Higher Field Bundles for Gauge Fields

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talk — based on [Schreiber13] — at

Operator and Geometric Analysis on Quantum theory
Levico Terme (Trento), Italy, 15-19 September 2014

www.science.unitn.it/∼moretti/convegno/convegno.html

The world is governed by

quantum

field theory

The world is governed by

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The world is governed by

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gauge

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A well kept secret is:

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A well kept secret is:

locality principle + gauge redundancy = ...

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A well kept secret is:

locality principle + gauge equivalences! = ...

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A well kept secret is:

locality principle + gauge principle = ...

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A well kept secret is:

locality principle + gauge principle = *stack* principle

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A well kept secret is:

locality principle + gauge principle = higher geometry

Goal today:

1. reveal secret by example:
   Field bundles for gauge fields are higher bundles (stacks).

2. indicate
   Local field theory with higher field bundles.

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   Local field theory with higher field bundles.

   5% of talk – research

the remaining 5% should be your questions!

A field is *local*
if
field configurations on a space(-time) X
are equivalently
field configurations on an atlas $\{U_i \hookrightarrow X\}$
with identifications on overlaps of charts.

Simplest example: plain scalar field.

A field configuration on X is a smooth function

$$\phi : X \to \mathbb{R} \, .$$

Restricts to local field configurations

$$\phi_i := \phi|_{U_i} : U_i \to \mathbb{R} \, .$$

With identification on overlaps

$$\phi_i = \phi_j \quad \text{on } U_i \cap U_j$$

Locality: $\{\{\phi_i\} + \text{identifications}\} \simeq \{\phi\}$.

In math jargon:
$\underline{\mathbb{R}} : U \mapsto \{\text{scalar fields on } U\} = C^\infty(U, \mathbb{R})$ is a **presheaf**.
Locality is the *sheaf condition*, $\underline{\mathbb{R}}$ is a **sheaf**.

A field configuration on X is a closed differential 2-form (Faraday tensor)

$$F \in \Omega^2_{\mathrm{cl}}(X) \qquad (\text{i.e. } \mathbf{d}F = 0)\,.$$

Restricts to local field configurations

$$F_i := \omega|_{U_i} \in \Omega^2_{\mathrm{cl}}(U_i)\,.$$

With identification on overlaps

$$F_i = F_j \quad \text{on} U_i \cap U_j\,.$$

Locality: $\{\{F_i\} + \text{identifications}\} \simeq \{F\}$

In math jargon:
$\Omega^2_{\mathrm{cl}} : U \mapsto \{\text{closed 2-forms on } U\} = \Omega^2_{\mathrm{cl}}(U)$ is a **presheaf**.
Locality is the *sheaf condition*, Ω^2_{cl} is a **sheaf**.

Let $E \to X$ be a bundle over X;
e.g. a *field bundle*, e.g. a spinor bundle for fermion fields.

Then

$$\Gamma(E) : U \mapsto \Gamma_U(E) := \{\text{sections of } E \text{ over } U\}$$

is a sheaf *on the given* X, which may be evaluated on charts $U \hookrightarrow X$.

But this is not yet a sheaf on *all* manifolds X.
Instead $E \to X$ is fixed background structure.

"**locality on background**"	"**covariant locality**"
sheaf on charts of fixed X	sheaf on all manifolds

quasi-example: sections of a bundle

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a subtly different kind of sheaf!

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kind of sheaf!
math jargon: *gros topos*

now: Gauge principle.

The meaning of “identification” changes!

Where scalar fields are equal (or not)

$$\phi_1 = \phi_2$$

gauge fields may be gauge equivalent without being equal

$$A_1 \sim A_2$$

But also the *choice* of gauge equivalence g matters

$$A_1 \xrightarrow{g} A_2$$

e.g. $$A \xrightarrow{g} g^{-1} A g + g^{-1} \mathbf{d} g$$

In math jargon: remembering choice of gauge equivalence means refining

equivalence relations

to

groupoids of gauge fields.

now: Gauge principle.

The meaning of "identification" changes! This is really deep...

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Given a chart $\mathbb{R}^n \hookrightarrow X$,
the *gauge equivalence relation* on $\mathbb{R}^n$ has
as "objects" the gauge field potentials

$$A \in \Omega^1(\mathbb{R}^n, \mathfrak{g})$$

as "morphisms" the *existence* of gauge transformations

$$A_1 \overset{\exists g}{\rightsquigarrow} A_2$$

$$A_1 \overset{\exists g}{\rightsquigarrow} g^{-1} A_1 g + g^{-1} \mathbf{d} g$$

transitivity is existence of iteration of gauge transformations:

$$\begin{array}{ccccc} & & A_2 & & \\ & {}^{\exists g}\nearrow & \| & \searrow^{\exists h} & \\ A_1 & & \underset{\exists (h \circ g)}{\rightsquigarrow} & & A_3 \end{array}$$

example: the groupoid of gauge fields on a chart

Given a chart $\mathbb{R}^n \hookrightarrow X$,
the *groupoid of gauge fields* on $\mathbb{R}^n$ has
as "objects" the gauge field potentials

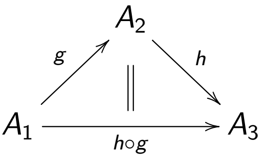
$$A \in \Omega^1(\mathbb{R}^n, \mathfrak{g})$$

as "morphisms" the *specific* gauge transformations

$$A_1 \xrightarrow{g} A_2$$

$$A_1 \xrightarrow{g} g^{-1} A_1 g + g^{-1} \mathbf{d} g$$

composition is iteration of gauge transformations:



...means identifying local field configurations on overlaps (only) via gauge transformations:

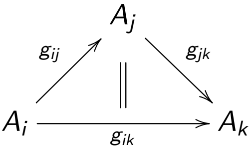
Local field configurations

$$A_i \quad \text{on } U_i$$

local identification via gauge equivalences

$$A_i \xrightarrow{g_{ij}} A_j \quad \text{on } U_i \cap U_j$$

identification of gauge equivalences on triple overlaps



$$\text{on } U_i \cap U_j \cap U_k$$

hence the familiar cocycle condition:

$$g_{ij}\, g_{jk} = g_{ik}$$

Example: The Dirac monopole.

On spacetime outside of a magnetic monopole

$$X \;=\; (\mathbb{R}^3 - \{0\}) \times \mathbb{R} \;\simeq\; S^2 \times (\mathbb{R}_+ \times \mathbb{R})$$

construct any electromagnetic field configuration:

- choose atlas by two hemispheres $U_\pm := S_\pm \times (\mathbb{R}_+ \times \mathbb{R})$
- choose local gauge fields $A_\pm \in \Omega^1(S_\pm)$
- choose identification-via-gauge-equivalence on equator

$$g : S^1 \longrightarrow U(1)$$

One finds that in the groupoid of local field data, g is characterized by its winding number

$$n_{\mathrm{monopole}} \;=\; \int_{S^2} F \;=\; \Phi_{\mathrm{mag}}$$

math jargon: **clutching construction** exhibiting **first Chern class**

Example: The Yang-Mills instanton.

On spacetime with fields appropriately "vanishing at infinity"

$$X = (\mathbb{R}^4)^+ \simeq S^4$$

construct $SU(2)$-gauge field configuration:

- choose atlas by two hemi-4-spheres $U_\pm := S_\pm$
- choose local gauge fields $A_\pm$
- choose identification-via-gauge-equivalence on equator

$$g : S^3 \longrightarrow SU(2)$$

One finds that in the groupoid of local field data, g is characterized by its winding number

$$n_{\mathrm{instanton}} = \int_{S^4} \mathrm{tr}(F \wedge F) \, .$$

math jargon: **clutching construction** exhibiting **2nd Chern class**

Punchline:

Choice of gauge transformation crucially matters.

locality principle + gauge relation $\Rightarrow$ no monopoles

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locality principle + gauge relation $\Rightarrow$ no topol. sector

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Choice of gauge transformation crucially matters.

locality principle + gauge groupoid ⇒ all topol. sectors

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locality+gauge principle captures global phenomena.

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locality+gauge principle captures global phenomena.

Conversely,

since there *are* topol. sectors (baryogenesis, QCD vacuum, ...):

Passing to gauge equivalence classes breaks locality.

see also A. Schenkel's talk at this meeting

Hence consider the "*sheaf of groupoids*"

$$X \mapsto \coprod_s \left\{ \begin{array}{c} \text{groupoid of} \\ \text{gauge fields on } X \\ \text{in topological sector } s \end{array} \right\}$$

Fact: This *is* covariantly local.
math jargon: this is a *stack* (a "higher sheaf of groupoids").

Fact: The naive

$$X \mapsto \left\{ \begin{array}{c} \text{groupoid of} \\ \text{gauge fields on } X \\ \text{in topological sector } 0 \end{array} \right\}$$

is *not* local.
math jargon: this is a *pre-stack*.

Fact: First case is universal way of making local the second.
math jargon: *stackification*

Neither of

- “stack”
- “sheaf of groupoids on all manifolds”
- “category fibered in groupoids over manifolds”

is great terminology.

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For formulating physics
it is useful to change perspective...

...and think of stacks on the "gros" site of all manifolds
as *smooth spaces* with refined gauge equivalence relation;
hence as *smooth groupoids*.

Like so:...

Think of arbitrary sheaf on manifolds

$$\mathbf{X} : U \mapsto \mathbf{X}(U)$$

as sending any manifold U
to the *set* of smooth maps

$$\text{“}\mathbf{X}(U) = \{U \to \mathbf{X}\} = \mathrm{Hom}(U, \mathbf{X})\text{”}$$

into a would-be smooth space $\mathbf{X}$.
Say “smooth space” for a sheaf regarded this way.

The *Yoneda embedding* says that

$$\{\text{smooth manifolds}\} \hookrightarrow \{\text{smooth spaces}\}$$

The *Yoneda lemma* says removing the quotation is consistent:

$$\left\{ \begin{array}{c} \text{maps of smooth spaces} \\ U \to \mathbf{X} \end{array} \right\} \simeq \mathbf{X}(U)$$

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Think of arbitrary stack on manifolds

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$$\text{``}\mathbf{X}(U) \simeq \{U \to \mathbf{X}\} \simeq \mathrm{Hom}(U, \mathbf{X})\text{''}$$

into a would-be smooth orbi-space $\mathbf{X}$.

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The *2-Yoneda embedding* says that

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The *2-Yoneda lemma* says quotation is consistent

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Fact. The category of smooth spaces is an excellent context for doing differential geometry.

Fact. The higher category of smooth groupoids is an excellent context for doing higher differential geometry.

(here excellent = cohesive homotopy theory [Schreiber13])

In particular: write $\mathbf{B}G_{\mathrm{conn}}$ for smooth groupoid of G-gauge fields.

Then: the covariantly local field 2-bundle for non-perturbative gauge fields is

$$\begin{array}{c} X \times \mathbf{B}G_{\mathrm{conn}} \\ \downarrow \\ X \end{array}$$

in the category of smooth groupoids.

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In particular: write $\mathbf{B}G_{\mathrm{conn}}$ for smooth groupoid of G-gauge fields.

Then: the covariantly local field 2-bundle for just topological sectors is

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in the category of smooth groupoids.
math jargon: *G-gerbe*

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Task: Formulate local gauge theory with higher field bundles!

Outlook.

Fact. For compact gauge group G

$$\{L \in H^{n+2}(BG,\mathbb{Z})\} \xrightarrow{\simeq} \left\{ \begin{array}{c} \text{maps of smooth higher groupoids} \\ \mathbf{B}G \xrightarrow{\mathbf{L}} \mathbf{B}^{n+1}U(1) \end{array} \right\}_{\sim}$$

Sends *level* L to fully local higher Chern-Simons Lagrangian.
Defines fully local pre-quantum gauge field theory

$$\exp(\tfrac{i}{\hbar}\int_{(-)}\mathbf{L}) \;:\; \mathrm{Bord}_n^{\mathrm{fr}} \longrightarrow \mathrm{Corr}_n(\mathrm{Sh}_\infty(\mathrm{Mfd})_{/\mathbf{B}^n U(1)})\,.$$

Sends closed n-manifold Σ to higher WZW θ-bundle

$$\exp(\tfrac{i}{\hbar}\int_{\Sigma}\mathbf{L}) \;:\; \mathrm{Loc}_G(\Sigma) \longrightarrow \mathbf{B}U(1)\,.$$

Quantize by pull-push in generalized cohomology...

more exposition in:

U. S.
What, and for what is higher geometric quantization?
ncatlab.org/schreiber/show/What,+and+for+what+is+Higher+geometric+quantization

details in:

U. S.,
Differential cohomology in a cohesive ∞-topos,
arXiv:1310.7930

in particular section 1.2 in there:

U. S.,
Classical field theory via Cohesive homotopy types,
ncatlab.org/schreiber/show/Classical+field+theory+via+Cohesive+homotopy+types