via cyclification
is unified IIA-brane cocycle

$$\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \xrightarrow{\quad \mu^{\mathrm{IIA}}_{F1/Dp} \quad} \mathrm{ku}/\!_{B^2\mathbb{Z}} \to \mathrm{KU}/\!_{B^2\mathbb{Z}}$$

Conclusion:

Double dimensional reduction
of unified M-brane cocycle
via cyclification
is unified IIA-brane cocycle

$$
\begin{tikzcd}
\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \arrow[rr, dashed, "(\mu^{\mathrm{IIA}}_{F1/Dp})"] \arrow[d, equal] & & \mathrm{KU}/_{B^2\mathbb{Z}} \\
\mathrm{Ext}_{\mathrm{IIA}}\big(\mathbb{T}^{8,1|\mathbf{16}+\overline{\mathbf{16}}}\big) & &
\end{tikzcd}
$$

we repeat the process:

and consider the double dimensional reduction of the IIA-cocycle

to 9d super-spacetime $\mathbb{T}^{8,1|\mathbf{16}+\mathbf{16}}$

$$
\begin{tikzcd}
\mathrm{Cyc}\big(\mathbb{T}^{9,1\vert \mathbf{16}+\overline{\mathbf{16}}}\big) \arrow[rr, dashed, "\mathrm{Cyc}(\mu^{\mathrm{IIA}}_{F1/Dp})"] \arrow[d, equal] & & \mathrm{Cyc}\big(\mathrm{KU}/_{B^2\mathbb{Z}}\big) \\
\mathrm{Cyc}\mathrm{Ext}_{\mathrm{IIA}}\big(\mathbb{T}^{8,1\vert \mathbf{16}+\overline{\mathbf{16}}}\big) & &
\end{tikzcd}
$$

hence apply cyclification

$$
\begin{tikzcd}
\mathrm{Cyc}\big(\mathbb{T}^{9,1\vert \mathbf{16}+\overline{\mathbf{16}}}\big) \arrow[rr, dashed, "\mathrm{Cyc}(\mu^{\mathrm{IIA}}_{F1/Dp})"] \arrow[d, equal] & & \mathrm{Cyc}\big(\mathrm{KU}/_{B^2\mathbb{Z}}\big) \\
\mathrm{CycExt}_{\mathrm{IIA}}\big(\mathbb{T}^{8,1\vert \mathbf{16}+\overline{\mathbf{16}}}\big) & & \\
{\color{blue}\mathbb{T}^{8,1\vert \mathbf{16}+\mathbf{16}}} \arrow[u, "{\color{blue}\eta^{\mathrm{IIA}}}"] & &
\end{tikzcd}
$$

and compose
with the adjunction unit

$$
\begin{tikzcd}
\mathrm{Cyc}\big(\mathbb{T}^{9,1\vert \mathbf{16}+\overline{\mathbf{16}}}\big) \arrow[rr, dashed, "\mathrm{Cyc}(\mu^{\mathrm{IIA}}_{F1/Dp})"] \arrow[d, equal] & & \mathrm{Cyc}\big(\mathrm{KU}/_{B^2\mathbb{Z}}\big) \\
\mathrm{CycExt}_{\mathrm{IIA}}\big(\mathbb{T}^{8,1\vert \mathbf{16}+\overline{\mathbf{16}}}\big) & & \\
\mathbb{T}^{8,1\vert \mathbf{16}+\mathbf{16}} \arrow[u, "\eta^{\mathrm{IIA}}"] \arrow[uurr, "\widetilde{\mu^{\mathrm{IIA}}_{F1/Dp}}"'] & &
\end{tikzcd}
$$

to obtain
the double dimensional reduction

$$
\begin{tikzcd}
\mathrm{Cyc}\big(\mathbb{T}^{9,1\vert \mathbf{16}+\overline{\mathbf{16}}}\big) \arrow[rr, dashed, "\mathrm{Cyc}(\mu^{\mathrm{IIA}}_{F1/Dp})"] \arrow[d, equal] & & \mathrm{Cyc}\big(\mathrm{KU}/{\scriptstyle B^2\mathbb{Z}}\big) \\
\mathrm{CycExt}_{\mathrm{IIA}}\big(\mathbb{T}^{8,1\vert \mathbf{16}+\overline{\mathbf{16}}}\big) & & \\
\mathbb{T}^{8,1\vert \mathbf{16}+\mathbf{16}} \arrow[u, "\eta^{\mathrm{IIA}}"] \arrow[uurr, "\widetilde{\mu^{\mathrm{IIA}}_{F1/Dp}}"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Ext}_{\mathrm{IIB}}\big(\mathbb{T}^{8,1\vert \mathbf{16}+\mathbf{16}}\big) \arrow[d, equal] \\
\mathbb{T}^{9,1\vert \mathbf{16}+\mathbf{16}}
\end{tikzcd}
$$

but there was also
the type IIB extension

$$
\begin{tikzcd}
\mathrm{Cyc}\big(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}\big) \arrow[rr, dashed, "\mathrm{Cyc}(\mu^{\mathrm{IIA}}_{F1/Dp})"] \arrow[d, equal] & & \mathrm{Cyc}\big(\mathrm{KU}/_{B^2\mathbb{Z}}\big) \\
\mathrm{CycExt}_{\mathrm{IIA}}\big(\mathbb{T}^{8,1|\mathbf{16}+\overline{\mathbf{16}}}\big) & & \\
\mathbb{T}^{8,1|\mathbf{16}+\mathbf{16}} \arrow[u, "\eta^{\mathrm{IIA}}"] \arrow[uurr, "\widetilde{\mu^{\mathrm{IIA}}_{F1/Dp}}"'] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathrm{Ext}_{\mathrm{IIB}}\big(\mathbb{T}^{8,1|\mathbf{16}+\mathbf{16}}\big) \arrow[d, equal] & & \\
\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} \arrow[rr, "\color{blue}{\mu}"'] & & (\qquad\qquad)
\end{tikzcd}
$$

whatever cocycle it carries

$$
\begin{tikzcd}
\mathrm{Cyc}(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \arrow[rr, dashed, "\mathrm{Cyc}(\mu^{\mathrm{IIA}}_{F1/Dp})"] \arrow[d, equal] & & \mathrm{Cyc}(\mathrm{KU}/_{B^2\mathbb{Z}}) \\
\mathrm{CycExt}_{\mathrm{IIA}}(\mathbb{T}^{8,1|\mathbf{16}+\overline{\mathbf{16}}}) & & \\
\mathbb{T}^{8,1|\mathbf{16}+\mathbf{16}} \arrow[u, "\eta^{\mathrm{IIA}}"] \arrow[d, "\eta^{\mathrm{IIB}}"'] \arrow[uurr, "\widetilde{\mu^{\mathrm{IIA}}_{F1/Dp}}"'] \arrow[ddrr, "\widetilde{\mu}"] & & \\
\mathrm{CycExt}_{\mathrm{IIB}}(\mathbb{T}^{8,1|\mathbf{16}+\mathbf{16}}) \arrow[d, equal] & & \\
\mathrm{Cyc}(\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}}) \arrow[rr, "\mathrm{Cyc}(\mu \qquad)"'] & & \mathrm{Cyc}(\qquad)
\end{tikzcd}
$$

has itself a double dimensional reduction

$$
\begin{tikzcd}
\mathrm{Cyc}\big(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}\big) \arrow[r, dashed, "\mathrm{Cyc}(\mu^{\mathrm{IIA}}_{F1/Dp})"] \arrow[d, equal] & \mathrm{Cyc}\big(\mathrm{KU}/_{B^2\mathbb{Z}}\big) \\
\mathrm{CycExt}_{\mathrm{IIA}}\big(\mathbb{T}^{8,1|\mathbf{16}+\overline{\mathbf{16}}}\big) & \\
\mathbb{T}^{8,1|\mathbf{16}+\mathbf{16}} \arrow[u, "\eta^{\mathrm{IIA}}"] \arrow[d, "\eta^{\mathrm{IIB}}"'] \arrow[uur, "\widetilde{\mu^{\mathrm{IIA}}_{F1/Dp}}"'] \arrow[ddr, "\widetilde{\mu}"] & {} \arrow[uu, "\simeq"'] \arrow[dd, "\simeq"'] \\
\mathrm{CycExt}_{\mathrm{IIB}}\big(\mathbb{T}^{8,1|\mathbf{16}+\mathbf{16}}\big) \arrow[d, equal] & \\
\mathrm{Cyc}\big(\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}}\big) \arrow[r, "\mathrm{Cyc}(\mu \qquad)"'] & \mathrm{Cyc}(\qquad\qquad)
\end{tikzcd}
$$

by adjunction
this defines μ
in terms of μ^{IIA}

$$
\begin{tikzcd}
\mathrm{Cyc}(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}) \arrow[rr, dashed, "\mathrm{Cyc}(\mu^{\mathrm{IIA}}_{F1/Dp})"] \arrow[d, equal] & & \mathrm{Cyc}(\mathrm{KU}/_{B^2\mathbb{Z}}) \\
\mathrm{CycExt}_{\mathrm{IIA}}(\mathbb{T}^{8,1|\mathbf{16}+\overline{\mathbf{16}}}) & & \\
\mathbb{T}^{8,1|\mathbf{16}+\mathbf{16}} \arrow[u, "\eta^{\mathrm{IIA}}"] \arrow[d, "\eta^{\mathrm{IIB}}"'] \arrow[uurr, "\widetilde{\mu^{\mathrm{IIA}}_{F1/Dp}}"'] \arrow[ddrr, "\color{blue}{\widetilde{\mu^{\mathrm{IIB}}_{F1/Dp}}}"] & & \color{blue}{(\mathrm{KU}\oplus\Sigma\mathrm{KU})/\mathcal{T}} \arrow[uu, "\simeq"'] \arrow[dd, "\simeq"] \\
\mathrm{CycExt}_{\mathrm{IIB}}(\mathbb{T}^{8,1|\mathbf{16}+\mathbf{16}}) \arrow[d, equal] & & \\
\mathrm{Cyc}(\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}}) \arrow[rr, "\mathrm{Cyc}(\color{blue}{\mu^{\mathrm{IIB}}_{F1/Dp}})"'] & & \mathrm{Cyc}(\color{blue}{\Sigma\mathrm{KU}/B^2\mathbb{Z}})
\end{tikzcd}
$$

Theorem A: (Fiorenza-Sati-Schreiber 17):
This is the cocycle in twisted K^1
for the F1/Dp-branes in type IIB

$$
\begin{tikzcd}
 & & \mathrm{Cyc}(\mathrm{KU}/_{B^2\mathbb{Z}}) \\
\mathbb{T}^{8,1|\mathbf{16}+\mathbf{16}} \arrow[urr, "\widetilde{\mu^{\mathrm{IIA}}_{F1/Dp}}"'] \arrow[drr, "\widetilde{\mu^{\mathrm{IIB}}_{F1/Dp}}"] & & (\mathrm{KU} \oplus \Sigma\mathrm{KU})/\mathcal{T} \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \\
 & & \mathrm{Cyc}(\Sigma\mathrm{KU}/B^2\mathbb{Z})
\end{tikzcd}
$$

Theorem B: (Fiorenza-Sati-Schreiber 17):

The commutativity of this diagram is equivalently the Buscher rules for the RR-fields

(Hori 99)

$$
\begin{tikzcd}
 & & \mathrm{Cyc}(\mathrm{KU}/_{B^2\mathbb{Z}}) \arrow[dr] & \\
\mathbb{T}^{8,1|\mathbf{16}+\mathbf{16}} \arrow[urr, "\widetilde{\mu^{\mathrm{IIA}}_{F1/Dp}}"'] \arrow[drr, "\widetilde{\mu^{\mathrm{IIB}}_{F1/Dp}}"] & & (\mathrm{KU} \oplus \Sigma\mathrm{KU})/\mathcal{T} \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \arrow[r] & B\mathcal{T} \\
 & & \mathrm{Cyc}(\Sigma\mathrm{KU}/B^2\mathbb{Z}) \arrow[ur] &
\end{tikzcd}
$$

Theorem C: (Fiorenza-Sati-Schreiber 17):
The commutativity of this diagram is equivalently
the rules of "topological T-duality"
(Bouwknegt-Evslin-Mathai 04, Bunke-Rumpf-Schick 08)
rationally

$$
\begin{tikzcd}
 & & \mathrm{Cyc}(\mathrm{KU}/B^2\mathbb{Z}) \arrow[dr] & \\
\mathbb{T}^{\widehat{8+(1,1),1}|32} \arrow[r, "\mathrm{hofib}"] & \mathbb{T}^{8,1|\mathbf{16}+\mathbf{16}} \arrow[ur, "\widetilde{\mu^{\mathrm{IIA}}_{F1/Dp}}"'] \arrow[dr, "\widetilde{\mu^{\mathrm{IIB}}_{F1/Dp}}"] & (\mathrm{KU}\oplus\Sigma\mathrm{KU})/\mathcal{T} \arrow[u, "\simeq"'] \arrow[d, "\simeq"] \arrow[r] & B\mathcal{T} \\
 & & \mathrm{Cyc}(\Sigma\mathrm{KU}/B^2\mathbb{Z}) \arrow[ur] &
\end{tikzcd}
$$

Theorem D: (Fiorenza-Sati-Schreiber 17):
The homotopy fiber
is the doubled
generalized geometry
10d super-spacetime

$$
\begin{tikzcd}
\mathbb{T}^{10+(1,1),1|32} \arrow[r] \arrow[d] \arrow[dr, phantom, "(\mathrm{pb})"] & \mathbb{T}^{10,1|\mathbf{32}} \arrow[d] & & \mathrm{Cyc}(\mathrm{KU}/_{B^2\mathbb{Z}}) \arrow[dr] & \\
\mathbb{T}^{9+(1,1),1|\mathbf{32}} \arrow[r] & \mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} & & & \\
\mathbb{T}^{\widehat{8+(1,1),1|32}} \arrow[u] \arrow[r, "\mathrm{hofib}"] & \mathbb{T}^{8,1|\mathbf{16}+\mathbf{16}} \arrow[uurr, "\widetilde{\mu^{\mathrm{IIA}}_{F1/Dp}}"'] \arrow[drr, "\widetilde{\mu^{\mathrm{IIB}}_{F1/Dp}}"] & & (\mathrm{KU}\oplus\Sigma\mathrm{KU})/\mathcal{T} \arrow[uu, "\simeq"'] \arrow[d, "\simeq"'] \arrow[r] & B\mathcal{T} \\
& & & \mathrm{Cyc}(\Sigma\mathrm{KU}/B^2\mathbb{Z}) \arrow[ur] &
\end{tikzcd}
$$

Theorem E: (Fiorenza-Sati-Schreiber 17):
The homotopy pullback
of type II doubled super-spacetime
back to 11d super-spacetime
is the local model for an F-theory fibration

Conclusion:

A fair bit of
the expected structure of M-theory
emerges out of the superpoint
in rational super-homotopy theory.

Evident Conjecture:

The full theory emerges
once passing beyond the rational approximation
in full super-geometric homotopy theory.
(arXiv:1310.7930).

Epilogue

In full super-geometric homotopy theory
the superpoint $\mathbb{R}^{0|1}$ itself
emerges from $\emptyset$

$$
\begin{array}{ccccccccc}
\mathrm{id} & \dashv & \mathrm{id} & & & & & & \\
\vee & & \vee & & & & & & \\
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \boxed{\color{blue}{\mathbb{R}^{0|1}}} & & & & \\
 & & \vee & & \vee & & & & \\
 & & \Re & \dashv & \boxed{\mathbb{D}} & \dashv & \text{Ét} & & \\
 & & & & \vee & & \vee & & \\
 & & & & \boxed{\mathbb{R}} & \dashv & \flat & \dashv & \sharp \\
 & & & & & & \vee & & \vee \\
 & & & & & & \emptyset & \dashv & \ast
\end{array}
$$

(Schreiber 16, FOMUS proceedings)