Supersymmetric homogeneous Quantum Cosmology

Urs Schreiber*

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Abstract

Canonical $N = 1$, $D = 4$ quantum supergravity is studied in a mode-amplitude basis, where its supersymmetry generators are found to be represented by deformed exterior derivatives on configuration space. Cosmological models in supergravity are shown to be governed by the covariant version of the Witten model of supersymmetric quantum mechanics. Properties of the latter are investigated and the results are applied to homogeneous supersymmetric models derived from 4- and 11-dimensional supergravity.

In the first part of this text, covariant supersymmetric quantum mechanics of a point, propagating on a pseudo-Riemannian manifold of arbitrary dimension, is studied in an operator-based framework with emphasis on differential geometric methods and the theory of Dirac operators. The indefiniteness of the underlying metric gives rise to constrained dynamics, which necessitates gauge fixing and the search for well defined scalar products and conserved probability currents. A method is proposed where gauge fixing is accomplished within the geometric framework by means of the cohomology of an operator naturally derived from the supercharge in a manner very similar to the method of BRST/coBRST-cohomology theory. Conserved probability currents and scalar products are found by generalizing respective results from Clifford algebraic formulations of Maxwell and Dirac theory, which are demonstrated to be formally intimately related to covariant supersymmetric quantum mechanics.

The next part focuses on canonically quantized supergravity. It is shown that the quantum supersymmetry generators, when transformed from the usual functional to a mode amplitude basis, are represented by deformed exterior (co-)derivatives on configuration space. Metric and connection of the latter are identified and the inner product on states is shown to be the Hodge inner product of differential forms. As a special case it follows that the supersymmetry constraints associated with homogeneous field modes govern the dynamics of homogeneous models and give rise to covariant supersymmetric quantum mechanics on mini-superspace.

In the last part, the above results are applied to example models in homogeneous quantum cosmology. Due to the identification of the supersymmetry generators with Dirac-Witten operators, the ordinary generator algebra of cosmological models may be systematically extended to the respective algebra of

*Urs.Schreiber@uni-essen.de

supergravity by identifying a superpotential function, which is determined by bosonic data, namely the DeWitt metric and the bosonic potential, alone. Using this method, which has been applied previously by Graham et al. to study Bianchi models in 4-dimensional supergravity, a recently proposed model deriving from $D = 11$ supergravity is investigated. It is found to have an essentially 20-dimensional configuration space without (super-)potential but with kinetic contributions, stemming from the presence of the homogeneous 3-form-field, which constitute an effective potential well for the moduli fields. Since generic classical solutions of this model are found to exhibit Mixmaster-like behavior, scenarios of localized wave packets scattering at these effective potential walls are studied. The respective probability currents confirm that supersymmetry, due to the appearance of (generalized) Dirac operators, adds an element of *zitterbewegung* to quantum cosmology.

Contents

1 Introduction

The subject of this text can be characterized in elementary terms roughly as follows:

What is supersymmetric homogeneous quantum cosmology?
Theoretical *cosmology* is concerned with making a (usually highly simplified) *Ansatz* for a solution to the gravitational field equations of the entire cosmos, a so called cosmological model, and studying its properties and physical predictions.

Quantum cosmology looks at these models by studying their quantized dynamics, which amounts to investigating the quantum mechanics of a point (the "universe point"), that propagates in *configuration space*, called mini-superspace[1] in this context.

Homogeneous quantum cosmology restricts attention to such cosmological models that are spatially homogeneous, thus exhibiting a high amount of symmetry. This greatly simplifies the analysis, but, of course, possibly at the cost of physical realism.

Supersymmetric homogeneous quantum cosmology finally considers extensions of (homogeneous) cosmological models by certain fermionic degrees of freedom in such a way, that their Hamiltonian, which is second order in the canonical momenta, is replaced by certain *first-order* operators, the supersymmetry generators.

This deserves further comment: Note that ordinary quantum cosmology is technically a quantum theory of a *single* point propagating on the pseudo-Riemannian manifold representing mini-superspace. Hence, formally, quantum cosmology is closely related to the single relativistic particle described by Klein-Gordon theory, and accordingly it inherits all the conceptual problems of the latter. One traditional way out of these problems, namely to switch to *many-particle* quantum theory, is not feasible in the context of cosmology. As is well known, another way is to replace the Klein-Gordon operator by its *square root*, the Dirac operator. This leads precisely to *supersymmetric quantum cosmology*[2].

However, it must be noted that this way of looking at things is in some contrast to another, probably more commonly stated way to express what should be the same: Usually, supersymmetry, and hence supergravity and supersymmetric cosmology, are motivated and defined by the requirement that the action of the given theory should be invariant under a certain exchange of bosonic and fermionic degrees of freedom. The special nature of this symmetry of the action is, after all, the reason for calling it a *super*symmetry. The difference in character between this formulation and the one emphasizing Dirac's square root leads directly to a central question of this text:

[1] Note that 'superspace' here has no relation to 'supersymmetry'.

[2] It is true that in the ordinary context the *single-particle* Dirac theory does *not* cure all the problems of the Klein-Gordon equation, since it suffers from the energy of the particle not being bounded from below. Hence, when interacting with its environment, for instance an electromagnetic field, the single Dirac electron is predicted by the Dirac equation to never find a stable ground state. But note that this problem is absent when applying the Dirac equation to dynamics in configuration space, because here no environment is present. The universe point in mini-superspace is truly isolated.

Motivation and Purpose. The purpose of this text is to investigate the application and applicability of supersymmetric quantum mechanics to (homogeneous) cosmological models derived from supergravity. One central motivation is the curious parallel existence of two different approaches to study such models:

On the one hand side, one has attempted to derive the dynamics of supersymmetric cosmologies by starting with the equations of full canonically quantized supergravity in functional Schrödinger representation and then trying to find (relatively) simple *Ansätze* for possible solutions. (Reviews are [83] and [197][3].) This has met with remarkable successes and has produced a couple of partly unexpected and deep insights. But it should be fair to say that it has also encountered difficulties and restrictions. These are ultimately due to the fact that, as opposed to ordinary quantum gravity, in the supersymmetric theory the *Ansatz* one makes is constrained by supersymmetry, so that *extending* known ordinary cosmological models to their supersymmetric versions, by finding a suitable model for the gravitino field, is non-trivial, or at least not automatic.

This is notably different in another, seemingly quite distinct method to approach the problem:

Once an ordinary quantum cosmological model has been constructed, one is left with the formal equivalent of a quantum mechanical system (albeit a covariant one being subject to a Hamiltonian constraint). This trivial fact makes it appear rather natural to apply methods known from the field of supersymmetric quantum mechanics to these models. In particular, it is well known how a given ordinary quantum mechanical system more or less uniquely extends (if it does at all) to one in supersymmetric quantum mechanics, a procedure that involves only straightforward manipulations. So this makes it appear quite compelling that the supersymmetric extension of a given ordinary cosmological has to be described, somehow, by the well known supersymmetric extension of the quantum mechanics of the ordinary model. And indeed, it turns out that such an extension is feasible for large classes of cosmological models. An account of its application to a variety systems is given in [25] and references therein.

In summary, one could perhaps roughly characterize the above two "routes" to supersymmetric quantum cosmology by saying that the first is based on the prescription: "First introduce supersymmetry into the full theory, then simplify to a cosmological model.", while the second instead follows the idea: "First simplify to a cosmological model, then introduce supersymmetry." Of course, in any particular case authors may use, and have used, a certain mixture of these prescriptions, but it seems worthwhile to identify and distinguish the two different principal ideas involved here. A diagrammatic account of the situation is given in figure 1 (p.13).

The remarkable advantage of the second method, namely that it admits a transparent, powerful, and elegant way to create and handle supersymmetric extensions of ordinary models *without* explicitly invoking elements of full-fledged supergravity, might at the same time be considered a fatal weakness: Namely doubts might be raised that the supersymmetric systems obtained this way have any direct relation to supergravity. Their quantum mechanics surely is super-

[3] These treat quantum supergravity mostly using the *vielbein* formulation. One can also study "loop quantum supergravity", by using the *connection* formalism instead (see §7 of [83]). Most investigations into supersymmetric cosmology, however, have been using the former approach and this is the one we shall stick to in the context of this text.

symmetric, by construction, but, in the light of all the simplifying assumptions involved in reducing the full bosonic field theory to a small finite number of degrees of freedom, it appears conceivable that one could loose too much information to be able to recover the correct supersymmetric version of a model, namely one which could also be obtained from full supergravity by the first of the above methods.

In other words the question is: "Does the diagram in figure 1 (p.13) commute?"

One goal of the present text is to attempt to shed some more light on this question. This is the content of §4 (p.181).

The other goal is to make use of the fact, that the answer seems in fact to be positive. Because if this is true (which, of course, is and has been expected and assumed by several researchers) it opens the possibility to study otherwise not as easily accessible aspects of supergravity by investigating the relevant properties of covariant supersymmetric quantum mechanics. In particular, one can try to study in covariant SQM issues such as:

- model building and model deformation
- conservation laws and conserved currents
- gauge transformations and gauge fixing
- scalar products and probability interpretation
- path integrals and stochastic models
- expectation values and statistical ensembles
- extended supersymmetry and central charges.

Attempts to approach some of these points are the content of §2 (p.14).

Finally, a purpose of this text is to try to apply, in §5 (p.255), theoretical insight to concrete cosmological models, in particular one deriving from 11-dimensional supergravity.

Outline. The organization of this text is as follows:

First, this introduction is completed (on p. 9) by a brief discussion of the existing literature on supersymmetric quantum cosmology, in order to put the present work in proper perspective.

Then, in §2 (p.14), we set out to discuss the theoretical framework of supersymmetric quantum theory of covariant relativistic point mechanics.

In §2.1 (p.15) selected basic concepts of supersymmetry are reviewed. Central to the further development is the close formal relationship of supersymmetric quantum mechanics (SQM) with differential geometry formulated by means of exterior and Clifford algebra, which is recalled in §2.1.1 (p.15). An overview of selected elements of supersymmetry, in §2.1.2 (p.37), then leads over to §2.1.3 (p.43), which concentrates on the graded extension of the $\mathbf{u}(1)$-algebra, being the basis of the next section.

There, in §2.2 (p.54), contact with quantum mechanics is made by discussing generalized Dirac operators and the Witten model of SQM (2.2.1 (p.55)). A maybe surprising formal relation of covariant SQM with classical electromagnetism, which will proof to be quite useful, is detailed in §2.2.3 (p.70). Insights gained here immediately open a way to obtain conservation laws and conserved currents in §2.2.4 (p.78). §2.2.5 (p.81) looks at a generalization of the Feynmann checkerboard model of the Dirac particle to SQM. This will help interpreting the numerical results given later on in §5 (p.255). A further crucial tool in theoretically understanding SQM systems are their symmetries. These are discussed in §2.2.7 (p.90). In order to gain also more concrete information about a system §2.2.8 (p.100) lists some general methods for finding formal and numeric solutions to covariant SQM systems.

While up to this point most of the discussion applies equally well to Riemannian as well as to pseudo-Riemannian configuration spaces, §2.3 (p.106) tries to come to terms with the covariant nature of "relativistic" supersymmetric quantum mechanics, which amounts to tackling the problem of gauge fixing. Since the procedure proposed here, though drawing heavily on well established concepts, may perhaps seem somewhat unorthodox, this section starts with a detailed outline of its main result in §2.3.1 (p.107). Stepping back again, §2.3.2 (p.115) reviews aspects of gauge theory and BRST-cohomology formalism as far as necessary for the present context. On this basis it is shown in §2.3.4 (p.134) that so called *graded operators of Dirac type* qualify as BRST-operators, and that, furthermore, such operators are naturally obtained from the present supercharges by modding out a generic symmetry. This result is then employed in §2.3.5 (p.140) to construct gauge-fixed expectation values and, in particular, gauge fixed scalar products. The technical definition and derivation of 'ghost' algebra within the superalgebra, necessary to make this approach work, is listed and derived in §E (p.330).

After being familiar with covariant supersymmetric quantum mechanics, §4 (p.181) sets out to see if related structures can be discovered in full-fledged quantum supergravity. After recalling basic concepts of ordinary quantum gravity in §4.1 (p.181), the canonical formalism of quantum supergravity is very briefly reviewed in §4.2 (p.187). As a preparation for the discussion to follow, the supersymmetric field theory consisting of two scalar and one Dirac field is studied in §3.1 (p.143) using the paradigm of the Witten model of SQM. This then motivates the investigation of canonical supergravity in a mode-amplitude basis, which is the content of §4.3.1 (p.193). Concentrating on only one mode of the supersymmetry constraint naturally leads in §4.3.2 (p.230) to homogeneous cosmological models as approximations to the full dynamics. It is shown in §4.3.3 (p.240) how from there one finally arrives at covariant SQM in mini-superspace. This result, being derived in the context of $N = 1, D = 4$ supergravity, is argued in §4.3.4 (p.250) to carry over to N-extended and higher dimensional theories.

After this purely theoretical material, §5 (p.255) is concerned with applications of supersymmetric quantum mechanics to quantum cosmology. §5.1 (p.255) makes contact with existing literature by re-examining well known models of 4-dimensional supergravity in the light of the above results. Since a more natural arena for supersymmetric cosmology might be 11-dimensional supergravity, §5.2 (p.266) is devoted to studying a Bianchi-I model in this higher-dimensional theory. The dimensional reduction analyzed in §5.2.1 (p.267) follows existing literature, but retains the general homogeneous 3-form field. Classical and quantum solutions to this model are presented in §5.2.2 (p.275).

Finally, §6 (p.291) gives a summary and conclusions and lists some open questions.

The appendices contain material which has been separated from the main text for convenience:

§B (p.297) and §C (p.319) continue the discussion of §2.1.1 (p.15) and §2.2.7 (p.90), respectively, giving more detailed definitions and results, which will be referred to from the main text as needed.

Similarly, §D (p.328) collects proofs and calculations which were previously omitted.

The construction of the ghost algebra representations discussed in §2.3 (p.106) is given in §E (p.330).

Finally, appendix §G (p.342) lists the spinor conventions necessary for canonical quantum gravity, which are used throughout §4 (p.181).

Historical overview: Lagrangian and Hamiltonian approaches to supersymmetric quantum mechanics and quantum cosmology.

Outline. The following brief (and certainly not exhaustive) overview of the existing literature on supersymmetric quantum cosmology should serve the purpose of putting the material of the following sections in proper perspective.

Two different routes to supersymmetric extensions of ordinary quantum mechanical systems are identified (*cf.* figure 1 (p.13)):

- Within the *Lagrangian approach*, as it shall be called here, one searches and studies supersymmetric extensions of the non-supersymmetric *action functional* of the system under consideration. This approach has received a lot of attention, its main proponents being D'Eath et al.

- In the context of what in the following shall loosely be called the *Hamiltonian approach* one instead searches and studies supersymmetric extensions at the level of quantum *operators*, i.e. one constructs supersymmetric extensions of the *Hamiltonian operator* without explicitly considering a supersymmetric action. The application of this technique to supersymmetric quantum cosmology has been developed by Graham et al. It is this approach, that §5 (p.255) will follow.

The *Lagrangian* (*functional*) and the *Hamiltonian* (*operator based*) method of supersymmetric quantization are complementary and equivalent, as has already been noticed shortly after the inception of supersymmetry in the mid 1970's, e.g. by Teitelboim (see below). In spite of its valuable advantages, the Hamiltonian approach has maybe not yet received due attention.

(Finally note that these issues of supersymmetric quantization are not at all specific to cosmology but apply to all kinds of covariant supersymmetric quantum mechanical systems.)

There are two complementary aspects to all quantum theories: One is their *global/integral* appearance in form of the action functional and path integral. The other is their *local/differential* realization by means of the operator formalism. A supersymmetric extension of a quantum theory can be realized in both setups, and both methods have been applied when supersymmetric cosmology was actively developed in the 1980s:

The approach via the action formalism has been studied mainly by D'Eath, as well as by Moniz, Macias, Obregon, Ryan, Hawking, and others: Based on the studies by D'Eath on canonically quantized supergravity ([80], see [83] and [197] for reviews), several supersymmetrically extended cosmological models have been considered (e.g. [81][84][7][198][49][196][199][200][48]). The supersymmetric extension of an ordinary cosmological model in the action formalism requires an ansatz for the *gravitino field*, the superpartner of the tetrad, such that the extended action of the cosmological model becomes invariant under supersymmetry transformations. However, finding such an ansatz is not straightforward, far less automatic. This might be one reason why it was common believe, for some time, that the only solutions to supersymmetric FRW-cosmology were of very restricted kind (namely residing either in the empty or in the completely filled *fermion sector*). When in [70] [69] more general solutions were discovered (*cf.* 4.35 (p.218)) it was recognized that the form of the fermionic wave function was taken to be unnecessarily restrictive in [84][81] [7].

The supersymmetric extensions of ordinary quantum mechanical systems and their solutions can often be treated more transparently when supersymmetry is implemented at the operator level:

The Hamiltonian approach to supersymmetric quantization of cosmological models, which has been followed mainly by Graham, as well as by Bene, Luckock, and others ([110][111][112][113][25]), concentrates on the Hamiltonian operator that governs the system's dynamics. Here supersymmetry is implemented by finding formal 'square roots' of this operator, the supersymmetry generators:

In [113] is says

> Quantizing the system in [this way] leads to a supersymmetric quantum cosmology which provides a new and perhaps more promising framework for posing the old questions of quantum cosmology. This new framework would seem to be the most natural one, if, indeed, it turns out that supersymmetry is a fundamental (but spontaneously broken) symmetry of nature. Therefore this approach to quantum cosmology is worth developing further.

While intuition strongly suggests a close connection, the exact relation between the Lagrangian and the Hamiltonian route to supersymmetric cosmology has remained somewhat vague, but probably merely due to lack of investigation. In [197] it is remarked that:

> It is our thinking, that a differential operator representation for the fermionic variables constitute the rightfull approach. [...] There are, however, other approaches. Ashtekar and loop variables were used in [141] [239] [206] [39] [40] [191] [78] [238] [6].[4] The method employed in [110] [113] [112] [114] is based on a σ-model approach to supersymmetric quantum mechanics. Finally, a matrix representation for the gravitino was used in ref. [179] [207] [245] [180]. All these approaches share some similarities but also have specific differences in the method and results. Moreover, a clear analysis establishing *if* (up to any extent) and *how* they are related is yet to be achieved.[5]

[4] Our reference numbering.

[5] The "differential operator representation for fermionic variables", stemming from *superanalysis* (*cf.* §B.3 (p.314)), refers to the functional Schrödinger representation of canonical quantum supergravity (*cf.* §4.2 (p.187)). The "σ-model" approach refers to articles by Graham which feature what is called the Hamiltonian approach here.

In this context it may be worth looking at some early papers on the subject:

In 1972, slightly before the advent of supergravity, [234] already mentions the possibility of taking a Dirac-like square root of the Wheeler-DeWitt-equation, but concludes (p. 37):

> [...] the fact that there are the two spinor degrees of freedom is disturbing [...] the interpretation of the "spin" states ψ_+ and ψ_- is not straightforward in terms of any known physical attributes of the universe.

In 1975 ([235]) it still reads (p. 197)

> We will only mention that the Dirac method[6] leads to a linear two-component spinor equation, and at present we lack an experimental quantity to associate with the spinor components [...].

But in 1977 Teitelboim recognizes, that the spinning point particle (i.e. Dirac's electron), the spinning string (i.e. the superstring), and supergravity (the 'spinning brane') all arise by the same square root process: In [256] and [251] he discusses the relation between supersymmetry at the Hamiltonian and the Lagrangian level and shows how supersymmetry is elegantly implemented by taking the square root of the constraints, which yields a result equivalent to writing down a supersymmetrically extended Lagrangian:

> It is important to emphasize that it is the Hamiltonian form of the supersymmetric theory which is obtained directly from the Hamiltonian form of the original theory [...] without going through a Lagrangian. Therefore, although finding a Lagrangian which yields the constraints (1a) and (1b)[7] is an interesting problem, it is not a necessary step.
>
> [...]
>
> Taking the square root of the constraints of the spinless string is not quite the way in which the theory of the spinning string was originally arrived at. However, again here the Hamiltonian form of the theory is the one most immediately accessible. In this case too, finding a Lagrangian which will yield all the constraints of the new theory is an interesting problem but is not a necessary step.
>
> [...]
>
> General relativity fits very well and naturally in the above scheme.
>
> [...]
>
> It is therefore natural to ask whether it is possible to take the square root of the generators of surface deformations[8] and endow in this way each point of space with a new degree of freedom associated with an intrinsic spin structure. One may attack the problem directly in terms of the deformation generators $\mathcal{H}_\mu$. [...] However, it is one of the purposes of this letter to point out that the answer already exists and is given by supergravity theory as formulated by Freedman, van Nieuwenhuizen, and Ferrara and by Deser and Zumino.

[6]Applied to the Wheeler-DeWitt equation.

[7]This refers to the supersymmetry constraints and their square.

[8]In the Hamiltonian formulation of general relativity, see §4.1 (p.181).

(Quoted from [256].)

Teitelboim also remarks that the operator approach features several advantages, among them the fact that the behavior of all fields under supersymmetry transformations immediately follows from the supersymmetry constraint operator (*cf.* note 2.56 (p.58)):

> Again here the Fermi generators $\mathcal{S}_+, \mathcal{S}_-$ mix Bose and Fermi variables and therefore generate supersymmetry transformations (this indeed appears to be where the idea of supersymmetry originally came from!).

(Quoted from [256].)

The 'square root'-idea caught some attention. An important paper in quantum cosmology dedicated to this concept is [179]. But remarkably, even though [179] has the *supersymmetric square root* in its title, the authors actually do extend the *action* supersymmetrically by constructing a spinor field on spacetime, which renders this reference an example of the Lagrangian approach. As Teitelboim emphasizes and demonstrates, such an explicit construction of the supersymmetric action is not necessarily required. In [251] it is noted:

> In this note we take the view that the natural way to introduce spin into a physical system is to take the square root à la Dirac, of the Hamiltonian constraints which generate spacetime evolution of the system without spin.
>
> [...]
>
> Hence from our point of view supersymmetry and the introduction of spin are both consequences of the square root process.

As [25] shows, and as shall be exhibited in §5 (p.255), model building with respect to the fermionic sector is much more systematic in the Hamiltonian approach. This is established in detail in §2.2.1 (p.55) and §4.3.3 (p.240).

A schematic diagram illustrating the two possible routes to arrive at a supersymmetric quantum mechanics starting from an ordinary action function is displayed in figure 1 (p.13). In §4 (p.181) it is shown that this diagram does indeed commute in at least some sense.

Figure 1

$$\begin{tikzcd}
 & \text{ordinary} & & \text{supersymmetric} \\
\text{full Lagrangian} & \mathcal{L}\big(e^a{}_i, p_a{}^i, \mathcal{H}\big) \arrow[rr, "{[83][197]}"] \arrow[d] & & \mathcal{L}\Big(e^a{}_i, p_a{}^i, \psi^A{}_i, \bar{\psi}^{A'}{}_i, \mathcal{H}, \mathcal{S}, \bar{\mathcal{S}}\Big) \arrow[d] \\
\text{reduced Lagrangian} & L(q, p, H) \arrow[rr, dashed, "{[166]\ [9]\ [178]}"] \arrow[d] & & L\big(q, p, \psi, \bar{\psi}, H, S, \bar{S}\big) \arrow[d] \\
\text{reduced Hamiltonian} & H(q, p) \arrow[rr, dashed] \arrow[d] & & H\big(q, p, \psi, \bar{\psi}\big) = \big\{S, \bar{S}\big\} \arrow[d] \\
\text{quantum Hamiltonian} & \hat{H}(\hat{q}, \hat{p}) \arrow[rr, "{[25][193]}"] & & \hat{H}\Big(\hat{q}, \hat{p}, \hat{\psi}, \hat{\bar{\psi}}\Big) = \Big\{\hat{S}, \hat{\bar{S}}\Big\}
\end{tikzcd}$$

Routes to dimensionally reduced supersymmetric quantum theory. The diagram indicates (very schematically, all expressions are supposed to be merely suggestive) several ways, mentioned on p. 9, to arrive at a supersymmetric quantum theory starting from a non-supersymmetric field theory. Following the thin solid arrows, the ordinary Lagrangian $\mathcal{L}$ can be made supersymmetric by incorporating fermionic fields and supersymmetry generators $\mathcal{S}$, $\bar{\mathcal{S}}$. Dimensional reduction then leads to a super-mechanical system (*cf.* [85]) which may be quantized to give supersymmetric quantum mechanics described by the Hamiltonian operator $\hat{H} = \Big\{\hat{S}, \hat{\bar{S}}\Big\}$. Alternatively, following the fat solid arrows, the original (bosonic) Lagrangian may be dimensionally reduced and quantized. Supersymmetry can then be implemented (*cf.* [25]) by finding the Dirac square-roots of the Hamiltonian operator. This text will mainly follow Graham et al. along the latter route. See §4 (p.181) and in particular §4.3.2 (p.230) and §4.3.3 (p.240) for a discussion of the relation between the two different routes in the context of supergravity.

2 Covariant Supersymmetric Quantum Mechanics

Outline. This section is concerned with the formal structure of supersymmetric quantum theory of covariant relativistic point mechanics. Well known material from various maybe seemingly unrelated fields (e.g. Clifford analysis, Witten model, Dirac electron, BRST theory) is accompanied by some original results and arranged in a coherent fashion in order to give a self-contained picture of the theory necessary to follow the "Hamiltonian route" to supersymmetric quantization (as outlined in the introduction). The motivation for this entire development is the success of the method discussed in [113] and [25], which consists of applying the Witten model of supersymmetric quantum theory to covariant systems governed by Hamiltonian constraints[9].

Some basic mathematical background is given in §2.1 (p.15). The underlying idea is to extend an ordinary $\mathbf{u}(1)$-algebra generated by a Hamiltonian $\mathbf{H}$ to a graded superalgebra containing formal square roots $\mathbf{D}$ of $\mathbf{H}$, $\mathbf{D}^2 = \mathbf{H}$. Special emphasis is put on concepts of Clifford and exterior calculus (§2.1.1 (p.15)), which are essential in implementing the formal square root process and which give rise to fruitful relations with differential geometry.

The step from formal superalgebras to supersymmetric quantum mechanics is done in §2.2 (p.54) by concentrating on Hamiltonian generators $\mathbf{H}$ of the form of generalized Laplace (or rather d'Alembert) operators on (pseudo-)Riemannian manifolds. The resulting square roots are generalized Dirac operators (§2.2.1 (p.55)). Their properties are investigated in §2.2.4 (p.78)- §2.2.7 (p.90). Some more technical material has been moved to the appendix (in particular to §B (p.297)). The reader will be pointed to definitions and results found there as need arises.

In §2.2.3 (p.70) it is established that it is useful to recognize the appearance of the formal SQM algebra in ordinary classical electromagnetism.

§2.3 (p.106) is concerned with the problem of gauge fixing in covariant supersymmetric quantum mechanics. The main idea here is to recognize on the one hand side the close relationship between BRST operators and supersymmetry generators, and on the other hand the geometric implication of switching from Riemannian to pseudo-Riemannian geometry, whereby the ordinary Laplace operator ceases to be elliptic. It is shown that the nilpotent components of the Dirac operator may essentially serve as BRST operators, which geometrically implies finding an elliptic substitute for the hyperbolic pseudo-Laplacian.

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[9] Note, for sake of comparison, that the superalgebra in these papers is presented in its 'polar' form with *nilpotent* charges $Q^2 = 0$ and ${Q^\dagger}^2 = 0$, satisfying

$$\left\{Q, Q^\dagger\right\} = \mathbf{H}\,.$$

By a simple linear transformation $\mathbf{D}_1 = Q + Q^\dagger$, $\mathbf{D}_2 = i\left(Q - Q^\dagger\right)$, one obtains the 'diagonal' form of this algebra:

$$\{\mathbf{D}_i, \mathbf{D}_j\} = 2\delta_{ij}\mathbf{H}\,,$$

which will be mostly used in this section. See §2.1.3 (p.43) and in particular 2.36 (p.46) for more details).

2.1 The idea of supersymmetry

Despite its fancy name, supersymmetry is a very simple and natural concept. The prefix 'super' merely indicates that one is concerned with *graded* mathematical objects, like graded vector spaces or graded algebras. Well known (and very important) examples of such graded structures are Grassmann and Clifford algebras, which are graded in the sense that elements of the algebra are products of either an *even* or an *odd* number of algebra generators. 'Supersymmetry', which could just as well be called 'graded symmetry', is then nothing but a graded (Lie-)algebra of symmetry generators of some physical system.

While the supersymmetric extension of the *standard model* still awaits its experimental verification (at time of this writing, *cf.* [96][162]), it is noteworthy that some well known physical systems do indeed already supersymmetry (for instance *cf.* 2.2.3 (p.70)).

Literature. A nice recent survey of supersymmetry with emphasis on supersymmetric quantum mechanics as well as BRST theory is [229]. The relation of supersymmetric quantum mechanics to differential geometry, which plays a central role in the following development, is particularly emphasized in [101]. Some introductory textbooks are [146], which for the most part consideres supersymmetric quantum mechanics, and [100], as well as [247], which are more concerned with field theory. A helpful recent review is also [167].

2.1.1 Differential Geometry with Exterior and Clifford algebra

Outline. The purpose of this section is to present notation, concepts, techniques and results needed for the treatment of supersymmetric quantum mechanics in differential geometric form. In 2.2 below, a good deal of notation and basic concepts, which will be used freely throughout this text, are briefly and roughly introduced. More details and further definitions and results are given in §B (p.297). The main text will refer to the material there as needed.

2.1 (Literature) Introductions to differential geometry and exterior calculus are for instance [98], [92], and [91]. Exterior and Clifford calculus as tools in supersymmetric physics are discussed e.g. in [101], [227]. The emphasis on the geometric aspect of supersymmetry goes back to [275], [274]. The identification of Fermions with differential forms (see 3.1 (p.143) for an example of how this identification arises in field theory) is referred to as a *basic tautology* in §11.9.1 of [57]: Both, Fermions and differential forms, are represented by antisymmetric tensors. How exactly the differential geometric structure emerges for instance in the case of the $N = 2$ supersymmetric non-linear σ-model is shown in detail in [53]. For the more field theoretic identification of the exterior bundle (the bundle of differential forms) with the fermionic Fock space see [233] and [160].

In the context of "Geometric Algebra" (*cf.* [125], [130], [128]), where one focuses on expressing physics in terms of Clifford algebra and Clifford calculus, there has also been work on supersymmetric mechanics and its relation to Dirac operators and differential geometry, which seems to be rather independent of the more standard texts mentioned above. See for instance [14] [15] [16] [17]. Lagrangians for Dirac equations in Clifford formalism are discussed in [77], while general field theory Lagrangians are discussed in the Geometric Algebra framework in [164]. The parallels between supercalculus (*cf.* §B.3 (p.314)) and Clifford techniques are expanded on in [263] [226].

2.2 (Some Elements of Differential Geometry with Exterior and Clifford Algebra) The basic setting for all of the following is an orientable (semi-)Riemannian manifold $(\mathcal{M}, g)$ of *dimension* D with *metric* $g = (g_{\mu\nu})$ (and inverse metric $g^{-1} = (g^{\mu\nu})$), having s negative and $D - s$ positive eigenvalues.

2.3 (Differential forms)

Let $\{x^\mu\}_{\mu\in\{0,1,\cdots,D-1\}}$ be coordinates of a coordinate chart on $\mathcal{M}$ and let ∂_μ be the partial derivative operators on functions $f : \mathcal{M} \to \mathbb{R}$ with respect to x^μ. These derivative operators span at each point $x \in \mathcal{M}$ the *tangent space* $T_x(\mathcal{M})$. Its dual is the *cotangent space*, $T^*(\mathcal{M})$, spanned by the dual basis of *differential forms* dx^μ, which satisfy the relation

$$dx^\mu(\partial_\nu) = \delta^\mu_\nu \,.$$

The manifold of tangent spaces over $\mathcal{M}$ is the *tangent bundle* $T(\mathcal{M})$, and similarly for the *cotangent bundle* $T^*(\mathcal{M})$. *Vector fields* v over $\mathcal{M}$ are sections of $T(\mathcal{M})$ and locally have the form $v = v^\mu(x)\,\partial_\mu$. *Covector fields*, or *1-forms* for short, are analogously sections of $T^*(\mathcal{M})$ with local form $\alpha = \alpha_\mu(x)\, dx^\mu$. The *wedge product* $\alpha \wedge \beta$ of two 1-forms α and β is defined on each $T_x(M) \otimes T_x(M)$ by

$$\alpha \wedge \beta(v, w) = \alpha(v)\,\beta(w) - \beta(v)\,\alpha(w)$$

and called a *2-form*. Similarly p-forms of higher degree are completely antisymmetrized tensor products of forms of lower degree. The space of p-forms will here be denoted by

$$\Lambda^p(\mathcal{M}) = \bigwedge_p T^*(\mathcal{M}) \,. \tag{1}$$

A basis for these are the coordinate p-forms, so that section of $\Lambda^p(\mathcal{M})$ can be written locally as[10]

$$\omega = \omega_{\mu_1\mu_2\cdots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} \,.$$

[10] In practice there always arises the issue of wether to sum over all indices, thereby counting every term carrying a set of p indices $p!$ times, or to instead sum over distinct index sets only:

$$\omega_{\mu_1\mu_2\cdots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} \;=\; p! \sum_{0\leq\mu_1<\mu_2<\cdots<\mu_p<D} \omega_{\mu_1\mu_2\cdots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} \,.$$

If one wants to identify forms with their coefficient tensors then the latter is more convenient, since for instance

$$\begin{aligned}
& \sum_{0\leq\mu_1<\cdots\mu_p<D} \omega_{\mu_1\mu_2\cdots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} \left(v^\lambda\partial_\lambda, \cdots\right) \\
= \;& \frac{1}{p!}\omega_{\mu_1\mu_2\cdots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} \left(v^\lambda\partial_\lambda, \cdots\right) \\
= \;& \frac{1}{(p-1)!} v^\lambda \omega_{\lambda\mu_2\cdots\mu_p} dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} \\
= \;& \sum_{0\leq\mu_2<\cdots\mu_p<D} v^\lambda \omega_{\lambda\mu_2\cdots\mu_p} dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} \,,
\end{aligned}$$

and hence the actions of forms on vectors is given by the usual index contraction with their summation-ordered coefficients. Now let α be a p-form and β a q-form and write for the

A 0-form $f \in \Lambda^0(\mathcal{M})$ is identified with an ordinary function on $\mathcal{M}$. A $(p = D)$-form (sometimes called a "top form")

$$\begin{aligned} \omega &= \omega_{\mu_1\mu_2\cdots\mu_D} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_D} \\ &= D!\, \omega_{012\cdots D-1} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{D-1} \end{aligned} \tag{4}$$

naturally defines a measure on $\mathcal{M}$ as follows:

$$\begin{aligned} & \int\limits_{\mathcal{M}} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{D-1} := \int\limits_{\mathcal{M}} dx^0 dx^1 \cdots dx^{D-1} \\ \Leftrightarrow \quad & \int\limits_{\mathcal{M}} \omega := D! \int\limits_{\mathcal{M}} \omega_{12\cdots D}\, dx^0\, dx^1 \cdots dx^{D-1}\,. \end{aligned} \tag{5}$$

(Here the left hand side is defined by the right hand side, which is ordinary Lebesgue integration over the given coordinate chart.) The integral over any non-top form may be defined to vanish

$$\int\limits_{\mathcal{M}} dx^{\mu_1} \wedge dx^{\mu_2} \cdots \wedge dx^{\mu_p} := 0, \quad \forall\, p < D\,. \tag{6}$$

The sum of all p-form bundles will be called the *exterior bundle*, denoted by

$$\Lambda(\mathcal{M}) = \bigoplus_{p=0\cdots D} \Lambda^p(\mathcal{M})\ . \tag{7}$$

Sections of this bundle are *inhomogeneous forms*, i.e. objects that locally read

$$\omega = \omega^0 + \omega^1_\mu dx^\mu + \omega^2_{\mu\nu} dx^\mu \wedge dx^\nu + \cdots\,.$$

For reasons that will become clear below, we will frequently, but not compulsorily, adopt Dirac "ket"-notation and write interchangeably

$$\omega = |\omega\rangle \tag{8}$$

---

ordered sum over coefficients for brevity

$$\alpha = \sum_{\vec{I}} \alpha_I dx^I,\ \beta = \sum_{\vec{J}} \beta_J dx^J\,,$$

with multi-indices I, J. In terms of the coefficients of ordered summation the wedge product reads:

$$\begin{aligned} \alpha \wedge \beta &= \sum_{\vec{I}} \alpha_I \sum_{\vec{J}} \beta_J dx^I \wedge dx^J \\ &= \frac{1}{p!q!} \alpha_I \beta_J dx^I \wedge dx^J \\ &= \frac{1}{p!q!} \alpha_{[I} \beta_{J]} dx^I \wedge dx^J \\ &= \frac{(p+q)!}{p!q!} \sum_{\vec{IJ}} \alpha_{[I} \beta_{J]} dx^I \wedge dx^J\,. \end{aligned} \tag{2}$$

Therefore in terms of antisymmetric tensors the wedge product may be defined by

$$(\alpha \wedge \beta)_{\mu_1\mu_2\cdots\mu_{(p+q)}} = \frac{(p+q!)}{p!q!} \alpha_{[\mu_1\cdots\mu_p} \beta_{\mu_{p+1}\cdots\mu_{p+q}]}\,. \tag{3}$$

This definition is used for instance in [269], appendix B.

for $\omega \in \Lambda(\mathcal{M})$.

On each tangent space the metric g induces an inner product, $(v,w) = v^\mu g_{\mu\nu} w^\mu$, and similarly for the cotangent space: $(\alpha,\beta) = \alpha_\mu g^{\mu\nu}\beta_\nu$. It is often useful to (pseudo-)orthonormalize these spaces by introducing a *vielbein* 1-form $e^a = e^a{}_\mu dx^\mu$, which satisfies

$$e^a{}_\mu g^{\mu\nu} e^b{}_\nu = \eta^{ab}\,.$$

Here

$$\eta = (\eta_{ab}) = \left(\eta^{ab}\right) = \operatorname{diag}\left(\underbrace{-,-,\cdots,-}_{s},\underbrace{+,+,\cdots,+}_{D-s}\right)$$

is sometimes called the *spin frame metric*, where *spin frame* refers to the vielbein frame. (This is because a vielbein frame is useful in order to define spinor fields on $\mathcal{M}$.) Conversely this gives

$$e^a{}_\mu \eta_{ab} e^b{}_\nu = g_{\mu\nu}\,.$$

The dual to e^a is denoted $\tilde{e}_a = \tilde{e}_a{}^\mu \partial_\mu$ and defined by

$$\begin{aligned} e^a(\tilde{e}_b) &= \delta^a_b \\ \Leftrightarrow e^a{}_\mu\, \tilde{e}^\mu{}_b &= \delta^a_b\,. \end{aligned} \tag{9}$$

The top-form obtained by wedging together all vielbein forms

$$\begin{aligned} \mathrm{vol} &:= e^0 \wedge e^1 \wedge \cdots \wedge e^{D-1} \\ &= \sqrt{|g|}dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{D-1} \end{aligned} \tag{10}$$

is called the *volume form* of $\mathcal{M}$.

2.4 (Creator/annihilator algebra)

The exterior bundle $\Lambda(\mathcal{M})$ may be regarded as the "Fock space" of forms: Define a coordinate basis of *form creation operators*

$$\hat{\mathrm{c}}^{\dagger\mu} : \Lambda(\mathcal{M}) \to \Lambda(\mathcal{M}) \tag{11}$$

(the dagger † is so far pure notation, its meaning will become clear below) by exterior multiplication:

$$\hat{\mathrm{c}}^{\dagger\mu}\omega \;:=\; dx^\mu \wedge \omega\,, \tag{12}$$

so that

$$\Lambda^p(\mathcal{M}) \xrightarrow{\hat{\mathrm{c}}^{\dagger\mu}} \Lambda^{p+1}(\mathcal{M}) \tag{13}$$

for $p < D$. The 0-forms are hence "form vacua" and the constant unit 0-form, denoted by $|0\rangle \in \Lambda^0(\mathcal{M})$, will be simply called the *vacuum*. Hence every p-form may be written as

$$\begin{aligned} \omega &= \omega_{\mu_1\mu_2\cdots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} \\ &= \omega_{\mu_1\mu_2\cdots\mu_p} \hat{\mathrm{c}}^{\dagger\mu_1}\hat{\mathrm{c}}^{\dagger\mu_2}\cdots\hat{\mathrm{c}}^{\dagger\mu_p}\,|0\rangle\,. \end{aligned} \tag{14}$$

It follows that the $\hat{c}^{\dagger\mu}$ anticommute among each other:

$$\left\{\hat{c}^{\dagger\mu}, \hat{c}^{\dagger\nu}\right\} = 0 \,. \tag{15}$$

Form creators with respect to the (pseudo-)orthonormal vielbein frame are defined by

$$\begin{aligned} \hat{e}^{\dagger a} &:= e^a{}_\mu \, \hat{c}^{\dagger\mu} \\ \Leftrightarrow \hat{c}^{\dagger\mu} &= \left(e^{-1}\right)^\mu{}_a \hat{e}^{\dagger a} \,. \end{aligned} \tag{16}$$

With form creators there should also be form annihilators (operators of "inner multiplication"). Hence define operators $\hat{c}_\mu$

$$\hat{c}_\mu : \Lambda(\mathcal{M}) \to \Lambda(\mathcal{M}) \tag{17}$$

by the relations

$$\hat{e}_\mu \left|0\right\rangle := 0 \tag{18}$$

and

$$\left\{\hat{c}_\mu, \hat{c}^{\dagger\nu}\right\} := \delta_\mu^\nu \,. \tag{19}$$

The index of these operators is shifted as usual,

$$\hat{c}^\mu := g^{\mu\nu} \hat{c}_\nu \,, \tag{20}$$

so that

$$\left\{\hat{c}^\mu, \hat{c}^{\dagger\nu}\right\} = g^{\mu\nu} \,, \tag{21}$$

and hence these operators do indeed reproduce the inner product, e.g.

$$\begin{aligned} \alpha_\mu \hat{c}^\mu \, \beta_\mu dx^\mu &= \alpha_\mu \hat{c}^\mu \, \beta_\mu \hat{c}^{\dagger\mu} \left|0\right\rangle \\ &\overset{(18)}{=} \alpha_\mu \beta_\nu \left\{\hat{c}^\mu, \hat{c}^{\dagger\nu}\right\} \left|0\right\rangle \\ &\overset{(19)}{=} \alpha_\mu g^{\mu\nu} \beta_\nu \left|0\right\rangle \\ &= \alpha_\mu g^{\mu\nu} \beta_\nu \,. \end{aligned} \tag{22}$$

Therefore, for $p > 0$,

$$\Lambda^p(\mathcal{M}) \xrightarrow{\hat{c}^\mu} \Lambda^{p-1}(\mathcal{M}) \,. \tag{23}$$

As before, the (pseudo-)orthonormal version of these operators is obtained by means of the vielbein:

$$\hat{e}^a := e^a{}_\mu \hat{c}^\mu \,, \tag{24}$$

and one has

$$\left\{\hat{e}^a, \hat{e}^{\dagger b}\right\} = \eta^{ab} \,. \tag{25}$$

The application of a differential form on a vector field may now be expressed algebraically by operator manipulations: For a form $\omega = \omega_{\mu_1\mu_2\cdots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}$ and a vector field $v = v^\mu \partial_\mu$ the expression $\omega(v, \cdots) = p\, v^{\mu_1} \omega_{\mu_1\mu_2\cdots\mu_p} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}$ corresponds to

$$\begin{aligned} \left[v^\nu \hat{c}_\nu, \omega\right]_\iota &= \left[v^\nu \hat{c}_\nu, \omega_{\mu_1\mu_2\cdot\mu_p} \hat{c}^{\dagger\mu_1} \hat{c}^{\dagger\mu_2} \cdots \hat{c}^{\dagger\mu_p}\right]_\iota \\ &= p\, v^{\mu_1} \omega_{\mu_1\mu_2\cdot\mu_p} \hat{c}^{\dagger\mu_2} \cdots \hat{c}^{\dagger\mu_p} \,. \end{aligned} \tag{26}$$

(Here $[\cdot,\cdot]_\iota$ denotes the *supercommutator*, which is the anticommutator if both its arguments are of odd degree and the usual commutator otherwise. See 2.1.2 (p.37) for more details.)

The form *number operator* is defined by

$$\begin{aligned} \hat{N} &:= \hat{c}^{\dagger\mu} \hat{c}_\mu \\ &= \hat{e}^{\dagger a} \hat{e}_a \,. \end{aligned} \tag{27}$$

It has as eigenvalues the degree of forms:

$$(\alpha \in \Lambda^p(\mathcal{M})) \Rightarrow \left(\hat{N}\alpha = p\,\alpha\right) , \tag{28}$$

which follows, for instance, from the relations

$$\begin{aligned} \left[\hat{N}, \hat{c}^{\dagger\mu}\right] &= \hat{c}^{\dagger\mu} \\ \Leftrightarrow \left[\hat{N}, \hat{c}^{\mu}\right] &= -\hat{c}^{\mu} \end{aligned} \tag{29}$$

and

$$\hat{N}\,|0\rangle = 0\,. \tag{30}$$

2.5 (Inner product)

There is a natural local inner product $\langle\cdot|\cdot\rangle_{\mathrm{loc}}$ on $\Lambda^p_x(\mathcal{M})$ with respect to which $\hat{c}^\mu$ and $\hat{c}^{\dagger\mu}$ are mutually adjoint: Let $\alpha, \beta \in \Lambda^p(\mathcal{M})$ with local form

$$\begin{aligned} \alpha &= \sum_{0\leq\mu_1<\mu_2<\cdots<\mu_p} \alpha_{\mu_1\mu_2\cdots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \\ \beta &= \sum_{0\leq\mu_1<\mu_2<\cdots<\mu_p} \beta_{\mu_1\mu_2\cdots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \,, \end{aligned} \tag{31}$$

and define

$$\langle\alpha|\beta\rangle_{\mathrm{loc}} := \sum_{0\leq\mu_1<\mu_2<\cdots<\mu_p} \alpha_{\mu_1\mu_2\cdots\mu_p} \beta^{\mu_1\mu_2\cdots\mu_p} \mathrm{vol}\,. \tag{32}$$

The factor vol is there for later convenience, as will become clear shortly. Because of it this local inner product is not scalar valued, but pseudo-scalar valued.

It is clear that

$$\langle\alpha|\beta\rangle_{\mathrm{loc}} = \langle\beta|\alpha\rangle_{\mathrm{loc}} \tag{33}$$

and furthermore that, for $\alpha \in \Lambda^p(\mathcal{M})$ and $\beta \in \Lambda^{p+1}(\mathcal{M})$,

$$\left\langle \hat{c}^{\dagger\mu}\alpha|\beta\right\rangle_{\mathrm{loc}} = \langle\alpha|\hat{c}^{\nu}\beta\rangle_{\mathrm{loc}}\,. \tag{34}$$

The local inner product defines the *Hodge star* operator $*$ by the requirement that $*$ be the unique operator on $\Lambda(\mathcal{M})$ for which

$$\langle\alpha|\beta\rangle_{\mathrm{loc}} := \alpha\wedge *\beta\,. \tag{35}$$

Because of (32) it is clear that

$$\Lambda^p(\mathcal{M}) \xrightarrow{*} \Lambda^{D-p}(\mathcal{M})\ . \tag{36}$$

Note that

$$*\,|0\rangle = \mathrm{vol}\,. \tag{37}$$

Further properties and local representations of the Hodge star operator are discussed in some detail in §B (p.297) (see in particular B.15 (p.305)) and are not further considered here.

Now define the global *Hodge inner product* by

$$\begin{aligned}\langle\alpha|\beta\rangle &:= \int\limits_{\mathcal{M}} \langle\alpha|\beta\rangle_{\mathrm{loc}} \\ &= \int\limits_{\mathcal{M}} \alpha\wedge *\beta\,.\end{aligned} \tag{38}$$

So far we have assumed that the degree of α and β is the same. But since in this case, and only in this case, $\alpha\wedge *\beta$ is a top form, and since only top forms give a non-vanishing contribution under the integral (*cf.* (6)), we can trivially, by using the last line of (38), extend the definition of $\langle\cdot|\cdot\rangle$ to arbitrary forms. Hence we have, for α a p form and β a q form:

$$\langle\alpha|\beta\rangle = \left\{\begin{array}{ll}\int\limits_{\mathcal{M}} \sqrt{|g|}\alpha\cdot\beta\, d^Dx & \text{if } p=q \\ 0 & \text{otherwise}\end{array}\right., \tag{39}$$

where $\alpha\cdot\beta$ is convenient shorthand for the index contraction displayed in (32). Analogously, $\langle\cdot|\cdot\rangle_{\mathrm{loc}}$ is extended to all forms by

$$\langle\alpha|\beta\rangle_{\mathrm{loc}} := \left\{\begin{array}{ll}\alpha\wedge *\beta & \text{if } p=q \\ 0 & \text{otherwise}\end{array}\right.. \tag{40}$$

The adjoint of an operator $\hat{A}$ with respect to $\langle\cdot|\cdot\rangle$ is denoted by $\hat{A}^\dagger$, as usual:

$$\left\langle \hat{A}\cdot|\cdot\right\rangle = \left\langle \cdot|\hat{A}^\dagger\cdot\right\rangle. \tag{41}$$

In particular one has

$$\begin{aligned}\left(\hat{c}^{\dagger\mu}\right)^\dagger &= \hat{c}^{\mu} \\ \left(\hat{e}^{\dagger a}\right)^\dagger &= \hat{e}^{a}\,.\end{aligned} \tag{42}$$

It is useful at this point to borrow some notions and notations from fermionic field theory in Fock representation:

Denote the local dual of the vacuum by $\langle 0|$:

$$
\begin{aligned}
\langle 0|(\alpha) &:= \langle |0\rangle | \alpha\rangle_{\mathrm{loc}} \\
&:= \langle 0|\, |\alpha\rangle_{\mathrm{loc}} \,.
\end{aligned}
\tag{43}
$$

Let α and β be created from the vacuum by operators $\hat{A}$ and $\hat{B}$:

$$
\begin{aligned}
\alpha &= \hat{A}\,|0\rangle \\
\beta &= \hat{B}\,|0\rangle \,.
\end{aligned}
\tag{44}
$$

Then

$$
\begin{aligned}
\langle \alpha|\beta\rangle_{\mathrm{loc}} &:= \left\langle \hat{A}\,|0\rangle \middle| \hat{B}\,|0\rangle \right\rangle_{\mathrm{loc}} \\
&= \langle 0|\, \hat{A}^\dagger \hat{B}\,|0\rangle_{\mathrm{loc}} \,.
\end{aligned}
\tag{45}
$$

Hence one can evaluate inner products in the way known from field theory. For instance:

$$
\begin{aligned}
\langle dx^\mu | dx^\nu\rangle_{\mathrm{loc}} &= \left\langle \hat{\mathrm{c}}^{\dagger\mu}\,|0\rangle | \hat{\mathrm{c}}^{\dagger\nu}\,|0\rangle \right\rangle_{\mathrm{loc}} \\
&= \langle 0|\, \hat{\mathrm{c}}^\nu \hat{\mathrm{c}}^{\dagger\mu}\,|0\rangle_{\mathrm{loc}} \\
&= g^{\mu\nu}\mathrm{vol}\,.
\end{aligned}
\tag{46}
$$

2.6 (Clifford algebra)

Consider now the operators

$$
\begin{aligned}
\hat{\gamma}_{g\pm}^{\;\mu} &:= \hat{\mathrm{c}}^{\dagger\mu} \pm \hat{\mathrm{c}}^\mu \\
\hat{\gamma}_{\pm}^{a} &:= \hat{\mathrm{e}}^{\dagger a} \pm \hat{\mathrm{e}}^a \,.
\end{aligned}
\tag{47}
$$

They generate a Clifford algebra (see B.1 (p.297)), satisfying

$$
\begin{aligned}
\left\{ \hat{\gamma}_{g\pm}^{\;\mu}, \hat{\gamma}_{g\pm}^{\;\nu} \right\} &= \pm 2 g^{\mu\nu} \\
\left\{ \hat{\gamma}_{g\pm}^{\;\mu}, \hat{\gamma}_{g\mp}^{\;\nu} \right\} &= 0
\end{aligned}
\tag{48}
$$

and

$$
\begin{aligned}
\left\{ \hat{\gamma}_{\pm}^{a}, \hat{\gamma}_{\pm}^{b} \right\} &= \pm 2 \eta^{ab} \\
\left\{ \hat{\gamma}_{\pm}^{a}, \hat{\gamma}_{\mp}^{b} \right\} &= 0\,,
\end{aligned}
\tag{49}
$$

as well as

$$
\left(\hat{\gamma}_{\pm}^{\cdot}\right)^\dagger = \pm \hat{\gamma}_{\pm}^{\cdot}\,.
\tag{50}
$$

Because of

$$
\begin{aligned}
\hat{\gamma}_{\pm}^{a}\,|0\rangle &= \hat{\mathrm{e}}^{\dagger a}\,|0\rangle \\
\langle 0|\, \hat{\gamma}_{\pm}^{a} &= \pm \langle 0|\, \hat{\mathrm{e}}^{a}\,,
\end{aligned}
\tag{51}
$$

every form $\omega = \omega_{a_1 a_2 \cdots a_p} \hat{\mathrm{e}}^{\dagger a_1} \hat{\mathrm{e}}^{\dagger a_2} \cdots \hat{\mathrm{e}}^{\dagger a_p} |0\rangle$ may be identified with an element

$$\hat{\gamma}_\pm(\omega) = \omega_{[a_1 a_2 \cdots a_p]} \hat{\gamma}_\pm^{a_1} \hat{\gamma}_\pm^{a_2} \cdots \hat{\gamma}_\pm^{a_p}$$

of the Clifford algebra, often called a "(Clifford) p-vector" or "(Clifford) multivector". Applying the latter to the vacuum recovers the original form (this is called the *symbol map*):

$$\omega \quad = \quad \hat{\gamma}_\pm(\omega)\, |0\rangle \ . \tag{52}$$

It can be useful in calculations to switch between Grassmann and Clifford operators this way. For instance define the bracket

$$\left\langle \omega^0 + \omega^1_a \hat{\gamma}^a + \omega^2_{[ab]} \hat{\gamma}^a_\pm \hat{\gamma}^b_\pm + \cdots \right\rangle_0 \quad := \quad \omega^0 \tag{53}$$

to be the projection on Clifford scalars. This is related to the above inner product on forms simply by

$$\left\langle \hat{A} \right\rangle_0 \mathrm{vol} \quad = \quad \langle 0|\, \hat{A}\, |0\rangle_{\mathrm{loc}} \ . \tag{54}$$

The Clifford inner product has the cyclic proprty[11]

$$\left\langle \hat{\gamma}^a_\pm \hat{\gamma}_\pm(\omega) \right\rangle_0 \quad = \quad \left\langle \hat{\gamma}_\pm(\omega)\, \hat{\gamma}^a_\pm \right\rangle_0 \tag{55}$$

(which is related to the cyclic property of the trace when the Clifford algebra is represented by matrices) and which translates to

$$\begin{aligned} \langle 0|\, \hat{\gamma}^a_\pm \hat{\gamma}_\pm(\omega)\, |0\rangle \quad &= \quad \langle 0|\, \hat{\gamma}_\pm(\omega)\, \hat{\gamma}^a_\pm\, |0\rangle \\ \Leftrightarrow \pm \langle 0|\, \hat{\mathrm{e}}^a \hat{\gamma}_\pm(\omega)\, |0\rangle \quad &= \quad \langle 0|\, \hat{\gamma}_\pm(\omega)\, \hat{\mathrm{e}}^{\dagger a}\, |0\rangle \ . \end{aligned} \tag{56}$$

2.7 (Hodge Duality operators)

Using the Clifford algebra one can conveniently discuss the operation of *Hodge duality*, induced by the operator $*$, by using the modified duality operator

$$\bar{*} \quad := \quad i^{D(D-1)/2+s} \left\{ \begin{array}{ll} \hat{\gamma}^0_- \hat{\gamma}^1_- \cdots \hat{\gamma}^{D-1}_- & \text{if } D \text{ is even} \\ \hat{\gamma}^0_+ \hat{\gamma}^1_+ \cdots \hat{\gamma}^{D-1}_+ & \text{if } D \text{ is odd} \end{array} \right. , \tag{57}$$

which is conveniently normalized so as to satisfy the relations[12]

$$\begin{aligned} (\bar{*})^\dagger \quad &= \quad (-1)^s \bar{*} && (59) \\ (\bar{*})^2 \quad &= \quad 1 && (60) \\ \bar{*}\, \hat{\mathrm{e}}^{\dagger a} \quad &= \quad \hat{\mathrm{e}}^a\, \bar{*} \,. && (61) \end{aligned}$$

---

[11] *Proof*: Consider the scalar part $\left\langle \hat{\gamma}^{a_1}_\pm \hat{\gamma}^{a_2}_\pm \cdots \hat{\gamma}^{a_p}_\pm \right\rangle_0$ of a p-vector consisting of orthonormal basis form. Obviously for p an odd number this expression vanishes, and hence for this case the cyclic property is trivially true. So let p be even. Then the obvious condition for the scalar part of our p-vector not to vanish is that every distinct basis element appears an even number of times. If this is not the case then the cyclic property again holds trivially. So assume it is true. But then $\hat{\gamma}^{a_1}_\pm$ commutes with $\hat{\gamma}^{a_2}_\pm \cdots \hat{\gamma}^{a_p}_\pm$ and may hence be moved to the right.

[12] To verify this note that

$$\begin{aligned} \left(\hat{\gamma}^0_\pm \hat{\gamma}^1_\pm \cdots \hat{\gamma}^{D-1}_\pm\right)^\dagger \quad &= \quad (\pm)^D (-1)^{D(D-1)/2}\, \hat{\gamma}^0_\pm \hat{\gamma}^1_\pm \cdots \hat{\gamma}^{D-1}_\pm \\ \left(\hat{\gamma}^0_\pm \hat{\gamma}^1_\pm \cdots \hat{\gamma}^{D-1}_\pm\right)^2 \quad &= \quad (\pm)^D (-1)^{D(D-1)/2+s} \\ \hat{\gamma}^0_\pm \hat{\gamma}^1_\pm \cdots \hat{\gamma}^{D-1}_\pm\, \hat{\mathrm{e}}^{\dagger a} \quad &= \quad \pm(-1)^{D-1} \hat{\mathrm{e}}^a\, \hat{\gamma}^0_\pm \hat{\gamma}^1_\pm \cdots \hat{\gamma}^{D-1}_\pm \,. \end{aligned} \tag{58}$$

We note here the important relation

$$\begin{aligned} \bar{*}\hat{N} &= \hat{\mathrm{e}}_a \hat{\mathrm{e}}^{\dagger a} \bar{*} \\ &= (D - \hat{N})\bar{*} \,. \end{aligned} \tag{62}$$

(The relation between $*$ and $\bar{*}$ is discussed in appendix §B (p.297).)

2.8 (Differential operators)

We now turn to differential operators on $\Lambda(\mathcal{M})$. Let $\hat{\nabla}_\mu$, which is called the *covariant derivative operator* with respect to the Levi-Civita-connetion ${\Gamma_\mu}^\alpha{}_\beta$ of $g_{\mu\nu}$, be defined by

$$\begin{aligned} \hat{\nabla}_\mu \, |0\rangle &= 0 \\ \left[\hat{\nabla}_\mu, f\right] &= (\partial_\mu f) \,, \quad f \in \Lambda^0(\mathcal{M}) \\ \left[\hat{\nabla}_\mu, \hat{\mathrm{c}}^{\dagger \alpha}\right] &= -{\Gamma_\mu}^\alpha{}_\beta \hat{\mathrm{c}}^{\dagger \beta} \,. \end{aligned} \tag{63}$$

If ${\omega_\mu}^a{}_b$ is the Levi-Civita connection in the orthonormal vielbein frame

$${\omega_\mu}^a{}_b \;:=\; e^a{}_\alpha \left(\delta^\alpha{}_\beta \partial_\mu + {\Gamma_\mu}^\alpha{}_\beta\right) (e^{-1})^\beta{}_b \,, \tag{64}$$

then the last line is equivalent to[13]

$$\left[\hat{\nabla}_\mu, \hat{\mathrm{e}}^{\dagger a}\right] \;=\; -{\omega_\mu}^a{}_b \hat{\mathrm{e}}^{\dagger b} \,. \tag{67}$$

This way one has:

$$\begin{aligned} \hat{\nabla}_\mu \left(\omega_{\alpha_1 \cdots \alpha_p} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p}\right) &= \left(\nabla_\mu \omega_{\alpha_1 \cdots \alpha_p}\right) dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p} \\ &= \left(\nabla_{[\mu} \omega_{\alpha_1 \cdots \alpha_p]}\right) dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p} \,. \end{aligned} \tag{68}$$

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[13] This follows by the standard calculation, but it may be worth spelling it out in the present context:

$$\begin{aligned} \left[\hat{\nabla}_\mu, \hat{\mathrm{e}}^{\dagger a}\right] &\overset{(16)}{=} \left[\hat{\nabla}_\mu, e^a{}_\alpha \hat{\mathrm{c}}^{\dagger \alpha}\right] \\ &= \left[\hat{\nabla}_\mu, e^a{}_\alpha\right] \hat{\mathrm{c}}^{\dagger \alpha} + e^a{}_\alpha \left[\hat{\nabla}_\mu, \hat{\mathrm{c}}^{\dagger \alpha}\right] \\ &\overset{(63)}{=} \left(\left(\partial_\mu e^a{}_\beta\right) - e^a{}_\alpha {\Gamma_\mu}^\alpha{}_\beta\right) \hat{\mathrm{c}}^{\dagger \beta} \\ &\overset{(16)}{=} \left(\left(\partial_\mu e^a{}_\beta\right) - e^a{}_\alpha {\Gamma_\mu}^\alpha{}_\beta\right) (e^{-1})^\beta{}_b \hat{\mathrm{e}}^{\dagger b} \\ &= -\left(e^a{}_\beta \left(\partial_\mu (e^{-1})^\beta{}_b\right) + e^a{}_\alpha {\Gamma_\mu}^\alpha{}_\beta (e^{-1})^\beta{}_b\right) \hat{\mathrm{e}}^{\dagger b} \\ &\overset{(64)}{=} -{\omega_\mu}^a{}_b \hat{\mathrm{e}}^{\dagger b} \,. \end{aligned} \tag{65}$$

In the last line the relation

$$\begin{aligned} & \partial_\mu \left(e^a{}_\beta (e^{-1})^\beta{}_b\right) = \partial_\mu \delta^a_b = 0 \\ \Leftrightarrow \quad & \left(\partial_\mu e^a{}_\beta\right) (e^{-1})^\beta{}_b = -e^a{}_\beta \left(\partial_\mu (e^{-1})^\beta{}_b\right) \end{aligned} \tag{66}$$

has been used.

The commutator of the covariant derivative operators with themselves gives the *Riemann curvature operator*[14]:

$$\begin{aligned}\left[\hat{\nabla}_\mu, \hat{\nabla}_\nu\right] &:= \mathbf{R}_{\mu\nu} \\ &:= R_{\mu\nu\alpha\beta}\hat{c}^{\dagger\alpha}\hat{c}^{\beta}\,. \end{aligned} \tag{70}$$

From its definition by a commutator it is clear, that

$$\mathbf{R}_{\mu\nu} = \mathbf{R}_{[\mu\nu]}\,. \tag{71}$$

By the relation $(\hat{\nabla}_\mu)^\dagger = -\frac{1}{\sqrt{|g|}}\hat{\nabla}_\mu\sqrt{|g|}$ (which is derived below, see (85)) it follows that $\mathbf{R}_{\mu\nu}$ is skew-self-adjoint

$$(\mathbf{R}_{\mu\nu})^\dagger = -\mathbf{R}_{\mu\nu}\,, \tag{72}$$

which, by the last line of (70), is equivalent to the statement

$$R_{\mu\nu\alpha\beta} = R_{\mu\nu[\alpha\beta]}\,. \tag{73}$$

Below, in (91), it is furthermore shown that $\hat{c}^{\dagger\mu}\hat{c}^{\dagger\nu}\mathbf{R}_{\mu\nu} = 0$, which implies

$$R_{[\mu\nu\alpha]\beta} = 0\,. \tag{74}$$

Finally the Jacobi identity

$$\left[\hat{\nabla}_\kappa, \left[\hat{\nabla}_\mu, \hat{\nabla}_\nu\right]\right] + \left[\hat{\nabla}_\nu, \left[\hat{\nabla}_\kappa, \hat{\nabla}_\mu\right]\right] + \left[\hat{\nabla}_\mu, \left[\hat{\nabla}_\nu, \hat{\nabla}_\kappa\right]\right] = 0 \tag{75}$$

implies the *Bianchi Identity*

$$\nabla_{[\kappa}R_{\mu\nu]\alpha\beta} = 0\,. \tag{76}$$

From the covariant derivative operator one can construct two flavours of partial derivative operators, distinguished by which of the basis forms they respect as constants, i.e. with which set of basis forms they commute. Introducing the operators

$$\begin{aligned}\partial_\mu &:= \hat{\nabla}_\mu + \omega_\mu{}^a{}_b\hat{e}^{\dagger b}\hat{e}_a \\ \partial^c_\mu &:= \hat{\nabla}_\mu + \Gamma_\mu{}^\alpha{}_\beta\hat{c}^{\dagger\beta}\hat{e}_\alpha \end{aligned} \tag{77}$$

one finds

$$\begin{aligned}\partial_\mu\left|0\right\rangle &= 0 \\ \left[\partial_\mu, f\right] &= (\partial_\mu f)\,, \quad f \in \Lambda^0(\mathcal{M}) \\ \left[\partial_\mu, \hat{e}^{\dagger a}\right] &= 0 \\ \left[\partial_\mu, \hat{e}_a\right] &= 0\,. \end{aligned} \tag{78}$$

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[14]With the signature convention following [269], (see e.g.eq. (3.4.3)) one has

$$\begin{aligned}\left[\hat{\nabla}_\mu, \hat{\nabla}_\nu\right] &= \left[\partial^c_\mu - \Gamma_\mu{}^\alpha{}_\beta\hat{c}^{\dagger\beta}{}_{\hat{c}_\alpha}, \partial^c_\nu - \Gamma_\nu{}^\gamma{}_\delta\hat{c}^{\dagger\delta}{}_{\hat{c}_\gamma}\right] \\ &= -2\left(\partial_{[\mu}\Gamma_{\nu]}{}^\alpha{}_\beta + \Gamma_{[\mu}{}^\alpha{}_{|\gamma|}\Gamma_{\nu]}{}^\gamma{}_\beta\right)\hat{c}^{\dagger\beta}\hat{c}_\alpha \\ &= R_{\mu\nu\beta}{}^\alpha\hat{c}^{\dagger\beta}\hat{c}_\alpha\,. \end{aligned} \tag{69}$$

and

$$\begin{aligned} \partial^c_\mu \left|0\right\rangle &= 0 \\ \left[\partial^c_\mu, f\right] &= (\partial_\mu f)\,, \quad f \in \Lambda^0(\mathcal{M}) \\ \left[\partial^c_\mu, \hat{\mathrm{c}}^{\dagger\alpha}\right] &= 0 \\ \left[\partial^c_\mu, \hat{\mathrm{c}}_\alpha\right] &= 0\,. \end{aligned} \tag{79}$$

(Note the position of the indices in the last two lines.) By acting with the partial derivative operators on an arbitrary form in a given basis one also verifies that

$$\begin{aligned} \left[\partial_\mu, \partial_\nu\right] &= 0 \\ \left[\partial^c_\mu, \partial^c_\nu\right] &= 0\,. \end{aligned} \tag{80}$$

Using (77), (78), and (79) it is now easy to establish the transformation properties of all creators and annihilators starting from (63)[15]:

$$\begin{aligned} \left[\hat{\nabla}_\mu, \hat{\mathrm{e}}^\dagger{}_a\right] &= +\omega_\mu{}^b{}_a \hat{\mathrm{e}}^\dagger{}_b \\ \left[\hat{\nabla}_\mu, \hat{\mathrm{e}}^a\right] &= -\omega_\mu{}^a{}_b \hat{\mathrm{e}}^b \\ \left[\hat{\nabla}_\mu, \hat{\mathrm{e}}_a\right] &= +\omega_\mu{}^b{}_a \hat{\mathrm{e}}_b \\ \left[\hat{\nabla}_\mu, \hat{\mathrm{c}}^\dagger{}_\alpha\right] &= +\Gamma_\mu{}^\beta{}_\alpha \hat{\mathrm{c}}^\dagger{}_\beta \\ \left[\hat{\nabla}_\mu, \hat{\mathrm{c}}^\alpha\right] &= -\Gamma_\mu{}^\beta{}_\alpha \hat{\mathrm{c}}^\alpha \\ \left[\hat{\nabla}_\mu, \hat{\mathrm{c}}_\alpha\right] &= +\Gamma_\mu{}^\beta{}_\alpha \hat{\mathrm{c}}_\beta\,. \end{aligned} \tag{82}$$

That is, all basis operators transform as usual according to the index they carry.

Another useful fact is that ∂_μ and $\hat{\nabla}_\mu$ commute with the duality operation:

$$\begin{aligned} \left[\partial_\mu, \bar{*}\right] &= 0 \\ \left[\hat{\nabla}_\mu, \bar{*}\right] &= 0\,, \end{aligned} \tag{83}$$

which follows straightforwardly by using the respective definitions.

Regarding the adjoints of the partial derivative operators it follows from the explicit form (39) of the inner product that

$$(\partial_\mu)^\dagger = -\frac{1}{\sqrt{|g|}}\partial_\mu \sqrt{|g|}\,. \tag{84}$$

[15]The relations involving the ONB are immediate. For the coordinate basis one uses for instance

$$\begin{aligned} \left[\hat{\nabla}_\mu, \hat{\mathrm{c}}^\dagger{}_\alpha\right] &= \left[\hat{\nabla}_\mu, g_{\alpha\beta}\hat{\mathrm{c}}^{\dagger\beta}\right] \\ &= \underbrace{\left(\partial_\mu g_{\alpha\beta} - \Gamma_{\mu\alpha\beta}\right)}_{=\Gamma_{\mu\beta\alpha}} \hat{\mathrm{c}}^{\dagger\beta} \\ &= +\Gamma_{\mu\beta\alpha}\hat{\mathrm{c}}^{\dagger\beta} \\ &= +\Gamma_\mu{}^\beta{}_\alpha \hat{\mathrm{c}}^\dagger{}_\beta\,. \end{aligned} \tag{81}$$

On the other hand the operator ∂^c_μ satisfies no such simple formula. Using (42) and the antisymmetry of $\omega_{\mu ab} = \omega_{\mu[ab]}$ one finds from (84) and (77) the analogous relation

$$\left(\hat{\nabla}_\mu\right)^\dagger = -\frac{1}{\sqrt{|g|}}\hat{\nabla}_\mu\sqrt{|g|}\,. \tag{85}$$

Next, it is of interest to have differential operators without free indices which map forms to forms. Such are obtained by contracting $\hat{\nabla}_\mu$ with some Grassmann or Clifford operator:

2.9 (Exterior derivative)

The *exterior derivative* is defined by

$$\mathbf{d} \;:=\; \hat{c}^{\dagger\mu}\hat{\nabla}_\mu\,. \tag{86}$$

Due to the special symmetry of the Levi-Civita connection in the coordinate basis, the exterior derivative here has the simple action

$$\mathbf{d}\omega_{\mu_1\cdots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} \;=\; \partial_{[\nu}\omega_{\mu_1\cdots\mu_p]}dx^\nu\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\,. \tag{87}$$

This can be made manifest by noting that

$$\mathbf{d} \;=\; \hat{c}^{\dagger\mu}\partial^c_\mu\,, \tag{88}$$

which follows by the definition of ∂^c_μ in (77) and the symmetry $\Gamma_\mu{}^\alpha{}_\beta = \Gamma_{(\mu}{}^\alpha{}_{\beta)}$. (Another way to say the same is

$$\begin{aligned} \left\{\mathbf{d},\hat{c}^{\dagger\mu}\right\} &= 0 \\ \Leftrightarrow \left\{\mathbf{d},\hat{e}^{\dagger a}\right\} &= -\hat{c}^{\dagger\mu}\omega_\mu{}^a{}_b\hat{e}^{\dagger b}\,. \end{aligned} \tag{89}$$

The second line is known as the *first structural equation*.)

Therefore $\mathbf{d}$ is *nilpotent*:

$$\begin{aligned} \mathbf{d}^2 &= \hat{c}^{\dagger\mu}\hat{c}^{\dagger\nu}\partial^c_\mu\partial^c_\nu \\ &= 0\,. \end{aligned} \tag{90}$$

Evaluating the same nilpotency equation using the covariant derivative shows that

$$\begin{aligned} \{\mathbf{d},\mathbf{d}\} &= -\hat{c}^{\dagger\mu}\hat{c}^{\dagger\nu}\left[\hat{\nabla}_\mu,\hat{\nabla}_\nu\right] \\ \Rightarrow\; \hat{c}^{\dagger\mu}\hat{c}^{\dagger\nu}\left[\hat{\nabla}_\mu,\hat{\nabla}_\nu\right] &= 0\,, \end{aligned} \tag{91}$$

which proves (74).

The adjoint of $\mathbf{d}$ with respect to $\langle\cdot|\cdot\rangle$ can be shown to be[16]

$$\mathbf{d}^\dagger \;=\; -\hat{c}^\mu\hat{\nabla}_\mu\,. \tag{93}$$

[16] Namely with the relations derived above one has:

$$\mathbf{d}^\dagger \;=\; \left(\hat{c}^{\dagger\mu}\hat{\nabla}_\mu\right)^\dagger$$

We will mostly refer to this "inner" derivative as the *exterior co-derivative*. It acts on $p > 0$-forms as the covariant divergence[17]:

$$\mathbf{d}^\dagger \omega_{\mu_1\cdots\mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} = -p \left(\nabla_\mu \omega^\mu{}_{\alpha_2\cdots\alpha_p}\right) dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_p} \,. \quad (95)$$

The exterior co-derivative, being the adjoint of a nilpotent operator, is itself nilpotent:

$$\mathbf{d}^{\dagger 2} = 0 \,. \quad (96)$$

A remarkable fact is that the differential operator $\mathbf{d}$ and its adjoint $\mathbf{d}^\dagger$ are dual to each other under Hodge duality[18]:

$$\mathbf{d}^\dagger = -\bar{*}\mathbf{d}\bar{*} \,. \quad (98)$$

2.10 (Lie derivative)

From $\mathbf{d}$ one revovers a directional derivative $\mathcal{L}_v$ along a vector field $v = v^\mu \partial_\mu$ by performing a contraction:

$$\mathcal{L}_v := \{\mathbf{d}, \hat{c}_\mu v^\mu\} \,. \quad (99)$$

$$\begin{aligned}
&= -\frac{1}{\sqrt{|g|}} \hat{\nabla}_\mu \sqrt{|g|}\hat{c}^\mu \\
&= -\hat{c}^\mu \hat{\nabla}_\mu - \frac{1}{\sqrt{|g|}} \left[\hat{\nabla}_\mu, \sqrt{|g|}\hat{c}^\mu\right] \\
&= -\hat{c}^\mu \hat{\nabla}_\mu - \hat{c}^\nu \underbrace{\left(\frac{1}{\sqrt{|g|}}(\partial_\mu \sqrt{|g|}) - \Gamma_\mu{}^\mu{}_\nu\right)}_{=0} \\
&= -\hat{c}^\mu \hat{\nabla}_\mu \,. \quad (92)
\end{aligned}$$

[17]The factor p may seem odd here, but it is perfectly sensible: In order to compensate for the $p!$-fold summation over permutations of the same indices, due to

$$\begin{aligned}
&\omega_{\mu_1\mu_2\cdots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} \\
= \; & p! \sum_{0\leq\mu_1<\mu_2<\cdots\mu_p<D} \omega_{\mu_1\mu_2\cdots\mu_p} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} \,, \quad (94)
\end{aligned}$$

it is usual to pull out a factor $1/p!$ from the coefficient functions of a p form. The factor p in (95) thus accounts for the lowering of the degree of the form:

$$\mathbf{d}^\dagger \frac{1}{p!} \omega_{\alpha_1\cdots\alpha_p} dx^{\alpha_1} \cdots dx^{\alpha_p} = -\frac{1}{(p-1)!} \left(\nabla_\mu \omega^\mu{}_{\alpha_2\cdots\alpha_p}\right) dx^{\alpha_2} \cdots dx^{\alpha_p} \,.$$

[18]For instance use

$$\begin{aligned}
-\bar{*}\mathbf{d}\bar{*} &= -\bar{*}\hat{c}^{\dagger\mu} \hat{\nabla}_\mu \bar{*} \\
&\overset{(83)}{=} -\bar{*}\hat{c}^{\dagger\mu} \bar{*} \hat{\nabla}_\mu \\
&\overset{(57)}{=} -\hat{c}^\mu \hat{\nabla}_\mu \\
&= \mathbf{d}^\dagger \,. \quad (97)
\end{aligned}$$

This is the *Lie derivative* of differential forms along v. More explicitly it reads

$$\begin{aligned}\{\mathbf{d}, \hat{\mathrm{c}}_\mu v^\mu\} &= \left\{\hat{\mathrm{c}}^\dagger{}_\mu \partial^c_\mu, \hat{\mathrm{c}}_\mu v^\mu\right\} \\ &= v^\mu \partial^c_\mu + (\partial_\mu v^\nu)\hat{\mathrm{c}}^{\dagger\mu}\hat{\mathrm{c}}_\nu\,,\end{aligned} \tag{100}$$

or alternatively

$$\begin{aligned}\{\mathbf{d}, \hat{\mathrm{e}}_\mu v^\mu\} &= \left\{\hat{\mathrm{c}}^\dagger{}_\mu \hat{\nabla}_\mu, \hat{\mathrm{e}}_\mu v^\mu\right\} \\ &= v^\mu \hat{\nabla}_\mu + (\nabla_\mu v^\nu)\hat{\mathrm{c}}^{\dagger\mu}\hat{\mathrm{c}}_\nu\,.\end{aligned} \tag{101}$$

The latter form is convenient for computing the adjoint:

$$\begin{aligned}(\mathcal{L}_v)^\dagger &= -\frac{1}{\sqrt{g}}\hat{\nabla}_\mu \sqrt{g} v^\mu + (\nabla_\nu v^\mu)\hat{\mathrm{c}}^\dagger{}_\mu \hat{\mathrm{c}}^\nu \\ &= -\mathcal{L}_v - (\nabla_\mu v^\mu) + 2(\nabla_{(\mu} v_{\nu)})\hat{\mathrm{c}}^{\dagger\mu}\hat{\mathrm{c}}^\nu\,.\end{aligned} \tag{102}$$

Obviously the Lie derivative $\mathcal{L}_v$ is skew-self-adjoint if and only if

$$\begin{aligned}\nabla_{(\mu} v_{\nu)} &= 0 \\ \Rightarrow \nabla_\mu v^\mu &= 0\,,\end{aligned} \tag{103}$$

i.e. if and only if v is a Killing vector field. From its definition (99) and the duality relation (98) it follows furthermore that the adjoint can be expressed as[19]

$$\mathcal{L}_v^\dagger = -\bar{*}\mathcal{L}_v\bar{*}\,. \tag{105}$$

From this we find the useful equivalence

$$v \text{ is Killing} \quad \Leftrightarrow \quad [\mathcal{L}_v, \bar{*}] = 0\,. \tag{106}$$

Also any $\mathcal{L}_v$ obviously preserves the form degree,

$$\left[\mathcal{L}_v, \hat{N}\right] = 0\,, \tag{107}$$

and because $*$ and $\bar{*}$ are proportional to each other up to a function of the number operator this implies in addition

$$v \text{ is Killing} \quad \Leftrightarrow \quad [\mathcal{L}_v, *] = 0\,. \tag{108}$$

[19] In components this follows from:

$$\begin{aligned}-\bar{*}\mathcal{L}_v\bar{*} &= -\bar{*}\left(v^\mu\hat{\nabla}_\mu + (\nabla_\mu v^\nu)\hat{\mathrm{c}}^{\dagger\mu}\hat{\mathrm{c}}_\nu\right)\bar{*} \\ &\overset{(83)(61)}{=} -v^\mu\hat{\nabla}_\mu - (\nabla_\mu v^\nu)\hat{\mathrm{c}}^\mu\hat{\mathrm{c}}^\dagger{}_\nu \\ &= -v^\mu\hat{\nabla}_\mu - (\nabla_\mu v_\nu)\left(g^{\mu\nu} - \hat{\mathrm{c}}^{\dagger\nu}\hat{\mathrm{c}}^\mu\right) \\ &= -v^\mu\hat{\nabla}_\mu - (\nabla_\mu v^\mu) + 2(\nabla_{(\mu}v_{\nu)})\hat{\mathrm{c}}^{\dagger\mu}\hat{\mathrm{c}}^\nu - (\nabla_\mu v_\nu)\hat{\mathrm{c}}^{\dagger\mu}\hat{\mathrm{c}}^\nu \\ &\overset{(102)}{=} (\mathcal{L}_v)^\dagger\,.\end{aligned} \tag{104}$$

Since from the very definition (99) and the nilpotency of $\mathbf{d}$ one has

$$[\mathcal{L}_v, \mathbf{d}] \quad = \quad 0 \tag{109}$$

identically, still another way to state (108) is

$$v \text{ is Killing} \quad \Leftrightarrow \quad \left[\mathcal{L}_v, \mathbf{d}^\dagger\right] = 0\,. \tag{110}$$

Another useful fact is that the partial derivative operators ∂^c_μ, defined in (77), are obviously (using (88)) Lie derivatives:

$$\partial^c_\mu \quad = \quad \{\mathbf{d}, \hat{c}_\mu\}\ , \tag{111}$$

i.e.[20]

$$\partial^c_\mu \quad = \quad \mathcal{L}_{\partial_\mu} \ := \ \mathcal{L}_\mu\,. \tag{113}$$

Accordingly the exterior derivative (88) can also be written as

$$\mathbf{d} \quad = \quad \hat{c}^{\dagger\mu}\mathcal{L}_{\partial_\mu} \tag{114}$$

and hence with (98) and (105) the exterior coderivative can also be written as

$$\mathbf{d}^\dagger \quad = \quad \hat{c}^\mu \mathcal{L}^\dagger_\mu\,. \tag{115}$$

2.11 (Dirac and Laplace-Beltrami operators)

Further non-nilpotent derivative operators on the exterior bundle are obtained by taking linear combinations of $\mathbf{d}$ and $\mathbf{d}^\dagger$. Therefore define the operator

$$\begin{aligned} \mathbf{D}_\pm &:= \mathbf{d} \pm \mathbf{d}^\dagger \\ &= \hat{\gamma}_{g_\mp}{}^\mu \hat{\nabla}_\mu\,, \end{aligned} \tag{116}$$

called the *Dirac operator* on $\Lambda(\mathcal{M})$.

In the presence of Killing vector fields tangent to coordinate lines this can be written in another useful form: Let $\{\partial_r | r = 1, \cdots n\}$ be the set of coordinate derivatives along Killing directions and $\{\partial_s | s = n+1, \cdots D\}$ the set of the remaining coordinate derivatives. Due to the results of 2.10 (p.28) one can then write

$$\begin{aligned} \mathbf{D}_\pm &= \hat{c}^{\dagger\mu}\mathcal{L}_\mu \pm \hat{c}^\mu \mathcal{L}^\dagger_\mu \\ &= \hat{\gamma}_{g_\mp}{}^r \mathcal{L}_{\partial_r} + \left(\hat{c}^{\dagger s}\mathcal{L}_j \pm \hat{c}^s \mathcal{L}^\dagger_j\right) \\ &= \hat{\gamma}_{g_\mp}{}^\mu \mathcal{L}_\mu + \text{non-derivative terms}\,. \end{aligned} \tag{117}$$

[20] Setting $v = v^\nu \partial_\nu = \partial_\mu$, i.e. $v^\nu = \delta^\nu_\mu$, one checks in components, that

$$\begin{aligned} \mathcal{L}_v &\overset{(101)}{=} v^\alpha \hat{\nabla}_\alpha + (\nabla_\beta v^\alpha)\hat{c}^{\dagger\beta}\hat{c}_\alpha \\ &= \hat{\nabla}_\mu + \Gamma_\beta{}^\alpha{}_\mu \hat{c}^{\dagger\beta}\hat{c}_\alpha \\ &= \hat{\nabla}_\mu + \Gamma_\mu{}^\alpha{}_\beta \hat{c}^{\dagger\beta}\hat{c}_\alpha \\ &\overset{(77)}{=} \partial^c_\mu\,. \end{aligned} \tag{112}$$

In particular, if ∂_0 is a timelike Killing vector, then one defines the *Dirac Hamiltonian*

$$\begin{aligned} H_{D_\pm} &:= \pm\frac{1}{g^{00}}\hat{\gamma}^0_\mp \mathbf{D}_\pm + \mathcal{L}_0 \\ &= \pm\frac{1}{g^{00}}\hat{\gamma}_{g\mp}^{\ 0}\hat{\gamma}_{g\mp}^{\ i}\mathcal{L}_i + \text{non-derivative terms} \end{aligned} \tag{118}$$

(where the summation is over $i > 0$). States $|\psi\rangle$ annihilated by $\mathbf{D}_\pm$,i,e. $\mathbf{D}_\pm |\psi\rangle = 0$, are then solutions of the *Dirac-Schrödinger-equation*

$$\mathcal{L}_0 |\psi\rangle = H_D |\psi\rangle \,. \tag{119}$$

One thing that makes the Dirac operator special is that it squares to a second order differential operator: The square

$$\begin{aligned} \mathbf{\Delta} &:= \mathbf{D}_+^2 \\ &= \left(\mathbf{d} + \mathbf{d}^\dagger\right)^2 \\ &= \left\{\mathbf{d}, \mathbf{d}^\dagger\right\} \end{aligned} \tag{120}$$

is called the *Laplace-Beltrami operator*. Similarly one has

$$\mathbf{D}_-^2 = -\mathbf{\Delta}\,. \tag{121}$$

The Laplace-Beltrami operator reads explicitly:

$$\begin{aligned} \mathbf{\Delta} &= \mathbf{D}_+^2 \\ &= \frac{1}{2}\left\{\gamma_-^\mu \nabla_\mu, \gamma_-^{\mu'}\nabla_{\mu'}\right\} \\ &= \frac{1}{2}\left(\left\{\gamma_-^\mu, \gamma_-^{\mu'}\right\}\nabla_\mu\nabla_{\mu'} + \gamma_-^\mu\left[\nabla_\mu, \gamma_-^{\mu'}\right]\nabla_{\mu'} + \gamma_-^{\mu'}\left[\nabla_{\mu'}, \gamma_-^\mu\right]\nabla_\mu + \gamma_-^\mu\gamma_-^{\mu'}\left[\nabla_\mu, \nabla_{\mu'}\right]\right) \\ &= \frac{1}{2}\left(-2g^{\mu\mu'}\nabla_\mu\nabla_{\mu'} + \underbrace{\gamma_-^\mu\gamma_-^\kappa\Gamma_\mu{}^{\mu'}{}_\kappa\nabla_{\mu'} + \gamma_-^{\mu'}\gamma_-^\kappa\Gamma_{\mu'}{}^\mu{}_\kappa\nabla_\mu}_{=-2\Gamma_\mu{}^{\mu'\mu}\nabla_{\mu'}} + \underbrace{\gamma_-^\mu\gamma_-^{\mu'}R_{\mu\mu'\kappa\lambda}e^{\dagger\kappa}e^\lambda}_{=-2R_{\mu\mu'\kappa\lambda}e^{\dagger\mu}e^{\mu'}e^{\dagger\kappa}e^\lambda}\right) \\ &= -\left(g^{\mu\mu'}\nabla_\mu\nabla_{\mu'} + \Gamma_\mu{}^{\mu'\mu}\nabla_{\mu'} + R_{\mu\mu'\kappa\lambda}e^{\dagger\mu}e^{\mu'}e^{\dagger\kappa}e^\lambda\right) \\ &= -\left(g^{\mu\mu'}\nabla_\mu\nabla_{\mu'} + \Gamma_\mu{}^{\mu'\mu}\nabla_{\mu'} - R_{\mu\mu'\kappa\lambda}e^{\dagger\mu}e^{\dagger\kappa}e^\nu e^\lambda - R_{\mu\lambda}e^{\dagger\mu}e^\lambda\right)\,. \end{aligned} \tag{122}$$

This expression is known as the *Weitzenböck formula* (*cf.* [31], p.130).

If one defines

$$\begin{aligned} \mathbf{D}_1 &:= \mathbf{D}_+ \\ \mathbf{D}_2 &:= i\mathbf{D}_- \end{aligned} \tag{123}$$

one finds the following superalgebra (definitions and details of such superalgebras are given in §2.1.3 (p.43) below):

$$\{\mathbf{D}_i, \mathbf{D}_j\} = 2\delta_{ij}\mathbf{\Delta}\,, \qquad i,j \in \{1,2\}\,. \tag{124}$$

Note that this is indeed a superalgebra: The respective involution ι is given by the *Witten operator*

$$\iota \;:=\; (-1)^{\hat{N}}\,, \tag{125}$$

which has eigenvalues $+1$ (-1) on even (odd) forms.

While the $\mathbf{D}_i$ are (anti-)self-dual under the action of $\bar{*}$, they satisfy the following intertwining relation (see 2.52 (p.56)):

$$\mathbf{D}_1(-1)^{\hat{N}(\hat{N}-1)/2} \;=\; i(-1)^{\hat{N}(\hat{N}+1)/2}\mathbf{D}_2\,. \tag{126}$$

Because of

$$\begin{aligned} \left[\hat{N},\mathbf{d}\right] &= \mathbf{d} \\ \left[\hat{N},\mathbf{d}^\dagger\right] &= -\mathbf{d}^\dagger \end{aligned} \tag{127}$$

all the above derivative operators are indeed odd graded with respect to ι:

$$\begin{aligned} \{\mathbf{d},\iota\} &= 0 \\ \left\{\mathbf{d}^\dagger,\iota\right\} &= 0 \\ \{\mathbf{D}_\pm,\iota\} &= 0\,. \end{aligned} \tag{128}$$

Further material related to this section is assembled in the following paragraphs as well as in §B (p.297) and will be referred to as needed.

2.12 (Conformal transformations)

Assume that the manifold $\mathcal{M}$ is equipped with two metric tensors $g_{\mu\nu}$, $\tilde{g}_{\mu\nu}$ related by

$$\tilde{g}_{\mu\nu}(p) = e^{2\Phi(p)} g_{\mu\nu}(p) \tag{129}$$

for some real function $\Phi : \mathcal{M} \to \mathbb{R}$.

In the following all objects associated with $\tilde{g}_{\mu\nu}$ are written under a tilde, $\tilde{\cdot}$, while all other objects are associated with $g_{\mu\nu}$.

The coordinate basis forms are obviously related by

$$\begin{aligned} \tilde{\hat{\mathrm{c}}^{\dagger}}{}^{\mu} &= e^{-\Phi}\hat{\mathrm{c}}^{\dagger\mu} \\ \tilde{\hat{\mathrm{c}}}_{\mu} &= e^{\Phi}\hat{\mathrm{c}}_{\mu} \end{aligned} \tag{130}$$

and we may choose

$$\begin{aligned} \tilde{\hat{\mathrm{e}}^{\dagger}}{}^{a} &= \hat{\mathrm{e}}^{\dagger a} \\ \tilde{\hat{\mathrm{e}}}_{a} &= \hat{\mathrm{e}}_{a}\,. \end{aligned} \tag{131}$$

Also obvious is the transformation of ∂^c_μ:

$$\tilde{\partial}^c_\mu = \partial^c_\mu + (\partial_\mu \Phi)\hat{N}\,, \tag{132}$$

because this is what satisfies the definition (79). With (88) it follows that[21]

$$\begin{aligned} \tilde{\mathbf{d}} &= \tilde{\hat{\mathrm{c}}^{\dagger}}{}^{\mu}\tilde{\partial}^c_\mu \\ &= e^{-\Phi}\left(\mathbf{d} + [\mathbf{d},\Phi]\,\hat{N}\right)\,. \end{aligned} \tag{138}$$

[21] This simple result may be checked by using component calculations: As discussed for instance in appendix D of [269] the Levi-Civita connections of the original and the transformed metric are related by

$$\tilde{\Gamma}_\mu{}^\alpha{}_\beta = \Gamma_\mu{}^\alpha{}_\beta + C_\mu{}^\alpha{}_\beta\,, \tag{133}$$

where the tensor $C_\mu{}^\alpha{}_\beta$ is defined by

$$C_\mu{}^\alpha{}_\beta = 2\delta^\alpha{}_{(\mu}\nabla_{\beta)}\Phi - g_{\mu\beta}g^{\alpha\gamma}\nabla_\gamma\Phi\,. \tag{134}$$

Accordingly one has for the ONB connection

$$\begin{aligned} \tilde{\omega}_\mu{}^a{}_b &= \tilde{e}^a{}_\alpha\left(\delta^a{}_\beta\partial_\mu + \tilde{\Gamma}_\mu{}^\alpha{}_\beta\right)\tilde{e}^\beta{}_b \\ &= e^\Phi e^a{}_\alpha\left(\delta^a{}_\beta\partial_\mu + \Gamma_\mu{}^\alpha{}_\beta + C_\mu{}^\alpha{}_\beta\right)e^\beta{}_b e^{-\Phi} \\ &= \omega_\mu{}^a{}_b + C_\mu{}^a{}_b - (\partial_\mu\Phi)\delta^a{}_b \end{aligned} \tag{135}$$

and hence

$$\begin{aligned} \tilde{\mathbf{d}} &= \tilde{\hat{\mathrm{c}}^{\dagger}}{}^{\mu}\left(\partial_\mu - \tilde{\omega}_\mu{}^a{}_b\right)\hat{\mathrm{e}}^{\dagger b}\hat{\mathrm{e}}_a \\ &= \tilde{\hat{\mathrm{c}}^{\dagger}}{}^{\mu}\left(\partial_\mu - \omega_\mu{}^a{}_b + (\partial_\mu\Phi)\delta^a{}_b\right)\hat{\mathrm{e}}^{\dagger b}\hat{\mathrm{e}}_a \\ &= e^{-\Phi}\mathbf{d} + \tilde{\hat{\mathrm{c}}^{\dagger}}{}^{\mu}(\partial_\mu\Phi)\hat{N}\,, \end{aligned} \tag{136}$$

where the term containing $C_\mu{}^\alpha{}_\beta$ disappears due to the symmetry

$$C_\mu{}^\alpha{}_\beta = C_{(\mu}{}^\alpha{}_{\beta)}\,. \tag{137}$$

The conformally transformed Lie derivative operators are also readily found, for instance from (99):

$$\begin{aligned}
\tilde{\mathcal{L}}_v &= \left\{\tilde{\mathbf{d}}, \tilde{\hat{c}_\mu} v^\mu\right\} \\
&= \left\{e^{-\Phi}\left(\mathbf{d} + \hat{c}^{\dagger\nu}(\partial_\nu\Phi)\hat{N}\right), e^{\Phi}\hat{c}_\mu v^\mu\right\} \\
&= \mathcal{L}_v + v^\mu(\partial_\mu\Phi)\hat{N}\,. \qquad (139)
\end{aligned}$$

The relation between the above operators and their conformal transformations is in fact a similarity transformation:

$$\begin{aligned}
\tilde{\hat{c}^{\dagger\mu}} &= e^{-\Phi\hat{N}}\,\hat{c}^{\dagger\mu}\,e^{\Phi\hat{N}} \\
\tilde{\hat{c}_\mu} &= e^{-\Phi\hat{N}}\,\hat{c}_\mu\,e^{\Phi\hat{N}} \\
\tilde{\partial}^c_\mu &= e^{-\Phi\hat{N}}\,\partial^c_\mu\,e^{\Phi\hat{N}} \\
\tilde{\mathbf{d}} &= e^{-\Phi\hat{N}}\,\mathbf{d}\,e^{\Phi\hat{N}} \\
\tilde{\mathcal{L}}_v &= e^{-\Phi\hat{N}}\,\mathcal{L}_v\,e^{\Phi\hat{N}}\,. \qquad (140)
\end{aligned}$$

But such is not true for every operator:

To find $\tilde{\mathbf{d}}^\dagger$ one can for instance use (98) and write[22]

$$\begin{aligned}
\tilde{\mathbf{d}}^\dagger &= -\bar{*}\,\tilde{\mathbf{d}}\,\bar{*} \\
&= -\bar{*}\,e^{-\Phi\hat{N}}\mathbf{d}\,e^{\Phi\hat{N}}\bar{*} \\
&= -\,e^{-\Phi(D-\hat{N})}\,\bar{*}\,\mathbf{d}\,\bar{*}\,e^{\Phi(D-\hat{N})} \\
&= e^{-\Phi(D-\hat{N})}\,\mathbf{d}^\dagger\,e^{\Phi(D-\hat{N})}\,, \qquad (144)
\end{aligned}$$

or

$$\tilde{\mathbf{d}}^\dagger = e^{-\Phi}\left(\mathbf{d}^\dagger - \left[\mathbf{d}^\dagger, \Phi\right](D-\hat{N})\right)\,. \qquad (145)$$

This is also a similarity transformation, but a different one. However, it coincides with that in (140) when evaluated on forms with eigenvalue n of $\hat{N}$ equal to

[22]This can again be checked in components:

$$\begin{aligned}
\tilde{\mathbf{d}}^\dagger &= -\tilde{\hat{c}^\mu}\,(\partial_\mu - \tilde{\omega}_\mu{}^a{}_b)\,\hat{e}^{\dagger b}\hat{e}_a \\
&= -\tilde{\hat{c}^\mu}\,(\partial_\mu - \omega_\mu{}^a{}_b - C_\mu{}^a{}_b + (\partial_\mu\Phi)\delta^a{}_b)\,\hat{e}^{\dagger b}\hat{e}_a\,. \qquad (141)
\end{aligned}$$

The term involving $C_\mu{}^\alpha{}_\beta$ gives

$$\begin{aligned}
C_\mu{}^\alpha{}_\beta\hat{c}^\mu\hat{c}^{\dagger\beta}\hat{c}_\alpha &= \left(2\delta^\alpha{}_{(\mu}(\partial_{\beta)}\Phi) - g_{\mu\beta}g^{\alpha\gamma}(\partial_\gamma\Phi)\right)\hat{c}^\mu\hat{c}^{\dagger\beta}\hat{c}_\alpha \\
&= \hat{c}^\mu(\partial_\mu\Phi)\hat{N} + \left(\delta^\alpha{}_\mu(\partial_\beta\Phi) - g_{\mu\beta}g^{\alpha\gamma}(\partial_\gamma\Phi)\right)\hat{c}^\mu\left(\delta^\beta_\alpha - \hat{c}_\alpha\hat{c}^{\dagger\beta}\right) \\
&= \hat{c}^\mu(\partial_\mu\Phi)\hat{N} + g_{\mu\beta}g^{\alpha\gamma}(\partial_\gamma\Phi)\hat{c}^\mu\hat{c}_\alpha\hat{c}^{\dagger\beta} \\
&= \hat{c}^\mu(\partial_\mu\Phi)\hat{N} - g_{\mu\beta}g^{\alpha\gamma}(\partial_\gamma\Phi)\hat{c}_\alpha\left(g^{\mu\beta} - \hat{c}^{\dagger\beta}\hat{c}^\mu\right) \\
&= \hat{c}^\mu(\partial_\mu\Phi)\hat{N} - \hat{c}^\mu(\partial_\mu\Phi)(D-\hat{N}) \\
&= -\hat{c}^\mu(\partial_\mu\Phi)(D-2\hat{N}) \qquad (142)
\end{aligned}$$

and reinserting this in (142) yields

$$\tilde{\mathbf{d}}^\dagger = e^{-\Phi}\mathbf{d}^\dagger - e^{-\phi}\hat{c}^\mu(\partial_\mu\Phi)(D-\hat{N})\,. \qquad (143)$$

$n = D/2$. An immediate consequence of this result is that, for D even and when acting on forms $|\psi\rangle$ of degree $n = D/2$, the equations

$$\begin{aligned} \mathbf{d}\,|\psi\rangle &= 0 \\ \mathbf{d}^\dagger\,|\psi\rangle &= 0 \end{aligned} \tag{146}$$

are conformally invariant[23] in the sense that, with

$$|\tilde{\psi}\rangle \;:=\; e^{-\Phi D/2}\,|\psi\rangle\,, \tag{148}$$

they are equivalent to

$$\begin{aligned} \tilde{\mathbf{d}}|\tilde{\psi}\rangle &= 0 \\ \tilde{\mathbf{d}}^\dagger|\tilde{\psi}\rangle &= 0\,. \end{aligned} \tag{149}$$

(Physically this means that p-form electromagnetism is conformally invariant in $D = 2p$ dimensions (see §2.2.3 (p.70)). In particular, ordinary electromagnetism is conformally invariant.)

2.13 (Deformations by Killing vectors)

In the presence of a Killing vector $k = k^\mu \partial_\mu$ one sometimes considers a deformation $\mathbf{d}_k$ of the exterior derivative defined by[24]

$$\mathbf{d}_k \;:=\; \mathbf{d} + i\hat{c}_\mu k^\mu\,. \tag{150}$$

[23] In components this is shown for $D = 4$ in (for instance) appendix D of [269]. The calculation presented there may be generalized as follows:

Proposition: In $2p$ dimensions the equation $g^{\alpha\mu}\nabla_\mu F_{\alpha\alpha_2\alpha_3\cdots\cdots\alpha_p} = 0$ is conformally invariant, more specifically:

$$\tilde{g}^{\alpha\mu}\tilde{\nabla}_\mu F_{\alpha\alpha_2\cdots\alpha_p} \;=\; \Omega^{-2}\left(g^{\alpha\mu}\nabla_\mu F_{\alpha\alpha_2\cdots\alpha_p} + (D-2p)g^{\alpha\mu}F_{\alpha\alpha_2\cdots\alpha_p}\nabla_\mu \ln\Omega\right)\,, \tag{147}$$

where $\Omega = e^{2\Phi}$.

Proof by induction:

- $p = 1$:

$$\begin{aligned} \tilde{g}^{\alpha\mu}\tilde{\nabla}_\mu F_\alpha &= \Omega^{-2}g^{\alpha\mu}\left(\nabla_\mu F_\alpha - C_\mu{}^\beta{}_\alpha F_\beta\right) \\ &= \Omega^{-2}\left(g^{\alpha\mu}\left(\nabla_\mu F_\alpha - \left(2\delta^\beta{}_{(\mu}\nabla_{\alpha)}\ln\Omega - g_{\mu\alpha}g^{\beta\gamma}\nabla_\gamma \ln\Omega\right)F_\beta\right)\right) \\ &= \Omega^{-2}\left(g^{\alpha\mu}\nabla_\mu F_\alpha + (D-2)g^{\alpha\mu}F_\alpha\nabla_\mu \ln\Omega\right) \end{aligned}$$

- $p = q + 1$:

$$\begin{aligned} &\tilde{g}^{\alpha\mu}\tilde{\nabla}_\mu F_{\alpha\alpha_2\cdots\alpha_q\alpha_p} \\ =\;& \Omega^{-2}\left(g^{\alpha\mu}\nabla_\mu F_{\alpha\alpha_2\cdots\alpha_q\alpha_p} + (D-2q)g^{\alpha\mu}F_{\alpha\alpha_2\cdots\alpha_q\alpha_p}\nabla_\mu \ln\Omega - g^{\alpha\mu}C_\mu{}^\beta{}_{\alpha_p}F_{\alpha_1\alpha_2\cdots\alpha_q\beta}\right) \\ =\;& \Omega^{-2}\left(g^{\alpha\mu}\nabla_\mu F_{\alpha\alpha_2\cdots\alpha_q\alpha_p} + (D-2q)g^{\alpha\mu}F_{\alpha\alpha_2\cdots\alpha_q\alpha_p}\nabla_\mu \ln\Omega\right) \\ &-\Omega^{-2}g^{\alpha\mu}\left(2\delta^\beta{}_{(\mu}\nabla_{\alpha_p)}\ln\Omega - g_{\mu\alpha_p}g^{\beta\gamma}\nabla_\gamma \ln\Omega\right)F_{\alpha\alpha_2\cdots\alpha_q\beta} \\ =\;& \Omega^{-2}\left(g^{\alpha\mu}\nabla_\mu F_{\alpha\alpha_2\cdots\alpha_q\alpha_p} + (D-2q)g^{\alpha\mu}F_{\alpha\alpha_2\cdots\alpha_q\alpha_p}\nabla_\mu \ln\Omega - g^{\alpha\mu}C_\mu{}^\beta{}_{\alpha_p}F_{\alpha_1\alpha_2\cdots\alpha_q\beta}\right) \\ =\;& \Omega^{-2}\left(g^{\alpha\mu}\nabla_\mu F_{\alpha\alpha_2\cdots\alpha_q\alpha_p} + (D-2q)g^{\alpha\mu}F_{\alpha\alpha_2\cdots\alpha_q\alpha_p}\nabla_\mu \ln\Omega\right) \\ &-\Omega^{-2}\left(g^{\alpha\mu}F_{\alpha\alpha_2\cdots\alpha_q\alpha_p}\nabla_\mu \ln\Omega - g^{\alpha\mu}F_{\alpha_p\alpha_2\cdots\alpha_q\alpha}\nabla_\mu \ln\Omega\right) \\ =\;& \Omega^{-2}\left(g^{\alpha\mu}\nabla_\mu F_{\alpha\alpha_2\cdots\alpha_q\alpha_p} + (D-2q-2)g^{\alpha\mu}F_{\alpha\alpha_2\cdots\alpha_q\alpha_p}\nabla_\mu \ln\Omega\right)\,. \end{aligned}$$

[24] This is discussed in [275] in the context of supersymmetric field theories. The supersymmetry constraints of the NSR superstring (*cf.* §3.3 (p.153)) are also of this form, as will be discussed elsewhere.

The associated adjoint operator is

$$\mathbf{d}^\dagger{}_k \quad := \quad \mathbf{d}^\dagger - i\hat{c}^\dagger{}_\mu k^\mu \,. \tag{151}$$

By the definition of the Lie-derivative one finds

$$\mathbf{d}^2 \quad = \quad i\mathcal{L}_k \,, \tag{152}$$

and, since k is Killing, also

$$\mathbf{d}^{\dagger 2} \quad = \quad i\mathcal{L}_k \,. \tag{153}$$

Defining

$$\begin{aligned} \mathbf{D}_{k,\pm} &= \mathbf{d}_k \pm \mathbf{d}^\dagger{}_k \\ &= \gamma^\mu_\mp \left(\hat{\nabla}_\mu \mp i k_\mu\right) \end{aligned} \tag{154}$$

one has

$$\{\mathbf{D}_{k,A}, \mathbf{D}_{k,B}\} \quad = \quad 2\delta_{AB} \left(\pm \mathbf{\Delta}_k + i\mathcal{L}_k\right) \,, \tag{155}$$

where the deformed Laplace-Beltrami operator is

$$\begin{aligned} \mathbf{\Delta}_k &:= \left\{\mathbf{d}_k, \mathbf{d}^\dagger{}_k\right\} \\ &= \mathbf{\Delta} + k^2 + i\left(\left\{\mathbf{d}^\dagger, \hat{c}_\mu k^\mu\right\} - \left\{\mathbf{d}, \hat{c}^\dagger{}_\mu k^\mu\right\}\right) \\ &= \mathbf{\Delta} + k^2 - i(\partial_{[\mu} k_{\nu]}) \left(\hat{c}^{\dagger\mu}\hat{c}^{\dagger\nu} + \hat{c}^\mu \hat{c}^\nu\right) \,. \end{aligned} \tag{156}$$

The deformed exterior differential operators still statisfy the duality relation (98):

$$\mathbf{d}^\dagger{}_k \quad = \quad -\bar{*}\, \mathbf{d}_k\, \bar{*} \,, \tag{157}$$

but the intertwining relation (126) is modified to

$$\mathbf{D}_{k,+}(-1)^{\hat{N}(\hat{N}-1)/2} \quad = \quad -(-1)^{\hat{N}(\hat{N}+1)/2}\mathbf{D}^*_{k,-} \,, \tag{158}$$

where $\mathbf{D}^*_{k,-}$ is the complex conjugate of $\mathbf{D}_{k,-}$

2.1.2 Super-mathematics: $\mathbb{Z}(2)$ gradings

Outline. This section lists some selected definitions and results of the theory of graded structures that are basic to all of the considerations of this text. After introducing graded vector spaces and graded algebras, attention is concentrated on nilpotent graded operators, their associated complexes, and their cohomology. The latter, which is closely related with the *Witten index* in Riemannian supersymmetry where supercharges are *elliptic* operators, will be seen to be related to *gauge fixing* in pseudo-Riemannian supersymmetry (*cf.* §2.3 (p.106)), where supercharges are no longer elliptic.

2.14 (Graded vector space) *A* $\mathbb{Z}(2)$-graded vector space (super vector space) (V, ι) *is a vector space* V *together with an* involutive linear mapping $\iota : V \to V,\ \iota^2 = 1$. *The grading of* V *corresponds to its decomposition into eigenspaces of* ι*:*

$$\begin{aligned} V &= V_+ \oplus V_- \\ \iota V_\pm &= \pm V_\pm \,. \end{aligned}$$

Example 2.15 The exterior algebra, regarded as a vector space, is a graded vector space, the $\mathbb{Z}(2)$-grading being induced by the 'Witten operator'

$$\iota := (-1)^{\hat{N}}\,,$$

which acts on homogeneous forms as

$$(-1)^{\hat{N}} \hat{\mathrm{e}}^{\dagger a_1} \cdots \hat{\mathrm{e}}^{\dagger a_p} |0\rangle = (-1)^p \hat{\mathrm{e}}^{\dagger a_1} \cdots \hat{\mathrm{e}}^{\dagger a_p} |0\rangle \,. \tag{159}$$

Hence the exterior algebra decomposes into subspaces of forms of even and of odd degree, respectively

$$\Lambda = \Lambda_+ \oplus \Lambda_- \,.$$

Only Λ^+ is also a sub*algebra*.

2.16 (Graded algebra) *A* $\mathbb{Z}(2)$-graded algebra (super algebra) (A, ι) *is an algebra* A *with an involutive element* $\iota, \iota^2 = 1$. Elements $a \in A$ with the property

$$\iota\, a\, \iota = \pm a$$

are called *even* $(+)$ or *odd* $(-)$ with respect to the grading induced by ι. Every element of A is the sum of an even and an odd element:

$$A \ni a = \frac{1}{2}(a + \iota a \iota) + \frac{1}{2}(a - \iota a \iota)\,.$$

If $(A, \iota) := (A, V, \iota)$ is an algebra of linear operators of a graded vector space (V, ι), then even elements of A preserve and odd elements switch the grading of a vector in V:

$$\begin{aligned} (a + \iota a \iota) &: V_\pm \to V_\pm \\ (a - \iota a \iota) &: V_\pm \to V_\mp \,. \end{aligned} \tag{160}$$

Example 2.17 Every Clifford algebra (see B.1 (p.297) for a definition and compare (47) and the discussion following it) is a graded algebra. The grading corresponds to multivectors being of even or odd degree. Let the dimension of the underlying vector space (the number of Clifford generators $\hat{\gamma}_-^a$, *cf.* observation **??** (p.**??**)) be even, $D = 2n$, then the chirality operator (definition B.16 (p.307)) serves as the involution which induces the grading:

$$\iota_{D=2n} := \bar{\gamma}_- = k\,\hat{\gamma}_-^0 \hat{\gamma}_-^1 \cdots \hat{\gamma}_-^{D-1}\,.$$

Here k is a complex number so that

$$\bar{\gamma}_-^2 = 1\,.$$

It is easily seen that the chirality operator anticommutes with the generators of the Clifford algebra:

$$\left\{\bar{\gamma}_-, \hat{\gamma}_-^a\right\} = 0\,.$$

Hence it anticommutes with multivectors of odd and commutes with multivector of even degree:

$$\bar{\gamma}_- \left(\hat{\gamma}_-^{a_1} \cdots \hat{\gamma}_-^{a_p}\right) \bar{\gamma}_- = (-1)^p\, \hat{\gamma}_-^{a_1} \cdots \hat{\gamma}_-^{a_p}\,.$$

In an odd number of dimensions, $D = 2n + 1$, the same is accomplished by setting

$$\iota_{D=2n+1} := \bar{\gamma}_+\,.$$

2.18 For two elements $a, b \in A$ of definite grading with respect to the involution ι, let $\epsilon(a, b)$ be defined by

$$\epsilon(a,b) \quad := \quad \left\{ \begin{array}{ll} -1 & a,b \text{ odd} \\ +1 & \text{otherwise} \end{array} \right. \,. \tag{161}$$

2.19 (Graded commutator) *The graded commutator (supercommutator)*

$$[\cdot,\cdot]_\iota : (A, V, \iota) \to (A, V, \iota)$$

on a graded algebra A is defined by

$$\begin{aligned} [a,b]_\iota &:= ab - \epsilon(a,b)\, ba \\ &= \left\{ \begin{array}{ll} \{a,b\} & a,b \text{ odd} \\ [a,b] & \text{otherwise} \end{array} \right. \,. \end{aligned} \tag{162}$$

It is sometimes useful to extend the superalgebra A by elements θ that commute with all of A but anticommute among themselves. For this purpose define $\theta_a,\ \ a \in A$, to be the unit element if a is even and to be a distinct odd generator of G if a is odd graded, so that (*cf.* (161))

$$\theta_a \theta_b = \epsilon(a,b)\, \theta_b \theta_a\,. \tag{163}$$

Then the above supercommutator is automatically obtained by multiplying every generator $a \in A$ with θ_a and considering ordinary commutators between these objects:

$$[\theta_a a, \theta_b b] \quad = \quad \theta_a \theta_b\, [a,b]_\iota\,. \tag{164}$$

2.20 (Graded Jacobi identity) *The supercommutator satisfies a Leibnitz rule called the super Jacobi identity:*

$$[a,[b,c]_\iota]_\iota \;=\; [[a,b]_\iota,c]_\iota + \epsilon(a,b)\,[b,[a,c]_\iota]_\iota\,. \tag{165}$$

Proof: This is conveniently shown by means of the ordinary Jacobi identity

$$[a,[b,c]] \;=\; [[a,b],c] + [b,[a,c]]\;, \tag{166}$$

by using relation (164):

$$\begin{aligned} & [\theta_a a,[\theta_b b,\theta_c c]] \stackrel{(166)}{=} [[\theta_a a,\theta_b b],\theta_c c] + [\theta_b b,[\theta_a a,\theta_c c]] \\ \Leftrightarrow\quad & \theta_a\theta_b\theta_c\,[a,[b,c]_\iota]_\iota = \theta_a\theta_b\theta_c\,[[a,b]_\iota,c]_\iota + \underbrace{\theta_b\theta_a\theta_c}_{=\epsilon(a,b)\theta_a\theta_b\theta_c}\,[b,[a,c]_\iota]_\iota\,. \end{aligned} \tag{167}$$

□

2.21 (Nilpotent operators, closed and exact elements) In a graded algebra (A,V,ι) an important class of operators are the nilpotent elements $\mathbf{q}\in A$ of odd grade

$$\begin{aligned} \{\mathbf{q},\iota\} &= 0 \\ \mathbf{q}^2 &= 0\,. \end{aligned} \tag{168}$$

Elements $|v\rangle\in V$ of the form $|v\rangle = \mathbf{q}\,|w\rangle$ are called $\mathbf{q}$-*exact*, while elements $|v\rangle$ with the property $\mathbf{q}\,|v\rangle = 0$ are called $\mathbf{q}$-*closed*. Analogously, operators $a\in A$ of the form $a = [\mathbf{q},b]_\iota$ are called $\mathbf{q}$-*exact* and operators a with the property $[\mathbf{q},a]_\iota = 0$ are called $\mathbf{q}$-*closed*. Note that this is consistent since

$$[\mathbf{q},[\mathbf{q},\cdot]_\iota]_\iota = 0$$

identically.

For more on exact and closed operators see §2.2.7 (p.90).

Example 2.22 The standard example are the exterior derivative and coderivative (see 2.2 (p.16)), eqs. (86), p. 27 and (93), p. 27):

$$\begin{aligned} \mathbf{q} &= \mathbf{d} \\ \mathbf{q}^\dagger &= \mathbf{d}^\dagger\,. \end{aligned}$$

Next consider the cohomology of nilpotent operators, which is the central tool of gauge theory in the BRST-formulation and which is needed in §2.3.2 (p.115):

2.23 (Cohomology.) *Let* $\mathbf{q}$, $\mathbf{q}^2 = 0$ *be any nilpotent operator in the graded algebra* (A,V,ι). *The equivalence class of* $\mathbf{q}$*-closed elements* V *modulo* $\mathbf{q}$*-exact elements is called the* cohomology $\mathrm{H}_c(\mathbf{q})$ *of* $\mathbf{q}$*:*

$$\begin{aligned} \mathrm{H}_c(\mathbf{q}) &:= \mathrm{Ker}\,(\mathbf{q})\,/\mathrm{Im}(\mathbf{q}) \\ &= \{[|\alpha\rangle + \mathbf{q}\,|\beta\rangle]\,|\,|\alpha\rangle\,,|\beta\rangle\in V;\,\mathbf{q}\,|\alpha\rangle = 0\} \end{aligned} \tag{169}$$

Theorem 2.24 (Hodge decomposition) Let the vector space V on which a graded algebra (A, V, ι) acts, be equipped with a scalar product (positive definite, non-degenerate inner product) $\langle\cdot|\cdot\rangle_{\hat{\eta}}$ and let $\mathbf{q} \in A$, $\mathbf{q}^2 = 0$ be any nilpotent operator and $\mathbf{q}^{\dagger_{\hat{\eta}}} \in A$; $\left(\mathbf{q}^{\dagger_{\hat{\eta}}}\right)^2 = 0$ its adjoint with respect to $\langle\cdot|\cdot\rangle_{\hat{\eta}}$.[25] *The Hilbert space then decomposes as a direct sum of* $\mathbf{q}$*-exact elements,* $\mathbf{q}^{\dagger_{\hat{\eta}}}$*-exact elements (also called* $\mathbf{q}$*-coexact), as well as* $\mathbf{q}$*-harmonic elements:*

$$\mathcal{H} \quad = \quad \overline{\mathrm{Im}(\mathbf{q})} \oplus \overline{\mathrm{Im}\left(\mathbf{q}^{\dagger_{\hat{\eta}}}\right)} \oplus \mathrm{Ker}\left(\mathbf{q}\right) \cap \mathrm{Ker}\left(\mathbf{q}^{\dagger_{\hat{\eta}}}\right) \ . \tag{170}$$

The subspace of $\mathbf{q}$*-harmonic elements can be characterized in several useful ways:*

$$\begin{aligned}
\mathrm{Ker}\left(\mathbf{q}\right) \cap \mathrm{Ker}\left(\mathbf{q}^{\dagger_{\hat{\eta}}}\right) &= \mathrm{Ker}\left(\mathbf{q} \pm \mathbf{q}^{\dagger_{\hat{\eta}}}\right) \\
&= \mathrm{Ker}\left(\left(\mathbf{q} \pm \mathbf{q}^{\dagger_{\hat{\eta}}}\right)^2\right) \\
&= \mathrm{Ker}\left(\mathbf{q}\mathbf{q}^{\dagger_{\hat{\eta}}} + \mathbf{q}^{\dagger_{\hat{\eta}}}\mathbf{q}\right) \\
&\simeq \mathrm{Ker}\left(\mathbf{q}\right)/\mathrm{Im}(\mathbf{q}) = \mathrm{H}_c(\mathbf{q}) \\
&\simeq \mathrm{Ker}\left(\mathbf{q}^{\dagger_{\hat{\eta}}}\right)/\mathrm{Im}\left(\mathbf{q}^{\dagger_{\hat{\eta}}}\right) = \mathrm{H}_c\left(\mathbf{q}^{\dagger_{\hat{\eta}}}\right) \ .
\end{aligned} \tag{171}$$

The proof can be found in the standard literature, e.g. [98]. □

2.25 (Picking a representative from the cohomology) By (171) *any* operator $\mathbf{q}^{\dagger_{\hat{\eta}}}$ that is adjoint to a nilpotent operator $\mathbf{q}$ with respect to *some* scalar product $\langle\cdot|\cdot\rangle_{\hat{\eta}}$ on V defines a unique representative $|\alpha\rangle \in [|\alpha\rangle]$, $\mathbf{q}\,|\alpha\rangle = 0$ of each equivalence class $[|\alpha\rangle] \in \mathrm{H}_c(\mathbf{q})$ in the cohomology of a nilpotent operator $\mathbf{q}$. Hence, each scalar product $\langle\cdot|\cdot\rangle_{\hat{\eta}}$ that can be defined on the vector space V induces a choice of representatives of the cohomology of $\mathbf{q}$.

Since the cohomology is characterized by $\mathbf{q} + \mathbf{q}^{\dagger_{\hat{\eta}}}$, this demonstrates the central importance of the **q-***Laplace* operator:

2.26 (q-Laplace operator) *For any nilpotent operator* $\mathbf{q}$ *as above, the operator*

$$\begin{aligned}
\mathbf{\Delta}_{\mathbf{q},\hat{\eta}} &:= \left(\mathbf{q} + \mathbf{q}^{\dagger_{\hat{\eta}}}\right)^2 \\
&= \left\{\mathbf{q}, \mathbf{q}^{\dagger_{\hat{\eta}}}\right\}
\end{aligned} \tag{172}$$

is called a $\mathbf{q}$-Laplace operator*. It is a positive operator, self-adjoint with respect to the scalar product* $\langle\cdot|\cdot\rangle_{\hat{\eta}}$*.*

Note that the point here is that $\mathbf{q}$ and $\mathbf{q}^{\dagger_{\hat{\eta}}}$ are mutually adjoint with respect to a positive definite scalar product. This is in general *not* the case for $\mathbf{d}$ (86) and $\mathbf{d}^{\dagger}$ (93), which are mutually adjoint with respect to the Hodge inner product (38). The latter is not positive definite for semi-Riemannian metrics. Hence the Laplace-Beltrami operator (120) is not a positive operator in this case. This difference is all important in covariant supersymmetric quantum mechanics, which does take place on a semi-Riemannian manifold $\mathcal{M}$. The introduction of a positive definite scalar product in addition to the indefinite Hodge inner prodcut is discussed in §2.3 (p.106)).

[25] The use of the index $\hat{\eta}$ will be justified in §2.3.2 (p.115).

2.27 (Graded trace) *The graded trace (supertrace) on a graded algebra* (A, ι) *is the alternating trace with respect to the grading* ι*:*

$$\begin{aligned} \mathrm{sTr}(\cdot)_\iota : A &\to \mathrm{R} \\ a &\mapsto \mathrm{sTr}(a)_\iota \\ &:= \mathrm{Tr}(\iota\, a) \,. \end{aligned} \tag{173}$$

Formally, this is the difference of the traces over the eigenspaces of the grading operator:

$$\mathrm{Tr}(\iota\, a) = \mathrm{Tr}\left(\frac{1}{2}(1+\iota)\, a\right) - \mathrm{Tr}\left(\frac{1}{2}(1-\iota)\, a\right) . \tag{174}$$

But these reorderings (and similar ones, *cf.* §2.3.2 (p.115)) have only formal meaning, since the graded trace will in general involve infinite sums, even in the quantum mechanical setting. To make sense of such formal relations the trace needs to be regulated in order to make the reordering well defined. This is discussed in 2.28 (p.41) below.

Theorem 2.28 (Regulated supertrace) Let (A, V, ι) be a graded algebra A of linear operators on the graded vector space V, which contains a graded nilpotent operator $\mathbf{q} \in A$, $\{\mathbf{q}, \iota\} = 0$, $\mathbf{q}^2 = 0$. Let $\mathbf{q}^{\dagger_{\hat\eta}}$ be the adjoint of $\mathbf{q}$ with respect to some scalar product on V.
The alternating trace regulated by $e^{-\left(\mathbf{q}+\mathbf{q}^{\dagger_{\hat\eta}}\right)^2}$ *is equal to the alternating trace over* $\mathrm{Ker}\,(q) \cap \mathrm{Ker}\left(q^{\dagger_{\hat\eta}}\right)$*:*

$$\mathrm{Tr}\left(e^{-\left(\mathbf{q}+\mathbf{q}^{\dagger_{\hat\eta}}\right)^2}\iota\, a\right) = \mathrm{Tr}(\iota\, a)_{\mathrm{Ker}\left(\mathbf{q}+\mathbf{q}^{\dagger_{\hat\eta}}\right)} \,. \tag{175}$$

Proof: The proof is given under point 1 (p.328) of §D (p.328).

Example 2.29

1. **The index of the exterior Dirac operator on closed Riemannian manifolds**

   This is the standard example (see e.g. [109][4]): Let $V := \Lambda(\mathcal{M})$ be the exterior bundle over a closed Riemannian manifold $\mathcal{M}$, i.e. $\mathcal{M}$ has *positive definite* metric (note that this is *not* the case of interest in covariant SQM), and let the nilpotent graded operators be the exterior derivative and coderivative:

   $$\begin{aligned} \mathbf{q} &:= \mathbf{d} \\ \mathbf{q}^{\dagger_{\hat\eta}} &:= \mathbf{d}^\dagger \,, \end{aligned} \tag{176}$$

   where the adjoint is with respect to the ordinary Hodge inner product

   $$\langle \alpha | \beta \rangle_{\hat\eta} := \int_{\mathcal{M}} \alpha \wedge * \beta$$

   and the grading is taken to be with respect to the involution

   $$\iota := (-1)^{\hat N}$$

which has eigenvalue +1 on forms of even degree and eigenvalues −1 on forms of odd degree.

The regulated trace over the identity

$$\begin{aligned} \mathrm{sTr}(\mathrm{I})_{\iota,\mathbf{d},\mathbf{d}^\dagger} &= \mathrm{Tr}\Big(\iota e^{\left(\mathbf{d}-\mathbf{d}^\dagger\right)^2}\Big) \\ &= \mathrm{Tr}(\mathrm{I})_+ - \mathrm{Tr}(\mathrm{I})_- \\ &= \chi(\mathcal{M}) \end{aligned} \tag{177}$$

gives the number of harmonic forms of even degree minus those of odd degree, which is known to be equal to the Euler characteristic $\chi(\mathcal{M})$ of the manifold $\mathcal{M}$.

2. In §2.3.4 (p.134) there will appear other examples and applications, since there the regulated supertrace is used construct a well-defined ('gauge-fixed') scalar product on solutions of a gauge theory with supersymmetric gauge generator.

2.1.3 The graded $\mathbf{u}(1)$ algebra

Introduction. The restriction in §5 (p.255) to *homogeneous* cosmologies means that the gauge symmetry of these systems (see §2.3.2 (p.115) for a brief discussion of gauge theories) is generated by a *single* operator, the Hamiltonian $\mathbf{H}/i\hbar$, which is thus the single element of a $\mathbf{u}(1)$ Lie algebra. This is obviously the simplest case of a gauge algebra that one can consider. It is also the algebra of ordinary non-relativistic quantum mechanics, where $\mathbf{H}/i\hbar$ is not a constraint but the generator of time evolution. For this reason, the various graded versions of $\mathbf{u}(1)$ are usually simply called *supersymmetric quantum mechanics* (abbreviated *SQM*). Even though we are here interested in *covariant* SQM, where there is no explicit time evolution, the following considerations apply to either situation.

This section presents some basic facts on super-$\mathbf{u}(1)$ that will be needed in §2.2 (p.54).

2.30 ($N = 1$ graded extension of $\mathbf{u}(1)$) Let the single element of $\mathbf{u}(1)$ be $\mathbf{H}/i\hbar$,[26]

$$\begin{aligned} \mathbf{H}^\dagger &= \mathbf{H} && (178)\\ [\mathbf{H},\mathbf{H}] &= 0\,. && (179)\end{aligned}$$

By the above discussion of graded algebras (see §2.1.2 (p.37) and in particular 2.16 (p.37)), the simplest $\mathbb{Z}(2)$ graded extension of $\mathbf{u}(1)$, is obtained by adding an involution ι and a ι-odd generator $\mathbf{D}_1$ to the ordinary $\mathbf{u}(1)$ algebra, such that:

$$\begin{aligned} {\mathbf{D}_1}^\dagger &= \mathbf{D}_1 && (180)\\ \{\mathbf{D}_1,\iota\} &= 0 && (181)\\ [\mathbf{D}_1,\mathbf{D}_1]_\iota &= \{\mathbf{D}_1,\mathbf{D}_1\}\\ &= 2\mathbf{H}\,. && (182)\end{aligned}$$

It follows that

$$\begin{aligned} [\mathbf{H},\iota] &= 0 && (183)\\ [\mathbf{D}_1,\mathbf{H}]_\iota &= [\mathbf{D}_1,\mathbf{H}]\\ &= 0\,. && (184)\end{aligned}$$

Example 2.31 The example of central importance in §2.2 (p.54) is that where the original $\mathbf{u}(1)$ generator is a (pseudo-)Laplace operator (*cf.* definition 2.48 (p.55)) on some (pseudo-)Riemannian manifold $(\mathcal{M}, g)$:

$$\begin{aligned} H : C^\infty(\mathcal{M}) &\to C^\infty(\mathcal{M})\\ f &\mapsto \nabla^\mu\nabla_\mu f\,. \end{aligned}$$

This can be *extended* to the Laplace-Beltrami operator $\mathbf{H}$ on the exterior bundle $\Lambda(\mathcal{M})$ over $\mathcal{M}$ (2.2 (p.16))

$$\begin{aligned} \mathbf{H} : \Lambda(\mathcal{M}) &\to \Lambda(\mathcal{M})\\ f &\mapsto \left(\mathbf{d}\mathbf{d}^\dagger + \mathbf{d}^\dagger\mathbf{d}\right) f\,, && (185)\end{aligned}$$

[26] Where it is understood that $\mathbf{H}^\dagger = \mathbf{H}$ is to be read as: $\mathbf{H}$ is *essentially* self-adjoint.

which admits the square root

$$\begin{aligned} \mathbf{D_1} : \Lambda(\mathcal{M}) &\rightarrow \Lambda(\mathcal{M}) \\ |\alpha\rangle &\mapsto \left(\mathbf{d} + \mathbf{d}^\dagger\right)|\alpha\rangle \,. \end{aligned} \tag{186}$$

The associated grading is induced by the Witten operator $(-1)^{\hat{N}}$ on $\Lambda(\mathcal{M})$ (*cf.* example 2.15 (p.37)).

Note that in the above extension one starts out with the ordinary Laplace operator $H = \nabla^\mu \nabla_\mu$ and then *extends* it to the Laplace-Beltrami operator $\mathbf{H} = \left(\mathbf{d} + \mathbf{d}^\dagger\right)^2$, which then admits the formal square root leading to the superalgebra. This is a general aspect of supersymmetric extensions. In physical applications the difference of the ordinary and the extended bosonic operator will be or order $\hbar$:

$$\mathbf{H} - H = \mathcal{O}(\hbar) \,.$$

(See 2.62 (p.61) for more details.)

This graded extension of $\mathbf{u}(1)$ is called $(N=1)$-supersymmetric because it contains $N=1$ odd generators $\mathbf{D}_1$. Extensions with $N > 1$ supercharges are called *extended SQM*. Generically, an algebra contains further supercharges if it also contains certain even symmetry generators (*cf.* §2.2.7 (p.90)).

2.32 (N-extended graded extension of $\mathbf{u}(1)$) The N-extended graded $\mathbf{u}(1)$-algebra contains N odd generators $\mathbf{D}_i,\ i \in \{1,\ldots,N\}$ so that

$$\mathbf{D}_i{}^\dagger = \mathbf{D}_i \tag{187}$$

$$\{\mathbf{D}_i, \iota\} = 0 \tag{188}$$

$$\begin{aligned} \left[\mathbf{D}_i, \mathbf{D}_j\right]_\iota &= \{\mathbf{D}_i, \mathbf{D}_j\} \\ &= 2\delta_{ij}\mathbf{H} \end{aligned} \tag{189}$$

(for all $i, j \in \{1,\ldots,N\}$), from which it follows that

$$\begin{aligned} \left[\mathbf{D}_i, \mathbf{H}\right]_\iota &= [\mathbf{D}_i, \mathbf{H}] \\ &= 0\,. \end{aligned} \tag{190}$$

Example 2.33 The exterior Laplace operator

$$\mathbf{H} = \mathbf{d}\mathbf{d}^\dagger + \mathbf{d}^\dagger\mathbf{d}$$

(*cf.* example 2.31 (p.43)) generically admits $N = 4$ supersymmetry: Start with

$$\mathbf{D_1} = \mathbf{d} + \mathbf{d}^\dagger$$

and choose the involution (*cf.* B.16 (p.307))

$$\iota = \bar{\gamma}_{(-1)^{D+1}}\,, \tag{191}$$

where D is the number of dimensions of the underlying manifold $\mathcal{M}$. The relation $\{\mathbf{D}_1, \iota\} = 0$ follows from (*cf.* §2.1.1 (p.15) and §B (p.297)):

$$\begin{aligned} \{\mathbf{D}_1, \iota\} &\stackrel{(1206)}{=} \left\{\hat{\gamma}^\mu_- \hat{\nabla}_\mu, \iota\right\} \\ &= \underbrace{\left\{\hat{\gamma}^\mu_-, \iota\right\}}_{\stackrel{(191)}{=}0} \hat{\nabla}_\mu + \hat{\gamma}^\mu_- \underbrace{\left[\hat{\nabla}_\mu, \iota\right]}_{\stackrel{(1220)}{=}0} \\ &= 0\,. \end{aligned}$$

Then a second supercharge is always given by

$$\mathbf{D}_3 := i\bar{\gamma}_{(-1)^{D+1}}\mathbf{D}_1 \,. \tag{192}$$

$\mathbf{D}_1, \mathbf{D}_2$ already satisfy ($N = 2$)-extended supersymmetry:

$$\{\mathbf{D}_i, \mathbf{D}_j\} \;=\; 2\delta_{ij}\mathbf{H} \qquad i,j \in \{1,3\} \;.$$

But there are further generic supercharges which are associated with a generic symmetry of $\mathbf{H}$ (*cf.* §2.2.7 (p.90), theorem 2.93 (p.93)): One has

$$\begin{aligned} \left[\hat{N}, \mathbf{H}\right] &= 0 \\ \left[\hat{N}, \mathbf{D}_1\right] &= \left[\hat{N}, \mathbf{d} + \mathbf{d}^\dagger\right] \\ &= \mathbf{d} - \mathbf{d}^\dagger \,. \end{aligned}$$

Normalizing by a factor of i this is easily seen to yield a third supercharge:

$$\begin{aligned} \mathbf{D}_2 &= i\left(\mathbf{d} - \mathbf{d}^\dagger\right) \\ &= i\hat{\gamma}_+^\mu \hat{\nabla}_\mu \,, \end{aligned} \tag{193}$$

which anticommutes with $\mathbf{D}_1$ and $\mathbf{D}_3$. (Note that $\left[\hat{\gamma}_+^a, \iota\right] = 0$.) Analogously to (192) a fourth supercharge is found to be

$$\mathbf{D}_4 = i\bar{\gamma}_{(-1)^D}\mathbf{D}_2 \,. \tag{194}$$

This finally gives ($N = 4$)-extended supersymmetry:

$$\{\mathbf{D}_i, \mathbf{D}_j\} \;=\; 2\delta_{ij}\mathbf{H} \qquad i,j \in \{1,2,3,4\} \;.$$

This example shows that there is a certain ambiguity in counting supercharges. The superalgebra on $\Lambda(\mathcal{M})$ is usually referred to as ($N = 2$) instead of ($N = 4$) (e.g. in [101]), counting only $\mathbf{D}_1 = \mathbf{d} + \mathbf{d}^\dagger$ and $\mathbf{D}_2 = i\left(\mathbf{d} - \mathbf{d}^\dagger\right)$. The is related to the fact that two of the supercharges are redundant when defining supersymmetric *states*, i.e. elements of $\Lambda(\mathcal{M})$ annihilated by $\mathbf{D}_i,\ i \in \{1,2,3,4\}$. This is because of the equivalences:

$$\begin{aligned} \mathbf{D}_1 |\phi\rangle = 0 &\quad\Leftrightarrow\quad \mathbf{D}_3 |\phi\rangle = 0 \\ \mathbf{D}_2 |\phi\rangle = 0 &\quad\Leftrightarrow\quad \mathbf{D}_4 |\phi\rangle = 0 \,. \end{aligned}$$

This situation will be analyzed in some detail in C.1 (p.319), C.2 (p.320), and C.4 (p.326).

2.34 (Generalized supersymmetry.) Often, the graded $\mathbf{u}(1)$ algebra under consideration is a subalgebra of a larger algebra that comes equipped with several commuting involutions ι_i, $i \in \{1, \ldots, k\}$ instead of only a single one:

$$\begin{aligned} (\iota_i)^2 &= 1 \\ [\iota_i, \iota_j] &= 0 \\ \{\mathbf{D}_i, \iota_j\} &= 0 \,. \end{aligned} \tag{195}$$

Such a situation is called *generalized supersymmetric quantum mechanics* in [35][87][88].

Example 2.35 If there is a Casimir operator c in the algebra that squares to the identity:

$$\begin{aligned} [c,\mathbf{X}] &= 0, \quad \mathbf{X} \in \{\mathbf{D}_1, \cdots, \mathbf{D}_N, \iota\} \\ c^2 &= 1\,, \end{aligned}$$

then

$$\begin{aligned} \iota_1 &:= \iota \\ \iota_2 &:= \iota c \end{aligned}$$

are two involutions that give rise to a generalized supersymmetry algebra (195) with $k = 2$. (This happens in every Clifford algebra in odd dimensions (see definition B.16 (p.307)).

2.36 (Nilpotent linear combinations of SQM generators) A special role is played by the following linear combinations of $(N = 2)$-SQM generators:

$$\mathbf{d} := \frac{1}{2}(\mathbf{D}_1 + i\mathbf{D}_2) \tag{196}$$

$$\mathbf{d}^\dagger := \frac{1}{2}(\mathbf{D}_1 - i\mathbf{D}_2)\,. \tag{197}$$

They satisfy

$$\begin{aligned} \mathbf{D}_1 &= \mathbf{d} + \mathbf{d}^\dagger \\ \mathbf{D}_2 &= -i\left(\mathbf{d} - \mathbf{d}^\dagger\right) && (198) \\ [\mathbf{d},\mathbf{d}]_\iota &= \{\mathbf{d},\mathbf{d}\} \\ &= 0 \\ \left[\mathbf{d}^\dagger,\mathbf{d}^\dagger\right]_\iota &= \left\{\mathbf{d}^\dagger,\mathbf{d}^\dagger\right\} \\ &= 0 && (199) \\ \left[\mathbf{d},\mathbf{d}^\dagger\right]_\iota &= \left\{\mathbf{d},\mathbf{d}^\dagger\right\} \\ &= \mathbf{H}\,. && (200) \end{aligned}$$

As the notation suggests, the exterior derivative and coderivative are of this form, see §2.2 (p.54).

To easily distinguish between the two versions of the superalgebra introduce the following terminology:

2.37 (Polar and diagonal superalgebra) A set of n mutually adjoint nilpotent operators

$$\mathbf{d}_i, \mathbf{d}^\dagger{}_i \qquad i \in \{1, \dots n\}$$

satisfying

$$\begin{aligned} \{\mathbf{d}_i, \mathbf{d}_j\} &= 0 \\ \left\{\mathbf{d}^\dagger{}_i, \mathbf{d}^\dagger{}_j\right\} &= 0 \\ \left\{\mathbf{d}_i, \mathbf{d}^\dagger{}_j\right\} &= \delta_{ij}\mathbf{H} && (201) \end{aligned}$$

define the *polar* superalgebra, whereas the corresponding set

$$\mathbf{D}_i, \qquad i \in \{1, \dots 2n\}$$

obtained by the change of basis

$$\begin{aligned} \mathbf{D}_{2i} &= \mathbf{d}_i + \mathbf{d}^\dagger{}_i \\ \mathbf{D}_{2i+1} &= i\left(\mathbf{d}_i - \mathbf{d}^\dagger{}_i\right) \end{aligned}$$

define the *diagonal* superalgebra

$$\{\mathbf{D}_i, \mathbf{D}_j\} = 2\delta_{ij}\mathbf{H}\,.$$

Note that this is completely analogous to the respective situation in Clifford algebra/exterior algebra (*cf.* §2.1.1 (p.15)), where from the polar (Grassmann) algebra of creators and annihilators

$$\begin{aligned} \left\{\hat{\mathrm{e}}^a, \hat{\mathrm{e}}^b\right\} &= 0 \\ \left\{\hat{\mathrm{e}}^{\dagger a}, \hat{\mathrm{e}}^{\dagger b}\right\} &= 0 \\ \left\{\hat{\mathrm{e}}^a, \hat{\mathrm{e}}^{\dagger b}\right\} &= \eta^{ab} \end{aligned} \tag{202}$$

one obtains by linear combination

$$\hat{\gamma}^a_\pm := \hat{\mathrm{e}}^{\dagger a} \pm \hat{\mathrm{e}}^a$$

the respective Clifford algebra

$$\begin{aligned} \left\{\hat{\gamma}^a_\pm, \hat{\gamma}^b_\mp\right\} &= 0 \\ \left\{\hat{\gamma}^a_\pm, \hat{\gamma}^b_\pm\right\} &= \pm 2\eta^{ab}\,. \end{aligned}$$

2.38 (Central charges) A generalization of the N-extended superalgebra of definition 2.32 (p.44) is obtained by considering even-graded operators $\mathbf{Z}_i, \quad i \in \{1, \dots, N\}$, that are in the center of the algebra, i.e.

$$\begin{aligned} &[\mathbf{Z}_i, \mathcal{X}] = 0 \\ &\mathcal{X} \in \{\mathbf{D}_1, \dots, \mathbf{D}_N, \mathbf{Z}_1, \dots, \mathbf{Z}_N\}\ , \end{aligned}$$

where

$$\mathbf{Z}_1 := \mathbf{H}\,,$$

and replacing (189) by

$$\{\mathbf{D}_i, \mathbf{D}_j\} = 2\delta_{ij}\mathbf{Z}_i\,. \tag{203}$$

Example 2.39

1. The operators $\mathbf{D}_i, \quad i \in \{1, 2, 3, 4\}$ of example 2.33 (p.44) are all selfadjoint in Riemannian geometry. But, according to (1229), p. 308, equations (192) and (194) define *anti*-selfadjoint operators in a geometry with

Lorentzian signature. Requiring these to be selfadjoint leads to a simple case of extended supersymmetry with central charges: Define the supercharges as

$$\begin{aligned}
\mathbf{D}_1 &:= \mathbf{d} + \mathbf{d}^\dagger \\
\mathbf{D}_2 &:= i\left(\mathbf{d} - \mathbf{d}^\dagger\right) \\
\mathbf{D}_3 &:= \bar{\gamma}_{(-1)^{D+1}}\mathbf{D}_1 \\
\mathbf{D}_4 &:= \bar{\gamma}_{(-1)^{D}}\mathbf{D}_2 \,.
\end{aligned} \tag{204}$$

and introduce the central charges

$$\begin{aligned}
\mathbf{Z}_1 &= \mathbf{H} \\
\mathbf{Z}_2 &= \mathbf{H} \\
\mathbf{Z}_3 &= -\mathbf{H} \\
\mathbf{Z}_4 &= -\mathbf{H} \,.
\end{aligned}$$

(Trivially, the $\mathbf{Z}_i$ commute with everything in sight.) These operators satisfy

$$\begin{aligned}
&{\mathbf{D}_i}^\dagger = \mathbf{D}_i \\
&\{\mathbf{D}_i, \mathbf{D}_j\} = 2\delta_{ij}\mathbf{Z}_i \,.
\end{aligned} \tag{205}$$

2. More interesting example arise when non-generic ('hidden') symmetries are present in the underlying geometry. See 5.19 (p.284) (p. 284) for an example of how Killing-Yano tensors give rise to superalgebras with central charges.

An important notion in supersymmetric quantum mechanics, introduced in [275], is that of *deforming* or *perturbing* (*cf.* [219]) the super-algebra, i.e. to continuously modify the system under consideration while preserving its supersymmetry. This turns out to be an important tool for finding supersymmetric extensions of non-supersymmetric systems. Formally, such a deformation is accomplished by a one-parameter group of $\mathbb{C}^*$-algebra homomorphisms:

Theorem 2.40 (Algebra homomorphisms of super-u(1)) *A one-parameter family $h(\cdot)$ of algebra homomorphism continuously connected to the identity will map the odd generators of $(N = 2)$-graded $\mathbf{u}(1)$ according to*

$$\mathbf{D} = \mathbf{d} + \mathbf{d}^\dagger \overset{h(\epsilon)}{\mapsto} \mathbf{D}^{(\epsilon)} = e^{-\epsilon A}\,\mathbf{d}e^{\epsilon A} + e^{\epsilon A^\dagger}\,\mathbf{d}^\dagger e^{-\epsilon A^\dagger}\,, \tag{206}$$

where A is any even graded operator.

Proof and construction:
Consider a one-parameter group of $\mathbb{C}^*$-algebra homomorphisms $h(\cdot)$

$$\begin{aligned}
&h(\cdot) : (A, V, \iota) \to (A', V, \iota) \\
&h(\epsilon_1) \circ h(\epsilon_2) = h(\epsilon_1 + \epsilon_2)
\end{aligned} \tag{207}$$

of the $(N=2)$-SQM algebra

$$h(\epsilon) : \mathbf{D}_i \mapsto \mathbf{D}_i^{(\epsilon)},\ i \in \{1,2\}\ .$$

The *-involution respected by $h(\epsilon)$ is the operator adjoint $(\cdot)^\dagger$. It follows that $\mathbf{d}$ and $\mathbf{d}^\dagger$ remain mutually adjoint under the action of $h(\epsilon)$:

$$\begin{aligned} \mathbf{d} &\mapsto \mathbf{d}^{(\epsilon)} \\ \mathbf{d}^\dagger &\mapsto \left(\mathbf{d}^{(\epsilon)}\right)^\dagger , \end{aligned} \tag{208}$$

and one can restrict attention to the transformation of one of these operators, say $\mathbf{d}$:
The identity homomorphism

$$h(0) : \mathbf{d} \mapsto \mathbf{d}$$

requires that

$$\mathbf{d}^{(\epsilon)} = \mathbf{d} + \epsilon \mathbf{d}' + \mathcal{O}\big(\epsilon^2\big)\ ,$$

so that the requirement (199) of nilpotency

$$\begin{aligned} \left(\mathbf{d}^{(\epsilon)}\right)^2 &= 0 \\ \Rightarrow \mathbf{d}^2 + \epsilon\,\{\mathbf{d},\mathbf{d}'\} + \mathcal{O}\big(\epsilon^2\big) &= 0 \\ \Rightarrow \{\mathbf{d},\mathbf{d}'\} &= 0 \end{aligned} \tag{209}$$

says that $\mathbf{d}'$ must be a $\mathbf{d}$-closed operator (see 2.21 (p.39)). Hence, choosing $\mathbf{d}'$ to be $\mathbf{d}$-exact

$$\mathbf{d}' = [\mathbf{d}, A]$$

(for some even operator A, $[A,\iota] = 0$) one arrives at

$$\begin{aligned} \mathbf{d}^{(\epsilon)} &= \mathbf{d} - \epsilon\,[A,\mathbf{d}] + \mathcal{O}\big(\epsilon^2\big) \\ \Rightarrow \mathbf{d}^{(\epsilon)} &= e^{-\epsilon[A,\cdot]}\mathbf{d} \\ &= e^{-\epsilon A}\mathbf{d}e^{\epsilon A}\,, \end{aligned} \tag{210}$$

so that the general form of a continuous super-$(N=2)$-$\mathbf{u}(1)$ homomorphism $h^{(\epsilon)}$ is

$$\begin{aligned} \mathbf{d} &\stackrel{h}{\mapsto} e^{-A}\,\mathbf{d}e^{A} \\ \mathbf{d}^\dagger &\stackrel{h}{\mapsto} e^{A^\dagger}\,\mathbf{d}^\dagger e^{-A^\dagger} \\ \mathbf{D} &\stackrel{h}{\mapsto} e^{-A}\,\mathbf{d}e^{A} + e^{A^\dagger}\,\mathbf{d}^\dagger e^{-A^\dagger}\,. \end{aligned} \tag{211}$$

Example 2.41 (The Witten model) In [275] Witten originated the above method by considering the scalar *superpotential* function $A := W$

$$\begin{aligned} \mathbf{d} &\mapsto e^{-W}\,\mathbf{d}\,e^{W} = \mathbf{d} + [\mathbf{d}, W] \\ \mathbf{d}^\dagger &\mapsto e^{W}\,\mathbf{d}^\dagger\,e^{-W} = \mathbf{d}^\dagger - \left[\mathbf{d}^\dagger, W\right]\ . \end{aligned} \tag{212}$$

For more on the Witten model see definition 2.2.2 (p.61).

Note 2.42 (Algebra homomorphisms as gauge transformations) In the special case where e^A in (211) is *unitary*, the homomorphism induced by e^A is a simple unitary transformation:

$$\left(\left(e^A\right)^\dagger = e^{-A}\right) \Rightarrow h^{(\epsilon)} : \mathbf{X} \mapsto e^{-A}\,\mathbf{X}\,e^A\,, \tag{213}$$

where $\mathbf{X} \in \{\mathbf{D}_1, \mathbf{D}_2, \mathbf{H}\}$ is any of the operators of the algebra. Hence, for unitary e^A any solution $|\phi\rangle$ to $\mathbf{D}\,|\phi\rangle = 0$ transforms to a solution $|\phi'\rangle := e^{-A}\,|\phi\rangle$ of the transformed equation ${}^{-A}\mathbf{D}e^A\,|\phi'\rangle = 0$. Such invariance under $U(1)$ transformations of the Dirac operator $\mathbf{D}$ are referred to as *gauge transformations*.[27] (For example, in [93], p.31, such gauge transformations are discussed with respect to Dirac operators on Riemannian manifolds. This is also the point of view expressed in [187][186][21] with respect to BRST operators of the form $\mathbf{Q} = \mathbf{d} + \mathbf{d}^\dagger$, *cf.* remark 2.127 (p.138).)

But if unitary e^A give rise to gauge equivalence classes of algebras, then *non-unitary* e^A transform between different gauge-*inequivalent* classes, i.e. they lead to transformation of the algebra that cannot be 'gauged away' (by a unitary transformation).

For example: In §2.2 (p.54) (see especially theorem 2.58 (p.58) and note 2.61 (p.60)) it is shown that Dirac operators $\mathbf{D} = \mathbf{d} + \mathbf{d}^\dagger$ on the exterior bundle of a (pseudo-)Riemannian manifold $(\mathcal{M}, g)$ can be transformed to a Dirac operator $\mathbf{D}' = \mathbf{d}' + \mathbf{d}^{\dagger\prime} = S^{-1}\mathbf{d}S + S\mathbf{d}^\dagger S^{-1}$ on $(\mathcal{M}, g')$, i.e. on the same manifold but with a different metric, by means of an invertible operator $e^A = S$ that transforms the vielbein frame. Since there are many vielbein frames associated to the same metric, which differ by (pseudo-)orthonormal transformations, there is *gauge freedom* in the vielbein field. Hence, as one should expect, when the transformation operator A describes a (pseudo-)orthonormal transformation of the vielbein it will be a unitary operator, inducing transformations between different explicit representations of one and the same Dirac operator, while otherwise it will be non-unitary and transform $\mathbf{D} \to \mathbf{D}'$ to another metric, a transformation that cannot be 'gauged away'.

Note 2.43 (Homomorphisms that preserve the even generator) If

$$e^A \;:=\; U$$

is unitary

$$U^\dagger \;=\; U^{-1} \tag{214}$$

and in addition commutes with the even graded generator $\mathbf{H}$

$$[\mathbf{H}, U] \;=\; 0\,, \tag{215}$$

then, by (213), it induces a homomorphism of the superalgebra which respects the even generator:

$$\begin{aligned} \mathbf{D}_{(i)} &\stackrel{h}{\mapsto} U^\dagger \mathbf{D}_{(i)} U \\ \mathbf{H} &\stackrel{h}{\mapsto} \mathbf{H}\,. \end{aligned} \tag{216}$$

[27] These are, though, not to be confused with the gauge transformations $e^{\tau\mathbf{D}}$ induced by the Dirac operator itself, in cases where it is considered as a gauge generator (*cf.* §2.3 (p.106)).

It follows that

$$\left(\left\{\mathbf{H},\, U^\dagger \mathbf{D}_{(i)} U\right\},\, U^\dagger \iota U\right)$$

is a supersymmetric extension of the $\mathbf{u}(1)$ algebra generated by $\mathbf{H}$ if

$$\left(\left\{\mathbf{H}, \mathbf{D}_{(i)}\right\},\iota\right)$$

is. This way entire families of supersymmetric extensions arise.

Example 2.44 Let $\mathbf{H} = \mathbf{\Delta} = \left(\mathbf{d} + \mathbf{d}^\dagger\right)^2$ be the exterior Laplace operator of some pseudo-Riemannian manifold. It commutes with the anti-self-adjoint involutions $\bar\gamma_\pm$ (*cf.* definition B.16 (p.307)):

$$\begin{aligned}
\bar\gamma_\pm^\dagger &= -\bar\gamma_\pm \\
\bar\gamma_\pm^2 &= 1 \\
[\mathbf{\Delta}, \bar\gamma_\pm] &= 0\,.
\end{aligned}$$

These give rise to the unitary operators

$$\begin{aligned}
U(\alpha) &:= e^{\alpha\bar\gamma_\pm} \\
U^\dagger(\alpha) &= e^{-\alpha\bar\gamma_\pm} \\
&= U^{-1}(\alpha)\;,
\end{aligned}$$

(for real numbers α) which induce *duality rotations*

$$U(\alpha) \;=\; \cosh(\alpha) + \sinh(\alpha)\,\bar\gamma_\pm\,. \tag{217}$$

Hence, e.g. in even dimensions,

$$\begin{aligned}
\mathbf{D} &= \mathbf{d} + \mathbf{d}^\dagger \\
\mathbf{D}^2 &= \mathbf{H}
\end{aligned}$$

is a Dirac operator with respect to $\mathbf{H}$, and so is

$$\begin{aligned}
\mathbf{D}' &:= e^{-\alpha\bar\gamma_-}\mathbf{D}e^{+\alpha\bar\gamma_-} \\
&= e^{-2\alpha\bar\gamma_-}\mathbf{D} \\
\mathbf{D}'^2 &= \mathbf{H}
\end{aligned} \tag{218}$$

It is of importance for some developments in §4 (p.181) (see in particular 4.34 (p.218)) that the well known Poincaré lemma has a straightforward generalization to deformed exterior derivatives. Therefore the following briefly recalls the ordinary Poincaré lemma and shows how it extends to more than one and to deformed exterior derivatives:

Theorem 2.45 (Poincaré lemma and homotopy operator)
Every closed form is either locally exact or of degree zero.

(See for instance [91].) More precisely, to every closed form $|\phi\rangle \in \Lambda(\mathcal{M})$, with

$$\begin{aligned}
\mathbf{d}\,|\phi\rangle &= 0 \\
\hat N\,|\phi\rangle &\neq 0\,,
\end{aligned}$$

and every point $p \in \mathcal{M}$, there is a starshaped neighborhood $U_p \subset \mathcal{M}$ of p such that there exists a form $|\psi\rangle$ on U_p satisfying

$$|\phi\rangle_{|U_p} \quad = \quad \mathbf{d}\,|\psi\rangle_{|U_p} \ . \tag{219}$$

Of course, this $|\psi\rangle$ is unique only up to a ‘gauge‘” $|\psi\rangle \to |\psi\rangle + \mathbf{d}\,|\psi'\rangle$. $|\psi\rangle$ may be obtained from $|\phi\rangle$ by applying the (equally non-unique) *homotopy operator* $\mathbf{K}$ on $\Lambda(\mathcal{M})$, which satisfies

$$\begin{aligned} \{\mathbf{d}, \mathbf{K}\} &= 1 \\ \left[\hat{N}, \mathbf{K}\right] &= -\mathbf{K}\,, \end{aligned} \tag{220}$$

so that

$$\begin{aligned} & \mathbf{d}\,|\phi\rangle = 0, \quad \hat{N}\,|\phi\rangle \neq 0 \\ \Rightarrow \quad & \mathbf{d}\,(\mathbf{K}\,|\phi\rangle) = \{\mathbf{d}, \mathbf{K}\}\,|\phi\rangle = |\phi\rangle \ . \end{aligned} \tag{221}$$

From (220) it follows that the homotopy operator decreases the form degree by one and hence it annihilates 0-forms

$$\left(\hat{N}\,|\phi\rangle = 0\right) \quad \Leftrightarrow \quad (\mathbf{K}\,|\phi\rangle = 0) \ . \tag{222}$$

The Poincaré lemma extends to the case where several anticommuting exterior derivatives are present: Let $\mathbf{d}_1$ and $\mathbf{d}_2$ be two anticommuting operators with

$$\left[\hat{N}, \mathbf{d}_i\right] \quad = \quad \mathbf{d}_i, \quad i \in \{1, 2\} \tag{223}$$

that each have an associated homotopy operator:

$$\begin{aligned} \{\mathbf{d}_i, \mathbf{d}_j\} &= 0 \\ \{\mathbf{d}_i, \mathbf{K}_i\} &= 1\,. \end{aligned}$$

Let $|\phi\rangle$ be an element in the kernel of both

$$\mathbf{d}_i\,|\phi\rangle = 0\,,$$

with

$$\hat{N}\,|\phi\rangle = n\,|\phi\rangle\,, \quad n \geq 2\,.$$

Applying the Poincaré lemma with respect to $\mathbf{d}_1$ yields (locally)

$$|\phi\rangle = \mathbf{d}_1\,|\psi'\rangle \ ,$$

where $\hat{N}\,|\psi'\rangle \neq 0$. Since $\mathbf{d}_1$ and $\mathbf{d}_2$ anticommute, $\mathbf{d}_1$ swaps eigenspaces of $\mathbf{d}_2$ with eigenvalues of opposite sign. Hence

$$\begin{aligned} & \mathbf{d}_2\,|\phi\rangle = 0 \\ \Leftrightarrow \quad & \mathbf{d}_2\mathbf{d}_1\,|\psi'\rangle = 0 \\ \Leftrightarrow \quad & \mathbf{d}_2\,|\psi'\rangle = 0 \\ \Leftrightarrow \quad & |\psi'\rangle = \mathbf{d}_2\,|\psi\rangle \quad \text{(locally)} \\ \Leftrightarrow \quad & |\phi\rangle = \mathbf{d}_1\mathbf{d}_2\,|\psi\rangle \quad \text{(locally)}\,. \end{aligned} \tag{224}$$

Therefore in general, with a set of anticommuting nilpotent operators

$$\{\mathbf{d}_i | i \in I\}$$
$$\left[\hat{N}, \mathbf{d}_i\right] = \mathbf{d}_i \,,$$

that each have a homotopy operator

$$\{\mathbf{d}_i, \mathbf{K}_i\} = 1,\; i \in I$$
$$\left[\hat{N}, \mathbf{K}_i\right] = -\mathbf{K}\,, \tag{225}$$

every state in the kernel of all $\mathbf{d}_i$ is locally of the form

$$\mathbf{d}_{i\in I} |\phi\rangle = 0 \;\Rightarrow\; |\phi\rangle = \left(\prod_{i\in I} \mathbf{d}_i\right) |\psi\rangle \;. \tag{226}$$

(See [216] for examples.)

These facts immediately carry over to deformed exterior derivatives as used in theorem 2.40 (p.48):

2.46 (Deformed homotopy)
Let A be an invertible operator preserving the form degree, i.e. $\left[\hat{N}, A\right] = 0$. Then: *Every $p > 0$-form closed under the A-deformed exterior derivative*

$$\mathbf{d}_A := A^{-1}\mathbf{d}A$$

is locally exact with respect to $\mathbf{d}_A$.

This simply follows from the existence of the deformed homotopy operator

$$\mathbf{K}_A \;:=\; A^{-1}\mathbf{K}A \tag{227}$$

satisfying

$$\{\mathbf{d}_A, \mathbf{K}_A\} \;=\; 1\,. \tag{228}$$

Before closing this section, an important remark is in order:

2.47 (Supersymmetry as a formal tool) Supersymmetry can be useful even if the system one is studying is truly bosonic. Consider a bosonic Hamiltonian H and a supersymmetric extension $\mathbf{H} = \mathbf{D}^2 = \left(\mathbf{q} + \mathbf{q}^\dagger\right)^2$. One usually has

$$Hf = \mathbf{q}^\dagger \mathbf{q} f$$

for f a bosonic state, i.e. $f = f\,|0\rangle$. This implies that the bosonic sector of every supersymmetric solution $|\phi\rangle$

$$\mathbf{D}\,|\phi\rangle = 0 \tag{229}$$

is also a solution to bosonic theory

$$H\,|\phi\rangle_0 = 0\,. \tag{230}$$

The point is that (229), which is first order, may be easier to solve than (230), which is second order. (229) replaces a single second order equations by a system of first order equations.

2.2 Supersymmetric (relativistic) quantum mechanics

This section considers the ($N = 2$)-supersymmetric quantum mechanics of a relativistic point particle propagating on a Lorentzian manifold of arbitrary dimension. While the analogous setup for non-relativistic quantum mechanics on Riemannian manifolds has been studied extensively ([53][275][274][101][102][176] [177] [280]), there is at present, to the best of our knowledge, no comparably exhaustive treatment of the indefinite metric case (but see [123]). On the other hand, many of the results and methods of non-relativistic *SQM* are unaffected by a change of signature of the metric, and the first part of the present section (§3.1, §3.2) will equally apply to either case. However, the indefiniteness of a pseudo-Riemannian metric has subtle but profound consequences for the supersymmetry formalism:

1. Most importantly from a mathematical point of view is the fact that the pseudo-Laplace operator on a Lorentzian manifold is no longer an *elliptic* differential operators, so that a large body of theory is not available.

2. Most importantly from a physical point of view is the fact that a physically interesting Lorentzian manifold is generically *non-compact* and that physically relevant fields on that manifold are not integrable (they do not vanish in the 'time-like' direction).

Section §2.3 (p.106) will present a way to deal with both of these issues:

In physics, the standard technique to handle problems like 2., above, is known as *gauge fixing* and a powerful formalism to handle this is *cohomology theory* (known as *BRST theory* in this context). Incidentally, by general results of cohomology theory, 'fixing a gauge' is tantamount to defining an *elliptic* operator (the *BRST Laplacian*), which is a modification of the hyperbolic D'Alembert operator and a substitute for the elliptic Laplacian on Riemannian manifolds. By means of this BRST Laplacian (or rather its adaption to the present supersymmetry context) one can construct, as is done in section §2.3.5 (p.140), a finite scalar product on physical fields and well defined expectation values of physical observables.

Notation. The following section makes free use of the notation concerning differential geometry and exterior and Clifford algebra introduced in §2.1.1 (p.15).

2.2.1 Taking the square root: Dirac operators

Outline. This section discusses realizations of the $\mathbf{u}(1)$-superalgebra (as introduced in §2.1.3 (p.43)) with the even generator $\mathbf{H}$ represented by a *generalized Laplace operator* $\mathbf{H} = \mathbf{\Delta}$ on a (pseudo-)Riemannian manifold. In this case the odd generators $\mathbf{D}_{(i)}$ with $\mathbf{D}_{(i)}^2 = \mathbf{\Delta}$ are called *generalized Dirac operators*. Generalized Laplace operators $\mathbf{\Delta}$ arise in quantum mechanics as *extensions* of quantum mechanical Hamiltonian (constraint) operators $\hat{H}$, that themselves do not admit any 'square root'. For this reason the representation of graded-$\mathbf{u}(1)$ by means of generalized Laplace and Dirac operators is called *supersymmetric quantum mechanics.*

Literature. Dirac operators on Riemannian manifolds are treated for instance in [109][93].

2.48 (Generalized Laplace and Dirac operators) Let $\mathcal{B}$ be a fiber bundle on a (pseudo-)Riemannian manifold $(\mathcal{M}, g)$ with metric tensor $g = (g_{\mu\nu})$ and inverse metric $g^{-1} = (g^{\mu\nu})$.

- A *generalized Laplace operator* is a linear *second order* differential operator

  $$\Delta : \mathcal{B} \to \mathcal{B}$$

  with local realization

  $$\Delta = g^{\mu\nu}\partial_\mu\partial_\nu + a^\mu\partial_\mu + A_{\mathrm{L}}\,, \tag{231}$$

  where $a = a^\mu\partial_\mu$ is any $\mathrm{End}(\mathcal{B}) - valued$ vector field and A_{L} any $\mathrm{End}(\mathcal{B})$-valued function on $\mathcal{M}$.

- A *generalized Dirac operator* is a linear *first order* differential operator

  $$\mathbf{D} : \mathcal{B} \to \mathcal{B}$$

  with local representation

  $$\mathbf{D} = \hat{\gamma}^\mu\partial_\mu + A_{\mathrm{D}} \tag{232}$$

  (where $\hat{\gamma}^\mu$ is a representation of the Clifford algebra on $\mathcal{B}$ (see B.1 (p.297)) and A_{D} are $\mathrm{End}(B)$-valued functions), which squares to a generalized Laplace operator on $(\mathcal{M}, g)$:

  $$\mathbf{D}^2 = \Delta\,. \tag{233}$$

Note 2.49 (Terminology) Usually, in the literature, the term 'generalized Laplacian' is restricted to operators on *Riemannian* manifolds. In the present context, however, it is necessary to consider operators with the local representation (231) but on *pseudo-Riemannian* manifolds, i.e. for *indefinite* metric tensors g. In analogy with the terminology for flat metrics, these should probably be called *generalized D'Alembert* operators. But since many of the following considerations are actually insensitive to the signature of the metric it would be inappropriately restrictive to use the latter term in these cases. Therefore the term 'generalized Laplacian' will in the following be understood to refer to arbitrary signatures, unless explicitly stated otherwise.

2.50 (Supersymmetric quantum mechanics (SQM)) Any representation of the $\mathbf{u}(1)$-superalgebra (defined in §2.1.3 (p.43)), where the even generator $\mathbf{H}$ is a generalized Laplace operator for some manifold and the odd generators $\mathbf{D}_{(i)}$ are the respective generalized Dirac operators, defines the evolution/constraint algebra of a *supersymmetric quantum mechanical* system.

Example 2.51 Supersymmetric quantum mechanics describes particles with spin: Consider a spinless relativistic quantum point particle propagating on a pseudo-Riemannian manifold $(\mathcal{M}, g)$. Its scalar wave function is annihilated by H, the Klein-Gordon operator

$$\begin{aligned} H\,|\phi\rangle &= 0 \\ \Leftrightarrow -\nabla^\mu \nabla_\mu\,|\phi\rangle &= 0\,, \end{aligned} \tag{234}$$

with ∇ the Levi-Civita connection. Demanding $N = 1$ worldline supersymmetry calls for the Dirac operator on the Spin bundle (see §B (p.297) and in particular §B.2 (p.311)):

$$\begin{aligned} D &= \hat{\gamma}^\mu \nabla^S_\mu \\ D^2 &= -\Delta^S - \frac{1}{4}R\,, \end{aligned} \tag{235}$$

(where $\nabla^S_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu ab}\hat{\gamma}^a\hat{\gamma}^b$ is the usual Levi-Civita compatible spin connection, *cf.* [109], §5) describing a massless relativistic spin-1/2 particle on $\mathcal{M}$.

Hence ordinary spinor particles are governed by supersymmetric $N = 1$ generator algebras (compare for instance [263] [17] [260][178]) In the present context, however, we need to deal with $N = 2$ generator algebras, where the spinor bundle is replaced by the exterior bundle. Both setups are quite different, but intimately related. This is discussed in §B.2 (p.311). Also see [101], where the $(N = 2)$-SQM on Riemannian manifolds is referred to as describing "positronium", namely a system of two spinor particles. Such an interpretation is possible because the $N = 2$ system essentially consists of the product of two $N = 1$ systems. However, on pseudo-Riemannian manifolds the $N = 2$ algebra is perhaps more naturally recognized as that of ordinary classical electromagnetism (see §2.2.3 (p.70)), which, of course, again involves the product of two spinors, namely in so far as the photon, being a vector, can be considered the square of a spinor.

In this context the following relation is noteworthy:

Theorem 2.52 *The kernels of* $\mathbf{D} = \mathbf{d}+\mathbf{d}^\dagger$ *and* $\bar{\mathbf{D}} = i\left(\mathbf{d} - \mathbf{d}^\dagger\right)$ *are isomorphic:*

$$\mathrm{Ker}\,(\mathbf{D}) = (-1)^{\hat{N}(\hat{N}-1)/2}\,\mathrm{Ker}\left(\bar{\mathbf{D}}\right) \tag{236}$$

Proof: Consider any operator $\hat{A}_\pm$ that increases (decreases) the form degree by one:

$$\hat{N}\hat{A}_\pm = \hat{A}_\pm\left(\hat{N} \pm 1\right)\,.$$

Then

$$(-1)^{\hat{N}\left(\hat{N}+1\right)/2+1}\,\hat{A}_\pm = \hat{A}_\pm\,(-1)^{\left(\hat{N}\pm 1\right)\left(\hat{N}\pm 1+1\right)/2+1}$$

$$
\begin{aligned}
&= \hat{A}_\pm \, (-1)^{\left(\hat{N}^2 \pm 2\hat{N}+1+\hat{N}\pm 1\right)/2+1} \\
&= \pm\hat{A}_\pm \, (-1)^{\left(\hat{N}^2+\hat{N}\right)/2\pm\hat{N}} \\
&= \pm\hat{A}_\pm \, (-1)^{\hat{N}\left(\hat{N}-1\right)/2+\hat{N}\pm\hat{N}} \\
&= \pm\hat{A}_\pm \, (-1)^{\hat{N}\left(\hat{N}-1\right)/2} \,,
\end{aligned}
$$

so that

$$
\begin{aligned}
\left(\mathbf{d}+\mathbf{d}^\dagger\right)|\alpha\rangle &= 0 \\
\Leftrightarrow \quad (-1)^{\hat{N}\left(\hat{N}+1\right)/2+1}\left(\mathbf{d}+\mathbf{d}^\dagger\right)|\alpha\rangle &= 0 \\
\Leftrightarrow \quad \left(\mathbf{d}-\mathbf{d}^\dagger\right)\,(-1)^{\hat{N}\left(\hat{N}-1\right)/2}|\alpha\rangle &= 0 \,.
\end{aligned}
$$

□

2.53 It follows that a state $|\alpha\rangle$ is $N=2$ supersymmetric, i.e.

$$
\begin{aligned}
\mathbf{D}\,|\alpha\rangle &= \bar{\mathbf{D}}\,|\alpha\rangle = 0 \\
\Leftrightarrow \qquad \mathbf{d}\,|\alpha\rangle &= \mathbf{d}^\dagger\,|\alpha\rangle = 0 \,,
\end{aligned}
\tag{237}
$$

if

$$\mathbf{D}\,|\alpha\rangle = 0$$

and

$$(-1)^{\hat{N}(\hat{N}+1)/2}\,|\alpha\rangle = \pm\alpha \,. \tag{238}$$

Example 2.54 Note that the states satisfying (238) with the $(+)$ sign are those that have only p-form components of degree $0,3,4,7,8,\cdots$, while those corresponding to the $(-)$ sign have only p-form components in the $1,2,5,6,9,10,\cdots$-form sectors. It is obvious that such states are annihilated by $\mathbf{d}+\mathbf{d}^\dagger$ exactly if they are annihilated by $\mathbf{d}$ and $\mathbf{d}^\dagger$ alone, because the images of both operators can never coincide on these states. The most well known example is classical source-free electromagnetism (*cf.* §2.2.3 (p.70)), where the state in question, the Faraday form, is a pure 2-form.

The strategy of the following presentation is to find various SQM representations by continuously deforming the trivial SQM algebra for flat manifolds by means of an algebra homomorphism (207), p. 48. This way arbitrary (non-flat) metrics and non-vanishing potentials ('superpotentials') arise.

2.55 (Standard $(N=2)$-SQM algebra for flat metrics) Let $\Gamma(\Lambda(\mathcal{M}))$ be the space of square integrable sections (forms) of the exterior bundle $\Lambda(\mathcal{M})$ on flat D-dimensional Euclidean or Minkowski space $\mathcal{M}$ with metric

$$\eta \;:=\; \mathrm{diag}(\pm,+,\ldots,+) \,. \tag{239}$$

The generalized Laplace and Dirac operators, which in this case are simply the D'Alembert and ordinary Dirac operator trivially extended to $\Lambda(\mathcal{M})$, explicitly read

$$
\begin{aligned}
\mathbf{H} &:= -\partial^\mu\partial_\mu \\
\mathbf{D}_1 &= \hat{\gamma}^\mu_-\partial_\mu \\
\mathbf{D}_2 &= i\hat{\gamma}^\mu_+\partial_\mu \,.
\end{aligned}
\tag{240}
$$

Note that *by definition* (see B.13 (p.304))

$$\left[\partial_a, \hat{\gamma}^b_\pm\right] = 0\,.$$

These are essentially self-adjoint operators on $\Gamma(\Lambda(\mathcal{M}))$ that form a super $\mathbf{u}(1)$-algebra

$$(A, V, \iota) = \left(\{\mathbf{H}, \mathbf{D}_1, \mathbf{D}_2\}, \Gamma(\Lambda(\mathcal{M})), (-1)^{\hat{N}}\right)\,.$$

Note 2.56 (Bosons and Fermions) Often, supersymmetry is introduced as a symmetry between bosonic and fermionic fields, because this is how supersymmetry transformations have to be defined in the Lagrangian approach (see 2.67 (p.65) for an example). By Noether's theorem one can derive a conserved charge associated with the invariance of the Lagrangian under supersymmetry transformations (*cf.* [53]). This is the supercharge, which in turn generates supersymmetry transformations. Since in the Hamiltonian approach the supercharge is obtained immediately by means of the square root process, one can easily derive the supersymmetry transformations of the bosonic (x^μ) and fermionic ($\hat{\gamma}^\mu$) variables by taking the supercommutator with the supercharge. For example, for the simple supercharge (240) the transformations read:

$$\begin{aligned}
\delta_{\text{susy}} x^\mu &:= [\mathbf{D}_1, x]_\iota = \hat{\gamma}^\mu \\
\delta_{\text{susy}} \hat{\gamma}^\mu &:= [\mathbf{D}_1, \hat{\gamma}^\mu]_\iota = -\partial^\mu = -ip^\mu\,.
\end{aligned} \tag{241}$$

In this sense supersymmetry interchanges bosonic and fermionic fields.

2.57 (The standard SQM as exterior differential calculus) Noting that for the flat metrics considered above one has

$$\begin{aligned}
\mathbf{d} &= \hat{\mathrm{e}}^{\dagger\mu}\partial_\mu \\
\mathbf{d}^\dagger &= -\hat{\mathrm{e}}^{\mu}\partial_\mu \\
\left\{\mathbf{d}, \mathbf{d}^\dagger\right\} &= -\partial^\mu\partial_\mu
\end{aligned} \tag{242}$$

and hence

$$\begin{aligned}
\mathbf{d}\,\phi_{\mu_1\ldots\mu_k}\hat{\mathrm{e}}^{\dagger\mu_1}\hat{\mathrm{e}}^{\dagger\mu_2}\cdots\hat{\mathrm{e}}^{\dagger\mu_k}\left|0\right\rangle &= \partial_\nu\phi_{\mu_1\ldots\mu_k}\hat{\mathrm{e}}^{\dagger\nu}\hat{\mathrm{e}}^{\dagger\mu_1}\hat{\mathrm{e}}^{\dagger\mu_2}\cdots\hat{\mathrm{e}}^{\dagger\mu_k}\left|0\right\rangle \\
\mathbf{d}^\dagger\,\phi_{\mu_1\ldots\mu_k}\hat{\mathrm{e}}^{\dagger\mu_1}\hat{\mathrm{e}}^{\dagger\mu_2}\cdots\hat{\mathrm{e}}^{\dagger\mu_k}\left|0\right\rangle &= -\partial_{\mu_1}\phi^{\mu_1}{}_{\mu_2\ldots\mu_k}\hat{\mathrm{e}}^{\dagger\mu_2}\cdots\hat{\mathrm{e}}^{\dagger\mu_k}\left|0\right\rangle
\end{aligned} \tag{243}$$

one sees that the $(N = 2)$-SQM on flat space is nothing but the algebra of exterior differential geometry on flat space. But since the *Hodge Laplacian* $\left\{\mathbf{d}, \mathbf{d}^\dagger\right\}$ is a generalized Laplace operator on the exterior bundle for *arbitrary metrics* with Dirac operators $\mathbf{D}_1 = \mathbf{d} - \mathbf{d}^\dagger$ and $\mathbf{D}_2 = i\left(\mathbf{d} + \mathbf{d}^\dagger\right)$ it is clear that standard SQM algebras for arbitrary metrics are found by deforming the flat algebra in such a way that the exterior differential algebra is preserved.

Next, some relations are established concerning transformations of the exterior superalgebra (in the sense of observation 2.40 (p.48)) that are related to geometric transformations of the underlying manifold. (*cf.* [8])

Theorem 2.58 (Algebra homomorphism for arbitrary metrics) *There is an algebra homomorphism (207) $h_{g\to g'}$ which, according to observation 2.40* *(p.48), deforms the operator representation* $\mathbf{d}_g$ *of the exterior derivative on* $(\mathcal{M}, g)$ *to the operator representation* $\mathbf{d}_{g'}$ *of the exterior derivative on* $(\mathcal{M}, g')$*:*

$$\mathbf{d}_{g'} = (A_{g\to g'})^{-1}\,\mathbf{d}_g\,A_{g\to g'}\ ,$$

and it is given by

$$A_{g\to g'} \quad = \quad \sum_{n=0}^{D-1} \underbrace{\hat{c}^{\dagger\mu_1}\hat{c}^{\dagger\mu_2}\cdots\hat{c}^{\dagger\mu_n}}_{a)}\underbrace{\hat{c}'_{\mu_{n+1}}\hat{c}'_{\mu_{n+2}}\cdots\hat{c}'_{\mu_D}\hat{c}'^{\dagger\mu_{n+1}}\hat{c}'^{\dagger\mu_{n+2}}\cdots\hat{c}'^{\dagger\mu_D}}_{b)}\underbrace{\hat{c}'_{\mu_1}\hat{c}'_{\mu_2}\cdots\hat{c}'_{\mu_n}}_{c)} \tag{244}$$

(where $\hat{c}'$ *are the coordinate basis annihilators with respect to* g'*).*

Note that the operator $A_{g\to g'}$ really transforms between different vielbein frames (*cf.* note 2.42 (p.50) and note 2.61 (p.60)).
Proof: $A_{g\to g'}$ acts on coordinate basis states of the primed metric by substituting unprimed creators for primed creators:

$$A_{g\to g'} : \hat{c}'^{\dagger\mu_1}\hat{c}'^{\dagger\mu_2}\cdots\hat{c}'^{\dagger\mu_n}\,|0\rangle \mapsto \hat{c}^{\dagger\mu_1}\hat{c}^{\dagger\mu_2}\cdots\hat{c}^{\dagger\mu_n}\,|0\rangle\ , \tag{245}$$

so that the action of $\mathbf{d}_{g'}$ is

$$\begin{aligned}
&\mathbf{d}_{g'}\hat{c}'^{\dagger\mu_1}\hat{c}'^{\dagger\mu_2}\cdots\hat{c}'^{\dagger\mu_n}\alpha_{\mu_1,\mu_2,\ldots,\mu_n}\,|0\rangle\\
=\ & {A_{g\to g'}}^{-1}\,\mathbf{d}\,A_{g\to g'}\hat{c}'^{\dagger\mu_1}\hat{c}'^{\dagger\mu_2}\cdots\hat{c}'^{\dagger\mu_n}\alpha_{\mu_1,\mu_2,\ldots,\mu_n}\,|0\rangle\\
=\ & {A_{g\to g'}}^{-1}\,\mathbf{d}\,\hat{c}^{\dagger\mu_1}\hat{c}^{\dagger\mu_2}\cdots\hat{c}^{\dagger\mu_n}\alpha_{\mu_1,\mu_2,\ldots,\mu_n}\,|0\rangle\\
=\ & h^{-1}\hat{c}^{\dagger\mu_n+1}\hat{c}^{\dagger\mu_1}\hat{c}^{\dagger\mu_2}\cdots\hat{c}^{\dagger\mu_n}\partial_{\mu_{n+1}}\alpha_{\mu_1,\mu_2,\ldots,\mu_n}\,|0\rangle\\
=\ & \hat{c}'^{\dagger\mu_n+1}\hat{c}'^{\dagger\mu_1}\hat{c}'^{\dagger\mu_2}\cdots\hat{c}'^{\dagger\mu_n}\partial_{\mu_{n+1}}\alpha_{\mu_1,\mu_2,\ldots,\mu_n}\,|0\rangle
\end{aligned} \tag{246}$$

which is indeed the correct action of the exterior derivative in the primed metric. □

Corollary 2.59 *The operator representation of the exterior derivative* $\mathbf{d}_g$ *on* $(\mathcal{M}, g)$ *can be written:*

$$\mathbf{d} \quad = \quad A^{-1}_{(g)}\,\hat{e}^{\dagger a}\partial_a\,A_{(g)}\ . \tag{247}$$

Example 2.60 To find the explicit local representation of $\mathbf{d}_g$ on a manifold with metric

$$g' \quad = \quad \mathrm{diag}\left(e^{2x^2}, e^{2(x^1+x^2)}\right) \tag{248}$$

one can calculate $A_{\eta\to g'}$ and its inverse as given by the above theorem,

$$\begin{aligned}
A_{g\to g'} \quad &= \quad 1 + \left(-1+e^{x^1+x^2}\right)\hat{e}^{\dagger 2}\hat{e}^2 + \left(-1+e^{x^2}\right)\hat{e}^{\dagger 1}\hat{e}^1\\
&\quad\ - \left(-1+e^{x^2}\right)\left(-1+e^{x^1+x^2}\right)\hat{e}^{\dagger 1}\hat{e}^{\dagger 2}\hat{e}^1\hat{e}^2\\
(A_{g\to g'})^{-1} \quad &= \quad 1 + \left(-1+e^{-(x^1+x^2)}\right)\hat{e}^{\dagger 2}\hat{e}^2 + \left(-1+e^{-x^2}\right)\hat{e}^{\dagger 1}\hat{e}^1\\
&\quad\ -e^{-x^1-2x^2\left(-1+e^{x^2}\right)}\left(-1+e^{x^1+x^2}\right)\hat{e}^{\dagger 1}\hat{e}^{\dagger 2}\hat{e}^1\hat{e}^2\,,
\end{aligned} \tag{249}$$

yielding

$$\begin{aligned}
\mathbf{d}_{g'} &= \left(A_{\eta\to g'}\right)^{-1} \hat{\mathrm{e}}^{\dagger a}\partial_a A_{\eta\to g'} \\
&= e^{-x^2}\hat{\mathrm{e}}^{\dagger 1}\partial_1 + e^{-(x^1+x^2)}\hat{\mathrm{e}}^{\dagger 2}\partial_2 - e^{-(x^1+x^2)}\hat{\mathrm{e}}^{\dagger 1}\hat{\mathrm{e}}^{\dagger 2}\hat{\mathrm{e}}^1 + e^{-x^2}\hat{\mathrm{e}}^{\dagger 1}\hat{\mathrm{e}}^{\dagger 2}\hat{\mathrm{e}}^2 \,,
\end{aligned}$$

which is indeed the same result one arrives at via the ordinary formula (see B.11 (p.301), eq. (1199))

$$\begin{aligned}
\mathbf{d}_{g'} &= \hat{\mathrm{c}}^{\dagger\mu}\hat{\nabla}_\mu \\
&= \hat{\mathrm{c}}^{\dagger\mu}\left(\partial_\mu - \omega_{\mu ab}\hat{\mathrm{e}}^{\dagger b}\hat{\mathrm{e}}^a\right)
\end{aligned}$$

by calculating the spin connection ω of g'.

Note 2.61 *(Pseudo-)orthonormal transformations of the vielbein field e^a are described by* unitary *transformation operators A in (244):*

$$A^\dagger = A^{-1}\,. \tag{250}$$

Proof: According to (245) the operator A implements the change of basis on the form basis. Hence, by definition of a (pseudo-)orthonormal transformation

$$\begin{aligned}
\langle\alpha|\beta\rangle_{\mathrm{loc}} &= \langle A\alpha|A\beta\rangle_{\mathrm{loc}} \\
\Leftrightarrow A^\dagger A &= 1
\end{aligned}$$

□

2.2.2 The Witten model

In [275] Witten considered the deformed generalized Dirac operator

$$\begin{aligned}
\mathbf{D}_{(\epsilon)} &:= e^{-\epsilon W}\,\mathbf{d}\,e^{\epsilon W} + e^{\epsilon W}\,\mathbf{d}^\dagger\,e^{-\epsilon W} \\
&= \mathbf{d} + \mathbf{d}^\dagger + [\mathbf{d}, \epsilon W] - [\mathbf{d}, \epsilon W] \\
&= \hat{\mathrm{c}}^{\dagger\mu}\hat{\nabla}_\mu - \hat{\mathrm{c}}^\mu\hat{\nabla}_\mu + \epsilon\hat{\mathrm{c}}^{\dagger\mu}\,(\partial_\mu W) + \epsilon\hat{\mathrm{c}}^\mu\,(\partial_\mu W) \\
&= \hat{\gamma}_{g_-}{}^\mu\,\hat{\nabla}_\mu + \epsilon\hat{\gamma}_{g_+}{}^\mu\,(\partial_\mu W)
\end{aligned} \tag{251}$$

to study Morse theory by means of SQM. (See [219] for a nice review of this approach and further background material.)

Theorem 2.62 (Generalized Laplacian of the Witten model) *The generalized Laplacian of the Witten model is*

$$\begin{aligned}
\mathbf{D}^2_{(\epsilon)} &= \mathbf{D}^2_{(0)} + \epsilon^2\,(\partial_\mu W)\,(\partial^\mu W) + \epsilon\hat{\gamma}_{g_-}{}^\mu\,\hat{\gamma}_{g_+}{}^\nu\,(\nabla_\mu\nabla_\nu W) \\
&= \mathbf{D}^2_{(0,\hbar)} + \epsilon^2\,(\partial_\mu W)\,(\partial^\mu W) + \epsilon\left[\hat{\mathrm{c}}^{\dagger\mu}, \hat{\mathrm{c}}^\nu\right](\nabla_\mu\nabla_\nu W)\,.
\end{aligned} \tag{252}$$

Proof:

$$\left(\hat{\gamma}_{g_-}{}^\mu\,\hat{\nabla}_\mu + \epsilon\hat{\gamma}_{g_+}{}^\mu\,(\partial_\mu W)\right)^2 = \mathbf{D}^2_{(0)} + \left(\epsilon\hat{\gamma}_{g_+}{}^\mu\,(\partial_\mu W)\right)^2 + \left\{\hat{\gamma}_{g_-}{}^\mu\,\hat{\nabla}_\mu, \epsilon\hat{\gamma}_{g_+}{}^\mu\,(\partial_\mu W)\right\}.$$

The result follows with eq. (1207), p. 303. □

What makes this Laplacian interesting is that is contains a scalar term $(\partial_\mu W)\,(\partial^\mu W)$ that acts like a potential term in all 'Fermion sectors'. Most applications of SQM make use this Laplacian as a supersymmetric extention of an ordinary Hamilton operator of the form

$$H = -\partial_\mu\partial^\mu + V\,.$$

(see [146][53][144][56][274] for general treatments and [110][111][112][113][25] for applications to cosmology).

2.63 (The super-oscillator) Sometimes supersymmetric quantum mechanics is motivated by means of the concept of the so-called "super-oscillator" (see for instance [146]). This is actually a special case of the general Witten model presented above:

The basic idea is as follows: Consider a single bosonic oscillator described by creation and annihilation operators $\hat{a}$, $\hat{a}^\dagger$, which satisfy the canonical Bose commutation relation $\left[\hat{a}, \hat{a}^\dagger\right] = 1$. The Hamiltonian is defined by

$$\hat{H}_b = \frac{1}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) = \hat{a}^\dagger\hat{a} + \frac{1}{2}\,.$$

A Hamiltonian of analogous form, but for fermionic creation and annihilation operators $\hat{c}$, $\hat{c}^\dagger$, which satisfy Fermi anticommutation relations $\left\{\hat{c}, \hat{c}^\dagger\right\} = 1$, is

$$\hat{H}_f = \frac{1}{2}\left(\hat{c}^\dagger\hat{c} - \hat{c}\hat{c}^\dagger\right) = \hat{c}^\dagger\hat{c} - \frac{1}{2}\,.$$

By adding both Hamiltonians the zero point energies cancel

$$\hat{H}_b + \hat{H}_f = \hat{a}^\dagger \hat{a} + \hat{c}^\dagger \hat{c}$$

to give an operator with manifestly non-negative spectrum. By defining the supercharges

$$\begin{aligned} \hat{d} &= \hat{c}^\dagger \hat{a} \\ \hat{d}^\dagger &= \hat{c} \hat{a}^\dagger \,, \end{aligned} \tag{253}$$

which annihilate a boson and create a fermion, or vice versa, the total Hamiltonian may be expressed as

$$\hat{H}_s = \hat{H}_b + \hat{H}_f = \left\{ \hat{d}, \hat{d}^\dagger \right\} \,,$$

which is said to describe the "super-oscillator". Some essential aspects of general supersymmetry are nicely visible in this toy system, such as the exchange symmetry between Bosons and fermions and the vanishing of the vacuum energy.

The super-oscillator can be seen to be a special case of the general Witten model as follows:

Consider the Witten model on a manifold with a constant metric tensor

$$\hat{\nabla}_\mu = \partial_\mu \tag{254}$$

and define another constant symmetric tensor

$$\begin{aligned} & w_{\alpha\beta} = w_{\beta\alpha} \\ & [\partial_\mu, w_{\alpha\beta}] = 0 \,, \end{aligned} \tag{255}$$

all with respect to some fixed coordinate system. Defining the superpotential W by

$$W := \frac{1}{4} w_{\alpha\beta} x^\alpha x^\beta \tag{256}$$

gives the following deformed derivative operators:

$$\begin{aligned} \hat{a}_\mu &:= e^{-W}\, \partial_\mu \, e^{W} \\ &= \partial_\mu + \frac{1}{2} w_{\mu\alpha} x^\alpha \\ \hat{a}^\dagger_\mu &:= (\hat{a}_\mu)^\dagger = -e^{W}\, \partial_\mu \, e^{-W} \\ &= -\partial_\mu + \frac{1}{2} w_{\mu\alpha} x^\alpha \,, \end{aligned} \tag{257}$$

which satisfy the canonical commutation algebra

$$\begin{aligned} [\hat{a}_\mu, \hat{a}_\nu] &= 0 \\ \left[\hat{a}^\dagger_\mu, \hat{a}^\dagger_\nu\right] &= 0 \\ \left[\hat{a}_\mu, \hat{a}^\dagger_\nu\right] &= w_{\mu\nu} \,. \end{aligned} \tag{258}$$

Hence the deformed exterior derivatives are

$$\begin{aligned} \mathbf{d}^W &= e^{-W}\,\hat{c}^{\dagger\mu}\partial_\mu\, e^W \\ &= \hat{c}^{\dagger\mu}\hat{a}_\mu \\ \mathbf{d}^{\dagger W} &= -e^W\hat{c}^{\dagger\mu}\partial_\mu\, e^{-W} \\ &= \hat{c}^\mu\hat{a}^\dagger_\mu \end{aligned} \tag{259}$$

and the associated generalized Laplace operator is indeed that of D superoscillators:

$$\left\{\mathbf{d}^W, \mathbf{d}^{\dagger W}\right\} = g^{\mu\nu}\,\hat{a}^\dagger_\mu\hat{a}_\nu + w_{\mu\nu}\,\hat{c}^{\dagger\mu}\hat{c}^\nu\,. \tag{260}$$

This construction, though very simple, is at the heart of supersymmetric field theory as well as the first-quantized superstring. This is discussed in §3 (p.143).

Note 2.64 (Semiclassical limit of the Witten model) The Witten model has remarkable properties with respect to its semiclassical limit. To discuss these, $\hbar$ needs to be reinserted into the equations via:

$$\begin{aligned} \mathbf{d} &\to \hbar\mathbf{d} \\ W &\to W/\hbar\,, \end{aligned}$$

so that the Laplacian (252) reads

$$\mathbf{D}^2_{(\epsilon,\hbar)} = \mathbf{D}^2_{(0,\hbar)} + \epsilon^2\left(\partial_\mu W\right)\left(\partial^\mu W\right) + \epsilon\hbar\hat{\gamma}_{g_-}{}^{\mu}\,\hat{\gamma}_{g_+}{}^{\nu}\left(\nabla_\mu\nabla_\nu W\right)\,, \tag{261}$$

and the semiclassical limit is found to be

$$\mathbf{D}^2_{(\epsilon,\hbar\to 0)} = \mathbf{D}^2_{(0,\hbar\to 0)} + \epsilon^2\left(\partial_\mu W\right)\left(\partial^\mu W\right)\,. \tag{262}$$

2.65 (Closed-form solutions of the Witten model) The factorization of a generalized Laplacian $\mathbf{\Delta}$ by a generalized Dirac operator $\mathbf{\Delta} = \mathbf{D}^2$ allows to characterize the kernel of the *second order* differential operator $\mathbf{\Delta}$ by a *first order* differential constraint. This greatly increases the probability to find solutions in closed form.

In particular, the Witten model (251), given by

$$\mathbf{D} = e^{-W}\,\mathbf{d}\,e^W + e^W\,\mathbf{d}^\dagger\,e^{-W} \tag{263}$$

always has formal analytic solutions in the full and empty form sector: Let $|\phi_0\rangle$ be a zero form and $|\phi_D\rangle$ a D-form, then, trivially

$$\begin{aligned} e^{-W}\,\mathbf{d}\,e^W\,|\phi_D\rangle &= 0 \\ e^W\,\mathbf{d}^\dagger\,e^{-W}\,|\phi_0\rangle &= 0 \end{aligned}$$

identically. Hence only the conditions

$$\begin{aligned} e^{-W}\,\mathbf{d}\,e^W\,|\phi_0\rangle &\stackrel{!}{=} 0 \\ e^W\,\mathbf{d}^\dagger\,e^{-W}\,|\phi_D\rangle &\stackrel{!}{=} 0 \end{aligned}$$

remain to be solved, which is immediate:

$$\begin{aligned} |\phi_0\rangle &= e^{-W}\,|0\rangle \\ |\phi_D\rangle &= e^{W}\,|\mathrm{vol}\rangle \,. \end{aligned} \tag{264}$$

If one of these is normalizable, it is a solution of the SQM system described by (263).

This obvious construction, which has received a lot of attention in the context of quantum cosmology (e.g. [25][89]) is actually a special case of a more general class of exact solutions to the Witten model:

2.66 (Further exact solutions)

As discussed in §A (p.293) (see in particular A.1 (p.295)) one may enter the Hamiltonian constraint (252)

$$\left(\hbar^2\left(\mathbf{d}+\mathbf{d}^\dagger\right)^2 + (\nabla_\mu W)(\nabla^\mu W) + \hbar\left[\hat{c}^{\dagger\mu},\hat{c}^\nu\right](\nabla_\mu\nabla_\nu W)\right)|\psi\rangle = 0$$

with the 0-form

$$|\psi\rangle = e^{(R-iS)/\hbar}\,|0\rangle \tag{265}$$

to obtain the coupled equations

$$\begin{aligned} &(\nabla_\mu S)(\nabla^\mu S) + V_W + V_{\mathrm{QM}} = 0 \\ &\nabla_\mu(\rho\nabla^\mu S) = 0\,, \end{aligned} \tag{266}$$

where V_W is the superpotential in the 0-form sector

$$V_W = (\partial_\mu W)(\partial^\mu W) - \hbar\,(\nabla_\mu\nabla^\mu W)\,,$$

and V_{QM} is the so-called quantum potential

$$V_{\mathrm{QM}} = -(\partial_\mu R)(\partial^\mu R) - \hbar\,(\nabla_\mu\nabla^\mu R)\,. \tag{267}$$

The upper line of (264) obviously corresponds to the choice

$$W = -R \tag{268}$$

$$S = 0\,. \tag{269}$$

Since for $W=-R$ the two potential functions mutually cancel

$$(W=-R) \Rightarrow (V_W + V_{\mathrm{QM}} = 0) \tag{270}$$

the equations (266) in this case become

$$\begin{aligned} W = -R \quad \Rightarrow & \\ &(\nabla_\mu S)(\nabla^\mu S) = 0 \\ &\nabla_\mu(\rho\nabla^\mu S) = 0\,. \end{aligned} \tag{271}$$

This is the equation for a classical relativistic null-current ∇S which is conserved with respect to the density $\rho = e^{2R}$. The trivial solution (268) with $S=0$

recovers (264), but there may in general be non-trivial solutions. Each of them thus gives an exact solution to the Hamiltonian constraint of the Witten model.

To obtain a supersymmetric state (i.e. one that is annihilated by $e^{-W}\mathbf{d}e^{W}$ and its adjoint) from such a solution with non-vanishing S, one can follow the general constructions discussed in §2.2.7 (p.90) and *close* $|\psi\rangle = e^{-(W+iS)/\hbar}|0\rangle$ by acting on it with $e^{-W}\mathbf{d}e^{W}$. This gives (we multiply with the imaginary unit to make the result real):

$$\begin{aligned} |\phi\rangle &:= i\, e^{-W/\hbar}\,\hbar\mathbf{d}\, e^{W/\hbar}\, e^{-(W+iS)/\hbar}\,|0\rangle \\ &= i\, e^{-W/\hbar}\,\hbar\mathbf{d}\, e^{-iS/\hbar}\,|0\rangle \\ &= \hat{c}^{\dagger\mu}\,(\nabla_\mu S)\, e^{-(W+iS)/\hbar}\,|0\rangle \,. \end{aligned} \tag{272}$$

It is readily checked that this state is indeed annihilated by both $e^{-W/\hbar}\hbar\mathbf{d}e^{W\hbar}$ and $e^{W/\hbar}\mathbf{d}^\dagger e^{-W/\hbar}$.

2.67 (Lagrangian of the Witten model) The Witten model can be derived from the following Lagrangian (*cf.* [274],§10; also see [53],§3 for more details):

$$L = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu + ig_{\mu\nu}\bar{c}^\mu Dc^\nu + \frac{1}{2}R_{\mu\nu\kappa\lambda}\bar{c}^\mu c^\nu\bar{c}^\kappa c^\lambda - \frac{1}{2}g^{\mu\nu}\nabla_\mu W\nabla_\nu W - \nabla_\mu\nabla_\nu W\bar{c}^\mu c^\nu \,. \tag{273}$$

Here

$$x^\mu = x^\mu(t)$$

are the coordinates of a point propagating on a Riemannian manifold $(\mathcal{M}, g = (g_{\mu\nu}))$. The point carries Grassmannian degrees of freedom parameterized by the complex Grassmann coordinates c^μ, $\bar{c}^\mu$. D is the covariant derivative of these (along the path of the point) defined by

$$Dc^\mu := \dot{c}^\mu + \dot{x}^\mu\Gamma_\kappa{}^\mu{}_\lambda c^\lambda \,.$$

Here $\Gamma_\mu{}^\kappa{}_\lambda$ is the Levi-Civita connection of $g_{\mu\nu}$ and $R_{\mu\nu}{}^\kappa{}_\lambda$ the associated Riemann curvature tensor. Associated with the supersymmetry of this Lagrangian are two mutually adjoint classical Noether charges

$$\begin{aligned} Q &:= c^\mu\,(g_{\mu\nu}\dot{x}^\nu + i\nabla_\mu W) \\ \bar{Q} &:= \bar{c}^\mu\,(g_{\mu\nu}\dot{x}^\nu - i\nabla_\mu W) \,. \end{aligned} \tag{274}$$

which, after quantization with the canonical substitutions

$$\begin{aligned} x^\mu &\to \hat{x}^\mu \\ p_\mu = g_{\mu\nu}\dot{x}^\mu + i\Gamma_{\mu\kappa\lambda}\bar{c}^\kappa c^\lambda &\to -i\hbar\,\partial_{x^\mu} \\ \bar{c}^\mu &\to \hat{c}^{\dagger\mu} \\ c^\mu &\to \hat{c}^\mu \,, \end{aligned}$$

become, with the right factor ordering, just the deformed exterior derivatives of 2.2.2 (p.61):

$$\begin{aligned} Q &\to \hat{c}^\mu\left(-i\hbar\,\hat{\nabla}_\mu + i\partial_\mu W\right) \\ &= i\, e^{W/\hbar}\,\hbar\mathbf{d}^\dagger\, e^{-W/\hbar} \\ \bar{Q} &\to \hat{c}^{\dagger\mu}\left(-i\hbar\,\hat{\nabla}_\mu - i\partial_\mu W\right) \\ &= -i\, e^{-W/\hbar}\,\hbar\mathbf{d}\, e^{W/\hbar} \,. \end{aligned} \tag{275}$$

Next we consider extensions of the Witten model to higher N supersymmetry by means of Kähler structures of the underlying manifold. (*cf.* §2.2.7 (p.90)). As is shown in §3.1 (p.143) the resulting formalism describes supersymmetric quantum field theory in the Schrödinger representation, and hence we naturally recover the super-Poincaré algebra within our SQM-based development. Note that this approach differs from that used for instance in [275], where Lie derivative operators are added to the supersymmetry generators in order to represent the translation generators of the Poincaré algebra.

2.68 ($N = 4$/Kähler version of the Witten model) To find higher supersymmetry in SQM, the underlying manifold must admit Kähler structures. Since the main point of the following discussion is to work out the algebra induced by a non-vanishing superpotential for higher N-extended SQM, we first ignore possible non-trivial geometries and assume that $\mathcal{M}$ is a ($2d$ dimensional real or d dimensional complex) manifold with trivial metric.

The complex coordinates and their derivatives are:

$$
\begin{aligned}
z^i &= x^i + iy^i \\
\bar{z}^{\bar{i}} &= x^i - iy^i \\
\partial_{z^i} &= \frac{1}{2}\left(\partial_{x^i} - i\partial_{y^i}\right) \\
\partial_{\bar{z}^{\bar{i}}} &= \frac{1}{2}\left(\partial_{x^i} + i\partial_{y^i}\right) .
\end{aligned} \tag{276}
$$

The holomorphic ($J+$) and antiholomorphic ($J-$) exterior (co-)derivatives are:

$$
\begin{aligned}
\mathbf{d}^{J+} &:= \hat{\mathrm{c}}^{\dagger i}\partial_i \\
\mathbf{d}^{J-} &:= \left(\mathbf{d}^{J+}\right)^* \\
&= \hat{\mathrm{c}}^{\dagger \bar{i}}\partial_{\bar{i}} \\
\mathbf{d}^{\dagger J+} &= \left(\mathbf{d}^{J+}\right)^\dagger \\
&= -\hat{\mathrm{c}}^{\bar{i}}\partial_{\bar{i}} \\
\mathbf{d}^{\dagger J-} &= \left(\mathbf{d}^{\dagger J+}\right)^* = \left(\mathbf{d}^{J-}\right)^\dagger \\
&= -\hat{\mathrm{c}}^{i}\partial_i \, ,
\end{aligned} \tag{277}
$$

where

$$
\begin{aligned}
\hat{\mathrm{c}}^{\dagger i} &= \hat{\mathrm{c}}^{\dagger x^i} + i\hat{\mathrm{c}}^{\dagger x^i} \\
\hat{\mathrm{c}}^{\dagger \bar{i}} &= \hat{\mathrm{c}}^{\dagger x^i} - i\hat{\mathrm{c}}^{\dagger x^i} \\
\hat{\mathrm{c}}^{i} &= \hat{\mathrm{c}}^{x^i} - i\hat{\mathrm{c}}^{x^i} \\
\hat{\mathrm{c}}^{\bar{i}} &= \hat{\mathrm{c}}^{x^i} - i\hat{\mathrm{c}}^{x^i} .
\end{aligned} \tag{278}
$$

Note that

$$
\left\{\mathbf{d}^{J\pm}, \mathbf{d}^{\dagger J\pm}\right\} = \frac{1}{2}\left\{\mathbf{d}, \mathbf{d}^\dagger\right\} . \tag{279}
$$

Now introduce a *real* superpotential

$$W^* = W \tag{280}$$

and deform the exterior derivatives with this function:

$$\begin{aligned}
\mathbf{d}^{WJ+} &:= e^{-W}\mathbf{d}^{J+}e^{W} \\
&= \hat{c}^{\dagger i}\left(\partial_i + (\partial_i W)\right) \\[1em]
\mathbf{d}^{WJ-} &:= \left(\mathbf{d}^{WJ+}\right)^* \\
&= e^{-W}\mathbf{d}^{J-}e^{W} \\
&= \hat{c}^{\dagger \bar{i}}\left(\partial_{\bar{i}} + (\partial_{\bar{i}} W)\right) \\[1em]
\mathbf{d}^{\dagger WJ+} &= \left(\mathbf{d}^{WJ+}\right)^\dagger \\
&= e^{W}\mathbf{d}^{J+}e^{-W} \\
&= -\hat{c}^{\bar{i}}\left(\partial_{\bar{i}} - (\partial_{\bar{i}} W)\right) \\[1em]
\mathbf{d}^{\dagger WJ-} &= \left(\mathbf{d}^{\dagger WJ+}\right)^* = \left(\mathbf{d}^{WJ-}\right)^\dagger \\
&= e^{W}\mathbf{d}^{J-}e^{-W} \\
&= -\hat{c}^{i}\left(\partial_i - (\partial_i W)\right) .
\end{aligned} \tag{281}$$

It is straightforward to compute the supercommutators of these operators. Noting that

$$\begin{aligned}
\left\{\hat{c}^{\dagger i}\partial_i, \hat{c}^{\bar{j}}\left(\partial_{\bar{j}}W\right)\right\} &= \hat{c}^{\dagger i}\hat{c}^{\bar{j}}\left(\partial_i\partial_{\bar{j}}W\right) + g^{i\bar{j}}\left(\partial_{\bar{j}}W\right)\partial_i \\
\left\{\hat{c}^{\bar{j}}\partial_{\bar{j}}, \hat{c}^{\dagger i}\left(\partial_i W\right)\right\} &= \hat{c}^{\bar{j}}\hat{c}^{\dagger i}\left(\partial_{\bar{j}}\partial_i W\right) + g^{i\bar{j}}\left(\partial_i W\right)\partial_{\bar{j}}
\end{aligned} \tag{282}$$

one finds

$$\begin{aligned}
&\left\{\mathbf{d}^{WJ+}, \mathbf{d}^{\dagger WJ+}\right\} \\
=\; & \frac{1}{2}\left\{\mathbf{d}, \mathbf{d}\right\} + g^{i\bar{j}}\left(\partial_i W\right)\left(\partial_{\bar{j}}W\right) + \left\{\hat{c}^{\dagger i}\partial_i, \hat{c}^{\bar{j}}\left(\partial_{\bar{j}}W\right)\right\} - \left\{\hat{c}^{\bar{j}}\partial_{\bar{j}}, \hat{c}^{\dagger i}\left(\partial_i W\right)\right\} \\
=\; & \frac{1}{2}\left\{\mathbf{d}, \mathbf{d}^\dagger\right\} + g^{i\bar{j}}\left(\partial_i W\right)\left(\partial_{\bar{j}}W\right) + \hat{c}^{\dagger i}\hat{c}^{\bar{j}}\left(\partial_i\partial_{\bar{j}}W\right) - \hat{c}^{\bar{j}}\hat{c}^{\dagger i}\left(\partial_{\bar{j}}\partial_i W\right) + g^{i\bar{j}}\left(\left(\partial_{\bar{j}}W\right)\partial_i - \left(\partial_i W\right)\partial_{\bar{j}}\right) \\
=\; & \frac{1}{2}\left\{\mathbf{d}, \mathbf{d}^\dagger\right\} + g^{i\bar{j}}\left(\partial_i W\right)\left(\partial_{\bar{j}}W\right) + 2\hat{c}^{\dagger i}\hat{c}^{\bar{j}}\left(\partial_i\partial_{\bar{j}}W\right) - g^{i\bar{j}}\left(\partial_i\partial_{\bar{j}}W\right) + g^{i\bar{j}}\left(\left(\partial_{\bar{j}}W\right)\partial_i - \left(\partial_i W\right)\partial_{\bar{j}}\right)
\end{aligned} \tag{283}$$

as well as

$$\left\{\mathbf{d}^{WJ+}, \mathbf{d}^{\dagger WJ-}\right\} = 2\hat{c}^{\dagger i}\hat{c}^{j}\left(\partial_i\partial_j W\right) . \tag{284}$$

By complex conjugation it follows that

$$\begin{aligned}
&\left\{\mathbf{d}^{WJ-}, \mathbf{d}^{\dagger WJ-}\right\} = \left\{\mathbf{d}^{WJ+}, \mathbf{d}^{\dagger WJ+}\right\}^* \\
=\; & \frac{1}{2}\left\{\mathbf{d}, \mathbf{d}^\dagger\right\} + g^{i\bar{j}}\left(\partial_i W\right)\left(\partial_{\bar{j}}W\right) + 2\hat{c}^{\dagger \bar{i}}\hat{c}^{j}\left(\partial_{\bar{i}}\partial_j W\right) - g^{i\bar{j}}\left(\partial_i\partial_{\bar{j}}W\right) - g^{i\bar{j}}\left(\left(\partial_{\bar{j}}W\right)\partial_i - \left(\partial_i W\right)\partial_{\bar{j}}\right)
\end{aligned} \tag{285}$$

and

$$\begin{aligned}\left\{\mathbf{d}^{WJ-},\mathbf{d}^{\dagger WJ+}\right\} &= \left\{\mathbf{d}^{WJ+},\mathbf{d}^{\dagger WJ-}\right\}^* \\ &= 2\hat{c}^{\dagger\bar{i}}\hat{c}^{\bar{j}}\left(\partial_{\bar{i}}\partial_{\bar{j}}W\right) .\end{aligned} \tag{286}$$

It turns out that a case of special importance is that where W is of the form

$$W := w_{i\bar{j}}z^i z^{\bar{j}} , \tag{287}$$

with $w_{i\bar{j}}$ some real constants. With such a W the above supercommutators simplify to:

$$\begin{aligned}\left\{\mathbf{d}^{WJ+},\mathbf{d}^{\dagger WJ+}\right\} &= \frac{1}{2}\left\{\mathbf{d},\mathbf{d}^\dagger\right\} + g^{i\bar{j}}w_{i\bar{k}}w_{l\bar{j}}z^l z^{\bar{k}} + 2w_{i\bar{j}}\hat{c}^{\dagger i}\hat{c}^{\bar{j}} - g^{i\bar{j}}w_{i\bar{j}} + g^{i\bar{j}}\left(z^k w_{k\bar{j}}\partial_i - z^{\bar{l}}w_{i\bar{l}}\partial_{\bar{j}}\right) \\ \left\{\mathbf{d}^{WJ-},\mathbf{d}^{\dagger WJ-}\right\} &= \frac{1}{2}\left\{\mathbf{d},\mathbf{d}^\dagger\right\} + g^{i\bar{j}}w_{i\bar{k}}w_{l\bar{j}}z^l z^{\bar{k}} + 2w_{i\bar{j}}\hat{c}^{\dagger\bar{j}}\hat{c}^{i} - g^{i\bar{j}}w_{i\bar{j}} - g^{i\bar{j}}\left(x^k w_{k\bar{j}}\partial_i - z^{\bar{l}}w_{i\bar{l}}\partial_{\bar{j}}\right) \\ \left\{\mathbf{d}^{WJ-},\mathbf{d}^{\dagger WJ+}\right\} &= 0 \\ \left\{\mathbf{d}^{WJ+},\mathbf{d}^{\dagger WJ-}\right\} &= 0 .\end{aligned} \tag{288}$$

The above is the super-Poincaré algebra in $1+1$ dimensions. To exhibit this more explicitly introduce the notation

$$\begin{aligned}\mathbf{S}^J_1 &:= \mathbf{d}^{WJ+} \\ \mathbf{S}^J_2 &:= \mathbf{d}^{WJ-}\end{aligned} \tag{289}$$

and

$$\begin{aligned}\sigma^{J0} &:= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \sigma^{J1} &:= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} .\end{aligned} \tag{290}$$

Then

$$\begin{aligned}\left\{\mathbf{S}^J_A,\mathbf{S}^{J\dagger}_B\right\} &= \sigma^{J\mu}_{AB}\mathbf{H}^J_\mu \\ &= \begin{bmatrix} \mathbf{H}^J_0+\mathbf{H}^J_1 & 0 \\ 0 & \mathbf{H}^J_0-\mathbf{H}^J_1 \end{bmatrix} .\end{aligned} \tag{291}$$

Hence, while the original $N=2$ supersymmetric Witten model can be regarded as giving the supersymmetry algebra of a $D=1+0$ dimensional field theory, the extension to $N=4$ gives rise to a $D=1+1$ dimensional field theory. The generator of time translations in this field theory (the Hamiltonian $\mathbf{H}$) and the generator of translations along the single spatial dimension $\mathbf{P}$ can be found from (291) as

$$\begin{aligned}\mathbf{H}^J_0 &= \frac{1}{2}\left(\left\{\mathbf{d}^{WJ+},\mathbf{d}^{\dagger WJ+}\right\} + \left\{\mathbf{d}^{WJ-},\mathbf{d}^{\dagger WJ-}\right\}\right) \\ &= \frac{1}{2}\left\{\mathbf{d},\mathbf{d}^\dagger\right\} + g^{i\bar{j}}\left(\partial_i W\right)\left(\partial_{\bar{j}}W\right) + \left(\hat{c}^{\dagger i}\hat{c}^{\bar{j}} + \hat{c}^{\dagger\bar{j}}\hat{c}^{i}\right)\left(\partial_j\partial_{\bar{j}}W\right) - g^{i\bar{j}}\left(\partial_i\partial_{\bar{j}}W\right) \\ \\ \mathbf{H}^J_1 &= \frac{1}{2}\left(\left\{\mathbf{d}^{WJ+},\mathbf{d}^{\dagger WJ+}\right\} - \left\{\mathbf{d}^{WJ-},\mathbf{d}^{\dagger WJ-}\right\}\right) \\ &= g^{i\bar{j}}\left(\left(\partial_{\bar{j}}W\right)\partial_i - \left(\partial_i W\right)\partial_{\partial_j}\right) + \left(\hat{c}^{\dagger i}\hat{c}^{\bar{j}} - \hat{c}^{\dagger\bar{j}}\hat{c}^{i}\right)\left(\partial_i\partial_{\bar{j}}W\right) .\end{aligned} \tag{292}$$

For making contact with quantum physics it is helpful to (re-)introduce Planck's constant

$$\begin{aligned} \mathbf{d} &\to \hbar \mathbf{d} \\ W &\to W/\hbar \end{aligned} \tag{293}$$

so that

$$\begin{aligned} \mathbf{H}_0^J &= \hbar^2 \frac{1}{2} \left\{ \mathbf{d}, \mathbf{d}^\dagger \right\} + g^{i\bar{j}} \left(\partial_i W \right) \left(\partial_{\bar{j}} W \right) + \left(\hat{c}^{\dagger i} \hat{c}^{\bar{j}} + \hat{c}^{\dagger \bar{j}} \hat{c}^{i} \right) \hbar \left(\partial_j \partial_{\bar{j}} W \right) - g^{i\bar{j}} \hbar \left(\partial_i \partial_{\bar{j}} W \right) \\ \mathbf{H}_1^J &= g^{i\bar{j}} \hbar \left(\left(\partial_{\bar{j}} W \right) \partial_i - \left(\partial_i W \right) \partial_{\partial_j} \right) + \left(\hat{c}^{\dagger i} \hat{c}^{\bar{j}} - \hat{c}^{\dagger \bar{j}} \hat{c}^{i} \right) \hbar \left(\partial_i \partial_{\bar{j}} W \right) \,. \end{aligned} \tag{294}$$

That this are indeed the temporal and spatial translation generators, respectively, of a supersymmetric quantum field theory on 1+1 dimensional spacetime is shown in detail in 3.2 (p.150).

2.2.3 SQM algebra of ordinary classical electromagnetism

Introduction. The title of this section may seem odd. The purpose of this section is to show that it is instead well motivated.

Maxwell's equations without macroscopic sources read

$$\left(\mathbf{d} \pm \mathbf{d}^\dagger\right) F \quad = \quad 0\,, \tag{295}$$

where

$$F = F_{\mu\nu} \hat{c}^{\dagger\mu} \hat{c}^{\dagger\nu} \left|0\right\rangle$$

is the *Faraday tensor* describing the electromagnetic field. According to the discussion in §2.2 (p.54) this is exactly the standard $(N=2)-$SQM constraint equation restricted to 2-forms.

(See also §2.2.4 (p.78), where it is shown how conservations laws of classical electromagnetism generalize to SQM, and see §4.38 (p.220), which discusses how the formalism of canonical quantum supergravity may be regarded as a generalization of that of classical electromagnetism.)

This does not mean that ordinary classical source-free electromagnetism is a supersymmetric theory, much less, of course, a supersymmetric *quantum* theory. But it does mean that the formal structure of the equations governing classical electromagnetism are mathematically very similar to, indeed a special case of, those governing covariant supersymmetric quantum systems. One may gain some insight into supersymmetric quantum mechanics by generalizing results known in classical electromagnetism. This shall be done below.

The task is greatly facilitated by formulating electromagnetism in Clifford algebraic language (e.g. [23] [128] [22] [125]).

The following observation, standard in electromagnetism, is stated merely in order to emphasize of the corresponding construction in general $(N=2)$-SQM:

2.69 (The Vector potential) Since F is a two-form, $\left(\mathbf{d} + \mathbf{d}^\dagger\right) F = 0$ implies that $\mathbf{d}F = 0$ and $\mathbf{d}^\dagger F = 0$ vanish separately. Hence, by the Poincaré lemma (see 2.45 (p.51)), F can always be chosen to be *exact*

$$F = \mathbf{d}A$$

on a starshaped region (*cf.* e.g. [91]). Choosing A so that

$$\mathbf{d}^\dagger A \stackrel{!}{=} 0\,,$$

(i.e. the Lorentz gauge) one can write

$$F = \left(\mathbf{d} + \mathbf{d}^\dagger\right) A\,.$$

So that Maxwell's equations imply the wave equation for A:

$$\begin{aligned} \left(\mathbf{d} + \mathbf{d}^\dagger\right) F &= 0 \\ \Leftrightarrow \left(\mathbf{d} + \mathbf{d}^\dagger\right)^2 A &= 0\,. \end{aligned} \tag{296}$$

This is an important fact in general SQM: If $\left|\alpha\right\rangle$ is a solution to the second order Hamiltonian constraint

$$\mathbf{H}\left|\alpha\right\rangle = 0$$

and $\mathbf{D}$ is the supercharge

$$\mathbf{D}^2 = \mathbf{H}\,,$$

then $|\beta\rangle = \mathbf{D}\,|\alpha\rangle$ is a solution to

$$\mathbf{D}\,|\beta\rangle = 0\,.$$

This is essentially the same method by which solutions in the nontrivial Fermion sector of canonical supergravity have been found in [69][70]. See 4.34 (p.218), 4.35 (p.218), and 4.38 (p.220) for a discussion and see §2.2.7 (p.90), and in particular 2.91 (p.92) for general considerations.

The central observation of this section is that the energy momentum tensor of the electromagnetic field is obtained from the Faraday tensor F in a way familiar from expectation values in quantum mechanics, with F playing the role of the wave function. This way of writing the energy-momentum tensor goes back to Riesz [224].

2.70 (Energy-Momentum tensor) *The energy-momentum tensor $T^{\mu\nu}$ of the electromagnetic field F is*[28]

$$T^{\mu\nu}\mathrm{vol} \;=\; \frac{1}{2}\big\langle F|\hat\gamma_-^\mu\hat\gamma_+^\nu\, F\big\rangle_{\mathrm{loc}}\,. \tag{297}$$

To see that this is equivalent to the traditional definition let

$$\mathbf{F} := \frac{1}{2}F_{\mu\nu}\hat\gamma^\mu\hat\gamma^\nu$$

be the Faraday tensor in Clifford bivector notation, note that

$$[\hat\gamma^\mu, \mathbf{F}] = -2F^\mu{}_\kappa\hat\gamma^\kappa\,,$$

and rearrange:

$$\begin{aligned}
\frac{1}{2}\big\langle F|\hat\gamma_-^\mu\hat\gamma_+^\nu\, F\big\rangle_{\mathrm{loc}} &= \frac{1}{2}\,\langle 0|\,\mathbf{F}^\dagger\hat\gamma_-^\mu\hat\gamma_+^\nu\mathbf{F}\,|0\rangle_{\mathrm{loc}}\\
&= -\frac{1}{2}\,\langle 0|\,\mathbf{F}\hat\gamma_-^\mu\hat\gamma_+^\nu\mathbf{F}\,|0\rangle_{\mathrm{loc}}\\
&= -\frac{1}{2}\,\langle 0|\,\mathbf{F}\hat\gamma_-^\mu\mathbf{F}\hat\gamma_+^\nu\,|0\rangle_{\mathrm{loc}}\\
&= -\frac{1}{2}\,\langle 0|\,\mathbf{F}\hat\gamma_-^\mu\mathbf{F}\hat\gamma_-^\nu\,|0\rangle_{\mathrm{loc}}\\
&= -\frac{1}{2}\,\langle 0|\,\mathbf{F}\left(\left[\hat\gamma_-^\mu,\mathbf{F}\right] + \mathbf{F}\hat\gamma_-^\mu\right)\hat\gamma_-^\nu\,|0\rangle_{\mathrm{loc}}\\
&= \left(F^{\mu\kappa}F^\nu{}_\kappa - \frac{1}{4}g^{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}\right)\mathrm{vol}\,.
\end{aligned} \tag{298}$$

(The last line (*cf.* [269], p. 70) follows[29] by noting that $\langle 0|\cdot|0\rangle_{\mathrm{loc}}$ projects out the Clifford scalar part of its argument, see (54), p. 23.))

[28] Recall that $\langle\alpha|\beta\rangle_{\mathrm{loc}} := \alpha\wedge *\beta$ is the *local* inner product over the Grassmann variables involving *no integration*. See 2.2 (p.16) and especially the discussion before and following eq. (45), p. 22.

[29] For instance this way:

$$\cdots \;=\; \langle\mathbf{F}F^\mu{}_\gamma\hat\gamma_-^\gamma\hat\gamma_-^\nu\rangle - \frac{1}{2}\langle\mathbf{F}\mathbf{F}\hat\gamma_-^\mu\hat\gamma_-^\nu\rangle$$

2.71 (Energy-momentum (density) vector) With respect to an observer associated with $\hat{\gamma}^0$, the *energy-momentum density* 4-vector of the electromagnetic field is

$$P^\mu \quad := \quad T^{0\mu}\,, \tag{301}$$

with components

$$\begin{aligned} P^0 &= \frac{1}{2}\left(|E|^2 + |B|^2\right) \geq 0 \\ P^\mu &= (E \times B)^\mu \quad \mu \neq 0\,. \end{aligned} \tag{302}$$

The *total* energy momentum P^μ_{tot} is obtained by integrating over a spacelike hypersurface perpendicular to $\hat{\gamma}^0$:

$$\begin{aligned} P^\mu_{\text{tot}}(\tau) &:= \int \delta\big(X^0 = \tau\big)\,\frac{1}{2}\big\langle F|\hat{\gamma}^\mu_- \hat{\gamma}^0_+\, F\big\rangle_{\text{loc}} \\ &= \frac{1}{2}\big\langle F|\delta\big(X^0 = \tau\big)\,\hat{\gamma}^\mu_- \hat{\gamma}^0_+\, F\big\rangle\,, \end{aligned} \tag{303}$$

where X^0 is the coordinate along the integral lines of the timelike unit vector field $\hat{\gamma}^\mu$[30]

In order to be able to proceed by analogy in the general framework of $(N=2)$-supersymmetric quantum mechanics (*cf.* §2.2.4 (p.78)), the well known conservation laws for the electromagnetic field are now rederived in Clifford notation:

Theorem 2.72 (Conservation laws) *In the absence of sources, the energy-momentum tensor T is conserved*

$$\nabla_\mu T^{\mu\nu} \quad = \quad 0\,. \tag{304}$$

In particular, the energy-momentum vector is a conserved current

$$\nabla_\mu P^\mu \quad = \quad 0\,. \tag{305}$$

Proof: Choose *Riemannian normal coordinates* x^μ, so that

$$\hat{\nabla}_\mu \quad = \quad \partial_\mu \tag{306}$$

at one point and denote objects with respect to a spin frame by Latin indices, as usual:

$$\begin{aligned} \partial_a &:= e^\mu{}_a \partial_\mu \\ \hat{\nabla}_a &:= e^\mu{}_a \hat{\nabla}_\mu\,. \end{aligned} \tag{307}$$

$$= \quad F^\mu{}_\gamma F^{\nu\gamma} - \frac{1}{2}\langle \mathbf{F}\mathbf{F}\hat{\gamma}^\mu_- \hat{\gamma}^\nu_- \rangle \tag{299}$$

The original expression is, due to the cyclic property of the Clifford inner product, symmetric in μ, ν. It follows that

$$\begin{aligned} \cdots &= F^\mu{}_\gamma F^{\nu\gamma} + g^{\mu\nu}\frac{1}{2}\langle \mathbf{F}\mathbf{F}\rangle \\ &= F^\mu{}_\gamma F^{\nu\gamma} - \frac{1}{4}g^{\mu\nu}F_{\gamma\lambda}F^{\gamma\lambda}\,. \end{aligned} \tag{300}$$

[30] Such integrals restricted to hypersurfaces will appear automatically in SQM theory when gauge fixing is applied, see §2.3.1 (p.107), especially eq. (448), p. 114.

Applying Maxwell's equation at that (fixed but arbitrary) point one finds, using the representation (297),

$$
\begin{aligned}
& \left(\mathbf{d}+\mathbf{d}^\dagger\right)\mathbf{F}\,|0\rangle = 0 \\
\Rightarrow\quad & \hat\gamma_-^a\partial_a\,\mathbf{F}\,|0\rangle = 0 \\
\Rightarrow\quad & \hat\gamma_-^a\hat\gamma_+^b\partial_a\,\mathbf{F}\,|0\rangle = 0 \\
\Rightarrow\quad & \langle 0|\,\mathbf{F}\hat\gamma_-^a\hat\gamma_+^b\partial_a\,\mathbf{F}\,|0\rangle = 0 \\
\Rightarrow\quad & \langle 0|\,\mathbf{F}^\dagger\,\hat\gamma_-^a\hat\gamma_+^b\,(\partial_a\,\mathbf{F})\,|0\rangle + \langle 0|\,(\partial_\mu\mathbf{F})^\dagger\,\hat\gamma_-^a\hat\gamma_+^b\,\mathbf{F}\,|0\rangle = 0 \\
\Rightarrow\quad & \left(\partial_a\left(\left\langle\mathbf{F}|\hat\gamma_-^a\hat\gamma_+^b\mathbf{F}\right\rangle\right)_{\mathrm{sp}}\right)\mathrm{vol} = 0 \\
\Leftrightarrow\quad & \nabla_a\left(\left\langle\mathbf{F}|\hat\gamma_-^a\hat\gamma_+^b\mathbf{F}\right\rangle\right)_{\mathrm{sp}} = 0\,.
\end{aligned}
\tag{308}
$$

(Where $(\cdot)_{\mathrm{sp}}$ denotes the scalar part multiplying the volume form, i.e. $(c\,\mathrm{vol})_{\mathrm{sp}} := c$.)

Since the chosen point was arbitrary and the result is manifestly covariant it follows by (297) that $\nabla_\mu F^{\mu\nu} = 0$. □

Note that the validity of the above proof does not depend on any property of $\mathbf{F}$. It is easily generalized to supercharges $\mathbf{D}$ more general than $\mathbf{D} = \mathbf{d} + \mathbf{d}^\dagger$, e.g. to supercharges with a Witten-superpotential. This is the content of §2.2.4 (p.78).

2.73 (Generalized electromagnetism) There is a straightforward generalization of ordinary electromagnetism, which describes point charges and the associated 1-form vector potential, to general p-form electromagnetism with brane-like charges. Furthermore, all p-form electromagnetism theories are neatly united in a single theory of *inhomogeneous*-form potentials governed by the Dirac operator on the exterior bundle. This generalized electromagnetism is very interesting in its own right (with intimate relations to string theory and supergravities) but here it serves, as ordinary electromagnetism already did above, to further illustrate the structure of constrained supersymmetric quantum mechanics, which shares very similar formal structures with it.

2.74 (General p-form electromagnetism) Let $(\mathcal{M}, g)$ be a (pseudo-)Riemannian manifold of dimension D. In ordinary electromagnetism the electromagnetic field strength $F^{(2)}$ and current $J^{(1)}$ are represented by *homogeneous* forms of degree 2 and 1, respectively:

$$
\begin{aligned}
F^{(2)} &= F^{(2)}_{\mu\nu}dx^\mu \wedge dx^\nu \\
J^{(1)} &= J^{(1)}_\mu dx^\mu\,,
\end{aligned}
\tag{309}
$$

satisfying Maxwell's equations:

$$
\begin{aligned}
\mathbf{d}F^{(2)} &= 0 \\
\mathbf{d}^\dagger F^{(2)} &= J^{(1)}\,,
\end{aligned}
\tag{310}
$$

which may, due to (309), equivalently be rewritten as a single equation

$$
\left(\mathbf{d}+\mathbf{d}^\dagger\right)F^{(2)} = J^{(1)}\,. \tag{311}
$$

General p-form electromagnetism is obtained from the ordinary theory when lifting the restriction (309) by allowing general inhomogeneous fields and currents:

$$\begin{aligned} F &:= F^{(0)} + F^{(1)}_\mu dx^\mu + F^{(2)}_{\mu\nu} dx^\mu dx^\nu + \cdots + F^{(D)}\mathrm{vol} \\ J &= J^{(0)} + J^{(1)}_\mu dx^\mu + J^{(2)}_{\mu\nu} dx^\mu dx^\nu \cdots + J^{(D)}\mathrm{vol} \end{aligned} \tag{312}$$

satisfying the generalized Maxwell equations

$$\begin{aligned} \mathbf{d}F &= 0 \\ \mathbf{d}^\dagger F &= J\,. \end{aligned} \tag{313}$$

From the second line it follows that the current J is divergence free,

$$\mathbf{d}^\dagger J = 0\,,$$

and hence, due to the Poincaré lemma (see 2.45 (p.51), now in its “dual” form) J is locally coexact:

$$J = \mathbf{d}^\dagger K\,. \tag{314}$$

(This is of course compoletely analogous to ordinary electromagnetism.) Because of the properties of $\mathbf{d}$ and $\mathbf{d}^\dagger$ one finds that (311) must hold in each sector separately:

$$\left(\mathbf{d} + \mathbf{d}^\dagger\right) F^{(p)} = J^{(p-1)}, \qquad p > 0\,. \tag{315}$$

The 0-form sector gives no non-trivial equations, since

$$\mathbf{d}F^{(0)} = 0 \tag{316}$$

says that $F^{(0)}$ must be a constant. Also, (315) yields no condition on $J^{(D)}$. Hence it is sensible to drop the components $F^{(0)}$ and $J^{(D)}$:

$$\begin{aligned} F &:= F^{(1)}_\mu dx^\mu + F^{(2)}_{\mu\nu} dx^\mu dx^\nu + \cdots + F^{(D)}\mathrm{vol} \\ J &= J^{(0)} + J^{(1)}_\mu dx^\mu + J^{(2)}_{\mu\nu} dx^\mu dx^\nu \cdots + J^{(D-1)}_{\mu_1\mu_2\cdots\mu_{D-1}} dx^{\mu_1} dx^{\mu_2} \cdots dx^{\mu_{D-1}}\,. \end{aligned} \tag{317}$$

2.75 (Solving generalized EM by means of the exterior Dirac equation) The generalized Maxwell equations are solved by a generalized vector potential

$$A = A^{(0)} + A^{(1)}_\mu dx^\mu + \cdots$$

satisfying the inhomogeneous exterior Dirac equation

$$(\mathbf{d} + \mathbf{d}^\dagger) A = K \tag{318}$$

(where K is given by (314)).

This is easily demonstrated:

$$\begin{aligned}
\mathbf{d}F &= \mathbf{d}\mathbf{d}A \\
&= 0 \\
\mathbf{d}^\dagger F &= \mathbf{d}^\dagger \mathbf{d}A \\
&\overset{(318)}{=} \mathbf{d}^\dagger \left(K - \mathbf{d}^\dagger A\right) \\
&= \mathbf{d}^\dagger K \\
&\overset{(314)}{=} J\,. \qquad (319)
\end{aligned}$$

2.76 (Physical interpretation of p-form electromagnetism) Supergravities arise in the context of string theory as admissible target spaces of super-p-branes (*cf.* [252] and reference therein). To every p-brane (assume $p < D - 3$) there is always also a *dual* p'-brane. The p-brane couples to a $(p+1)$-form $A^{(p+1)}$, the generalized 'vector' potential, via

$$L^{(p)} := q \int\limits_{V^{(p+1)}} A^{(p+1)}\,, \qquad (320)$$

where $V^{(p+1)}$ is the $(p+1)$-dimensional worldvolume of the p-brane. Let

$$F^{(p+2)} := \mathbf{d}A^{(p+1)}$$

be the associated field strength. In the absence of p-brane sources F is coclosed

$$\mathbf{d}^\dagger F = 0$$

and hence locally coexact

$$F^{(p+2)} = \mathbf{d}^\dagger A'^{(p+3)}\,. \qquad (321)$$

As usual, this can be dualized to yield (see (1228), p.308 for the use of $\bar{\gamma}_+$ instead of $*$):

$$\bar{\gamma}_+ F^{(p+2)} = (-1)^D \mathbf{d}\bar{\gamma}_+ A'^{(p+3)}\,, \qquad (322)$$

and hence

$$\bar{A}^{(D-(p+3))} := (-1)^D \bar{\gamma}_+ A'^{(p+3)} \qquad (323)$$

is the dual vector potential. It has to couple to the $(D-(p+3))$-dimensional worldvolume $V^{(D-(p+3))}$ of $(D-(p+4))$-branes via

$$L^{(D-(p+4))} := q' \int\limits_{V^{(D-(p+3))}} \bar{A}^{(D-(p+3))}\,. \qquad (324)$$

In summary, Hodge duality in higher electromagnetism relates p-branes to $(D-(p+4))$ branes.

Example 2.77

1. *Ordinary electromagnetism:* In the ordinary case one has a $(p = 0)$-brane source in $D = 4$: the electric monopole. As is well known, under electric/magnetic duality this is associated to the magnetic monopole. This is consistent with the above formula which demands a dual brane of degree $D - (p + 4) = 4 - 4 = 0$, i.e. another point source.

2. $11D$ *supergravity:* 11 dimensional supergravity arises as the target space of the super-$(p = 2)$-brane, which couples to the 3-form field $\mathcal{A}$ (*cf.* definition 5.8 (p.267)). By remark 2.76 (p.75) there is also a $(D-(p+4) = 11-6 = 5)$-brane in the theory.

3. In general, one finds the following *dyadic*, i.e. self-dual, p-branes, which satisfy

$$\begin{aligned} p &= D - (p + 4) \\ \Leftrightarrow p &= (D - 4)/2, \quad p \text{ even}: \end{aligned}$$

spacetime dimension	degree of dyadic brane
4	0
6	1
8	2
10	3

2.78 (Relation of p-form electromagnetism to SQM) Generalized p-form electromagnetism (definition 2.74 (p.73)) without sources (but possibly with inhomogeneous media, *cf.* [208]) is *formally* equivalent to $(N = 2)$-SQM (*cf.* §2.1.3 (p.43)): The constraint algebra contains the even-graded Hamiltonian constraint

$$\mathbf{H} = \mathbf{d}\mathbf{d}^\dagger + \mathbf{d}^\dagger\mathbf{d}\,,$$

and the two odd-graded supercharges

$$\begin{aligned} \mathbf{D}_1 &:= \mathbf{d} + \mathbf{d}^\dagger \\ \mathbf{D}_2 &:= i\left(\mathbf{d} - \mathbf{d}^\dagger\right) \end{aligned}$$

satisfying the defining $(N = 2)$ algebra relation

$$\{\mathbf{D}_i, \mathbf{D}_j\} = 2\delta_{ij}\mathbf{H}\,. \tag{325}$$

The grading is induced by the Witten operator:

$$\begin{aligned} &\iota := (-1)^{\hat{N}} \\ &[\iota, \mathbf{H}] = 0 \\ &\{\iota, \mathbf{D}_i\} = 0\,. \end{aligned} \tag{326}$$

The generalized Maxwell equations (313) are easily seen to be equivalent to the condition that the inhomogeneous field strength F be $(N = 2)$-supersymmetric under the above algebra:

$$\mathbf{D}_i F = 0\,, \quad i, j \in \{1, 2\}\,. \tag{327}$$

The usual technique for solving Maxwell's equations in terms of a potential A such that $\mathbf{d}A = F$ can be regarded as a special case of the general situation in SQM, as detailed in §2.2.7 (p.90), 2.91 (p.92): According to 2.75 (p.74) the potential A may be chosen to be formally ($N = 1$) supersymmetric, i.e. satisfying the single constraint

$$\begin{aligned} \mathbf{D}_1 A &= \left(\mathbf{d} + \mathbf{d}^\dagger\right) A \\ &= 0 \end{aligned} \tag{328}$$

(in the sourceless case). It proofs useful to think of this as

$$\mathbf{d}A = -\mathbf{d}^\dagger A\,. \tag{329}$$

By the general strategy (2.90 (p.92) and 2.91 (p.92)) an ($N = 2$)-supersymmetric field is obtained from A by acting on it with the remaining supercharge $\mathbf{D}_2$. But this is tantamount to the usual construction:

$$\begin{aligned} \frac{-i}{2}\mathbf{D}_2 A &= \frac{1}{2}\left(\mathbf{d} - \mathbf{d}^\dagger\right) A \\ &\stackrel{(329)}{=} \frac{1}{2}\left(\mathbf{d} + \mathbf{d}\right) \\ &= \mathbf{d}A \\ &= F\,. \end{aligned} \tag{330}$$

2.2.4 Conservation laws.

Outline. As shown in §2.2.3 (p.70) (source-free) electromagnetism is formally governed by an $(N = 2)$-SQM algebra. It is thus no surprise that conservation laws in $(N = 2)$-SQM turn out to be generalizations of the respective laws in electromagnetism. This is shown below.

(In the Lagrangian approach to supersymmetric quantization such conservation laws are not as transparent, *cf.* [201]).

Theorem 2.79 (Conserved currents) Let $\mathbf{D}$ be a generalized Dirac operator (232) of the form

$$\mathbf{D} = \hat{\gamma}^\mu \hat{\nabla}_\mu + A$$

(where, by definition, $A^\dagger = A$) and let $|\phi\rangle$ be an element in its kernel,

$$\mathbf{D}\,|\phi\rangle = 0\,.$$

Then: *All quantities of the form*

$$J^{\mu\nu_1\cdots\nu_k}\,\mathrm{vol} \quad := \quad \left\langle \phi | \hat{\gamma}_{g_-}{}^{\mu}\, \hat{\gamma}_{g_+}{}^{\nu_1} \cdots \hat{\gamma}_{g_+}{}^{\nu_k}\,\phi \right\rangle_{\mathrm{loc}} \tag{331}$$

are conserved, i.e.

$$\nabla_\mu J^{\mu\nu_1\cdots\nu_k} \quad = \quad 0\,, \tag{332}$$

if the ter, A satisfies

$$A\,\hat{\gamma}_+^{\nu_1} \cdots \hat{\gamma}_+^{\nu_k} = (-1)^k \hat{\gamma}_+^{\nu_1} \cdots \hat{\gamma}_+^{\nu_k}\,A\,. \tag{333}$$

Proof: The proof is a simple generalization of (304) (see there). All that remains to be shown is that the term involving A drops out:

$$\begin{aligned}
\left\langle \phi | \hat{\gamma}_+^{b_1} \cdots \hat{\gamma}_+^{b_k} \mathbf{D}\,\phi \right\rangle_{\mathrm{loc}} &= 0 \\
\Rightarrow \left\langle \phi | \hat{\gamma}_-^{a}\, \hat{\gamma}_+^{b_1} \cdots \hat{\gamma}_+^{b_k}\,\partial_a \phi \right\rangle_{\mathrm{loc}} + \left\langle \phi | A \hat{\gamma}_+^{b_1} \cdots \hat{\gamma}_+^{b_k}\,\phi \right\rangle_{\mathrm{loc}} &= 0 \\
\stackrel{(\cdot)^\dagger}{\Rightarrow} -\left\langle \partial_a \phi | \hat{\gamma}_-^{a}\, \hat{\gamma}_+^{b_1} \cdots \hat{\gamma}_+^{b_k}\,\phi \right\rangle_{\mathrm{loc}} + \left\langle \phi | A\, \hat{\gamma}_+^{b_1} \cdots \hat{\gamma}_+^{b_k}\,\phi \right\rangle_{\mathrm{loc}} &= 0\,,
\end{aligned}$$

which it does by assumption (333). The difference of the last two equations yields:

$$\left\langle \partial_a \phi | \hat{\gamma}_-^{a}\, \hat{\gamma}_+^{b_1} \cdots \hat{\gamma}_+^{b_k}\,\phi \right\rangle_{\mathrm{loc}} + \left\langle \phi | \hat{\gamma}_-^{a}\, \hat{\gamma}_+^{b_1} \cdots \hat{\gamma}_+^{b_k}\,\partial_\mu \phi \right\rangle_{\mathrm{loc}} \quad = \quad 0\,.$$

Following now the exact same steps as in the proof of (304) gives the desired result. $\Box$

Corollary 2.80 (Energy momentum and probability density) By (301) the conserved energy-momentum current in electromagnetism is

$$P^\mu \mathrm{vol} \quad := \quad \left\langle \phi | \hat{\gamma}_{g_-}{}^{\mu}\, \hat{\gamma}_{g_+}{}^{0}\,\phi \right\rangle_{\mathrm{loc}}\,. \tag{334}$$

According to the above theorem this will be a conserved current for a generalized Dirac operator $\mathbf{D} = \hat{\gamma}_{g_-}^{\ \mu} \hat{\nabla}_\mu + A$ and $\mathbf{D} \left|\phi\right\rangle = 0$ if

$$\left\{A, \hat{\gamma}_{g_+}^{\ 0}\right\} = 0\,.$$

And since

$$P^0 = \left\langle \phi | \hat{\gamma}_{g_-}^{\ 0} \hat{\gamma}_{g_+}^{\ 0}\, \phi \right\rangle_{\text{loc}}$$

is the non-negative 0-component of a conserved current, it can consistently be interpreted as a probability density.

Example 2.81 (Conserved current in the Witten model) It follows that in the Witten model (251) with Dirac operator

$$\mathbf{D} = \hat{\gamma}_{g_-}^{\ \mu} \hat{\nabla}_\mu + \hat{\gamma}_{g_+}^{\ \mu} \left(\partial_\mu W\right)$$

the current P^μ is conserved if

$$\partial_0 W = 0\,,$$

i.e. if the potential is time-independent in the observer's frame. This is exactly what one would expect on physical grounds.

Literature. Very recently, a comparable result has been (independently) reported in a different but closely related context: [203] analyses inner products conserved under the time evolution induced by a Klein-Gordon type equation. This is done by rewriting the Klein-Gordon equation as a system of first order differential equations (but without reference to the Dirac equation) and by introducing inner products $\langle\cdot|\cdot\rangle_\eta$ derived from the ordinary L^2 inner product by

$$\langle\cdot|\cdot\rangle_\eta := \langle\cdot|\eta\,\cdot\rangle\ ,$$

for some invertible linear operator η (this construction is also used in §2.3 (p.106), see 2.111 (p.117))). An essentially unique η is found such that $\langle\cdot|\cdot\rangle_\eta$ is positive semi-definite and conserved in time if the potential term entering the respective Klein-Gordon equation is time independent.

As an application, [203] considers the Wheeler-DeWitt equation of an FRW cosmology minisuperspace model with a real massive scalar field, which is of course of Klein-Gordon type. Since the potential term in this equations does depend on the time parameter in minisuperspace (namely the scale factor of the universe), the respective scalar product is found not to be conserved with respect to evolution along this time parameter. This parallels the findings in 5.4 (p.259) and 5.5 (p.260) for the Kantowski-Sachs model (also see figures 4 (p.262), 6 (p.264), and 7 (p.265)). The conserved scalar product found in [203] depends on an explicit splitting of spacetime into space and time. It is a global quantity which involves integration over all of space and no local currents are being associated with it. On the other hand, the method of theorem 2.79 (p.78), using the Dirac square root of Klein-Gordon-type operators, has the advantage that it yields conserved quantities that are covariant and local. Conserved, gauge-fixed scalar products can be obtained from these currents by taking their suitably gauge fixed expectation value, which, usually, amounts to integrating them over all spatial variables (*cf.* §2.3 (p.106), and §2.3.1 (p.107),

§2.3.5 (p.140) in particular). This stronger result is probably exactly due to the fact that being a solution of the Dirac equation is a stronger condition on a state than being a solution to only its square, the associated Klein-Gordon equation.

Finally we note that there may be more than one conserved current:

It is known that in an N-extended superalgebra there are N conserved (super-) currents. According to §2.2.7 (p.90) further supersymmetries are associated with covariantly constant Killing-Yano tensors

$$\begin{aligned} f_{\mu\nu} &= f_{(\mu\nu)} \\ \nabla_\kappa f_{\mu\nu} &= 0 \end{aligned} \tag{335}$$

which give rise to Dirac operators of the form

$$\mathbf{D}_f = f^\mu{}_a \hat{\gamma}^a_- \hat{\nabla}_\mu \,. \tag{336}$$

Each of these Dirac operators gives rise to a further conserved current:

Theorem 2.82 (Hidden conserved supercurrents) Let f and $\mathbf{D}_f$ be as above and $\mathbf{D}_f \left|\phi\right\rangle = 0$, then conserved currents arise as

$$\begin{aligned} J_f^{\mu\nu_1\cdots\nu_r}\mathrm{vol} &= \left\langle\phi\right| f^\mu{}_a \hat{\gamma}^a_- \hat{\gamma}^{\nu_1}_+ \cdots \hat{\gamma}^{\nu_r}_- \left|\phi\right\rangle_{\mathrm{loc}} \\ \nabla_\mu J_f^{\mu\nu_1\cdots\nu_r} &= 0\,. \end{aligned} \tag{337}$$

Poof: The proof is completely analogous to that of theorem 2.79 (p.78), making use of the fact that f is covariantly constant.

2.2.5 Checkerboard models

Outline. In this section the general local character of the dynamics of relativistic supersymmetric quantum mechanics is investigated. Due to the close relationship to the ordinary relativistic Dirac-particle, Feynman's checkerboard model plays a prominent role. This model is generalized to supersymmetric quantum mechanics with non-vanishing superpotential and some interesting effects are pointed out. The propagation of a supersymmetric relativistic particle in the presence of a constant potential is simulated numerically and graphical representations of the probability density and the probability current are given, which show in detail some of the discussed features. This is of relevance for the interpretation and understanding of the probability densities and currents obtained from supersymmetric cosmological models in §5 (p.255). Indeed, the probability currents calculated there (see figures 4 (p.262), 6 (p.264), 7 (p.265), and 10 (p.283)) show properties discussed below.

Introduction. It is known that the propagator for the Dirac electron can be obtained from a kind of path integral over zig-zag paths, that are lightlike everywhere and stochastically reverse direction with a probability proportional to the particle's mass. This idea is know as the *checkerboard model*.

In the following it is shown how the checkerboard model generalizes to supersymmetric quantum mechanics. This is done by first rederiving the $1+1$ dimensional checkerboard model from the knowledge of the Dirac operator, while making use of algebraic spinor representations. Generalizations that replace the ordinary Dirac operator by any SQM-generator, i.e. any generalized Dirac operator (*cf.* 2.48 (p.55), eq. (232)), will then be seen to be straightforward.

Literature. The checkerboard model originates with Feynman and Hibbs [97] who considered the $1+1$ dimensional case. The underlying stochastic process, basically the difference of two Poisson processes, has been discussed in more detail in [142][105][147]. As shown by Gersch [106], it turns out to be formally equivalent to an Ising model. One can also consider paths that reverse in time, as has been done in [212] [209][210] (*cf.* example 2.86 (p.84)).

Generalizations to $1+3$-dimensions have been discussed by Ord and McKeon [213][192]. With regard of the fact that the Dirac equation and Maxwell's equations are closely related, it should not be surprising that one can also construct checkerboard models for the electromagnetic field, see [211]. A formal generalization of Feynman's path-sum to $1+3$-dimensions is also given in [243], but it remains unclear if this reproduces the proper Dirac propagator.

Example 2.83 (Massive Dirac particles in flat $1+1$ dimensions.) Dirac spinors in $1+1$ dimension have $2^{D/2} = 2^1 = 2$ complex components. They are most elegantly represented as *operator spinors* living in minimal left ideals (*cf.* [172] [171][268]) of (recall the notation of 2.2 (p.16), especially following eq. (47), p. 22)

$$P_+ := \frac{1}{2}\left(1 + \hat{\gamma}^0_-\right)|0\rangle \ . \tag{338}$$

A simple inspection shows that this ideal is spanned, for example, by the elements

$$|\psi_\pm\rangle \;:=\; \frac{1}{2}\left(1 \pm \hat{\gamma}^0_- \hat{\gamma}^1_-\right) P_+ \tag{339}$$

so that a generic spinor state looks like

$$|\psi\rangle = c_+ |\psi_+\rangle + c_- |\psi_-\rangle \;, \tag{340}$$

where $c_\pm$ are complex coefficients.

The free Dirac equation in $D = 1+1$ Minkowski space is

$$\begin{aligned} &\left(\hat{\gamma}^0_- \partial_0 + \hat{\gamma}^1_- \partial_1\right) |\psi\rangle = im\, |\psi\rangle \\ \Leftrightarrow\quad &\partial_0 |\psi\rangle = \left(-\hat{\gamma}^0_- \hat{\gamma}^1_- \partial_1 + im \hat{\gamma}^0_-\right) |\psi\rangle \;, \end{aligned} \tag{341}$$

yielding a generator of time evolution

$$H \;:=\; -\hat{\gamma}^0_- \hat{\gamma}^1_- \partial_1 + im \hat{\gamma}^0_- \;. \tag{342}$$

Observing that

$$\begin{aligned} \hat{\gamma}^0_- \hat{\gamma}^1_- |\psi_\pm\rangle &= \pm |\psi_\pm\rangle \\ \hat{\gamma}^0_- |\psi_\pm\rangle &= |\psi_\mp\rangle \end{aligned} \tag{343}$$

one can read off the infinitesimal time evolution: In each discrete time step

- $|\psi_+\rangle$ is translated at the speed of light in positive x^1-direction,
- $|\psi_-\rangle$ is translated at the speed of light in negative x^1-direction,
- with amplitude im the left- and right-going components reverse direction ($c_\pm \leftrightarrow c_\mp$).

This gives Feynman's prescription [97] for the propagator U in $D = 1+1$ by way of a path integral in the checkerboard model:

$$U(x, x', \sigma, \sigma') = \lim_{N\to\infty} \sum_{\mathcal{P}_N} (im)^{R(\mathcal{P}_N)} \;. \tag{344}$$

That is: The amplitude to go from $x = (x^0, x^1)$ to x', starting in state $\sigma = \pm 1$ and ending in state σ', is the continuum limit of the sum of $(im)^{R(\mathcal{P}_N)}$ over all possible lightlike paths of N discrete time steps, where $R(\mathcal{P}_N)$ is the number of bends in each such path.

2.84 (Higher dimensions) The direct extension of this simple rule to higher dimensions is hampered by the fact that there are no simultaneous eigenstates for translations. In [213][192] this is circumvented by considering *plane wave* solutions, i.e. states that only depend on one coordinate, so that one is left again with essentially the $(1+1)$-dimensional case. However, analysis of the general structure of the path integral for a system described by a Dirac operator show that the general features of the 1+1 dimensional checkerboard model carry over to arbitrary dimensions. But this requires further investigation and no more details will be presented here. See item 1 in 6.2 (p.291).

2.85 (Checkerboard model with superpotential) Now introduce a *superpotential* as in the Witten model (251):

$$\mathbf{D} = \hat{\gamma}_-^\mu \partial_\mu + \hat{\gamma}_+^\mu (\partial_\mu W) \ .$$

This gives the time propagator

$$\begin{aligned} \mathbf{D}\,|\psi\rangle &= 0 \\ \Leftrightarrow \partial_0\,|\psi\rangle &= H\,|\psi\rangle \\ &= \left(-\hat{\gamma}_-^0 \hat{\gamma}_-^1 \partial_1 - \hat{\gamma}_-^0 \hat{\gamma}_+^0 (\partial_0 W) - \hat{\gamma}_-^0 \hat{\gamma}_+^1 (\partial_1 W)\right)|\psi\rangle \ . \end{aligned} \tag{345}$$

Since the minimal left ideal $\mathrm{Cl}(\mathcal{M})_-\,P_+$ is not an ideal with respect to the action of $\mathrm{Cl}(\mathcal{M})_+$, one now has to include the complementary ideal generated by

$$P_- \ := \ \frac{1}{2}\left(1 - \hat{\gamma}_-^0\right)|0\rangle \ , \tag{346}$$

thereby obtaining a 4-dimensional basis for the entire exterior algebra in $D = 1+1$ (*cf.* §B.2 (p.311)):

$$\begin{aligned} |\psi_\pm\rangle &:= \frac{1}{2}\left(1 \pm \hat{\gamma}_-^0 \hat{\gamma}_-^1\right) P_+ \\ |\phi_\pm\rangle &:= \frac{1}{2}\left(1 \pm \hat{\gamma}_-^0 \hat{\gamma}_-^1\right) P_- \ . \end{aligned} \tag{347}$$

The action of the various operators in the propagator H on this basis is

$$\begin{aligned} \hat{\gamma}_-^0 \hat{\gamma}_-^1\,|\psi_\pm\rangle &= \pm\,|\psi_\pm\rangle \\ \hat{\gamma}_-^0 \hat{\gamma}_-^1\,|\phi_\pm\rangle &= \pm\,|\phi_\pm\rangle \\ \hat{\gamma}_-^0 \hat{\gamma}_+^1\,|\psi_\pm\rangle &= \mp\,|\psi_\mp\rangle \\ \hat{\gamma}_-^0 \hat{\gamma}_+^1\,|\phi_\pm\rangle &= \mp\,|\phi_\mp\rangle \\ \hat{\gamma}_-^0 \hat{\gamma}_+^0\,|\psi_\pm\rangle &= |\phi_\mp\rangle \\ \hat{\gamma}_-^0 \hat{\gamma}_+^0\,|\phi_\pm\rangle &= |\psi_\mp\rangle \ . \end{aligned} \tag{348}$$

Note that

$$\begin{aligned} \left(\hat{\gamma}_-^0 \hat{\gamma}_-^1\right)^2 &= 1 \\ \left(\hat{\gamma}_-^0 \hat{\gamma}_+^0\right)^2 &= 1 \\ \left(\hat{\gamma}_-^0 \hat{\gamma}_+^1\right)^2 &= -1 \,. \end{aligned}$$

Hence the spatial part of the superpotential $\hat{\gamma}_-^0 \hat{\gamma}_+^1$ plays the role of an imaginary unit in the real Dirac-Witten equation (*cf.* 2.86 (p.84)). Again reading off the discrete dynamics for a 'time step', one finds

- $|\psi_+\rangle$ and $|\phi_+\rangle$ are translated at the speed of light in negative x^1-direction.
- $|\psi_-\rangle$ and $|\phi_+\rangle$ are translated at the speed of light in positive x^1-direction.
- With amplitude $(\partial_1 W)$ the left- and right-going components reverse direction ($c_\pm \leftrightarrow c_\mp$).

- With amplitude $-(\partial_0 W)$ left- and right-going components reverse direction *and* representations are interchanged $|\psi\rangle \leftrightarrow |\phi\rangle$.

Apart from giving a precise description of how to do the checkerboard path-integral for this supersymmetric system, this has a nice physical interpretation: First, a time dependent potential $\partial_0 W \neq 0$ induces particle creation and annihilation (non-conserved energy). Then, the larger the spatial potential $(\partial_1 W)$, the higher the probability for the particle to change direction. Since in the checkerboard model the frequency of direction changes is what slows the particle down, a high potential decelerates the particle (just as a mass does), which is as it should be. Also, the probability that the particle moves opposite to the gradient of the potential, when averaged over a large number of 'checkerboard moves', is higher than that to move with the gradient, since the probability to return decreases while the particle is heading towards lower potential.

It should be noted that, with the entire equation (345) being real, the checkerboard path-integral of the Witten model sums over *real*, not imaginary, 'amplitudes', or rather: probability weights.

It is known (e.g. [53], [114]) that the Witten model in the full and empty form sector is given by Fokker-Planck dynamics, i.e. by true diffusion. In the above checkerboard model the empty and full sectors are, respectively

$$
\begin{aligned}
|0\rangle &= |\psi_+\rangle + |\psi_-\rangle + |\phi_+\rangle + |\phi_-\rangle \\
|\mathrm{vol}\rangle &= |\psi_-\rangle - |\psi_+\rangle - |\phi_-\rangle + |\phi_+\rangle \,. \qquad (349)
\end{aligned}
$$

Example 2.86 (Constant spatial superpotential in $D = 1 + 1$) For a constant spatial superpotential

$$W := mx^1$$

the Dirac-Witten operator reads

$$\mathbf{D}_m = \hat{\gamma}^0_- \partial_0 + \hat{\gamma}^1_- \partial_1 + m\hat{\gamma}^1_+ \,. \qquad (350)$$

Its square is the ordinary Klein-Gordon operator for a particle of mass m:

$$(\mathbf{D}_m)^2 = \partial_0^2 - \partial_1^2 + m^2 \,. \qquad (351)$$

Hence $\mathbf{D}_m$ may be identified with the operator of the free relativistic electron in 1+1 flat Minkowski dimensions. (Actually, since $\mathbf{D}_m$ is an $N = 2$ Dirac operator acting on the exterior bundle instead of on the spin bundle, $\mathbf{D}_m |\phi\rangle = 0$ is really a version of the Kähler equation, *cf.* [29] §8.3.)

Figures 2 (p.86) and 3 (p.88) show the result of a numeric solution of $\mathbf{D}_m |\phi\rangle = 0$ with the initial condition

$$\left|\phi(x^0 = 0)\right\rangle = e^{-(x^1)^2} \frac{1}{2}\left(1 + \hat{\gamma}^0_- \hat{\gamma}^0_+\right) \frac{1}{2}\left(1 + \hat{\gamma}^0_-\right) |0\rangle \,,$$

i.e. the time evolution of an initial purely left going wave packet located at $x^1 = 0$.

The left going Gaussian can be seen to propagate at the speed of light, albeit eventually diminishing. The diagram displaying the conserved probability current (figure 2 (p.86), below) shows that this is due to 'particles' scattering off the constant potential (i.e. their mass) and dropping from the left going into the right going component, which thereby gains a non-vanishing amplitude (figure 3 (p.88), below). But the right going component scatters again with the result that a *second* wave packet appears in the left going component, also propagating at the speed of light but being somewhat behind the original one. While the solution is only plotted up to $x^0 \approx 4$, one can already clearly see how the checkerboard path integral prescription translates into the propagation of a wave packet of finite size and how a mean subluminal velocity arises from 'waves' of lightspeed wavepackets chasing each other.

Particles and anti-particles Ord has argued ([212]) that the imaginary unit in the ordinary Dirac equation

$$\hat{\gamma}_-^\mu \partial_\mu \left|\phi\right\rangle = im \left|\phi\right\rangle$$

is 'really' due to the physical requirement to *subtract* contributions by antiparticles, i.e. by paths going backwards in time, from amplitudes of paths going forward in time and he gives summing prescriptions that directly implement this idea and reproduce the correct propagator. It turns out that, with the above *real* Dirac-Kähler equation, such a negative weight of time-reversed paths is *explicit* in the equation itself:

The totally covariant constraint

$$\left(\hat{\gamma}_-^0 \partial_0 + \hat{\gamma}_-^1 \partial_1 + m\hat{\gamma}_+^1\right) \left|\phi\right\rangle = 0$$

singles out no direction in space-time. It can be written in the form of an x^0-propagation

$$\partial_0 \left|\phi\right\rangle = \left(-\hat{\gamma}_-^0 \hat{\gamma}_-^1 \partial_1 - m\hat{\gamma}_-^0 \hat{\gamma}_+^1\right) \left|\phi\right\rangle , \tag{352}$$

as in (345), but it can just as well be equivalently reordered to yield x^1-propagation:

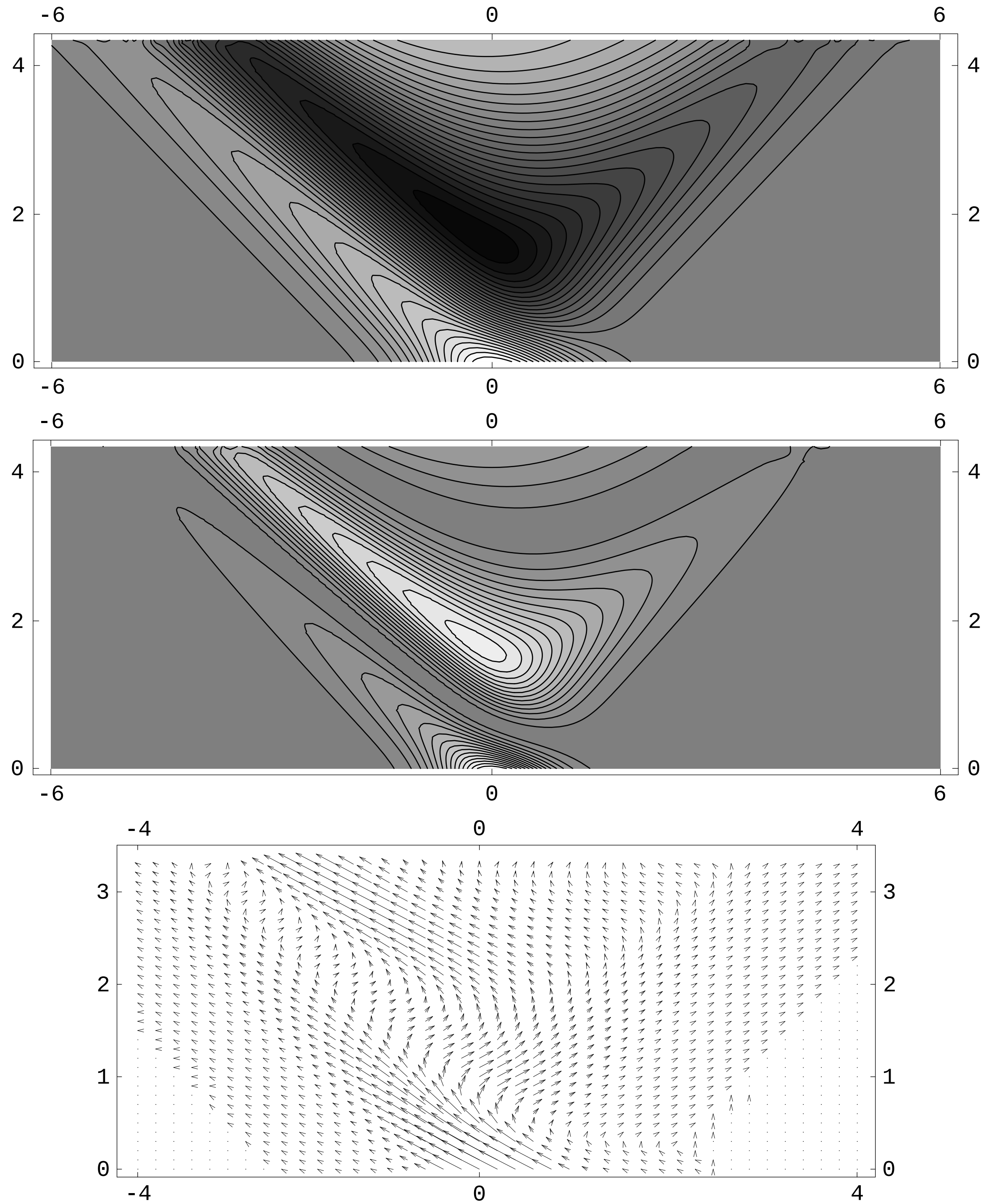
$$\partial_1 \left|\phi\right\rangle = \left(\hat{\gamma}_-^0 \hat{\gamma}_-^1 \partial_0 - m\hat{\gamma}_-^1 \hat{\gamma}_+^1\right) \left|\phi\right\rangle . \tag{353}$$

While (**??**) describes paths that move forward in time while wiggling in space, (**??**) describes paths that move forward in space while wiggling in time. That is, (**??**) explicitly describes paths that also move backwards in time. Since it is a real equation, it should, according to Ord, assign an explicit factor of -1 to such 'anti-paths'. And indeed, it does: While $\hat{\gamma}_-^0 \hat{\gamma}_-^1 \partial_0$ propagates ϕ_+ and ψ_+ forward and ϕ_- and ψ_- backward in time, the term $m\hat{\gamma}_-^1 \hat{\gamma}_+^1$ switches the time direction of paths and inserts a sign:

$$\begin{aligned} \hat{\gamma}_-^1 \hat{\gamma}_+^1 \left|\psi_\pm\right\rangle &= -\left|\psi_\mp\right\rangle \\ \hat{\gamma}_-^1 \hat{\gamma}_+^1 \left|\phi_\pm\right\rangle &= -\left|\phi_\mp\right\rangle . \end{aligned} \tag{354}$$

This immediately gives the summing prescription proposed by Ord.

Figure 2



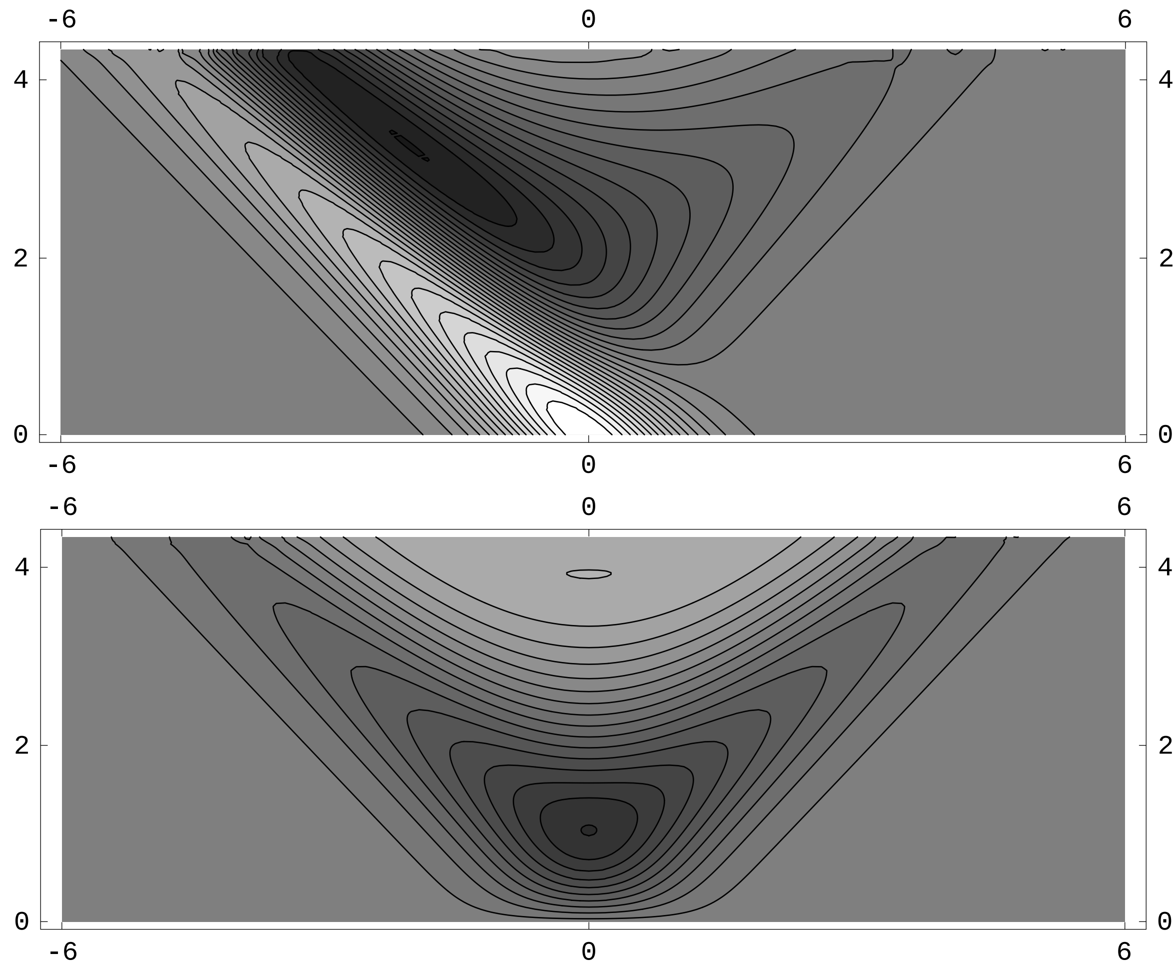
Total amplitude, density, and current in flat $D = 1 + 1$ **with superpotential** $W = x^1$**.** Displayed is a numeric solution of the Witten-Dirac equation $\mathbf{D}\,|\psi\rangle = \left(\hat{\gamma}^0_-\partial_0 + \hat{\gamma}^1_-\partial_1 + \hat{\gamma}^1_+\right)|\psi\rangle = 0$ with initial value $\left|\psi\left(x^0 = 0\right)\right\rangle :=$

$e^{-\left(x^1\right)^2} \frac{1}{2}\left(1+\hat{\gamma}_-^0 \hat{\gamma}_-^1\right) \frac{1}{2}\left(1+\hat{\gamma}_-^0\right)|0\rangle$ The 'spatial' coordinate x^1 varies along the horizontal axis, while x^0 varies along the vertical axis. For more details see example 2.86 (p.84) and also figure 3 (p.88).

Note that the *zitterbewegung*-type of the dynamics can be very clearly seen: The particle starts out in a "left-moving" state and propagates at unit speed ("speed of light") to the left. But eventually it scatters at the constant potential (which acts similarly as a mass term) and drops from the left moving into the right moving component. Hence the right moving component gradually builds up while the left moving one diminishes. But the particle keeps scattering at the potential, so that the whole process reverses and the left moving amplitude builds up again. Hence, while locally moving at constant maximal speed ("light-like'"), due to random changes of direction the *effective* speed of the particle is the mean of its zig-zag path, which is slower that unity in proportion to the effective "mass" of the particle.

A similar type of dynamics is found in the supersymmetric cosmological models discussed in §5 (p.255), where the dynamics in configuration space is governed by exactly the type of Dirac-Witten operator that is considered here. Compare figures 6 (p.264) and 10 (p.283).

Figure 3



Left and right going amplitude in flat $D = 1+1$ with superpotential $W = x^1$. Displayed is the amplitude of the left going component $\phi_{\mathrm{left}} \frac{1}{2}\left(1+\hat{\gamma}_-^0\hat{\gamma}_-^1\right)\frac{1}{2}\left(1+\hat{\gamma}_-^0\right)|0\rangle$ (above) and right moving component $\phi_{\mathrm{right}} \frac{1}{2}\left(1-\hat{\gamma}_-^0\hat{\gamma}_-^1\right)\frac{1}{2}\left(1+\hat{\gamma}_-^0\right)|0\rangle$ (below). See 2.85 (p.83) and figure 2 (p.86) for details.

2.2.6 Complex structures

Several authors have investigated the physical interpretation of the requirement in quantum mechanics to work over the complex number field instead of over the field of real numbers (e.g. [117][13],[150]). At first sight, this may perhaps seem a question of little real scientific value. However, in the context of quantum cosmology, where the probabilistic content of the state vector is not as clear as in ordinary quantum mechanics, this question gains a more concrete importance.

The Hamiltonian (the generalized Laplace operator) and its square root, the exterior Dirac operator, both are *real* operators and in covariant theories they appear in *real* equations

$$\begin{aligned} \mathbf{\Delta}\,|\psi\rangle &= 0 \\ \mathbf{D}\,|\psi\rangle &= 0\,. \end{aligned}$$

The fact that it is therefore not at all obvious why a quantum theory of cosmology should need to involve complex numbers is stressed in [13] and [150].

But remarkably, in supersymmetric quantum mechanics no need for the imaginary unit arises, because the ordinary momentum operator

$$\begin{aligned} p_\mu &= -i\partial\mu \\ {p_\mu}^\dagger &= p_\mu \end{aligned} \tag{355}$$

is replaced by the Dirac operator

$$\begin{aligned} \mathbf{D} &= \hat{\gamma}^\mu_- \partial_\mu \\ \mathbf{D}^\dagger &= \mathbf{D} \end{aligned} \tag{356}$$

(where a trivial Minkowski setup is here assumed for convenience), which remains selfadjoint due to $\hat{\gamma}_-$ being anti-selfadjoint. In fact, from the discussion in §2.2.5 (p.81) it can be seen that the Clifford structure available in SQM is responsible for the appearance of complex structures that are ordinarily implemented by means of the scalar imaginary unit. Compare the ordinary $1+1$-dimensional free massive Dirac electron equation

$$\partial_0\,|\psi\rangle = \left(-\hat{\gamma}^0_-\hat{\gamma}^1_-\partial_1 + im\hat{\gamma}^0_-\right)|\psi\rangle$$

with its super-Kähler version (see example 2.86 (p.84))

$$\partial_0\,|\psi\rangle = \left(-\hat{\gamma}^0_-\hat{\gamma}^1_-\partial_1 + m\hat{\gamma}^0_-\hat{\gamma}^1_+\right)|\psi\rangle$$

and note that

$$\left(\hat{\gamma}^0_-\hat{\gamma}^1_+\right)^2 = -1\,. \tag{357}$$

Hence a complex structure J on the quantum Hilbert space is in the latter case realized by the operator

$$J = \hat{\gamma}^0_-\hat{\gamma}^1_+\,. \tag{358}$$

In a similar way, all the cosmological quantum models discussed in §5 (p.255) are governed by real quantum constraints that nevertheless induce the usual wave-like behavior on their solutions.

2.2.7 Symmetries

Introduction In order to understand and find solutions to the equation $\mathbf{D}\,|\psi\rangle = 0$, it is advantageous to have a good understanding of the symmetries of the generalized Dirac operator $\mathbf{D}$, i.e. of all those operators Σ that commute with $\mathbf{D}$:

$$[\Sigma, \mathbf{D}] = 0\,.$$

Symmetries allow to apply algebraic methods when solving differential equations. In the ideal case a complete set of commuting operators can be found and used to exhaustively label all solutions. Generalized raising and lowering operators may be used to construct one physical state from another, thus 'walking' through the entire space of solutions. This process is standard in quantum mechanics, prominent examples being the harmonic oscillator and the hydrogen atom. However, this method was not applied to supersymmetric quantum cosmology until 1995, when in [70] [69] (see also [68] [25]) Csordás and Graham for the first time found nontrivial fermion states in supersymmetric quantum cosmology by using the nilpotent supercharges as generalized raising and lowering operators (see [197] for a historical review). Paraphrasing their original result in the formal language used here (also see [25]), it amounts to the following:

Let $\mathbf{H}$ be the supersymmetrically extended Wheeler-DeWitt Hamiltonian and $\mathbf{q}$, $\mathbf{q}^\dagger$ the two associated nilpotent supercharges (in practice obtained via the Witten model by $\mathbf{q} := e^{-W}\mathbf{d}e^{W}$, *cf.* definition 2.2.2 (p.61)):

$$\begin{aligned}
\mathbf{q}^2 &= 0\\
\mathbf{q}^{\dagger 2} &= 0\\
\mathbf{D}_1 &= \mathbf{q} + \mathbf{q}^\dagger\\
\mathbf{D}_2 &= i\left(\mathbf{q} - \mathbf{q}^\dagger\right)
\end{aligned}$$

$$\{\mathbf{D}_i, \mathbf{D}_j\} = 2\delta_{ij}\mathbf{H} \qquad i,j \in \{1,2\}$$

(*cf.* definition 2.32 (p.44), observation 2.36 (p.46) and §2.2.1 (p.55)). Because of the obvious relation

$$\begin{aligned}
\mathrm{Ker}\,(\mathbf{H}) &\supset \mathrm{Ker}\,(\mathbf{D}_i)\\
\mathbf{D}_i\mathrm{Ker}\,(\mathbf{H}) &\subset \mathrm{Ker}\,(\mathbf{D}_i)
\end{aligned} \tag{359}$$

one can find elements in the kernel of the 'diagonal' supercharges $\mathbf{D}_i$ (*cf.* definition 2.37 (p.46)) by the following algorithm (valid for any N-extended superalgebra (*cf.* definition 2.32 (p.44)), in particular for $N = 2$, as in the present case):

1. Choose any element $|\phi\rangle \in \mathrm{Ker}\,(\mathbf{H})$.
2. For all supercharges $\mathbf{D}_i$, $\quad i \in \{1, \ldots, N\}$ repeat:
    - If $\mathbf{D}_i\,|\phi\rangle \neq 0$ then replace:

$$|\phi\rangle \to \mathbf{D}_i\,|\phi\rangle\ .$$

The resulting state $|\phi\rangle$ is annihilated by all supercharges:

$$\begin{aligned}
\mathbf{D}_{1,2}\,|\phi\rangle &= 0 \\
&\Leftrightarrow \\
\mathbf{q}\,|\phi\rangle &= 0 \\
\mathbf{q}^\dagger\,|\phi\rangle &= 0\,.
\end{aligned} \tag{360}$$

This is essentially the method that has been applied in [70] to find solutions in intermediate fermion-number sectors of supersymmetric Bianchi cosmologies. (A more detailed discussion of some technical details related to the special nature of the supersymmetry generators in this case (see 4.35 (p.218) and 4.36 (p.219)) is postponed until full supergravity is discussed in §4.3.1 (p.193).)

The following section tries to generalize this construction a little. A key observation (2.90 (p.92), 2.91 (p.92)) is that solutions to N-extended SQM systems can effectively be found by solving a *single* generalized Dirac equation (one of the 'diagonal' supersymmetry constraints) and applying the other supercharges to the solution if necessary. This is in general easier than solving the system of equations given by the nilpotent generators.

The first basic observation is that the essence of the formalism of nilpotent operators (*cf.* 2.21 (p.39)) can be carried over to the non-nilpotent operator $\mathbf{D}$ when this is restricted to *harmonic* operators:

2.87 (D-harmonic states and operators) *Let $\mathbf{\Delta_D} = \mathbf{D}^2$ be the generalized Laplace operator associated with the generalized Dirac operator $\mathbf{D}$ (cf. 2.48 (p.55)), then:*

- *Any state $|\phi\rangle$ annihilated by $\mathbf{\Delta_D}$*

  $$\mathbf{\Delta_D}\,|\phi\rangle = 0$$

  is called $\mathbf{D}$-harmonic.

- *Any operator A commuting with $\mathbf{\Delta_D}$*

  $$[\mathbf{\Delta_D}, A] = 0\,,$$

  is called a $\mathbf{D}$-harmonic operator.

With respect to $\mathbf{D}$-harmonicity some notions familiar from nilpotent operators make good sense:

2.88 (D-exact and D-coexact states and operators)

- *If $|\phi\rangle$ is a $\mathbf{D}$-harmonic state, then $\mathbf{D}\,|\phi\rangle$ is called* $\mathbf{D}$-exact.
- *If A is a $\mathbf{D}$-harmonic operator, then $[\mathbf{D}, A]_\iota$ is called* $\mathbf{D}$-exact.
- *If $\mathbf{D}\,|\psi\rangle = 0$ then $|\psi\rangle$ is called* $\mathbf{D}$-closed.
- *If $[\mathbf{D}, A]_\iota = 0$ then A is called a $\mathbf{D}$-closed operator.*

Pairs of operators $(A, \{\mathbf{D}, A\})$ with A a $\mathbf{D}$-harmonic operator are also called *supermultiplets* of operators (or of 'constants of motion') and $\mathbf{D}$-closed but not $\mathbf{D}$-exact operators are correspondingly termed *supersinglets* (*cf.* [108] §3).

By construction of this suggestive analogy it follows that

2.89 *Every* $\mathbf{D}$*-exact state is* $\mathbf{D}$*-closed and also every* $\mathbf{D}$*-exact operator is* $\mathbf{D}$*-closed.*

2.90 (Closing harmonic states in N-extended SQM) Let

$$\left\{\mathbf{D}_1, \ldots, \mathbf{D}_N, \mathbf{H} = \mathbf{D}_i^2\right\} \tag{361}$$

be an N-extended SQM algebra (*cf.* definition 2.32 (p.44)). From any state $|\phi\rangle$, which is annihilated by any *one* of the generators (361) a state can be constructed that is annihilated by *all* generators: Let $I = \{i_1, i_2, \ldots\}$ be the set of integers so that

$$\mathbf{D}_i\,|\phi\rangle = \left\{ \begin{array}{ll} \neq 0, & i \in I \\ = 0, & i \notin I \end{array} \right. ,$$

then the state

$$|\phi'\rangle := \mathbf{D}_{i_1}\mathbf{D}_{i_2}\cdots|\phi\rangle\,,$$

obtained from $|\phi\rangle$ by "closing" with respect to the $\mathbf{D}_i$ is in the kernel of all generators:

$$\begin{aligned} \mathbf{D}_i\,|\phi'\rangle &= 0 \qquad i \in \{1, \ldots, N\} \\ \mathbf{H}\,|\phi'\rangle &= 0\,. \end{aligned} \tag{362}$$

Proof: Let $j \notin I$. Then:

$$\begin{aligned} \mathbf{D}_j\,|\phi'\rangle &= \mathbf{D}_j\,\mathbf{D}_{i_1}\mathbf{D}_{i_2}\cdots|\phi\rangle \\ &= \pm\mathbf{D}_{i_1}\mathbf{D}_{i_2}\cdots\underbrace{\mathbf{D}_j\,|\phi\rangle}_{=0} \\ &= 0\,, \end{aligned} \tag{363}$$

and

$$\begin{aligned} D_{i_1}\,|\phi'\rangle &= \mathbf{D}_{i_1}\,\mathbf{D}_{i_1}\mathbf{D}_{i_2}\cdots|\phi\rangle \\ &= \mathbf{H}\,\mathbf{D}_{i_2}\cdots|\phi\rangle \\ &= \mathbf{D}_j\mathbf{D}_j\,\mathbf{D}_{i_2}\cdots|\phi\rangle \\ &\overset{(363)}{=} 0\,. \end{aligned} \tag{364}$$

Analogously for $i_2, i_3, \cdots$. $\square$

This observation is central for the method of solving supersymmetry constraints as presented here:

2.91 (Dirac-operator based strategy to solve SQM constraints) Observation 2.90 shows that in order to find states invariant under an N-extended SQM algebra it is sufficient to concentrate on solving *one* of the diagonal (*cf.* definition 2.37 (p.46)) supersymmetry constraints. If, as assumed here, the diagonal supercharges are generalized Dirac operators (*cf.* definition 2.48 (p.55)), these can *always* be formally solved (*cf.* §2.2.8 (p.100)).

This is the method by which solutions to the supersymmetry constraints of the cosmological models in §5 (p.255) are found (see in particular 5.5 (p.260) and §5.2.2 (p.275)).

The remaining part of this subsection is concerned with how additional symmetries arise. It is common practice to make the following distinction between classes of symmetries:

2.92 (Generic and hidden symmetries) Symmetries which are found in every setup are called *generic*, those that only arise in certain special cases are called *hidden*.

Example 2.93 Let $\mathbf{D} = \mathbf{d} + \mathbf{d}^\dagger$. One always has (*cf.* (27), p. 20 and B.10 (p.300)):

$$\begin{aligned} \left[\mathbf{D}^2, *\right] &= 0 \\ \left[\mathbf{D}^2, \hat{N}\right] &= 0\,, \end{aligned} \tag{365}$$

so $*$ and $\hat{N}$ are generic symmetries of $\Delta_{\mathbf{D}} = \mathbf{D}^2$. They are however not $\mathbf{D}$-closed. Closing the $\hat{N}$-symmetry yields a further symmetry that has to anticommute with $\mathbf{D}$:

$$i\left[\hat{N}, \mathbf{D}\right] = i\left(\mathbf{d} - \mathbf{d}^\dagger\right) .. \tag{366}$$

This is the second supercharge from 2.37 (p.46).

A systematic analysis along these lines is given in C.1 (p.319) and C.2 (p.320) in appendix §C (p.319).

An important class of hidden symmetries are associated with Killing vectors of the underlying manifold:

Theorem 2.94 (Killing Lie derivatives are symmetries of $\boldsymbol{\Delta}$) *Every Lie derivative operator $\hat{\mathcal{L}}_\xi$ (see (1203), p. 302) with respect to a Killing vector ξ^μ, i.e. $\nabla_{(\mu}\xi_{\nu)} = 0$, is a symmetry of $\boldsymbol{\Delta}$*

$$\left[\hat{\mathcal{L}}_\xi, \boldsymbol{\Delta}\right] = 0\,.$$

Proof: The proof is given in the appendix, point **??** (p.**??**) of D (p.328).

The generalization from Killing vectors to Killing tensors gives rise to further "hidden" Dirac operators:

2.95 (Killing vectors and tensors) Any vector field ξ satisfying

$$\nabla_{(\mu}\xi_{\nu)} = 0 \tag{367}$$

is called a *Killing vector* and known to generate isometries. The generalization to higher-rank tensors is:

- *Stäckel-Killing tensors* are totally *symmetric* tensors that are weakly covariantly constant:

$$\begin{aligned} K_{\mu_1\mu_2\ldots\mu_r} &= K_{(\mu_1\mu_2\ldots\mu_r)} \\ \nabla_{(\nu}K_{\mu_1)\mu_2\ldots\mu_r} &= 0\,. \end{aligned} \tag{368}$$

- *Killing-Yano tensors* are totally *antisymmetric* tensors that are weakly covariantly constant:

$$\begin{aligned} f_{\mu_1\mu_2\ldots\mu_r} &= f_{[\mu_1\mu_2\ldots\mu_r]} \\ \nabla_{(\nu} f_{\mu_1)\mu_2\ldots\mu_r} &= 0\,. \end{aligned} \tag{369}$$

The way symmetry operators are associated with Killing tensors is analogous to the way in which the ordinary Laplace and Dirac operators are associated with the metric and the vielbein field:

The metric tensor itself is obviously a generic Stäckel-Killing tensor

$$\nabla_\kappa g_{\mu\nu} = 0\,.$$

It is associated to second order differential operators, namely the generalized Laplace operators, which are of the form (*cf.* definition 2.48 (p.55))

$$g^{\mu\nu}\partial_\mu\partial_\nu + \cdots\,.$$

As a tensor, the metric has a formal square root, the vielbein

$$e_\mu \cdot e_\nu = g_{\mu\nu}\,.$$

To this is associated a generalized Dirac operator, namely a first order differential operator which constitutes the formal square root of the generalized Laplace operator:

$$\mathbf{D} = e^\mu{}_a \hat{\gamma}^a_- \partial_\mu + \cdots\,.$$

This pattern can be continued when further Stäckel-Killing tensors and their square-root Killing-Yano tensors are present:

2.96 (Hidden symmetries and associated operators) Each second-rank symmetric Killing tensor

$$K_{\mu\nu}\,,$$

which may be regarded as a *dual metric* tensor (*cf.* [225]), gives rise to a further Laplace-like operator of the form

$$\mathbf{\Delta}_K = K^{\mu\nu}\partial_\mu\partial_\nu + \cdots\,,$$

which commutes with the original Laplacian

$$[\mathbf{\Delta}, \mathbf{\Delta}_K] = 0\,.$$

Accordingly, if there are further Killing-Yano square roots of K:

$$f_\mu \cdot f_\nu = K_{\mu\nu}$$

then further Dirac-like operators can be constructed

$$\mathbf{D}_f = f^\mu{}_a \hat{\gamma}^a_- \partial_\mu + \cdots \tag{370}$$

that square to the respective Laplacians:

$$(\mathbf{D}_f)^2 = \mathbf{\Delta}_K\,.$$

2.97 (Literature) A very in-depth discussion of hidden symmetries is found in [218]. N-extended supersymmetric quantum mechanics and its relation to geometrical symmetries is treated in [133] [193]. The notion of *geometric duality* is introduced in [225]. The relation between hidden supersymmetries and Lie-algebra invariances is discussed in [10]. Applications of geometric symmetry closely related to the present cosmological problems have been studied in string theory. In particular, the moduli space of static extremal black holes exhibits hidden supersymmetry (for special parameter values): [107] [215] [38] [237]. Also the magnetic monopole field gives rise to interesting hidden symmetries [94], [222]. The Taub-NUT model has a rich structure of hidden supersymmetries which has been investigated in [138], [260], [261]. The series of papers [58] [59] [60] [63] [61] [62] [64] is concerned with the Dirac operator and eigenmode solutions in Taub-NUT background. [266] discusses a necessary condition that a Stäckel-Killing tensor admits a Killing-Yano square root. A generalization of Killing's equation adapted to fermionic symmetries is given in [108] and applied to the Taub-NUT metric. With respect to these generalized Killing equations also see [265]. A powerful method based on exterior calculus with applications to electromagnetism is presented in [28] [27] [156]. [12] discusses Killing-Yano symmetries in phase space. The index theorem is generalized to hidden Dirac operators in [139]. A discussion of hidden symmetries with explicit emphasis on supersymmetric sigma models can be found in [175]. Remarkable for the present context is that [253] proposes to apply the technique of finding further supersymmetries by means of Killing-Yano tensors to supersymmetric quantum cosmology and in particular to the mini-superspace metrics presented in [110] [84] [7] [25] [114]. But respective results have apparently not been published. (5.19 (p.284) will give an example of the application of hidden supersymmetry to quantum cosmology.)

Theorem 2.98 (Construction of hidden symmetry operators) Let $f_{\mu\nu}$ be a covariantly constant antisymmetric (i.e. Killing-Yano) tensor

$$\nabla_\kappa f_{\mu\nu} = 0\,,$$

then symmetries of $\mathbf{\Delta}$ are given by the following contractions with fermionic elements:

$$\begin{aligned} \left[\mathbf{\Delta}, f_{ab}\hat{\mathrm{e}}^{\dagger a}\hat{\mathrm{e}}^{\dagger b}\right] &= 0 \\ \left[\mathbf{\Delta}, f_{ab}\hat{\mathrm{e}}^{a}\hat{\mathrm{e}}^{b}\right] &= 0 \\ \left[\mathbf{\Delta}, f_{ab}\hat{\mathrm{e}}^{\dagger a}\hat{\mathrm{e}}^{b}\right] &= 0 \\ \left[\mathbf{\Delta}, f_{ab}\hat{\gamma}^a_\pm\hat{\gamma}^b_\pm\right] &= 0\,. \end{aligned} \tag{371}$$

Closing these symmetries yields hidden supercharges in the form of further Dirac operators:

$$\begin{aligned} \mathbf{D}_f &= \frac{1}{2}\left[\mathbf{D}, \frac{1}{2}f_{ab}\hat{\gamma}^a_-\hat{\gamma}^b_-\right] \\ &= f^\mu{}_a\hat{\gamma}^a_-\,\hat{\nabla}_\mu\,. \end{aligned} \tag{372}$$

Proof: The first line of (371) is one of the Hodge identities known from the theory of Kähler manifolds (see for instance [101]). The second arises by taking the adjoint of the first. The third line expresses the fact that the holomorphic degree of a form is respected by the exterior Laplace operator on a Kähler manifold (e.g. [52]). The last line follows by linear combinations of the above identities. Finally the form of $\mathbf{D}_f$ follows because $f_{\mu\nu}$ is covariantly constant.

Note 2.99 (Complex structures and extended supersymmetry) Under certain conditions, Dirac operators constructed from Killing-Yano tensors $f^{(i)}$ as in remark 2.96 (p.94) give rise to N-*extended* supersymmetry, i.e. to the algebra (*cf.* 2.32 (p.44))

$$\begin{aligned} \left\{\mathbf{D}^{(i)}, \mathbf{D}^{(j)}\right\} &= 2\delta^{ij}\mathbf{H} \\ &= 2\delta^{ij}\left(-g^{\mu\nu}\partial_\mu\partial_\nu + \cdots\right), \quad i,j \in \{1,\ldots,N\}. \end{aligned} \tag{373}$$

Consider N operators of the form (372):

$$\mathbf{D}^{(i)} := f^{(i)\mu}{}_a \hat{\gamma}_-^a \partial_a + \cdots .$$

Their anticommutator has the form

$$\begin{aligned} \left\{\mathbf{D}^{(i)}, \mathbf{D}^{(j)}\right\} &= f^{(i)\mu}{}_\kappa f^{(j)\nu}{}_\lambda \left\{\hat{\gamma}_-^\kappa, \hat{\gamma}_-^\lambda\right\}\partial_\mu\partial_\nu + \cdots \\ &= -2 f^{(i)\mu}{}_\kappa f^{(j)\nu}{}_\lambda g^{\kappa\lambda}\partial_\mu\partial_\nu + \cdots \\ &= 2 f^{(i)\mu}{}_\kappa f^{(j)\kappa\nu}\partial_\mu\partial_\nu + \cdots \\ &= 2\left(f^{(i)} \cdot f^{(j)}\right)^{\mu\nu}\partial_\mu\partial_\nu + \cdots \\ &= 2\left(f^{(i)} \cdot f^{(j)}\right)^{\mu\nu}\partial_{(\mu}\partial_{\nu)} + \cdots \\ &= \left(f^{(i)} \cdot f^{(j)} + f^{(i)} \cdot f^{(j)}\right)^{\mu\nu}\partial_{(\mu}\partial_{\nu)} + \cdots . \end{aligned} \tag{374}$$

A necessary condition for (374) to give rise to (373) is that

$$f^{(i)} \cdot f^{(j)} + f^{(i)} \cdot f^{(j)} = -2g, \tag{375}$$

which says that the Killing-Yano tensors $f^{(i)}$ must be (almost) complex structures[31] satisfying a Clifford algebra (see B.1 (p.297)). If, furthermore, the $f^{(i)}$ are covariantly constant

$$\nabla f^{(i)} = 0,$$

then (according to [133], §5.1) they commute with the holonomy group of the connection of ∇, and this implies that they form an associative division algebra. This in turn means that, except in degenerate cases where the metric is trivial, there can be at most 7 covariantly constant complex structures on a manifold, since this is the number of square roots of -1 that corresponds to the division algebra of the octonions, which is the largest division algebra possible (*cf.* [215], [107] and see 4.3.4 (p.250)).

[31] An almost complex structure $I^\mu{}_\nu$ is a second grade tensor which squares to minus the identity, i.e. $I^\mu{}_\nu I^\nu{}_\lambda = -\delta^\mu_\lambda$. (See for instance [52].)

Example 2.100 ((Hidden) symmetries of a $D = 1 + 1$ model) Consider the coordinate chart

$$\begin{aligned} g = (g_{\mu\nu}) &:= \begin{bmatrix} -1 & 0 \\ 0 & e^{2\left(c^{(0)}x^0 + 2c^{(1)}x^1\right)} \end{bmatrix} \\ x^0,\, x^1 &\in (-\infty, \infty) \end{aligned} \tag{376}$$

with real non-vanishing constants $c^{(i)}$.

1. *Vielbein:* The natural choice for the vielbein and inverse vielbein is

$$\begin{aligned} e = (e^a{}_\mu) &:= \begin{bmatrix} 1 & 0 \\ 0 & e^{c^{(0)}x^0 + c^{(1)}x^1} \end{bmatrix} \\ \tilde{e} = (\tilde{e}^\mu{}_a) &:= \begin{bmatrix} 1 & 0 \\ 0 & e^{-c^{(0)}x^0 - c^{(1)}x^1} \end{bmatrix} . \end{aligned} \tag{377}$$

2. *Connection:* One finds the coefficients of the Levi-Civita connection $\omega_\mu = \left(\omega_\mu{}^a{}_b\right)$ to be

$$\begin{aligned} \omega_1 = (\omega_1{}^a{}_b) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \omega_2 = (\omega_2{}^a{}_b) &= \begin{bmatrix} 0 & a e^{c^{(1)}x^1 + c^{(2)}x^2} \\ a e^{c^{(1)}x^1 + c^{(2)}x^2} & 0 \end{bmatrix} . \end{aligned} \tag{378}$$

3. *Dirac operators:* The Dirac operator on the exterior ($\mathbf{D}$) and on the spin bundle ($\mathbf{D}^{(S)}$) are (*cf.* §B.2 (p.311)):

$$\begin{aligned} \mathbf{D} &= \hat{\gamma}^a_- \tilde{e}^\mu{}_a \hat{\nabla}_\mu \\ &= \hat{\gamma}^0_- \partial_0 + e^{-c^{(0)}x^0 - c^{(1)}x^1} \hat{\gamma}^1_- \partial_1 + \hat{\gamma}^2_- c^{(0)} \left(\hat{\mathrm{e}}^{\dagger 1} \hat{\mathrm{e}}^{\dagger 0} - \hat{\mathrm{e}}^{\dagger 0} \hat{\mathrm{e}}^{\dagger 1} \right) \\ \mathbf{D}^{(S)} &= \hat{\gamma}^a_- \tilde{e}^\mu{}_a \hat{\nabla}^{(S)}_\mu \\ &= \hat{\gamma}^0_- \partial_0 + e^{-c^{(0)}x^0 - c^{(1)}x^1} \hat{\gamma}^1_- \partial_1 + \frac{1}{2} c^{(0)} \hat{\gamma}^0_- . \end{aligned} \tag{379}$$

4. *Killing vectors:* The Killing equation

$$\nabla_{(\mu}\xi_{\nu)} \stackrel{!}{=} 0$$

for Killing vectors ξ yields the conditions

$$\begin{aligned} \partial_0 \xi_0 &= 0 \\ \partial_0 \xi_1 &= 2c^{(0)} \xi_1 - \partial_1 \xi_0 \\ \partial_1 \xi_1 &= c^{(1)} \xi_1 + c^{(0)} e^{2c^{(0)}x^0 + 2c^{(1)}x^1} \xi_0 . \end{aligned}$$

The first two equations imply

$$\begin{aligned} \xi_0\left(x^0, x^1\right) &= \xi_0\left(x^1\right) \\ \xi_1\left(x^0, x^1\right) &= e^{2c^{(0)}} + \frac{1}{2c^{(0)}} \partial_1 \xi_0 . \end{aligned} \tag{380}$$

Inserting this into the third equation yields

$$\partial_1^2\xi_0 - c^{(1)}\partial_1\xi_0 - 2\left(c^{(0)}\right)^2 e^{2c^{(0)}x^0+2c^{(1)}x^1}\xi_0 - 2c^{(1)}c^{(2)}e^{2c^{(0)}x^0} = 0\,,$$

which has nontrivial solutions only for $c^{(1)} = 0$. Therefore one finds *no* Killing vectors.

5. *Killing-Yano tensors:* In 2 dimensions the general antisymmetric second rank tensor is

$$f = (f_{\mu\nu}) = \phi\big(x^0, x^1\big)\,\epsilon_{\mu\nu}\,.$$

The Killing-Yano equation

$$\nabla_{(\kappa} f_{\mu)\nu} \stackrel{!}{=} 0$$

leads to the condition

$$\begin{aligned}\partial_0\phi &= c^{(0)}\phi\\ \partial_1\phi &= c^{(1)}\phi\,.\end{aligned}$$

This means that there is, up to a factor, exactly one second-rank Killing-Yano tensor (i.e. a Killing-Yano tensor of valence 2):

$$\begin{aligned}f_{\mu\nu} &= e^{c^{(0)}x^0+c^{(1)}x^1}\epsilon_{\mu\nu}\\ \left(f^\mu{}_a\right) &= \begin{bmatrix} 0 & -1\\ -e^{-c^{(0)}x^0-c^{(1)}x^1} & 0\end{bmatrix}\,,\end{aligned} \tag{381}$$

which can be checked to be covariantly constant:

$$\nabla_\kappa f_{\mu\nu} = 0\,. \tag{382}$$

Since it squares to the metric tensor,

$$f_{\mu\kappa}g^{\kappa\lambda}f_{\lambda\nu} = g_{\mu\nu}\,, \tag{383}$$

it indicates a complex structure of g and gives rise to a hidden Dirac operator (hidden supercharge), which turns (379) into a $N = 4$ extended superalgebra.

6. *Hidden supercharge.* According to the general formula (372) the hidden Dirac operator associated with the Killing-Yano tensor (381) reads

$$\begin{aligned}\mathbf{D}_f &:= \hat{\gamma}_-^a f^\mu{}_a \hat{\nabla}_\mu\\ &= -\hat{\gamma}_-^0 e^{-c^{(0)}x^0-c^{(1)}x^1}\partial_2 - \hat{\gamma}_-^1\partial_0 + \hat{\gamma}_-^0 c^{(0)}\left(\hat{\mathrm{e}}^{\dagger 0}\hat{\mathrm{e}}^1 - \hat{\mathrm{e}}^{\dagger 1}\hat{\mathrm{e}}^0\right)\\ \mathbf{D}_f^{(S)} &:= \hat{\gamma}_-^a f^\mu{}_a \hat{\nabla}_\mu^{(S)}\\ &= -\hat{\gamma}_-^0 e^{-c^{(0)}x^0-c^{(1)}x^1}\partial_2 - \hat{\gamma}_-^1\partial_0 - \frac{1}{2}c^{(0)}\hat{\gamma}_-^1\,.\end{aligned} \tag{384}$$

It can be checked that

$$\begin{aligned}\{\mathbf{D}, \mathbf{D}_f\} &= 0\\ \{\mathbf{D}, \mathbf{D}\} &= \{\mathbf{D}_f, \mathbf{D}_f\}\\ \left\{\mathbf{D}^{(S)}, \mathbf{D}_f^{(S)}\right\} &= 0\\ \left\{\mathbf{D}^{(S)}, \mathbf{D}^{(S)}\right\} &= \left\{\mathbf{D}_f^{(S)}, \mathbf{D}_f^{(S)}\right\}\,.\end{aligned} \tag{385}$$

A systematic account of the Dirac operators that can be constructed by the methods discussed in this section, as well as a brief discussion of symmetries as related to the Witten model, is given in appendix C (p.319).

2.2.8 Solutions

Introduction. This section considers some methods for finding solutions to the equations of covariant SQM. The focus is on formal and numerical solutions to the constraints for specified data on an initial spatial hypersurface. This is needed for the investigation of the models considered in §5 (p.255) (see in particular 5.5 (p.260) and §5.2.2 (p.275)). We also briefly mention the possibility to consider statistical solutions.

First, for completeness and to introduce our notation, we recall the very well known formal solution of Schrödinger's equation:

2.101 (Parameter evolution) Let $\hat{A}(\lambda)$ be a linear operators on some space of states and let $|\phi(\lambda)\rangle$ be be such a state, where λ is a real parameter. Assume that $\hat{A}$ generates evolution of these states with respect to λ, i.e. that

$$\partial_\lambda |\phi(\lambda)\rangle = \hat{A}(\lambda) |\phi(\lambda)\rangle . \tag{386}$$

Of course, for $\lambda = t$ and $\hat{A} = \hat{H}/i\hbar$ a Hamiltonian operator this is the Schrödinger equation. Introduction of the constraint operator $\hat{C}$ defined by

$$\hat{C} := \partial_\lambda - \hat{A} \tag{387}$$

allows to trivially rewrite (386) in the form of a constraint:

$$\hat{C} |\phi\rangle = 0 . \tag{388}$$

Quite independent of the precise nature of $\hat{A}$, this equation is formally solved by constructing the *propagator* $\hat{U}(\lambda, \lambda_0)$, which satisfies

$$\begin{aligned} \hat{U}(\lambda_0, \lambda_0) &= 1 \\ \hat{C}\hat{U}(\lambda, \lambda_0) &= 0 . \end{aligned} \tag{389}$$

Hence, given any *initial state* at $\lambda = \lambda_0$

$$\langle x|\phi(\lambda_0)\rangle = \phi_0(x) , \tag{390}$$

equation (386) and (388) are solved by

$$|\phi(\lambda)\rangle = \hat{U}(\lambda, \lambda_0) |\phi_0\rangle . \tag{391}$$

As is well known, the propagator $\hat{U}$ can be formally given either by the "parameter-ordered" exponential

$$\hat{U}(\lambda, \lambda_0) = \mathcal{T}\left(\exp \int_{\lambda_0}^{\lambda} \hat{A}(\lambda') \, d\lambda'\right) , \tag{392}$$

(where $\mathcal{T}$ indicates parameter ordering with respect to λ) or, equivalently, by

$$\hat{U}(\lambda, \lambda_0) = \lim_{n\to\infty} \left(1 + \Delta\lambda \, \hat{A}(\lambda_{n-1})\right) \left(1 + \Delta\lambda \, \hat{A}(\lambda_{n-2})\right) \cdots \left(1 + \Delta\lambda \, \hat{A}(\lambda_0)\right) , \tag{393}$$

where

$$\begin{aligned}\Delta\lambda &:= (\lambda - \lambda_0)/n \\ \lambda_m &:= \lambda_0 + m\,\Delta\lambda\,.\end{aligned} \tag{394}$$

Of course, for a λ-independent generator $\hat{A}(\lambda) = \hat{A}(\lambda_0) = \hat{A}$ this simplifies to

$$U(\lambda, \lambda_0) = \exp\Big((\lambda - \lambda_0)\hat{A}\Big) = \lim_{n\to\infty}\left(1 + \frac{1}{n}(\lambda - \lambda_0)\hat{A}\right)^n\,. \tag{395}$$

Now apply this formalism to covariant constraints given by a Dirac operator:

2.102 (Generator associated with a Dirac operator constraint) Consider a generalized Dirac operator (*cf.* 2.48 (p.55))

$$\mathbf{D} \quad = \hat{\gamma}^\mu_- \partial_\mu + \hat{B}\,, \tag{396}$$

and the constraint given by

$$\mathbf{D}\,|\phi\rangle \quad = \quad 0\,. \tag{397}$$

Now identify any one of the x^μ with the parameter λ, say $\lambda = x^0$. Since the associated Clifford element is invertible, $\left(\hat{\gamma}^0_-\right)^2 = 1$, we have

$$\begin{aligned} & \mathbf{D}\,|\phi\rangle = 0 \\ \Leftrightarrow\quad & \hat{\gamma}^0_- \mathbf{D}\,|\phi\rangle = 0 \\ \Leftrightarrow\quad & \Big(\partial_0 + \hat{\gamma}^0_-\hat{\gamma}^i_-\partial_i + \hat{\gamma}^0_-\hat{B}\Big)\,|\phi\rangle = 0\,, \end{aligned} \tag{398}$$

(where the index i runs over non-zero values, as usual). This has the form of the constraint (388):

$$\hat{C} \quad = \quad \hat{\gamma}^0_- \mathbf{D} \tag{399}$$

and one identifies the x^0-generator as

$$\hat{A}\big(x^0\big) \quad = \quad -\hat{\gamma}^0_-\hat{\gamma}^i_-\partial_i - \hat{\gamma}^0_-\hat{B}\big(x^0\big)\,. \tag{400}$$

Hence, any constraint (397) induced by a generalized Dirac operator is formally solved by equation (391).

One might wonder how an 'evolution' equation like (391) conceptually fits into a completely covariant theory, like that which is described by the constraint $\mathbf{D}\,|\phi\rangle = 0$. This is discussed in detail in [232], [231] and references therein (see also [189]):

2.103 (Group averaging) One may regard (392) as a special case of a method known as *group averaging*: From the initial state $|\phi_0\rangle$, which itself does not depend on the parameter λ, one obtains the parameter-dependent state

$$|\phi'(\lambda)\rangle \quad := \quad \delta(\lambda - \lambda_0)\,|\phi_0\rangle\,, \tag{401}$$

which is supported only at $\lambda = \lambda_0$. The state $|\phi'(\lambda)\rangle$ is a kinematical state, i.e. one representing a configuration of the model whose evolution is described by $\hat{A}$. But, because in this configuration the model is restricted to be found at $\lambda = \lambda_0$, this is generally not a physically realizable configuration, since it does not satisfy the condition $\hat{C}\,|\phi\rangle = 0$. (For example such a state describes a point particle localized in a certain instance of time, not being present before or after.) The operator $\hat{C}$, being the constraint, also generates gauge transformations. $\hat{C}\,|\phi'\rangle \neq 0$ says that $|\phi'\rangle$ is not gauge invariant. The idea of group averaging is that one can get a gauge invariant state from $|\phi'\rangle$ by "smearing" it over its entire gauge orbit, i.e. by superposing it with all its gauge transformations $\exp\left(-\tau\hat{C}\right)|\phi'\rangle$. This amounts to "averaging" $|\phi'\rangle$ over the entire gauge group, to obtain the physical state

$$|\phi\rangle = \int_{\mathbb{R}} \exp\left(-\tau\hat{C}\right)|\phi'\rangle\; d\tau\,. \tag{402}$$

That this state is indeed gauge invariant, i.e. annihilated by $\hat{C}$, is shown as follows:

$$\begin{aligned}
\hat{C}\,|\phi\rangle &= \hat{C}\int_{\mathbb{R}} \exp\left(-\tau\hat{C}\right)|\phi'\rangle\; d\tau \\
&= \int_{\mathbb{R}} \hat{C}\exp\left(-\tau\hat{C}\right)|\phi'\rangle\; d\tau \\
&= -\int_{\mathbb{R}} \frac{\partial}{\partial\tau}\exp\left(-\tau\hat{C}\right)|\phi'\rangle\; d\tau \\
&= -\exp\left(-\tau\hat{C}\right)|\phi'\rangle\Big|_{\tau=-\infty}^{\tau=\infty}\,.
\end{aligned} \tag{403}$$

The last line indeed vanishes for $|\phi'\rangle$ which are supported on a compact interval in λ, like the $|\phi'\rangle$ we started with. Intuitively this is because the last line is the sum of two states which are localized at λ- infinity, i.e. they vanish at every finite λ. This can be easily made more precise for the case where $\hat{A}$ does not depend on λ, i.e, where $\left[\partial_\lambda, \hat{A}\right] = 0$, because then

$$\begin{aligned}
\exp\left(-\tau\hat{C}\right)|\phi'\rangle\Big|_{\tau=-\infty}^{\tau=\infty} &\stackrel{(387)}{=} \exp\left(-\tau\left(\partial_\lambda - \hat{A}\right)\right)|\phi'\rangle\Big|_{\tau=-\infty}^{\tau=\infty} \\
&= \exp\left(\tau\hat{A}\right)\exp(-\tau\partial_\lambda)\,|\phi'\rangle\Big|_{\tau=-\infty}^{\tau=\infty} \\
&\stackrel{(401)}{=} \exp\left(\tau\hat{A}\right)\exp(-\tau\partial_\lambda)\,\delta(\lambda-\lambda_0)\,|\phi_0\rangle\Big|_{\tau=-\infty}^{\tau=\infty} \\
&= \exp\left(\tau\hat{A}\right)\delta(\lambda-(\lambda_0+\tau))\,|\phi_0\rangle\Big|_{\tau=-\infty}^{\tau=\infty}
\end{aligned} \tag{404}$$

and

$$\lim_{\tau\to\pm\infty}\delta(\lambda+\tau) = 0\,. \tag{405}$$

Therefore $\int_{\mathbb{R}} \exp\left(-\tau\hat{C}\right)|\phi'\rangle\; d\tau$ is a physical state: It is indeed equal to $|\phi\rangle$ in eq. (391) as can be shown as follows (still for the case of λ-independent $\hat{A}$):

$$\int_{\mathbb{R}} \exp\left(-\tau\hat{C}\right)|\phi'\rangle\; d\tau = \int_{\mathbb{R}} \exp\left(\tau\hat{A}\right)\exp(-\tau\partial_\lambda)\,\delta(\lambda-\lambda_0)\,|\phi_0\rangle\; d\tau$$

$$
\begin{aligned}
&= \int_{\mathrm{I\!R}} \exp\Big(\tau \hat{A}\Big)\, \delta(\lambda - \lambda_0 - \tau)\, |\phi_0\rangle\; d\tau \\
&= \exp\Big((\lambda - \lambda_0)\hat{A}\Big)\, |\phi_0\rangle \\
&\overset{(395)}{=} |\phi\rangle\ .
\end{aligned}
\tag{406}
$$

2.104 (Numerical propagation) The form (393) of the propagator can be used to numerically propagate states by choosing a finite value for the interval number n and setting

$$
|\phi(\lambda)\rangle \;\approx\; \Big(1 + \Delta\lambda\, \hat{A}(\lambda_{n-1})\Big)\Big(1 + \Delta\lambda\, \hat{A}(\lambda_{n-2})\Big)\cdots\Big(1 + \Delta\lambda\, \hat{A}(\lambda_0)\Big)\, |\phi_0\rangle\ .
\tag{407}
$$

The right hand side will be a good approximation to the exact $|\phi\rangle$ for small enough λ and large enough n.

This is essentially the method used in §5 (p.255) to numerically calculate quantum states of supersymmetric cosmological models. However, there is a further subtlety, since, according to the discussion in 4.3.3 (p.240), these states have to satisfy $N = 2$ independent supersymmetry constraints, not only one. Namely the supersymmetry generators obtained from supergravity are deformed exterior and co-exterior derivatives of the form

$$
\begin{aligned}
\hat{\bar{S}} &= e^{-W}\mathbf{d}e^{W} \\
\hat{S} &= e^{W}\mathbf{d}^{\dagger}e^{-W}\ .
\end{aligned}
\tag{408}
$$

For the present purpose of finding solutions to the constraints

$$
\hat{\bar{S}}\,|\phi\rangle = 0 = \hat{S}\,|\phi\rangle
$$

we can take linear combinations and construct the usual deformed Dirac operators by setting

$$
\begin{aligned}
\mathbf{D}_1 &:= \hat{\bar{S}} + \hat{S} \\
\mathbf{D}_2 &:= i\left(\hat{\bar{S}} - \hat{S}\right)\ ,
\end{aligned}
\tag{409}
$$

so that equivalently

$$
\mathbf{D}_1\,|\phi\rangle = 0 = \mathbf{D}_2\,|\phi\rangle\ .
$$

In such a case, a solution $|\phi\rangle$ to the first constraint obtained by the above method (2.101 (p.100), 2.102 (p.101) and 2.103 (p.101)) will in general *not* be a solution to the second constraint. This can be remedied as follows:

2.105 (Solving more than one supersymmetry constraint) Consider the case of a supersymmetric system constrained by N generators $\mathbf{D}_i,\ i \in \{1, 2, \cdots, N\}$, satisfying

$$
\{\mathbf{D}_i, \mathbf{D}_j\} \;=\; 2\delta_{ij}\mathbf{H}, \quad \forall\, i, j \in \{1, 2, \cdots, N\}\ ,
\tag{410}
$$

where a physical state $|\phi\rangle$ has to satisfy

$$
\mathbf{D}_i\,|\phi\rangle = 0, \quad \forall\, i \in \{1, 2, \cdots, N\}\ .
\tag{411}
$$

According to 2.91 (p.92), a solution to this set of constraints can be constructed by starting with a solution $|\phi_1\rangle$ to any *one* of these constraints, say $\mathbf{D}_1$:

$$\mathbf{D}_1 |\phi_1\rangle = 0 \,,$$

and applying to this solution all the other supersymmetry generators to obtain a solution $|\phi\rangle$

$$|\phi\rangle \quad := \quad \left(\prod_{i=2}^{N} \mathbf{D}_i\right) |\phi_1\rangle \ . \tag{412}$$

to the full set of constraints (410).
Assuming that, for a given initial condition

$$|\phi(0)\rangle = |\phi_0\rangle \tag{413}$$

one can find a state $\left|\tilde{\phi}\right\rangle$ such that

$$|\phi_0\rangle \quad = \quad \left(\prod_{i=2}^{N} \mathbf{D}_i\right) \left|\tilde{\phi}\right\rangle \ , \tag{414}$$

then, obviously, a solution $|\phi\rangle$ to the constraints (411) which satisfies (413) is

$$|\phi\rangle \quad = \quad \left(\prod_{i=2}^{N} \mathbf{D}_i\right) U_1\!\left(x^0, 0\right) \left|\tilde{\phi}\right\rangle \ , \tag{415}$$

where $U_1\!\left(x^0, 0\right)$ is the generator (392) associated with $\mathbf{D}_1$ (as described in 2.102 (p.101)).

On the other hand, if the exact form of the initial condition is for some reason not crucial, one can solve for $|\phi_1\rangle$, apply the other generators to obtain $|\phi\rangle$ and then read off the value of

$$|\phi(0)\rangle = \left(\prod_{i=2}^{N} \mathbf{D}_n\right) |\phi_1(0)\rangle \ .$$

See §5 (p.255) for examples of applications of this method.

2.106 (Literature) Another approach to numerical solutions of supersymmetric quantum mechanics is described in [278].

We now turn to a different method to obtain solutions to the supersymmetry constraints. One of the unsettled issues of quantum cosmology is that of finding physically appropriate *boundary conditions* for the wave function of the universe (*cf.* §4.8 (p.185)). Many authors have investigated this problem and the related problem of the (cosmological) arrow of time. See for instance [279] [120] [202] [161] [149] [153] [152] [19] [18] [20]. In particular, it is not clear which initial states $|\phi_0\rangle$ should be chosen for the very early universe. An alternative to specifying such conditions explicitly is to instead regard an entire ensemble of states satisfying various boundary conditions.

2.107 (Canonical ensemble for covariant systems.) In ordinary statistical physics the notion of time evolution plays a central role as the prerequisite for thermalization. Therefore it might be unclear how exactly statistical physics for completely covariant systems should be formulated, since these do in general not admit the identification of a well defined time evolution.
One method recently proposed to resolve this problem is that presented in [202]. Here translational symmetry operators $\hat{P}_\mu$ of a completely covariant system are considered, and the density operator $\hat{\rho}$ and partition function Z of a canonical ensemble of gauge covariant systems are defined by

$$\begin{aligned} \hat{\rho} &:= Z^{-1} \exp\left(-\beta^\mu \hat{P}_\mu\right) \\ Z &:= \mathrm{Tr}_{\mathrm{ph}}\left[\hat{\rho}\right] , \end{aligned} \tag{416}$$

where β^μ are thermodynamical parameters, and $\mathrm{Tr}_{\mathrm{ph}}[\cdot]$ is the gauge fixed trace over physical states.
For instance, in flat spacetime the $\hat{P}_\mu$ are simply the conserved spacetime momenta, i.e. energy and the spatial momenta. When one switches to the restframe of the system, say a relativistic gas, the only non-vanishing contribution comes from $\hat{P}_0 = \hat{H}$, the energy operator, and one recovers the ordinary canonical ensemble. (See [202] for more details.)

2.108 (Canonical ensemble in supersymmetric cosmology) With the results of §2.3.5 (p.140) it is straightforward to adapt the construction 2.107 (p.105) to covariant systems governed by supersymmetry generators in the form of generalized Dirac operators acting as constraints. With the gauge fixed trace over physical states as in §2.3.5 (p.140), all we need to identify are the translational symmetry operators $\hat{P}_\mu$. It is clear that these must be identified with operators $\hat{\mathcal{L}}_{\xi_i}$ (see B.11 (p.301) equation (1203), p. 302) that implement the Lie derivative on $\Lambda(\mathcal{M})$ with respect to Killing vectors ξ_i of $\mathcal{M}$.

Hence for a supersymmetric system described by the exterior Dirac operator $\mathbf{D} = \mathbf{d} + \mathbf{d}^\dagger$ on configuration space we set

$$\hat{\rho} := Z^{-1} \exp\left(-\beta^i \hat{\mathcal{L}}_{\xi_i}\right) . \tag{417}$$

As discussed in 2.2.7 (p.90) this density operator commutes with $\mathbf{D}$ and hence with any BRST operator $\mathbf{Q}$ constructed from $\mathbf{D}$ as described in §2.3 (p.106).

This construction in principle makes it possible to discuss statistical ensembles in the context of the cosmological models considered in §5 (p.255). For instance the Killing vectors ξ_i and the associated Lie derivative operators $\hat{\mathcal{L}}_{\xi_i}$ of the configuration space of the model 5.2 (p.266) (Bianchi I model from $N = 1$, $D = 11$ supergravity) have been calculated (this is not reported here) and could be used to study the density operator (417) of this model. However, this will not be done here, see instead point 2 (p.291) of 6.2 (p.291).

2.109 (Literature) Another approach to cosmology via statistical ensembles can be found in [134].

2.3 Dealing with the constraint: gauge fixing

Outline. The following is concerned with the problem of *gauge fixing* in a theory whose single gauge generator $\mathbf{D}$ is a Dirac operator, i.e. a first order odd-graded differential operator. Gauge theories are all about handling *quotient spaces* of *physical states modulo gauge transformations*:

$$\mathrm{Ker}\,(\mathbf{D})\,/\mathrm{Im}(\mathbf{D})\ .$$

The key idea, presented in §2.3.4 (p.134), is to analyse this quotient space by making use of the special nature of $\mathbf{D}$: Generically, a Dirac operator allow a decomposition as the sum of two *nilpotent graded operators* $\mathbf{D}_\pm$ *"of Dirac type"* (*cf.* [109]§4.2)

$$\mathbf{D} = \mathbf{D}_+ + \mathbf{D}_-$$

that map between two disjoint subspaces:

$$\mathbf{D}_\pm : \Lambda(\mathcal{M})_\pm \to \Lambda(\mathcal{M})_\mp$$

and are 'mirror images' of each other in the sense that there exists an intertwiner ι' such that

$$\iota'\,\mathbf{D}_+ = \pm\mathbf{D}_-\,\iota'\ .$$

Hence, the study of $\mathrm{Ker}\,(\mathbf{D})\,/\mathrm{Im}(\mathbf{D})$ can be reduced to that of, say,

$$\mathrm{Ker}\,(\mathbf{D}_+)\,/\mathrm{Im}(\mathbf{D}_+) = \mathrm{H}_c(\mathbf{D}_+)\ .$$

But $\mathrm{H}_c(\mathbf{D}_+)$ is the *cohomology* of the nilpotent operator $\mathbf{D}_+$ (see 2.23 (p.39)) and thus allows the use of powerful tools of cohomology theory. In particular, the task of 'gauge fixing', that is choosing in each gauge equivalence class $[|\alpha\rangle] \in \mathrm{H}_c(\mathbf{D}_+)$ a unique representative element $|\alpha\rangle \in [|\alpha\rangle]$, is tantamount to defining $\mathbf{D}_+$-*harmonic* elements by means of a co-operator $\mathbf{D}_+^{\mathrm{co}}$ (which implements the *gauge condition*):

$$\mathbf{D}_+\,|\alpha\rangle = \mathbf{D}_+^{\mathrm{co}}\,|\alpha\rangle = 0 \Leftrightarrow \left\{\mathbf{D}_+, \mathbf{D}_+^{\mathrm{co}}\right\}|\alpha\rangle = 0$$

by making use of the *Hodge decomposition* associated with $\mathbf{D}_+$ (see 2.24 (p.40) and 2.25 (p.40)).

It will become clear that the operator $\mathbf{D}_+$, at least formally, plays a role perfectly equivalent to what is known in gauge theory as the *BRST operator*, which is a nilpotent graded extension of the gauge generator and ordinarily operates on an *unphysical* graded extension of the physical Hilbert space (see §(2.3.2) for details). From the ordinary gauge theoretic viewpoint one might object against dubbing $\mathbf{D}_+$ a 'BRST operator', as will be done in the following for conceptual convenience, but since $\mathbf{D}_+$ satisfies all the formal relations demanded of a BRST charge we can nonetheless make use of several results of BRST cohomology theory, for example in order to define a gauge fixed scalar product. A detailed discussion of this point is given in §2.3.4 (p.134).

Literature. Incidentally, there are numerous approaches to relate BRST symmetry with supersymmetry, see for instance [2]. For another, unrelated proposal to do gauge fixing in quantum string cosmology see [45]. Detailed literature on the concrete techniques used here are are given in the respective subsections below.

2.3.1 Outline of the main result.

Introduction. This section gives an overview of the gauge fixing process in a covariant SQM theory as proposed here.

The theory of the SQM system under consideration is formulated in terms of a pseudo-Riemannian manifold (*cf.* 2.2 (p.16))

$$(\mathcal{M}, g)$$

of dimension D, with signature

$$(-, +, +, \cdots, +) \,,$$

coordinates

$$x^0, x^1, \ldots, x^{D-1}$$

(with respect to some fixed but arbitrary coordinate chart), as well as a further parameter, the *Lagrange multiplier*, which one can view as a further coordinate

$$x^{(\lambda)} := \lambda \,.$$

One may imagine $\mathcal{M}$ to be a submanifold of $\mathcal{M} \otimes \mathbb{R}$, where λ is the coordinate varying on the factor $\mathbb{R}$. For later convenience we chose the metric on $\mathcal{M} \otimes \mathbb{R}$ to be the product metric

$$g_{\mathcal{M}\otimes\mathbb{R}} := \begin{bmatrix} g & 0 \\ 0 & -1 \end{bmatrix} . \tag{418}$$

In the *bosonic* version of the theory *states* ('wave functions') are (not necessarily globally integrable) functions

$$\{|f\rangle \mid |f\rangle : \mathcal{M} \otimes \mathbb{R} \to \mathbb{C}\} \tag{419}$$

and *physical* states $|f_{\text{phys}}\rangle$ are those states annihilated by the *Hamiltonian* $\tilde{\mathbf{H}}$ which are furthermore independent of the Lagrange multiplier λ:

$$\begin{aligned} \tilde{\mathbf{H}} \, |f_{\text{phys}}\rangle &= 0 \\ \partial_{(\lambda)} \, |f_{\text{phys}}\rangle &= 0 \,. \end{aligned} \tag{420}$$

$\tilde{\mathbf{H}}$ is a *generalized pseudo-Laplace operator* (*cf.* definition 2.48 (p.55)) on (419), i.e. it locally reads

$$\tilde{\mathbf{H}} = g^{\mu\nu}\partial_\mu\partial_\nu + a^\mu\partial_\mu + V \,,$$

where a is any vector field and V a real scalar valued function on $\mathcal{M}$.

Without further qualification, (419) is not a Hilbert space, but the usual theory of *quantum gauge systems* could be used to implement a notion of *gauge fixing* and to construct a well defined physical theory from the above ingredients (*cf.* e.g. [124]).

In supersymmetric quantum mechanics this step is postponed until a supersymmetric extension of the above setup has been established (*cf.* §2.1 (p.15)). This consists of extending the ordinary states (419) from sections of a line bundle to sections of the *exterior bundle*

$$\Lambda(\mathcal{M} \otimes \mathbb{R}) \,,$$

over $\mathcal{M} \otimes \mathrm{I\!R}$ (see §2.1.1 (p.15)), extending $\tilde{\mathbf{H}}$ to a generalized pseudo-Laplace operator $\mathbf{H}$

$$\tilde{\mathbf{H}} \to \mathbf{H} \tag{421}$$

on $\Lambda(\mathcal{M} \otimes \mathrm{I\!R})$, and replacing the condition (420) for physical states by the stronger condition

$$\begin{aligned} \mathbf{D}\,|\psi_{\mathrm{phys}}\rangle &= 0 \\ \partial_{(\lambda)}\,|\psi_{\mathrm{phys}}\rangle &= 0\,, \end{aligned} \tag{422}$$

where $\mathbf{D}$ is a generalized Dirac operator (*cf.* definition 2.48 (p.55)) on $\Lambda(\mathcal{M})$, compatible with $\mathbf{H}$, i.e. an operator satisfying

$$\mathbf{D}^2 = \mathbf{H}\,. \tag{423}$$

Together with an involutive operator ι

$$\begin{aligned} \iota &: \Lambda(\mathcal{M}) \to \Lambda(\mathcal{M}) \\ \iota^2 &= 1 \\ \{\mathbf{D}, \iota\} &= 0 \end{aligned} \tag{424}$$

the triple

$$\{\mathbf{H}, \mathbf{D}, \iota\} \tag{425}$$

defines the supersymmetric extension of the original theory (419) (420) (*cf.* 2.50 (p.56)).

In order to treat a quantum gauge theory by the BRST method, one ordinarily (see §2.3.2 (p.115)) extends the space of states to a graded vector space in order to construct an operator, the *BRST charge*, that carries the same information as the constraint $\tilde{\mathbf{H}}$, but, as opposed to the latter, is *nilpotent*. The method proposed here (§2.3.4 (p.134)) instead finds a nilpotent equivalent of the constraint $\mathbf{D}$ (422) without further extensions of the space of states. This turns out to be possible because the supersymmetric theory is already equipped with a grading induced by the involution ι and because of the presence of certain symmetries which make some degrees of freedom redundant in a suitable sense. This works as follows:

Of the two constraints (422) first regard the *dynamic* constraint

$$\mathbf{D}\,|\psi_{\mathrm{phys}}\rangle = 0\,.$$

Since $\mathbf{D}$ is of odd grade with respect to ι (424), it can be decomposed into two terms that each carry one of the eigenspaces of ι into the other (*cf.* 2.123 (p.134)):

$$\begin{aligned} \mathbf{D} &= \mathbf{D}\frac{1}{2}(1+\iota) + \mathbf{D}\frac{1}{2}(1-\iota) \\ &= \frac{1}{2}(1-\iota)\,\mathbf{D}\frac{1}{2}(1+\iota) + \frac{1}{2}(1+\iota)\,\mathbf{D}\frac{1}{2}(1-\iota) \\ &:= \mathbf{D}_+ + \mathbf{D}_-\,. \end{aligned} \tag{426}$$

Note that $\mathbf{D}_\pm$ is nilpotent. If a further symmetry were present that related the $\pm$ eigenspaces of ι, then clearly either of $\mathbf{D}_+$, $\mathbf{D}_-$ would carry the entire information of the kernel of $\mathbf{D}$, in a sense to be made precise shortly.

It turns out that generically such a further symmetry is present in form of a further involution ι', which *anti-commutes* with the original involution ι and commutes or anti-commutes with $\mathbf{D}$ (*cf.* 2.124 (p.136)):

$$\begin{aligned} (\iota')^2 &= 1 \\ \{\iota', \iota\} &= 0 \\ [\iota', \mathbf{D}]_\pm &= 0\,. \end{aligned} \tag{427}$$

For example, consider the simple case of a free supersymmetric particle in odd dimensions, where the constraint is the standard Dirac operator on the exterior bundle (see §2.2.1 (p.55) and (1200)(1202))

$$\begin{aligned} \mathbf{D} &= \mathbf{d} + \mathbf{d}^\dagger \\ &= \hat{\gamma}_-^\mu \hat{\nabla}_\mu\,. \end{aligned} \tag{428}$$

In odd dimensions, $D = 2n + 1$, it is natural to choose (see definition B.16 (p.307))

$$\begin{aligned} \iota &= \bar{\gamma}_+ \\ \iota' &= \bar{\gamma}_-\,. \end{aligned} \tag{429}$$

By theorems B.15 (p.305) and B.17 (p.307) the chirality operators $\bar{\gamma}_\pm$ are versions of the *Hodge duality* on the exterior bundle. The redundancy associated with this duality is here seen to allow to find the entire information of the kernel of $\mathbf{d} + \mathbf{d}^\dagger$ in one of its nilpotent graded components $\mathbf{D}_\pm$. (This example and further cases are discussed in detail in §E (p.330).)

To make this explicit, consider the following derivation (this is the content of theorem 2.124 (p.136)): Every state can be decomposed into parts of definite parity with respect to $\iota' = \bar{\gamma}_-$:

$$\begin{aligned} |\psi\rangle &= \frac{1}{2}(1 + \bar{\gamma}_-)\,|\psi\rangle + \frac{1}{2}(1 - \bar{\gamma}_-)\,|\psi\rangle \\ &:= |\psi_{+\iota'}\rangle + |\psi_{-\iota'}\rangle\,. \end{aligned}$$

Since $\bar{\gamma}_-$ commutes with $\mathbf{D} = \mathbf{d} + \mathbf{d}^\dagger$ (in odd dimensions) both, $|\psi_{+\iota'}\rangle$, and $|\psi_{-\iota'}\rangle$ have to separately satisfy the dynamic constraint

$$\mathbf{D}\,|\psi_{\pm\iota'}\rangle = 0\,,$$

which is equivalent to $|\psi_{\pm\iota'}\rangle$ being in the kernel of *both* $\mathbf{D}_+$ and $\mathbf{D}_-$:

$$\mathbf{D}\frac{1}{2}(1 + \iota)\,|\psi_{\pm\iota'}\rangle = 0 = \mathbf{D}\frac{1}{2}(1 - \iota)\,|\psi_{\pm\iota'}\rangle\,.$$

But this is really only one condition, since

$$\begin{aligned} \mathbf{D}\frac{1}{2}(1 + \iota)\,|\psi_{\pm\iota'}\rangle &= \pm\mathbf{D}\frac{1}{2}(1 + \iota)\,\iota'\,|\psi_{\pm\iota'}\rangle \\ &= \pm\iota'\mathbf{D}\frac{1}{2}(1 - \iota)\,|\psi_{\pm\iota'}\rangle\,. \end{aligned} \tag{430}$$

This says that on eigenspaces of definite ι'-parity one has the equivalences

$$\begin{aligned} \mathbf{D} \left|\psi_{\pm\iota'}\right\rangle &= 0 \\ \Leftrightarrow \mathbf{D}_+ \left|\psi_{\pm\iota'}\right\rangle &= 0 \\ \Leftrightarrow \mathbf{D}_- \left|\psi_{\pm\iota'}\right\rangle &= 0\,. \end{aligned} \tag{431}$$

Hence on these eigenspaces one can replace the constraint $\mathbf{D}\left|\psi\right\rangle = 0$ by, say, the nilpotent constraint $\mathbf{D}_+ \left|\psi\right\rangle = 0$.

This is the key to constructing a BRST charge from the graded nilpotent component $\mathbf{D}_+$ (or $\mathbf{D}_-$) of $\mathbf{D}$. It can be shown that one can identify auxiliary gradings on the space of states that allow to express $\mathbf{D}_+$ in the form known from BRST theory:

$$\mathbf{D}_+ := \mathcal{C}\mathbf{p}\,,$$

where $\mathcal{C}$ is the nilpotent creator of a so-called *ghost* degree of freedom. Constructions of such ghost algebras within the superalgebra of $\Lambda(\mathcal{M} \otimes \mathbb{R})$ are presented in §E (p.330) for various cases.

All that remains to be done to acquire a full-fledged BRST operator is to incorporate the constraint concerning the Lagrange multiplier:

$$\partial_{(\lambda)} \left|\psi\right\rangle = 0\,.$$

This can be achieved as usual by adding a suitable term to $\mathbf{D}_+$ to finally give the total BRST operator of our supersymmetric constrained quantum mechanics:

$$\begin{aligned} \mathbf{Q} &:= \mathbf{D}_+ + \bar{\mathcal{P}}\partial_{(\lambda)} \\ &= \mathcal{C}\mathbf{p} + \bar{\mathcal{P}}\partial_{(\lambda)}\,. \end{aligned} \tag{432}$$

Here $\bar{\mathcal{P}}$ is a nilpotent creator of so-called *anti-ghost* degrees of freedom, which can, in the present setup, be constructed in terms of the Clifford elements $\hat{\gamma}^{(\lambda)}$ (or, equivalently, the differential forms), associated with the Lagrange multiplier λ, and which (as is shown in §E (p.330)) satisfies (anti-)commutation relations so as to keep the total BRST charge nilpotent:

$$\mathbf{Q}^2 = 0\,. \tag{433}$$

Relation (430) translates into ghost language as

$$\begin{aligned} \left(\mathcal{C}\mathbf{p} + \bar{\mathcal{P}}\partial_{(\lambda)}\right)\left|\psi_{\pm\iota'}\right\rangle &= \pm\left(\mathcal{C}\mathbf{p} + \bar{\mathcal{P}}\partial_{(\lambda)}\right)\iota'\left|\psi_{\pm\iota'}\right\rangle \\ &= \pm\iota'\left(\mathcal{P}\mathbf{p} + \bar{\mathcal{C}}\partial_{(\lambda)}\right)\left|\psi_{\pm\iota'}\right\rangle\,, \end{aligned} \tag{434}$$

where $\mathcal{P}$ and $\bar{\mathcal{C}}$ are the annihilators associated with $\mathcal{C}$, $\bar{\mathcal{P}}$, respectively. This shows explicitly how the two constraints (422) are subsumed into one:

$$\begin{aligned} &\left(\mathcal{C}\mathbf{p} + \bar{\mathcal{P}}\partial_{(\lambda)}\right)\left|\psi_{\pm\iota'}\right\rangle = 0 \\ \Leftrightarrow\quad &\mathbf{p}\left|\psi_{\pm\iota'}\right\rangle = 0,\;\; \partial_{(\lambda)}\left|\psi_{\pm\iota'}\right\rangle = 0\,. \end{aligned} \tag{435}$$

In summary, this means that within the superalgebra (425) an operator can be identified that perfectly well serves the purpose of a BRST operator when applied to subspaces of definite ι' parity. Definition 2.123 (p.134) shows that this is not merely a physically motivated oddity, but can be related to the search for a remedy of the non-ellipticity of the pseudo-Laplace operator on pseudo-Riemannian manifolds: The *BRST-Laplacian* associated with $\mathbf{Q}$ will be seen to not only define physical gauge fixed states but also to be an elliptic substitute for the non-elliptic pseudo-Laplace operator. The need to fix gauges in a physical theory is closely related to the non-Riemannian signature of the metric on configuration space.

With a nilpotent constraint $\mathbf{Q}$ at hand, it is now straightforward to make use of standard BRST-cohomology theory in order to 'fix a gauge'. For that purpose, as is described in detail in §2.3.2 (p.115) (in particular 2.111 (p.117), 2.112 (p.117), and 2.113 (p.118)) one needs to pick elements from each gauge equivalence class of states, which are nothing but the cohomology classes of $\mathbf{Q}$ (*cf.* definition 2.23 (p.39)). This is most conveniently done by means of a *co-BRST* operator $\mathbf{Q}^{\dagger_{\hat\eta}}$ (*cf.* 2.24 (p.40)). In order to define $\mathbf{Q}^{\dagger_{\hat\eta}}$ one first needs to say a word about the inner product spaces with which we have to do here:

The scalar product $\langle\cdot|\cdot\rangle_{\mathrm{phys}}$ on physical states is what will eventually be defined but is not at our disposal at this point. In order to make progress let

$$\mathcal{K} := \Gamma(\Lambda(\mathcal{M}\otimes\mathbb{R}))$$

be the space of sections of the exterior bundle of $\mathcal{M}\otimes\mathbb{R}$ that are square integrable with respect to the usual inner product (*cf.* §2.1.1 (p.15))

$$\langle\phi|\psi\rangle_{\mathcal{K}} := \int\limits_{\mathcal{M}\otimes\mathbb{R}} \phi\wedge *\psi\,, \qquad |\phi\rangle\,,|\psi\rangle \in \Gamma(\Lambda(\mathcal{M}\otimes\mathbb{R}))\;. \tag{436}$$

Elements of $\mathcal{K}$ are in general not physical, since they are square integrable in all directions, even the timelike one. Physical states instead live in the dual, $\mathcal{K}^*$, of $\mathcal{K}$.

The inner product $\langle\cdot|\cdot\rangle_{\mathcal{K}}$ (which is *indefinite* because of the indefinite metric (418) on $\mathcal{M}\otimes\mathbb{R}$) turns $\mathcal{K}$ into an *inner product space*

$$\mathcal{K} = \{\Gamma_S(\Lambda(\mathcal{M}\otimes\mathbb{R}))\,,\langle\cdot|\cdot\rangle_{\mathcal{K}}\}\;.$$

$\mathbf{D}$ and $\mathbf{Q}$ are essentially self-adjoint with respect to $\langle\cdot|\cdot\rangle_{\mathcal{K}}$:

$$\begin{aligned}\mathbf{D}^\dagger &= \mathbf{D}\\ \mathbf{Q}^\dagger &= \mathbf{Q}\,.\end{aligned} \tag{437}$$

In order to use the Hodge decomposition (*cf.* 2.24 (p.40)) to pick an element from the cohomology of $\mathbf{Q}$, it is necessary to find a *hermitian metric operator* $\hat\eta$ that induces a positive definite and non-degenerate inner product (i.e. a scalar product) $\langle\cdot|\cdot\rangle_{\hat\eta}$ via

$$\langle\cdot|\cdot\rangle_{\hat\eta} := \langle\cdot|\hat\eta\,\cdot\rangle_{\mathcal{K}}$$

(see §2.3.2 (p.115)). Equipped with this scalar product $\mathcal{K}$ becomes a *Krein space* (*cf.* 2.111 (p.117))

$$\mathcal{K} = \Big\{\Gamma_S(\Lambda(\mathcal{M}\otimes\mathbb{R}))\,,\langle\cdot|\cdot\rangle_{\mathcal{K}}\,,\langle\cdot|\cdot\rangle_{\hat\eta}\,,\Big\}\;. \tag{438}$$

This is a central construction of general BRST-cohomology theory. The implication for the present context is that the general form of $\hat\eta$ on our Krein space

$\mathcal{K}$ can be easily given. It is

$$\begin{aligned}
\hat{\eta} &= A^\dagger f \hat{\eta}^{(0)} A \\
\hat{\eta}^{-1} &= A^{-1} f^{-1} \hat{\eta}^{(0)} A^{\dagger\,-1} \\
\hat{\eta}^{(0)} &= \hat{\gamma}^0_- \hat{\gamma}^0_+ \hat{\gamma}^{(\lambda)}_- \hat{\gamma}^{(\lambda)}_+ \,, \quad f(x) > 0\,,
\end{aligned} \tag{439}$$

where A is any invertible operator and f is a positive scalar function. The point here is that the operator $\hat{\eta}^{(0)}$ switches the signs of exactly those states that would otherwise give a negative contribution to $\langle\cdot|\cdot\rangle_{\mathcal{K}}$ (436). The adjoint of $\mathbf{Q}$ with respect to the scalar product $\langle\cdot|\cdot\rangle_{\hat{\eta}}$ is

$$\mathbf{Q}^{\dagger_{\hat{\eta}}} = \hat{\eta}^{-1}\mathbf{Q}\hat{\eta}\,. \tag{440}$$

Note that

$$\begin{aligned}
\hat{\eta}^{(0)}\mathbf{Q}\hat{\eta}^{(0)} &= \hat{\eta}^{(0)}\left(\mathcal{C}\mathbf{p} + \bar{\mathcal{P}}\partial_{(\lambda)}\right)\hat{\eta}^{(0)} \\
&= \mathcal{P}\tilde{\mathbf{p}} + \bar{\mathcal{C}}\partial_{(\lambda)}\,,
\end{aligned} \tag{441}$$

with

$$\tilde{\mathbf{p}} = \hat{\eta}^{(0)}\mathbf{p}\hat{\eta}^{(0)}\,, \tag{442}$$

which follows from the general ghost algebra (see 2.115 (p.118), part 3 (p.119).

Choosing

$$\begin{aligned}
A &= 1 \\
f(x) &= e^{i\lambda x^0}\,,
\end{aligned}$$

in (439) gives the following coBRST operator:

$$\begin{aligned}
\mathbf{Q}^{\dagger_{\hat{\eta}}} &= e^{-i\lambda x^0}\hat{\eta}^{(0)}\mathbf{Q}\hat{\eta}^{(0)}e^{i\lambda x^0} \\
&= \hat{\eta}^{(0)}\mathbf{Q}\hat{\eta}^{(0)} + i\left[\hat{\eta}^{(0)}\mathbf{Q}\hat{\eta}^{(0)}, \lambda x^0\right] \\
&= \hat{\eta}^{(0)}\mathbf{Q}\hat{\eta}^{(0)} + i\left(\left[\mathbf{p}, x^0\right]\mathcal{P}\lambda + \bar{\mathcal{C}}x^0\right)\,.
\end{aligned} \tag{443}$$

Here the last two terms constitute precisely the standard so-called *gauge-fixing fermion* (*cf.* part 6 (p.120) of 2.115 (p.118) and see the introduction of §2.3.2 (p.115) for references to the literature) used to fix gauges in BRST theory.

Note that the hermitian metric operator inducing the gauge fix has corresponding bosonic, $e^{i\lambda x^0}$, and fermionic, $\hat{\gamma}^0_- \hat{\gamma}^0_+ \hat{\gamma}^{(\lambda)}_- \hat{\gamma}^{(\lambda)}_+$, parts[32]. Also note that[33] $i\left(\mathcal{P}\lambda + \bar{\mathcal{C}}x^0\right)$ is reminiscent of a Fourier transform of $\mathbf{Q}$ with differential

[32] It is possible to choose other functions than $f(x) = e^{\lambda x^0}$, as long as they are positive and lead to a finite scalar product. The choice here is motivated by the fact that it reproduces the standard gauge fixing fermion.

But one cannot for instance choose $f(x) = \text{const}$, because then the resulting gauge condition would be essentially empty, i.e. the resulting coBRST operator would not completely single out a unique representative from the gauge-equivalence classes (*cf.* 2.113 (p.118)).

[33] The expression $\left[\mathbf{p}, x^0\right]$ is in BRST theory usually set to unity, by definition. In the present framework it is instead proportional to $\hat{\gamma_g}^0_-$. But since this is invertible and commuting with the ghosts, this is just as fine and does not affect the general formalism. In particular, in equations like (444) in can be entirely dropped for notational convenience.

operators replaced by 'conjugate' multiplication operators. Hence $\left(\mathcal{P}\lambda + \bar{\mathcal{C}}x^0\right)$ implements the 'co-constraint' to (435):

$$\begin{aligned} & \left(\mathcal{P}\lambda + \bar{\mathcal{C}}x^0\right) |\psi_{\pm\iota'}\rangle = 0 \\ \Leftrightarrow \quad & \lambda\, |\psi_{\pm\iota'}\rangle = 0, \quad x^0\, |\psi_{\pm\iota'}\rangle = 0\,. \end{aligned} \tag{444}$$

So we now know the dynamic constraint

$$\mathbf{Q}\, |\psi_{\pm\iota'}\rangle = 0\,,$$

as well as the gauge condition

$$\mathbf{Q}^{\dagger_{\hat{\eta}}}\, |\psi_{\pm\iota'}\rangle = 0\,,$$

and the gauge fixed physical scalar product can finally be defined (*cf.* §2.3.5 (p.140) and definition 2.129 (p.141) in particular) as the gauge fixed expectation value of the 0-component of a conserved probability current. According to §2.2.4 (p.78) such a probability density on $\mathcal{M}$ is given by the expectation value of the operator

$$J^0_{\mathcal{M}} = \left\langle \hat{\gamma}^0_- \hat{\gamma}^0_+ \right\rangle\,.$$

To keep this density positive in the presence of the spurious Lagrange multiplier fermions on $\mathcal{M} \otimes \mathrm{I\!R}$ one simply has to take the operator product with $\hat{\gamma}^{(\lambda)}_- \hat{\gamma}^{(\lambda)}_+$, so that

$$\begin{aligned} J^0_{\mathcal{M}\otimes\mathrm{I\!R}} &= \left\langle \hat{\gamma}^0_- \hat{\gamma}^0_+ \hat{\gamma}^{(\lambda)}_- \hat{\gamma}^{(\lambda)}_+ \right\rangle \\ &\overset{(439)}{=} \left\langle \hat{\eta}^{(0)} \right\rangle\,. \end{aligned} \tag{445}$$

Hence the physical scalar product is:

$$\begin{aligned} \langle\phi|\psi\rangle_{\mathrm{phys}} &:= \left\langle \phi|\hat{\eta}^0\, \psi\right\rangle_{\text{gauge fixed}} \\ &:= \sum_{\sigma_1=\pm,\sigma_2=\pm} \mathrm{Tr}_{\mathcal{K}} \left[P_{(\mathbf{Q}=0)}\, |\psi_{\sigma_1\iota'}\rangle \langle\phi_{\sigma_2\iota'}|\, \hat{\eta}^{(0)} P_{(\mathbf{Q}^{\dagger_{\hat{\eta}}}=0)} \right]. \end{aligned} \tag{446}$$

Here $\mathrm{Tr}_{\mathcal{K}}$ indicates the trace over the Krein space $\mathcal{K}$ (438), and $P_{(\mathbf{Q}=0)}$ and $P_{(\mathbf{Q}^{\dagger_{\hat{\eta}}}=0)}$ are the projectors onto the kernels of $\mathbf{Q}$ and $\mathbf{Q}^{\dagger_{\hat{\eta}}}$, respectively, that restrict the trace to gauge fixed physical states.

In this form the scalar product is quite general and defined on all states, physical or not. For special cases the formal expression (446) can be written more explicitly. For example, when both $|\phi\rangle$ and $|\psi\rangle$ are physical, i.e.

$$\begin{aligned} \mathbf{Q}\, |\phi_{\pm\iota'}\rangle &= 0 \\ \mathbf{Q}\, |\psi_{\pm\iota'}\rangle &= 0 \end{aligned} \tag{447}$$

then the projector $P_{\mathbf{Q}=0}$ can be dropped entirely and the projector $P_{\mathbf{Q}^{\dagger_{\hat{\eta}}}=0}$ reduces to the projector $P_{\lambda=0,x^0=0}$ onto states satisfying the 'co-constraint' (444) since

$$\begin{aligned} \mathbf{Q}^{\dagger_{\hat{\eta}}}\, \hat{\eta}^{(0)}\, |\phi_{\pm\iota'}\rangle &\overset{(443)}{=} \left(\hat{\eta}^{(0)} \mathbf{Q} \hat{\eta}^{(0)} + \left(\left[\mathbf{p}, x^0\right] \mathcal{P}\lambda + \bar{\mathcal{C}}x^0 \right) \right) \hat{\eta}^{(0)}\, |\phi_{\pm\iota'}\rangle \\ &= \underbrace{\hat{\eta}^{(0)} \mathbf{Q}\, |\phi_{\pm\iota'}\rangle}_{=0} + \left(\left[\mathbf{p}, x^0\right] \mathcal{P}\lambda + \bar{\mathcal{C}}x^0 \right) \hat{\eta}^{(0)}\, |\phi_{\pm\iota'}\rangle\,. \end{aligned}$$

Assuming furthermore, for simplicity of presentation, that both $|\phi\rangle$ and $|\psi\rangle$ are independent of $\hat{\gamma}_{\pm}^{(\lambda)}$, then (446) gives (*cf.* §2.1.1 (p.15))

$$\begin{aligned}\langle\phi|\psi\rangle_{\mathrm{phys}} &= \sum_{\sigma=\pm,\sigma'=\pm} \mathrm{Tr}_{\mathcal{K}}\left[|\psi_{\sigma\iota'}\rangle\,\langle\phi_{\sigma'\iota'}|\,\hat{\eta}^{(0)}\delta(\lambda)\,\delta\big(x^0\big)\right] \\ &= \int\limits_{\lambda=0,x^0=0} \left\langle\phi|\hat{\gamma}^0_-\hat{\gamma}^0_+\psi\right\rangle_{\mathrm{loc}}.\end{aligned} \tag{448}$$

Hence the gauge fixed scalar product over physical states is taken by integrating the positive 0-component of the conserved current over the spacelike hypersurface $x^0 = 0$.

So this procedure of gauge fixing reproduces the usual integration prescription (familiar from electrodynamics and the Dirac electron, *cf.* §2.2.3 (p.70)) when applied to physical states, i.e. to states that satisfy the dynamical equation $\mathbf{D}\,|\phi\rangle = 0$ (i.e the equations of motion). But the present procedure is more general than that, since it can in principle be used to compute gauge fixed scalar products also of non-physical states.

2.3.2 Gauge theory and BRST-cohomology

BRST-cohomology is a powerful technique in gauge theory. This paragraph presents some definitions and results as they are needed later on in §2.3.4 (p.134). Attention is restricted to the present special case of interest, namely gauge theories with a single constraint. Also, particular emphasis will be put on the coBRST-method.

Literature. The standard introduction to quantum gauge theory and BRST methods is [124], a recent elementary introduction is [132]. The coBRST method has been studied in [246][145][103][104][259]. The *standard gauge fixing fermion* (a certain BRST Laplacian, see below) has been studied extensively by Marnelius, e.g. in [21] [187] [186].

Introduction. The Hamiltonian of any mechanical system may be cast into reparametrization invariant form:

$$\mathcal{H} \;=\; \lambda^i p_i \,, \tag{449}$$

where the p_i are the generators of gauge transformations and the λ^i are Lagrange multipliers. (In the completely covariant formulation one of these gauge transformations is that giving time evolution. For a nice elementary discussion of this fact with an eye on quantum gravity see §2.2 of [83].) Let us concentrate on the case where there is only a single such generator and denote its quantum version by $\mathbf{p}$.

Any kinematical quantum state $|\phi\rangle$ is gauge transformed to a new state $|\phi'\rangle$ by acting on it with an element of the gauge group:

$$\begin{aligned} |\phi'\rangle &= \exp(\xi\mathbf{p})\,|\phi\rangle \\ &= |\phi\rangle + \sum_{n=1}^{\infty}\frac{1}{n!}\,(\xi\mathbf{p})^n\;|\phi\rangle \\ &= |\phi\rangle + \mathbf{p}\left(\xi\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\,(\xi\mathbf{p})^n\;|\phi\rangle\right) . \end{aligned} \tag{450}$$

(ξ is a real parameter.) Hence $|\phi\rangle$ and its gauge transformed version differ by an element in the image of $\mathbf{p}$. But since they must be physically indistinguishable, describing the same configuration of the quantum system in two different gauges, one has to identify states modulo elements in $\mathrm{Im}(\mathbf{p})$.

Furthermore, states $|\phi_{\rm phys}\rangle$ which describe physically realizable configurations are completely characterized by being gauge invariant, i.e. they satisfy the *dynamical constraint*

$$\begin{aligned} &\exp(\xi\mathbf{p})\,|\phi_{\rm phys}\rangle = |\phi_{\rm phys}\rangle \\ \Leftrightarrow\quad &\mathbf{p}\,|\phi_{\rm phys}\rangle = 0\,, \end{aligned} \tag{451}$$

and are therefore elements in the kernel $\mathrm{Ker}\,(\mathbf{p})$ of $\mathbf{p}$.

Therefore in gauge theory one is studying the kernel of some operator modulo its image. The mathematical tool of choice to use in such a situation is cohomology theory (*cf.* 2.23 (p.39)), which however requires a *nilpotent* operator. Since $\mathbf{p}$ is not nilpotent, the approach of BRST theory is to embed the theory into one where a suitable nilpotent substitute for $\mathbf{p}$, namely the *BRST operator* $\mathbf{Q}$, can be constructed.

This is ordinarily accomplished by enlarging the space of states so that it can carry further anticommuting operators, the so-called ghosts, $\mathcal{C}_i$, (associated with the constraints) and anti-ghosts, $\bar{\mathcal{P}}_i$, (associated with the Lagrange multipliers), as well as their respective canonical conjugates $\mathcal{P}^i$ and $\bar{\mathcal{C}}^i$. These are taken so satisfy anticommutation relations with the following non-vanishing brackets:

$$\begin{aligned} \left\{\mathcal{C}^i, \mathcal{P}_j\right\} &= \delta^i_j \\ \left\{\bar{\mathcal{P}}^i, \bar{\mathcal{C}}_j\right\} &= -\delta^i_j \,. \end{aligned} \tag{452}$$

The nilpotent BRST charge is then given by

$$\mathbf{Q} = \mathcal{C}^i p_i + \bar{\mathcal{P}}^i \hat{p}_{\lambda i} \,, \tag{453}$$

where

$$\hat{p}_{\lambda i} := \frac{\partial}{\partial \lambda^i} \tag{454}$$

is the (anti-hermitian) momentum associated with the lagrange multiplier λ^i.

In general, any BRST operator has to satisfy three conditions:

2.110 (BRST operator) An operator $\mathbf{Q}$ is called a *BRST operator* associated with the Abelian gauge algebra generated by the single constraint $\mathbf{p}$, if it satisfies the following conditions:

1. *(essential) self-adjointness*

$$\mathbf{Q}^\dagger = \mathbf{Q} \tag{455}$$

2. *nilpotency:*

$$\mathbf{Q}^2 = 0 \tag{456}$$

3. *physical relevance:*

$$\mathrm{H}_c(\mathbf{Q}) \leftrightarrow \mathrm{Ker}\left(\mathbf{p}\right)/\mathrm{Im}(\mathbf{p}) \,. \tag{457}$$

Hence the cohomology $\mathrm{H}_c(\mathbf{Q})$ (*cf.* 2.23 (p.39)) will be of central importance. It is most conveniently studied by means of a Hodge decomposition (*cf.* 2.24 (p.40)), which however, requires a non-degenerate, positive definite inner product, i.e. a scalar product. Since spaces of physical states in relativistic settings are generically merely inner product spaces, not Hilbert spaces[34], one needs to find suitable scalar products on these inner product spaces. This leads to the notion of Krein spaces:

[34] This is in particular true for any space that carries a BRST operator as above. Since by self-adjointness and nilpotency of $\mathbf{Q}$ it follows that every $\mathbf{Q}$-exact state has vanishing norm:

$$\langle \mathbf{Q}\,|\phi\rangle | \mathbf{Q}\,|\phi\rangle\rangle = \left\langle \phi | \mathbf{Q}^2 \phi \right\rangle = 0 \,.$$

2.111 (Krein space and hermitian metric operators) A *Krein space* $(\mathcal{K}, \langle\cdot|\cdot\rangle, \hat{\eta})$ is an inner product space $(\mathcal{K}, \langle\cdot|\cdot\rangle)$, on which an invertible *hermitian metric operator* $\hat{\eta}$ is defined, which induces a positive definite scalar product $\langle\cdot|\cdot\rangle_{\hat{\eta}}$ on $\mathcal{K}$ via:

$$\langle\cdot|\cdot\rangle_{\hat{\eta}} \quad := \quad \langle\cdot|\hat{\eta}\,\cdot\rangle\ , \tag{458}$$

so that $\langle\cdot|\cdot\rangle_{\hat{\eta}}$ is *positive definite*

$$\langle\phi|\phi\rangle_{\hat{\eta}} \geq 0$$

and *non degenerate*, i.e.

$$\langle\psi|\phi\rangle_{\hat{\eta}} = 0 \forall\, |\psi\rangle \quad \Rightarrow \quad |\phi\rangle = 0\,.$$

On a Krein space there are therefore two different notions of 'adjoint': Let A be a linear operator on $\mathcal{K}$, then $A^\dagger$ and $A^{\dagger_{\hat{\eta}}}$ are defined by

$$\begin{aligned} \left\langle A^\dagger\cdot|\cdot\right\rangle &:= \langle\cdot|A\,\cdot\rangle \\ \left\langle A^{\dagger_{\hat{\eta}}}\cdot|\cdot\right\rangle_{\hat{\eta}} &:= \langle\cdot|A\,\cdot\rangle_{\hat{\eta}}\ . \end{aligned} \tag{459}$$

It follows at once that for operators, which are self-adjoint with respect to the indefinite inner product, the adjoint with respect to the scalar product simply reads:

$$A^\dagger = A \quad \Rightarrow \quad A^{\dagger_{\hat{\eta}}} = \hat{\eta}^{-1} A\, \hat{\eta}\,. \tag{460}$$

By means of a nilpotent BRST operator defined on a Krein space one can give the following elegant definition of *gauge fixing*:

2.112 (Gauge fixing) *Fixing a gauge* amounts to specifying a unique representative element $|\phi\rangle \in [|\phi\rangle] \in \mathrm{H}_c(\mathbf{Q})$ in each equivalence class of gauge equivalent states (*cf.* 2.25 (p.40)). With the BRST-cohomology at hand and by the Hodge decomposition theorem (*cf.* 2.24 (p.40)), such a unique representative in each equivalence class can conveniently be given by defining a *coBRST* charge $\mathbf{Q}^{\dagger_{\hat{\eta}}}$ by means of some hermitian metric operator $\hat{\eta}$. The unique representatives, i.e. the gauge fixed physical states, are then those elements $|\phi\rangle$ that are *harmonic* with respect to $\mathbf{Q}$ and $\mathbf{Q}^{\dagger_{\hat{\eta}}}$, i.e. those that satisfy $\mathbf{Q}\,|\phi\rangle = \mathbf{Q}^{\dagger_{\hat{\eta}}}\,|\phi\rangle = 0$. While $\mathbf{Q}\,|\phi\rangle = 0$ is the condition for physical states (the dynamical equation), $\mathbf{Q}^{\dagger_{\hat{\eta}}}\,x = 0$ is the *gauge condition*:

- *dynamical equation:*

$$\mathbf{Q}\,|\phi\rangle = 0$$

- *gauge condition:*

$$\mathbf{Q}^{\dagger_{\hat{\eta}}}\,|\phi\rangle = 0$$

- *condition for gauge fixed physical states:*

$$\begin{aligned} & \left(\mathbf{Q} + \mathbf{Q}^{\dagger_{\hat{\eta}}}\right)|\phi\rangle = 0 \\ \Leftrightarrow \quad & \mathbf{Q}\,|\phi\rangle = \mathbf{Q}^{\dagger_{\hat{\eta}}}\,|\phi\rangle = 0\,. \end{aligned} \tag{461}$$

This establishes a close connection between fixing a gauge and finding a *hermitian metric operator*:

Note 2.113 (Gauge fixing induced by hermitian metric operator) *A positive definite, non-degenerate scalar product gives rise to a coBRST operator and thus to a certain gauge fixing. (However, not every such scalar product will necessarily yield a* complete *gauge fixing.)*

Since the coBRST operator is uniquely determined by $\mathbf{Q}$ and a hermitian metric $\hat{\eta}$ on $\mathcal{K}$, it follows that *every hermitian metric operator $\hat{\eta}$ defines a certain gauge.* This way one can fix gauges by finding hermitian metric operators.

Note that the scalar products thus obtained with the sole purpose of finding coBRST operators, have no direct relation with the *physical scalar product* that will be defined in §2.3.5 (p.140).

Usually, BRST operators for gauge theories are obtained by introducing so-called *ghost operators*:

2.114 (Ghosts) None of the operators that can be constructed in non-graded quantum gauge theory (with Hilbert space $\mathcal{H}$ and operator algebra $(A, \mathcal{H})$) can satisfy conditions (456) and (457) for a BRST operator. Hence, in order to construct a BRST operator, one defines a *graded extension* $\left(\mathcal{H}', (-1)^{\hat{N}_{\mathcal{G}}}\right)$ and $\left(A, \mathcal{H}', (-1)^{\hat{N}_{\mathcal{G}}}\right)$ of $\mathcal{H}$ and $(A, \mathcal{H})$ with involution $(-1)^{\hat{N}_{\mathcal{G}}}$ (see §2.1.2 (p.37)) induced by a so-called *total ghost number operator* $\hat{N}_{\mathcal{G}}$ which has integer eigenvalues on the various graded copies of $\mathcal{H}$ that make up $\mathcal{H}'$. The elements of $\mathcal{H}'$ that are eigenvectors of $\hat{N}_{\mathcal{G}}$ to nonvanishing eigenvalues are called *ghosts*, in order to emphasize that they represent spurious degrees of freedom that have been added only for formal reasons and have no direct physical interpretation.

Note that from the point of view of BRST cohomology theory all one needs in order to find physical states and to fix gauges are appropriate BRST and coBRST operators. These *need not* be formulated in terms of spurious ghost degrees of freedom. This is established in §2.3.4 (p.134).

The following paragraph lists the basic elements of the usual setup of BRST theory for Abelian gauge groups with a single generator (*cf.* [21][187] [186]):

2.115 (Ghost algebra)

1. *Ghosts and anti-ghosts.* Ghost and anti-ghost creators and annihilators are linear operators
   - *ghost creator:* $\mathcal{C}$
   - *ghost annihilator:* $\mathcal{P}$
   - *anti-ghost creator:* $\bar{\mathcal{C}}$
   - *anti-ghost annihilator:* $\bar{\mathcal{P}}$

that can be chosen to be anti-self-adjoint: [35]

$$\begin{aligned}
\mathcal{C}^\dagger &= -\mathcal{C} \\
\mathcal{P}^\dagger &= -\mathcal{P} \\
\bar{\mathcal{C}}^\dagger &= -\bar{\mathcal{C}} \\
\bar{\mathcal{P}}^\dagger &= -\bar{\mathcal{P}}\,,
\end{aligned} \tag{462}$$

and that satisfy canonical anticommutation relations with the following non-vanishing brackets:

$$\begin{aligned}
\{\mathcal{C},\mathcal{P}\} &= 1 \\
\{\bar{\mathcal{C}},\bar{\mathcal{P}}\} &= -1\,.
\end{aligned} \tag{463}$$

2. *Ghost number operators.* As usual, number operators

$$\begin{aligned}
\hat{N}_G &= \mathcal{C}\mathcal{P} \\
\hat{N}_{\bar{G}} &= \bar{\mathcal{C}}\bar{\mathcal{P}} \\
\hat{N}_{\mathcal{G}} &= \hat{N}_G + \hat{N}_{\bar{G}}
\end{aligned} \tag{464}$$

are defined (*ghost number*, *anti-ghost number*, and *total ghost number*), having as eigenvalues the number of ghosts *minus* the number of antighosts: Let $\left|\mathcal{P}=0,\bar{\mathcal{P}}=0\right\rangle$ be the ghost vacuum, then, by the above anticommutation relations

$$\begin{aligned}
\hat{N}_G \left|\mathcal{P}=0,\bar{\mathcal{P}}=0\right\rangle &= 0 \\
\hat{N}_{\bar{G}} \left|\mathcal{P}=0,\bar{\mathcal{P}}=0\right\rangle &= 0 \\
\hat{N}_G\,\mathcal{C} \left|\mathcal{P}=0,\bar{\mathcal{P}}=0\right\rangle &= \mathcal{C}\left|\mathcal{P}=0,\bar{\mathcal{P}}=0\right\rangle \\
\hat{N}_{\bar{G}}\,\bar{\mathcal{C}} \left|\mathcal{P}=0,\bar{\mathcal{P}}=0\right\rangle &= -\bar{\mathcal{C}}\left|\mathcal{P}=0,\bar{\mathcal{P}}=0\right\rangle \\
\hat{N}_{\mathcal{G}}\,\mathcal{C}\bar{\mathcal{C}} \left|\mathcal{P}=0,\bar{\mathcal{P}}=0\right\rangle &= 0\,.
\end{aligned} \tag{465}$$

The adjoint operators are, by (462) and (463),

$$\begin{aligned}
\hat{N}_G^\dagger &= 1 - \hat{N}_G \\
\hat{N}_{\bar{G}}^\dagger &= -1 - \hat{N}_G \\
\hat{N}_{\mathcal{G}}^\dagger &= -\hat{N}_{\mathcal{G}}\,.
\end{aligned} \tag{466}$$

The anti-self-adjointness of $\hat{N}_{\mathcal{G}}$ is an essential feature of the theory (*cf.* 2.116 (p.121)).

3. $\hat{\eta}$*-adjoint:* The ghost operators cannot be proportional to their own $\hat{\eta}$-adjoint, instead one finds (*cf.* §E (p.330)):

$$\begin{aligned}
\mathcal{C}^{\dagger_{\hat{\eta}}} &= -\mathcal{P} \\
\mathcal{P}^{\dagger_{\hat{\eta}}} &= -\mathcal{C} \\
\bar{\mathcal{C}}^{\dagger_{\hat{\eta}}} &= \bar{\mathcal{P}} \\
\bar{\mathcal{P}}^{\dagger_{\hat{\eta}}} &= \mathcal{C}\,.
\end{aligned} \tag{467}$$

[35] Several different conventions for which of these operators are self-adjoint and which are anti-self-adjoint can be found in the literature. The important point, however, is merely that all these operators are proportional to their own adjoints. The choice presented here complies with the ghost representations as they are found in §E (p.330).

But this implies that

$$\begin{aligned} \hat{N}_G^{\dagger_{\hat\eta}} &= \hat{N}_G^{\dagger} \\ \hat{N}_{\bar G}^{\dagger_{\hat\eta}} &= \hat{N}_{\bar G}^{\dagger} \\ \hat{N}_{\mathcal{G}}^{\dagger_{\hat\eta}} &= \hat{N}_{\mathcal{G}}^{\dagger} \,. \end{aligned} \tag{468}$$

4. *Gauge generator.* In the simple setting considered here, there is a single gauge generator $\mathbf{p}$

$$\mathbf{p}^\dagger = -\mathbf{p}\,, \tag{469}$$

which commutes with all ghost operators

$$[\mathbf{p}, \mathcal{X}] = 0, \quad \mathcal{X} \in \left\{\mathcal{C}, \mathcal{P}, \bar{\mathcal{C}}, \bar{\mathcal{P}}\right\} \tag{470}$$

and generates the $U(1)$ group

$$\left\{e^{\lambda \mathbf{p}} \mid \lambda \in \mathbb{R}\right\}$$

of *gauge transformation operators.* Physical states are defined as being exactly those states, that are in the trivial representation of this group, i.e. that are invariant under the action of $e^{\lambda \mathbf{p}}$:

$$\begin{aligned} |\psi(\lambda)\rangle &:= e^{\lambda \mathbf{p}}\,|\psi\rangle \\ &\stackrel{!}{=} |\psi\rangle \\ \Leftrightarrow \partial_{(\lambda)}\,|\psi(\lambda)\rangle &= \mathbf{p}\,|\psi(\lambda)\rangle \\ &\stackrel{!}{=} 0\,. \end{aligned} \tag{471}$$

5. *BRST operator.* Condition 4 (p.120). leads to the definition of the BRST operator:

$$\mathbf{Q} := \mathcal{C}\mathbf{p} + \bar{\mathcal{P}}\partial_{(\lambda)}\,, \tag{472}$$

which singles out physical states that are invariant under gauge transformations and independent of λ. By the above properties of the ghost algebra, $\mathbf{Q}$ satisfies:

$$\begin{aligned} \mathbf{Q}^\dagger &= \mathbf{Q} \\ \left[\hat{N}_{\mathcal{G}}, \mathbf{Q}\right] &= \mathbf{Q}\,. \end{aligned} \tag{473}$$

The last property implies that $\mathbf{Q}$ is an odd operator with respect to the involution $(-1)^{\hat{N}_{\mathcal{G}}}$:

$$\begin{aligned} \iota &:= (-1)^{\hat{N}_{\mathcal{G}}} \\ \Rightarrow \{\iota, \mathbf{Q}\} &= 0\,. \end{aligned} \tag{474}$$

6. *Standard gauge fixing operator.* As shown in [104] and [228], the operator

$$\mathcal{P}\lambda + \bar{\mathcal{C}}\chi \tag{475}$$

is a coBRST operator to $\mathbf{Q}$ above (often called a "gauge fixing fermion"). Here, χ is a *gauge condition* operator satisfying

$$\begin{aligned} \chi^\dagger &= \chi \\ [\mathbf{p}, \chi] &= 1 \\ [\mathcal{X}, \chi] &= 0 \quad \mathcal{X} \in \left\{\mathcal{C}, \mathcal{P}, \bar{\mathcal{C}}, \bar{\mathcal{P}}\right\} . \end{aligned} \tag{476}$$

Note that (475) arises as

$$\begin{aligned} e^{-\lambda\chi}\,\hat{\eta}\mathbf{Q}\hat{\eta}\,e^{\lambda\chi} &= e^{-\lambda\chi}\left(\mathcal{P}\mathbf{p} + \bar{\mathcal{C}}\partial_{(\lambda)}\right)e^{\lambda\chi} \\ &= \left(\mathcal{P}\mathbf{p} + \bar{\mathcal{C}}\partial_{(\lambda)}\right) + \mathcal{P}\lambda + \bar{\mathcal{C}}\chi \\ &= \hat{\eta}\mathbf{Q}\hat{\eta} + \mathcal{P}\lambda + \bar{\mathcal{C}}\chi \end{aligned} \tag{477}$$

and that $e^{\lambda\chi}\hat{\eta}$ is an admissible hermitian metric operator if $\hat{\eta}$ is[36].

7. *Gauge fixed physical expectation values.* As discussed in detail in §2.3.5 (p.140) (*cf.* [228]), the gauge fixed expectation value of observables is given by

$$\left\langle (-1)^{\hat{N}_{\mathcal{G}}} A \right\rangle_\phi := \mathrm{Tr}\Big(e^{-\left(\mathbf{Q}+\mathbf{Q}^{\dagger_{\hat{\eta}}}\right)^2} (-1)^{\hat{N}_{\mathcal{G}}} A \left|\phi\right\rangle \left\langle\phi\right| \Big) ,$$

for gauge invariant observables A and states $|\phi\rangle$

$$[\mathbf{Q}, A\left|\phi\right\rangle\left\langle\phi\right|] = 0 \,.$$

Looking at the exponent of the regulator term

$$\begin{aligned} \left(\mathbf{Q} + \mathbf{Q}^{\dagger_{\hat{\eta}}}\right)^2 &= \left\{\mathbf{Q}, \mathbf{Q}^{\dagger_{\hat{\eta}}}\right\} \\ &= \lambda\mathbf{p} - \chi\partial_{(\lambda)} + \mathcal{C}\bar{\mathcal{C}}\underbrace{[\mathbf{p},\chi]}_{=1} - \mathcal{P}\bar{\mathcal{P}}\underbrace{\left[\partial_{(\lambda)}, \lambda\right]}_{=1} . \end{aligned} \tag{478}$$

one finds terms familiar from Lagrangian formulations of gauge theory:

- *physical projector:* $e^{\lambda\mathbf{p}}$ acts, when integrated over λ, as a projector on physical states by a mechanism known as *group averaging* ([188][232]).
- *Gauge fixing term:* $e^{\chi\partial_{(\lambda)}}$ acts as a projector on states satisfying the gauge condition $\chi\left|\phi\right\rangle = 0$ when integrated over $\partial_{(\lambda)}$.
- *Fadeev-Popov term:* $e^{\mathcal{C}^i\bar{\mathcal{C}}_j\left[\mathbf{p}_i,\chi^j\right]}$ gives the *Fadeev-Popov* determinant

  $$\det{}_{\mathrm{FP}} := \det\left(\left[\mathbf{p}_i, \chi^j\right]\right)$$

  under Berezin integration over the (anti-) ghosts. (In the simplified case considered here this is trivially constant: $\det_{\mathrm{FP}} = 1$. However, a non-trivial Fadeev-Popov like term will arise when in §2.3.4 (p.134) the above method is modified to allow graded gauge generators $\mathbf{p}$.)

2.116 (Important relations following from the ghost algebra)

[36]Both will define a positive definite inner product, but not both of them will in general yield a finite trace over physical states, see below.

1. *The inner product of states with non-reciprocal ghost number vanishes.* *Proof:* Let

$$\begin{aligned} \hat{N}_{\mathcal{G}} \left|\alpha\right\rangle &= n \left|\alpha\right\rangle \\ \hat{N}_{\mathcal{G}} \left|\beta\right\rangle &= n' \left|\beta\right\rangle \ , \end{aligned}$$

then, due to (466),

$$\begin{aligned} &\left\langle \alpha | \hat{N}_{\mathcal{G}}\, \beta \right\rangle = - \left\langle \hat{N}_{\mathcal{G}}\, \alpha | \beta \right\rangle \\ \Rightarrow \qquad & (n' + n) \left\langle \alpha | \beta \right\rangle = 0 \ . \end{aligned}$$

2. *Representations of the ghost algebra: Singlets, doublet, and quartets.* From the relations

$$\begin{aligned} \left\{\hat{\eta}, \hat{N}_{\mathcal{G}}\right\} &= 0 \\ \left[\mathbf{Q}\mathbf{Q}^{\dagger_{\hat{\eta}}}, \hat{N}_{\mathcal{G}}\right] &= 0 \end{aligned} \tag{479}$$

it follows that $\hat{\eta}$ inverts the ghost number

$$\left|-N\right\rangle \quad := \quad \hat{\eta} \left|N\right\rangle \tag{480}$$

and that for physical states,

$$\mathbf{\Delta_Q} \left|N\right\rangle = 0 \,,$$

one has

$$\mathbf{Q} \left|N\right\rangle = 0, \ \mathbf{Q} \left|-N\right\rangle = 0 \ .$$

It follows that $\{\left|N\right\rangle, \left|-N\right\rangle\}$ is a *BRST singlet* for $N = 0$ and a *BRST doublet* for $N \neq 0$.
Furthermore, $\mathbf{Q}\hat{\eta}\mathbf{Q}\hat{\eta}$ has simultaneous eigenvalues with $\hat{N}_{\mathcal{G}}$,

$$\mathbf{Q}\hat{\eta}\mathbf{Q}\hat{\eta} \left|N\right\rangle \sim \left|N\right\rangle \ ,$$

so that for non-physical states one has the following mappings:

$$\left|N\right\rangle \xmapsto{\mathbf{Q}} \left|N+1\right\rangle \xmapsto{\hat{\eta}} \left|-N-1\right\rangle \xmapsto{\mathbf{Q}} \left|-N\right\rangle \xmapsto{\hat{\eta}} \sim \left|N\right\rangle \ .$$

Here $\{\left|N\right\rangle, \left|N+1\right\rangle, \left|-N-1\right\rangle, \left|-N\right\rangle\}$ is called a *BRST quartet.*

2.3.3 BRST-cohomology of an ordinary Hamiltonian

Introduction. In order to work out the central ideas of quantum BRST cohomology the formalism is in the following applied to the simplest imaginable problem, namely ordinary bosonic quantum mechanics in covariant formulation. The adaption of the techniques discussed below to a covariant quantum system whose single constraint is a generalized Dirac operator is then straightforward.

2.117 (Ordinary Schrödinger dynamics as constrained dynamics) Consider some ordinary quantum mechanical system described by a Hilbert space $\mathcal{H}$ with scalar product

$$\langle\cdot|\cdot\rangle_{\mathcal{H}}$$

carrying a selfadjoint Hamiltonian operator

$$\hat{H} : \mathcal{H} \to \mathcal{H} \,. \tag{481}$$

Let x^0 be the time parameter and assume for simplicity that $\hat{H}$ satisfies

$$\begin{aligned} \left[x^0, \hat{H}\right] &= 0 \\ \left[\partial_{x^0}, \hat{H}\right] &= 0 \,. \end{aligned} \tag{482}$$

We will also need a further parameter, λ, and so we assume that

$$\begin{aligned} \left[\lambda, \hat{H}\right] &= 0 \\ \left[\partial_{(\lambda)}, \hat{H}\right] &= 0 \,, \end{aligned} \tag{483}$$

too. The ordinary Schrödinger equation of the system then is

$$\partial_{x^0} |\phi_{x^0}\rangle = \frac{1}{i\hbar}\hat{H}\, |\phi_{x^0}\rangle \,, \quad |\phi_{x^0}\rangle \in \mathcal{H}\; \forall\, x^0 \,. \tag{484}$$

Here x^0 is merely a parameter labeling elements in $\mathcal{H}$. Next we want to extend $\mathcal{H}$ to a larger Hilbert space $\mathcal{H}'$ of functions on spacetime, for which x^0 and λ are not merely parameters. It is convenient to choose $\mathcal{H}'$ to be the Schwartz space $\mathcal{S}_{(x^0,\lambda)}$ of rapidly decreasing functions with respect to x^0 and λ:

$$\mathcal{H}' = \left\{ \left|\phi(x^0,\lambda)\right\rangle \mid \left\langle \phi(x^0,\lambda)|\phi(x^0,\lambda)\right\rangle_{\mathcal{H}} \in \mathcal{S}_{(x^0,\lambda)} \right\} \,. \tag{485}$$

Hence the scalar product on $\mathcal{H}'$ is the scalar product on $\mathcal{H}$ combined with the usual L^2 scalar product on $\mathbb{R}^2$:

$$\left\langle \phi(x^0,\lambda)|\psi(x^0,\lambda)\right\rangle_{\mathcal{H}'} := \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \left\langle \phi(x^0,\lambda)|\psi(x^0,\lambda)\right\rangle_{\mathcal{H}} \, dx^0\, d\lambda \,. \tag{486}$$

Obviously, solutions to the Schrödinger equation (484), referred to as *physical states* in the following, are not elements of $\mathcal{H}'$, since they are not rapidly decreasing, and in particular not square integrable, with respect to x^0 and λ. But physical states $\left|\phi_{\rm phys}(x^0,\lambda)\right\rangle$ are elements of the dual $\mathcal{H}'^*$ of $\mathcal{H}'$:

$$\left|\phi_{\rm phys}(x^0,\lambda)\right\rangle \in \mathcal{H}'^* \,. \tag{487}$$

To single out physical states define the operator

$$\mathbf{p} : \mathcal{H}' \to \mathcal{H}' \tag{488}$$

given by

$$\mathbf{p} \;:=\; \partial_{x^0} - \frac{1}{i\hbar}\hat{H}\,, \tag{489}$$

and extend it as usual to an operator on $\mathcal{H}'^*$. Note that $\mathbf{p}$ is (essentially) anti-selfadjoint:

$$\mathbf{p}^\dagger \;=\; -\mathbf{p}\,. \tag{490}$$

Physical states are precisely those elements of $\mathcal{H}'^*$ which are annihilated by $\mathbf{p}$:

$$\begin{aligned} &\mathbf{p}\,|\phi_{\text{phys}}\rangle = 0 \\ \Leftrightarrow\quad &\exp(-\xi\mathbf{p})\,|\phi_{\text{phys}}\rangle = |\phi_{\text{phys}}\rangle\ . \end{aligned} \tag{491}$$

The second line expresses that physical states can equivalently be characterized as those transforming trivially under the Lie group generated by $\mathbf{p}$. (ξ is here a real parameter.) To better understand the nature of this group, which we call the *gauge group* generated by the *gauge generator* $\mathbf{p}$, consider an element $|\psi_0\rangle \in \mathcal{H}$ in the original Hilbert space and promote it to an element $\tilde{\psi}(x^0,\lambda) \in \mathcal{H}'^*$ given by

$$\left|\tilde{\psi}(x^0,\lambda)\right\rangle \;:=\; \delta(x^0)\,\delta(\lambda)\,|\psi_0\rangle\ . \tag{492}$$

This is clearly an *unphysical* state, since it is concentrated at one instant of time $x^0 = 0$. The gauge group generated by $\mathbf{p}$ acts non-trivially on this state, yielding:

$$\begin{aligned} \exp(-\xi\mathbf{p})\left|\tilde{\psi}(x^0,\lambda)\right\rangle &= \delta(\lambda)\left(e^{-\xi\partial_{x^0}}\delta(x^0)\right)\left(e^{\xi\frac{1}{i\hbar}\hat{H}}\left|\tilde{\psi}_0\right\rangle\right) \\ &= \delta(\lambda)\,\delta(x^0-\xi)\,|\psi_\xi\rangle\ . \end{aligned} \tag{493}$$

(Note that in the notation introduced with (484) the state $|\psi_\xi\rangle$ is the state $|\psi_0\rangle$ propagated by an amount ξ of coordinate time by the ordinary Hamiltonian $\hat{H}$.) It can be checked that it is justified to call this a gauge transformation by observing that $\left|\tilde{\psi}(x^0\lambda)\right\rangle$ and its transformed version (493) have the same projection on a given physical state:

$$\left\langle\phi_{\text{phys}}|\tilde{\psi}\right\rangle_{\mathcal{H}'} \;=\; \left\langle\phi_{\text{phys}}|\exp(-\xi\mathbf{p})\,\tilde{\psi}\right\rangle_{\mathcal{H}'}\ . \tag{494}$$

This is trivial but conceptually important. It shows that it is justified to call $\exp(-\xi\mathbf{p})$ a gauge transformation and in particular it allows to interpret the failure of physical states to be elements of $\mathcal{H}'$ in terms of gauge theory. Namely, the integral over x^0 in (486) sums over all gauge equivalent states (493) for all values of ξ. Clearly, a physically meaningful scalar product on physical states needs to restrict integration to a limited number of gauge equivalent states, i.e. to *fix a gauge*. This is precisely what is accomplished by means of BRST cohomology.

Before going into that we first note for completeness that a similar reasoning applies to transformations generated by $\partial_{(\lambda)}$

$$\begin{aligned}
&\partial_{(\lambda)} : \mathcal{H}' \to \mathcal{H}' \\
&\partial_{(\lambda)}^\dagger = -\partial_{(\lambda)} \,.
\end{aligned} \tag{495}$$

Since the original Schrödinger equation makes no reference to λ, one requires that physical states be independent of this parameter:

$$\begin{aligned}
& \partial_{(\lambda)} \left|\phi_{\text{phys}}\right\rangle = 0 \\
\Leftrightarrow \quad & \exp\bigl(-\xi \partial_{(\lambda)}\bigr) \left|\phi_{\text{phys}}\right\rangle \stackrel{!}{=} \left|\phi_{\text{phys}}\right\rangle \,.
\end{aligned} \tag{496}$$

2.118 (Applying BRST formalism to Schrödinger dynamics in constrained form) The idea of BRST gauge fixing is to employ the cohomology of a nilpotent operator which is closely related to the gauge generators $\mathbf{p}$ and $\partial_{(\lambda)}$. In ordinary bosonic quantum mechanics such an operator is not available. Therefore one again enlarges the space of states under consideration to a space $\mathcal{K}$, which contains several copies of the original state space

$$\mathcal{K} \;\; := \;\; \mathcal{H}' \oplus \mathcal{H}' \oplus \cdots \oplus \mathcal{H}' \tag{497}$$

in such a way, that it can also carry four ghost and anti-ghost creation and annihilation operators

$$\mathcal{C},\, \mathcal{P},\, \bar{\mathcal{C}},\, \bar{\mathcal{P}} : \mathcal{K} \to \mathcal{K}\,, \tag{498}$$

which are nilpotent and satisfy relations (462) and (463) from 2.115 (p.118). These imply in particular that the inner product $\langle\cdot|\cdot\rangle_{\mathcal{K}}$ on $\mathcal{K}$ cannot be positive definite: Due to the nilpotency and (anti-)selfadjointness of the ghost operators there are 0-norm states

$$\begin{aligned}
\langle \mathcal{X}\phi|\mathcal{X}\phi\rangle_{\mathcal{K}} \;\; &\propto \;\; \left\langle \phi|\mathcal{X}^2\phi\right\rangle_{\mathcal{K}} \\
&= \;\; 0, \qquad\qquad \mathcal{X} \in \left\{\mathcal{C},\, \mathcal{P},\, \bar{\mathcal{C}},\, \bar{\mathcal{P}}\right\} \,.
\end{aligned} \tag{499}$$

Furthermore, letting $|\phi\rangle\,, |\psi\rangle \in \mathcal{K}$ be two such 0-norm states $\langle\phi|\phi\rangle_{\mathcal{K}} = \langle\psi|\psi\rangle_{\mathcal{K}} = 0$, one finds that the norm squares of $|\phi\rangle \pm |\psi\rangle$ and cannot both be positive

$$|\,|\psi\rangle \pm |\phi\rangle\,|^2_{\mathcal{K}} \;\; = \;\; \pm\left(\langle\psi|\phi\rangle_{\mathcal{K}} + \langle\psi|\phi\rangle_{\mathcal{K}}\right) \,. \tag{500}$$

But there is an operator

$$\hat{\eta} : \mathcal{K} \to \mathcal{K} \tag{501}$$

on $\mathcal{K}$ such that

$$\langle\phi|\hat{\eta}\phi\rangle_{\mathcal{K}} \geq 0, \;\; \forall \phi \in \mathcal{K}\,. \tag{502}$$

This *hermitian metric operator* will play a central role below.

From the ghost operators one can also construct an operator counting total ghost number, $\hat{N}_{\mathcal{G}}$, and the associated involutive grading operator $(-1)^{\hat{N}_{\mathcal{G}}}$. There is some freedom in the definition of $\hat{N}_{\mathcal{G}}$, but of importance for the following constructions is only that all ghost operators are odd graded with respect to $(-1)^{\hat{N}_\mathcal{G}}$:

$$\left\{(-1)^{\hat{N}_\mathcal{G}}, \mathcal{X}\right\} = 0, \qquad \forall \mathcal{X} \in \left\{\mathcal{C}, \mathcal{P}, \bar{\mathcal{C}}, \bar{\mathcal{P}}\right\} . \tag{503}$$

Next introduce the *BRST-operator*

$$\mathbf{Q} : \mathcal{K} \to \mathcal{K} \tag{504}$$

which incorporates both gauge generators $\mathbf{p}$, $\partial_{(\lambda)}$ multiplied by ghost operators in order to achieve nilpotency:

$$\mathbf{Q} := \mathcal{C}\mathbf{p} + \bar{\mathcal{P}}\partial_{(\lambda)} . \tag{505}$$

Clearly one has

$$\begin{aligned} &\mathbf{Q}^2 = 0 \\ &\mathbf{Q}^\dagger = \mathbf{Q} \\ &\left\{(-1)^{\hat{N}_\mathcal{G}}, \mathbf{Q}\right\} = 0 . \end{aligned} \tag{506}$$

This gives rise to the following formal argument of central importance: Let $\hat{A} : \mathcal{K} \to \mathcal{K}$ be any operator that commutes with $\mathbf{Q}$, then

$$\begin{aligned} & \left[\mathbf{Q}, \hat{A}\right] = 0 \\ \Rightarrow \quad & \mathrm{Tr}_\mathcal{K}\left(\hat{A}\,(-1)^{\hat{N}_\mathcal{G}}\right) \text{``}=\text{''} \mathrm{Tr}_{\mathrm{H}_c(\mathbf{Q})}\left(\hat{A}\,(-1)^{\hat{N}_\mathcal{G}}\right) \end{aligned} \tag{507}$$

reduces to the trace over states that are annihilated by $\mathbf{Q}$ but not in the image of $\mathbf{Q}$. (This is because of the factor $(-1)^{\hat{N}_\mathcal{G}}$ which leads to a cancellation of contributions from summing over pairs of states $|\phi\rangle , \mathbf{Q}\,|\phi\rangle \neq 0$ in the trace. See 2.28 (p.41).) The above expression is exactly the gauge fixed inner product that is needed to get rid of the infinities arising from summing over gauge equivalent states. But it is formal in that for the cancellation to occur the trace itself, which involves an infinite sum, must converge. This will in general *not* be the case for some operator $\hat{A}$, and hence the whole problem of finding a gauge fixed inner product reduces to finding proper regularizations of the above formal expression. Hence one needs a regularization operator $\hat{R}$ which commutes with $\mathbf{Q}$, is the identity on $\mathrm{H}_c(\mathbf{Q})$ and has eigenvalues tending to zero such that the following expression, which shall be called the *physical trace*, converges to a finite value:

$$\mathrm{Tr}_{\mathrm{phys}}\left(\hat{A}\right) := \mathrm{Tr}_\mathcal{K}\left((-1)^{\hat{N}_\mathcal{G}}\hat{R}\hat{A}\right) < \infty. \tag{508}$$

A convenient way to find such a regularization operator $\hat{R}$ is by means of another operator, the so-called *gauge-fixing fermion* $\hat{\rho}$, via

$$\hat{R} := \exp(-\left\{\mathbf{Q}, \hat{\rho}\right\}) . \tag{509}$$

Such an $\hat{R}$ automatically commutes with $\mathbf{Q}$. One way to ensure that it acts as the identity on $\mathrm{H}_c(\mathbf{Q})$ and has eigenvalues tending to zero is to choose for $\hat{\rho}$ the adjoint of $\mathbf{Q}$ with respect to some positive definite inner product.

Assume a suitable $\hat{\rho}$ has been found. The physical trace then is

$$\mathrm{Tr}_{\mathrm{ph}}\left[\hat{A}\right] \quad := \quad \mathrm{Tr}_{\mathcal{K}}\left[e^{-\{\mathbf{Q},\hat{\rho}\}}\hat{A}\,(-1)^{\hat{N}_{\mathcal{G}}}\right]. \tag{510}$$

This expression is in general non-trivial to evaluate. However, for certain observables $\hat{A}$ it may occur that the right hand side of (510) reduces simply to the ordinary trace on $\mathcal{K}$:

2.119 (Gauge fixed expectation values) One may reverse the reasoning and regard the physical trace as an object that tells us which conditions one has to impose on an observable so that its trace gives a meaningful physical result.

For instance, if $\hat{A}'$ satisfies

$$\begin{aligned}
&\mathbf{Q}\hat{A}' = 0\\
&\hat{\rho}\hat{A}' = 0\\
&\hat{A}'(-1)^{\hat{N}_{\mathcal{G}}} = \hat{A}',
\end{aligned} \tag{511}$$

then clearly the physical trace is just the ordinary trace

$$\mathrm{Tr}_{\mathrm{ph}}\left[\hat{A}'\right] \quad = \quad \mathrm{Tr}_{\mathcal{K}}\left[\hat{A}'\right], \tag{512}$$

since

$$e^{-\{\mathbf{Q},\hat{\rho}\}}\hat{A}'(-1)^{\hat{N}_{\mathcal{G}}} \quad = \quad \hat{A}'. \tag{513}$$

Usually one wants to evaluate the expectation value of an observable $\hat{A}$ with respect to some state $|\phi\rangle$. If this state is such that

$$\begin{aligned}
&\mathbf{Q}\,|\phi\rangle = 0 = \hat{\rho}\,|\phi\rangle\\
&\mathbf{Q}\hat{A}\,|\phi\rangle = 0 = \hat{\rho}\hat{A}\,|\phi\rangle\ ,
\end{aligned}$$

then clearly

$$\Rightarrow \quad \mathrm{Tr}_{\mathrm{ph}}\left[\hat{A}\,|\phi\rangle\,\langle\phi|\right] = \mathrm{Tr}_{\mathcal{K}}\Big(\hat{A}\,|\phi\rangle\,\langle\phi|\Big) = \left\langle\phi|\hat{A}\phi\right\rangle. \tag{514}$$

This is a useful result. It tells us that when a state $|\phi\rangle$ satisfies the *dynamical constraint* $\mathbf{Q}\,|\phi\rangle = 0$ as well as a *gauge constraint* $\hat{\rho}\,|\phi\rangle = 0$, then the gauge fixed physical inner product reduces just to the ordinary inner product.

To illustrate this consider the following example, which actually completely solves the problem of gauge fixing for the setup considered in this section with gauge generator $\mathbf{p}$ given by (489):

Example 2.120 As gauge fixing fermion choose the adjoint of $\mathbf{Q}$ with respect to the inner product induced by $\hat{\eta} = \hat{\eta}^{(0)}e^{i\lambda x^0}$, that is

$$\begin{aligned}
\hat{\rho} \quad &:= \quad \mathbf{Q}^{\dagger_{\hat{\eta}}}\\
&= \quad e^{-i\lambda x^0}\hat{\eta}\mathbf{Q}\hat{\eta}e^{i\lambda x^0}.
\end{aligned} \tag{515}$$

Note that

$$\left\{\mathbf{Q},\mathbf{Q}^{\dagger_{\hat{\eta}}}\right\} \quad = \quad \left(\mathbf{Q}+\mathbf{Q}^{\dagger_{\hat{\eta}}}\right)^2 \tag{516}$$

and that the operator in brackets on the right is self-adjoint with respect to $\langle\cdot|\cdot\rangle_{\mathcal{K}}$:

$$\left(\mathbf{Q}+\mathbf{Q}^{\dagger_{\hat\eta}}\right)^{\dagger} \;=\; \left(\mathbf{Q}+\mathbf{Q}^{\dagger_{\hat\eta}}\right) . \tag{517}$$

It follows that this operator has real eigenvalues so that its square has non-negative real eigenvalues and hence $\exp\!\left(-\left\{\mathbf{Q},\mathbf{Q}^{\dagger_{\hat\eta}}\right\}\right)$ qualifies as a regulator term. The most general state $|\phi\rangle$ which satisfies both the dynamical and the gauge constraint is obviously

$$|\phi\rangle \;=\; \phi_1\,|0_{\mathcal{G}}\rangle + e^{-i\lambda x^0}\hat\eta^{(0)}\phi_2\,|0_{\mathcal{G}}\rangle \;, \tag{518}$$

where $|0_{\mathcal{G}}\rangle := \left|\mathcal{P}=0,\bar{\mathcal{C}}=0\right\rangle$ is the ghost vacuum and where ϕ_i are gauge invariant elements of $\mathcal{H}'^{*}$:

$$\mathbf{p}\phi_i = 0 = \partial_{(\lambda)}\phi_i\,. \tag{519}$$

Since $\hat\eta^{(0)}$ swaps the ghost vacuum with the completely filled ghost sector the above state is a physical state in the non-ghost sector and a physical state times $e^{-i\lambda x^0}$ in the ghost sector. Its gauge fixed norm square may now be computed to be (assume $\phi_1=\phi_2$ for simplicity)

$$\begin{aligned}
\mathrm{Tr}_{\mathrm{ph}}\left[|\phi\rangle\langle\phi|\right] &= \langle\phi|\phi\rangle_{\mathcal{K}}\\
&= 2\,\langle 0_{\mathcal{G}}|\,e^{-i\lambda x^0}\phi_1^*\hat\eta\phi_1\,|0_{\mathcal{G}}\rangle_{\mathcal{K}}\\
&= 2\,\langle 0_{\mathcal{G}}|\,\hat\eta\,|0_{\mathcal{G}}\rangle_{\mathcal{K}}\left\langle\phi_1|e^{-i\lambda x^0}\phi_1\right\rangle_{\mathcal{H}'^*}\\
&= 2\,\langle 0_{\mathcal{G}}|\,\hat\eta\,|0_{\mathcal{G}}\rangle_{\mathcal{K}}\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} e^{-i\lambda x^0}\,\langle\phi_1|\phi_1\rangle_{\mathcal{H}}\;dx^0 d\lambda\\
&= 4\pi\,\langle 0_{\mathcal{G}}|\,\hat\eta\,|0_{\mathcal{G}}\rangle_{\mathcal{K}}\,\langle\phi_1|\phi_1\rangle_{\mathcal{H}}\;.
\end{aligned} \tag{520}$$

Hence this reproduces, up to a constant factor, the ordinary scalar product $\langle\phi_1|\phi_1\rangle_{\mathcal{H}}$ on $\mathcal{H}$ as usually defined.

Because the result must not depend on the regularization, one may choose other regulators $\hat R$. The popular *standard gauge fixing fermion* may be recovered as follows:

2.121 (The standard gauge fixing fermion) Consider the hermitian metric operator

$$\hat\eta \;=\; \hat\eta^{(0)}\,e^{-\lambda x^0}\,, \tag{521}$$

and the adjoint of $\mathbf{Q}$ with respect to $\langle\cdot|\hat\eta\cdot\rangle_{\mathcal{K}}$:

$$\begin{aligned}
\mathbf{Q}^{\dagger_{\hat\eta}} &= \hat\eta^{-1}\mathbf{Q}\hat\eta\\
&= e^{\lambda x^0}\left(-\mathcal{P}\mathbf{p}+\bar{\mathcal{C}}\partial_{(\lambda)}\right)e^{-\lambda x^0}\\
&= \underbrace{-\mathcal{P}\mathbf{p}+\bar{\mathcal{C}}\partial_{(\lambda)}}_{\text{(a)}}+\underbrace{\mathcal{P}\lambda-\bar{\mathcal{C}}x^0}_{\text{(b)}}\,.
\end{aligned} \tag{522}$$

The term (b) is the so-called *standard gauge fixing fermion*. The anticommutator in (509) is:

$$\left\{\mathbf{Q}, \mathbf{Q}^{\dagger_{\hat\eta}}\right\} = \underbrace{-\mathbf{p}^2 - \partial_{(\lambda)}^2}_{\text{(a)}} + \underbrace{\lambda\mathbf{p} + x^0\partial_{(\lambda)} - \mathcal{C}\bar{\mathcal{C}} + \bar{\mathcal{P}}\mathcal{P}}_{\text{(b)}} . \tag{523}$$

Here, again, (b) denotes the term obtained usually from the standard gauge fixing fermion. The term (a) is obviously zero on physical states and has positive eigenvalues. Adapting a discussion in [228] the eigenvalues of the term (b) may be found as follows: First regard $\lambda\mathbf{p} + x^0\partial_{(\lambda)}$ and rewrite it in terms of some suitable ladder operators. For that purpose introduce the notation

$$\begin{aligned}
\hat{b}_1 &:= \mathbf{p} + x^0 \\
\bar{\hat{b}}_1 &:= -\mathbf{p} + x^0 \\
\hat{b}_2 &:= \partial_{(\lambda)} + \lambda \\
\bar{\hat{b}}_2 &:= -\partial_{(\lambda)} + \lambda .
\end{aligned} \tag{524}$$

(Here the bar on $\bar{\hat{b}}_i$ is pure notation and does not refer to any operation on $\hat{b}_i$.) One checks that

$$\left[\hat{b}_i, \bar{\hat{b}}_j\right] = 2\delta_{ij} \tag{525}$$

and that

$$\begin{aligned}
\lambda\mathbf{p} + x^0\partial_{(\lambda)} &= \frac{1}{2}\left(\bar{\hat{b}}_1\bar{\hat{b}}_2 - \hat{b}_1\hat{b}_2\right) \\
&= \frac{1}{2}\left(\bar{\hat{b}}_1 - \hat{b}_2\right)\frac{1}{2}\left(\bar{\hat{b}}_2 + \hat{b}_1\right) + \frac{1}{2}\left(\bar{\hat{b}}_1 + \hat{b}_2\right)\frac{1}{2}\left(\bar{\hat{b}}_2 - \hat{b}_1\right) .
\end{aligned} \tag{526}$$

The last line motivates the definitions

$$\begin{aligned}
\bar{\hat{a}}_1 &:= \frac{1}{2}\left(\bar{\hat{b}}_1 - \hat{b}_2\right) \\
\hat{a}_1 &:= \frac{1}{2}\left(\bar{\hat{b}}_2 + \hat{b}_1\right) \\
\bar{\hat{a}}_2 &:= \frac{1}{2}\left(\bar{\hat{b}}_1 + \hat{b}_2\right) \\
\hat{a}_1 &:= \frac{1}{2}\left(\bar{\hat{b}}_2 - \hat{b}_1\right) .
\end{aligned} \tag{527}$$

These operators satisfy

$$\left[\hat{a}_i, \bar{\hat{a}}_j\right] = \delta_{ij} \tag{528}$$

and one finally has

$$\lambda\mathbf{p} + x^0\partial_{(\lambda)} = \bar{\hat{a}}_1\hat{a}_1 + \bar{\hat{a}}_2\hat{a}_2 . \tag{529}$$

This shows that $\lambda\mathbf{p} + x^0\partial_{(\lambda)}$ has eigenvalues ≥ 0. A similar reformulation applies for the remaining ghost term:

$$-\mathcal{C}\bar{\mathcal{C}} + \bar{\mathcal{P}}\mathcal{P} = \frac{1}{\sqrt{2}}\left(\mathcal{C} + \bar{\mathcal{P}}\right)\frac{1}{\sqrt{2}}\left(-\bar{\mathcal{C}} + \mathcal{P}\right) + \frac{1}{\sqrt{2}}\left(-\mathcal{C} + \bar{\mathcal{P}}\right)\frac{1}{\sqrt{2}}\left(\bar{\mathcal{C}} + \mathcal{P}\right) . \tag{530}$$

With the definition

$$\begin{aligned}
\mathcal{D}_1 &:= \frac{1}{\sqrt{2}}\left(\mathcal{C}+\bar{\mathcal{P}}\right)\\
\bar{\mathcal{D}}_1 &:= \frac{1}{\sqrt{2}}\left(-\bar{\mathcal{C}}+\mathcal{P}\right)\\
\mathcal{D}_2 &:= \frac{1}{\sqrt{2}}\left(-\mathcal{C}+\bar{\mathcal{P}}\right)\\
\bar{\mathcal{D}}_2 &:= \frac{1}{\sqrt{2}}\left(\bar{\mathcal{C}}+\mathcal{P}\right)
\end{aligned} \tag{531}$$

one has

$$\begin{aligned}
\left\{\mathcal{D}_1,\bar{\mathcal{D}}_1\right\} &= 1\\
\left\{\mathcal{D}_2,\bar{\mathcal{D}}_2\right\} &= -1\,,
\end{aligned} \tag{532}$$

and hence the eigenvalues of (530) are $0, \pm 1$. In summary, this shows that the standard gauge fixing fermion alone is an admissible gauge fixing fermion. Setting

$$\hat{\rho}_{\mathrm{s}} := \mathcal{P}\mathbf{p} - \bar{\mathcal{C}}x^0 \tag{533}$$

one can again find the most general state annihilated by $\mathbf{Q}$ and $\hat{\rho}_{\mathrm{s}}$. It consists of a physical state in the ghost vacuum and of a non-physical state supported at $x^0 = 0 = \lambda$ in the filled ghost sector.

In the following sections we want to apply this formalism to quantum mechanical systems which contain bosonic and fermionic degrees of freedom and thus already come equipped with a graded Hilbert space. The key observation is that in such a case the requirements (497) and (498) may already be satisfied even without extending the existing Hilbert space by ghost degrees of freedom. That is, it may happen that the states of the system naturally live in a graded Krein space of states which does by itself support operators that fulfill the requirements of being "ghost" operators. One may regard all the "ghost" terminology as being due to the historical development and note that what we really only need here in order to do gauge fixing is a suitably regulated alternating trace (510). The following sections §2.3.4 (p.134) and §2.3.5 (p.140) as well as appendix §E (p.330) provide some details as to how this may be accomplished in the case of supersymmetric quantum mechanics.

The essential idea, however, is already captured in the following simple example:

Example 2.122 Consider supersymmetric quantum mechanics (or Dirac spinors) on flat 1+1 dimensional Minkowski spacetime as in 2.83 (p.81).

With the notation of 2.2 (p.16). the spacetime Clifford algebra is given by

$$\left\{\hat{\gamma}_-^a, \hat{\gamma}_-^b\right\} = -2\eta^{ab} = -2\left[\begin{array}{cc} -1 & 0\\ 0 & 1\end{array}\right]^{ab}, \tag{534}$$

where the Clifford generators $\hat{\gamma}_-^a$ are anti-hermitian (see (50), p. 22):

$$\left(\hat{\gamma}_-^a\right)^\dagger = -\hat{\gamma}_-^a\,. \tag{535}$$

According to B.18 (p.308) there is the chirality operator

$$\bar{\gamma}_- = \hat{\gamma}_-^0 \hat{\gamma}_-^1 \tag{536}$$

which satisfies

$$\begin{aligned} \bar{\gamma}^2 &= 1 \\ \bar{\gamma}^\dagger &= -\bar{\gamma} \end{aligned} \tag{537}$$

and anticommutes with all Clifford generators:

$$\left\{ \bar{\gamma}_-, \hat{\gamma}_-^a \right\} = 0 \,. \tag{538}$$

The projectors on its eigenspaces are (*cf.* B.19 (p.308))

$$\hat{h}'_\pm = \frac{1}{2} \left(1 \pm \bar{\gamma}_- \right) \tag{539}$$

and they satisfy

$$\begin{aligned} \left(\hat{h}'_\pm \right)^2 &= \hat{h}'_\pm \\ \hat{h}'_\pm \hat{h}'_\mp &= 0 \\ \left(\hat{h}'_\pm \right)^\dagger &= \hat{h}'_\mp \,. \end{aligned} \tag{540}$$

It follows that each Clifford generator swaps the eigenspaces of $\bar{\gamma}_-$:

$$\hat{\gamma}_-^a \hat{h}'_\pm = \hat{h}'_\mp \hat{\gamma}_-^a \,. \tag{541}$$

Now to the dynamics: The free, massless Dirac operator in 1+1 dimensions is

$$\mathbf{D} = \hat{\gamma}_-^0 \partial_0 + \hat{\gamma}_-^1 \partial_1 \,. \tag{542}$$

It is self-adjoint,

$$\mathbf{D}^\dagger = \mathbf{D} \,, \tag{543}$$

and maps between the two eigenspaces of $\bar{\gamma}_-$:

$$\begin{aligned} & \mathbf{D} \hat{h}'_\pm = \hat{h}'_\mp \mathbf{D} \\ \Rightarrow \quad & \mathbf{D} = \hat{h}'_- \mathbf{D} \hat{h}'_+ + \hat{h}'_+ \mathbf{D} \hat{h}'_- \,. \end{aligned} \tag{544}$$

Hence a candidate for a BRST operator is one of its nilpotent components, for instance:

$$\mathbf{Q} := \hat{h}'_- \mathbf{D} \hat{h}'_+ \,, \tag{545}$$

which, by the above relations, does satisfy

$$\begin{aligned} \mathbf{Q}^2 &= 0 \\ \mathbf{Q}^\dagger &= \mathbf{Q} \,, \end{aligned} \tag{546}$$

as it should. This way the -1 eigenspace of $\bar{\gamma}_-$ now *plays the role* of the "ghost" sector.

Next we need a hermitian metric operator. Our inner product is the Hodge inner product $\langle\cdot|\cdot\rangle$ (38) on the exterior bundle over spacetime (which coincides with configuration space here). It is indefinite due to the indefiniteness of the Minkowski metric η. Since $\hat{\gamma}_-^0\hat{\gamma}_+^0$ swaps the sign of states containing the offending temporal fermions, the expression $\langle\psi|\,\hat{\gamma}_-^0\hat{\gamma}_+^0\,|\psi\rangle$ is positive definite, and so our standard hermitian metric operator is

$$\hat{\eta}^{(0)} \;=\; \hat{\gamma}_-^0\hat{\gamma}_+^0\,. \tag{547}$$

This operator may be multiplied by any positive function and still yield a scalar product. One choice among many that will produce a finite physical trace is

$$\hat{\eta} \;=\; e^{(x^0)^2}\hat{\eta}^{(0)}\,. \tag{548}$$

So now a co-BRST operator may be defined as the $\hat{\eta}$-adjoint of $\mathbf{Q}$, which gives

$$\begin{aligned}\mathbf{Q}^{\dagger_{\hat{\eta}}} &= \hat{\eta}^{-1}\mathbf{Q}\hat{\eta}\\ &= e^{-(x^0)^2}\hat{h}'_+\left(-\hat{\gamma}_-^0\partial_0+\hat{\gamma}_+^1\partial_1\right)\hat{h}'_-e^{(x^0)^2}\,.\end{aligned} \tag{549}$$

Because $\hat{\gamma}_-^0\hat{\gamma}_+^0$ anticommutes with $\bar{\gamma}_-$, the co-BRST operator maps the "ghost" sector ($\bar{\gamma}_-=-1$) into the ghost vacuum, as it should be.

Now a physical and gauge fixed state $|\psi\rangle$ is defined by demanding

$$\mathbf{Q}\,|\psi\rangle = \mathbf{Q}^{\dagger_{\hat{\eta}}}\,|\psi\rangle = 0\,. \tag{550}$$

To find the general solution to these two equations choose any solution $|\phi\rangle$ of the Dirac equation $\mathbf{D}\,|\phi\rangle = 0$ residing in the $\bar{\gamma}_- = +1$ sector:

$$\begin{aligned}&\mathbf{D}\,|\phi\rangle = 0\\ &\hat{h}'_+\,|\phi\rangle = |\phi\rangle\ .\end{aligned} \tag{551}$$

Now, obviously, a state $|\psi\rangle$ which satisfies (550) is

$$|\psi\rangle \;=\; \frac{1}{\sqrt{2}}\left(|\phi\rangle + e^{-(x^0)^2}\hat{\gamma}_-^0\hat{\gamma}_+^0\,|\phi\rangle\right)\ . \tag{552}$$

By construction, this is a solution to the Dirac equation in the "ordinary" sector and a solution times the hermitian metric operator in the "ghost" sector:

$$\begin{aligned}\hat{h}'_+\,|\psi\rangle &= \frac{1}{\sqrt{2}}\,|\phi\rangle\\ \hat{h}'_-\,|\psi\rangle &= \frac{1}{\sqrt{2}}e^{-(x^0)^2}\hat{\gamma}_-^0\,|\phi\rangle\ .\end{aligned} \tag{553}$$

Next it is essential that due to relation $\left(\hat{h}'_+\right)^\dagger = \hat{h}'_-$ the ordinary and the ghost sector both contain only 0-norm states:

$$\begin{aligned}\langle\phi|\phi\rangle &= \left\langle\hat{h}'_+\phi|\hat{h}'_+\phi\right\rangle\\ &= \left\langle\phi|\hat{h}'_-\hat{h}'_+\phi\right\rangle\\ &= 0\end{aligned} \tag{554}$$

and similarly

$$\left\langle \hat{\gamma}^0_- \hat{\gamma}^0_+ \phi | \hat{\gamma}^0_- \hat{\gamma}^0_+ \phi \right\rangle = 0\,. \tag{555}$$

(In passing it may be noted that this corresponds to the fact that ghosts are subject to Berezin integration: A ghost creator sandwiched between the ghost vacuum gives a contribution, but the ghost vacuum alone has zero norm.)

It follows that in the product $\langle \psi | \psi \rangle$ only the cross-terms contribute, so that

$$\langle \psi | \psi \rangle \quad = \quad \langle \phi |\, e^{-(x^0)^2} \hat{\gamma}^0_- \hat{\gamma}^0_+ \,| \phi \rangle \ . \tag{556}$$

We hence recover the positive definite scalar product $\left\langle \cdot | \hat{\gamma}^0_- \hat{\gamma}^0_+ \cdot \right\rangle$ regularized by $e^{-(x^0)^2}$.

2.3.4 BRST-cohomology of operators of Dirac type

Introduction. In the present supersymmetric setting the gauge generator is a Dirac operator $\mathbf{D}$ (see 2.48 (p.55)). The understanding of its gauge equivalence classes $\mathrm{Ker}(\mathbf{D})/\mathrm{Im}(\mathbf{D})$ is facilitated by the graded nature of $\mathbf{D}$, which allows it to be decomposed as the sum of two nilpotent operators that automatically serve the purpose of BRST-operators on pseudo-Riemannian manifolds.

Literature. The following constructions is a little different from other methods, in that it does not introduce extra ghost degrees of freedom, but models these on the already existing fermionic ones (which is possible due to a certain generic redundancy). Hence there is no specific literature to refer to, except for that on BRST theory in general. But there is one rather similar construction in a different but closely related context:

By its emphasis on Dirac operators, SQM has deep connections with noncommutative geometry [262][55] (this is the general theme of [101] [102]). Just like SQM, noncommutative geometry was and essentially is restricted to Riemannian geometry. But now a generalization of noncommutative geometry to semi-Riemannian geometry was rather recently proposed in [248]. Remarkably, this generalization is based on essentially the same principle that is tried to be used here to generalize SQM from Riemannian to semi-Riemannian manifolds, namely one based on Krein spaces and scalar products derived therefrom.

Recall that on Riemannian manifolds we have the following central notion:

2.123 (Operators of Dirac type induced by involutions.) (*cf.* [109],§4.2) The restriction of a Dirac operator $\mathbf{D}$ acting on a bundle $\mathcal{B}$

$$\mathbf{D} : \mathcal{B} \to \mathcal{B}$$

to a subspace $\mathcal{A} \subset \mathcal{B}$ of $\mathcal{B}$

$$\mathbf{D}_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$$

is called an *operator of Dirac type*. Important restrictions arise from eigenspaces of involutive linear mappings

$$\iota,\ \iota^2 = 1\,,$$

with respect to which $\mathbf{D}$ is an odd operator

$$\{\mathbf{D}, \iota\} = 0\,.$$

These induce the decomposition

$$\begin{aligned} \mathbf{D} &= \mathbf{D}_{+\iota} + \mathbf{D}_{-\iota} \\ &= \mathbf{D}\frac{1}{2}(1+\iota) + \mathbf{D}\frac{1}{2}(1-\iota)\,, \end{aligned} \tag{557}$$

where

$$\mathbf{D}_{\pm\iota} : \{|\phi\rangle \in \mathcal{B} \,|\, \iota\,|\phi\rangle = \pm 1\,|\phi\rangle\} \to \{|\phi\rangle \in \mathcal{B} \,|\, \iota\,|\phi\rangle = \mp 1\,|\phi\rangle\}$$

are called *graded operators of Dirac type.*

- *Riemannian case.*

  In a Riemannian setting one usually requires the involution, ι, which induces the decomposition, to be self-adjoint

$$\iota^\dagger = \iota\,, \tag{558}$$

  so that

$$(\mathbf{D}_{\pm\iota})^\dagger = \mathbf{D}_{\mp\iota}$$

  and the corresponding *Laplacian* reads in terms of $\mathbf{D}_\pm$

$$\begin{aligned}\mathbf{D}^2 &= \mathbf{D}_{+\iota}\mathbf{D}_{-\iota} + \mathbf{D}_{-\iota}\mathbf{D}_{+\iota}\\ &= \{\mathbf{D}_{+\iota}, \mathbf{D}_{-\iota}\}\\ &= \left\{\mathbf{D}_{\pm\iota}, (\mathbf{D}_{\pm\iota})^\dagger\right\}\\ &= \left(\mathbf{D}_{\pm\iota} + (\mathbf{D}_{\pm\iota})^\dagger\right)^2\,.\end{aligned}$$

- *Pseudo-Riemannian case.* In the Pseudo-Riemannian case the operator $\left(\mathbf{D}_{\pm\iota} + (\mathbf{D}_{\pm\iota})^\dagger\right)^2$ will not enjoy the important positivity property of a Riemannian Laplace operator, since the inner product $\langle\cdot|\cdot\rangle$ with respect to which the adjoint $(\cdot)^\dagger$ is taken is not positive definite. In order to preserve as much of the Riemannian theory as possible one can instead define a positive definite inner product $\langle\cdot|\cdot\rangle_{\hat\eta} := \langle\cdot|\hat\eta\,\cdot\rangle$ and consider the adjoint $(\cdot)^{\dagger_{\hat\eta}}$ with respect to this scalar product. This means that one needs $\hat\eta$-self-adjoint involutions

$$\iota^{\dagger_{\hat\eta}} = \iota \tag{559}$$

  instead of (558). Note that, since

$$A^{\dagger_{\hat\eta}} = \hat\eta^{-1}A^\dagger\hat\eta\,,$$

  this is in particular fullfilled if

$$\iota^\dagger = \iota,\ \ [\iota,\hat\eta] = 0$$

  or

$$\iota^\dagger = -\iota,\ \ \{\iota,\hat\eta\} = 0\,.$$

  The pseudo-Riemannian substitute for a positive Laplace operator is then the positive operator

$$\left\{\mathbf{D}_{\pm\iota}, (\mathbf{D}_{\pm\iota})^{\dagger_{\hat\eta}}\right\}\,.$$

As shall be shown below, the naturally motivated search for a positive analog of the Laplace operator in the pseudo-Riemannian setting automatically yields all the ingredients of BRST cohomology theory.

The following argument establishes the fact that in the presence of a certain symmetry all the information about the kernel of $\mathbf{D}$ is contained in the kernel of its graded nilpotent compotents $\mathbf{D}_{\pm\iota}$. This observation is the key to using $\mathbf{D}_{\pm\iota}$ as BRST operators.

2.124 (The cohomology of graded components of D)
Let $\mathbf{D}$ by any Dirac operator that can be written as the sum of two graded nilpotent operators $\mathbf{D}_\pm$. Assume the presence of an involution ι' which relates $\mathbf{D}_\pm$ via

$$\begin{aligned} (\iota')^2 &= 1 \\ \mathbf{D} &= \mathbf{D}_+ + \mathbf{D}_- \\ &= \mathbf{D}_+ \pm \iota' \mathbf{D}_+ \iota' \,, \end{aligned}$$

which implies that

$$[\iota', \mathbf{D}]_\pm = 0 \,,$$

(where the last line is supposed to say that *either* the commutator *or* the anti-commutator vanishes).

Now consider the following argument: Since

$$[\mathbf{D}, \iota']_\pm = 0 \,, \tag{560}$$

the kernel of $\mathbf{D}$ is the direct sum of spaces of eigenstates of ι':[37]

$$\mathrm{Ker}\,(\mathbf{D}) = \mathrm{Ker}\,(\mathbf{D})_+ \oplus \mathrm{Ker}\,(\mathbf{D})_- \,, \tag{562}$$

where

$$\mathrm{Ker}\,(\mathbf{D})_\pm := \left\{ |\phi_\pm\rangle \in \mathrm{Ker}\,(\mathbf{D}) \,|\iota' \,|\phi_\pm\rangle = \pm\,|\phi_\pm\rangle \right\} \,. \tag{563}$$

But on eigenstates of ι' the kernels of $\mathbf{D}_{+\iota}$ and $\mathbf{D}_{-\iota}$ coincide,

$$\mathbf{D}_+ \,|\phi_\pm\rangle = 0 \Leftrightarrow \mathbf{D}_- \,|\phi_\pm\rangle = 0 \,, \tag{564}$$

since

$$\begin{aligned} \mathbf{D}_+ \,|\phi_\pm\rangle &= 0 \\ \Leftrightarrow \iota' \mathbf{D}_+ \,|\phi_\pm\rangle &= 0 \\ \Leftrightarrow \mathbf{D}_- \iota' \,|\phi_\pm\rangle &= 0 \\ \Leftrightarrow \mathbf{D}_- \,|\phi_\pm\rangle &= 0 \,. \end{aligned}$$

Hence:

$$\mathbf{D}\,|\phi\rangle = 0 \Leftrightarrow \mathbf{D}\,|\phi_\pm\rangle = 0 \Leftrightarrow \mathbf{D}_+ \,|\phi_\pm\rangle = 0 \Leftrightarrow \mathbf{D}_- \,|\phi_\pm\rangle = 0 \,.$$

It follows that the space $\mathrm{Ker}\,(\mathbf{D})\,/\mathrm{Im}(\mathbf{D})$ of *gauge equivalence classes* of $\mathbf{D}$ is isomorphic to the cohomologies of its two graded and nilpotent components

$$\mathrm{Ker}\,(\mathbf{D})\,/\mathrm{Im}(\mathbf{D}) \simeq \mathrm{H}_c(\mathbf{D}_{+\iota}) \simeq \mathrm{H}_c(\mathbf{D}_{-\iota}) \,. \tag{565}$$

[37]This is trivial when $[\mathbf{D}, \iota'] = 0$, since then $\mathbf{D}$ and ι' are simultaneously diagonalizable. But when $\{\mathbf{D}, \iota'\} = 0$ one still has

$$\begin{aligned} \mathbf{D}\,|\phi\rangle &= 0 \\ \Rightarrow \frac{1}{2}(1 \mp \iota')\mathbf{D}\,|\phi\rangle &= 0 \\ \Rightarrow \mathbf{D}\frac{1}{2}(1 \pm \iota')\,|\phi\rangle &= 0 \\ \Leftrightarrow \mathbf{D}\,|\phi_\pm\rangle &= 0 \,. \end{aligned} \tag{561}$$

Example 2.125 The following gives two examples of this constructions. The first is for a Riemannian manifold (and reproduces well known results), the second for a pseudo-Riemannian manifold.

1. The standard example is the deRahm cohomology, where

$$\begin{aligned} \mathbf{D} &= \mathbf{d} + \mathbf{d}^\dagger \\ &= \mathbf{d} + (-1)^D \bar{\gamma}_+ \mathbf{d} \bar{\gamma}_+ \end{aligned} \tag{566}$$

(*cf.* (1228), p. 308) on a *compact Riemannian manifold without boundary*. In this special case the cohomology of $\mathbf{D}$ is even *equal* to its kernel:

$$\mathrm{Ker}\left(\mathbf{d} + \mathbf{d}^\dagger\right)/\mathrm{Im}\left(\mathbf{d} + \mathbf{d}^\dagger\right) = \mathrm{Ker}\left(\mathbf{d} + \mathbf{d}^\dagger\right) .$$

But this kernel is just the space of harmonic forms, i.e. the cohomology of $\mathbf{d}$ and $\mathbf{d}^\dagger$:

$$\mathrm{Ker}\left(\mathbf{d} + \mathbf{d}^\dagger\right) \simeq \mathrm{H}_c(\mathbf{d}) \simeq \mathrm{H}_c\left(\mathbf{d}^\dagger\right) ,$$

so that

$$\mathrm{Ker}\left(\mathbf{d} + \mathbf{d}^\dagger\right)/\mathrm{Im}\left(\mathbf{d} + \mathbf{d}^\dagger\right) \simeq \mathrm{H}_c(\mathbf{d}) \simeq \mathrm{H}_c\left(\mathbf{d}^\dagger\right) .$$

2. Now consider the standard exterior Dirac operator $\mathbf{D} = \mathbf{d} + \mathbf{d}^\dagger$ on an *odd* dimensional *pseudo-Riemannian* manifold. One has (see §B (p.297))

$$\begin{aligned} \{\bar{\gamma}_-, \bar{\gamma}_+\} &= 0 \\ [\mathbf{D}, \bar{\gamma}_-] &= 0 \\ \{\mathbf{D}, \bar{\gamma}_+\} &= 0 . \end{aligned} \tag{567}$$

Hence one can set

$$\begin{aligned} \iota &:= \bar{\gamma}_+ \\ \iota' &:= \bar{\gamma}_- . \end{aligned} \tag{568}$$

The Dirac operator is decomposed into the nilpotent components

$$\mathbf{D} = \mathbf{D}\frac{1}{2}\left(1 + \bar{\gamma}_+\right) + \mathbf{D}\frac{1}{2}\left(1 - \bar{\gamma}_+\right) . \tag{569}$$

Since $\bar{\gamma}_-$ commutes with $\mathbf{D}$ one can assume without restriction of generality that $|\phi\rangle$ is an eigenstate of ι', $|\phi\rangle = \frac{1}{2}\left(1 \pm \bar{\gamma}_-\right)|\phi\rangle$. But then

$$\begin{aligned} \mathbf{D}\frac{1}{2}\left(1 + \bar{\gamma}_+\right)|\phi\rangle &= \pm\mathbf{D}\frac{1}{2}\left(1 + \bar{\gamma}_+\right)\bar{\gamma}_-|\phi\rangle \\ &= \pm\bar{\gamma}_-\mathbf{D}\frac{1}{2}\left(1 - \bar{\gamma}_+\right)|\phi\rangle , \end{aligned} \tag{570}$$

so that

$$\begin{aligned} \mathbf{D}|\phi\rangle &= 0 \\ \Leftrightarrow \mathbf{D}\frac{1}{2}\left(1 + \bar{\gamma}_+\right)|\phi\rangle &= 0 \\ \Leftrightarrow \mathbf{D}\frac{1}{2}\left(1 - \bar{\gamma}_+\right)|\phi\rangle &= 0 . \end{aligned} \tag{571}$$

Without loss of generality assume $[\mathbf{D}, \iota'] = 0$ in the following. Consider an element $|\phi_\pm\rangle$ in the kernel of $\mathbf{D}$ of the form

$$|\phi_\pm\rangle = |\alpha_\pm\rangle + \mathbf{D}\,|\beta_\pm\rangle \;.$$

It can be rewritten as

$$\begin{aligned} |\phi_\pm\rangle &= |\alpha_\pm\rangle + (\mathbf{D}_+ + \mathbf{D}_-)\,|\beta_\pm\rangle \\ &= |\alpha_\pm\rangle + \mathbf{D}_+\,|\beta_\pm\rangle + \iota'\mathbf{D}_+\iota'\,|\beta_\pm\rangle \\ &= |\alpha_\pm\rangle + (1 \pm \iota')\,\mathbf{D}_+\,|\beta_\pm\rangle \;. \end{aligned}$$

So one finds a one-to-one relationship between these elements and the elements in the cohomology of $\mathbf{D}_+$:

$$|\alpha_\pm\rangle + \mathbf{D}_+\,|\beta_\pm\rangle \;.$$

Hence a graded component $\mathbf{D}_{+\iota}$ of $\mathbf{D}$, as in the above theorem, can be used in place of $\mathbf{D}$ itself to identify gauge equivalence classes of physical states. But since it is also nilpotent and (essentially) self-adjoint, $\mathbf{D}_{+\iota}$ qualifies as a BRST operator according to equations (455)-(457), p. 116.

All that needs to be added in order to get a full BRST operator of the form (453) from $\mathbf{D}_+$ is a term of the form $\bar{\mathcal{C}}\partial_{(\lambda)}$:

2.126 (BRST operator from graded nilpotent components of D) *Given involutions ι, ι' satisfying*

$$\begin{aligned} \{\iota, \iota'\} &= 0 \\ \iota'\mathbf{D} &\sim \mathbf{D}\iota' \end{aligned} \tag{572}$$

the operator

$$\mathbf{Q} := \mathbf{D}\,\frac{1}{2}\,(1+\iota) + \bar{\mathcal{C}}\partial_{(\lambda)} \tag{573}$$

will be a BRST operator *for the SQM gauge theory with gauge generator* $\mathbf{D}$ *if*

$$\left\{\mathbf{D}\,\frac{1}{2}\,(1+\iota), \bar{\mathcal{P}}\right\} = 0\,, \tag{574}$$

because then

$$\mathbf{Q}^2 = 0\,. \tag{575}$$

2.127 (Literature) Some comments will relate the BRST method presented in this section with the relevant literature:

Marnelius has worked extensively on the BRST formalism, see [21] [187] [186] [184] [183] [182] [181] [185].

One of the techniques applied in these papers (*cf.* [187], eq. (2.2)) is to decompose the BRST operator $\mathbf{Q}$ as the sum of two mutually adjoint nilpotent operators

$$\begin{aligned} \mathbf{Q} &= \delta + \delta^\dagger \\ \delta^2 &= 0 \\ \left\{\delta, \delta^\dagger\right\} &= 0\,. \end{aligned} \tag{576}$$

It is noteworthy that such a decomposition is very natural with respect to the graded Dirac operators considered here. According to 2.36 (p.46) every such Dirac operator $\mathbf{D}$ in an $(N = 2)$ theory can be written as

$$\mathbf{D} = \mathbf{d} + \mathbf{d}^\dagger \,.$$

Now, if a graded, nilpotent component

$$\begin{aligned} \mathbf{Q} &:= \mathbf{D}\,\frac{1}{2}\,(1+\iota) \\ &= \frac{1}{2}\,(1-\iota)\,\mathbf{D}\,\frac{1}{2}\,(1+\iota) \end{aligned}$$

of such a Dirac operator is used as a BRST operator, with involution ι

$$\begin{aligned} \iota^2 &= 1 \\ \iota^\dagger &= -\iota \\ \{\mathbf{D},\iota\} &= 0\,, \end{aligned}$$

as discussed in (572), then it is natural to write

$$\begin{aligned} \mathbf{Q} &= \frac{1}{2}\,(1-\iota)\,\mathbf{D}\,\frac{1}{2}\,(1+\iota) \\ &= \frac{1}{2}\,(1-\iota)\,\left(\mathbf{d}+\mathbf{d}^\dagger\right)\,\frac{1}{2}\,(1+\iota) \\ &= \frac{1}{2}\,(1-\iota)\,\mathbf{d}\,\frac{1}{2}\,(1+\iota) + \frac{1}{2}\,(1-\iota)\,\mathbf{d}^\dagger\,\frac{1}{2}\,(1+\iota) \\ &= \delta + \delta^\dagger\,, \end{aligned} \tag{577}$$

where

$$\delta := \frac{1}{2}\,(1-\iota)\,\mathbf{d}\,\frac{1}{2}\,(1+\iota)$$

automatically satisfies conditions (576).

With respect to this δ one could now discuss, following [187], unitary transformations $U = e^A$ of the type (213) in 2.42 (p.50):

$$\delta \quad \to \quad U^{-1}\,\delta\,U\,.$$

2.3.5 Gauge fixed expectation values

Introduction. In order for integration to be well defined, the inner products considered so far (e.g. $\langle\cdot|\cdot\rangle$,$\langle\cdot|\cdot\rangle_{\hat\eta}$) were all defined on square integrable sections of the exterior bundle. But the physical states that are ultimately of interest, i.e. the solutions $|\phi\rangle$ to $\mathbf{D}\,|\phi\rangle = 0$ will in general not belong to this restricted function space, since they will not vanish along the 'timelike' direction. Hence it is important to define and construct a physically sensible method that yields a *finite* trace over projectors on physical states and thus defines a well defined *physical scalar product* $\langle\cdot|\cdot\rangle_{\mathrm{phys}}$.

2.128 (Physical gauge fixed expectation value) *Every tuple $(\hat\eta, \iota)$ of a hermitian metric operator $\hat\eta$ (which defines the fixed gauge by way of 2.112 (p.117) and 2.113 (p.118)) and an involution ι (with respect to which $\mathbf{Q}$ is odd graded) induces a notion of* expectation value $\langle\iota A\rangle_{\hat\eta}$ *of a gauge invariant operator*

$$A,\ [\mathbf{Q}, A] = 0$$

defined by

$$\langle\iota A\rangle_{\hat\eta} \quad := \quad \mathrm{Tr}\Big(e^{\left\{\mathbf{Q},\mathbf{Q}^{\dagger_{\hat\eta}}\right\}}\iota A\Big)_{\mathcal{S}} \,. \tag{578}$$

It follows from 2.28 (p.41) that this is equal to the trace over 'physical states' in the cohomology of $\mathbf{Q}$, represented by $\mathbf{Q}$-harmonic states with respect to the hermitian metric operator $\hat\eta$:

$$\langle\iota A\rangle_{\hat\eta} \quad = \quad \sum_{\left\{\mathbf{Q},\mathbf{Q}^{\dagger_{\hat\eta}}\right\}|\phi\rangle=0} \langle\phi|\iota A\,\phi\rangle \ . \tag{579}$$

The expectation value of an operator should be independent of the gauge $\hat\eta$ it is evaluated in. Indeed:

Theorem: *The gauge fixed expectation value of gauge invariant operators is independent of the fixed gauge, i.e.*

$$\langle\iota A\rangle_{\hat\eta_1} \quad = \quad \langle\iota A\rangle_{\hat\eta_2} \tag{580}$$

for all admissible ι, A, and $\hat\eta_{1,2}$.
Proof: The proof is essentially based on the argument that

$$\mathbf{Q}\psi = 0 \Rightarrow \langle\phi + \mathbf{Q}\phi'|\psi\rangle \quad = \quad \langle\phi|\psi\rangle + \langle\phi'|\underbrace{\mathbf{Q}\psi}_{=0}\rangle = \langle\phi|\psi\rangle \ , \tag{581}$$

namely:

$$\begin{aligned}
\langle\iota A\rangle_{\hat\eta_1} \quad &= \quad \sum_{\left\{\mathbf{Q},\mathbf{Q}^{\dagger_{\hat\eta_1}}\right\}|\phi\rangle=0} \langle\phi|\iota A\,\phi\rangle \\
&= \quad \sum_{\left\{\mathbf{Q},\mathbf{Q}^{\dagger_{\hat\eta_2}}\right\}|\psi\rangle=0} \langle(\psi + \mathbf{Q}\psi')|\iota A\ (\psi + \mathbf{Q}\psi')\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\left\{\mathbf{Q},\mathbf{Q}^{\dagger_{\hat\eta_2}}\right\}|\psi\rangle=0} \left(\langle\psi|\iota A\,\psi\rangle - \left\langle \psi|\iota A\,\underbrace{\mathbf{Q}\,(\psi+\mathbf{Q}\psi')}_{=0} \right\rangle - \left\langle \underbrace{\mathbf{Q}\,(\psi+\mathbf{Q}\psi')}_{=0}|\iota A\,\psi \right\rangle \right) \\
&= \sum_{\left\{\mathbf{Q},\mathbf{Q}^{\dagger_{\hat\eta_2}}\right\}|\psi\rangle=0} \langle\psi|\iota A\,\psi\rangle \\
&= \langle\iota A\rangle_{\hat\eta_2}
\end{aligned}
\tag{582}
$$

□

Note: By 2.28 (p.41) one can equivalently write

$$
\langle\iota A\rangle_{\hat\eta} \;=\; \mathrm{Tr}\Big(\mathrm{P}_{(\mathbf{Q}=0)}\,\iota A\,\mathrm{P}_{(\mathbf{Q}^{\dagger_{\hat\eta}}=0)}\Big) \tag{583}
$$

where $P_{(\cdot)}$ are projectors on the subspaces indicated by their arguments. Also note that $\mathrm{P}_{(\mathbf{Q}=0)}$ can essentially be expressed by the usual action functional while $\mathrm{P}_{(\mathbf{Q}^{\dagger_{\hat\eta}}=0)}$ can be expressed by the Fadeev-Popov ghost action.

Now it is straightforward to define a gauge fixed scalar product on physical states. By 2.80 (p.78) a conserved local probability density is given by

$$
P^0 = \left\langle\phi|\hat\gamma^0_-\hat\gamma^0_+\,\phi\right\rangle_{\mathrm{loc}}.
$$

This implies that a physical scalar product of two states $|\phi\rangle$, $|\psi\rangle$ is obtained by taking the gauged fixed trace over the projector

$$
|\phi\rangle\,\langle\psi|\,\hat\gamma^0_-\hat\gamma^0_+\hat\gamma^{(\lambda)}_-\hat\gamma^{(\lambda)}_+\,.
$$

With the gauge being fixed by the standard hermitian metric operator (*cf.* §2.3.1 (p.107))

$$
\begin{aligned}
\hat\eta &= \hat\eta^{(0)}e^{\lambda X^0} \\
&= = \hat\gamma^0_-\hat\gamma^0_+\hat\gamma^{(\lambda)}_-\hat\gamma^{(\lambda)}_+e^{\lambda X^0}
\end{aligned}
$$

this leads to the following definition:

2.129 (Gauge fixed scalar product) The gauge fixed physical scalar product

$$
\langle\phi|\psi\rangle_{\mathrm{phys}}
$$

is given by

$$
\langle\phi|\psi\rangle_{\mathrm{phys}} \;:=\; \mathrm{Tr}\Big(P_{(\mathbf{Q}=0)}\;|\psi\rangle\,\langle\phi|\,\hat\eta^{(0)}\,P_{(\mathbf{Q}^{\dagger_{\hat\eta}}=0)}\Big)\,. \tag{584}
$$

2.130 (Effective gauge condition) Recall that (see (477), p. 121 and

$$
\begin{aligned}
\mathbf{Q}^{\dagger_{\hat\eta}} &= \hat\eta^{-1}\mathbf{Q}\hat\eta \\
&= \hat\eta^{-1}\left(\mathcal{C}\mathbf{p}+\bar{\mathcal{P}}\partial_{(\lambda)}\right)\hat\eta \\
&= e^{-\lambda X^0}\hat\eta^{(0)}\left(\mathcal{C}\mathbf{p}+\bar{\mathcal{P}}\partial_{(\lambda)}\right)\hat\eta^{(0)}e^{\lambda X^0} \\
&= e^{-\lambda X^0}\left(\mathcal{P}\mathbf{p}+\bar{\mathcal{C}}\partial_{(\lambda)}\right)e^{\lambda X^0} \\
&= \left(\mathcal{P}\mathbf{p}+\bar{\mathcal{C}}\partial_{(\lambda)}\right)+\mathcal{P}\lambda\left[\mathbf{p},X^0\right]+\bar{\mathcal{C}}X^0 \\
&= \hat\eta^{(0)}\mathbf{Q}\hat\eta^{(0)}+\mathcal{P}\lambda\left[\mathbf{p},X^0\right]+\bar{\mathcal{C}}X^0\,.
\end{aligned}
\tag{585}
$$

Hence if $|\phi\rangle$ in (584) is physical, $\mathbf{Q}\,|\phi\rangle = 0$, the only remaining gauge condition is

$$\left(\mathcal{P}\lambda\left[\mathbf{p}, X^0\right] + \bar{\mathcal{C}}X^0\right)|\phi\rangle \quad \stackrel{!}{=} \quad 0\,, \tag{586}$$

where the operator on the left hand side is essentially the standard ‘gauge fixing fermion’, see point 475 (p.120) of 2.115 (p.118).

Hence we have the following general result:

2.131 (Physical scalar product) The gauge fixed scalar product over physical states is computed by integrating $\left\langle\phi|\hat{\eta}^{(0)}\psi\right\rangle_{\mathrm{loc}}$ over the $x^0 = 0$ hypersurface.

3 Supersymmetric fields and strings in Q-representation

3.1 Supersymmetric scalar and Dirac fields

Outline. The free scalar field with spatially periodic boundary conditions may be quantized in the Schrödinger representation[38] (*cf.* e.g. [169][170]) using normal coordinates. In this representation the action describes point particle mechanics in infinite dimensional configuration space $\mathcal{M}^{(\mathrm{conf})}$. The time evolution is generated by a Hamiltonian which is a generalized Laplace operator on $\mathcal{M}^{(\mathrm{conf})}$ (*cf.* 2.48 (p.55)). Hence this representation of the free bosonic field theory lends itself to a supersymmetric extension following the generalized Dirac square-root process, discussed in §2.2.1 (p.55), and, in particular, the Witten model of supersymmetric quantum mechanics (*cf.* 2.2.2 (p.61)). Such an extension amounts to extending the Hilbert space of states, $L_2\big(\mathcal{M}^{(\mathrm{conf})}\big)$, to a *super Hilbert space* $\Gamma\big(\Lambda\big(\mathcal{M}^{(\mathrm{conf})}\big)\big)$ (*cf.* [233] [160]) of square integrable section of the exterior bundle of $\mathcal{M}^{(\mathrm{conf})}$.
Most of the conceptual features of the above scheme carry over to spin-2 fields, i.e. gravity (see §4.3.1 (p.193) and 4.24 (p.203), 4.25 (p.205) in particular). The main differences being that the latter has constrained dynamics instead of ordinary time evolution and that instead of a Hilbert space of states there is merely a Krein space which, in order to be promoted to a proper Hilbert space, requires a notion of gauge fixing (*cf.* §2.3 (p.106)).

The action of the free complex scalar field of mass m is

$$S = \frac{1}{2}\int\limits_M \left(\eta^{ab}\partial_a\phi^*\partial_b\phi - m\phi^*\phi\right)\, d^4x\,. \tag{587}$$

Consider the locally Minkowkian spacetime

$$M = \Sigma \otimes \mathbb{R}\,,$$

where

$$\Sigma = T^3 = \mathbb{R}^3/\mathbb{Z}^3$$

is flat Euclidean space with periodic boundary conditions, and $\mathbb{R}$ is the time axis. On Σ the Fourier modes

$$B_n(x) \quad := \quad N e^{i2\pi n_j x^j}, \quad n = (n_1, n_2, n_3) \in \mathbb{Z}^3 \tag{588}$$

(with N the normalization constant) constitute a complete set of orthonormal modes into which an arbitrary complex field ϕ may be expanded as

$$\phi = \sum_{n\in\mathbb{Z}^3} \left(b^{n,1}(t) + i b^{n,2}(t)\right) B_n(x)\;, \tag{589}$$

with real, time dependent amplitudes $b^{n,r}(t)$. Inserting this ansatz into the above action and integrating over Σ yields the ("dimensionally reduced") action:

$$S \quad = \quad \int_{\mathbb{R}} L\, dt$$

[38] Also known as the Q-representation or Itô-Segal-Wiener or real wave representation. See [11] for a rigorous treatment and [119] for an elementary introduction.

$$= \int_{\mathbb{R}} \sum_{\substack{n\in\mathbb{Z}^3 \\ r\in\{1,2\}}} \left(\frac{1}{2}\left(\dot{b}^{n,r}(t)\right)^2 - \frac{1}{2}\left(4\pi^2|n|^2+m^2\right)\left(b^{n,r}(t)\right)^2\right) dt\,. \quad (590)$$

The canonical momenta are

$$\begin{aligned} p_{n,r} &:= \frac{\partial L}{\partial \dot{b}^{n,r}} \\ &= \dot{b}^{n,r} \end{aligned} \quad (591)$$

and hence the Hamiltonian reads

$$H = \sum_{\substack{n\in\mathbb{Z}^3 \\ r\in\{1,2\}}} \left(\frac{1}{2}\left(p_{n,r}(t)\right)^2 + \frac{1}{2}\left(4\pi^2|n|^2+m^2\right)\left(b^{n,r}(t)\right)^2\right). \quad (592)$$

This is the Hamiltonian of a mechanical system with countable degrees of freedom.

The configuration space $\mathcal{M}^{(\mathrm{conf})}$ of this system is coordinatized by the amplitudes $b_{n,r}$. One may view H as the Hamiltonian of a point particle propagating on flat Euclidean $\mathcal{M}^{(\mathrm{conf})}$. Canonical quantization of this system is straightforward: Denote the trivial metric on $\mathcal{M}^{(\mathrm{conf})}$ by

$$\begin{aligned} G_{(n,r)(n',r')} &= 2\,\mathrm{diag}(1,1,\ldots) \\ \Leftrightarrow G^{(n,r)(n',r')} &= \frac{1}{2}\,\mathrm{diag}(1,1,\ldots)\,. \end{aligned} \quad (593)$$

The quantum Hamiltonian then reads

$$\hat{H} = -\hbar^2 G^{(n,r)(n',r')}\partial_{b^{n,r}}\partial_{b^{n',r'}} + V\,, \quad (594)$$

where the potential is

$$V = \sum_{\substack{n\in\mathbb{Z}^3 \\ r\in\{1,2\}}} \frac{1}{2}\left(4\pi^2|n|^2+m^2\right)\left(b^{n,r}\right)^2\,. \quad (595)$$

As usual, this can be regarded as describing a (countable infinite) collection of uncoupled harmonic oscillators with

$$\begin{aligned} \omega_{n,r} := \omega_n &:= \sqrt{4\pi|n|^2+m^2} \\ E_{n,r} := E_n &:= \hbar\omega_n \\ E^{(0)}_{n,r} := E^{(0)}_n &:= \frac{1}{2}E_n\,, \end{aligned} \quad (596)$$

where $\omega_{n,r}$ is the frequency, $E_{n,r}$ the energy quantum and $E^{(0)}_{n,r}$ is the ground state energy of the oscillator associated with the field mode indexed by (n,r). Of course, the total ground state energy

$$E^{(0)} := \frac{1}{2}\sum_{n,r} E_n \quad (597)$$

of $\hat{H}$ diverges. This is remedied by extending the system supersymmetrically, as will be done now: Let us first neglect the generators of spatial translations and concentrate on finding a “square root” of the time translation generator, the Hamiltonian. Since $\hat{H}$ has the form of a generalized Laplace operator on $\mathcal{M}^{(\mathrm{conf})}$ (*cf.* definition 2.48 (p.55)) one may look for generalized Dirac operators $\mathbf{D}$ associated with it. $\hat{H}$ has standard form, so that one is led to the Witten-Dirac operator (*cf.* definition 2.2.2 (p.61)). Its existence is guaranteed if one can (locally) find a superpotential $W = W(b^{n,r})$ satisfying

$$G^{(n,r)(n',r')}\,(\partial_{b^{n,r}}W)\,(\partial_{b^{n',r'}}W) = V\,. \tag{598}$$

In the present simple case W is globally defined by (*cf.* 2.63 (p.61)):

$$\begin{aligned} W &= \sum_{\substack{n\in\mathbf{Z}^3\\ r\in\{1,2\}}} \frac{1}{2}\sqrt{4\pi^2|n|^2+m^2}\;\left(b^{n,r}\right)^2 \\ &= \sum_{\substack{n\in\mathbf{Z}^3\\ r\in\{1,2\}}} \frac{1}{2}\omega_n\left(b^{n,r}\right)^2\,. \end{aligned} \tag{599}$$

Hence, formally following [275] [274] [53] (*cf.* §2.2.1 (p.55)), the system may be rendered supersymmetric by replacing the bosonic configuration space $\mathcal{M}^{(\mathrm{conf})}$ by the associated superspace

$$\Lambda\left(\mathcal{M}^{(\mathrm{conf})}\right)\,,$$

i.e. the exterior bundle (the bundle of differential forms) over $\mathcal{M}^{(\mathrm{conf})}$. The bosonic degrees of freedom, namely the mode amplitudes $b^{n,r}$ that act as coordinates on $\mathcal{M}^{(\mathrm{conf})}$, are then accompanied by fermionic degrees of freedom represented by the Grassmannian differential forms $\mathbf{d}b^{n,r}$ over $\mathcal{M}^{(\mathrm{conf})}$. More precisely, after quantization one has the bosonic multiplication and differentiation operators $\hat{b}^{n,r}, \partial_{b^{n,r}}$ satisfying

$$\left[\partial_{b^{n,r}}, \hat{b}^{n',r'}\right] = \delta_n^{n'}\delta_r^{r'}\,,$$

as well as the fermionic operators $\hat{c}^{\dagger r,s}$, $\hat{c}_{r,s}$ that create and annihilate differential forms by means of the outer product (wedge product) and the inner product (contraction), and which satisfy

$$\left\{\hat{c}_{n,r}, \hat{c}^{\dagger n',r'}\right\} = \delta_n^{n'}\delta_r^{r'}\,.$$

All other supercommutators between these operators vanish. (See §2.1.1 (p.15) for conventions and notations with respect to differential geometry in terms of exterior algebra. In particular see the brief introduction 2.2 (p.16).)

One can now construct the two nilpotent supercharges

$$\hat{\mathbf{Q}}_i : \Gamma\left(\Lambda\left(\mathcal{M}^{(\mathrm{conf})}\right)\right) \to \Gamma\left(\Lambda\left(\mathcal{M}^{(\mathrm{conf})}\right)\right) \tag{600}$$

($\Gamma(\mathcal{B})$ denotes the space of square integrable sections of the bundle $\mathcal{B}$) given by the deformed exterior derivatives

$$\hat{\mathbf{Q}}_1 \;:=\; e^{-W/\hbar}\,\hbar\mathbf{d}_{\mathcal{M}^{(\mathrm{conf})}}\,e^{W/\hbar}$$

$$
\begin{aligned}
&= \hbar \mathbf{d}_{\mathcal{M}^{(\mathrm{conf})}} + [\mathbf{d}_{\mathcal{M}^{(\mathrm{conf})}}, W] \\
&= \hat{\mathrm{c}}^{\dagger n,r} \left(\hbar \partial_{b_{n,r}} + \frac{\partial W}{\partial b_{n,r}} \right) \\
\hat{\mathbf{Q}}_2 &:= e^{W/\hbar}\, \hbar \mathbf{d}^{\dagger}{}_{\mathcal{M}^{(\mathrm{conf})}}\, e^{-W/\hbar} \\
&= \hbar \mathbf{d}^{\dagger}{}_{\mathcal{M}^{(\mathrm{conf})}} + [\mathbf{d}_{\mathcal{M}^{(\mathrm{conf})}}, -W] \\
&= -\hat{\mathrm{c}}^{n,r} \left(\hbar \partial_{b_{n,r}} - \frac{\partial W}{\partial b_{n,r}} \right) ,
\end{aligned}
\tag{601}
$$

as well their anticommutator

$$
\begin{aligned}
\hat{\mathbf{H}} &:= \left\{ \hat{\mathbf{Q}}_1, \hat{\mathbf{Q}}_2 \right\} \\
&= -\hbar^2 \left\{ \mathbf{d}, \mathbf{d}^\dagger \right\} + \left\{ \hat{\mathrm{c}}^{\dagger n,r}, \hat{\mathrm{c}}^{n',r'} \right\} \frac{\partial W}{\partial b^{n,r}} \frac{\partial W}{\partial b^{n',r'}} + \hbar \left[\hat{\mathrm{c}}^{\dagger n,r}, \hat{\mathrm{c}}^{n',r'} \right] \frac{\partial^2 W}{\partial_{b^{n,r}} \partial_{b^{n',r'}}} \\
&= -\hbar^2 G^{(n,r)(n',r')} \partial_{b^{n,r}} \partial_{b^{n',r'}} + V + \sum_{n,r} \hbar \omega_n \left(2 \hat{\mathrm{c}}^{\dagger n,r} \hat{\mathrm{c}}^{n,r} - \frac{1}{2} \right) \\
&= \hat{H} + \sum_{n,r} E_n \left(\hat{\mathrm{c}}^{\dagger n,r} \hat{\mathrm{c}}_{n,r} - \frac{1}{2} \right) ,
\end{aligned}
\tag{602}
$$

which defines the supersymmetric extension of $\hat{H}$ to $\Lambda\big(\mathcal{M}^{(\mathrm{conf})}\big)$. (See §2.2 (p.54), §2.2.1 (p.55) for details on this method of making supersymmetric extensions on the level of quantum operators.) By construction, the so defined Hamiltonian operator $\hat{\mathbf{H}}$ is supersymmetric:

$$
\left[\hat{\mathbf{H}}, \hat{\mathbf{Q}}_i \right] = 0 .
$$

This has been achieved by adding to the original bosonic Hamiltonian $\hat{H}$ the term

$$
\begin{aligned}
\hat{H}_{\mathrm{f}} &:= \sum_{n,r} E_n \left(\hat{\mathrm{c}}^{\dagger n,r} \hat{\mathrm{c}}_{n,r} - \frac{1}{2} \right) \\
&= \sum_{n,r} E_n \hat{N}_{n,r} - \underbrace{\sum_{n,r} \frac{1}{2} E_n}_{=E^{(0)}} .
\end{aligned}
\tag{603}
$$

This term describes a (countable infinite) collection of uncoupled systems that are, in the context of supersymmetric quantum mechanics, sometimes called "Fermi oscillators" (e.g. [146]). The number operator $\hat{N}_{n,r}$ has eigenvalue 1 on states that are proportional to the Grassmann element $\mathbf{d}b^{n,r}$, and on all other states it has eigenvalue 0. Algebraically the new term $\hat{H}_{\mathrm{f}}$ is hence the exact Grassmannian analog of the bosonic operator $\hat{H}$ and, most notably, it contains a diverging sum that exactly cancels that of $\hat{H}$, so that the ground state energy of $\hat{\mathbf{H}}$ vanishes[39]

$$
\left\langle \psi_0 | \hat{\mathbf{H}} | \psi_0 \right\rangle = 0 .
$$

[39] We purposefully refrain here from what might seem most natural at this point, namely introducing bosonic creation and annihilation operators

$$
\hat{a}^{\dagger n,r} = \frac{1}{\sqrt{2}} \left(\sqrt{\omega} \hat{b}^{n,r} - \frac{1}{\sqrt{\omega}} \partial_{b^{n,r}} \right)
$$

Here

$$|\psi_0\rangle = |0\rangle_{\mathrm{b}} \otimes |0\rangle_{\mathrm{f}}$$

is given by the bosonic vacuum defined by the oscillator ground state wave functions

$$\langle\{b^{n,r}\}|\psi_0\rangle_{\mathrm{b}} = \prod_{n,r} \left(\frac{2}{\omega_n}\right)^{\frac{1}{4}} \exp\Big(-\omega_n \left(b^{n,r}\right)^2/2\Big) \tag{604}$$

and by the fermionic vacuum defined by the relation

$$\hat{\mathrm{c}}^{n,r}\,|0\rangle_{\mathrm{f}} = 0, \quad \forall\, n, r\,. \tag{605}$$

Now that the original system, the free scalar field, has been successfully turned supersymmetric at the level of its quantum mechanical operator description (this way of introducing supersymmetry is what is called the "Hamiltonian route" on p. 9 of the introduction) we can reobtain the supersymmetric action functional that is associated with the new supersymmetric system by taking the (pseudo-)classical limit of the new Hamiltonian $\hat{\mathbf{H}}$. This gives the classical Hamiltonian density $\mathcal{H}$, from which the Lagrangian density $\mathcal{L}$ is obtained by taking the Legendre transform[40]. In order to find the usual form of the action, the mode decomposition has to be reversed. In the present simple example this (straightforward but not very illuminating) procedure can be avoided by recognizing the fermionic Hamiltonian $\hat{H}_{\mathrm{f}}$ as the Hamiltonian of the free Dirac spinor field: The free Dirac particle is described by a four-component Dirac spinor field ψ with the following action (e.g. [119]§4,§10.3):

$$\begin{aligned}
S_{\mathrm{D}} &= \int\limits_{\mathbb{R}} L_{\mathrm{D}}\; dt \\
&= \int\limits_{\mathbb{R}} \int\limits_{\Sigma} \bar{\psi}\left(\gamma^\mu\, i\partial_\mu - m\right)\psi\; d^3x\; dt \\
&= \int\limits_{\mathbb{R}} \int\limits_{\Sigma} \psi^\dagger\left(\gamma^0\gamma^\mu\, i\partial_\mu - \gamma^0 m\right)\psi\; d^3x\; dt
\end{aligned}$$

$$\hat{a}_{n,r} = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\hat{b}^{n,r} + \frac{1}{\sqrt{\omega}}\partial_{b^{n,r}}\right)$$

with

$$\left[\hat{a}_{n,r}, \hat{a}^{\dagger n',r'}\right] = \delta_n^{n'}\,\delta_r^{r'}\,.$$

Such a notation is very natural in the simple case considered in this example, but it becomes clumsy in more general cases. On the other hand, the geometrically motivated representation that goes into (601) is generally of good use and will be used throughout.

[40] In fact, from 2.67 (p.65), we already know that the action, in the mode basis used above, looks formally like that of the $D = 1$, $N = 2$ supersymmetric sigma model:

$$\begin{aligned}
L &= \frac{1}{2}G_{(n,r)(n',r')}\dot{b}^{n,r}\dot{b}^{n',r'} + iG_{(n,r)(n',r')}\bar{c}^{n,r}Dc^{n',r'} \\
&\quad -\frac{1}{2}G^{(n,r)(n',r')}\left(\partial_{b^{n,r}}W\right)\left(\partial_{b^{n',r'}}W\right) - \bar{c}^{n,r}c^{n',r'}\left(\nabla_{b^{n,r}}\partial_{b^{n',r'}}W\right) \\
&= \sum_{\substack{n\in\mathbb{Z}^3 \\ r\in\{1,2\}}} \left(\dot{b}^{n,r}\right)^2 + 2i\bar{c}^{n,r}\dot{c}_{n,r} - \frac{1}{2}V - \omega_n\bar{c}^{n,r}c_{n,r}\,.
\end{aligned} \tag{606}$$

$$= \int\limits_{\mathbb{R}} \int\limits_{\Sigma} \left(i\psi^\dagger \dot{\psi} + \psi^\dagger \left(\gamma^0 \gamma^j \, i\partial_j - \gamma^0 m \right) \psi \right) \, d^3x \, dt \,. \qquad (607)$$

The canonical momentum associated with $\psi(x)$ is

$$p_\psi(x) = \frac{\delta L}{\delta \dot{\psi}(x)} = i\psi^\dagger(x) \,.$$

This gives the Hamiltonian

$$\begin{aligned} H_{\mathrm{D}} &= \int\limits_{\Sigma} i\psi^\dagger \dot{\psi} \, d^3x - L_{\mathrm{D}} \\ &= \int\limits_{\Sigma} \psi^\dagger \left(-i\gamma^0 \gamma^j \partial_j + \gamma^0 m \right) \psi \, d^3x \,. \end{aligned} \qquad (608)$$

The usual mode decomposition of ψ on $\Sigma = T^3$

$$\psi(x,t) = \sum_{\substack{n \in \mathbf{Z}^3 \\ r \in \{1,2\}}} \sqrt{\frac{m}{E_n}} \left(c_{n,r}(t) \, u^{n,r} e^{-i2\pi n \cdot x} + d^{*n,r}(t) \, v_{n,r} e^{i2\pi n \cdot x} \right) \qquad (609)$$

with constant basis spinors[41] $u^{n,r}$, $v_{n,r}$, leads to the quantum field operator

$$\hat{\psi}(x) = \sum_{\substack{n \in \mathbf{Z}^3 \\ r \in \{1,2\}}} \sqrt{\frac{m}{E_n}} \left(\hat{c}_{n,r} u^{n,r} e^{-i2\pi n \cdot x} + \hat{d}^{\dagger n,r} v_{n,r} e^{i2\pi n \cdot x} \right) , \qquad (610)$$

and the following nonvanishing anticommutation relations between the mode creators and annihilators:

$$\begin{aligned} \left\{ \hat{c}_{n,r}, \hat{c}^{\dagger n',r'} \right\} &= \delta_n^{n'} \delta_r^{r'} \\ \left\{ \hat{d}_{n,r}, \hat{d}^{\dagger n',r'} \right\} &= \delta_n^{n'} \delta_r^{r'} \,. \end{aligned} \qquad (611)$$

It follows that the Hamilton operator becomes

$$\begin{aligned} \hat{H}_{\mathrm{D}} &= \sum_{\substack{n \in \mathbf{Z}^3 \\ r \in \{1,2\}}} E_n \left(\hat{\mathrm{c}}^{\dagger n,r} \hat{\mathrm{c}}_{n,r} - \hat{d}_{n,r} \hat{d}^{\dagger n,r} \right) \\ &= \sum_{\substack{n \in \mathbf{Z}^3 \\ r \in \{1,2\}}} E_n \left(\hat{\mathrm{c}}^{\dagger n,r} \hat{\mathrm{c}}_{n,r} + \hat{d}^{\dagger n,r} \hat{d}_{n,r} - 1 \right) . \end{aligned} \qquad (612)$$

This is evidently the sum of two copies of $\hat{H}_{\mathrm{f}}$ defined in (603). Hence the supersymmetric action that we are looking for contains two complex scalar fields and one Dirac spinor field:

$$S_{\mathrm{susy}} = \frac{1}{2} \int\limits_{\mathcal{M}} \left(\partial^\mu \phi_1^* \partial_\mu \phi_1 - m \phi_1^* \phi_1 + \partial^\mu \phi_2^* \partial_\mu \phi_2 - m \phi_2^* \phi_2 + \bar{\psi} \left(\gamma^\mu \partial_\mu - m \right) \psi \right) \, d^4x \qquad (613)$$

[41] The position of the mode indices n, r is here chosen so as to reproduce the index convention used above.

This coincides with the result obtained by the usual superfield formalism. See for instance [131], where it is shown that *one* complex scalar field is the superpartner to one *Weyl* spinor field.

The above derivation of the supersymmetric action shows physically, why the ground state energy of the supersymmetrically extended system vanishes: The operator $\hat{\mathrm{c}}^\dagger$ creates a Dirac particle of positive energy, $\hat{\mathrm{c}}$ annihilates one. On the other hand, $\hat{d}$ creates a Dirac particle in a negative energy state and $\hat{d}^\dagger$ annihilates it. The physical vacuum $|0\rangle$ (605) with

$$\hat{\mathrm{c}}^{n,r}\,|0\rangle = \hat{d}^{n,r}\,|0\rangle = 0, \;\; \forall\, n, r$$

is therefore filled with negative energy states. The diverging negative energy of this physical fermionic vacuum cancels exactly with the positive diverging ground state energy of the bosonic scalar fields.

Next we consider the full supersymmetry algebra including spatial translation generators.

3.2 Scalar field and superpartner in $D = 1 + 1$

Let spacetime be given by

$$M := \mathbb{R} \otimes \Sigma := \mathbb{R} \otimes S^1 \,, \tag{614}$$

with coordinates

$$(x^0, x^1) = (ct, x) \in \mathbb{R} \otimes [0, L] \;. \tag{615}$$

The action of a free and massles complex scalar field propagating on $\mathcal{M}$ is

$$S = \frac{1}{2} \int\limits_M \eta^{ab} \partial_a \phi^* \partial_b \phi \; d^2x \,. \tag{616}$$

The field ϕ may be expanded as

$$\begin{aligned} \phi(t,x) &= \sum_{n \in \mathbb{N}} z^n(t)\, N e^{ik_n x} \\ \Leftrightarrow \phi^*(t,x) &= \sum_{n \in \mathbb{N}} \bar{z}^{\bar{n}}(t)\, N e^{-ik_n x} \,, \end{aligned} \tag{617}$$

where

$$k_n \;:=\; \frac{2\pi}{L} n \tag{618}$$

is the wave vector along the single space-like direction and where z^n and $\bar{z}^{\bar{n}}$ are complex coordinates on the infinite dimensional configuration space of the system.

After inserting this expansion into (616) one finds the following Lagrangian:

$$L = \int\limits_{\mathbb{R}} \eta^{ab} \partial_a \phi^* \partial_b \phi \; dx \;=\; \sum_{n \in \mathbb{N}} \left(\frac{1}{2} \frac{1}{c^2} \dot{\bar{z}}^{\bar{n}}(t)\, \dot{z}^n(t) - \frac{1}{2} k_n^2\, \bar{z}^{\bar{n}}(t)\, z^n(t) \right) \tag{619}$$

The canonical momenta are

$$\begin{aligned} p_{z^n} &= \frac{\delta L}{\delta \dot{z}^n} \\ &= \frac{1}{2c^2} \dot{\bar{z}}^{\bar{n}} \\ p_{\bar{z}^{\bar{n}}} &= \frac{1}{2c^2} \dot{z}^n \,. \end{aligned} \tag{620}$$

The energy-momentum tensor

$$\begin{aligned} T_{ab} &= (\partial_a \phi) \frac{\delta L}{\delta \partial_b \phi} + (\partial_a \phi^*) \frac{\delta L}{\delta \partial_b \phi^*} - \eta_{ab} L \\ &= \frac{1}{2} \partial_a \phi^* \partial_b \phi + \frac{1}{2} \partial_a \phi \partial_b \phi^* - \eta^{ab} L \end{aligned} \tag{621}$$

has the components

$$\begin{aligned} T_{00} &= \frac{1}{2c^2} \dot{\phi}^* \dot{\phi} + \frac{1}{2} \partial_1 \phi^* \partial_1 \phi \\ T_{10} &= \frac{1}{2c} (\partial_1 \phi)\, \dot{\phi}^* + \frac{1}{2c} (\partial_1 \phi^*)\, \dot{\phi} \,. \end{aligned} \tag{622}$$

Integrating this over space gives the Hamiltonian and the momentum functions:

$$\begin{aligned}
H_0 := H \quad &:= \int_\Sigma T_{00}\, dx \\
&= \sum_n \left(\frac{1}{2}\frac{1}{c^2}\dot{\bar{z}}^{\bar{n}}\dot{z}^n + \frac{1}{2}k_n^2 \bar{z}^{\bar{n}}(t)\, z^n(t) \right) \\
&= \sum_n \left(2c^2 p_{z^n} p_{\bar{z}^{\bar{n}}} + \frac{1}{2}k_n^2 \bar{z}^{\bar{n}}(t)\, z^n(t) \right) \\
H_1 := P \quad &:= \int_\Sigma T_{10}\, dx \\
&= \sum_n i\frac{1}{2c}k_n \left(z^n \dot{\bar{z}}^{\bar{n}} - \bar{z}^{\bar{n}} \dot{z}^n \right) \\
&= \sum_n c k_n \left(z^n p_{z^n} - \bar{z}^{\bar{n}} p_{\bar{z}^{\bar{n}}} \right) .
\end{aligned} \tag{623}$$

Canonical quantization by the rule

$$\begin{aligned}
p_{z^n} \quad &\to \quad \hat{p}_{z^n} = -i\hbar\partial_{z^n} \\
p_{\bar{z}^{\bar{n}}} \quad &\to \quad \hat{p}_{\bar{z}^{\bar{n}}} = -i\hbar\partial_{\bar{z}^{\bar{n}}}
\end{aligned} \tag{624}$$

yields the operators

$$\begin{aligned}
H \quad &\to \quad \hat{H} = \sum_{n\in\mathbb{N}} \left(-2c^2\hbar^2 \partial_{z^n}\partial_{\bar{z}^{\bar{n}}} + \frac{1}{2}k_n^2\, \bar{z}^{\bar{n}} z^n \right) \\
P \quad &\to \quad \hat{P} = \sum_{n\in\mathbb{N}} c\hbar k_n \left(z^n \partial_{z^n} - \bar{z}^{\bar{n}} \partial_{\bar{z}^{\bar{n}}} \right) .
\end{aligned} \tag{625}$$

From this purely bosonic system one finds the metric $G^{n\bar{m}}$ and superpotential W on configuration space

$$\begin{aligned}
G^{n\bar{m}} \quad &:= \quad 2c^2\delta^{nm} \\
W \quad &:= \quad \sum_{n\in\mathbb{N}} \frac{k_n}{2c} z^n \bar{z}^{\bar{n}}
\end{aligned} \tag{626}$$

so that

$$\begin{aligned}
\hat{H} \quad &= \quad -\hbar^2 G^{n\bar{m}} \partial_{z^n}\partial_{\bar{z}^{\bar{m}}} + G^{n\bar{m}} \left(\partial_{z^n} W\right)\left(\partial_{z^{\bar{m}}} W\right) \\
\hat{P} \quad &= \quad G^{n\bar{m}} \left(\left(\partial_{\bar{m}} W\right)\partial_n - \left(\partial_n W\right)\partial_{\bar{m}} \right)
\end{aligned} \tag{627}$$

The main point of this derivation is that $\hat{H}$ and $\hat{P}$ are indeed of the form required to apply the results of 2.68 (p.66). Comparison with (294) shows that the supersymmetric extension of this system is described by the following extended Hamiltonian and momentum operators:

$$\begin{aligned}
\mathbf{H} \quad &= \quad \sum_{n\in\mathbb{N}} \left(-2c^2\hbar^2 \partial_{z^n}\partial_{\bar{z}^{\bar{n}}} + \frac{1}{2}k_n^2 \bar{z}^{\bar{n}} z^n + \hbar c k_n \left(\hat{c}^{\dagger n}\hat{c}_n + \hat{c}^{\dagger \bar{n}}\hat{c}_{\bar{n}} \right) - \hbar c k_n \right) \\
\mathbf{P} \quad &= \quad c\sum_{n\in\mathbb{N}} \left(\hbar k_n \left(z^n \partial_{z^n} - \bar{z}^n \partial_{z^{\bar{n}}} \right) + \hbar k_n \left(\hat{c}^{\dagger n}\hat{c}_n - \hat{c}^{\dagger \bar{n}}\hat{c}_{\bar{n}} \right) \right) .
\end{aligned} \tag{628}$$

Here we have shifted the index on the fermionic annihilators by means of

$$\begin{aligned}\hat{\mathrm{c}}^{n} &= G^{n\bar{m}}\hat{\mathrm{c}}_{\bar{m}}\\ \hat{\mathrm{c}}^{\bar{m}} &= G^{n\bar{m}}\hat{\mathrm{c}}_{n}\,.\end{aligned} \tag{629}$$

The following constituents of these operators: can be identified:

- *total bosonic "kinetic" energy:* $-2c^2\hbar^2\partial_{z^n}\partial_{\bar{z}^{\bar{n}}}$
- *total bosonic "potential" energy:* $\frac{1}{2}k_n^2\bar{z}^{\bar{n}}z^n$
- *"renormalized fermionic energy":* $\hbar c k_n\left(\hat{\mathrm{c}}^{\dagger n}\hat{\mathrm{c}}_n + \hat{\mathrm{c}}^{\dagger\bar{n}}\hat{\mathrm{c}}_{\bar{n}}\right)$
- *fermionic ground state energy:* $-\hbar c k_n$
- *total bosonic momentum:* $\hbar k_n\left(z^n\partial_{z^n} - \bar{z}^n\partial_{z^{\bar{n}}}\right)$
- *total fermionic momentum:* $\hbar k_n\left(\hat{\mathrm{c}}^{\dagger n}\hat{\mathrm{c}}_n - \hat{\mathrm{c}}^{\dagger\bar{n}}\hat{\mathrm{c}}_{\bar{n}}\right)$

The negative fermionic ground state energy again exactly cancels the positive bosonic ground state energy.

3.3 The NSR superstring

Outline. The Polyakov action with non-trivial background fields is treated in Hamiltonian formalism (following [122]) which brings out the close formal relation to canonical gravity and provides a particularly convenient means to introduce worldsheet supersymmetry. The system is then canonically quantized in the mode amplitude Schrödinger representation. It is shown that the 0-mode of the supercurrent in this representation is a linear combination of deformed exterior derivatives on configuration space (and in fact precisely of the form of the supersymmetry generators discussed in 2.68 (p.66) and §3.2 (p.150)). All other modes can be seen to be formally "hidden supercharges" in the terminology of §2.2.7 (p.90). Formal relations between the canonical treatment of the Polyakov action and quantum gravity and in particular quantum cosmology are discussed.

Introduction. Formally the first quantized superstring is equivalent to 1+1 dimensional quantum supergravity coupled to supermatter. This fact gives rise to many useful analogies to our treatment of quantum supergravity in §4 (p.181). In particular, the center-of-mass motion of the (super-)string corresponds formally to the dynamics of the homogeneous modes of (super-) quantum gravity (*cf.* §4.3.2 (p.230)), i.e. to quantum cosmology. Hence looking at string theory as a quantum supergravity theory in 1+1 dimensions is a particular fruitful point of view with respect to understanding quantum cosmology. Among other things, the preference of normal mode decomposition over functional formulations that is common practice in analysing the Polyakov action also proves worthwhile in quantum supergravity, where it is however less commonly used (see §4.2 (p.187)).

Literature Standard textbooks on string theory are [115] and [223]. Several useful introductory lecture notes are also available, for instance [122] (valuable details on Hamiltonian and BRST formalism), [37] (emphasis on light-cone gauge), [250] (brief outline including D-branes but skipping conformal field theory), [154] (detailed exposition of perturbative string theory), [157] and [158] (concise presentation of the bosonic string and conformal field theory).

The starting point of string theory is the generalization of the usual action that describes a relativistic (0-dimensional) point particle (see §A (p.293), eq. (1158), p. 294) to one describing a relativistic 1-dimensional "line-particle", the string.

3.1 (The Polyakov action) Let $\mathcal{M} = \mathbb{R} \otimes \Sigma$ with $\Sigma = S^1$ and let the metric on $\mathcal{M}$ be

$$ds^2 = g_{\alpha\beta} d\sigma^\alpha \otimes d\sigma^\beta$$

$$\begin{aligned} \sigma^0 &:= \tau \in \mathbb{R} \\ \sigma^1 &:= \sigma \in [0, 2\pi] \,. \end{aligned} \tag{630}$$

With $\mathcal{M}$ the world-sheet the bosonic Polyakov action for a string moving in a gravitational and RR-form background is given by ([223], eq. (3.7.6))

$$\begin{aligned} S &= \int d\tau d\sigma \; \mathcal{L} = \int d\tau \; L \\ &= -\frac{T}{2} \int d\tau d\sigma \; \sqrt{g} \left(g^{\alpha\beta} G_{\mu\nu} + \epsilon^{\alpha\beta} B_{\mu\nu} \right) \partial_\alpha X^\mu \partial_\beta X^\nu \, . \end{aligned} \tag{631}$$

3.2 (Planck length, String tension, and Regge slope)

$$T = \frac{1}{2\pi\alpha'} \tag{632}$$

$$\sqrt{\alpha'} = l_{\mathrm{s}} \tag{633}$$

In order to quantize this action we apply standard Hamiltonian formalism, following [122] (also see [159]):

3.3 (Hamiltonian treatment of the bosonic string) The canonical dynamical coordinates are the X^μ. Their canonical momenta are

$$\begin{aligned} P_\mu &= \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \\ &= -T\sqrt{g} \left(g^{0a} G_{\mu\nu} + \epsilon^{0a} B_{\mu\nu} \right) \partial_a X^\nu \, , \end{aligned} \tag{634}$$

so that[42]

$$G_{\mu\nu} \dot{X}^\nu = \frac{1}{g^{00}} \left(-\frac{1}{T\sqrt{g}} P_\mu - \left(g^{01} G_{\mu\nu} + \frac{1}{\sqrt{g}} B_{\mu\nu} \right) X'^\nu \right) . \tag{636}$$

(In *conformal gauge*, where $g_{\alpha\beta} = e^\phi \eta_{\alpha\beta}$, this gives $G_{\mu\nu}\dot{X}^\nu = \frac{1}{T} P_\mu - B_{\mu\nu} X'^\nu$.) To evaluate the Hamiltonian

$$H = \int d\sigma \; \left(P_\mu \dot{X}^\mu - \mathcal{L} \right) , \tag{637}$$

one makes the usual ADM decomposition (see §4.1 (p.181) for details) of the world-sheet metric in lapse and shift functions and "spatial" parts. For any d-dimensional manifold with spatial coordinates x^i and temporal coordinate t the ADM form of the metric is

$$ds^2 = -N^2 dt \otimes dt + \left(dx^i + N^i dt \right) \otimes \left(dx^j + N^j dt \right) \tilde{g}_{ij} \, . \tag{638}$$

[42] Here ϵ_{ab} is the antisymmetric *tensor*, so that in particular $\epsilon^{01} = -\epsilon^{10} = 1/\sqrt{g}$ (*cf.* [223], p. 105). This is because the RR-term comes from the integration of the potential B over the worldsheet:

$$\begin{aligned} \int B &= \frac{1}{2} \int B_{\mu\nu} dX^\mu \wedge dX^\nu \\ &= \frac{1}{2} \int B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu d\sigma^a \wedge d\sigma^b \\ &= \int d\sigma d\tau \; B_{\mu\nu} \partial_{[a} X^\mu \partial_{b]} X^\nu \, . \end{aligned} \tag{635}$$

Due to the special Weyl invariance of the Polyakov action it turns out to be convenient to multiply the lapse by the "spatial" volume element

$$N \to \sqrt{\tilde{g}}N \tag{639}$$

(this may be compared with the treatment of the general p-brane in 3.21 (p.176)) so that one sets

$$ds^2 = -N^2\tilde{g}\,d\tau\otimes d\tau + \left(dx^i + N^i dt\right)\otimes\left(dx^j + N^j dt\right)\tilde{g}_{ij} \tag{640}$$

Here $\tilde{g}_{ij}$ is the spatial metric whose only component is $\tilde{g}_{11}$ and whose determinant is also $\tilde{g} = \tilde{g}_{11}$. The components of the world-sheet metric in ADM parameterization now read

$$\begin{aligned} g_{00} &= -N^2\tilde{g} + \left(N^1\right)^2\tilde{g}_{11} \\ g_{01} &= N^1\tilde{g}_{11} \\ g_{11} &= \tilde{g}_{11}\,. \end{aligned} \tag{641}$$

The determinant is

$$\det(g) = -N^2\tilde{g}_{11}^2 \tag{642}$$

and the elements of the inverse metric tensor are

$$\begin{aligned} g^{00} &= -\frac{1}{N^2\tilde{g}} \\ g^{01} &= \frac{N^1}{N^2\tilde{g}} \\ g^{11} &= \frac{1}{\tilde{g}_{11}} - \frac{\left(N^1\right)^2}{N^2\tilde{g}}\,. \end{aligned} \tag{643}$$

Using these expressions one finds for the Hamiltonian density $\mathcal{H}$

$$\begin{aligned} P_\mu\dot{X}^\mu - \mathcal{L} &= P_\mu\dot{X}^\mu - \left(\frac{1}{2}P_\mu\dot{X}^\mu - \frac{T}{2}\sqrt{g}g^{11}X'^\mu X'_\mu - \frac{T}{2}\sqrt{g}g^{01}\dot{X}^\mu X'_\mu - \frac{T}{2}B_{\mu\nu}\dot{X}^\mu X'^\nu\right) \\ &= \frac{1}{2}P_\mu\dot{X}^\mu + \frac{T}{2}\sqrt{g}g^{11}X'^\mu X'_\mu + \frac{T}{2}\sqrt{g}g^{01}\dot{X}^\mu X'_\mu + \frac{T}{2}B_{\mu\nu}\dot{X}^\mu X'^\nu \\ &= \frac{1}{2}P_\mu\frac{1}{g^{00}}\left(-\frac{1}{T\sqrt{g}}P^\mu - g^{01}X'^\mu - \frac{1}{\sqrt{g}}B^\mu{}_\nu X'^\nu\right) + \\ &\quad + \frac{T}{2}\sqrt{g}g^{01}X'^\mu\frac{1}{g^{00}}\left(-\frac{1}{T\sqrt{g}}P_\mu - g^{01}X'_\mu - \frac{1}{\sqrt{g}}B_{\mu\nu}X'^\nu\right) + \\ &\quad + \frac{T}{2}\sqrt{g}g^{11}X'^\mu X'_\mu + \frac{T}{2}B_{\mu\kappa}\frac{1}{g^{00}}\left(-\frac{1}{T\sqrt{g}}P^\mu - g^{01}X'^\mu - \frac{1}{\sqrt{g}}B^\mu{}_\nu X'^\nu\right)X'^\kappa \\ &= -\frac{1}{2Tg^{00}\sqrt{g}}P_\mu P^\mu - \frac{g^{01}}{g^{00}}P_\mu X'^\mu + \frac{T}{2}\sqrt{g}\left(-\frac{(g^{01})^2}{g^{00}} + g^{11}\right)X'_\mu X'^\mu + \\ &\quad - P_\mu\frac{1}{g^{00}\sqrt{g}}B^\mu{}_\nu X'^\nu - \frac{T}{2}\frac{1}{g^{00}\sqrt{g}}B_{\mu\kappa}B^\mu{}_\nu X'^\kappa X'^\nu \\ &= \frac{1}{2T}NG^{\mu\nu}\left(P_\mu + TB_{\mu\kappa}X'^\kappa\right)\left(P_\nu + TB_{\nu\kappa}X'^\kappa\right) + \frac{T}{2}NX'^\mu X'_\mu + N^1P_\mu X'^\mu \\ &= N\mathcal{H}_\perp + N^1\mathcal{H}_1\,. \end{aligned} \tag{644}$$

$\mathcal{H}_\perp$ is the Hamiltonian generator of τ translations and $\mathcal{H}_1$ is the generator of σ-translations:

$$\mathcal{H}_\perp = \frac{1}{2}\left(\frac{1}{T}G^{\mu\nu}\left(P_\mu + iTB_{\mu\kappa}X'^\kappa\right)\left(P_\nu + iTB_{\nu\kappa}X'^\kappa\right) + T\,X'^\mu X'^\nu\right)G_{\mu\nu}$$
$$\mathcal{H}_1 = P_\mu X'^\mu\,. \tag{645}$$

The total Hamiltonian is

$$H = \int d\sigma\left(N\mathcal{H}_\perp + N^1\mathcal{H}_1\right)\,. \tag{646}$$

In 1+1 dimensions it is very natural to consider "light cone" coordinates $\sigma^\pm = \sigma^0 \pm \sigma^1$. Accordingly one defines

$$\begin{aligned}\mathcal{H}_\pm &= \mathcal{H}_\perp \pm \mathcal{H}_1\\ &= \frac{1}{2}G^{\mu\nu}\left(\frac{1}{\sqrt{T}}\left(P_\mu + TB_{\mu\kappa}X'^\kappa\right) \pm \sqrt{T}G_{\mu\kappa}X'^\kappa\right)\left(\frac{1}{\sqrt{T}}\left(P_\nu + TB_{\nu\kappa}X'^\kappa\right) \pm \sqrt{T}G_{\nu\kappa}X'^\kappa\right)\,,\end{aligned} \tag{647}$$

where in the last line we have introduced a convenient factorization which expresses $\mathcal{H}_\pm$ as the square

$$\mathcal{H}_\pm = G^{\mu\nu}\mathcal{P}_{\mu\pm}\mathcal{P}_{\nu\pm} \tag{648}$$

of some sort of generalized momentum

$$\mathcal{P}_{\mu\pm} := \frac{1}{\sqrt{T}}P_\mu + \sqrt{T}\left(B_{\mu\kappa} \pm G_{\mu\kappa}\right)X'^\kappa\,. \tag{649}$$

Note that in this form the Hamiltonian constraint of the relativistic string in gravitational and RR-field background is an obvious generalization of the Hamiltonian constraint of the relativistic point particle in gravitational and electromagnetic backgrounds (*cf.* §A (p.293), in particular equation (1159)).

With the bosonic Hamiltonian of the theory identified, we can now consider supersymmetric extensions, i.e. the superstring. As is demonstrated in [122], §14.2.1, this is readily done (following the "Hamiltonian route", *cf.* p. 9 ff.) by the "square-root" method:

3.4 (Supersymmetric extension) Since the light-cone Hamiltonian $\mathcal{H}_\pm$ is a square of momentum-like expressions, $\mathcal{P}_{\mu\pm}$, its supersymmetric square root is necessarily a Dirac-operator like expression[43] in terms of the $\mathcal{P}_{\mu\pm}$. Hence one defines a fermionic pendant to the dynamical field X^μ, namely the field

$$\psi^\mu_A, \quad A \in \{+,-\}$$

which is a *real* (Majorana) worldsheet spinor, carrying a spinor index A (which we conveniently let take values not in $\{1,2\}$ but, equivalently, in $\{+,-\}$). It must satisfy the canonical super Poisson bracket algebra (see the remark at the end of 3.5 (p.157) with respect to the use of δ_{AB} on the right hand side)

$$\{\psi^\mu_A(\sigma), \psi^\nu_B(\sigma')\}_{\mathrm{PB}} = iG^{\mu\nu}\delta_{AB}\delta(\sigma,\sigma')\,. \tag{650}$$

[43] Note, though, that we are still discussing the classical mechanics of the string here.

(Here we deviate from the conventions used in [122], which has a factor of 4π on the right hand side.) We can switch to a more convenient basis by introducing the vielbein $E^\mu{}_a$ on spacetime

$$E^\mu{}_a E^\nu{}_b \eta^{ab} = G^{\mu\nu}$$

and defining the orthonormal fields $\psi^a_\pm(\sigma)$ so that

$$\psi^\mu_\pm(\sigma) \quad = \quad E^\mu{}_a \psi^a_\pm(\sigma) \ , \tag{651}$$

which gives the bracket (*cf.* [75], eq. 3.9)

$$\left\{\psi^a_A(\sigma), \psi^b_B(\sigma')\right\}_{\rm PB} \quad = \quad i\eta^{ab}\delta_{AB}\delta(\sigma,\sigma') \, . \tag{652}$$

Contracting the "Clifford generator" ψ^μ with the generalized momentum gives the desired Dirac-like square root $\mathcal{S}_A$ of the light-cone Hamiltonian:

$$\mathcal{S}_\pm \quad := \quad \psi^\mu_\pm \mathcal{P}_{\mu\pm} \, . \tag{653}$$

As always, this leads to a supersymmetric extension of the original bosonic Hamiltonian by further fermionic terms:

$$\begin{aligned}
\{\mathcal{S}_\pm(\sigma), \mathcal{S}_\pm(\sigma')\}_{\rm PB} &= \left\{\psi^\mu_\pm \mathcal{P}_{\mu\pm}(\sigma), \psi^\nu_\pm \mathcal{P}_{\nu\pm}(\sigma')\right\}_{\rm PB} \\
&= \left\{\psi^\mu_\pm(\sigma), \psi^\nu_\pm(\sigma)\right\}_{\rm PB} \mathcal{P}_{\mu\pm}(\sigma)\, \mathcal{P}_{\nu\pm}(\sigma') + (\text{terms containing } \psi^\mu_\pm) \\
&= i\delta(\sigma,\sigma')\, \mathcal{H}_\pm(\sigma) + (\text{terms containing } \psi^\mu_\pm) \, .
\end{aligned} \tag{654}$$

The supersymmetric Hamiltonian then reads

$$H = \int d\sigma \ \left(N\mathcal{H}^{\rm s}_\perp + N^1 \mathcal{H}^{\rm s}_1 + \chi^A_0 \mathcal{S}_A\right) \, . \tag{655}$$

Here χ_0 is the fermionic Lagrange multiplier associated with the supersymmetry. The superscript $^{\rm s}$ indicates the supersymmetric extensions of the original bosonic expressions. Since we will be concerned only with these supersymmetric objects, these superscripts will be suppressed in the following.

3.5 (NSR string as $D = 2$ supergravity) We have, following [122], obtained the constraints of the worldsheet supersymmetric extension of the bosonic string, called the Neveu-Schwarz-Ramond (NSR) model, by working in Hamiltonian formalism. For flat target space, $G_{\mu\nu} = \eta_{\mu\nu}$, and vanishing 2-form field, $B_{\mu\nu} = 0$, the Lagrangian density corresponding to this model has been found in [86] to be

$$L = -\frac{T}{2} \int d\sigma^2 \ \eta_{\mu\nu} e \left(g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu - \bar{\psi}\gamma^\alpha\partial_\alpha\psi - 2\bar{\chi}_\alpha\gamma^\alpha\gamma^\beta\psi^\mu \left(\partial_\beta X^\nu + \frac{1}{2}\bar{\psi}^\nu\chi_\beta\right)\right) \tag{656}$$

Here χ_α is the superpartner of the worldsheet vielbein and $\gamma^\alpha = e^\alpha{}_a\gamma^a$, where γ^a are the flat worldsheet Clifford generators. This action is formally that of $D = 2$ supergravity coupled to supermatter. In two dimensions the gravitational supermultiplet is non-dynamical and acts as Lagrange multipliers only.

Another specialty of supergravity in two dimensions is that the action only contains ordinary, as opposed to covariant, derivatives of the fermionic fields, because the terms involving the connection coefficients vanish identically. This has a remarkable consequence: For canonical supergravity in $D > 2$ the 0-component of the spin connection always serves as the Langrange multiplier for the Lorentz constraints, which enforce invariance of physical states under Lorentz transformations of the vielbein field (*cf.* §4.2 (p.187)). Such a constraint does not appear in $D = 2$. This in particular implies that in $D = 2$ physical states may have odd fermion number, something that is forbidden by Lorentz invariance in $D > 2$. It also justifies our definition (650) of the canonical Poisson bracket of the fermionic fields, which involves a term δ_{AB}, where A, B are the world-sheet spinor indices. Such a non-Lorentz-invariant notation implies that a Lorentz frame has been fixed.

3.6 (Remark on string theory and worldsheet quantum gravity) The formal equivalence of the free first quantized superstring with quantum supergravity on the worldsheet (minimally coupled to certain forms of supermatter) gives rise to a host of parallels between the present discussion and that of $N = 1, D = 4$ canonical supergravity in §4 (p.181). For instance, we there find it particularly convenient to make a mode expansion of all fields on spacetime (see §4.3 (p.192)), completely analogous to the mode expansion common in string theory (*cf.* 3.7 (p.158)). In both cases we find that in the mode amplitude basis the supersymmetry generators have the form of deformed exterior derivatives on configuration space.

With respect to understanding (supersymmetric or ordinary) quantum cosmology it is furthermore interesting to note the following: As is discussed in §4.3.2 (p.230), quantum cosmology is obtained from full quantum gravity by restricting attention to a certain subset of all the modes of the fields, namely to such modes which are *constant* over the spatial hyperslice Σ (with respect to some frame, see 4.45 (p.231) for details). *This means that the analog of cosmology in the theory of the free first-quantized string is precisely the center-of-mass motion of the string.* Or, to put it the other way round, quantum cosmology is the study of the "center-of-mass" motion of the spacetime "brane" in the gravitational configuration space (Wheeler's superspace). (A similar observation has been made in [5], p. 14.)

Note that these formal analogies of string theory with quantum gravity are quite distinct from the usual applications that string theory, when regarded as a theory of elementary strings propagating in spacetime, has to the study of quantum gravity . It is quite remarkable that consistent superstring theory describes quantum supergravity *both* on the string's worldsheet *and* on the string's target spacetime.

3.7 (Mode expansion of the fields) In order to yield a countably infinite dimensional configuration space all fields are now expanded with respect to the standard basis of Fourier modes on the string:

The bosonic fields and their momenta are

$$X^\mu(\tau,\sigma) \quad := \quad \frac{1}{\sqrt{2\pi T}} \left(X_0^\mu + \sum_{n=1}^{\infty} \left(Z_n^\mu(\tau)\, e^{in\sigma} + \bar{Z}_n^\mu(\tau)\, e^{-in\sigma} \right) \right)$$

$$P_\mu(\tau,\sigma) \quad := \quad \frac{\sqrt{T}}{\sqrt{2\pi}} \left(P_{0\mu}(\tau) + \sum_{n=1}^{\infty} \left(P_{\bar{Z}^\mu_n}(\tau)\, e^{in\sigma} + P_{Z^\mu_n}(\tau)\, e^{-in\sigma} \right) \right) \quad (657)$$

Here Z^μ_n and $\bar{Z}^\mu_n$ are mutually complex adjoints

$$\begin{aligned} Z^\mu_n &= C^\mu_n + iD^\mu_n \\ \bar{Z}^\mu_n &= C^\mu_n - iD^\mu_n\,, \end{aligned} \quad (658)$$

where $C^\mu_n, D^\mu_n \in \mathbb{R}$ are real in order for the fields to be real.

There are two different spin structures one can put on the circle. Accordingly the fermionic fields are either periodic $\psi^\mu_A(\sigma = 0) = \psi^\mu_A(\sigma = 2\pi)$ ("Ramond sector"), or antiperiodic $\psi^\mu_A(\sigma = 0) = -\psi^\mu_A(\sigma = 2\pi)$ ("Neveu-Schwarz sector"). The periodic fields are expanded as usual

$$\psi^\mu_\pm(\tau,\sigma) \quad = \quad \frac{1}{\sqrt{2\pi}} \sum_{n=0,1,\ldots}^{\infty} \left(c^{\mu n}_\pm(\tau)\, e^{in\sigma} + c^{*\mu n}_\pm(\tau)\, e^{-in\sigma} \right)\,, \quad (659)$$

while the antiperiodic fields have only half-integral mode numbers

$$\psi^\mu_\pm(\tau,\sigma) \quad = \quad \frac{1}{\sqrt{2\pi}} \sum_{r=\frac{1}{2},\frac{3}{2},\ldots}^{\infty} \left(c^{\mu r}_\pm(\tau)\, e^{ir\sigma} + c^{*\mu r}_\pm(\tau)\, e^{-ir\sigma} \right) \quad (660)$$

and in particular no constant component. Here, again, to ensure reality of the fields, $c_\pm$ and $c^*_\pm$ are complex conjugates of each other.

Expanding the Lagrange multipliers into Fourier modes along the string

$$Y(\tau,\sigma) = Y^{(n)}(\tau)\,\frac{1}{\sqrt{2\pi}} e^{in\sigma}, \quad Y \in \left\{N, N^1, \chi^A_0\right\} \quad (661)$$

and varying the action with respect to the coefficient functions $Y^n(\tau)$ gives the Fourier modes of the constraints of the system:

$$\begin{aligned} H_{(n)}(\tau) \quad &:= \quad \int \mathcal{H}_\perp(\tau,\sigma)\,\frac{1}{\sqrt{2\pi}} e^{in\sigma}\, d\sigma = 0 \\ H^1_{(n)}(\tau) \quad &:= \quad \int \mathcal{H}_1(\tau,\sigma)\,\frac{1}{\sqrt{2\pi}} e^{in\sigma}\, d\sigma = 0 \\ S_{A(n)}(\tau) \quad &:= \quad \int \mathcal{S}_A(\tau,\sigma)\,\frac{1}{\sqrt{2\pi}} e^{in\sigma}\, d\sigma = 0 \qquad \text{(R)} \\ S_{A(r)}(\tau) \quad &:= \quad \int \mathcal{S}_A(\tau,\sigma)\,\frac{1}{\sqrt{2\pi}} e^{ir\sigma}\, d\sigma = 0 \qquad \text{(NS)}\,. \end{aligned} \quad (662)$$

3.8 (Quantization) The canonical Poisson brackets at equal τ are

$$\begin{aligned} \left[X^\mu(\tau,\sigma), P_\nu(\tau,\sigma')\right]_{\mathrm{PB}} \quad &= \quad \delta^\mu_\nu\, \delta(\sigma,\sigma') \\ \left\{\psi^a_A(\tau,\sigma), \psi^b_B(\tau,\sigma')\right\}_{\mathrm{PB}} \quad &= \quad i\eta^{ab}\, \delta_{AB}\delta(\sigma,\sigma')\ . \end{aligned} \quad (663)$$

Upon quantization in the Schrödinger representation the bosonic fields become functional multiplication operators $\hat{X}^\mu(\sigma)$ and their momenta become functional differentiation operators

$$\hat{P}_\mu(\sigma) = -i\frac{\delta}{\delta X^\mu(\sigma)}\,,$$

satisfying

$$\left[\hat{X}^{\nu}(\sigma), \hat{P}_{\mu}(\sigma')\right] = i\delta^{\nu}_{\mu}\,\delta(\sigma,\sigma')\ . \tag{664}$$

Analogously the fermionic fields are promoted to functional Clifford algebra generators $\hat{\psi}^{\mu}_{A}(\sigma)$, satisfying

$$\left\{\hat{\psi}^{a}_{A}(\sigma), \hat{\psi}^{b}_{B}(\sigma')\right\} = \eta^{ab}\,\delta_{AB}\delta(\sigma,\sigma')\ . \tag{665}$$

The mode expansions of the quantum operators are

$$\begin{aligned}
\hat{X}^{\mu}(\sigma) &:= \frac{1}{\sqrt{2\pi T}}\left(\hat{X}^{\mu}_{0} + \sum_{n=1}^{\infty}\left(\hat{Z}^{\mu}_{n}e^{in\sigma} + \hat{\bar{Z}}^{\mu}_{n}e^{-in\sigma}\right)\right)\\
\hat{P}_{\mu}(\sigma) &:= \frac{\sqrt{T}}{\sqrt{2\pi}}\left(\hat{P}_{0\mu} + \sum_{n=1}^{\infty}\left(\hat{P}_{\bar{Z}^{\mu}_{n}}e^{in\sigma} + \hat{P}_{Z^{\mu}_{n}}e^{-in\sigma}\right)\right)\\
\hat{\psi}^{\mu}_{\pm}(\sigma) &= \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\left(\hat{c}^{\mu n}_{\pm}e^{in\sigma} + \hat{c}^{\dagger\mu n}_{\pm}e^{-in\sigma}\right), \qquad \text{(R)}\\
\hat{\psi}^{\mu}_{\pm}(\sigma) &= \frac{1}{\sqrt{2\pi}}\sum_{r=\frac{1}{2}}^{\infty}\left(\hat{c}^{\mu r}_{\pm}e^{ir\sigma} + \hat{c}^{\dagger\mu r}_{\pm}e^{-ir\sigma}\right), \qquad \text{(NS)}\,.
\end{aligned} \tag{666}$$

The converse relations expressing the mode amplitudes in terms of the continuum fields are ($n > 0$ throughout)

$$\begin{aligned}
\hat{X}^{\mu}_{0} &= \frac{\sqrt{T}}{\sqrt{2\pi}}\int d\sigma\ \hat{X}^{\mu}(\sigma)\\
\hat{P}_{0\mu} &= \frac{1}{\sqrt{2\pi T}}\int d\sigma\ \hat{P}_{\mu}(\sigma)
\end{aligned} \tag{667}$$

$$\begin{aligned}
\hat{Z}^{\mu}_{n} &= \sqrt{T}\int d\sigma\ \hat{X}^{\mu}(\sigma)\,\frac{1}{\sqrt{2\pi}}e^{-in\sigma}\\
\hat{\bar{Z}}^{\mu}_{n} &= \sqrt{T}\int d\sigma\ \hat{X}^{\mu}(\sigma)\,\frac{1}{\sqrt{2\pi}}e^{+in\sigma}\\
\hat{P}_{Z^{\mu}_{n}} &= \frac{1}{\sqrt{T}}\int d\sigma\ \hat{P}_{\mu}(\sigma)\,\frac{1}{\sqrt{2\pi}}e^{+in\sigma}\\
\hat{P}_{\bar{Z}^{\mu}_{n}} &= \frac{1}{\sqrt{T}}\int d\sigma\ \hat{P}_{\mu}(\sigma)\,\frac{1}{\sqrt{2\pi}}e^{-in\sigma}
\end{aligned} \tag{668}$$

$$\hat{c}^{\mu 0}_{\pm} + \hat{c}^{\dagger\mu 0}_{\pm} = \frac{1}{\sqrt{2\pi}}\int d\sigma\ \hat{\psi}^{\mu}_{\pm}(\sigma) \tag{669}$$

$$\begin{aligned}
\hat{c}^{\mu n}_{\pm} &= \frac{1}{\sqrt{2\pi}}\int d\sigma\ \hat{\psi}^{\mu}_{\pm}(\sigma)\,e^{-in\sigma}\\
\hat{c}^{\dagger\mu n}_{\pm} &= \frac{1}{\sqrt{2\pi}}\int d\sigma\ \hat{\psi}^{\mu}_{\pm}(\sigma)\,e^{in\sigma}\,.
\end{aligned} \tag{670}$$

This gives the following supercommutators for the mode amplitude operators:

$$\left[\hat{X}^\mu_0, \hat{P}^\mu_0\right] \quad = \quad i\delta^\mu_\nu \tag{671}$$

$$\begin{aligned}\left[\hat{Z}^\mu_n, \hat{P}_{Z^\nu_m}\right] &= i\delta^\mu_\nu \delta^n_m \\ \left[\hat{\bar{Z}}^\mu_n, \hat{P}_{\bar{Z}^\nu_m}\right] &= i\delta^\mu_\nu \delta^n_m\end{aligned} \tag{672}$$

$$\left\{\hat{c}^{\mu 0}_A + \hat{c}^{\dagger\mu 0}_A, \hat{c}^{\nu 0}_B + \hat{c}^{\dagger\nu 0}_B\right\} \quad = \quad G^{\mu\nu}\delta_{AB} \tag{673}$$

$$\left\{\hat{c}^{\mu n}_A, \hat{c}^{\dagger\nu m}_B\right\} \quad = \quad G^{\mu\nu}\delta^{nm}\delta_{AB}\,. \tag{674}$$

It follows in particular that the bosonic momentum mode operators can be represented as

$$\begin{aligned}\hat{P}_{Z^\mu_n} &= -i\partial_{Z^\mu_n} \\ \hat{P}_{\bar{Z}^\mu_n} &= -i\partial_{\bar{Z}^\mu_n}\,.\end{aligned} \tag{675}$$

A discussion of a natural representation of the fermionic mode amplitude operators is postponed until the special structure of the supersymmetry generator in Schrödinger representation is given in 3.14 (p.166).

3.9 (Lowest-order perturbation in background-field coupling) When the coupling of the string to the background fields is expanded in terms of derivatives of the background fields, higher order terms are negligible as long as the "radius of curvature", R, of these fields, the scale over which they vary appreciably, is small as compared to the intrinsic length $l_\mathrm{s} = \sqrt{\alpha'}$ of the string

$$\frac{\sqrt{\alpha'}}{R} \ll 1$$

(*cf.* [223], pp. 109-110). Consider the dependence of the background fields on the coordinate fields of the string:

$$G_{\mu\nu}(X^\kappa) \stackrel{(657)}{=} G_{\mu\nu}\left(X^\kappa_0 + \sum_{n=1}^{\infty}\left(Z^\kappa_n(\tau)\, e^{in\sigma} + \bar{Z}^\kappa_n(\tau)\, e^{-in\sigma}\right)\right)\,.$$

The center-of-mass coordinates X^κ_0 are not subject to an oscillator potential and take on large values as the string propagates. On the other hand, the oscillator amplitudes Z^κ_n, $\bar{Z}^\kappa_n$ are confined in oscillator potentials and are generally much smaller. In fact, the linear extension of the fundamental string, and hence the scale of its oscillation amplitudes, is of the order of the Planck length $\sqrt{\alpha'}$. But for the very concept of a purely massless background field in string theory to make any sense at all, these must not vary appreciably over distances comparable to the Planck length. It follows that by expanding the background fields around the center-of-mass position of the string

$$G_{\mu\nu}(X^\kappa(\tau,\sigma)) \quad = \quad G_{\mu\nu}(X^\kappa_0(\tau)) + \cdots \tag{676}$$

the first term, $G_{\mu\nu}(X_0^\kappa)$ is a very good approximation to $G_{\mu\nu}(X^\kappa)$. The same holds for $B_{\mu\nu}$.

In the following we now make use of this fact by inserting the mode expansion of the fields in all expressions, like the action and the constraints, and then replacing all occurrences of $G_{\mu\nu}(X^\kappa(\tau,\sigma))$, $B_{\mu\nu}(X^\kappa(\tau,\sigma))$ by $G_{\mu\nu}(X_0^\kappa(\tau))$ and $B_{\mu\nu}(X_0^\kappa(\tau))$, respectively. This will be the lowest order approximation of the quantum theory to the background-field coupling of the string.

When the background metric is flat Minkowski spacetime it makes sense to think of the X^μ fields, as well as of their mode amplitudes $X_0^\mu, Z_n^\mu, \bar{Z}_n^\mu$ as *coordinate fields*. However, the introduction of non-trivial gravitational backgrounds reveals that the zero mode X^μ, which describes the center-of-mass of the string, is not on an equal footing with the $Z_n^\mu, \bar{Z}_n^\mu$. The latter are, in the lowest order approximation discussed above, rather *infinitesimal* quantities that describe displacements of the string from it's center-of-mass position, which are, due to the smallness of the string, insensitive to spacetime curvature. This shows that, in lowest order perturbation theory, one should think of the $Z_n^\mu, \bar{Z}_n^\mu$ as being *tangent vectors* living in $T_{X_0^\mu}\left(\mathcal{M}^{\mathrm{ST}}\right)$, the tangent space to the spacetime manifold $\left(\mathcal{M}^{\mathrm{ST}}, G_{\mu\nu}\right)$ at the string's center-of-mass point X_0^μ. In this sense, when the background is not Minkowski space, one must think of the Greek index of X_0^μ as being the index of a coordinate function, not that of a vector, while the indices carried by $Z_n^\mu, \bar{Z}_n^\mu$ indicate proper vector quantities.

3.10 (The constraint modes in lowest order in the background interaction) When the lowest order of (676) is inserted into (662), (647) one finds the following expressions:

$$H_{\pm(0)} = \frac{1}{2}G^{\mu\nu}\left(P_{\mu 0}P_{\nu 0} + 2\sum_{n=1}^{\infty}\left(P_{\bar{Z}_n^\mu} - nB_{\mu\kappa}Z_n^\kappa \pm inG_{\mu\kappa}Z_n^\kappa\right)\left(P_{Z_n^\nu} + nB_{\nu\kappa}\bar{Z}_n^\kappa \mp inG_{\nu\kappa}\bar{Z}_n^\kappa\right)\right) \tag{677}$$

$$\begin{aligned}
H_{\perp(0)} &= G^{\mu\nu}\left(\frac{1}{2}P_{\mu 0}P_{\nu 0} + \sum_{n=1}^{\infty}\left(P_{\bar{Z}_n^\mu} - nB_{\mu\kappa}Z_n^\kappa\right)\left(P_{Z_n^\nu} + nB_{\nu\kappa}\bar{Z}_n^\kappa\right)\right) + \\
&\quad + \sum_{n=1}^{\infty} n^2 Z_n^\mu \bar{Z}_n^\nu G_{\mu\nu} \\
H_{\perp(m\geq 1)} &= G^{\mu\nu}\left(\frac{1}{2}mP_{\mu 0}B_{\nu\kappa}\bar{Z}_m^\kappa + \sum_{n=m+1}^{\infty}\left(P_{\bar{Z}^\mu_{(n-m)}} - nB_{\mu\kappa}Z^\kappa_{(n-m)}\right)\left(P_{Z_n^\nu} + nB_{\nu\kappa}\bar{Z}_n^\kappa\right)\right) + \\
&\quad + \sum_{n=m+1}^{\infty} n^2 Z^\mu_{(n-m)}\bar{Z}_n^\nu G_{\mu\nu} \\
H_{\perp(m\leq -1)} &= G^{\mu\nu}\left(-\frac{1}{2}mP_{\mu 0}B_{\nu\kappa}Z_m^\kappa + \sum_{n=m+1}^{\infty}\left(P_{\bar{Z}_n^\mu} - nB_{\mu\kappa}Z_n^\kappa\right)\left(P_{Z^\nu_{(n-m)}} + nB_{\nu\kappa}\bar{Z}^\kappa_{(n-m)}\right)\right) + \\
&\quad + \sum_{n=|m|+1}^{\infty} n^2 Z_n^\mu \bar{Z}^\nu_{(n-|m|)} G_{\mu\nu}
\end{aligned} \tag{678}$$

3.11 (Classical equations of motion in lowest order in background interaction) Varying the Lagrange multipliers N, N^1 has lead to the above constraints governing the strings's dynamics. After these have been found the string may be classically evolved along the τ coordinate with the Lagrange multipliers specified freely. It is most convenient to choose the gauge defined by

$$\begin{aligned} N(\tau,\sigma) &= 1 \\ N^1(\tau,\sigma) &= 0\,. \end{aligned}$$

It follows that the Hamiltonian which generates the τ-evolution is

$$H = H_{\perp(0)}\,.$$

From this Hamiltonian one obtains the following classical equations of motion:

$$\begin{aligned}
\dot X_0^\mu &= G^{\mu\nu} P_{\nu 0} \\
\dot Z_n^\mu &= G^{\mu\nu}\left(P_{\bar Z_n^\nu} - n B_{\nu\kappa} Z_n^\kappa\right) \\
\dot{\bar Z}_n^\mu &= G^{\mu\nu}\left(P_{Z_n^\nu} + n B_{\nu\kappa} \bar Z_n^\kappa\right) \\
\dot P_{\rho 0} &= -\left(\partial_\rho G^{\mu\nu}\right)\left(\frac{1}{2} P_{\mu 0} P_{\nu 0} + \sum_{n=1}^\infty \left(P_{\bar Z_n^\mu} - n B_{\mu\kappa} Z_n^\kappa\right)\left(P_{Z_n^\nu} + n B_{\nu\kappa}\bar Z_n^\kappa\right)\right) - \\
&\quad - \sum_{n=1}^\infty n^2 Z_n^\mu \bar Z_n^\nu \left(\partial_\rho G_{\mu\nu}\right) + \\
&\quad + G^{\mu\nu}\sum_{n=1}^\infty n\left(\left(\partial_\rho B_{\mu\kappa}\right) Z_n^\kappa \left(P_{Z_n^\nu} + n B_{\nu\kappa}\bar Z_n^\kappa\right) - \left(\partial_\rho B_{\mu\kappa}\right)\bar Z_n^\kappa\left(P_{\bar Z_n^\nu} - n B_{\nu\kappa} Z_n^\kappa\right)\right) \\
&= -\left(\partial_\rho G^{\mu\nu}\right)\left(\frac{1}{2}\dot X_{\mu 0}\dot X_{\nu 0} + \sum_{n=1}^\infty \dot Z_{\mu n}\dot{\bar Z}_{\nu n}\right) - \sum_{n=1}^\infty n^2 Z_n^\mu \bar Z_n^\nu\left(\partial_\rho G_{\mu\nu}\right) + \\
&\quad + G^{\mu\nu}\sum_{n=1}^\infty n\left(\left(\partial_\rho B_{\mu\kappa}\right) Z_n^\kappa \dot{\bar Z}_{\nu n} - \left(\partial_\rho B_{\mu\kappa}\right)\bar Z_n^\kappa \dot Z_{\nu n}\right) \\
&= \left(\partial_\rho G_{\mu\nu}\right)\left(\frac{1}{2}\dot X_0^\mu \dot X_0^\nu + \sum_{n=1}^\infty\left(\dot Z_n^\mu \dot{\bar Z}_n^\nu - n^2 Z_n^\mu \bar Z_n^\nu\right)\right) + \left(\partial_\rho B_{\mu\nu}\right)\sum_{n=1}^\infty n\left(\dot{\bar Z}_n^\mu Z_n^\nu - \dot Z_n^\mu \bar Z_n^\nu\right) \\
&= \left(\partial_\rho G_{\mu\nu}\right)\left(\frac{1}{2}\dot X_0^\mu \dot X_0^\nu + \sum_{n=1}^\infty\left(\dot Z_n^\mu \dot{\bar Z}_n^\nu - n^2 Z_n^\mu \bar Z_n^\nu\right)\right) + \left(\partial_\rho B_{\mu\nu}\right)\sum_{n=1}^\infty n\partial_\tau\left(\bar Z_n^\mu Z_n^\nu\right) \\
\dot P_{Z_n^\kappa} &= n G^{\mu\nu} B_{\mu\kappa}\left(P_{Z_n^\nu} + n B_{\nu\lambda}\bar Z_n^\lambda\right) - n^2 \bar Z_n^\nu G_{\nu\kappa} \\
&= n B_{\mu\kappa}\dot{\bar Z}_n^\mu - n^2 \bar Z_n^\nu G_{\nu\kappa} \\
\dot P_{\bar Z_n^\kappa} &= n G^{\mu\nu} B_{\mu\kappa}\left(P_{\bar Z_n^\nu} - n B_{\nu\lambda} Z_n^\lambda\right) - n^2 Z_n^\nu G_{\nu\kappa} \\
&= n B_{\mu\kappa}\dot Z_n^\mu - n^2 \bar Z_n^\nu G_{\nu\kappa}\,.
\end{aligned} \tag{679}$$

(The right hand side of some equations contains terms with ordinary (non-covariant) derivatives, that do not transform as vectors. That should be because the respective left hand side are not vectors, either, as has been discussed above. The free index is instead that of a coordinate field.)

One finds the following second derivative of the center-of-mass position:

$$\begin{aligned}
\ddot{X}_0^\lambda &= \dot{G}^{\lambda\rho} P_{\rho 0} + G^{\lambda\rho} \dot{P}_{\rho 0} \\
&= \underbrace{\left(\partial_\mu G^{\lambda\rho}\right) G_{\rho\kappa} \dot{X}_0^\mu \dot{X}_0^\kappa + \frac{1}{2} G^{\lambda\rho} \left(\partial_\rho G_{\mu\nu}\right) \dot{X}_0^\mu \dot{X}_0^\nu}_{=-\Gamma_\mu{}^\lambda{}_\nu \dot{X}_0^\mu \dot{X}_0^\nu} + \\
&\quad + G^{\lambda\rho} \left(\partial_\rho G_{\mu\nu}\right) \sum_{n=1}^\infty \left(\dot{Z}_n^\mu \dot{\bar{Z}}_n^\nu - n^2 Z_n^\mu \bar{Z}_n^\nu \right) + G^{\lambda\rho} \left(\partial_\rho B_{\mu\nu}\right) \sum_{n=1}^\infty n\, \partial_\tau \left(\bar{Z}_n^\mu Z_n^\nu \right) .
\end{aligned} \tag{680}$$

The first term, which is independent of the internal oscillations of the string, is the usual interaction of a relativistic point with the gravitational field. If all the $Z_n^\mu, \bar{Z}_n^\mu$ are set to zero, so that the string collapses to a point, then then above equation simply describes the usual geodesic motion of a point in spacetime (*cf.* (1156), p. 294).

3.12 (Translation between Schrödinger- and Fock representation) In the literature one mostly finds a Heisenberg picture and Fock representation quantization of the string oscillations. These are obtained by writing the classical solutions to the string's equations of motion (in conformal gauge and for Minkowski target space $G_{\mu\nu} = \eta_{\mu\nu}$) as

$$\begin{aligned}
& \hat{X}^\mu(\tau, \sigma) \\
= \; & \hat{x}^\mu + \frac{1}{2\pi T} \hat{p}^\mu \tau + \frac{i}{\sqrt{4\pi T}} \sum_{n \neq 0} \left(\frac{1}{n} \hat{\alpha}_n^\mu e^{-in\tau} e^{in\sigma} + \frac{1}{n} \hat{\tilde{\alpha}}_n^\mu e^{-in\tau} e^{-in\sigma} \right) \\
= \; & \hat{x}^\mu + \frac{1}{2\pi T} \hat{p}^\mu \tau + \frac{i}{\sqrt{4\pi T}} \sum_{n > 0} \frac{1}{n} \left(\left(\hat{\alpha}_n^\mu e^{-in\tau} - \hat{\tilde{\alpha}}_{-n}^\mu e^{+in\tau} \right) e^{in\sigma} + \left(\hat{\tilde{\alpha}}_n^\mu e^{-in\tau} - \hat{\alpha}_{-n}^\mu e^{+in\tau} \right) e^{-in\sigma} \right) \\
= \; & \hat{x}^\mu + \frac{1}{2\pi T} \hat{p}^\mu \tau + \frac{i}{\sqrt{4\pi T}} \sum_{n > 0} \frac{1}{n} \left(\left(\hat{\alpha}_n^\mu(\tau) - \hat{\tilde{\alpha}}_{-n}^\mu(\tau) \right) e^{in\sigma} + \left(\hat{\tilde{\alpha}}_n^\mu(\tau) - \hat{\alpha}_{-n}^\mu(\tau) \right) e^{-in\sigma} \right) .
\end{aligned} \tag{681}$$

(This oscillator expansion follows [154], p. 19, and can be obtained from [115], p. 66, by changing integration bounds from $[0, \pi]$ to $[0, 2\pi]$). The canonical momentum in this representation is thus

$$\begin{aligned}
& \hat{P}^\mu(\tau, \sigma) = T \partial_\tau \hat{X}^\mu(\tau, \sigma) \\
= \; & \frac{1}{2\pi} \hat{p}^\mu + \frac{\sqrt{T}}{\sqrt{4\pi}} \sum_{n > 0} \left(\left(\hat{\alpha}_n^\mu(\tau) + \hat{\tilde{\alpha}}_{-n}^\mu(\tau) \right) e^{in\sigma} + \left(\hat{\tilde{\alpha}}_n^\mu(\tau) + \hat{\alpha}_{-n}^\mu(\tau) \right) e^{-in\sigma} \right) .
\end{aligned} \tag{682}$$

Comparison of (681) and (682) with (657) gives the following relations between the Schrödinger and the Fock representation ($n \geq 1$ throughout):

$$
\begin{array}{|ccc|}
\hline
\text{Schrödinger} & & \text{Fock}\\
\hline
\hat{X}_0^\mu & = & \sqrt{2\pi T}\left(x^\mu + \frac{1}{2\pi T}\hat{p}^\mu \tau\right)\\
\hat{P}_0^\mu & = & \frac{1}{\sqrt{2\pi T}}\hat{p}^\mu\\
\frac{1}{\sqrt{2}}\hat{P}_0^\mu & = & \alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{1}{\sqrt{4\pi T}}\hat{p}^\mu\\
 & & \\
\hat{Z}_n^\mu & = & \frac{1}{\sqrt{2}}\frac{i}{n}\left(\hat{\alpha}_n^\mu - \hat{\bar{\alpha}}_{-n}^\mu\right)\\
\hat{\bar{Z}}_n^\mu & = & \frac{1}{\sqrt{2}}\frac{i}{n}\left(\hat{\bar{\alpha}}_n^\mu - \hat{\alpha}_{-n}^\mu\right)\\
\hat{P}_{\bar{Z}_{\mu n}} & = & \frac{1}{\sqrt{2}}\left(\hat{\alpha}_n^\mu + \hat{\bar{\alpha}}_{-n}^\mu\right)\\
\hat{P}_{Z_{\mu n}} & = & \frac{1}{\sqrt{2}}\left(\hat{\bar{\alpha}}_n^\mu + \hat{\alpha}_{-n}^\mu\right)\\
 & & \\
\frac{1}{\sqrt{2}}\left(\hat{P}_{\bar{Z}_{\mu n}} - in\hat{Z}_n^\mu\right) & = & \hat{\alpha}_n^\mu\\
\frac{1}{\sqrt{2}}\left(\hat{P}_{Z_{\mu n}} + in\hat{\bar{Z}}_n^\mu\right) & = & \hat{\alpha}_{-n}^\mu\\
\frac{1}{\sqrt{2}}\left(\hat{P}_{Z_{\mu n}} - in\hat{\bar{Z}}_n^\mu\right) & = & \hat{\bar{\alpha}}_n^\mu\\
\frac{1}{\sqrt{2}}\left(\hat{P}_{\bar{Z}_{\mu n}} + in\hat{Z}_n^\mu\right) & = & \hat{\bar{\alpha}}_{-n}^\mu\\
 & & \\
\begin{array}{rcl}
\left[\hat{Z}_n^\mu, \hat{P}_{Z_{\nu m}}\right] & = & iG^{\mu\nu}\delta_{nm}\\
\left[\hat{\bar{Z}}_n^\mu, \hat{P}_{\bar{Z}_{\nu m}}\right] & = & iG^{\mu\nu}\delta_{nm}
\end{array} & \Leftrightarrow & \begin{array}{rcl}
\left[\hat{\alpha}_n^\mu, \hat{\alpha}_{-m}^\nu\right] & = & nG^{\mu\nu}\delta_{nm}\\
\left[\hat{\bar{\alpha}}_n^\mu, \hat{\bar{\alpha}}_{-m}^\nu\right] & = & nG^{\mu\nu}\delta_{nm}
\end{array}\\
 & & \\
\begin{array}{rcl}
\left(\hat{Z}_n^\mu\right)^\dagger & = & \hat{\bar{Z}}_n^\mu\\
\left(\hat{P}_{Z_n^\mu}\right)^\dagger & = & \hat{P}_{\bar{Z}_n^\mu}
\end{array} & \Leftrightarrow & \begin{array}{rcl}
\left(\hat{\alpha}_n^\mu\right)^\dagger & = & \hat{\alpha}_{-n}^\mu\\
\left(\hat{\bar{\alpha}}_n^\mu\right)^\dagger & = & \hat{\bar{\alpha}}_{-n}^\mu
\end{array}\\
\hline
\end{array}
\tag{683}
$$

Let $W := -\sum_{n=1}^{\infty} n\hat{Z}_n^\mu \hat{\bar{Z}}_n^\nu G_{\mu\nu}$, then:

$$
\begin{array}{|ccc|}
\hline
\frac{1}{\sqrt{2}}\, e^{W}\, \hat{P}_{\bar{Z}_{\mu n}}\, e^{-W} & = & \hat{\alpha}_n^\mu\\
\frac{1}{\sqrt{2}}\, e^{-W}\, \hat{P}_{Z_{\mu n}}\, e^{W} & = & \hat{\alpha}_{-n}^\mu\\
\frac{1}{\sqrt{2}}\, e^{W}\, \hat{P}_{Z_{\mu n}}\, e^{-W} & = & \hat{\bar{\alpha}}_n^\mu\\
\frac{1}{\sqrt{2}}\, e^{-W}\, \hat{P}_{\bar{Z}_{\mu n}}\, e^{W} & = & \hat{\bar{\alpha}}_{-n}^\mu\\
\hline
\end{array}
$$

(The oscillator definitions should be compared to those of the *super-oscillator* in 2.63 (p.61)).

3.13 (State spaces in Schrödinger and Fock representation) The choice of the Schrödinger representation makes manifest a subtlety in the definition of the space of states for the quantized string. Consider a usual Hilbert space for systems in Schrödinger representation, say $L^2(M)$, the space of complex valued square interable functions over configuration space $\mathcal{M}^{(\mathrm{conf})}$ with the usual scalar product. The first thing to note is that this space, by construction, *is* a Hilbert space, not a Krein space, since its inner product is positive definite. This is in contrast to the Fock space used in the string theory literature. This Fock space is constructed by defining oscillator vacua $|p, 0\rangle$ by the relations

$$\hat{P}_0^\mu\,|p, 0\rangle = p^\mu$$

$$\hat{\alpha}^\mu_n \left|p,0\right\rangle = 0, \quad n \geq 1\,, \tag{684}$$

from which all states are obtained by acting with the creation operators $\hat{\alpha}^\mu_{-n}$, $n \geq 1$. The obvious inner product on this space is indefinite due to the Lorentzian signature of the commutator $\left[\hat{\alpha}^\mu_n, \hat{\alpha}^\nu_{-n}\right] = nG^{\mu\nu}$. From this indefiniteness and the resulting existence of null-norm states follows the essential existence of gauge degrees of freedom in the string's spectrum.

It is clear that the above Fock space cannot coincide with the Hilbert space $L^2\left(\mathcal{M}^{(\mathrm{conf})}\right)$. In particular, the vacuum states $\left|p,0\right\rangle$ cannot be elements of $L^2\left(\mathcal{M}^{(\mathrm{conf})}\right)$, because they cannot be square integrable with respect to $Z^0_n, \bar{Z}^0_n$:

$$\begin{aligned} & \alpha^0_n \left|p,0\right\rangle = 0 \\ \Leftrightarrow \quad & \left(\eta^{00}\hat{P}_{\bar{Z}^0_n} - in\hat{Z}^0_n\right)\left|p,0\right\rangle = 0 \\ \Leftrightarrow \quad & \left|p,0\right\rangle \propto e^{-\eta_{00} n Z^0_n \bar{Z}^0_n} = e^{+nZ^0_n\bar{Z}^0_n}\,. \end{aligned} \tag{685}$$

This problem has, for example, been noted in [26], in the context of 1+1 dimensional general relativity. One obvious way out might be to demand that the vacuum be annihilated by $\hat{\alpha}^0_{-n}$, $n \geq 1$ instead. This leads to a square integrable vacuum. However, this vacuum has the serious defect that it is obviously no longer Lorentz invariant.

The above problem, in another guise, is precisely that encountered in the path integral quantization of the string, where the path integral over X^0 has a wrong-sign gaussian weight. According to [223], p. 34 and pp. 82-83, this should be remedied by a contour deformation so that X^0 is integrated over the *imaginary* axis instead of over the real axis of the complex plane.

We will adopt this technique to define a Schrödinger-representation space of states $L^{2\prime}\left(\mathcal{M}^{(\mathrm{conf})}\right)$, which is obtained from the usual $L^2\left(\mathcal{M}^{(\mathrm{conf})}\right)$ by demanding that the integral in the scalar product, which thus becomes a mere inner product, be over imaginary X^0-values. This space then coincides, by construction, with the traditional Fock space of states defined above.

We can now give the quantum version of the supersymmetry generator of the NSR string in Schrödinger representation.

3.14 (The supersymmetry generators in Schrödinger representation) We want to find the quantum version of the expression (653) for the generator of world-sheet supersymmetry transformations. First consider the special case that all background fields vanish so that

$$G_{\mu\nu} = \eta_{\mu\nu}\,, \quad B_{\mu\nu} = 0$$

in the following. Then one has

$$\begin{aligned} S_{\pm(0)} &= \int d\sigma\; \psi^\mu_\pm \mathcal{P}_{\mu\pm} \\ &= \sqrt{2}\left(\left(c^{\mu 0}_\pm + c^{*\mu 0}_\pm\right)\dot{X}_{\mu 0} + \sum_{n=1}^\infty \left(c^{\mu n}_\pm\left(P_{Z^\mu_n} \mp in\,\bar{Z}_{\mu n}\right) + c^{*\mu n}_\pm\left(P_{\bar{Z}^\mu_n} \pm in\, Z_{\mu n}\right)\right)\right) \end{aligned} \tag{686}$$

In terms of oscillators this reads

$$
\begin{aligned}
J_{+(0)} &= \frac{1}{\sqrt{2}T}\left(\left(c_+^{\mu 0} + c_+^{*\mu 0}\right)\dot{X}_{\mu 0} + \sum_{n=1}^{\infty}\left(c_+^{\mu n}\hat{\alpha}_{\mu n} + c_+^{*\mu n}\hat{\bar{\alpha}}_{\mu -n}\right)\right) \\
J_{-(0)} &= \frac{1}{\sqrt{2}T}\left(\left(c_-^{\mu 0} + c_-^{*\mu 0}\right)\dot{X}_{\mu 0} + \sum_{n=1}^{\infty}\left(c_-^{\mu n}\hat{\alpha}_{\mu -n} + c_-^{*\mu n}\alpha_{\mu n}\right)\right) .
\end{aligned}
\tag{687}
$$

Quantization is straightforward since no operator ordering ambiguity is encountered:

$$
\begin{aligned}
\hat{J}_{\pm(0)} &= \frac{1}{\sqrt{2}T}\left(\left(\hat{c}_\pm^{\mu 0} + \hat{c}_\pm^{\dagger\mu 0}\right)(-i)\partial_{X_0^\mu} + \sum_{n=1}^{\infty}\left(\hat{c}_\pm^{\mu n}\left(-i\partial_{Z_n^\mu} \mp in\,\hat{\bar{Z}}_{\mu n}\right) + \hat{c}_\pm^{\dagger\mu n}\left(-i\partial_{\bar{Z}_n^\mu} \pm in\,\hat{Z}_{\mu n}\right)\right)\right) \\
&= \frac{-i}{\sqrt{2}T}\left(\left(\hat{c}_\pm^{\mu 0} + \hat{c}_\pm^{\dagger\mu 0}\right)\partial_{X_0^\mu} + \sum_{n=1}^{\infty}\left(\hat{c}_\pm^{\mu n}\left(\partial_{Z_n^\mu} \pm n\,\hat{\bar{Z}}_{\mu n}\right) + \hat{c}_\pm^{\dagger\mu n}\left(\partial_{\bar{Z}_n^\mu} \mp n\,\hat{Z}_{\mu n}\right)\right)\right) .
\end{aligned}
\tag{688}
$$

In order to make manifest the special nature of this operator in Schrödinger representation we define *complex coordinates on configuration space*: First, spacetime indices μ and mode indices n are united in a single multi-index i:

$$
(\mu n) \to i = i(\mu, n) . \tag{689}
$$

This allows to introduce the following modified notation:

$$
\begin{aligned}
Z_n^\mu &\to Z^i \\
\bar{Z}_n^\mu &\to Z^{\bar{i}} \\
P_{Z^{\mu n}} &\to P_i \\
P_{\bar{Z}_n^\mu} &\to P_{\bar{i}} \\
\partial_{Z_n^\mu} &\to \partial_i \\
\partial_{\bar{Z}^{\mu n}} &\to \partial_{\bar{i}} \\
\hat{c}_+^{\mu n} &\to \hat{c}^i \\
\hat{c}_-^{\mu n} &\to \hat{c}^{\dagger i} \\
c_+^{\dagger\mu n} &\to c^{\dagger\bar{i}} \\
c_-^{\dagger\mu n} &\to c^{\bar{i}} .
\end{aligned}
\tag{690}
$$

There is a natural involution $(\cdot)^*$ on these operators defined by

$$
\begin{aligned}
&\left(x^i\right)^* = x^{\bar{i}} \\
&\left(x^{\bar{i}}\right)^* = x^i, \qquad x \in \left\{X, P, c, c^*, \hat{X}, \hat{P}, \hat{c}, \hat{c}^\dagger\right\} .
\end{aligned}
\tag{691}
$$

The metric tensor on this manifold is defined to be

$$
\begin{aligned}
G_{(\mathrm{conf})}^{\bar{j}i} = G_{(\mathrm{conf})}^{i\bar{j}} &:= \left\{\hat{c}^i, \hat{c}^{\dagger\bar{j}}\right\} \\
&= \left\{\hat{c}^{\dagger i}, \hat{c}^{\bar{j}}\right\} = G^{\mu(i)\,\nu(\bar{j})}\delta^{n(i)\,m(\bar{j})} \\
G_{(\mathrm{conf})}^{ij} &= \left\{\hat{c}^i, \hat{c}^{\dagger j}\right\} = 0 \\
G_{(\mathrm{conf})}^{\bar{i}\bar{j}} &= \left\{\hat{c}^{\bar{i}}, \hat{c}^{\dagger\bar{j}}\right\} = 0 .
\end{aligned}
\tag{692}
$$

We furthermore define the superpotential

$$\begin{aligned} W &:= -\sum_{n=1}^{\infty} n Z_n^\mu \bar{Z}_{\mu n} \\ &:= w_{i\bar{j}} Z^i Z^{\bar{j}} , \end{aligned} \tag{693}$$

where $w_{i\bar{j}} = n(i) G_{(\mathrm{conf}) i\bar{j}}$ (*cf.* 2.63 (p.61)).

The zero modes of the fermions, which we will denote by

$$\hat{\Gamma}^\mu_\pm := \hat{c}^{\mu 0}_\pm + \hat{c}^{\dagger \mu 0}_\pm , \tag{694}$$

are special in that they generate two anticommuting copies of a Clifford algebra

$$\left\{ \hat{\Gamma}^\mu_s , \hat{\Gamma}^\nu_{s'} \right\} = G^{\mu\nu} \delta_{ss'} .$$

Any two such copies are isomorphic to the canonical creator/annihilator algebra via

$$\begin{aligned} \hat{c}^\mu &:= \frac{1}{\sqrt{2}} \left(\hat{\Gamma}^\mu_+ + i \hat{\Gamma}^\mu_- \right) \\ \hat{c}^{\dagger\mu} &:= \frac{1}{\sqrt{2}} \left(\hat{\Gamma}^\mu_+ - i \hat{\Gamma}^\mu_- \right) , \end{aligned} \tag{695}$$

so that

$$\begin{aligned} \left\{ \hat{c}^\mu , \hat{c}^\nu \right\} &= 0 \\ \left\{ \hat{c}^{\dagger\mu} , \hat{c}^{\dagger\nu} \right\} &= 0 \\ \left\{ \hat{c}^\mu , \hat{c}^{\dagger\nu} \right\} &= \frac{1}{2} G^{\mu\nu} . \end{aligned} \tag{696}$$

This allows us to identify, say, $\hat{c}^{\mu 0}_+ = \hat{c}^\mu$ and $\hat{c}^{\dagger\mu 0}_+ = \hat{c}^{\dagger\mu}$. Because of $\hat{\Gamma}^\mu_- = \frac{1}{\sqrt{2}} i \left(\hat{c}^{\dagger\mu} - \hat{c}^\mu \right)$ this then implies the identification $\hat{c}^{\mu 0}_- = -i \hat{c}^\mu$ and $\hat{c}^{\dagger \mu 0}_- = i \hat{c}^{\dagger\mu}$. We then make contact with the notation of 2.2 (p.16) by means of the relations

$$\begin{aligned} \hat{\gamma}^\mu_+ &= \hat{\mathrm{c}}^{\dagger\mu} + \hat{\mathrm{c}}^\mu = \sqrt{2}\, \hat{\Gamma}^\mu_+ \\ \hat{\gamma}^\mu_- &= \hat{\mathrm{c}}^{\dagger\mu} - \hat{\mathrm{c}}^\mu = -i\sqrt{2}\, \hat{\Gamma}^\mu_- . \end{aligned} \tag{697}$$

With this notation the supersymmetry generators now read

$$\begin{aligned} \hat{J}_{+(0)} &= \frac{-i}{\sqrt{2T}} \left(\frac{1}{\sqrt{2}} \left(\hat{c}^{\dagger\mu} + \hat{c}^\mu \right) \partial_{X_0^\mu} + \hat{c}^i \left(\partial_i - (\partial_i W) \right) + \hat{c}^{\dagger\bar{i}} \left(\partial_{\bar{i}} + (\partial_{\bar{i}} W) \right) \right) \\ \hat{J}_{-(0)} &= \frac{-i}{\sqrt{2T}} \left(\frac{i}{\sqrt{2}} \left(\hat{c}^{\dagger\mu} - \hat{c}^\mu \right) \partial_{X_0^\mu} + \hat{c}^{\dagger i} \left(\partial_i + (\partial_i W) \right) + \hat{c}^{\bar{i}} \left(\partial_{\bar{i}} - (\partial_{\bar{i}} W) \right) \right) . \end{aligned} \tag{698}$$

One recognizes the center-of-mass components as the two exterior derivative operators over (Minkowski) spacetime

$$\begin{aligned} \mathbf{d}_0 &= \hat{c}^{\dagger\mu} \partial_{X_0^\mu} \\ \mathbf{d}_0^\dagger &= -\hat{c}^\mu \partial_{X_0^\mu} \end{aligned} \tag{699}$$

and the other components as the holomorphic and antiholomorphic exterior derivatives over the oscillator configuration space

$$\begin{aligned}
\mathbf{d}^{J+} &= \hat{c}^{\dagger i}\partial_i \\
\mathbf{d}^{J-} &= \hat{c}^{\dagger \bar{i}}\partial_{\bar{i}} \\
\mathbf{d}^{\dagger J+} &= -\hat{c}^{\bar{i}}\partial_{\bar{i}} \\
\mathbf{d}^{\dagger J-} &= -\hat{c}^{i}\partial_i \,,
\end{aligned} \tag{700}$$

or rather their deformations by the superpotential W

$$\begin{aligned}
\mathbf{d}^{WJ+} &= e^{-W}\mathbf{d}^{J+}e^{W} \\
\mathbf{d}^{WJ-} &= e^{-W}\mathbf{d}^{J-}e^{W} \\
\mathbf{d}^{\dagger WJ+} &= e^{W}\mathbf{d}^{\dagger J+}e^{-W} \\
\mathbf{d}^{\dagger WJ-} &= e^{W}\mathbf{d}^{\dagger J-}e^{-W} \,.
\end{aligned} \tag{701}$$

This finally allows us to write

$$\begin{aligned}
\hat{J}_{+(0)} &= \frac{-i}{\sqrt{2T}}\left(\frac{1}{\sqrt{2}}\left(\mathbf{d}_0 + \mathbf{d}_0^\dagger\right) + \left(\mathbf{d}^{WJ-} - \mathbf{d}^{\dagger WJ-}\right)\right) \\
\hat{J}_{-(0)} &= \frac{-i}{\sqrt{2T}}\left(\frac{i}{\sqrt{2}}\left(\mathbf{d}_0 - \mathbf{d}_0^\dagger\right) + \left(\mathbf{d}^{WJ+} - \mathbf{d}^{\dagger WJ+}\right)\right) \,.
\end{aligned} \tag{702}$$

The fermionic center-of-mass modes are seen to give rise to the two exterior Dirac operators over spacetime. This expression has an obvious generalization to curved spacetime, i.e. to non-trivial gravitational background, by inserting the well known terms involving the Levi-Civita connection. From the resulting operator one may read off the associated functional supercharge, see 3.16 (p.170).

The non-zero modes of the supersymmetry generator is found to be ($m > 0$ in the following)

$$\begin{aligned}
\hat{J}_{\pm(+m)} &= \frac{-i}{\sqrt{2T}}\left(\left(\hat{c}^{\mu 0}_{\pm} + \hat{c}^{\dagger\mu 0}_{\pm}\right)\partial_{\bar{Z}^\mu_m} + \hat{c}^{\dagger\mu m}\partial_{X^\mu_0}\right) + \\
&+\frac{-i}{\sqrt{2T}}\sum_{n=m+1}^{\infty}\left(\hat{c}^{\mu(n-m)}_{\pm}\left(\partial_{Z^\mu_n} \pm n\,\hat{\bar{Z}}_{\mu n}\right) + \hat{c}^{\dagger\mu n}_{\pm}\left(\partial_{\bar{Z}^\mu_{(n-m)}} \mp n\,\hat{Z}_{\mu(n-m)}\right)\right)
\end{aligned} \tag{703}$$

$$\begin{aligned}
\hat{J}_{\pm(-m)} &= \frac{-i}{\sqrt{2T}}\left(\left(\hat{c}^{\mu 0}_{\pm} + \hat{c}^{\dagger\mu 0}_{\pm}\right)\partial_{Z^\mu_m} + \hat{c}^{\mu m}\partial_{X^\mu_0}\right) + \\
&+\frac{-i}{\sqrt{2T}}\sum_{n=m+1}^{\infty}\left(\hat{c}^{\mu n}_{\pm}\left(\partial_{Z^\mu_{(n-m)}} \pm n\,\hat{\bar{Z}}_{\mu(n-m)}\right) + \hat{c}^{\dagger\mu(n-m)}_{\pm}\left(\partial_{\bar{Z}^\mu_n} \mp n\,\hat{Z}_{\mu n}\right)\right) \,.
\end{aligned} \tag{704}$$

These may be obtained from the zero mode by means of the generalized number operators

$$\hat{N}_{(+m)} := \sum_{n=1}^{\infty}\hat{c}^{\dagger\mu(n+m)}\hat{c}^{n}_{\mu} + \hat{c}^{\dagger\mu(m)}\left(\hat{c}^0_\mu + \hat{c}^{\dagger 0}_\mu\right)$$

$$\hat{N}_{(-m)} \quad := \quad \sum_{n=m+1}^{\infty} \hat{c}^{\dagger\mu(n-m)} \hat{c}^{n}_{\mu} + \left(\hat{c}^{0}_{\mu} + \hat{c}^{\dagger 0}_{\mu}\right) \hat{c}^{\mu(m)} \,. \tag{705}$$

3.15 (Literature) The fact that the supersymmetry generators of the superstring can naturally be identified with Dirac operators has been particularly emphasized in [102], §7.

3.16 (NSR String in gravitational background via the Hamiltonian route) The zero mode component of (702), being the exterior Dirac operator over spacetime, is immediately generalized to a non-trivial gravitational background by replacing the ordinary derivative ∂_μ by the covariant derivative on the exterior bundle

$$\begin{aligned} \partial_\mu \quad &\rightarrow \quad \hat{\nabla}_\mu \\ &= \quad \partial_\mu - {\omega_\mu}^a{}_b {\hat{\mathrm{e}}^\dagger}{}^b \hat{\mathrm{e}}_a \\ &= \quad \partial_\mu + \omega_{\mu ab} {\hat{\mathrm{e}}^\dagger}{}^a \hat{\mathrm{e}}_b \,, \end{aligned} \tag{706}$$

where ${\omega_\mu}^a{}_b$ is the spin connection and ${\hat{\mathrm{e}}^\dagger}{}^a = e^a{}_\mu {\hat{\mathrm{c}}^\dagger}{}^\mu$. (All this is discussed in detail in 2.2 (p.16), see also the appendix, e.g. B.11 (p.301).) Re-expressing the covariant derivative in terms of the Clifford generators gives (where it is essential that $\omega_{\mu ab} = -\omega_{\mu ba}$)

$$\omega_{\mu ab} {\hat{\mathrm{e}}^\dagger}{}^a \hat{\mathrm{e}}^b \quad = \quad \frac{1}{4} \omega_{\mu ab} \left(\hat{\gamma}^a_+ \hat{\gamma}^b_+ - \hat{\gamma}^a_- \hat{\gamma}^b_- \right) \,. \tag{707}$$

All this applies to the zero modes of the fermionic fields, not to the oscillators. But from the knowledge of the zero modes alone we obtain the associated functional operator. It must read (using (649) and (653))

$$\mathcal{S}_\pm \quad = \quad \psi^\mu_\pm \left(\mathcal{P}_{\mu\pm} - i \frac{1}{2} \omega_{\mu ab} \left(\psi^a_+ \psi^b_+ - \psi^a_- \psi^b_- \right) \right) \,. \tag{708}$$

This result agrees with that reported in [75], which has been obtained by Lagrangian methods. (Note, when comparing with this reference, that there ψ_+ carries an additional phase factor i).

By again inserting the mode expansion of the fields into this expression and integrating over the string one finds the expression of the supersymmetry generators for arbitrary gravitational background in the mode representation.

3.4 The supermembrane

3.17 (Topology, metric and ADM split) We assume, as usual, that the brane has spatial topology Σ, where Σ is p-dimensional, and a $(p+1)$-dimensional world-volume

$$\mathcal{M}_p^{(\text{brane})} = \mathbb{R} \otimes \Sigma \,.$$

The coordinates x^μ range over the worldvolume, while coordinates x^i parameterize the spatial hypeslice Σ. According to the common ADM prescription, the metric $g_{\mu\nu}$ on $\mathcal{M}^{(\text{brane})}$ is split into a lapse function N, a shift vector N^i and a spatial metric $\tilde{g}_{ij}$ on Σ (see for instance [195], §21.4):

$$\begin{aligned}
g_{\mu\nu} &= \begin{bmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{bmatrix} = \begin{bmatrix} N_k N^k - N^2 & N_j \\ N_i & \tilde{g}_{ij} \end{bmatrix} \\
g^{\mu\nu} &= \begin{bmatrix} g^{00} & g^{0j} \\ g^{i0} & g^{ij} \end{bmatrix} = \frac{1}{N^2} \begin{bmatrix} -1 & N^j \\ N^i & N^2 \tilde{g}^{ij} - N^i N^k \end{bmatrix}
\end{aligned} \tag{709}$$

The determinant of the metric is

$$\sqrt{-g} = N\sqrt{\tilde{g}}\,, \tag{710}$$

where we use the convenient shorthand

$$g \;:=\; \det(g_{\cdot\cdot})\;. \tag{711}$$

It is understood that spatial indices, like those carried by the shift vector, are raised and lowered by means of the spatial metric tensor

$$N^i \;:=\; \tilde{g}^{ij} N_j\,. \tag{712}$$

The *unit timelike normal vector* $n = n^\mu \partial_\mu$ is defined by the requirement that it be orthogonal to all tangent vectors of Σ

$$n \cdot \partial_i = 0\,, \quad i > 0$$

and that it be of unit timelike length

$$n \cdot n = -1\,.$$

The first requirement is obviously satisfied for

$$n_\mu = g_{\mu\nu} n^\nu \propto \delta^0_\mu\,,$$

and using (709) one finds

$$\begin{aligned}
n_\mu &= N\delta^0_\mu \\
n^\mu &= \frac{1}{N}\begin{bmatrix} -1 \\ N^i \end{bmatrix}\,.
\end{aligned} \tag{713}$$

Next we need to introduce Lorentz frames and hence a vielbein field:

$$\begin{aligned}
e^a{}_\mu e^b{}_\nu \eta_{ab} &= g_{\mu\nu} \\
e^a{}_\mu e^b{}_\nu g^{\mu\nu} &= \eta^{ab}\,.
\end{aligned} \tag{714}$$

(Using the vielbein one can alternatively characterize n by its orthogonality to all $e^a = e^a{}_i dx^i$ in the sense that:

$$n_a e^a{}_i \quad = \quad 0\,.) \tag{715}$$

The ADM split furthermore requires to identify the object

$$\tilde{e}^a{}_i := e^a{}_{\mu=i}\,, \tag{716}$$

which is a p-covector on Σ and a Lorentz $p+1$-vector. As with all spatial indices, those of $\tilde{e}^a{}_i$ are raised and lowered with the spatial metric

$$\tilde{e}^{ai} = \tilde{g}^{ij} e^a{}_j\,. \tag{717}$$

It is often useful to have the following expressions for the Lorentz vector components of the unit timelike normal vector:

$$\begin{aligned} n^a &= e^a{}_\mu n^\mu \\ &= \frac{1}{N}\left(-e^a{}_0 + e^a{}_i N^i\right) \\ n_a &= e_a{}^\mu n_\mu \\ &= N e_a{}^0\,. \end{aligned} \tag{718}$$

Using these one shows that every world-covector may be split into tangential and normal Lorentz vectors as follows

$$\begin{aligned} v_a &= e_a{}^\mu v_\mu \\ &= -n_a \underbrace{(n^\mu v_\mu)}_{=v_\perp} + \tilde{e}_a{}^i v_i\,. \end{aligned} \tag{719}$$

This is essential in order to understand the superalgebra in the ADM split. Let P_μ be the generators of translations/reparameterizations and Q_A the generators of supersymmetry transformations. Because of (719) one has

$$\begin{aligned} \left\{Q_A, \bar{Q}_B\right\} &= \gamma^a_{AB} P_a \\ &= -\gamma^a_{AB}\, n_a P_\perp + \gamma^a_{AB}\, \tilde{e}_a{}^i P_i\,. \end{aligned} \tag{720}$$

In the case of reparametrization invariant systems $P_\perp$ is the Hamiltonian constraint which generates reparametrizations normal to the spatial hypersurfaces Σ, while P_i are the generators of reparametrizations tangential to Σ. Compare this with the canonical generator algebra of $N = 1, D = 4$ supergravity given in (803), p. 191 of 4.11 (p.189).

3.18 (Bosonic worldvolume action of the brane) The action of the free brane is, in generalization of the action for the free relativistic particle as well as the Nambo-Goto action of the free string, proportional to the proper worldvolume of the brane:

$$\begin{aligned} S_{(p)\mathrm{NG}} &= -T_p \int\limits_{\mathcal{M}_p^{(\mathrm{brane})}} \mathrm{dvol} \\ &= -T_p \int\limits_{\mathcal{M}_p^{(\mathrm{brane})}} \sqrt{g}\; d^{p+1}x\,, \end{aligned} \tag{721}$$

and hence the Lagrangian density reads

$$\mathcal{L}_{(p)\mathrm{NG}} = -T_p\sqrt{g}\,. \tag{722}$$

(Here T_p is the *tension* of the p-brane which has units of mass^{p+1}.) One may use the action in this form and proceed by canonical techniques. (For instance [205] take this form of the action as the starting point for the supermembrane in light-cone gauge. In [244] a covariant treatment of the bosonic theory is developed.) However, as with the relativistic point and the string, it is often desirable to use a form of the action which is quadratic in the propagating fields, i.e. to use the Polyakov action[44] (this is the approach used in [255])

$$\mathcal{L}_{(p)\mathrm{P}} = -\frac{T_p}{2}\sqrt{g}\left(g^{\mu\nu}\,\partial_\mu X\cdot\partial_\nu X - \Lambda\right)\,. \tag{723}$$

Here X^α are the target space coordinate fields (α is a target space index) and we assume the target space metric to be constant (independent of the X^α) for the moment. It is convenient to abbreviate

$$X^\alpha Y_\alpha := X\cdot Y\,. \tag{724}$$

The quantity

$$h_{\mu\nu} := \partial_\mu X\cdot\partial_\nu X \tag{725}$$

is called the *induced metric*. We have included a yet to be determined constant Λ, which formally plays the role of a *cosmological constant* on the brane. This constant is essential in order for $\mathcal{L}_{\mathrm{NG}}$ and $\mathcal{L}_{\mathrm{P}}$ to be really equivalent. Namely, assume that the metric on the brane is the induced metric, $g_{\mu\nu} = h_{\mu\nu}$, then it follows that

$$(g_{\mu\nu} = h_{\mu\nu}) \quad\Rightarrow\quad \left(\mathcal{L}_{\mathrm{P}} = -\frac{T}{2}\sqrt{g}\,(p+1-\Lambda)\right)\,. \tag{726}$$

Hence we have to set

$$\Lambda = p-1\,, \tag{727}$$

because then the right hand side of the above equation is equal to the Nambu-Goto action $\mathcal{L}_{\mathrm{NG}}$. To check that this is consistent, compute the variation of the Polyakov action with respect to the metric:

$$\begin{aligned} 0 \stackrel{!}{=} \frac{\delta\mathcal{L}_{\mathrm{NG}}}{\delta g^{\mu\nu}} &= -\frac{T}{2}\sqrt{g}\left(h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(h^\mu_\mu - \Lambda\right)\right) \\ &= -\frac{T}{2}\sqrt{g}\left(h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(h^\mu_\mu - p + 1\right)\right)\,. \end{aligned} \tag{728}$$

Which implies that indeed both tensors are equal. (In the case of the string, $d=1$, they only need to be equal up to conformal rescaling, since then the term $-p+1$ in the second line vanishes.) Therefore we have

$$\Lambda = p-1 \Rightarrow \mathcal{L}\sim\tilde{\mathcal{L}}\,, \tag{729}$$

[44] The terms "Nambu-Goto action" and "Polyakov action" are often understood to refer to the string, exclusively. In the following we will however use these terms for the generalization to any number of brane dimensions D.

where the " $\sim$ " means "equal when using the equations of motion", i.e. "equal on shell". We are led to refine our definition of the Polyakov action of the brane as

$$\mathcal{L}_{\mathrm{P}} = -\frac{T}{2}\sqrt{g}\,(g^{\mu\nu}\,\partial_\mu X \cdot \partial_\nu X + 2 - D)\;. \tag{730}$$

It is immediate that for $p = 1$ this coincides with the Polyakov action of the string (*cf.* 3.1 (p.153), eq. (631)), while for $D = 2$ this is the Polyakov action of the membrane as given in [255], eq. (8). Also, for $p = 0$ the above correctly reproduces the action of the relativistic point particle with mass $T = m$ (*cf.* §A (p.293)). This becomes obvious from inspection of the Hamiltonian of the Polyakov action, which is obtained below.

3.19 (Dimensional reduction of branes) From the Nambu-Goto action (721) of the bosonic brane it is obvious that any p-brane may be obtained from the respective $p+1$-brane by a special kind of dimensional reduction:

Suppose that the volume density $\sqrt{-g_p} = \sqrt{-g_p(x^0,\ldots,x^p)}$ of the brane factors, in some suitable coordinates, as

$$\sqrt{-g_p(x^0,\ldots,x^p)} = \sqrt{-g_{p-1}(x^0,\ldots,x^{p-1})}\sqrt{\gamma(x^p)}\,. \tag{731}$$

This means that the extension of the brane along its x^p coordinate is independent of all other coordinates (and in particular independent of the timelike coordinate). Hence the topology of the brane must factor as

$$\mathcal{M}_p^{(\text{brane})} = \mathcal{M}_{p-1}^{(\text{brane})} \otimes G\,,$$

where G is the line segment or the circle S^1. The linear volume of the brane along x^p, to be denoted by R_p is then

$$R_p = \int \sqrt{\gamma(x^p)}\; dx^p\,, \tag{732}$$

and the Nambu-Goto may be rewritten as

$$\begin{aligned} S_{(p)\mathrm{NG}} &= -T_p \int\limits_{\mathcal{M}_{p-1}^{(\text{brane})}\otimes G} \sqrt{-g_{p-1}}\,\sqrt{\gamma}\; d^px\, dx^p \\ &= -R_pT_p \int\limits_{\mathcal{M}_{p-1}^{(\text{brane})}} \sqrt{-g_{p-1}}\; d^px\,. \end{aligned} \tag{733}$$

This is the Nambu-Goto action of a $p-1$-brane with tension

$$T_{p-1} = R_pT_p\,. \tag{734}$$

An important example is the so-called *double dimensional reduction* where the the target space of the brane, spacetime, is compactified on a circle and the brane wraps around that circle. In this case R_p is the radius of the compact dimension and the $p-1$-brane tension is hence related to the compactification scale of spacetime.

See for instance [272], in particular §7.1, for a general brief discussion of the relation between various p-branes and their tension. Also see [76] for a discussion of *double dimensional reduction* in the context of supermembranes and superstrings.

3.20 (Canonical formalism from the Nambu-Goto action) The Nambu-Goto Lagrangian may be written explicitly as

$$\begin{aligned}
\mathcal{L} &= -T\sqrt{-g} \\
&= -T\sqrt{\frac{-1}{(p+1)!}\left(\bar{\epsilon}^{\mu_0\mu_1\cdots\mu_p}\partial_{\mu_1}X^{\alpha_0}\partial_{\mu_1}X^{\alpha_1}\cdots\partial_{\mu_p}X^{\alpha_p}\right)\left(\bar{\epsilon}^{\mu'_0\mu'_1\cdots\mu'_p}\partial_{\mu'_1}X_{\alpha_0}\partial_{\mu'_1}X_{\alpha_1}\cdots\partial_{\mu'_p}X_{\alpha_p}\right)}\,.
\end{aligned} \tag{735}$$

Therefore its canonical momenta are

$$\begin{aligned}
P_\alpha &= T\frac{1}{\sqrt{-g}}\frac{1}{(p+1)!}\left(\frac{\delta}{\delta\dot{X}_\alpha}\,\bar{\epsilon}^{\mu_0\mu_1\cdots\mu_p}\partial_{\mu_0}X^{\alpha_0}\partial_{\mu_1}X^{\alpha_1}\cdots\partial_{\mu_p}X^{\alpha_p}\right)\left(\bar{\epsilon}^{\mu'_0\mu'_1\cdots\mu'_p}\partial_{\mu'_0}X_{\alpha_0}\partial_{\mu'_1}X_{\alpha_1}\cdots\partial_{\mu'_p}X_{\alpha_p}\right) \\
&= T\frac{1}{\sqrt{-g}}\left(\frac{\delta}{\delta\dot{X}_\alpha}\,\partial_0X^{\alpha_0}\partial_1X^{\alpha_1}\cdots\partial_pX^{\alpha_p}\right)\left(\bar{\epsilon}^{\mu'_0\mu'_1\cdots\mu'_p}\partial_{\mu'_0}X_{\alpha_0}\partial_{\mu'_1}X_{\alpha_1}\cdots\partial_{\mu'_p}X_{\alpha_p}\right) \\
&= T\frac{1}{\sqrt{-g}}\left(\partial_1X^{\alpha_1}\cdots\partial_pX^{\alpha_p}\right)\left(\bar{\epsilon}^{\mu_0\mu_1\cdots\mu_p}\partial_{\mu_0}X_{\alpha}\partial_{\mu_1}X_{\alpha_1}\cdots\partial_{\mu_p}X_{\alpha_p}\right)\,,
\end{aligned} \tag{736}$$

so that the respective Hamiltonian density vanishes identically:

$$\begin{aligned}
\mathcal{H} &= P_\alpha\dot{X}^\alpha - \mathcal{L} \\
&= T\frac{1}{\sqrt{-g}}\underbrace{\left(\partial_0X^{\alpha}\partial_1X^{\alpha_1}\cdots\partial_pX^{\alpha_p}\right)\left(\bar{\epsilon}^{\mu_0\mu_1\cdots\mu_p}\partial_{\mu_0}X_{\alpha}\partial_{\mu_1}X_{\alpha_1}\cdots\partial_{\mu_p}X_{\alpha_p}\right)}_{=g} + T\sqrt{-g} \\
&= 0\,.
\end{aligned} \tag{737}$$

This vanishing is familiar from the Nambu-Goto action of the relativistic point particle. The reason is that the dynamics is fully determined by constraints which restrict the brane's motion to a hypersurface in phase space. The points of this hypersurface satisfy the equation[45]

$$P\cdot P = -T^2\,\tilde{g}\,. \tag{739}$$

[45] This can be derived as follows:

$$\begin{aligned}
P\cdot P &= T^2\frac{1}{|g|}\left(\partial_1X^{\alpha_1}\cdots\partial_pX^{\alpha_p}\right)\left(\bar{\epsilon}^{\mu_0\mu_1\cdots\mu_p}\partial_{\mu_0}X_{\alpha}\partial_{\mu_1}X_{\alpha_1}\cdots\partial_{\mu_p}X_{\alpha_p}\right) \\
&\quad \left(\partial_1X^{\alpha'_1}\cdots\partial_pX^{\alpha'_p}\right)\left(\bar{\epsilon}^{\mu'_0\mu'_1\cdots\mu'_p}\partial_{\mu'_0}X^{\alpha}\partial_{\mu'_1}X_{\alpha'_1}\cdots\partial_{\mu'_p}X_{\alpha'_p}\right) \\
&= T^2\frac{1}{|g|}\underbrace{\left(\partial_1X^{\alpha_1}\cdots\partial_pX^{\alpha_p}\right)\left(\bar{\epsilon}^{0\mu_1\cdots\mu_p}\partial_{\mu_1}X_{\alpha_1}\cdots\partial_{\mu_p}X_{\alpha_p}\right)}_{=\tilde{g}} \\
&\quad \underbrace{\partial_0X_\alpha\left(\partial_1X^{\alpha'_1}\cdots\partial_pX^{\alpha'_p}\right)\left(\bar{\epsilon}^{\mu'_0\mu'_1\cdots\mu'_p}\partial_{\mu'_0}X^{\alpha}\partial_{\mu'_1}X_{\alpha'_1}\cdots\partial_{\mu'_p}X_{\alpha'_p}\right)}_{=g} + \\
&\quad +T^2\frac{1}{|g|}\left(\partial_1X^{\alpha_1}\cdots\partial_pX^{\alpha_p}\right)\left(\bar{\epsilon}^{i\mu_1\cdots\mu_p}\partial_{\mu_1}X_{\alpha_1}\cdots\partial_{\mu_p}X_{\alpha_p}\right)
\end{aligned}$$

For the point particle, $p = 0$, this is, with $\tilde{g} = 1$, the usual relation

$$P \cdot P = -T^2 = -m^2\,, \tag{740}$$

while for $p = 1$, where $\tilde{g} = \partial_1 X \cdot \partial_1 X$, this is the Hamiltonian constraint of the string (*cf.* (645))

$$P \cdot P + T^2\, \partial_1 X \cdot \partial_1 X \quad = \quad 0\,. \tag{741}$$

For $p \geq 1$ there are further constraints, expressed by the identities

$$P \cdot \partial_i X \quad = \quad 0\,, \qquad 1 \leq i \leq p\,, \tag{742}$$

which are due to the fact that

$$\partial_i X^\alpha \left(\partial_1 X^{\alpha_1} \cdots \partial_p X^{\alpha_p}\right) \left(\bar{\epsilon}^{\mu_0\mu_1\cdots\mu_p} \partial_{\mu_0} X_\alpha \partial_{\mu_1} X_{\alpha_1} \cdots \partial_{\mu_p} X_{\alpha_p}\right) \quad = \quad 0\,, \qquad 1 \leq i \leq p\,, \tag{743}$$

because of the antisymmetry of the right factor and the double occurrence of ∂_i in the left factor.

These constraints are related to the generators of diffeomorphism perpendicular and parallel to Σ. This becomes obvious in the Hamiltonian formulation of the Polyakov form of the brane's action in 3.21 (p.176).

3.21 (Hamiltonian of the Polyakov action) The canonical momenta are

$$\begin{aligned} P &= -T\sqrt{g} g^{0\mu}\, \partial_\mu X \\ &= -T\sqrt{g} g^{00}\, \dot{X} - T\sqrt{g} g^{0i}\, \partial_i X \\ \Leftrightarrow \dot{X} &= -\frac{1}{T\sqrt{g} g^{00}}\, P - \frac{g^{0i}}{g^{00}}\, \partial_i X\,. \end{aligned} \tag{744}$$

(Here, as in our treatment of the string, a dot indicates the derivative with respect to x^0, i.e. $\dot{X} = \partial_0 X$.)

The Hamiltonian density is

$$\begin{aligned} \mathcal{H} &= \dot{X} \cdot P - \mathcal{L} \\ &= T\sqrt{-g}\left(-g^{00}\, \dot{X} \cdot \dot{X} - g^{0i}\, \dot{X} \cdot \partial_i X + \frac{1}{2} g^{00} \dot{X} \cdot \dot{X} + g^{0i}\, \dot{X} \cdot \partial_i X + \frac{1}{2} g^{ij}\, \partial_i X \cdot \partial_j X - \frac{1}{2}\Lambda\right) \\ &= \frac{T}{2}\sqrt{g}\left(-g^{00}\, \dot{X} \cdot \dot{X} + g^{ij}\, \partial_i X \cdot \partial_j X - \Lambda\right) \\ &= \frac{T}{2}\sqrt{g}\left(-g^{00}\left(\frac{1}{T\sqrt{g} g^{00}}\, P + \frac{g^{0i}}{g^{00}}\, \partial_i X\right)^2 + g^{ij}\, \partial_i X \cdot \partial_j X - \Lambda\right) \end{aligned}$$

$$\underbrace{\partial_i X_\alpha \left(\partial_1 X^{\alpha'_1} \cdots \partial_p X^{\alpha'_p}\right) \left(\bar{\epsilon}^{\mu'_0\mu'_1\cdots\mu'_p} \partial_{\mu'_0} X^\alpha \partial_{\mu'_1} X_{\alpha'_1} \cdots \partial_{\mu'_p} X_{\alpha'_p}\right)}_{\overset{(743)}{=} 0}$$

$$= \quad -T^2 \tilde{g} \tag{738}$$

$$
\begin{aligned}
&= \frac{T}{2}\left(-\frac{1}{T^2\sqrt{g}g^{00}}P\cdot P - \frac{\sqrt{g}}{g^{00}}\left(g^{0i}\,\partial_i X\right)^2 - 2\frac{1}{T}\frac{g^{0i}}{g^{00}}\,P\cdot\partial_i X + \sqrt{g}g^{ij}\,\partial_i X\cdot\partial_j X - \sqrt{g}\Lambda\right) \\
&= \frac{T}{2}\left(\frac{N}{\sqrt{\tilde g}}\frac{1}{T^2}P\cdot P + \frac{\sqrt{\tilde g}}{N}\left(N^i\,\partial_i X\right)^2 + 2\frac{1}{T}N^i\,P\cdot\partial_i X + N\sqrt{\tilde g}\tilde g^{ij}\,\partial_i X\cdot\partial_j X - \right. \\
&\quad \left. -\frac{\sqrt{\tilde g}}{N}\left(N^i\,\partial_i X\right)^2 - N\sqrt{\tilde g}\Lambda\right) \\
&= N\,\frac{1}{2}\left(\frac{1}{T}\frac{1}{\sqrt{\tilde g}}\,P\cdot P + T\sqrt{\tilde g}\tilde g^{ij}\,\partial_i X\cdot\partial_j X - T\sqrt{\tilde g}\,\Lambda\right) + N^i\;\left(P\cdot\partial_i X\right)\,. \qquad (745)
\end{aligned}
$$

In the last lines the special form of the ADM metric (709) has been inserted.

From looking at this expression it seems reasonable to also consider a transformed lapse function

$$\tilde N \;=\; \frac{1}{\sqrt{\tilde g}}N \qquad (746)$$

(and in fact this is what was used in the treatment of the string in 3.3 (p.154), eq. (639)). In terms of this the Hamiltonian density reads:

$$\mathcal{H} \;=\; \tilde N\,\frac{1}{2}\left(\frac{1}{T}P\cdot P + T\tilde g\tilde g^{ij}\,\partial_i X\cdot\partial_j X - T\tilde g\Lambda\right) + N^i\;\left(P\cdot\partial_i X\right)\,. \qquad (747)$$

At this point one can again use the equations of motion to set

$$\partial_i X\cdot\partial_j X \;\sim\; \tilde g_{ij} \qquad (748)$$

so that

$$\tilde g^{ij}\,\partial_i X\cdot\partial_j X \;\sim\; p\,, \qquad (749)$$

where, as above, "$\sim$" denotes on-shell equality. This finally leads to the simple form

$$\mathcal{H} \;\sim\; \tilde N\frac{1}{2}\left(\frac{1}{T}\,P\cdot P + T\,\tilde g\right) + N^i\;\left(P\cdot\partial_i X\right)\,. \qquad (750)$$

Since $\tilde g_{ij}$ is an induced metric tensor, its determinant can, for $p \geq 2$, be expressed in terms of *Nambu brackets* $\{X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_d}\}$ of the embedding coordinates. These are defined in 3.24 (p.179). In terms of these brackets the determinant of the spatial metric reads

$$\tilde g \;=\; \frac{1}{p!}\,\{X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_p}\}\,\{X_{\mu_1}, X_{\mu_2}, \ldots, X_{\mu_p}\}\,. \qquad (751)$$

This allows to express the Hamiltonian of the brane as

$$\mathcal{H} \;\sim\; \tilde N\frac{1}{2}\left(\frac{1}{T}\,P\cdot P + \frac{1}{p!}T\,\{X^{\mu_1}, X^{\mu_2}, \ldots, X^{\mu_p}\}\,\{X_{\mu_1}, X_{\mu_2}, \ldots, X_{\mu_p}\}\right) + N^i\;\left(P\cdot\partial_i X\right)\,. \qquad (752)$$

Of special importance in this context is the case $p = 2$, since then the Nambu bracket can be approximated by an ordinary matrix commutator. This is how matrix mechanics appears in the theory of the 2-brane (see [255] [254] [205] [32] for reviews).

3.22 (The ADM Hamiltonian of the point, the string, and the membrane) The Hamiltonian we have obtained is quite general. It applies to all p-brane dimensions $p \geq 0$. The first few values of p, which are of special importance, are summarized in the following table:

p	$\Lambda = p-1$		$\mathcal{H}$	
$1+0$	-1	$N^i = 0$, $\partial_i X = 0$, $\tilde{g} \to 1$	$N\frac{1}{2}\left(\frac{1}{T}P\cdot P + T\right)$	$T = m$
$1+1$	0	$\tilde{g} = \tilde{g}_{11}$, $\tilde{g}^{11} = 1/\tilde{g}_{11}$	$\tilde{N}\frac{1}{2}\left(\frac{1}{T}P\cdot P + T\,X'\cdot X'\right) + N^1\,P\cdot X'$	$T = \frac{1}{2\pi\alpha'}$
$1+2$	1	$\tilde{g} = \frac{1}{2}\left\{X^\alpha, X^\beta\right\}\left\{X_\alpha, X_\beta\right\}$	$\tilde{N}\frac{1}{2}\left(\frac{1}{T}P\cdot P + \frac{T}{2}\left\{X^\alpha, X^\beta\right\}\left\{X_\alpha, X_\beta\right\}\right) + N^i\left(P\cdot\partial_i X\right)$	

(753)

1. The first line, $p=0$, is the relativistic point. Comparison with §A (p.293), e.g. eq. (1154), shows that here T is the *mass* of the point(-particle), $T=m$.
2. The second line, $p=1$, gives the Hamiltonian of the relativistic string (*cf.* (644), p. 155) and T is, as usual, the string tension $T = 1/2\pi\alpha'$.
3. The third line, $p=2$, is the membrane, or 2-brane with the determinant of the spatial metric expressed in terms of the Nambu bracket (*cf.* 3.24 (p.179)).

3.23 (Comparison with the literature) In [255], eq. (11), the action of the membrane is given in the gauge

$$g_{0i} = 0, \qquad g_{00} = -\frac{4}{\nu^2}\tilde{g}$$

(this are three gauge choices, obtainable by using transformations in the three coordinates). In the presently used ADM formulation this means that

$$N^i = 0, \qquad N = \frac{2}{\nu}\sqrt{\tilde{g}}, \quad \tilde{N} = \frac{2}{\nu}\,.$$

In this gauge the canonical momentum (744) reads

$$P = \frac{T}{\tilde{N}}\dot{X}$$

and we find

$$\begin{aligned}
\mathcal{L} &= P\cdot\dot{X} - \mathcal{H} \\
&= \frac{T}{\tilde{N}}\dot{X}\cdot\dot{X} - \frac{\tilde{N}}{2}\left(\frac{T}{\tilde{N}^2}\dot{X}\cdot\dot{X} + T\tilde{g}\right) \\
&= \frac{T}{2}\frac{1}{\tilde{N}}\left(\dot{X}\cdot\dot{X} - \tilde{N}^2\,\tilde{g}\right) \\
&= \frac{T\nu}{4}\left(\dot{X}\cdot\dot{X} - \frac{4}{\nu^2}\,\tilde{g}\right)\,,
\end{aligned} \tag{754}$$

which is the form of the action given in [255].

3.24 (Nambu Brackets) Let $\{X^\alpha(x^\mu)\}$ be functions of p coordinates x^μ, $\mu = 1 \dots p$, where $p \geq 2$. Let $\eta_{\alpha\beta}$ be *any* constant matrix (i.e. independent of the x^μ) and set $X_\mu := \eta_{\mu\nu} X^\nu$. (The notation insinuates that the X^μ are target space coordinate fields, but that is not essential for the definition of the Nambu bracket.)

The *Nambu bracket of order p* is defined by

$$\begin{aligned} \{X^{\mu_1}, X^{\mu_2}, \cdots, X^{\mu_p}\} &:= \bar{\epsilon}^{\alpha_1\alpha_2\cdots\alpha_p} (\partial_{\alpha_1} X^{\mu_1}) (\partial_{\alpha_2} X^{\mu_2}) \cdots (\partial_{\alpha_d} X^{\mu_p}) \\ &= p! \left(\partial_{[\alpha_1} X^{\mu_1}\right) (\partial_{\alpha_2} X^{\mu_2}) \cdots \left(\partial_{\alpha_d]} X^{\mu_p}\right) , \end{aligned} \tag{755}$$

where the last line expresses that here $\bar{\epsilon}$ is the completely antisymmetric *symbol*.

The essential property of the Nambu bracket is the following relation to the determinant of the “induced metric”:

$$\{X^{\mu_1}, X^{\mu_2}, \cdots, X^{\mu_p}\} \left\{X_{\mu_1}, X_{\mu_2}, \cdots, X_{\mu_p}\right\} = p! \det_{\alpha,\beta} \left[(\partial_\alpha X^\mu) (\partial_\beta X_\mu)\right] . \tag{756}$$

The proof is immediate due to the standard expression for the determinant of any matrix $h_{\alpha\beta}$

$$\det(h_{\alpha\beta}) = \bar{\epsilon}^{\alpha_1\alpha_2\cdots\alpha_p} h_{1\alpha_1} h_{2\alpha_2} \cdots h_{d\alpha_p} ,$$

which implies that

$$\epsilon^{\alpha_1\alpha_2\cdots\alpha_p} h_{\beta_1\alpha_1} h_{\beta_2\alpha_2} \cdots h_{\beta_d\alpha_p} = \epsilon_{\beta_1\beta_2\cdots\beta_p} \det(h_{\alpha\beta}) .$$

3.25 (Worldvolume supersymmetry)

$$\begin{aligned} \gamma^0 &= \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \gamma^1 &= \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \\ \gamma^2 &= \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \end{aligned} \tag{757}$$

3.26 (The super-2-brane in unit gauge)

$$\tilde{g}_{ij} := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{ij} \tag{758}$$

(709)

$$\begin{aligned} g_{\mu\nu} &= \begin{bmatrix} -(N)^2 + (N_1)^2 + (N_2)^2 & N_1 & N_2 \\ N_1 & 1 & 0 \\ N_2 & 0 & 1 \end{bmatrix} \\ g^{\mu\nu} &= \frac{1}{(N)^2} \begin{bmatrix} -1 & N_1 & N_2 \\ N_1 & (N)^2 - (N_1)^2 & N_1 N_2 \\ N_2 & N_1 N_2 & (N_2)^2 \end{bmatrix} \end{aligned} \tag{759}$$

$$
\begin{aligned}
{e_\mu}^a &= \begin{bmatrix} N & N_1 & N_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
{e_a}^\mu &= \begin{bmatrix} 1/N & -N_1/N & -N_2/N \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{760}
$$

$$
{e_a}^\mu = \frac{1}{N} \begin{bmatrix} -1 & N_1 & N_2 \\ N_1 & (N)^2 - (N_1)^2 & N_1 N_2 \\ N_2 & N_1 N_2 & (N)^2 - (N_2)^2 \end{bmatrix}^{\mu\nu} \tag{761}
$$

4 Quantum Supergravity and Quantum Cosmology

Outline. This section first briefly reviews some aspects of ordinary canonical quantum gravity (§4.1 (p.181)) and of canonical quantum supergravity (§4.2 (p.187)). The main content, in §4.3 (p.192), is then the reformulation of parts of canonical quantum supergravity using a basis of modes to parameterise physical fields. It is found that the supersymmetry generators in this formulation have the form of deformed exterior derivatives on configuration space. Supersymmetric cosmological models are discussed in the context of solving the 0-mode of the supersymmetry constraints (§4.3.2 (p.230) and §4.3.3 (p.240)). The mode representation also seems to point a way to treat higher dimensional and higher-N-extended supergravity theories. Tentative steps in this direction are finally sketched in §4.3.4 (p.250).

4.1 Ordinary canonical quantum gravity

Introduction. The method of canonical quantum gravity is to take the Einstein-Hilbert action as is and look for a way to consistently quantize it. Canonical quantum gravity is a demanding field and we naturally cannot go into much detail here (the reader is instead referred to the valuable review [42]), but some basic ideas and notation will be introduced with an eye on laying a modest conceptual and notational foundation for the supersymmetric theory to be discussed in the following section §4.2 (p.187). Under the same token quantum cosmology is only very roughly introduced, with an emphasis on open technical and interpretational questions. Details are postponed until supersymmetric quantum cosmology is discussed in §4.3.2 (p.230) and §4.3.3 (p.240).

First we state some conventions that are used throughout:

4.1 (Conventions) Four dimensional spacetime is assumed to be a globally hyperbolic Lorentzian manifold

$$\mathcal{M} = \mathrm{I\!R} \otimes \Sigma \tag{762}$$

with *compact* spatial hyperslices Σ and the time "axis" $\mathrm{I\!R}$. The metric tensor $g_{\mu\nu}$ on $\mathcal{M}$ is taken to have signature $(-+++)$.
At least as soon as supergravity in the functional representation enters the picture, vector and spinor index conventions become essential. The following are the index conventions used here:

- λ, μ, ν: Lower-case Greek letters from the middle of the alphabet are "world" indices with respect to a coordinate patch on $\mathcal{M}$ which take the values $\{0, 1, 2, 3\}$.
- i, j, k: Lower-case Latin letters from the middle of the alphabet are "world" indices, with respect to a coordinate patch on Σ, that range in $\{1, 2, 3\}$.
- a, b, c: Lower-case Latin letters from the beginning of the alphabet are Lorentz-vector indices in the range $\{0, 1, 2, 3\}$.

- A, B, C: Upper-case Latin indices from the beginning of the alphabet are 2-component Weyl-spinor indices in the range $\{1, 2\}$. They transform under the $(1/2, 0)$ representation of the Lorentz group. A prime (A', B', C') indicates transformation in the $(0, 1/2)$ representation. (For more on spinor conventions see §G.1 (p.342).)

- $(l)(m)(n)$: Lower case Latin letters from the middle of the alphabet which are written in parentheses label modes of the graviton and the gravitino fields. They range over the natural numbers $\mathbb{N}$. (These modes are introduced in 4.15 (p.194).)

- α, β, γ: Lower-case Greek letters from the beginning of the alphabet indicate components with respect to a certain (generally non-holonomic) coframe basis on $T^*(\Sigma)$ with range $\{1, 2, 3\}$. (See 4.43 (p.227) for details.)

We very briefly review some fundamental aspects of ordinary canonical quantum gravity. Selected aspects are discussed in more detail in the sections §4.2 (p.187) and §4.3 (p.192) on supersymmetric quantum gravity.

4.2 (Literature) Standard references for the classical aspects (e.g. the classical Hamiltonian ADM formulation) are [269] and [195]. A useful elementary introduction to canonical quantization of gravity with emphasis on quantum cosmology is [273]. A general review of the field of quantum gravity is given in [42]. We will mostly follow the notation and conventions used in [83].

4.3 (Hamiltonian form of general relativity) The Einstein-Hilbert action for the metric tensor field $g_{\mu\nu}$ on $\mathcal{M}$ is

$$S \;=\; \frac{1}{2\kappa^2} \int\limits_{\Sigma} \sqrt{g}\; {}^4R \; d^4x \,, \tag{763}$$

where g is the determinant of the metric and ${}^4R$ its scalar curvature. The constant factor is determined by

$$\kappa^2 := 8\pi \tag{764}$$

where units have been chosen such that the speed of light and Newton's constant are $c = G = 1$. For a canonical Hamiltonian treatment of this field theory one has to consider a space-time split and decompose the action into an integral over spatial coordinates and one over the temporal coordinate. One therefore writes the metric as

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu \;=\; -N^2 dt \otimes dt + h_{ij} \left(dx^i + N^i dt\right) \otimes \left(dx^j + N^j dt\right) \,, \tag{765}$$

where t it the coordinate time function along the factor $\mathbb{R}$ in (762) and $\left\{x^i\right\}_{i\in\{1,2,3\}}$ is the set of spatial coordinates on Σ. h_{ij} is the metric tensor on *space* Σ. N is called the *lapse* and N^i the *shift* function.
The Hamiltonian density turns out to be

$$\mathcal{H} \;=\; N\mathcal{H}_\perp + N^i \mathcal{H}_i \,, \tag{766}$$

where the *Hamiltonian constraint* $\mathcal{H}$ and the *diffeomorphism constraints* $\mathcal{H}_i$ are given by

$$\begin{aligned} \mathcal{H}_\perp &= G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{h}\,{}^3R \\ \mathcal{H}^i &= -2{\pi_i}^k{}_{|k}\,. \end{aligned} \tag{767}$$

Here

$$\pi^{ij} = \frac{1}{2\kappa^2}\sqrt{h}\left(K^{ij} - Kh^{ij}\right) \tag{768}$$

is the canonical momentum associated with g_{ij}. (K^{ij} is the extrinsic curvature tensor on Σ and ${}^3R$ the spatial scalar curvature. The index ${}_{|k}$ denotes a covariant spatial derivative.) The configuration space metric G_{ijkl}, known as the DeWitt metric, is given by

$$G_{ijkl} = \frac{\kappa^2}{\sqrt{h}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right)\,. \tag{769}$$

Variation of the action with respect to the Lagrange multipliers N, N^i gives the dynamical equations of the theory, which in this case are the Hamiltonian and the diffeomorphism constraints,

$$\begin{aligned} \mathcal{H}_\perp &= 0 \\ \mathcal{H}_i &= 0\,, \end{aligned} \tag{770}$$

which express the invariance of any physical configuration under time and spatial coordinate reparameterizations, respectively.

4.4 (Canonical quantization in metric formalism) Canonical quantization of the theory proceeds in the functional Schrödinger representation by substituting functional multiplication and functional derivation operators for g_{ij} and π^{ij}, respectively:

$$\begin{aligned} & g_{ij}(x) \to \hat{g}_{ij}(x) \\ & \pi^{ij}(x) \to \hat{\pi}^{ij} = -i\hbar\frac{\delta}{\delta g_{ij}(x)} \\ & \left[\frac{\delta}{\delta g_{ij}(x)}, \hat{g}_{kl}(x)\right] = \delta(x,y)\ \delta^i_k\,\delta^j_l\,. \end{aligned} \tag{771}$$

The quantum version of the classical Hamiltonian constraint is the *Wheeler-DeWitt* equation

$$\begin{aligned} & \hat{\mathcal{H}}_\perp(x)\,|\phi\rangle = 0 \\ \Leftrightarrow\ & \left(G_{ijkl}(x)\,\frac{\delta}{\delta g_{ij}(x)}\frac{\delta}{\delta g_{kl}(x)} - \sqrt{h}\,{}^3R + \mathcal{O}(\hbar)\right)|\phi\rangle = 0, \quad \forall\, x \in \Sigma \end{aligned} \tag{772}$$

(where $|\phi\rangle$ is a functional of g_{ij}, a kinematical state of the theory), which is analogous to a Klein-Gordon equation. The quantum diffeomorphism constraints

$$\hat{\mathcal{H}}_i(x)\,|\phi\rangle = 0, \quad \forall\, x \in \Sigma \tag{773}$$

can be shown to express the invariance of $|\phi\rangle$ under spatial coordinate transformations[46]. The restriction of configuration space to all diffeomorphism invariant states is called *superspace* (no relation to supersymmetry).

The Wheeler-DeWitt equation can be regarded as describing point particle propagation of the "universe point" in superspace. This will be made more precise in §4.3 (p.192). It is interesting to note the following:

4.5 (Scale factor dependence of the potential in superspace) The potential term

$$V = \sqrt{h}^3 R \tag{775}$$

in the Hamiltonian constraint (772) vanishes as the volume of the spatial hypersclice goes to zero:
When the spatial metric h_{ij} is conformally rescaled by $h_{ij} \to \Omega h_{ij}$, with Ω a constant factor, one has:

$$\begin{aligned}
h_{ij} &= \mathcal{O}(\Omega) \\
h^{ij} &= \mathcal{O}\big(\Omega^{-1}\big) \\
\sqrt{h} &= \mathcal{O}\Big(\Omega^{D/2}\Big) \\
{}^3\Gamma_i{}^k{}_l &= \mathcal{O}(h^{\cdot\cdot}\partial . h_{..}) \\
&= \mathcal{O}(1) \\
{}^3R_{ij}{}^k{}_l &= \mathcal{O}(\partial\Gamma + \Gamma\Gamma) \\
&= \mathcal{O}(1) \\
{}^3R_{ij} &= \mathcal{O}\big({}^3R_{ij}{}^k{}_k\big) \\
&= \mathcal{O}(1) \\
{}^3R &= \mathcal{O}\big(h^{ij\,3}R_{ij}\big) \\
&= \mathcal{O}\big(\Omega^{-1}\big) \\
\sqrt{h}\,{}^3R &= \mathcal{O}\Big(\Omega^{D/2-1}\Big) \,.
\end{aligned} \tag{776}$$

[46] Under a coordinate transformation generated by the vector field v the spatial metric transforms as

$$\begin{aligned}
\delta g_{ij}(x) &= \epsilon \mathcal{L}_v g_{ij}(x) \\
&= \epsilon D_{(i} v_{j)}(x) \,.
\end{aligned}$$

According to the chain rule, a functional $|\psi\,[g_{ij}]\rangle$ of g_{ij} thus transforms as

$$\begin{aligned}
\delta\,|\psi\rangle &= \epsilon \int_\Sigma \sqrt{g}\,\Big(D_{(i} v_{j)}(x)\Big) \frac{\delta}{\delta g_{ij}(x)}\,|\psi\rangle\; d^3x \\
&= \epsilon \int_\Sigma \sqrt{g}\,(D_i v_j(x))\,\frac{\delta}{\delta g_{ij}(x)}\,|\psi\rangle\; d^3x \\
&= -\epsilon \int_\Sigma \sqrt{g} v_j(x)\, D_i \frac{\delta}{\delta g_{ij}(x)}\,|\psi\rangle\; d^3x \,.
\end{aligned} \tag{774}$$

Hence when the diffeomorphism constraint is satisfied this expression vanishes for arbitrary v and $|\psi\rangle$ is invariant under a change of spatial coordinates.

Therefore, in dimensions $D > 2$, with vanishing spatial volume of the universe the superspace potential also vanishes:

$$V = \sqrt{h}\left(2\Lambda - {}^3R\right) = \mathcal{O}\left(\Omega^{D/2}\right)\left(\mathcal{O}(1) - \mathcal{O}(\Omega^{-1})\right) = \mathcal{O}\left(\Omega^{D/2}\right) - \mathcal{O}\left(\Omega^{(D-2)/2}\right) \stackrel{\Omega\to 0}{\longrightarrow} 0 \tag{777}$$

(*cf.* [163]). As discussed in detail in [257],§E, this means that the dynamics in configuration space is approximately free in the neighborhood of cosmological singularities.

This is interesting with regard to the fact, that, according to §2.2.4 (p.78), 2.79 (p.78), 2.80 (p.78), and 2.81 (p.79) (also [203]), a conserved probability current in quantum cosmology exists only if the superspace potential is independent of the time parameter in superspace, which is exactly the scale factor of the universe. See the discussion of the Kantowski-Sachs model (5.4 (p.259) and 5.5 (p.260)) for an example and further discussion (also compare the figures 4 (p.262), 6 (p.264), and 7 (p.265)). It is interesting to note that in higher dimensional supergravity (*cf.* §4.3.4 (p.250)) there are highly non-trivial cosmological models (*cf.* §5.2 (p.266)), exhibiting, in particular, chaotic Mixmaster-type behavior (*cf.* 5.14 (p.276) and figures 8 (p.278) and 9 (p.279)), which feature *no* potential term, as above, but where the kinetic contributions of the 3-form field constitute effective potentials in superspace. These kinetic terms do depend on the scale factor (*cf.* (1104),p. 272), but since theorem 2.79 (p.78) demands only that potential terms be scale factor independent, a conserved probability current is in this case exactly defined over the whole of configuration space (a slice through such a current is displayed in figure 10 (p.283)).

4.6 (Quantum cosmology) The basic idea of quantum cosmology is to attack the formidable set of constraints (772) of the full theory of quantum gravity by making some radically simplifying approximations. The hope is that thereby at least a minimal set of general characteristics of the space of solutions is preserved, which may give physical insight. More details on some aspects of quantum cosmology will be given in §4.3.2 (p.230) and §4.3.3 (p.240) below.

4.7 (Literature) General introductions to quantum cosmology are [54] [273]. Further texts with relevance to the homogeneous (and supersymmetric) cosmological models that are ultimately of interest here (see §5.2 (p.266)) are [137] [41] [46] [197] [13] [114] [43] [173] [174] [150] [151] [230] [148]. A mathematically founded argument that, despite their huge simplification, mini-superspace models are apparently a good approximation to the true dynamics of the cosmos, is given in [241].

4.8 (Conceptual questions of quantum cosmology) The conceptual issues of quantum cosmology, which are still more or less unsettled today[47], all

[47] This is emphasized in a particularly pointed way by Woodard in [277]:

> It hardly needs to be stated that the reliable extraction of *any* testable prediction from this murky subject would go a long way towards improving our understanding of it. [...] Even points as basic as the probabilistic interpretation of the wave functional and how to compute inner products were held to be

arise, in one way or another, from the question of how to make physical sense of the state vectors in the theory, i.e. of the 'wave function of the universe'. This gives rise to the following chain of problems:

1. *Physical interpretation of the 'wave function of the universe'.*
2. *Boundary conditions of the wave function.*
3. *Probability interpretation.*
4. *Conserved currents.*
5. *Scalar product.*
6. *Measure on configuration space.*
7. *Gauge fixing.*

It is noteworthy that the more technical of these problems are those that arise in any quantum theory of *constrained, relativistic, single-particle* mechanics. The traditional way to avoid problems of such sort is to instead switch to *many-particle theory*, an option that would, however, rather worsen the conceptual difficulties in the case of cosmology (but there are attempts to do so, e.g. [249]). It is on the technical issues that the formalism of supersymmetry is most likely to yield helpful results. However, better insight into formal aspects might inevitably make some answers to the more philosophical questions of quantum cosmology look more convincing than others.
One of the purposes of the present work (§2 (p.14)) is to see if from supersymmetric quantum mechanics, applied to constrained relativistic systems, some tentative hints on above problems 3-7 (p.186) can be deduced.

A general fact about the quantization of systems with constraints, which is of some importance, is the following:

4.9 (Quantization and conformal transformations.) Without further conditions imposed, naive quantization of a Hamiltonian $H(x,p)$ by the 'correspondence rule' $p \to -i\hbar\partial$ is ambiguous. As is discussed in §2 (p.14), (*cf.* 2.2.2 (p.61)) the condition of *supersymmetry* is sufficient to fix the operator ordering, so that to a Hamiltonian function $H(x,p)$ corresponds to a more or less *unique* supersymmetric Hamiltonian operator. But constrained dynamics introduces a further ambiguity that spoils this desirable uniqueness:

The *classical* Hamiltonian constraint

$$H = g^{\mu\nu} p_\mu p_\nu + U \stackrel{!}{=} 0$$

is obviously not affected by a non-zero *conformal transformation*

$$H(x,p) \to e^{\Omega(x,p)}\, H(x,p)\ .$$

---

unclear.

The author goes on to construct well defined inner products in quantum cosmology by means of projectors on gauge fixed states and Fadeev-Popov determinants. An adaption of this approach within BRST formalism is discussed in §2.3.2 (p.115) and applied to supersymmetric systems in §2.3.4 (p.134).

Even if one can associate a unique quantum operator with $H(x,p)$, this will in general differ by terms of order $\hbar$ from that associated with $e^{\Omega(x,p)}H(x,p)$[48]

Hence, *even with a well defined quantization rule, quantum constraint operators cannot be uniquely associated to classical constraints.* To fix a unique quantum constraint operator a further condition has to be imposed.

This problem, and its implication for quantum cosmology, is addressed in [118]. There a family of allowed quantum constraint operators is constructed, parameterized by a parameter ξ. The author concludes:

> Particular values of ξ may be preferred if there exist additional symmetries, such as conformal invariance or supersymmetry.

The author then opts for imposing the condition that the quantum constraint be invariant under conformal rescaling. It should be noted that this is a decision about which model to choose for describing physics, an issue that cannot be resolved *within* any model without further assumptions.

As shown in **??** (p.**??**), conformal invariance of the quantum constraint is incompatible with supersymmetric quantization.

4.2 Canonical quantum supergravity in the functional Schrödinger representation

Outline. Elements of canonical quantum supergravity as developed in [80] and §3 of [83] are reviewed, mainly in order to introduce the notation that will be needed in §4.3 (p.192).

4.10 (Basic definition and conventions) As in ordinary Hamiltonian formulations of canonical gravity (*cf.* [195]§21, [269]) spacetime is represented by a globally hyperbolic Lorentzian manifold

$$\mathcal{M} = \Sigma \otimes \mathrm{I\!R}\,,$$

with spatial Cauchy surfaces Σ and $\mathrm{I\!R}$ being the time axis. The action $\mathcal{S} = \int\limits_{\mathcal{M}} \mathcal{L}\, d^4x$ of the physical fields propagating on $\mathcal{M}$ is accordingly rewritten as

$$\begin{aligned} \mathcal{S} &= \int\limits_{\mathcal{M}} \mathcal{L}\, d^4x \\ &= \int\limits_{\mathrm{I\!R}} \int\limits_{\Sigma} \mathcal{L}\, d^3x\, dt \\ &= \int\limits_{\mathrm{I\!R}} (2T - H)\; dt\,, \end{aligned} \tag{778}$$

where

$$H := \int\limits_{\Sigma} \mathcal{H}\, d^3x$$

[48] Unless, of course, the conformal transformation is itself a quantum effect, e.g. $e^{\Omega} = e^{\omega/\hbar}$, in which case the terms will be of order 1. Such quantum transformation may give rise to additional potential terms, *cf.* theorem **??** (p.**??**).

is the Hamiltonian of the theory. In the case of $N = 1$, $D = 4$ supergravity the physical field content of the theory is the Lorentz vector valued 1-form $e^a = e^a{}_\mu dx^\mu$, known from ordinary gravity, as well as the Weyl-spinor valued form $\psi^A = \psi^A{}_\mu dx^\mu$ and its adjoint $\bar\psi^A = \bar\psi^A{}_\mu dx^\mu$. These represent the graviton (vielbein) and the gravitino field, respectively. The Lorentz-vector and Weyl-spinor indices are defined with respect to a fixed but arbitrary Lorentz frame bundle over $\mathcal{M}$ (the "spin-frame") equipped with the flat Lorentzian metric

$$\begin{aligned} \eta &:= (\eta_{ab})_{a,b\in\{0,1,2,3\}} := \mathrm{diag}(-1,1,1,1) \\ = \eta^{-1} &:= \left(\eta^{ab}\right) . \end{aligned} \tag{779}$$

The vielbein tensor $e^a{}_\mu$ defines the metric tensor $g_{\mu\nu}$ via

$$\begin{aligned} \eta_{ab} e^a{}_\mu e^b{}_\nu &:= e^a{}_\mu e_{a\nu} \\ &= g_{\mu\nu} \\ \Leftrightarrow g^{\mu\nu} e^a{}_\mu e^b{}_\nu &:= e^a{}_\mu e^{b\mu} \\ &= \eta^{ab} . \end{aligned} \tag{780}$$

The determinant is

$$g := \det(g_{\mu\nu}) . \tag{781}$$

These objects are again split into spatial and temporal parts: The spatial metric on Σ is given by

$$h_{ij} := e^a{}_i\, e_{aj}, \quad i,j \in \{1,2,3\} . \tag{782}$$

The embedding of the spatial hyperslice in $\mathcal{M}$ is characterized by a unit timelike normal vector n,

$$n_a n^a = -1 , \tag{783}$$

which satisfies

$$e^a{}_0 = N n^a + N^i e^a{}_i , \tag{784}$$

and which is orthogonal to all spatial elements of the vielbein

$$n_a e^a{}_i = 0, \quad i \in \{1,2,3\} . \tag{785}$$

Spinor and vector indices are related by the Pauli matrices ("Infeld-van der Waerden translation symbols"). Following [83], we specifically set:

$$\sigma = \left(\sigma_a{}^{AA'}\right) = \left(-\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) , \tag{786}$$

so that for instance

$$e^{AA'}{}_\mu := e^a{}_\mu \sigma_a{}^{AA'} . \tag{787}$$

(A summary of the 2-component spinor conventions used, as well as some useful relations, can be found in appendix §G.1 (p.342).)
A connection form

$$\omega = \omega_\mu dx^\mu$$

on the Lorentz frame bundle gives rise to a covariant derivative

$$D_\mu \quad := \quad \partial_\mu + \omega_\mu \tag{788}$$

acting, for instance, as

$$\begin{aligned} D_\mu v^a &= \partial_\mu v^a + \omega_\mu{}^a{}_b v^b \\ D_\mu \psi^A &:= \partial_\mu \psi^a + \omega_\mu{}^A{}_B \psi^B \,. \end{aligned} \tag{789}$$

The projection of the covariant derivative D onto the spatial hyperslice is denoted by ${}^3D$ and the torsion free part of this projection by ${}^{3s}D$.

4.11 (Hamiltonian formalism) The action of $D = 4$, $N = 1$ supergravity is (e.g. [44] eq. (III.2.18b))

$$\begin{aligned} \mathcal{S} &= \int_{\mathcal{M}} \mathcal{L}\, dx^4 \\ &= \frac{1}{8\kappa^2} \int_{\mathcal{M}} \left(*R + 4\bar{\Psi} \wedge \gamma_5 \gamma_a D\Psi \wedge e^a \right) \,, \end{aligned} \tag{790}$$

where

$$\Psi = \begin{bmatrix} \psi \\ \bar{\psi} \end{bmatrix} . \tag{791}$$

In Hamiltonian formalism this is rewritten as (*cf.* [221])

$$\mathcal{S} \quad = \quad \int_{\mathcal{M}} \Big(2\mathcal{T} - N\mathcal{H}_\perp - N^i \mathcal{H}_i - \psi_0^A \mathcal{S}_A + \bar{\psi}_0^{A'} \bar{\mathcal{S}}_{A'} + M^{ab} \mathcal{J}_{ab} \Big) \,, \tag{792}$$

where $\mathcal{H}_\mu$ are the usual Hamiltonian and diffeomorphism generators of ordinary gravity, $\mathcal{S}$, $\bar{\mathcal{S}}$ are the supersymmetry generators and $\mathcal{J}_{ab}$ the generators of Lorentz rotations of the spin frame. N, ψ_0, $\bar{\psi}_0$, M are Lagrange multipliers. (See also §2.3.2 (p.115).)

4.12 (Quantization) By canonical quantization, as described in [80], the spatial components of the vielbein and the gravitino fields are promoted to functional (bosonic and fermionic) multiplication operators

$$\begin{aligned} e^a{}_i(x) &\rightarrow \hat{e}^a{}_i(x) \\ \psi^A{}_i(x) &\rightarrow \hat{\psi}^A{}_i(x) \,. \end{aligned} \tag{793}$$

Together with the functional differentiation operators

$$\begin{aligned} \hat{e}^\#{}_a{}^i(x) &:= \frac{\delta}{\delta e^a{}_i(x)} \\ \hat{\psi}^\#{}_A{}^i(x) &:= \frac{\delta}{\delta \psi^A{}_i(x)} \,, \end{aligned} \tag{794}$$

these satisfy the canonical superalgebra

$$\begin{aligned}
\left[\hat{e}^a{}_i(x), \hat{e}^{\#}{}_b{}^j(y)\right] &= \delta^a_b\, \delta^j_i\, \delta(x,y) \\
\left[\hat{e}^a{}_i(x), \hat{e}^b{}_j(y)\right] &= 0 \\
\left[\hat{e}^{\#}{}_a{}^i(x), \hat{e}^{\#}{}_b{}^j(y)\right] &= 0 \\
\left\{\hat{\psi}^A{}_i(x), \hat{\psi}^{\#}{}_B{}^j(y)\right\} &= \delta^A_B\, \delta^j_i\, \delta(x,y) \\
\left\{\hat{\psi}^A{}_i(x), \hat{\psi}^B{}_j(y)\right\} &= 0 \\
\left\{\hat{\psi}^{\#}{}_A{}^i(x), \hat{\psi}^{\#}{}_B{}^j(y)\right\} &= 0\,,
\end{aligned} \tag{795}$$

all other commutators vanishing.
With the canonical coordinates $\hat{e}^a{}_i$, $\hat{\psi}^A{}_i$ are associated canonical momentum operators $\hat{p}_a{}^i(x)$ and $\hat{\bar{\psi}}^{A'}{}_i(x)$, respectively, which are found to be given by (*cf.* [83] (3.3.2))[49]:

$$\hat{\bar{\psi}}^{A'}{}_i(x) \;:=\; -iD^{AA'}{}_{ij}(x)\, \hat{\psi}^{\#}{}_A{}^j(x)\;, \tag{796}$$

and ([83](3.3.3))

$$\begin{aligned}
\hat{p}_{AA'}{}^i(x) &= -i\hbar\left(\frac{\delta}{\delta e^{AA'}{}_i(x)} + \frac{i}{2}\epsilon^{ijk}\hat{\psi}_{Aj}(x)\,\hat{\bar{\psi}}_{A'_k}(x)\right) \\
\Leftrightarrow \qquad \hat{p}_a{}^i(x) &= -i\hbar\left(\frac{\delta}{\delta e^a{}_i(x)} - \frac{i}{2}\epsilon^{ijk}\sigma_a{}^{AA'}\hat{\psi}_{Aj}(x)\,\hat{\bar{\psi}}_{A'_k}(x)\right)\,.
\end{aligned} \tag{797}$$

Here the coefficients $D^{AA'}{}_{jk}(x)$ are given by

$$D^{AA'}{}_{jk} \;=\; -2i\frac{1}{\sqrt{h}}e^{AB'}{}_k e_{BB'j} n^{BA'}\,. \tag{798}$$

This matrix has an inverse:

$$\begin{aligned}
&C_{AA'}{}^{ij}(x) := \epsilon^{ijk} e_{AA'k}(x) \\
&C_{AA'}{}^{ij} D^{AB'}{}_{jk} = \delta^{B'}_{A'}\,\delta^i_k\,.
\end{aligned} \tag{799}$$

In terms of these operators the quantum version of the primed supersymmetry generator reads:

$$\hat{\bar{\mathcal{S}}}_{A'}(x) \;=\; \frac{\kappa^2}{2}\hat{\psi}^A{}_i(x)\, i\hat{p}_{AA'}{}^i(x) + e^{ijk}e_{AA'i}\,{}^{3s}D_j\hat{\psi}^A{}_k(x)\;. \tag{800}$$

It turns out that in the expression $\hat{\psi}^A{}_i(x)\,\hat{p}_{AA'}{}^i(x)$ the terms trilinear in $\hat{\psi}$ and $\hat{\bar{\psi}}$ cancel (*cf.* 4.29 (p.210) below), so that

$$\hat{\bar{\mathcal{S}}}_{A'}(x) \;=\; \frac{\kappa^2}{2}\hat{\psi}^A{}_i(x)\;\hbar\frac{\delta}{\delta e^{AA'}{}_i(x)} + e^{ijk}e_{AA'i}\,{}^{3s}D_j\hat{\psi}^A{}_k(x)\;. \tag{801}$$

---

[49] This definition of $\hat{\bar{\psi}}^{A'}{}_i(x)$ differs from that given in [80][83] by a factor of $\hbar$. We here do not include this factor in the definition of $\hat{\bar{\psi}}^{A'}{}_i(x)$ but instead write it out explicitly. This way some of our formulas, which relate the supersymmetry generators to exterior derivatives on configuration space, obtain a more natural form.

Schematically one has simply:

$$\text{“}\ \bar{\mathcal{S}} = \psi \frac{\delta}{\delta e} + D\psi\ \text{”}\ .$$

Knowledge of this single generator is sufficient to determine all other generators (except for the Lorentz generators): The unprimed supersymmetry generator is the adjoint of this operator with respect to a suitable inner product

$$\hat{\mathcal{S}}_A \quad = \quad \left(\hat{\bar{\mathcal{S}}}_{A'}\right)^\dagger \tag{802}$$

and the Hamiltonian and diffeomorphism constraints are *defined* (*cf.* [83] pp. 106) by

$$\left\{\hat{\mathcal{S}}_A(x), \hat{\bar{\mathcal{S}}}_{A'}(y)\right\} \quad = \quad -\frac{\kappa^2}{2}\delta(x,y)\ \hat{\mathcal{H}}_{AA'}(x)\ , \tag{803}$$

where ([83](3.2.46))

$$\hat{\mathcal{H}}_{AA'}(x) \quad := \quad -n_{AA'}\mathcal{H}_\perp(x) + e_{AA'}{}^i\mathcal{H}_i(x)\ . \tag{804}$$

Note that there is no factor ordering ambiguity in (801). Hence, determination of the inner product also uniquely determines the factor ordering of (802) and that of the Hamiltonian and diffeomorphism generators (803).

4.3 Canonical quantum supergravity in the mode amplitude Schrödinger representation

Introduction In the following it is demonstrated how full-fledged canonically quantized supergravity can be reformulated as covariant supersymmetric quantum mechanics in infinite dimensional configuration space. The supersymmetry generators are shown to be deformed exterior derivatives on configuration space.

Our method is based on the usual Schrödinger representation of canonical supergravity (§4.2 (p.187)), but formulated in some set of 'normal coordinates', i.e. using a complete set of spatial modes into which the vielbein and the gravitino fields are expanded. The action in this representation describes constrained point mechanics in infinite dimensional configuration space, which is coordinatized by the field mode amplitudes. It is shown that local spacetime supersymmetry translates into the existence of Witten-Dirac square roots of the Hamiltonian generator(s). The approach is well suited for studying truncated models of supergravity.

4.3.1 From functional to mode representation

Outline. The following sketches the program of formulating $d = 4$, $N = 1$ canonical quantum supergravity (in the metric formalism as developed in [80]) in the usual Schrödinger representation but using normal mode field operators instead of pointwise field operators. This way, as in the above example 3.1 (p.143), the constraints formally resemble those of a countable infinite dimensional quantum mechanical system. In particular, the supersymmetry generators of quantum supergravity are shown to be deformed exterior derivatives on configuration space.

First recall the basis of all spacetime supersymmetry, the graded extension of the Poincaré algebra, known as the super-Poincaré algebra (e.g. [146]):

4.13 (Super-Poincaré algebra) The (extended) $d = 4$ super-Poincaré algebra (describing 'spacetime supersymmetry') is generated by the usual (even graded) Poincaré generators $\mathcal{P}_\mu$, $\mathcal{J}^{ab}$ (which respectively generate translations and rotations as usual) together with N pairs of spinorial odd graded (super-)generators $\mathcal{S}^{(i)}_A$, $\bar{\mathcal{S}}^{(i)}_{A'}$, $i \in \{1, \ldots, N\}$:

$$\left\{\mathcal{P}_\mu, \mathcal{J}^{ab}, S^{(i)}_A, \bar{\mathcal{S}}^{(i)}_{A'}\right\} . \tag{805}$$

The (super-)commutators between these generators are the usual Poincaré algebra brackets together with the further relations (e.g. [271]):

$$\begin{aligned} \left\{\mathcal{S}^{(i)}_A, \bar{\mathcal{S}}^{(j)}_{A'}\right\} &= \delta^{ij}\left(\sigma^\mu\right)_{AA'}\mathcal{P}_\mu \\ \left[\mathcal{J}^{ab}, \mathcal{S}_A\right] &= -\sigma^{ab}{}_A{}^B S_B \\ \left[\mathcal{J}^{ab}, \mathcal{J}^{cd}\right] &= \left(\eta^{ac}\mathcal{J}^{bd} - \eta^{bc}\mathcal{J}^{ad} + \eta^{bd}\mathcal{J}^{ac} - \eta^{ad}\mathcal{J}^{bc}\right) . \end{aligned} \tag{806}$$

For $d = 4$, the supercharges transform among each other in the automorphism group $U(N)$. This transformation is generated by operators $\mathcal{T}^l$ which are represented on the supercharges as the matrices $t^l = \left(t^{li}{}_j\right)_{ij}$ (e.g. [252]):

$$\begin{aligned} \left[\mathcal{T}^l, \mathcal{T}^m\right] &= f^{lm}{}_n \mathcal{T}^n \\ \left[\mathcal{T}^l, \mathcal{S}^{(i)}\right] &= t^{li}{}_j \mathcal{S}^{(j)} \end{aligned} \tag{807}$$

Here $f^{lm}{}_n$ are the structure constants of the automorphism group U(N).
Furthermore, for $N > 1$ one has 'central charges' $\mathcal{Z}^{ij} = -\mathcal{Z}^{ji}$ in the algebra, which satisfy:

$$\begin{aligned} \left[\mathcal{T}^A, \mathcal{Z}^{ij}\right] &= t^{Ai}{}_k \mathcal{Z}^{kj} + t^{Aj}{}_k \mathcal{Z}^{ik} \\ \left\{\mathcal{S}^{(i)}_A, \mathcal{S}^{(j)}_B\right\} &= \epsilon_{AB}\mathcal{Z}^{ij} \\ \left\{\bar{\mathcal{S}}^{(i)}_{A'}, \bar{\mathcal{S}}^{(j)}_{B'}\right\} &= \epsilon_{A'B'}\mathcal{Z}^{ij} . \end{aligned} \tag{808}$$

When the super Poincaré algebra is gauged, one obtains the generator algebra of supergravity:

Theorem 4.14 (Generator algebra of canonical supergravity) Gauging the super-Poincaré algebra leads to supergravity. It has been shown by Teitelboim ([256]) that the non-vanishing classical Dirac brackets[50] of the generator algebra of $d = 4$, $N = 1$ canonical supergravity read:

$$\begin{aligned}
\left\{\mathcal{S}_A(x), \bar{\mathcal{S}}_{A'}(x')\right\} &= \delta(x,x')\,\sigma^a{}_{AA'}\mathcal{P}_a(x) \\
\left[\mathcal{S}_A(x), \mathcal{P}_a(x')\right] &= \frac{1}{2}\delta(x,x')\,\Sigma_{Aabc}(x)\,\mathcal{J}^{bc}(x) \\
\left[\mathcal{S}_A(x), \mathcal{J}^{ab}(x')\right] &= -\delta(x,x')\,\sigma^{ab}{}_A{}^B\mathcal{S}_B(x) \\
\left[\mathcal{P}_a(x), \mathcal{P}_b(x')\right] &= \delta(x,x')\left(\frac{1}{2}\Omega_{abcd}(x)\,\mathcal{J}^{cd}(x) + \bar{H}^A{}_{ab}(x)\,\mathcal{S}_A(x)\right) \\
\left[\mathcal{P}_c(x), \mathcal{J}^{ab}(x')\right] &= \delta(x,x')\left(\delta_c{}^b\mathcal{P}^a(x) - \delta_c{}^a\mathcal{P}^b(x)\right) \\
\left[\mathcal{J}^{ab}(x), \mathcal{J}^{cd}(x')\right] &= \delta(x,x')\left(\eta^{ac}\mathcal{J}^{bd}(x) - \eta^{bc}\mathcal{J}^{ad}(x) + \eta^{bd}\mathcal{J}^{ac}(x) - \eta^{ad}\mathcal{J}^{bc}(x)\right) ,
\end{aligned} \tag{809}$$

where H, Σ, and Ω are objects measuring curvature of spacetime (see [256] for details). As remarked in [121] and stressed in [81], pp. 96, these relations in general only hold up to terms proportional to the Lorentz generators. However, the supercommutators

$$\begin{aligned}
\left\{\mathcal{S}_A(x), \mathcal{S}_B(y)\right\} &= 0 \\
\left\{\bar{\mathcal{S}}_{A'}(x), \bar{\mathcal{S}}_{B'}(y)\right\} &= 0
\end{aligned} \tag{810}$$

vanish exactly (i.e. "off shell") and $\mathcal{P}_a$ may be (re)defined (by using freedom encoded in the Lagrange multipliers, see [81], pp. 96) as the right hand side of

$$\left\{\mathcal{S}_A(x), \bar{\mathcal{S}}_{A'}(x')\right\} = \delta(x,x')\,\sigma^a{}_{AA'}\mathcal{P}_a(x) \; . \tag{811}$$

These are the essential relations for the following development.

The above algebra is stated with respect to a basis of Dirac-δ (generalized) functions. One may switch to another basis of more well behaved functions by introducing *modes*:

4.15 (Bosonic and fermionic modes) On the compact and unbounded spatial hyperslice Σ, assumed to be completely covered (possibly up to a set of measure zero) by a fixed but arbitrary coordinate patch $\left\{x^i\right\}_{i\in\{1,2,3\}}$, any scalar, spin-2 and spin-3/2 fields may be expressed (with respect to the coordinates $\left\{x^i\right\}$) in terms of the following mode functions:

- *Scalar modes:*

$$\begin{aligned}
C_{(n)} : \Sigma &\to \mathbb{R} \\
C'^{(n)} : \Sigma &\to \mathbb{R}, \qquad n \in \mathbb{N}
\end{aligned} \tag{812}$$

[50] All brackets between classical quantities, i.e. those that carry no operator "hats" are supposed to be Dirac brackets here, i.e. modified classical Poisson brackets. For more details, with which we need not be concerned here, see [221] and [80], [83]. We write $\{\cdot,\cdot\}$ for the Dirac bracket involving two Grassmann-odd quantities and write $[\cdot,\cdot]$ otherwise, adapting the quantum mechanical anticommutator and commutator notation.

- *Spin-2 modes:*

$$\begin{aligned} {B_{(n)}}^a{}_i : \Sigma &\rightarrow \mathbb{R} \\ {B'^{(n)}}_a{}^i : \Sigma &\rightarrow \mathbb{R}, \qquad n \in \mathbb{N},\ a \in \{0,1,2,3\}\,,\ i \in \{1,2,3\} \end{aligned} \tag{813}$$

- *Spin-3/2 modes;:*

$$\begin{aligned} {F_{(n)}}^A{}_i : \Sigma &\rightarrow \mathbb{C} \\ {F'^{(n)}}_A{}^i : \Sigma &\rightarrow \mathbb{C}, \qquad n \in \mathbb{N},\ A \in \{1,2\}\,,\ i \in \{1,2,3\} \end{aligned} \tag{814}$$

These functions shall be complete and orthonormal in the following sense:

1. *Completeness:*

$$\begin{aligned} {B_{(n)}}^a{}_i(x)\, {B'^{(n)}}_b{}^j(y) &= \delta^a_b\, \delta^j_i\, \delta(x,y) \\ {F_{(n)}}^A{}_i(x)\, {F'^{(n)}}_B{}^j(y) &= \delta^A_B\, \delta^j_i\, \delta(x,y) \\ C_{(n)}(x)\, C'^{(n)}(y) &= \delta(x,y) \end{aligned} \tag{815}$$

2. *Orthonormality:*

$$\begin{aligned} \int\limits_\Sigma C_{(n)}(x)\, C'^{(m)}(x)\ d^3x &= \delta^m_n \\ \int\limits_\Sigma {B_{(n)}}^a{}_i(x)\, {B'^{(m)}}_a{}^i(x)\ d^3x &= \delta^m_n \\ \int\limits_\Sigma {F_{(n)}}^A{}_i(x)\, {F'^{(m)}}_A{}^i(x)\ d^3x &= \delta^m_n \end{aligned} \tag{816}$$

Mode indices are set in parentheses (n) to distinguish them from spacetime indices a, A, i. A summation of all indices which are repeated upstairs and downstairs is implicit here and in the following (unless a different summation is indicated explicitly).

The primed functions will be called *dual modes*. For the following development the exact nature of the above modes is irrelevant, nothing else will be assumed about them than the orthonormality relations (815) (816) given above.

One further assumption will be very helpful: Let the scalar modes C, C' be chosen in such a way that their set of zeros has vanishing measure:

$$\begin{aligned} \mu\big(\{x \in \Sigma \,|\, C_{(n)}(x) = 0\}\big) &= 0 \\ \mu\Big(\Big\{x \in \Sigma \,|\, C'^{(n)}(x) = 0\Big\}\Big) &= 0, \quad \forall\, n \in \mathbb{N}\,. \end{aligned} \tag{817}$$

This is a natural requirement. For instance on $\Sigma = T^3$ the usual plane wave Fourier modes obviously satisfy it.

This way some important quantities to be introduced below will be invertible, since division by the scalar modes will be allowed under an integral. For this purpose define the following expression:

$$\left[\frac{1}{f(x)}\right] := \left\{ \begin{array}{cl} \frac{1}{f(x)} & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{array} \right. . \tag{818}$$

We furthermore assume, for later convenience only, that the scalar 0-mode is the constant mode:

$$C_{(0)}(x) = C'^{(0)}(x) = 1/\sqrt{V}\,, \tag{819}$$

where

$$V = \int\limits_{\Sigma} d^3x \tag{820}$$

is a normalization factor.

Now the field content of $D = 4$, $N = 1$ supergravity can be expanded with respect to the above mode basis:

4.16 (Field content of $D = 4$, $N = 1$ supergravity in the mode basis) The spatial vielbein $e^a{}_i(t,x)$ and gravitino fields $\psi^A{}_i(t,x)$ may be expanded as

$$\begin{aligned} e^a{}_i(t,x) &:= b^{(n)}(t)\, B^a_{(n)i}(x) \\ \psi^A{}_i(t,x) &:= f^{(n)}(t)\, F^A_{(n)i}(x) \end{aligned} \tag{821}$$

with coordinate-time dependent real amplitudes $b^{(n)}(t)$ and $f^{(n)}(t)$. One also needs a mode decomposition of the Lagrange multipliers $N^\mu(t,x)$ and $\psi^A{}_0(t,x)$:

$$\begin{aligned} N^\mu(t,x) &:= N^{(n)\mu}(t)\, C_{(n)}(x) \\ \psi^A{}_0(t,x) &:= \psi^{(n)A}{}_0(t)\, C_{(n)}(x) \\ \bar\psi^{A'}{}_0(t,x) &:= \bar\psi^{(n)A'}{}_0(t)\, C_{(n)}(x)\,. \end{aligned} \tag{822}$$

Due to relations (815) and (816), these expansions may be inverted to yield, for instance:

$$\begin{aligned} b^{(n)}(t) &= \int\limits_{\Sigma} B'^{(n)}{}_a{}^i(x)\, e^a{}_i(t,x)\ d^3x \\ f^{(n)}(t) &= \int\limits_{\Sigma} F'^{(n)}{}_A{}^i(x)\, \psi^A{}_i(t,x)\ d^3x\,. \end{aligned} \tag{823}$$

The gauged super Poincaré algebra 4.14 (p.194) can then be reformulated in the normal mode basis. Of importance here are merely the relations (810) and (811):

4.17 (Generator algebra in normal mode representation) Recall (see (792), p. 189) the part of the supergravity Hamiltonian associated with the translational and supersymmetry generators:

$$\int\limits_{\Sigma} N^0(t,x)\,\mathcal{H}_\perp(t,x) + N^i(t,x)\,\mathcal{H}_i(t,x) + \psi^A{}_0(t,x)\,\mathcal{S}_A(t,x) - \bar\psi^{A'}{}_0(t,x)\,\bar{\mathcal{S}}_{A'}(t,x)\ d^3x \tag{824}$$

Varying the Lagrange multipliers N^0, N^i, $\psi^A{}_0$, and $\bar{\psi}^{A'}{}_i$ *at every point of* Σ *separately* gives the usual pointwise constraints of supergravity

$$\begin{aligned} \mathcal{H}_\perp(t,x) &= 0 \\ \mathcal{H}_i(t,x) &= 0 \\ \mathcal{S}_A(t,x) &= 0 \\ \bar{\mathcal{S}}_{A'}(t,x) &= 0, \qquad \forall\, t \in \mathbb{R},\ x \in \Sigma\,. \end{aligned} \tag{825}$$

This may be reexpressed in the mode basis by expanding the Lagrange multipliers as in (822):

$$\begin{aligned} &\int\limits_\Sigma N^0(t,x)\,\mathcal{H}_\perp(t,x) + N^i(t,x)\,\mathcal{H}_i(t,x) + \psi^A{}_0(t,x)\,\mathcal{S}_A(t,x) - \bar{\psi}^{A'}{}_0(t,x)\,\bar{\mathcal{S}}_{A'}(t,x)\ d^3x \\ = \quad & N^{(n)0}(t) \int\limits_\Sigma C_{(n)}(x)\,\mathcal{H}_\perp(t,x)\ d^3x \\ & + N^{(n)i}(t) \int\limits_\Sigma C_{(n)}(x)\,\mathcal{H}_i(t,x)\ d^3x \\ & + \psi^{(n)A}{}_0(t) \int\limits_\Sigma C_{(n)}(x)\,\mathcal{S}_A(t,x)\ d^3x \\ & - \bar{\psi}^{(n)A'}{}_0(t) \int\limits_\Sigma C_{(n)}(x)\,\mathcal{S}_{A'}(t,x)\ d^3x\,. \end{aligned} \tag{826}$$

In this representation the independent degrees of freedom in the Lagrange multipliers are the amplitudes $N^{(n)\mu}$, $\psi^{(n)A}{}_0$, $\bar{\psi}^{(n)A'}{}_0$. Varying these *for each* $n \in \mathbb{N}$ *separately* gives the mode representation version of the constraints:

$$\begin{aligned} H_{(n)\perp}(t) &= 0 \\ H_{(n)i}(t) &= 0 \\ S_{(n)A}(t) &= 0 \\ \bar{S}_{(n)A'}(t) &= 0, \qquad \forall\, t \in \mathbb{R},\ n \in \mathbb{N}\,, \end{aligned} \tag{827}$$

where[51]

$$\begin{aligned} H_{(n)\perp} &:= \int\limits_\Sigma C_{(n)}(x)\,\mathcal{H}_\perp(t,x)\ d^3x \\ H_{(n)i} &:= \int\limits_\Sigma C_{(n)}(x)\,\mathcal{H}_i(t,x)\ d^3x \end{aligned}$$

[51] The integrals over spinor and vector quantities are here defined componentwise with respect to the fixed but arbitrary coordinate chart on Σ. These integrals have no coordinate-invariant meaning, and they need not have. The whole mode decomposition relies on fixing an arbitrary coordinate system in which to formulate the theory. This is inevitable and ultimately simply amounts to a choice of parameterization of configuration space. Choosing different coordinates with different mode functions gives rise to another parameterization of configuration space, which is just as acceptable. This does not affect the physics, since the diffeomorphism constraints will ensure that any physical state is independent of the coordinates chosen. Note that also the functional representation must fix a coordinate system on Σ in order to define the pointwise field $e^a{}_i(x)$, etc.

$$
\begin{aligned}
S_{(n)A} &:= \int_\Sigma C_{(n)}(x)\, \mathcal{S}_A(t,x)\ d^3x \\
\bar{S}_{(n)A'} &:= \int_\Sigma C_{(n)}(x)\, \mathcal{S}_{A'}(t,x)\ d^3x
\end{aligned} \tag{828}
$$

are the mode versions of the constraints. From (810), (811) one finds the following Dirac bracket algebra among these quantities:

$$
\begin{aligned}
\left\{S_{(n)A}, S_{(m)B}\right\} &= \int_\Sigma\int_\Sigma C_{(n)}(x)\, C_{(m)}(x')\, \underbrace{\{\mathcal{S}_A(x), \mathcal{S}_B(x')\}}_{=0}\ d^3x\, d^3x' = 0 \\
\left\{\bar{S}_{(n)A'}, \bar{S}_{(m)B'}\right\} &= \int_\Sigma\int_\Sigma C_{(n)}(x)\, C_{(m)}(x')\, \underbrace{\{\bar{\mathcal{S}}_{A'}(x), \bar{\mathcal{S}}_{B'}(x')\}}_{=0}\ d^3x\, d^3x' = 0 \\
\left\{S_{(n)A}, \bar{S}_{(m)B'}\right\} &= \int_\Sigma\int_\Sigma C_{(n)}(x)\, C_{(m)}(x')\, \underbrace{\{\mathcal{S}_A(x), \bar{\mathcal{S}}_{B'}(x')\}}_{=\sigma^a{}_{AA'}\delta(x,x')\mathcal{H}_a(x)}\ d^3x\, d^3x' = K_{(n)(m)}{}^{(p)}\sigma^a{}_{AB'}H_{(p)a} \\
\left[S_{(n)A}, H_{(m)a}\right] &= \int_\Sigma\int_\Sigma C_{(n)}(x)\ C_{(m)}(x')\, \underbrace{\left[\hat{\mathcal{S}}_A(x), \mathcal{H}_0(x')\right]}_{\approx 0}\ d^3x\, d^3x' \approx 0 \\
\left[\bar{S}_{(n)A'}, H_{(m)}\right] &= \int_\Sigma\int_\Sigma C_{(n)}(x)\ C_{(m)}(x')\, \underbrace{\left[\bar{\mathcal{S}}_{A'}(x), \mathcal{H}_0(x')\right]}_{\approx 0}\ d^3x\, d^3x' \approx 0\,,
\end{aligned} \tag{829}
$$

where "≈" stands for "up to terms proportional to the Lorentz constraints".
To summarize, the only non-vanishing (up to Lorentz generators) bracket is the central supersymmetry anticommutator:

$$
\left\{S_{(n)A}, \bar{S}_{(m)B'}\right\} = K_{(n)(m)}{}^{(p)}\sigma^a{}_{AB'}\, H_{(p)a}\,. \tag{830}
$$

Here the constants K are defined by

$$
K_{(n)(m)}{}^{(p)} = \int_\Sigma C_{(n)}(x)\, C_{(m)}(x)\, C'^{(p)}(x)\ d^3x\,. \tag{831}
$$

The system may now conveniently be quantized in the mode basis:

4.18 (Canonical quantization in the mode basis) In the functional (pointwise) representation the classical graviton and gravitino fields become functional multiplication and differentiation operators, respectively ([80][83]):

$$
\begin{aligned}
e^a{}_i(t,x) &\to \hat{e}^a{}_i(x) \\
e^{\#i}_a(t,x) &\to \hat{e}^{\#i}_a(x) = \frac{\delta}{\delta e^a{}_i(x)} \\
\psi^A{}_i(t,x) &\to \hat{\psi}^A{}_i(x) \\
\psi^{\#i}_A(t,x) &\to \hat{\psi}^{\#i}_A(x) = \frac{\delta}{\delta \psi^A{}_i(x)}\,.
\end{aligned} \tag{832}
$$

Here the fermionic operators are of Grassmann type so that

$$
\begin{aligned}
\left[\frac{\delta}{\delta e^a{}_i(x)}, e^b{}_j(y)\right] &= \delta^b_a\, \delta^i_j\, \delta(x,y) \\
\left\{\frac{\delta}{\delta \psi^A{}_i(x)}, \psi^B{}_j(y)\right\} &= \delta^B_A\, \delta^i_j\, \delta(x,y)\ ,
\end{aligned} \tag{833}
$$

with all other supercommutators vanishing. After expansion into modes one has

$$
\begin{aligned}
\hat{e}^a{}_i(x) &:= B_{(n)}{}^a{}_i(x)\,\hat{b}^{(n)} \\
\frac{\delta}{\delta e^a{}_i(x)} &:= B'^{(n)}{}_a{}^i \hat{b}^{\#}_{(n)} \\
\hat{\psi}^A{}_i(x) &:= F_{(n)}{}^A{}_i(x)\,\hat{f}^{(n)} \\
\frac{\delta}{\delta \psi^A{}_i(x)} &:= F'^{(n)}{}_A{}^i \hat{f}^{\#}_{(n)} ,
\end{aligned}
\tag{834}
$$

or conversely

$$
\begin{aligned}
\hat{b}^{(n)} &= \int_\Sigma B'^{(n)}{}_a{}^i(x)\,\hat{e}^a{}_i(x)\;d^3x \\
\hat{b}^{\#}_{(n)} &= \int_\Sigma B_{(n)}{}^a{}_i(x)\,\frac{\delta}{\delta e^a{}_i(x)}\;d^3x \\
\hat{f}^{(n)} &= \int_\Sigma F'^{(n)}{}_A{}^i(x)\,\hat{\psi}^A{}_i(x)\;d^3x \\
\hat{f}^{\#}_{(n)} &= \int_\Sigma F_{(n)}{}^A{}_i(x)\,\frac{\delta}{\delta \psi^A{}_i(x)}\;d^3x .
\end{aligned}
\tag{835}
$$

By construction (see eqs. (815) and (816) in 4.15 (p.194)) the following supercommutator algebra holds for the mode amplitude operators $\hat{b}$, $\hat{f}$:

$$
\begin{aligned}
\left[\hat{b}^{(n)}, \hat{b}^{(m)}\right] &= 0 \\
\left[\hat{b}^{\#}_{(n)}, \hat{b}^{\#}_{(m)}\right] &= 0 \\
\left[\hat{b}^{\#}_{(n)}, \hat{b}^{(m)}\right] &= \delta^m_n \\
\left\{\hat{f}^{(n)}, \hat{f}^{(m)}\right\} &= 0 \\
\left\{\hat{f}^{\#}_{(n)}, \hat{f}^{\#}_{(m)}\right\} &= 0 \\
\left\{\hat{f}^{\#}_{(n)}, \hat{f}^{(m)}\right\} &= \delta^m_n ,
\end{aligned}
\tag{836}
$$

all other commutators vanishing.

4.19 (Literature) A mode decomposition for the gravitino field in canonical supergravity was also considered in [43]. In contrast to the construction discussed above, the authors there choose a set of spinor modes $\rho_i^{(m)}(x)$, $\beta^{(M)}(x)$ which are dependent on the bosonic vielbein field $e^a{}_\mu$ (eq. (5.2)(5.3) of [43]):

$$
\begin{aligned}
\rho_i^{(m)}(x) &= \rho_i^{(m)}\left[x, e^{\cdot}{}_{\cdot}\right] \\
\beta^{(M)}(x) &= \beta^{(M)}\left[x, e^{\cdot}{}_{\cdot}\right] .
\end{aligned}
\tag{837}
$$

This might raise some subtle questions when it comes to quantizing the theory in this representation, because in the quantum theory the vielbein field no longer has a fixed value. For instance, in equation (5.ag) of [43],

$$\psi(x) = \sum_m r_m \rho_i^{(m)}(x) + \sum_m b_M \tilde{\sigma}_i \beta^{(M)}(x) \;, \tag{838}$$

which, with the functional dependencies of the field modes and the modified Pauli matrix $\tilde{\sigma}$ (*cf.* eq. (5.aac) of that paper) explicitly stated, reads

$$\psi(x) = \sum_m r_m \rho_i^{(m)}\left[x, e^{\cdot}.(x)\right] + \sum_m b_M \tilde{\sigma}_i \left[x, e^{\cdot}.(x)\right] \beta^{(M)}\left[x, e^{\cdot}.(x)\right] \;,$$

it is not obvious which value of the vielbein field $e^{\cdot}.$ is referred to. The r_m and r_M are Grassmann numbers (*cf.* eq. (4.26)(4.27)), but it seems they cannot be both linearly independent and independent of the vielbein field themselves. For suppose they are independent of the vielbein field, so that $\left[\frac{\delta}{\delta e^{\cdot}.}, r.\right] = 0$, then because of the requirement that ψ_i itself is independent of the vielbein (which follows from [80] eqs. (2.34),(3.3)) one has

$$\begin{aligned}
&\left[\frac{\delta}{\delta e^{\cdot}.(x')}, \psi_i(x)\right] = 0 \\
&\Rightarrow \sum_m r_m \left[\rho_i^{(m)}\left[x, e^{\cdot}.(x)\right], \frac{\delta}{\delta e^{\cdot}.(x')}\right] + \sum_m b_M \left[\tilde{\sigma}_i \left[x, e^{\cdot}.(x)\right] \beta^{(M)}(x) \left[x, e^{\cdot}.(x)\right], \frac{\delta}{\delta e^{\cdot}.(x')}\right] = 0 \,,
\end{aligned} \tag{839}$$

which says that the Grassmann numbers r are linearly dependent. The only way out is to make the Grassmann numbers r functionals of the vielbein, too. It might not be obvious how this then is supposed to be quantized.
Such potential problems are avoided when working with a fixed set of modes as described above. Of course, one thereby loses the useful property that the fermionic modes are eigenmodes of the Dirac operator with respect to the connection induced by the vielbein field. But we do not need this property for the present purpose.

Before looking for a representation of the mode amplitude operators (835) first consider the form of the primed supersymmetry constraint operator in the mode basis:

4.20 (Primed supersymmetry generator in the mode representation) As shown in [80], using the functional representation $\hat{e}^a{}_j$, $\frac{\delta}{\delta e^a{}_j}$, the following operator is obtained for the primed supersymmetry generator:

$$\begin{aligned}
\bar{\mathcal{S}}_{A'}(x) &= \frac{\hbar\kappa^2}{2} \hat{\psi}^A{}_i(x) \frac{\delta}{\delta \hat{e}^{AA'}{}_i(x)} + \epsilon^{ijk} \hat{e}^a{}_i(x)\, \sigma_{aAA'} \left(D_j \left[x, \hat{e}^{\cdot}.(x)\right] \hat{\psi}^A{}_k(x)\right) \\
&\overset{(1384)}{=} -\frac{\hbar\kappa^2}{2} \hat{\psi}^A{}_i(x)\, \sigma^a{}_{AA'} \frac{\delta}{\delta \hat{e}^{a'}{}_i(x)} + \epsilon^{ijk} \hat{e}^a{}_i(x)\, \sigma_{aAA'} \left(D_j \left[x, \hat{e}^{\cdot}.(x)\right] \hat{\psi}^A{}_k(x)\right) \,.
\end{aligned} \tag{840}$$

(In the last line use has been made of properties of the σ-matrices. These are summarized in the appendix, §G.1 (p.342)).
For clarity, we have here explicitly stated function dependencies on the coordinates x and functional dependencies on the vielbein field $e^a{}_i(x)$. In general

these explicit dependencies will be suppressed.
The above expression may be inserted into (828) to obtain the respective operators in the normal mode operator basis. One finds:

$$\bar{S}_{(n)A'} \quad = \quad \hat{f}^{(p)} \left(E_{(n)A'(p)}{}^{(q)} \, \hbar \hat{b}^{\#}_{(q)} + U_{(n)A'(p)} \left(\hat{b}^{\cdot} \right) \right) , \tag{841}$$

where $E_{(n)A'(p)}{}^{(q)}$ are constants:

$$E_{(n)A'(p)}{}^{(q)} \quad := \quad -\frac{1}{2}\kappa^2 \int_\Sigma C_{(n)} F_{(p)}{}^A{}_i \sigma^a{}_{AA'} B'^{(q)}{}_a{}^i \; d^3x \tag{842}$$

and $U_{(n)A'(p)} = U_{(n)A'(p)}\left(b^{(\cdot)}\right)$ are function of the bosonic amplitude operators $\hat{b}^{(n)}$:

$$U_{(n)A'(p)}\left(b^{(\cdot)}\right) \quad := \quad \int_\Sigma C_{(n)} \epsilon^{ijk} \hat{b}^{(q)} B_{(q)}{}^a{}_i \sigma_{aAA'} \, D_j \left(b^{(\cdot)}\right) F_{(p)}{}^A{}_k \; d^3x \, . \tag{843}$$

This is the result of the following straightforward calculation:

$$\begin{aligned}
\hat{\bar{S}}_{(n)A'} &\overset{(828)}{=} \int_\Sigma C_{(n)}(x) \, \bar{\mathcal{S}}_{A'}(x) \; d^3x \\
&\overset{(840)}{=} \int_\Sigma C_{(n)}(x) \left(-\frac{\hbar\kappa^2}{2} \sigma^a{}_{AA'} \hat{\psi}^A{}_i(x) \frac{\delta}{\delta \hat{e}^a{}_i(x)} + \epsilon^{ijk} \hat{e}^a{}_i(x) \, \sigma_{aAA'} \left(D_j \left[x, \hat{e}^{\cdot}{}_{\cdot}(x)\right] \hat{\psi}^A{}_k(x) \right) \right) \; d^3x \\
&\overset{(834)}{=} \int_\Sigma C_{(n)} \left(-\frac{\hbar\kappa^2}{2} \hat{f}^{(p)} F_{(p)}{}^A{}_i \sigma^a{}_{AA'} B'^{(q)}{}_a{}^i \frac{\partial}{\partial b^{(q)}} + \hat{f}^{(p)} \epsilon^{ijk} \hat{b}^{(q)} B_{(q)}{}^a{}_i \sigma_{aAA'} \, D_j \left(b^{(\cdot)}\right) F_{(p)}{}^A{}_k \right) \; d^3x \\
&= \quad \hat{f}^{(p)} \underbrace{\frac{1}{2}\kappa^2 \int_\Sigma C_{(n)} F_{(p)}{}^A{}_i \sigma^a{}_{AA'} B'^{(q)}{}_a{}^i \; d^3x}_{:= E_{(n)A'(p)}{}^{(q)}} \; \hbar \frac{\partial}{\partial b^{(q)}} + \\
&\qquad + \hat{f}^{(p)} \underbrace{\int_\Sigma C_{(n)} \epsilon^{ijk} \hat{b}^{(q)} B_{(q)}{}^a{}_i \sigma_{aAA'} \, D_j \left(b^{(\cdot)}\right) F_{(p)}{}^A{}_k \; d^3x}_{:= U_{(n)A'(p)}\left(\hat{b}_{(\cdot)}\right)} \\
&:= \quad \hat{f}^{(p)} \left(E_{(n)A'(p)}{}^{(q)} \hbar \frac{\partial}{\partial b^{(q)}} + U_{(n)A'(p)}\left(b^{(\cdot)}\right) \right) ,
\end{aligned} \tag{844}$$

It is sometimes convenient to make an integration by parts in the definition of the $U_{(n)A'(p)}$:

$$\begin{aligned}
U_{(n)A'(p)}\left(b^{(\cdot)}\right) &= \int_\Sigma C_{(n)} \epsilon^{ijk} \hat{b}^{(q)} B_{(q)}{}^a{}_i \sigma_{aAA'} \, D_j \left(b^{(\cdot)}\right) F_{(p)}{}^A{}_k \; d^3x \\
&= \int_\Sigma C_{(n)} \epsilon^{ijk} \hat{b}^{(q)} B_{(q)}{}^a{}_i \sigma_{aAA'} \, \left(\delta^A{}_B \partial_j + \omega_j{}^A{}_B \right) F_{(p)}{}^B{}_k \; d^3x \\
&= -\int_\Sigma F_{(p)}{}^B{}_k \left(\delta^A{}_B \partial_j - \omega_j{}^A{}_B \right) C_{(n)} \epsilon^{ijk} \hat{b}^{(q)} B_{(q)}{}^a{}_i \sigma_{aAA'} \; d^3x
\end{aligned}$$

$$= \quad -\int\limits_\Sigma F_{(p)}{}^A{}_k \left(\delta^B{}_A \partial_j - \omega_j{}^B{}_A\right) C_{(n)} \epsilon^{ijk} \hat{b}^{(q)} B_{(q)}{}^a{}_i \sigma_{aBA'} \; d^3x \,. \tag{845}$$

(In the last line we have merely renamed spinor indices $A \leftrightarrow B$.)

4.21 (Invertability of the $E_{(n)A'}$) With respect to finding a natural representation for the mode amplitude operators, it is essential to note that the matrix

$$E_{(n)A'} \quad := \quad \left(E_{(n)A'(p)}{}^{(q)}\right)_{p,q\in\mathbb{N}} \tag{846}$$

has a *right* inverse

$$\begin{aligned} \tilde{E}_{(n)}{}^{A'} &:= \left(\tilde{E}_{(n)}{}^{A'}{}_{(p)}{}^{(q)}\right)_{p,q\in\mathbb{N}} \\ E_{(n)A'(p)}{}^{(q)} \tilde{E}_{(n)}{}^{B'}{}_{(q)}{}^{(p')} &= \delta_p^{p'} \, \delta_{A'}^{B'} \end{aligned} \tag{847}$$

given by (see (818) for the definition of $\left[1/C_{(n)}(x)\right]$):

$$\tilde{E}_{(n)}{}^{A'}{}_{(p)}{}^{(q)} = \frac{2}{\kappa^2} \int\limits_\Sigma \left[\frac{1}{C_{(n)}(x)}\right] B_{(p)}{}^a{}_i(x)\, \sigma_a{}^{AA'} F'^{(q)}{}_A{}^i(x) \; d^3x \tag{848}$$

(this holds due to relation (1382) in §G (p.342): $\sigma_a{}^{BB'} \sigma^a{}_{AA'} = -\delta_A^B \, \delta_{A'}^{B'}$), but no *left* inverse (since the converse relation (1381), namely $\sigma_a{}^{AA'} \sigma^b{}_{AA'} = -\delta_{ab}$, is not applicable here, because the index A' is fixed in (847)). In order to get the analog of a left inverse one has to sum over the A' index:

$$\tilde{E}_{(n)}{}^{A'}{}_{(p)}{}^{(q)} E_{(n)A'(q)}{}^{(r)} \quad = \quad \delta_p^r \,. \tag{849}$$

4.22 (Remark.) However, when a certain Lorentz gauge is chosen, the coefficient matrices E, $\tilde{E}$ may be fully invertible (for fixed A') in a certain sense: One can then extract the object they are contracted with by a combination of linear transformations and complex conjugations. This is detailed in the appendix §D, point **??** (p.**??**), but won't be needed for the following constructions.

4.23 (Mode representation vs. functional representation) While ultimately it is just a question of representation whether one uses the pointwise form (840) of the supersymmetry generator or the mode-basis version (841), it may be noted that a crucial advantage of the latter is that it allows a *factorization* of the fermionic operators $\hat{f}^{(p)}$ from the purely bosonic operator $\left(E_{(n)A'(p)}{}^{(q)} \, \hbar \hat{b}^{\#}_{(q)} + U_{(n)A'(p)}\left(\hat{b}^{\cdot}\right)\right)$. A similar factorization is not possible in the functional representation (840), since the second term, $\epsilon^{ijk} \hat{e}^a{}_i \sigma_{aAA'} D_j \hat{\psi}^A{}_k$, is, in a sense, non-local, due to the derivative D_j. Attempting to separate the fermionic functional multiplication from the derivative by transforming to the mode basis, then taking the derivative and transforming back to the functional basis, leads to the following expression:

$$\begin{aligned} \epsilon^{ijk} \hat{e}^a{}_i(x)\, \sigma_{aAA'} D_j \hat{\psi}^A{}_k(x) &\overset{(834)}{=} \hat{f}^{(p)}\, \epsilon^{ijk} \hat{e}^a{}_i(x)\, \sigma_{aAA'} D_j F^A_{(p)i}(x) \\ &\overset{(835)}{=} \left(\int\limits_\Sigma \hat{\psi}^A{}_i(x') \; F'^{(p)}{}_A{}^i(x') \; d^3x'\right) \epsilon^{ijk} \hat{e}^a{}_i(x)\, \sigma_{aAA'} D_j F^A_{(p)i}(x) \,. \end{aligned} \tag{850}$$

This term is not proportional to $\hat{\psi}^{A}{}_{i}(x)$ at point x, but instead it contains contributions $\hat{\psi}^{A}{}_{i}(x')$ from all points $x' \in \Sigma$ on the entire spatial manifold Σ. This nonlocality is the reason why the functional representation, which is inherently local, cannot easily capture the more simple form (841) of the supersymmetry generator. It is shown below that the latter form allows to identify the supersymmetry generators with deformed exterior derivatives.

4.24 (Representation of the mode amplitude operators) We can now make contact with the differential geometry on $\mathcal{M}^{(\mathrm{conf})}$. See §2.1.1 (p.15) for details on differential geometry and its relation to supersymmetry, and see in particular the brief introduction 2.2 (p.16). The following construction closely parallels that of §3.1 (p.143).
The form of the primed supersymmetry generator (841) suggests that it is naturally represented as an exterior derivative (see (86), p. 27 for the general definition and (1214), p. 305 for the representation that applies here, as discussed in B.13 (p.304)). This can be made more precise as follows:

Let $\mathcal{M}^{(\mathrm{conf})}$ be the *bosonic configuration space*, i.e. the space of all vielbein fields on Σ. By the above mode decomposition, this space is fully coordinatized by the amplitudes $b^{(n)},\ n \in \mathbb{N}$, so $\mathcal{M}^{(\mathrm{conf})}$ is a manifold of countable infinite dimension. Each of the matrices $\tilde{E}_{(m)A'}$ defines a change of coordinates on $\mathcal{M}^{(\mathrm{conf})}$ from $\left\{b^{(p)}\right\}_{p\in\mathbb{N}}$ to some set $\left\{X^{(q)A'}_{(n)}\right\}_{q\in\mathbb{N}}$ by the linear coordinate transformation

$$X^{(q)A'}_{(n)} := \tilde{E}_{(n)}{}^{A'}{}_{(p)}{}^{(q)} b^{(p)} \,. \tag{851}$$

Since the $\tilde{E}_{(n)}{}^{A'}{}_{(p)}{}^{(q)}$ are in general complex, this implies that the $\left\{X^{(q)A'}_{(n)}\right\}_{q\in\mathbb{N}}$ are complex coordinates on $\mathcal{M}^{(\mathrm{conf})}$. (This is ultimately due to the fact that the gravitino spin is represented on a complex vector space, because it are the complex components of the Pauli matrices and of the spinor modes that make $\tilde{E}$ complex.) According to (849) the original coordinates are reobtained by taking linear combinations of the $A' = 1$ and the $A' = 2$ coordinates:

$$b^{(p)} \quad = \quad X^{(q)A'}_{(n)} E_{(n)A'(q)}{}^{(p)} \,, \tag{852}$$

(where a sum over A' is implicit, as always for indices appearing upstairs and downstairs).
With this interpretation it is easy to see that the operators $E_{(n)A'(p)}{}^{(q)} \hat{b}^{\#}_{(q)}$ are partial derivative operators on $\mathcal{M}^{(\mathrm{conf})}$ with respect to the coordinates $X^{(q)}_{(n)A'}$. In other words, they span the tangent bundle $T\big(\mathcal{M}^{(\mathrm{conf})}\big)$:

$$\begin{aligned}
\hat{X}^{(q)A'}_{(n)} \quad &:= \quad \tilde{E}_{(n)}{}^{A'}{}_{(p)}{}^{(q)} \hat{b}^{(p)} \\
\frac{\partial}{\partial \hat{X}^{(q)A'}_{(n)}} \quad &:= \quad E_{(n)A'(q)}{}^{(p)} \hat{b}^{\#}_{(p)} \\
\Rightarrow \left[\frac{\partial}{\partial \hat{X}^{(q)A'}_{(n)}}, \hat{X}^{(q')B'}_{(n)}\right] \quad &= \quad \left[E_{(n)A'(q)}{}^{(p)} \hat{b}^{\#}_{(p)}, \tilde{E}_{(n)}{}^{B'}{}_{(p')}{}^{(q')} \hat{b}^{(p')}\right]
\end{aligned}$$

$$
\begin{aligned}
&= E_{(n)A'(q)}{}^{(p)} \tilde{E}_{(n)}{}^{B'}{}_{(p')}{}^{(q')} \underbrace{\left[\hat{b}^{\#}_{(p)}, \hat{b}^{(p')}\right]}_{=\delta_p^{p'}} \\
&= E_{(n)A'(q)}{}^{(p)} \tilde{E}_{(n)}{}^{B'}{}_{(p)}{}^{(q')} \\
&= \delta_q^{q'} \, \delta_{A'}^{B'} \, . \qquad (853)
\end{aligned}
$$

(The $X^{(q)\,A'}_{(n)}$ are complex coordinates on $\mathcal{M}^{(\mathrm{conf})}$ and the $\partial/\partial X^{(q)\,A'}_{(n)}$ their partial derivatives. Due to lack of time (*cf.* item (3) in 6.2 (p.291)) it is not discussed here what the complex structure on $T\big(\mathcal{M}^{(\mathrm{conf})}\big)$ is and how these coordinates can be separated into holomorphic and antiholomorphic coordinates (*cf.* [50] and [52]§14). In fact, $\mathcal{M}^{(\mathrm{conf})}$ should be a Kähler manifold, as discussed in 4.27 (p.208), 4.28 (p.210), and 4.29 (p.210) below.)
For definiteness, fix one of the coordinate systems on $\mathcal{M}^{(\mathrm{conf})}$, which are induced by the matrices $\tilde{E}_{(n)}{}^{A'}$, say that associated with $\tilde{E}_{(n=0)}{}^{A'=1}$, and denote the respective coordinates by $X^{(n)}$ and the multiplication operators by these coordinate functions by $\hat{X}^{(n)}$:

$$
\begin{aligned}
X^{(n)} &:= \tilde{E}_{(0)}{}^{1'}{}_{(m)}{}^{(n)} b^{(m)} \\
\hat{X}^{(n)} &:= \tilde{E}_{(0)}{}^{1'}{}_{(m)}{}^{(n)} \hat{b}^{(m)} \, . \qquad (854)
\end{aligned}
$$

Then, obviously,

$$
\partial_{X^{(n)}} = E_{(0)1'(n)}{}^{(m)} \hat{b}^{\#}_{(m)} \, , \qquad (855)
$$

so that

$$
\hat{f}^{(n)} E_{(0)1'(n)}{}^{(m)} \hat{b}^{\#}_{(m)} = \hat{f}^{(n)} \partial_{X^{(n)}} \, . \qquad (856)
$$

Comparison with the discussion in B.13 (p.304) shows that, because of the relation

$$
\begin{aligned}
\left[\partial_{X^{(n)}}, \hat{f}^{(m)}\right] &= E_{(0)}{}^{1'}{}_{(n)}{}^{(p)} \underbrace{\left[\hat{b}^{\#}_{(p)}, \hat{f}^{(m)}\right]}_{=0} \\
&= 0 \qquad (857)
\end{aligned}
$$

the expression $\hat{f}^{(n)}\partial_{X^{(n)}}$ is an exterior derivative on $\mathcal{M}^{(\mathrm{conf})}$. This implies that the fermionic amplitude multiplication operators $\hat{f}^{(n)}$ can be identified with operators of exterior multiplication with differential forms, i.e. as spanning a basis of the cotangent bundle $T^*\big(\mathcal{M}^{(\mathrm{conf})}\big)$ of configuration space (see 2.2 (p.16)). The notation of §2.1.1 (p.15), introduced in 2.2 (p.16), is hence obtained by identifying (*cf.* equation (12), p. 18):

$$
\hat{\mathrm{c}}^{\dagger\,(n)} := \hat{f}^{(n)} \, . \qquad (858)
$$

Because of the anticommutation relations (*cf.* (19), p. 19)

$$
\left\{\hat{f}^{\#}_{(n)}, \hat{f}^{(m)}\right\} = \delta_n^m = \left\{\hat{\mathrm{c}}_{(n)}, \hat{\mathrm{c}}^{(m)}\right\}
$$

this implies the further identification (*cf.* (17), (18), p. 19)

$$
\hat{\mathrm{c}}_{(n)} := \hat{f}^{\#}_{(n)} \, . \qquad (859)
$$

(Note here the position of the index on $\hat{c}$, *cf.* 4.27 (p.208)).

The above exterior derivative now reads

$$\begin{aligned} \mathbf{d}_{\mathcal{M}^{(\mathrm{conf})},1} &= \hat{f}^{(n)}\partial_{X^{(n)}} \\ &= \hat{c}^{\dagger(n)}\partial_{X^{(n)}} \,. \end{aligned} \tag{860}$$

The index 1 here is to distinguish this exterior derivative from that obtained by choosing $A' = 2$ instead of $A' = 1$ in (854).

The kinematical space of states of canonical $D = 4$, $N = 1$ quantum supergravity on $\mathcal{M} = \Sigma \otimes \mathbb{R}$ can now be identified with that of (square integrable) sections of the exterior bundle $\Lambda\big(\mathcal{M}^{(\mathrm{conf})}\big)$ over configuration space. This space is usually denoted by $\Gamma\big(\Lambda\big(\mathcal{M}^{(\mathrm{conf})}\big)\big)$. Identification of a metric on $\mathcal{M}^{(\mathrm{conf})}$ induces the natural (Hodge) inner product (*cf.* (38), p. 21) on this space, which turns the kinematical space of states into a super Krein space[52] $\mathcal{K}$:

$$\mathcal{K} = \Gamma\Big(\Lambda\Big(\mathcal{M}^{(\mathrm{conf})}\Big)\Big) \tag{861}$$

(*cf.* [233][119]). This will be discussed in more detail in 4.27 (p.208) and 4.31 (p.213).

After these considerations it is clear that the primed supersymmetry generator may be identified with a deformed exterior derivative on $\mathcal{M}^{(\mathrm{conf})}$:

4.25 (Supersymmetry generator as a deformed exterior derivative) Consider first the 0-mode, $\bar{S}_{(0)1'}$ of the $A' = 1$ component of the barred supersymmetry constraint $\bar{\mathcal{S}}_{1'}$. According to (844) and (854), (855), (856), (858) it reads:

$$\bar{S}_{(0)1'} = \hat{c}^{\dagger(m)}\hbar\partial_{X^{(m)}} + \hat{c}^{\dagger(m)}V_{(0)(m)} \,, \tag{862}$$

where $V_{(0)m}$ are some function on configuration space. From the supersymmetry algebra (829) it follows that

$$\begin{aligned} & \big(\bar{S}_{(0)1'}\big)^2 = 0 \\ \Leftrightarrow\ & \Big(\hat{c}^{\dagger(m)}\hbar\partial'_{X^{(m)}} + \hat{c}^{\dagger(m)}V_{(0)(m)}\Big)^2 = 0 \\ \Leftrightarrow\ & \big(\partial_{X^{(m)}}V_{(0)(n)}\big)\,\hat{c}^{\dagger(m)}\hat{c}^{\dagger(n)} = 0 \\ \Leftrightarrow\ & \partial_{X^{[(m)}}V_{(0)(n)]} = 0 \,, \end{aligned} \tag{863}$$

which means, according to the Poincaré lemma (*cf.* 2.45 (p.51)), that locally

$$V_{(0)(m)} = \partial_{X^{(m)}}W_{(0)} \,, \tag{864}$$

[52] The manifold $\Lambda^1\big(\mathcal{M}^{(\mathrm{conf})}\big)$ may be regarded as the superspace $\mathcal{M}^{(\mathrm{conf})}(\infty|\infty)$ over bosonic base space $\mathcal{M}^{(\mathrm{conf})}$ (*cf.* e.g. [229], [57] §11.9.1) and elements in $\Gamma\Big(\Lambda\Big(\mathcal{M}^{(\mathrm{conf})}\Big)\Big)$ are *superfields* on this space (e.g. [57], §11.9.1). Hence we are dealing here, with "super-super-space", where the first "super" refers to supersymmetry and the Grassmannian coordinates dX^m, while the second "super" is in the cosmological sense of Wheeler, it refers to "the configuration space of space", which has bosonic coordinates X^m. While the term "super-super-space" is perfectly in agreement with currently accepted conventions on terminology, for obvious reasons we prefer to address this space as $\Lambda^1\big(\mathcal{M}^{(\mathrm{conf})}\big)$, the form bundle over the gravitational configuration space, i.e. the form bundle over the configuration space of the vielbein field.

for some function $W_{(0)}$ on $\mathcal{M}^{(\mathrm{conf})}$. Since

$$\mathbf{d}_{\mathcal{M}^{(\mathrm{conf})},1} \;=\; \hat{\mathrm{c}}^{\dagger\,(m)} \partial'_{X^{(m)}} \tag{865}$$

is the exterior derivative on configuration space (*cf.* §2.1.1 (p.15), (1214)), the supercharge $\bar{S}_{(0)1'}$ may be rewritten as

$$\begin{aligned} \bar{S}_{(0)1'} &= \hbar \mathbf{d}_{\mathcal{M}^{(\mathrm{conf})},1} + \hat{\mathrm{c}}^{\dagger\,(m)} \left(\partial_{(m)} W_{(0)}\right) \\ &= e^{-W_{(0)}/\hbar}\, \hbar \mathbf{d}_{\mathcal{M}^{(\mathrm{conf})}}\, e^{W_{(0)}/\hbar}\,, \end{aligned} \tag{866}$$

i.e. as the exterior derivative on configuration space deformed according to the Witten model of supersymmetric quantum mechanics (*cf.* §2.2.1 (p.55) and in particular 2.2.2 (p.61)). The function $W_{(0)}$ is known as a *superpotential*.

The form of the other modes of the primed supersymmetry constraint is most conveniently discussed after having first introduced a set of generalized number operators:

4.26 (Generalized number operators) Consider the set $\hat{N}_{(n)(m)}$ of operators defined by

$$\begin{aligned} \hat{N}_{(n)(m)} &:= \int_\Sigma C_{(n)}(x) \left[\frac{1}{C_{(m)}(x)}\right] \hat{\psi}^A{}_i(x)\, \frac{\delta}{\delta \psi^A{}_i(x)}\; d^3x \\ &= \int_\Sigma C_{(n)}(x) \left[\frac{1}{C_{(m)}(x)}\right] \hat{\mathrm{c}}^{\dagger\,(p)} F_{(p)}{}^A{}_i(x)\, F'^{(q)}{}_A{}^i(x)\, \hat{\mathrm{c}}_{(q)}\; d^3x \\ &:= N_{(n)(m)(p)}{}^{(q)}\, \hat{\mathrm{c}}^{\dagger\,(p)} \hat{\mathrm{c}}_{(q)}\,, \end{aligned} \tag{867}$$

where the constant matrices $N_{(n)(m)}$ are defined by the integrals[53]

$$N_{(n)(m)(p)}{}^{(q)} \;=\; \int_\Sigma C_{(n)}(x) \left[\frac{1}{C_{(m)}(x)}\right] F_{(p)}{}^A{}_i(x)\, F'^{(q)}{}_A{}^i(x)\; d^3x\,. \tag{868}$$

These will be called *generalized number operators*, since for $n = m$ one recovers the ordinary number operator on $\Lambda\left(\mathcal{M}^{(\mathrm{conf})}\right)$:

$$\begin{aligned} \hat{N}_{(0)(0)} &= \hat{\mathrm{c}}^{\dagger\,(p)} \hat{\mathrm{c}}_{(p)} \\ &= \hat{N}\,. \end{aligned} \tag{869}$$

The matrices $\left(N_{(n)(m)}\right)_{p,q}$ are invertible, because, for all functions $K_A{}^i(x)$, one has

$$\left[\hat{N}_{(n)(m)}, \left[\hat{N}_{(m)(n)}, \int_\Sigma C_{(n)}(x)\, \hat{\psi}^A{}_i(x)\, K_A{}^i(x)\; d^3x\right]\right] \;=\; \int_\Sigma C_{(n)}(x)\, \hat{\psi}^A{}_i(x)\, K_A{}^i(x)\; d^3x\,, \tag{870}$$

[53]For $m \neq 0$ the term $1/C_{(m)}$ will have poles and the integrals may have to be evaluated as principal value integrals around the poles to be well defined. Note that for the following constructions (see (876)) what is essential are really only the $N_{(n)(0)}$, where no poles occur.

but also

$$\begin{aligned}
&\left[\hat{N}_{(n)(m)}, \left[\hat{N}_{(m)(n)}, \int\limits_{\Sigma} C_{(n)}(x)\,\hat{\psi}^{A}{}_{i}(x)\,K_{A}{}^{i}(x)\ d^3x\right]\right] \\
= &\left[N_{(n)(m)(p)}{}^{(q)}\hat{\mathrm{c}}^{\dagger(p)}\hat{\mathrm{c}}_{(q)}, \left[N_{(n)(m)(r)}{}^{(s)}\hat{\mathrm{c}}^{\dagger(r)}\hat{\mathrm{c}}_{(s)}, \hat{\mathrm{c}}^{\dagger(t)}\right]\right]\int\limits_{\Sigma} C_{(n)}(x)\,\hat{F}_{(t)}{}^{A}{}_{i}(x)\,K_{A}{}^{i}(x)\ d^3x \\
= &\ N_{(n)(m)(p)}{}^{(t)}\,N_{(m)(n)(q)}{}^{(p)}\hat{\mathrm{c}}^{\dagger(q)}\int\limits_{\Sigma} C_{(n)}(x)\,\hat{F}_{(t)}{}^{A}{}_{i}(x)\,K_{A}{}^{i}(x)\ d^3x\,. \qquad (871)
\end{aligned}$$

Therefore

$$N_{(n)(m)(p)}{}^{(t)}\,N_{(m)(n)(q)}{}^{(p)} \quad = \quad \delta^{t}_{q} \qquad (872)$$

and hence

$$N_{(n)(m)} = \left(N_{(m)(n)}\right)^{-1}\,. \qquad (873)$$

These number operators relate the exterior derivatives associated with the supersymmetry generators among each other. The "non-deformed" part of the primed supersymmetry generator is

$$\begin{aligned}
\bar{S}_{(n)A'} \quad &= \quad \int\limits_{\Sigma} C_{(n)}(x)\,\hat{\psi}^{A}{}_{i}(x)\,\sigma^{a}{}_{AA'}\frac{\delta}{\delta e^{a}{}_{i}(x)}\ d^3x + \cdots \\
&= \quad \hat{\mathrm{c}}^{\dagger(p)}E_{(n)A'(p)}{}^{(q)}\hat{b}^{\#}_{(q)} + \cdots\,. \qquad (874)
\end{aligned}$$

Taking the commutator with $\hat{N}_{(m)(n)}$ yields

$$\begin{aligned}
\left[\hat{N}_{(m)(n)}, \bar{S}_{(n)A'}\right] \quad &= \quad -\int\limits_{\Sigma}\int\limits_{\Sigma} C_{(m)}(x)\left[\frac{1}{C_{(n)}(x)}\right]C_{(n)}(x')\underbrace{\left[\hat{\psi}^{B}{}_{j}(x)\,\frac{\delta}{\delta\psi^{B}{}_{j}(x)}, \psi^{A}{}_{i}(x')\right]}_{=\delta(x,x')\psi^{A}{}_{i}(x)}\sigma^{a}{}_{AA'}\frac{\delta}{\delta e^{a}{}_{i}(x')}\ d^3x\,d^3x' + \cdots \\
&= \quad -\int\limits_{\Sigma} C_{(m)}(x)\,\hat{\psi}^{A}{}_{i}(x)\,\bar{\sigma}^{a}{}_{AA'}\frac{\delta}{\delta e^{a}{}_{i}(x)}\ d^3x + \cdots\,, \qquad (875)
\end{aligned}$$

i.e. the first term of the constraint $\hat{\bar{S}}_{(m)A'}$ (*cf.* (828)) plus terms related to the superpotential. Hence, except for the part coming from the superpotential, the supersymmetry generators in the mode representation are related by the adjoint action of these generalized number operators:

$$\begin{aligned}
\bar{S}_{(n)1'} \quad &= \quad \left[\hat{N}_{(n)(0)}, \hbar\mathbf{d}_{\mathcal{M}^{(\mathrm{conf})},1}\right] + \hat{\mathrm{c}}^{\dagger(p)}U_{(n)1'(p)} \\
&= \quad \hat{\mathrm{c}}^{\dagger(p)}\,N_{(n)(0)(p)}{}^{(q)}\,\hbar\partial_{X^q} + \hat{\mathrm{c}}^{\dagger(p)}U_{(n)1'(p)} \\
&= \quad e^{-W_{(n)}/\hbar}\left(\hat{\mathrm{c}}^{\dagger(p)}\,N_{(n)(0)(p)}{}^{(q)}\,\hbar\partial_{X^q}\right)e^{W_{(n)}/\hbar} \\
&= \quad e^{-W_{(n)}/\hbar}\hbar\left[\hat{N}_{(n)(0)}, \mathbf{d}_{\mathcal{M}^{(\mathrm{conf})},1}\right]e^{W_{(n)}/\hbar}\,. \qquad (876)
\end{aligned}$$

Here the superpotentials $W_{(n)}$ are defined by the last line above, i.e.

$$N_{(n)(0)(p)}{}^{(q)}\,\partial_{X^p}W_{(n)} \quad = \quad U_{(n)1'(q)}\,. \qquad (877)$$

They exist locally, again according to the Poincaré lemma, since the $N_{(n)(m)}$ are invertible.

The primed supersymmetry generator $\bar{S}_{(n)A'}$ and the kinematical space of states $\mathcal{K}$ have been identified in their mode basis representation. According to the remarks following equation (801), page 190, the unprimed supersymmetry generator as well as the Hamiltonian and diffeomorphism generators are conveniently obtained from the primed supersymmetry generator and the knowledge of the inner product on $\mathcal{K}$. This inner product in turn is determined by identifying a metric on $\mathcal{M}^{(\mathrm{conf})}$, which is the content of the following paragraphs:

4.27 (Metric on configuration space) The classical Dirac bracket between $\psi^A{}_i$ and $\bar{\psi}^{A'}{}_i$ is (*cf.* [83] (3.2.32)):

$$\left\{\psi^A{}_i(x)\,,\bar{\psi}^{A'}{}_j(x)\right\} \quad = \quad -D^{AA'}{}_i j(x)\;\delta(x,x')\;, \tag{878}$$

where the coefficients D are defined by

$$D^{AA'}{}_{ij}(x) \quad := \quad -2ih^{-1/2}e^{AB'}{}_j(x)\,e_{BB'i}(x)\,n^{BA'}(x)\;. \tag{879}$$

In [80][83] the quantum version of (878) is taken to be

$$\left\{\hat{\psi}^A{}_i(x),\hat{\bar{\psi}}^{A'}{}_j(x')\right\} = i\hbar\left(-D^{AA'}{}_{ij}(x)\;\delta(x,x')\right)\;. \tag{880}$$

Here we will instead follow another convention according to which there is no factor of $\hbar$ in the quantum anticommutator of the fermionic degrees of freedom and we will have an explicit factor of $\hbar$ when needed. With this convention the geometric interpretation of some expressions becomes more natural (e.g. see 4.29 (p.210) below). Hence the definition of the above anticommutator is here taken to be:

$$\left\{\hat{\psi}^A{}_i(x),\hat{\bar{\psi}}^{A'}{}_j(x')\right\} := -iD^{AA'}{}_{ij}(x)\;\delta(x,x')\;. \tag{881}$$

From the analogous relation, (19), p. 19, of the fermionic operators on the exterior bundle

$$\left\{\hat{\mathrm{c}}^{\dagger n},\hat{\mathrm{c}}^m\right\} = g^{nm}$$

one expects $D^{AA'}{}_{ij}(x)$ to play the role of the inverse metric on configuration space in the pointwise basis. This is confirmed by transforming it to the mode basis: In the functional representation one obtains from (880) the relation (*cf.* [83](3.3.2)):

$$\bar{\psi}^{A'}{}_i(x) \quad = \quad -iD^{AA'}{}_{ji}(x)\,\frac{\delta}{\delta\psi^A{}_j(x)}\,. \tag{882}$$

The operators $\hat{\psi}^A{}_i$ and $\hat{\bar{\psi}}^{A'}{}_i$ are furthermore mutual adjoints with respect to the inner product $\langle\cdot|\cdot\rangle$ on $\mathcal{K}$. The adjoint of an operator A with respect to $\langle\cdot|\cdot\rangle$ will be denoted by $A^\dagger$. Hence one has

$$\left(\hat{\psi}^A{}_i(x)\right)^\dagger \quad = \quad \hat{\bar{\psi}}^A{}_i(x) \tag{883}$$

as well as

$$\begin{aligned}\left(\hat{\psi}^A{}_i(x)\right)^\dagger &= \left(F_{(n)}{}^A{}_i(x)\,\hat{f}^{(n)}\right)^\dagger \\ &= \bar{F}_{(n)}{}^{A'}{}_i(x)\left(\hat{f}^{(n)}\right)^\dagger \\ &= F^*_{(n)}{}^{A'}{}_i(x)\left(\hat{f}^{(n)}\right)^\dagger\,, \end{aligned} \tag{884}$$

and of course

$$\left(\hat{\mathrm{c}}^{\dagger\,(n)}\right)^\dagger = \hat{\mathrm{c}}^{(n)}\,. \tag{885}$$

This gives in terms of the mode basis:

$$\begin{aligned} & \bar{\psi}^{A'}{}_i(x) := -iD^{AA'}{}_{ji}(x)\,\frac{\delta}{\delta\psi^A{}_j(x)} \\ \overset{(834)}{\Leftrightarrow}\quad & \bar{F}_{(n)}{}^{A'}{}_i(x)\left(\hat{f}^{(n)}\right)^\dagger = -iD^{AA'}{}_{ji}(x)\,F'^{(m)}{}_A{}^i(x)\,\frac{\partial}{\partial f^m} \\ \overset{(858)}{=}\quad & \bar{F}_{(n)}{}^{A'}{}_i(x)\left(\hat{\mathrm{c}}^{\dagger\,(n)}\right)^\dagger = -iD^{AA'}{}_{ji}(x)\,F'^{(m)}{}_A{}^j(x)\,\hat{\mathrm{c}}_m \\ \overset{(816)}{\Leftrightarrow}\quad & \left(\hat{\mathrm{c}}^{\dagger\,(n)}\right)^\dagger = \underbrace{-i\int\limits_\Sigma \bar{F}'^{(n)}{}_{A'}{}^i(x)\,D^{AA'}{}_{ji}(x)\,F'^{(m)}{}_A{}^j(x)\ d^3x}_{:=G^{nm}}\ \hat{\mathrm{c}}_m \\ \Leftrightarrow\quad & \hat{\mathrm{c}}^n = G^{nm}\hat{\mathrm{c}}_m\,. \end{aligned} \tag{886}$$

Here G^{nm} is defined by

$$G^{nm} := -i\int\limits_\Sigma \bar{F}'^{(n)}{}_{A'}{}^i(x)\,D^{AA'}{}_{ji}(x)\,F'^{(m)}{}_A{}^j(x)\ d^3x\,. \tag{887}$$

Its inverse is the metric tensor

$$G_{nm} = -i\int\limits_\Sigma F_{(n)}{}^A{}_j(x)\,C_{AA'}{}^{ji}(x)\,\bar{F}_{(m)}{}^{A'}{}_i(x)\ d^3x\,. \tag{888}$$

As expected, when $D^{AA'}{}_{ij}$ is projected onto the normal modes it yields the respective metric tensor on configuration space. Note that (886) is the direct analog of (882).

It is remarkable that G_{mn} is a *linear* function of the configuration space coordinates:

$$\begin{aligned} G_{nm} &= -i\int\limits_\Sigma F_{(n)}{}^A{}_j(x)\,C_{AA'}{}^{ji}(x)\,\bar{F}_{(m)}{}^{A'}{}_i(x)\ d^3x \\ &\overset{(891)}{=} i\int\limits_\Sigma F_{(n)}{}^A{}_j(x)\,\epsilon^{ijk}\sigma^a{}_{AA'k}(x)\,\bar{F}_{(m)}{}^{A'}{}_i(x)\,e^a{}_k\ d^3x \\ &= \underbrace{i\int\limits_\Sigma F_{(n)}{}^A{}_j(x)\,\epsilon^{ijk}\sigma^a{}_{AA'k}(x)\,\bar{F}_{(m)}{}^{A'}{}_i(x)\,B^a_{p\,k}(x)\ d^3x}_{:=G_{pnm}}\ b^{(p)} \\ &:= b^{(p)}\,G_{pnm}\,. \end{aligned} \tag{889}$$

4.28 (Hermiticity of the metric on configuration space) *The metric tensor G_{nm} is a hermitian matrix:*

$$(G_{nm})^* = G_{mn} . \tag{890}$$

Proof: Recall that

$$\begin{aligned} C^{AA'}{}_{ij}(x) &= -\epsilon^{ijk} e^{AA'}{}_k(x) \\ &= -\epsilon^{ijk} e^a{}_k \sigma_a{}^{AA'} . \end{aligned} \tag{891}$$

Therefore

$$\begin{aligned} F_{(n)} \cdot C \cdot F^*_{(m)} &:= \left(F_{(n)}{}^A{}_j(x)\, C_{AA'}{}^{ji}(x)\, F^*_{(m)}{}^{A'}{}_i(x) \right)^* \\ &= \epsilon^{ijk} e^a{}_k \left(F_{(n)j}\, \sigma_a\, F^*_{(m)i} \right)^* \\ &= \epsilon^{ijk} e^a{}_k F_{(m)i}\, \sigma_a\, F^*_{(n)j} \\ &= -\epsilon^{ijk} e^a{}_k F_{(m)j}\, \sigma_a\, F^*_{(n)i} \\ &= -F_{(m)} \cdot C \cdot F^*_{(n)} , \end{aligned} \tag{892}$$

and hence

$$\begin{aligned} (G_{nm})^* &= \left(-i \int_\Sigma F_{(n)} \cdot C \cdot F_{(m)}\, d^3x \right)^* \\ &= i \int_\Sigma F_{(m)} \cdot C \cdot F_{(n)}\, d^3x \\ &= G_{mn} . \end{aligned} \tag{893}$$

□

4.29 (Covariant derivative on configuration space) The canonical momentum operator of supergravity can be interpreted as a covariant derivative operator on configuration space. It turns out that the respective affine connection is the Levi-Civita connection associated with the metric (888).
To see this, consider first the the functional representation of the canonical momentum, it reads ([83](3.3.3)):

$$\begin{aligned} \hat{p}_{AA'}{}^i(x) &= -i\hbar \left(\frac{\delta}{\delta e^{AA'}{}_i(x)} + \frac{i}{2} \epsilon^{ijk} \hat{\psi}_{Aj}(x)\, \hat{\bar{\psi}}_{A'_k}(x) \right) \\ \Leftrightarrow \qquad \hat{p}_a{}^i(x) &= -i\hbar \left(\frac{\delta}{\delta e^a{}_i(x)} - \frac{i}{2} \epsilon^{ijk} \sigma_a{}^{AA'} \hat{\psi}_{Aj}(x)\, \hat{\bar{\psi}}_{A'_k}(x) \right) \end{aligned} \tag{894}$$

(Recall that our convention differs from that in [83], as discussed in 4.27 (p.208) (see (882)), in that our $\hat{\bar{\psi}}^{A'}{}_i$ does not carry a factor of $\hbar$. Therefore we here have an explicit factor of $\hbar$ which can be factored out.)

In the primed supersymmetry generator $\hat{p}_a{}^i$ appears in the combination

$$\hat{\psi}^A{}_i(x)\, \hat{p}_{AA'}{}^i(x) = \hat{\mathrm{c}}^{\dagger(l)}\, F_{(l)}{}^A{}_i(x)\, \hat{p}_{AA'}{}^i(x) . \tag{895}$$

Hence consider the zero mode of ${F_{(l)}}^A{}_i(x)\,\hat{p}_{AA'}{}^i(x)$ (and divide by $-i\hbar$ for convenience):

$$
\begin{aligned}
&\frac{1}{-i\hbar}\int\limits_\Sigma C_{(0)}(x)\,{F_{(l)}}^A{}_i(x)\,\hat{p}_{AA'}(x)\;d^3x \\
=\;& \frac{1}{-i\hbar}\int\limits_\Sigma {F_{(l)}}^A{}_i(x)\,\hat{p}_{AA'}(x)\;d^3x \\
=\;& \frac{1}{-i\hbar}\int\limits_\Sigma {F_{(l)}}^A{}_i(x)\,\sigma^a{}_{AA'}\hat{p}_a{}^i(x)\;d^3x \\
=\;& \int\limits_\Sigma {F_{(l)}}^A{}_i(x)\,\sigma^a{}_{AA'}\left(\frac{\delta}{\delta e^a{}_i(x)} - \frac{i}{2}\epsilon^{ijk}\,\sigma_{aBB'}\,\hat{\psi}^B{}_j(x)\,\hat{\bar{\psi}}{}^{B'}_k(x)\right)\;d^3x \\
=\;& \int\limits_\Sigma \left({F_{(l)}}^A{}_i\,\sigma^a{}_{AA'}\,B'^{(m)}{}_a{}^i\hat{b}^{\#}_{(m)} - \frac{i}{2}\epsilon^{ijk}{F_{(l)}}^A{}_i\underbrace{\sigma^a{}_{AA'}\,\sigma_a{}^{BB'}}_{=-\delta^B_A\,\delta^{B'}_{A'}}F_{(n)Bj}F^*_{(m)B'k}\;\hat{\mathrm{c}}^{\dagger(n)}\hat{\mathrm{c}}^{(m)}\right)\;d^3x \\
=\;& \frac{\partial}{\partial X^{A'(l)}} - \frac{i}{2}\int\limits_\Sigma \epsilon^{ijk}{F_{(l)}}^A{}_i\,F_{(n)Aj}F^*_{(m)A'k}\;d^3x\;\;\hat{\mathrm{c}}^{\dagger(n)}\hat{\mathrm{c}}^{(m)} \\
:=\;& \frac{\partial}{\partial X^{A'(l)}} - \Gamma_{A'(l)(m)(n)}\;\hat{\mathrm{c}}^{\dagger(n)}\hat{\mathrm{c}}^{(m)}\,.
\end{aligned}
\tag{896}
$$

In the last line the coefficients $\Gamma_{A'(l)(m)(n)}$ have been introduced, defined by:

$$
\Gamma_{A'(l)(m)(n)} \;:=\; \frac{i}{2}\int\limits_\Sigma \epsilon^{ijk}{F_{(l)}}^A{}_i\,F_{(n)Aj}F^*_{(m)A'k}\;d^3x\,. \tag{897}
$$

As the notation suggests, $\Gamma_{A'(l)(m)(n)}$ can be identified with an affine connection on $\left(\mathcal{M}^{(\mathrm{conf})},G_{mn}\right)$, as will be detailed below. Following B.11 (p.301), eq. (1199), we call (896) the covariant derivative operator, denoted by

$$
\hat{\nabla}_{A'(l)} \;:=\; \frac{\partial}{\partial X^{A'(l)}} - \Gamma_{A'(l)(m)(n)}\;\hat{\mathrm{c}}^{\dagger(n)}\hat{\mathrm{c}}^{(m)}\,. \tag{898}
$$

Note that the expression

$$
\begin{aligned}
\epsilon^{ijk}{F_{(l)}}^A{}_i\,F_{(n)Aj} &= \epsilon^{ijk}\epsilon^{AB}F_{(l)Bi}\,F_{(n)Aj} \\
&= \epsilon^{ijk}\epsilon^{AB}F_{(l)Aj}\,F_{(n)Bi}
\end{aligned}
\tag{899}
$$

is symmetric with respect to $(l)\leftrightarrow(n)$, since both ϵ^{ijk} and ϵ^{AB} are completely antisymmetric. It follows that $\Gamma_{A'(l)(m)(n)}$ is symmetric in its first and last mode indices:

$$
\Gamma_{A'(l)(m)(n)} \;=\; \Gamma_{A'(n)(m)(l)}\,. \tag{900}
$$

This is of course mandatory if Γ is supposed to be the Levi-Civita connection on $\left(\mathcal{M}^{(\mathrm{conf})},G_{mn}\right)$.

One immediate consequence of this symmetry is that the Γ-term vanishes in the expression for the primed supersymmetry generator due to the antisymmetry $\hat{c}^{\dagger(n)}\hat{c}^{\dagger(l)} = -\hat{c}^{\dagger(l)}\hat{c}^{\dagger(n)}$:

$$
\begin{aligned}
\int_\Sigma \hat{\psi}^A{}_i(x)\ \hat{p}_{AA'}{}^i(x)\ d^3x &= \hat{c}^{\dagger(l)} \int_\Sigma F_{(l)}{}^A{}_i(x)\ \hat{p}_{AA'}{}^i(x)\ d^3x \\
&= \hat{c}^{\dagger(l)}\hat{\nabla}_{A'(l)} \\
&= \hat{c}^{\dagger(l)}\frac{\partial}{\partial X^{A'(l)}} - \underbrace{\hat{c}^{\dagger(l)}\Gamma_{A'(l)(m)(n)}\ \hat{c}^{\dagger(n)}\hat{c}^{(m)}}_{=0} \\
&= \hat{c}^{\dagger(l)}\frac{\partial}{\partial X^{A'(l)}}\,. \qquad (901)
\end{aligned}
$$

This is simply the usual formula (1214), p. 305, for the exterior derivative expressed in a coordinate basis (*cf.* B.13 (p.304)). In the context of canonical supergravity the vanishing of the second term in (901) has first been noticed in [80](see eqs.(4.5)-(4.6), or [83], eqs. (3.4.5)-(3.4.6)).

4.30 (The Levi-Civita connection on $\mathcal{M}^{(\mathrm{conf})}$) *One finds*

$$
\Gamma_{A'(l)(m)(n)} = \frac{1}{2}\frac{\partial}{\partial X^{A'(l)}}G_{mn}\,. \qquad (902)
$$

Proof:

$$
\begin{aligned}
&\frac{\partial}{\partial X^{A'(l)}}G_{mn} \\
=\ & E_{(0)A'(l)}{}^{(p)}\frac{\partial}{\partial b^{(p)}}G_{mn} \\
\overset{(888)}{=}\ & -iE_{(0)A'(l)}{}^{(p)}\frac{\partial}{\partial b^{(p)}}\int_\Sigma F_{(n)}{}^B{}_j(x)\, C_{BB'}{}^{ji}(x)\, F^*_{(m)}{}^{B'}{}_i(x) \\
\overset{(891)}{=}\ & iE_{(0)A'(l)}{}^{(p)}\frac{\partial}{\partial b^{(p)}}\int_\Sigma F_{(n)}{}^B{}_j(x)\ \epsilon^{ijk}\ e^a{}_k\, \sigma_{aBB'}\, F^*_{(m)}{}^{B'}{}_i(x) \\
\overset{(821)}{=}\ & iE_{(0)A'(l)}{}^{(p)}\frac{\partial}{\partial b^{(p)}}\int_\Sigma F_{(n)}{}^B{}_j(x)\ \epsilon^{ijk}\sigma_{aBB'}\, F^*_{(m)}{}^{B'}{}_i(x)\ B_{(q)}{}^a{}_k(x)\ b^{(q)} \\
=\ & iE_{(0)A'(l)}{}^{(p)}\int_\Sigma F_{(n)}{}^B{}_j(x)\ \epsilon^{ijk}\sigma_{aBB'}\, F^*_{(m)}{}^{B'}{}_i(x)\ B_{(p)}{}^a{}_k(x)\ d^3x \\
\overset{(842)}{=}\ & i\int_\Sigma\int_\Sigma F_{(n)}{}^B{}_j(x)\ \epsilon^{ijk}\sigma_{aBB'}\, F^*_{(m)}{}^{B'}{}_i(x)\ F_{(l)}{}^A{}_l(x')\,\sigma^b{}_{AA'}\,\underbrace{B'^{(p)}{}_b{}^l(x')\, B_{(p)}{}^a{}_i(x)}_{\overset{(815)}{=}\delta(x,x')\,\delta^a_b\,\delta^l_k}\ d^3x\, d^3x' \\
=\ & i\int_\Sigma F_{(n)Bj}(x)\ \epsilon^{ijk}\ \underbrace{\sigma_a{}^{BB'}\sigma^a{}_{AA'}}_{=-\delta^B_A\,\delta^{B'}_{A'}}\ F^*_{(m)B'i}(x)\ F_{(l)}{}^A{}_i(x')\ d^3x \\
=\ & i\int_\Sigma \epsilon^{ijk}F_{(l)}{}^A{}_i(x')\, F_{(n)Aj}(x)\ F^*_{(m)A'k}(x)\ d^3x \qquad (903)
\end{aligned}
$$

The configuration space of canonical supergravity should hence be a Kähler manifold (*cf.* [52]§14). See point (3) in 6.2 (p.291).

Now that the metric on $\mathcal{M}^{(\mathrm{conf})}$ is identified, one can define the Hodge star operator $*$ on $\Gamma\big(\mathcal{M}^{(\mathrm{conf})}\big)$ with respect to that metric, and by means of this operator the usual inner product on the exterior bundle over configuration space (see equation (1195) in B.10 (p.300)): Let

$$
\begin{aligned}
|f\rangle &= \sum_{0\leq m_1<m_2<\cdots} f_{(m_1)(m_2)\cdots}\hat{\mathrm{c}}^{\dagger (m_1)}\hat{\mathrm{c}}^{\dagger (m_2)}\cdots|0\rangle \\
|g\rangle &= \sum_{0\leq m_1<m_2<\cdots} g_{(m_1)(m_2)\cdots}\hat{\mathrm{c}}^{\dagger (m_1)}\hat{\mathrm{c}}^{\dagger (m_2)}\cdots|0\rangle
\end{aligned}
\tag{904}
$$

be two kinematical quantum supergravity states in $\Gamma\Big(\Lambda\big(\mathcal{M}^{(\mathrm{conf})}\big)\Big)$, then

$$
\begin{aligned}
\langle f|g\rangle_{\mathrm{loc}} &= \int\limits_{\mathcal{M}^{(\mathrm{conf})}} f^* \wedge *g \\
&= \int\limits_{\mathcal{M}^{(\mathrm{conf})}} \sqrt{|G|} \sum_{0\leq m_1<m_2<\cdots} f^*_{(m_1)(m_2)\cdots} g^{(m_1)(m_2)\cdots}\, dX^{(1)}\, dX^{(2)}\cdots
\end{aligned}
\tag{905}
$$

(Here we have trivially extended the inner product discussed in B.10 (p.300) to complex coefficient functions.) The inner product given in [80][83] coincides with the above inner product except for the weight $\sqrt{|G|}$. This is shown in the following paragraph.

4.31 (Inner product on the space of states) The inner product defined in [83](3.3.4) (following [95], §2.4) reads (due to our different convention (881) we here omit factors of $\hbar$ that appear in [83])

$$
\langle f, g\rangle_D = \int \bar{f}\, g\, \exp\left[i\int\limits_{\Sigma} \psi^A{}_i(x)\, C_{AA'}{}^{ij}(x')\, \bar{\psi}^{A'}{}_j(x')\ d^3x'\right] \prod_x \det\left(\frac{1}{i}C\right)^{-1}(x)\, de^{\cdot}{}_{\cdot}(x)\, d\psi^{\cdot}{}_{\cdot}(x)\, d\bar{\psi}^{\cdot}{}_{\cdot}(x)\ .
\tag{906}
$$

Recall that

$$
\bar{\psi}^{A'}{}_i(x) = -iD^{AA'}{}_{ji}\psi^{\# j}_A
\tag{907}
$$

and

$$
C_{AA'}{}^{ij} D^{AB'}{}_{jk} = \delta^{B'}_{A'}\delta^i_k\,.
\tag{908}
$$

The exponent may therefore be rewritten as

$$
\begin{aligned}
i\int\limits_{\Sigma} \psi^A{}_i(x')\, C_{AA'}{}^{ij}(x')\, \bar{\psi}^{A'}{}_j(x')\ d^3x' &= -\int\limits_{\Sigma} \psi^A{}_i(x')\, \psi^{\# i}_A(x')\ d^3x' \\
&= -c^{\dagger n} c_m \int\limits_{\Sigma} F_{(n)}{}^A{}_i(x')\, F'^{(m)}{}_A{}^i(x')\ d^3x' \\
&= -c^{\dagger n} c_n\,.
\end{aligned}
\tag{909}
$$

Due to the change of variables formula for anticommuting variables (B.24 (p.316)), the integration measure can be reexpressed as

$$\begin{aligned}
\prod_x \det\left(\frac{1}{i}C\right)^{-1} de^{\cdot}{}_{\cdot}(x)\, d\psi^{\cdot}{}_{\cdot}(x)\, d\bar{\psi}^{\cdot}{}_{\cdot}(x) &= \prod_x de^{\cdot}{}_{\cdot}(x)\, d\psi^{\cdot}{}_{\cdot}(x)\, d\psi^{\#\cdot}_{\cdot}(x) \\
&= \mathcal{D}e^{\cdot}{}_{\cdot}(x)\, \mathcal{D}\psi^{\cdot}{}_{\cdot}(x)\, \mathcal{D}\psi^{\#\cdot}_{\cdot}(x) \;.
\end{aligned} \tag{910}$$

(The last line simply restates pointwise variation as functional variation.) The integration over all $\psi^A{}_i(x)$ and $\psi^{\#i}_A(x)$ is equal to integration over all c^{*n} and c_n, because the fermionic Jacobeans mutually cancel. To (formally) see this explicitly introduce multi-indices:

$$\begin{aligned}
e^a{}_i(x) &:= e^{(a,i,x)} := e^I \\
\psi^A{}_i(x) &:= \psi^{(A,i,x)} := \psi^I \\
\psi^{\#i}_A(x) &:= \psi^{\#}_{(A,i,x)} := \bar{\psi}_I \;.
\end{aligned} \tag{911}$$

The transformations (834), (835) then symbolically read:

$$\begin{aligned}
e^I &= B_n{}^I b^n \\
e^{\#}_I &= B'^n{}_I b^{\#}_n \\
\psi^I &= F_n{}^I f^n \\
\psi^{\#I} &= F'^n{}_I f^{\#}_n
\end{aligned} \tag{912}$$

and

$$\begin{aligned}
b^n &= B'^n{}_I e^I \\
b^{\#}_n &= B_n{}^I e^{\#}_I \\
f^n &= F'^n{}_I \psi^I \\
f^{\#}_n &= F_n{}^I \psi^{\#}_I \;.
\end{aligned} \tag{913}$$

Hence formally one has (again by B.24 (p.316))

$$\begin{aligned}
\mathcal{D}\psi^{\cdot}{}_{\cdot}(x)\, \mathcal{D}\psi^{\#\cdot}_{\cdot}(x) &= \prod_{I,J} d\psi^I d\psi^{\#}_J \\
&= \prod_{I,J} d\left(F_n{}^I c^{*n}\right) d\left(F'^m{}_J c_m\right) \\
&= \underbrace{\det(F'^n{}_I)\det\left(F_m{}^J\right)}_{=\det(F'^n{}_I F_m{}^I)=\det(\delta^n_m)=1} \prod_{n,m} dc^{*n} dc_m \\
&= \prod_{n,m} dc^{*n} dc_m \;.
\end{aligned} \tag{914}$$

With these transformations the inner product becomes (and this agrees with eq. (4.31) on p. 53 of [95], as it should):

$$\langle f, g\rangle_D = \int \bar{f}\, g \exp(-c^{*n}c_n) \prod_{n,m} dc^{*n} dc_m \mathcal{D}e^{\cdot}{}_{\cdot}(x) \;. \tag{915}$$

According to B.26 (p.317), the fermionic integration amounts to contracting the indices of f and g. Let

$$\begin{aligned} f &= f_{m_1,m_2,\dots,m_p} c^{*m_1} c^{*m_2} \cdots c^{*m_p} \\ g &= g_{n_1,n_2,\dots,n_q} c^{*n_1} c^{*n_2} \cdots c^{*n_q} \,, \end{aligned}$$

then

$$\int \bar{f}\, g \exp(-c^{*n} c_n) \prod_{n,m} dc^{*n} dc_m = \begin{cases} f_{m_1,m_2,\dots,m_p} g^{m_1,m_2,\dots,m_p} & \text{if } p = q \\ 0 & \text{if } p \neq q \end{cases} \,, \tag{916}$$

where indices are raised with the inverse metric G^{nm} on configuration space (*cf.* 4.27 (p.208), (886)). It follows that (915) can be equivalently written as

$$\begin{aligned} \langle f|g\rangle_D &= \int\limits_{\mathcal{M}^{(\text{conf})}} * \left(\bar{f} \wedge * g\right) \mathcal{D}e^{\cdot}_{\cdot}(x) \\ &= \int\limits_{\mathcal{M}^{(\text{conf})}} * \left(\bar{f} \wedge * g\right) \prod_{n\in\mathbb{N}} \mu db^n \,. \end{aligned} \tag{917}$$

In the last line the functional measure has been replaced by a measure on the space of amplitudes b^n, following the discussion in [83], eq. (4.2.8b). The constant normalization factor μ depends on the exact choice of modes ${B_{(n)}}^a{}_i$.

Comparing the above expression with that for the Hodge inner product on $\mathcal{M}^{(\text{conf})}$:

$$\begin{aligned} \langle f|g\rangle &= \int\limits_{\mathcal{M}^{(\text{conf})}} \bar{f} \wedge * g \\ &= \int\limits_{\mathcal{M}^{(\text{conf})}} \sqrt{|G|} \bar{f}_{m_1 m_2 \cdots} g^{m^1 m^2 \cdots} \prod_{n\in\mathbb{N}} db^n \,, \end{aligned} \tag{918}$$

shows that the integrands differ by the factor $\sqrt{|G|}/\mu$. While μ is a constant that may be reabsorbed in the normalization, the metric G on configuration space of course depends on the coordinates b^n and cannot be "defined away". Its presence guarantees that the inner product is indeed invariant under a change of coordinates on configuration space and should therefore be included (as it actually is in [83], eqs. (2.7.40) and (2.7.41)) in order for the inner product to be well defined.

One important consequence of using the inner product $\langle\cdot|\cdot\rangle$ instead of $\langle\cdot|\cdot\rangle_D$ is that operator adjoints may differ by a term of order $\hbar$.

We can now define the unprimed supersymmetry generator as the adjoint of the primed supersymmetry generator with respect to the invariant inner product $\langle\cdot|\cdot\rangle$, following (802), p.191. The Hamiltonian and diffeomorphism generators are then obtained by means of the relation (803).

4.32 (Unprimed supersymmetry generator in the mode representation) According to (802), p. 191 we have for the unprimed supersymmetry generators in the mode basis

$$\hat{S}_{(n)A} := \left(\hat{\bar{S}}_{(n)A'}\right)^{\dagger} \,. \tag{919}$$

Inserting (841) one finds

$$\begin{aligned}\hat{S}_{(n)A} &= \left(\hat{c}^{\dagger\,(p)} E_{(n)A'(p)}{}^{(q)}\, \hbar \hat{b}^{\#}_{(q)} + U_{(n)A'(p)}\left(\hat{b}^{\cdot}\right)\right) \\ &= E^{*}_{(n)A(p)}{}^{(q)} \left(\hbar \hat{b}^{\#}_{(q)}\right)^{\dagger} \hat{c}^{(p)} + \hat{c}^{(p)} U^{*}_{(n)A(p)}\left(\hat{b}^{\cdot}\right) . \qquad (920)\end{aligned}$$

This is the mode representation of the unprimed supersymmetry constraint given in [83], eq. (3.4.10), except for a term of order $\hbar$. One may proceed with this expression analogously to the discussion of $\hat{\bar{S}}_{(n)A'}$ in 4.25 (p.205). It is more illuminating, however, to directly apply the adjoint operation to the result of that paragraph, namely the representation of $\hat{\bar{S}}_{(n)A'}$ as a deformed exterior derivative on $\mathcal{M}^{(\mathrm{conf})}$ (876):

$$\begin{aligned}\hat{S}_{(n)1} &= \left(e^{-W_{(n)}/\hbar}\hbar \left[\hat{N}_{(n)(0)}, \mathbf{d}_{\mathcal{M}^{(\mathrm{conf})},1}\right] e^{W_{(n)}/\hbar}\right)^{\dagger} \\ &= e^{W^{*}_{(n)}/\hbar}\hbar \left[\mathbf{d}^{\dagger}{}_{\mathcal{M}^{(\mathrm{conf})},1}, \hat{N}^{\dagger}_{(n)(0)}\right] e^{-W^{*}_{(n)}/\hbar} , \qquad (921)\end{aligned}$$

where $\mathbf{d}^{\dagger}{}_{\mathcal{M}^{(\mathrm{conf})},1}$ is the exterior coderivative on $\mathcal{M}^{(\mathrm{conf})}$ associated with $\mathbf{d}_{\mathcal{M}^{(\mathrm{conf})},1}$ (*cf.* (93), p.27). For $n = 0$ this yields simply (since $\hat{N}_{(0)(0)} = \hat{N}$)

$$\hat{S}_{(0)1} = e^{W^{*}_{(0)}/\hbar}\hbar \mathbf{d}^{\dagger}{}_{\mathcal{M}^{(\mathrm{conf})},1} e^{-W^{*}_{(0)}/\hbar} . \qquad (922)$$

The mere identification of the supersymmetry generators with deformed exterior derivatives and coderivatives on configuration space allows to draw some conclusions about the space of physical states. In particular, the following well known result can be easily rederived:

Theorem 4.33 (Fermion number of physical states) *Every solution* $|\phi\rangle$ *to the full set of constraints of canonically quantized* $D = 4$, $N = 1$ *supergravity*

$$\hat{\mathcal{S}}_{A}(x)\,|\phi\rangle = \hat{\bar{\mathcal{S}}}_{A'}(x)\,|\phi\rangle = \hat{\mathcal{J}}^{ab}\,|\phi\rangle = 0, \quad \forall\, x \in \Sigma,\ A, A' \in \{1,2\}\,,\ a,b \in \{0,1,2,3\} \qquad (923)$$

has infinite fermion number as measured by the fermion number operator

$$\begin{aligned}\hat{N} &= \hat{c}^{\dagger\,n}\hat{c}_{n} \\ &= \int\limits_{\Sigma} \hat{\psi}^{A}{}_{i}(x)\, \frac{\delta}{\delta \psi^{A}{}_{i}(x)}\, d^{3}x\, . \qquad (924)\end{aligned}$$

Literature. In [82] it was first argued that there are no states of finite but *non-zero* fermion number. [214] then noted that the same argument should also apply to 0-fermion states. Finally in [43] a proof was presented that every solution to the constraints must have infinite fermion number. Formal solutions with exactly this property were then given in [68] (see below).

The proof of theorem 4.33 in the present context relies on the fact that the supersymmetry generators are deformed exterior derivatives when expanded into modes, and that therefore the extended Poincaré lemma applies:

Proof of theorem 4.33 (p.216): The pointwise supersymmetry constraints in (923) are equivalent to the mode basis constraints

$$\hat{S}_{(n)A}\,|\phi\rangle = \hat{\bar{S}}_{(n)A'}\,|\phi\rangle \quad = \quad 0, \quad \forall\, n \in \mathbb{N}, A, A' \in \{1,2\}\ . \tag{925}$$

The $\hat{\bar{S}}_{(n)A'}$ are deformed exterior derivatives. They anticommute and increase the fermion number by one:

$$\begin{aligned} \left\{\hat{\bar{S}}_{(n)A'}, \hat{\bar{S}}_{(m)B'}\right\} &= 0 \\ \left[\hat{N}, \hat{\bar{S}}_{(n)A'}\right] &= \hat{\bar{S}}_{(n)A'}\ . \end{aligned} \tag{926}$$

Any state $|\phi\rangle$ which is annihilated by all the constraints is in particular *closed* with respect to all the $\hat{\bar{S}}_{(n)A'}$. Therefore it follows from the generalized Poincaré lemma 2.45 (p.51) that $|\phi\rangle$ is

- either locally exact with respect to all $\hat{\bar{S}}_{(n)A'}$, i.e.

$$\begin{aligned} |\phi\rangle &= \left(\prod_{n\in\mathbb{N}} 2\,\hat{\bar{S}}_{(n)1'}\hat{\bar{S}}_{(n)2'}\right)|\phi_0\rangle \\ &= \left(\prod_{n\in\mathbb{N}} \hat{\bar{S}}_{(n)A'}\hat{\bar{S}}_{(n)}{}^{A'}\right)|\phi_0\rangle\ , \end{aligned} \tag{927}$$

- or it is locally exact only with respect to a subset of all the $\hat{\bar{S}}_{(n)A'}$, say those with $(n, A') \notin I$ for some set $I \subset \mathbb{N} \otimes \{1,2\}$:

$$|\phi\rangle \quad = \quad \left(\prod_{(n,A')\notin I} \hat{\bar{S}}_{(n)A'}\right)|\phi_0\rangle\ , \tag{928}$$

  in which case $|\phi_0\rangle$ must be a 0-fermion state which is annihilated by the remaining generators:

$$\hat{\bar{S}}_{(n)A'}\,|\phi_0\rangle \quad = \quad 0, \quad \forall\,(n, A') \in I\ . \tag{929}$$

Since each $\hat{\bar{S}}_{(n)A'}$ increases the fermion number by one, it is obviously necessary for $|\phi\rangle$ to have finite fermion number that the latter case applies and that furthermore the set I is of infinite cardinality. But this leads to the following contradiction:

Choose any two elements $\hat{\bar{S}}_{(n)A'}$, $\hat{\bar{S}}_{(m)A'}$, with $(n, A'), (m, A') \in I$, $n \neq m$. Without restriction of generality assume that $A' = 1$. According to (876) one has

$$\begin{aligned} & \hat{\bar{S}}_{(n)1'}\,|\phi_0\rangle = 0 \\ \Leftrightarrow\quad & \mathbf{d}_{\mathcal{M}^{(\mathrm{conf})},1} e^{W_{(n)}/\hbar}\,|\phi_0\rangle = 0 \\ \Leftrightarrow\quad & |\phi_0\rangle = e^{-W_{(n)}}\,|\psi_0\rangle\ , \end{aligned} \tag{930}$$

where $|\psi_0\rangle$ is some 0-fermion state annihilated by $\mathbf{d}_{\mathcal{M}^{(\mathrm{conf})},1}$. Hence that part of the 0-fermion state $|\phi_0\rangle$ which depends on the coordinates to which $\mathbf{d}_{\mathcal{M}^{(\mathrm{conf})},1}$ is sensitive is uniquely defined already by *one* constraint. In particular

$$\begin{aligned}\hat{\bar{S}}_{(m)1'}\, e^{-W_{(n)}}\,|\psi_0\rangle &= e^{-W_{(m)}}\hat{N}_{(m)(0)}\mathbf{d}_{\mathcal{M}^{(\mathrm{conf})},1}\, e^{W_{(m)}-W_{(n)}}\,|\psi_0\rangle \\ &= e^{-W_{(m)}}\hat{N}_{(m)(0)}\underbrace{\left[\mathbf{d}_{\mathcal{M}^{(\mathrm{conf})},1},\, e^{W_{(m)}-W_{(n)}}\right]}_{\neq 0}|\psi_0\rangle \\ &\neq 0 \end{aligned} \tag{931}$$

and hence the requirement (929) cannot be fulfilled. It follows that every physical state must be locally exact with respect to all modes of the $\hat{\bar{S}}_{(n)A'}$ generator and thus have infinite fermion number.
□

In addition to stating the infinite fermion-number property of physical states this proof therefore implies the following important fact:

4.34 (Local form of solutions of quantum supergravity) Every solution to the supersymmetry constraints must be locally of the form

$$|\phi\rangle = \left(\prod_{n\in\mathbb{N}}\hat{\bar{S}}_{(n)A'}\hat{\bar{S}}_{(n)}{}^{A'}\right)|\phi_0\rangle\,. \tag{932}$$

This form of solutions of quantum supergravity has originally been given in [68] (also see [69] and [70]):

Theorem 4.35 (Csordás and Graham (1995)) *Every Lorentz invariant 0-fermion solution $|\phi_0\rangle$ to the Hamiltonian and diffeomorphism constraints*

$$\begin{aligned}&\hat{\mathcal{H}}_a(x)\,|\phi_0\rangle = 0, \qquad \forall\, x\in\Sigma,\ a\in\{0,1,2,3\}\\ \Leftrightarrow\ &\hat{H}_{(n)a}\,|\phi_0\rangle = 0, \qquad \forall\, n\in\mathbb{N},\ a\in\{0,1,2,3\}\end{aligned}$$

gives rise to a solution

$$|\phi\rangle := \left(\prod_{n\in\mathbb{N}}\hat{\bar{S}}_{(n)A'}\hat{\bar{S}}_{(k)}{}^{A'}\right)|\phi_0\rangle \tag{933}$$

of full quantum supergravity.

Proof: By construction $|\phi\rangle$ is annihilated by all the $\hat{\bar{S}}_{(n)A'}$. Also, since $|\phi_0\rangle$ is Lorentz invariant by assumption and since $\prod_{n\in\mathbb{N}}\hat{\bar{S}}_{(n)A'}\hat{\bar{S}}_{(k)}{}^{A'}$ is manifestly Lorentz invariant, $|\phi\rangle$ is annihilated by the Lorentz constraints

$$\mathcal{J}^{ab}\,|\phi\rangle = 0\,.$$

Finally the unprimed supersymmetry generator applied to ϕ give:

$$\hat{S}_{(n)A}\,|\phi\rangle = \hat{S}_{(n)A}\left(\prod_{n\in\mathbb{N}}\hat{\bar{S}}_{(n)A'}\hat{\bar{S}}_{(k)}{}^{A'}\right)|\phi_0\rangle$$

$$
\begin{aligned}
&= \underbrace{\left[\hat{S}_{(n)A}, \prod_{n\in\mathbb{N}} \hat{\bar{S}}_{(n)A'} \hat{\bar{S}}_{(k)}{}^{A'}\right]}_{\overset{(809)}{\propto}\ \hat{H}_{(\cdot)\cdot},\ \mathcal{J}^{\cdot\cdot}} |\phi_0\rangle \\
&= 0\,. \qquad (934)
\end{aligned}
$$

4.36 (Remark) In the last line of (934) use is made of the constraint algebra of supergravity (809) given in 4.14 (p.194). This assumes that the quantum operators obey the same algebra up to terms proportional to the Lorentz generators. In particular, it needs to be assumed that within the quantum commutator algebra the expression $\left[\hat{\mathcal{H}}_{AA'}, \hat{\bar{\mathcal{S}}}_{B'}\right]$ is proportional to Lorentz generators. A calculation directly checking this property has been reported in [68], albeit for a different factor ordering of the supersymmetry generators than that following from the prescription given in 4.20 (p.200) and 4.32 (p.215). In [68] it was found that $\left[\hat{\mathcal{H}}_{AA'}, \hat{\mathcal{S}}_{B}\right]$ is proportional to Lorentz generators while $\left[\hat{\mathcal{H}}_{AA'}, \hat{\bar{\mathcal{S}}}_{B'}\right]$ contains a term proportional to the Lorentz generators as well a a *diverging* term proportional to a linear combination of Lorentz generators and supersymmetry generators. (One can always interchange the role of the primed and the unprimed supersymmetry constraints in these relations by a Hodge duality operation, known in the context of the functional representation as going from the holomorphic to the antiholomorphic representation or vice versa (*cf.* [95]), see also footnote 20 in [69].) Note that both commutators are mutually adjoint and hence closely related:

$$
\left[\hat{\mathcal{H}}_{AA'}, \hat{\mathcal{S}}_{B}\right]^{\dagger} \propto \left[\hat{\mathcal{H}}_{AA'}, \hat{\bar{\mathcal{S}}}_{B'}\right] .
$$

This can be turned into an argument that in general, *except when one (and hence both) of these commutators vanish identically*, at least one of them contains a diverging term when Lorentz generators are ordered to the right:
Consider one commutator, symbolically of the general form

$$
\text{“}\ \left[\hat{\mathcal{H}}(x), \hat{\mathcal{S}}(y)\right] \propto \delta(x,y)\, D(x)\cdot \hat{\mathcal{J}}(x)\ \text{”}\,.
$$

Here the right hand side is assumed to be some linear combination of the Lorentz generators. The coefficient function $D(x)$ must be odd graded, since the Lorentz generators are even graded and the left hand side is odd graded. Hence $D(x)$ must be a linear combination of an uneven number of the fermionic operators $\hat{\psi}(x)$ and $\hat{\bar{\psi}}(x)$. It follows that D itself cannot commute with the Lorentz generators but instead

$$
\text{“}\ [\mathcal{J}(x), D(y)] \propto \delta(x,y)\, E(x)\ \text{“}\,,
$$

for some odd graded term E. This is the origin of the diverging term in the other commutator:

$$
\begin{aligned}
\text{“}\ \left[\hat{\mathcal{H}}(x), \hat{\bar{\mathcal{S}}}(y)\right] &\propto \left[\hat{\mathcal{H}}(x), \hat{\mathcal{S}}(y)\right]^{\dagger} \\
&\propto \delta(x,y)\left(D(x)\cdot \hat{\mathcal{J}}(x)\right)^{\dagger}
\end{aligned}
$$

$$
\begin{aligned}
&\propto \quad \hat{\mathcal{J}}^\dagger(x) \cdot D^\dagger(x)\, \delta(x,y) \\
&\propto \quad \delta(x,y) \left(D^\dagger(x) \cdot \hat{\mathcal{J}}^\dagger(x) + \left[\hat{\mathcal{J}}^\dagger(x), D^\dagger(x) \right] \right) \\
&\propto \quad \delta(x,y) \left(D^\dagger(x) \cdot \hat{\mathcal{J}}^\dagger(x) + \delta(x,x)\, E^\dagger(x) \right) \text{ "}\,.
\end{aligned}
\tag{935}
$$

So a formal term $\delta(x,x)$ is always picked up in one commutator when the Lorentz generators are ordered to the right, except, of course, when both commutators vanish, i.e. when $D(x) = 0$.

4.37 (Literature) Around 1993-96 there was some debate about whether or not (full fledged) canonically quantized supergravity admits exact solutions with only a finite number of fermion modes excited [214]. Claims that, on general grounds, indeed no such states could exist [43] came as a surprise in light of the fact that other researchers had reported [82] the very construction of exact solutions (in the metric representation of quantum supergravity [80]) containing no fermions. The existence of solutions with an infinite number of fermions, which had been shown constructively in [68] (*cf.* theorem 4.35 (p.218), p. 218), was not questioned by these results. Related insights from a quite different perspective came from the connection representation formalism, known as loop quantum supergravity [140][168]. Here exact bosonic Wilson loop states [143] were known, which solved all the supergravity quantum constraints – but these states turned out to be highly pathological and not physically relevant [190], so that the results of [43] might not apply. A little later it was found that also the Chern-Simons state with respect to the GSU(2)-connection representation of super LQG is an exact solution of all the constraints and, furthermore, not pathological at all [6]. (Its fermion content is not obvious and in fact not known to the author.)

From 4.33 (p.216) and 4.35 (p.218) it can be seen that the formal structure of quantum supergravity (in the vielbein formalism) is quite similar, and in fact a generalization of, the formal structure of classical source free electromagnetism (*cf.* §2.2.3 (p.70)):

4.38 (Formal analogy with source free electromagnetism) For emphasis, briefly recall the formal elements of classical electromagnetism (e.g. [98],[91]) in curved spacetime and in the the absence of electric and magnetic sources (*cf.* 2.2.3 (p.70)):
Kinematic states in classical electromagnetism are sections $\mathbf{F}$ of the 2-form bundle $\Lambda^2(\mathcal{M}^{ST})$ over a pseudo-Riemannian manifold $(\mathcal{M}^{\rm ST}, g)$ representing physical spacetime. The 'state'

$$
\mathbf{F} = F_{\mu\nu}\, dx^\mu \wedge dx^\nu = E_i\, dt \wedge dx^i + B^i \epsilon_{ijk}\, dx^j \wedge dx^k
$$

is of course the *Faraday 2-form* encoding the electromagnetic field. Physical states are singled out by two 'constraints'

$$
\begin{aligned}
\mathbf{dF} &= 0 \\
\mathbf{d}^\dagger \mathbf{F} &= 0\,,
\end{aligned}
\tag{936}
$$

(where $\mathbf{d}$ and $\mathbf{d}^\dagger$ are the exterior derivative and coderivative, respectively, on $(\mathcal{M}^{\mathrm{ST}}, g)$) which are equivalent to Maxwell's equations for vanishing electric and magnetic 4-currents. This means in particular that $\mathbf{F}$ is a *closed* form. The Poincaré lemma then states that $\mathbf{F}$ is locally exact and hence locally of the form

$$\mathbf{F} = \mathbf{dA}\,,$$

where $\mathbf{A} = A_\mu\, dx^\mu$ is the vector potential. Because $\mathbf{A}$ is unique only up to a gauge transformation, $\mathbf{A} \to \mathbf{A} + \mathbf{d}\chi$, one may assume that $\mathbf{A}$ satisfies the Lorentz gauge

$$\mathbf{d}^\dagger \mathbf{A} = 0\,.$$

Therefore, in terms of $\mathbf{A}$, the constraints $\mathbf{dF} = \mathbf{d}^\dagger\mathbf{F} = 0$ read

$$\begin{aligned} \mathbf{d}^\dagger\mathbf{dA} &= 0 \\ \Leftrightarrow \left\{\mathbf{d}, \mathbf{d}^\dagger\right\}\mathbf{A} &= 0\,, \end{aligned} \tag{937}$$

which says that $\mathbf{A}$ must be a solution of the wave equation $\Delta\mathbf{A} = 0$, where $\Delta = \left\{\mathbf{d}, \mathbf{d}^\dagger\right\}$ is the wave operator in curved spacetime. In other words, every vector potential which is annihilated by the wave operator is associated with an electromagnetic field solving the free Maxwell's equations, and, *vice versa*, every electromagnetic field solving the free Maxwell equations is locally associated with a 1-form solving the wave equation.

The analogy with the above formulation of quantum supergravity is immediate. It is summarized in the following dictionary:

source-free electromagnetism	**canonical quantum supergravity**
spacetime: $\left(\mathcal{M}^{\rm ST}, g = (g_{\mu\nu})\right)$	bosonic configuration space: $\left(\mathcal{M}^{(\rm conf)}, G = (G_{mn})\right)$
coordinates: $\{x^\mu\}_{\mu\in\{0,1,2,3\}}$	graviton mode amplitudes: $\left\{b^{(n)}\right\}_{n\in\mathbb{N}}$
differential forms: $\{dx^\mu\}_{\mu\in\{0,1,2,3\}}$	gravitino mode amplitudes: $\left\{\hat{c}^{\dagger(n)}\right\}_{n\in\mathbb{N}}$
exterior bundle over $\mathcal{M}^{\rm ST}$: $\Lambda\left(\mathcal{M}^{\rm ST}\right)$	superspace over $\mathcal{M}^{(\rm conf)}$: $\Lambda\left(\mathcal{M}^{(\rm conf)}\right)$
kinematical state space: $\Gamma\left(\Lambda^{(2)}\left(\mathcal{M}^{\rm ST}\right)\right)$	kinematical state space[54]: $\Gamma\left(\Lambda\left(\mathcal{M}^{(\rm conf)}\right)\right)$
kinematical states: $\mathbf{F} = F_{\mu\nu}\, dx^\mu dx^\nu$	kinematical states: $\lvert\phi\rangle = \phi_{m_1 m_2, m_3\cdots}\, \hat{c}^{\dagger(m_1)}\hat{c}^{\dagger(m_2)}\hat{c}^{\dagger(m_3)}\cdots$
exterior derivative: $\mathbf{d}$	primed supersymmetry generator: $\hat{\bar{S}}_{(n)A'} = e^{-W_{(n)}/\hbar}\hbar\left[\hat{N}_{(n)(0)}, \mathbf{d}_{\mathcal{M}^{(\rm conf)},A'}\right] e^{W_{(n)}/\hbar}$
exterior coderivative: $\mathbf{d}^\dagger$	unprimed supersymmetry generator: $\hat{S}_{(n)A} = e^{W^*_{(n)}/\hbar}\hbar\left[\mathbf{d}^\dagger{}_{\mathcal{M}^{(\rm conf)},A}, \hat{N}^\dagger_{(n)(0)}\right] e^{-W^*_{(n)}/\hbar}$
wave operator: $\Delta = \left\{\mathbf{d}, \mathbf{d}^\dagger\right\}$	Hamiltonian generator and diffeom. generators: $K_{(n)(m)}{}^{(p)}\hat{H}_{(p)AA'} = \left\{\hat{S}_{(n)A}, \hat{\bar{S}}_{(m)A'}\right\}$
vector potential: $\mathbf{A},\ \ \Delta\mathbf{A} = 0$	0-fermion solution to Hamilt. and diffeo. constraints.: $\lvert\phi_0\rangle,\ \ \hat{H}_{(n)AA'}\lvert\phi_0\rangle = 0$
locally exact form of $\mathbf{F}$: $\mathbf{F} = \mathbf{dA}$	locally exact form of $\lvert\phi\rangle$: $\lvert\phi\rangle = \left(\prod_{n\in\mathbb{N}} \hat{\bar{S}}_{(n)A'}\hat{\bar{S}}_{(n)}{}^{A'}\right)\lvert\phi_0\rangle$
homogeneous Maxwell equations: $\mathbf{dF} = 0$	primed supersymmetry constraint: $\hat{\bar{S}}_{(n)A'}\lvert\phi\rangle = 0$
‘inhomogeneous’ Maxwell equations: $\mathbf{d}^\dagger\mathbf{F} = 0$	unprimed supersymmetry constraint: $\hat{S}_{(n)A}\lvert\phi\rangle = 0$
wave equation: $\Delta\mathbf{A} = 0$	Hamiltonian and diffeomorphism constraints: $\hat{H}_{(n)AA'}\lvert\phi_0\rangle = 0$

While these analogies are rather strong, it should be stressed that it is by no means implied that quantum supergravity is *physically* related to classical electromagnetism. What shows up here instead is the universal importance of geometry in general, and differential geometry in particular, in physical theories. The differential geometric formulation of classical electromagnetism has been known for a rather long time. The geometric character of supersymmetric theories has been stressed in more recent times by Witten (e.g. [275],[274]) and others (e.g. [101], [102],[133], [218],[216]).

It may be noted that one important ingredient of canonical quantum supergravity has no analogy in the formalism of classical electromagnetism, namely the Lorentz constraints[55]. The next paragraphs focus on some aspects of the role played by the Lorentz constraints in quantum supergravity:

[54]Due to gauge invariances there are some subtleties in the definition of the kinematical state space. See 4.24 (p.203) and §2.3 (p.106) for details.

[55]Of course one can consider Lorentz transformations on spacetime, $\mathcal{M}^{\rm ST}$, in classical elec-

4.39 (Supersymmetry constraints on Lorentz invariant states) *When restricted to Lorentz invariant states, the two spin components of the supersymmetry constraints are equivalent.* More precisely, let $|\phi\rangle$ be a Lorentz invariant state so that

$$\hat{\mathcal{J}}^{ab}(x)\,|\phi\rangle = 0, \quad \forall\, x \in \Sigma,\ a,b \in \{0,1,2,3\}\ ,$$

then the following equivalences hold:

$$\begin{aligned}
\hat{\bar{S}}_{(n)1'}\,|\phi\rangle = 0 \quad &\Leftrightarrow \quad \hat{\bar{S}}_{(n)2'}\,|\phi\rangle = 0\\
\hat{S}_{(n)1}\,|\phi\rangle = 0 \quad &\Leftrightarrow \quad \hat{S}_{(n)2}\,|\phi\rangle = 0\\
\\
\hat{\bar{\mathcal{S}}}_{1'}(x)\,|\phi\rangle = 0 \quad &\Leftrightarrow \quad \hat{\bar{\mathcal{S}}}_{2'}(x)\,|\phi\rangle = 0\\
\hat{\mathcal{S}}_{1}(x)\,|\phi\rangle = 0 \quad &\Leftrightarrow \quad \hat{\mathcal{S}}_{2}(x)\,|\phi\rangle = 0\,.
\end{aligned} \tag{938}$$

Proof: According to 4.14 (p.194), eqn. (809), the quantum generator of Lorentz rotations in the $3-1$ plane, $\hat{\mathcal{J}}^{31}(x)$, satisfies

$$\left[\hat{\mathcal{J}}^{31}(y), \hat{\mathcal{S}}_A(x)\right] \;=\; -\delta(y,x)\,\sigma^{31}{}_A{}^B \hat{\mathcal{S}}_B(x)\ . \tag{939}$$

Because of

$$\sigma^{31} = \sigma^3\bar{\sigma}^1 = i\sigma^2 = \begin{bmatrix} 0 & 1\\ -1 & 0\end{bmatrix},$$

it follows that

$$\begin{aligned}
\left[\hat{\mathcal{J}}^{31}(y), \hat{\mathcal{S}}_1(x)\right] &= -\delta(y,x)\,\sigma^{31}{}_A{}^B \hat{\mathcal{S}}_B(x)\\
&= \delta(y,x)\,\hat{\mathcal{S}}_2(x)\\
\left[\hat{\mathcal{J}}^{31}(y), \hat{\mathcal{S}}_2(x)\right] &= -\delta(y,x)\,\hat{\mathcal{S}}_1(x)\ .
\end{aligned} \tag{940}$$

Define the operator

$$\hat{J}^{31} \;:=\; \int_\Sigma \hat{\mathcal{J}}^{31}(x)\ d^3x\,. \tag{941}$$

It satisfies:

$$\begin{aligned}
\left[\hat{J}^{31}, \hat{\mathcal{S}}_1(x)\right] &= \mathcal{S}_2(x)\\
\left[\hat{J}^{31}, \hat{\mathcal{S}}_2(x)\right] &= -\mathcal{S}_1(x)\ ,
\end{aligned} \tag{942}$$

and hence

$$\begin{aligned}
e^{\frac{\pi}{2}\hat{J}^{31}}\ \hat{\mathcal{S}}_1(x)\ e^{-\frac{\pi}{2}\hat{J}^{31}} &= \hat{\mathcal{S}}_2(x)\\
e^{\frac{\pi}{2}\hat{J}^{31}}\ \hat{\mathcal{S}}_2(x)\ e^{-\frac{\pi}{2}\hat{J}^{31}} &= -\hat{\mathcal{S}}_1(x)\ .
\end{aligned} \tag{943}$$

---

tromagnetism. But these are, in the context of the above 'dictionary', analogous to Lorentz transformation on configuration space, $\mathcal{M}^{(\mathrm{conf})}$, in supergravity and *not* to Lorentz transformations on physical supergravity spacetime.

Because $e^{\frac{\pi}{2}\hat{J}^{31}}$ is invertible, it follows that on Lorentz invariant states $|\phi\rangle$ the following equations are equivalent:

$$\begin{aligned}
& \hat{\mathcal{S}}_1(x)\,|\phi\rangle = 0 \\
\Leftrightarrow\quad & e^{\frac{\pi}{2}\hat{J}^{31}}\,\hat{\mathcal{S}}_1(x)\,|\phi\rangle = 0 \\
\Leftrightarrow\quad & e^{\frac{\pi}{2}\hat{J}^{31}}\,\hat{\mathcal{S}}_1(x)\,\underbrace{e^{-\frac{\pi}{2}\hat{J}^{31}}\,|\phi\rangle}_{=|\phi\rangle} = 0 \\
\Leftrightarrow\quad & \hat{\mathcal{S}}_2(x)\,|\phi\rangle = 0\,.
\end{aligned} \tag{944}$$

The same reasoning applies to the other cases of (938).
□

There is another way to exhibit this redundancy induced by Lorentz symmetry on the supersymmetry constraints:

4.40 (Full solutions from solutions in one spin component) One may reduce the problem of solving the full set of supersymmetry constraints

$$\hat{S}_{(n)A}\,|\phi\rangle = \hat{\bar{S}}_{(n)A'}\,|\phi\rangle = 0, \forall\, n \in \mathbb{N},\ A, A' \in \{1,2\} \tag{945}$$

to that of solving for only one of the spin indices, say $A = A' = 1$:

$$\hat{S}_{(n)1}\,|\phi\rangle = \hat{\bar{S}}_{(n)1'}\,|\phi\rangle = 0, \forall\, n \in \mathbb{N}, \tag{946}$$

in the following sense:

Consider any solution $|\phi'\rangle$ to (946), i.e. to one spin component of the constraints. By the same argument as in 4.33 (p.216) $|\phi'\rangle$ must be locally of the form

$$|\phi'\rangle \quad = \quad \left(\prod_{n\in N} \hat{\bar{S}}_{(n)1'}\right)|\phi_0\rangle\ , \tag{947}$$

for some zero-fermion state $|\phi_0\rangle$ which solves the constraint

$$\hat{H}_{(n)11'}\,|\phi_0\rangle = 0, \quad \forall\, n \in \mathbb{N}\,.$$

Now, *if $|\phi_0\rangle$ is furthermore Lorentz invariant and diffeomorphism invariant*, i.e. if furthermore

$$\begin{aligned}
&\hat{H}_{(n)i}\,|\phi_0\rangle = 0, \quad \forall\, n \in \mathbb{N},\ A, A' \in \{1,2\}\,,\ i \in \{1,2,3\} \\
&\hat{\mathcal{J}}^{ab}(x)\,|\phi_0\rangle = 0, \quad\ \forall\, x \in \Sigma,\ a, b \in \{0,1,2,3\}\ ,
\end{aligned} \tag{948}$$

so that in particular

$$\hat{H}_{(n)AA'}\,|\phi_0\rangle = \sigma^a{}_{AA'}\hat{H}_{(n)a}\,|\phi_0\rangle = 0, \quad \forall\, n \in \mathbb{N},\ A, A' \in \{1,2\}\ , \tag{949}$$

then, by theorem 4.35 (p.218), $|\phi'\rangle$ gives rise to a solution $|\phi\rangle$ of the full set of supersymmetry constraints (945) by "closing" it with respect to the $\hat{\bar{S}}_{(n)2'}$ operators:

$$\begin{aligned}
|\phi\rangle \quad &= \quad \left(\prod_{n\in N} \hat{\bar{S}}_{(n)2'}\right)|\phi_0\rangle \\
&\propto \quad \left(\prod_{n\in N} \hat{\bar{S}}_{(n)A'}\hat{\bar{S}}_{(n)}{}^{A'}\right)|\phi_0\rangle\ ,
\end{aligned} \tag{950}$$

The point here is that the Lorentz and diffeomorphism constraints are usually relatively easy to solve and that the full physical information, the dynamics, is encoded in the Hamiltonian constraint alone. But the above shows that to solve the Hamiltonian constraint it is sufficient to solve only one spin component of the supersymmetry constraints.

A very simple example will serve as an illustration for 4.39 (p.223):

Example 4.41 (Lorentz symmetry in supersymmetric Friedmann cosmology) (This example is really a special case of the homogeneous cosmological models to be discussed further below in §4.3.2 (p.230) and §4.3.3 (p.240), see there for more details.) Consider the $N = 1$ supersymmetric extension of the $k = +1$ Friedmann cosmological model as discussed in [85] and [83]§5.2. Mini-superspace is 1-dimensional in this case, being parameterized by the scale factor a of the S^3 spatial sections of the Friedmann universe. The truncated supersymmetry generators given in the above references read (up to a factor of $i = \sqrt{-1}$)

$$\begin{aligned} \hat{S}_{(0)A} &= \psi_A\,(\hbar\partial_a + 6a) \\ \hat{\bar{S}}_{(0)A'} &= -\tilde{\psi}_{A'}\,(\hbar\partial_a - 6a)\;. \end{aligned} \tag{951}$$

The $\tilde{\psi}_{A'}$ are here quantum versions of homogeneous gravitino mode amplitudes. The notation is a little different from that used in the above paragraphs (a more detailed discussion is postponed until general Bianchi cosmologies are discussed in example 4.50 (p.237) below), but that and further details are of no concern for our present purpose, which is simply to point out the following:
The most general Lorentz invariant state in this setup is obviously

$$\begin{aligned} |\phi\rangle &= A(a)\,|0\rangle + B(a)\,\psi_A\psi^A\,|0\rangle \\ &= A(a)\,|0\rangle + 2B(a)\,\psi_1\psi_2\,|0\rangle\;, \end{aligned} \tag{952}$$

in a representation where the ψ_A are regarded as fermion creators and where $|0\rangle$ is the fermionic vacuum defined by $\bar{\psi}_{A'}\,|0\rangle = 0 = \partial_a\,|0\rangle$. Now consider the action of the supersymmetry operators on this general state. It is simply given by:

$$\begin{aligned} \hat{S}_{(0)A}\,|\phi\rangle &= (\hbar\partial_a + 6a)\,A(a)\,\psi_A\,|0\rangle \\ \hat{\bar{S}}_{(0)A'}\,|\phi\rangle &= -\,(\hbar\partial_a - 6a)\,B(a)\,\tilde{\psi}_{A'}\psi_A\psi^A\,|0\rangle\;. \end{aligned} \tag{953}$$

Obviously the vanishing of these expressions is equivalent to

$$(\hbar\partial_a + 6a)\,A(a) = 0$$

and

$$(\hbar\partial_a - 6a)\,B(a) = 0\,,$$

respectively. Up to a factor this is uniquely solved by

$$A(a) \propto \exp\bigl(-3a^2/\hbar\bigr)$$

and

$$B(a) \propto \exp\bigl(+3a^2/\hbar\bigr)\;.$$

But these conditions, and their solutions, do not at all depend on the Lorentz spin indices A, A', so that indeed in this example the relations

$$\begin{aligned} \hat{S}_{(0)1}\,|\phi\rangle = 0 \quad &\Leftrightarrow \quad \hat{S}_{(0)2}\,|\phi\rangle = 0 \\ \hat{\bar{S}}_{(0)1}\,|\phi\rangle = 0 \quad &\Leftrightarrow \quad \hat{\bar{S}}_{(0)2}\,|\phi\rangle = 0 \end{aligned} \tag{954}$$

are confirmed.

(It is noteworthy that the content of 4.40 (p.224) is not applicable to the above (possibly overly simplified) example, since, as a direct calculation shows, the expressions $\left[\hat{H}_{(0)AA'}, \hat{S}_{(0)B}\right] = \left[\left\{\hat{S}_{(0)A}, \hat{\bar{S}}_{(0)A'}\right\}, \hat{S}_{(0)B}\right]$ are nontrivial and in particular not proportional to Lorentz generators.)

What makes the above model particularly simple with respect to its behavior under Lorentz rotations is the fact that the terms in the supersymmetry generators $\hat{S}_{(0)A}$, $\hat{\bar{S}}_{(0)A'}$ which carry spinor indices *factor out*: Both generators have the abstract form

$$\text{“ } \hat{S}_{\cdot} = \hat{F}_{\cdot}\,\hat{B} \text{ ”}\,,$$

where $\hat{F}$ is a fermionic operator and $\hat{B}$ a bosonic operator. The latter does not carry any spinor indices (or, for that matter, Lorentz vector indices). It is, by itself, manifestly invariant under Lorentz transformations. The entire dependence on the spin frame over spacetime is carried by the fermionic term $\hat{F}_{\cdot}$ of the supersymmetry operator. Recall from 4.20 (p.200) that this is not in general so. In 4.42 (p.226) below it is shown that such a factoring occurs exactly when the fermionic modes are purely homogeneous.

4.42 (Factoring out of spinor-index carrying terms in the susy generators)

In order to analyse under which conditions the terms $\hat{\psi}^A{}_i\bar{\sigma}^a{}_{A'A}$ in the supersymmetry constraint $\hat{\bar{S}}_{(n)A'}$, given in 4.20 (p.200) equation (844), can be factored out, rewrite (844) by performing an integration by parts and inserting σ-matrices:

$$\begin{aligned}
\hat{\bar{S}}_{(n)A'} &= \int_\Sigma C_{(n)}\left(-\hat{\psi}^A{}_i\sigma^a{}_{AA'}\frac{\hbar\kappa^2}{2}\frac{\delta}{\delta e^a{}_i} + \epsilon^{ijk}\hat{e}_{ai}\sigma^a{}_{AA'}\left(\partial_j + \omega_j\right)\hat{\psi}^A{}_k\right)\,d^3x \\
&\overset{(845)}{=} -\int_\Sigma \left(\hat{\psi}^A{}_i\sigma^a{}_{AA'}\frac{\hbar\kappa^2}{2}C_{(n)}\frac{\delta}{\delta e^a{}_i} + \hat{\psi}^A{}_k\left(\delta^B{}_A\partial_j - \omega_j{}^B{}_A\right)C_{(n)}\epsilon^{ijk}\hat{e}_{ai}\sigma^a{}_{BA'}\right)\,d^3x \\
&= -\int_\Sigma \left(\hat{\psi}^A{}_i\sigma^a{}_{AA'}\frac{\hbar\kappa^2}{2}C_{(n)}\frac{\delta}{\delta e^a{}_i} + \hat{\psi}^A{}_k\underbrace{\left(-\sigma^a{}_{AA'}\,\sigma_a{}^{CC'}\right)}_{\overset{(1382)}{=}\delta^C_A\,\delta^{C'}_{A'}}\left(\delta^B{}_C\partial_j - \omega_j{}^B{}_C\right)C_{(n)}\epsilon^{ijk}\hat{e}_{bi}\sigma^b{}_{BC'}\right)\,d^3x \\
&= -\int_\Sigma \hat{\psi}^A{}_i\sigma^a{}_{AA'}\left(\frac{\hbar\kappa^2}{2}C_{(n)}\frac{\delta}{\delta e^a{}_i} + \sigma_a{}^{CC'}\left(\delta^B{}_C\partial_j - \omega_j{}^B{}_C\right)C_{(n)}\epsilon^{ijk}\hat{e}_{bk}\sigma^b{}_{BC'}\right)\,d^3x\,.
\end{aligned} \tag{955}$$

(Note that in the second term in the last line the indices $i \leftrightarrow k$ have been interchanged, which gives a factor of -1.)

In order to make further progress at this point we first need to consider non-coordinate tangent frames on Σ:

4.43 (Change of spatial tangent frame) Up to now we have only considered the "holonomic" coordinate frame on the spatial tangent bundle $T(\Sigma)$, namely that spanned by the basis tangent vectors $\{\partial_i\}_{i\in\{1,2,3\}}$, where the ∂_i are the partial derivatives with respect to our fixed but arbitrary coordinate patch $\left\{x^i\right\}_{i\in\{1,2,3\}}$ (see 2.2 (p.16)). Correspondingly, on the cotangent bundle $T^*(\Sigma)$ the dual frame is spanned by the coordinate differentials $\left\{\mathbf{d}x^i\right\}_{i\in\{1,2,3\}}$, so that $\mathbf{d}x^i(\partial_j) = \delta^i_j$. The vielbein field ${e^a}_i$ and the gravitino field ${\psi^A}_i$ define the differential forms $\{\mathbf{e}^a\}_{a\in\{0,1,2,3\}}$ and $\left\{\boldsymbol{\psi}^A\right\}_{A\in\{1,2\}}$, where

$$\begin{aligned} \mathbf{e}^a &:= {e^a}_i\,\mathbf{d}x^i \\ \boldsymbol{\psi}^A &:= {\psi^A}_i\,\mathbf{d}x^i\,. \end{aligned} \tag{956}$$

For applications, especially those in cosmology, one may want to express the vielbein and gravitino degrees of freedom with respect to some other cotangent frame, possibly an "anholonomic" one, i.e. one not deriving from a coordinate system.
Hence let $\{\boldsymbol{\omega}^\alpha\}_{\alpha\in\{1,2,3\}}$ be any set of 1-forms on Σ constituting a covector basis. With respect to our original basis $\left\{\mathbf{d}x^i\right\}$ these may be expressed as

$$\boldsymbol{\omega}^\alpha := {\omega^\alpha}_i\mathbf{d}x^i\,, \tag{957}$$

for some coefficient matrices ${\omega^\alpha}_i = {\omega^\alpha}_i(x)$. Because the $\boldsymbol{\omega}^\alpha$ are supposed to form a basis, the ${\omega^\alpha}_i$ are nondegenerate and have an inverse, ${\omega^{-1i}}_\alpha = {\omega^{-1i}}_\alpha(x)$, satisfying

$$\begin{aligned} {\omega^{-1j}}_\alpha {\omega^\alpha}_i &= \delta^j_i \\ {\omega^\alpha}_i {\omega^{-1i}}_\beta &= \delta^\alpha_\beta\,. \end{aligned} \tag{958}$$

In particular, the tangent frame is given by the set $\{\partial_\alpha\}_{\alpha\in\{1,2,3\}}$ with

$$\partial_\alpha = {\omega^{-1i}}_\alpha\,\partial_i\,, \tag{959}$$

so that

$$\boldsymbol{\omega}^\alpha(\partial_\beta) = \delta^\alpha_\beta\,,$$

and the $\mathbf{d}x^i$ read in terms of the $\boldsymbol{\omega}^\alpha$:

$$\mathbf{d}x^i = {\omega^{-1i}}_\alpha\,\boldsymbol{\omega}^\alpha\,. \tag{960}$$

The physical fields can be expressed in the new basis as

$$\begin{aligned} \mathbf{e}^a &= {e^a}_i\,{\omega^{-1i}}_\alpha\,\boldsymbol{\omega}^\alpha \\ \boldsymbol{\psi}^A &= {\psi^A}_i\,{\omega^{-1i}}_\alpha\,\boldsymbol{\omega}^\alpha\,. \end{aligned} \tag{961}$$

It is usual practice to define:

$$\begin{aligned} {e^a}_\alpha &:= {e^a}_i\,{\omega^{-1i}}_\alpha \\ {\psi^A}_\alpha &:= {\psi^A}_i\,{\omega^{-1i}}_\alpha\,, \end{aligned} \tag{962}$$

so that

$$\begin{aligned} \mathbf{e}^a &= e^a{}_\alpha \omega^\alpha \\ \boldsymbol{\psi}^A &= \psi^A{}_\alpha \omega^\alpha \,. \end{aligned} \tag{963}$$

Now recall the mode expansion (821) in 4.16 (p.196). It may be reexpressed in terms of the new frame $\omega^\alpha{}_i$ as:

$$\begin{aligned} e^a{}_i(t,x) &= b^{(n)}(t)\, B_{(n)}{}^a{}_i(x) \\ &:= b^{(n)}(t)\, B_{\omega(n)}{}^a{}_\alpha\, \omega^\alpha{}_i(x) \\ \psi^A{}_i(t,x) &= f^{(n)}(t)\, F_{(n)}{}^A{}_i(x) \\ &:= f^{(n)}(t)\, F_{\omega(n)}{}^A{}_\alpha\, \omega^\alpha{}_i(x) \ , \end{aligned} \tag{964}$$

or, for short,

$$\begin{aligned} e^a{}_\alpha &= b^{(n)} B_{(n)}{}^a{}_\alpha \\ \psi^A{}_\alpha &= f^{(n)} F_{(n)}{}^a{}_\alpha \,, \end{aligned} \tag{965}$$

where the label ω has been suppressed. The dual modes similarly read

$$\begin{aligned} \frac{\delta}{\delta e^a{}_i(x)} &= B'^{(n)}{}_a{}^i(x)\, \frac{\partial}{\partial b^{(n)}} \\ &= \omega^{-1i}{}_\alpha B'^{(n)}{}_a{}^\alpha(x)\, \frac{\partial}{\partial b^{(n)}} \\ \frac{\delta}{\delta \psi^A{}_i(x)} &= F'^{(n)}{}_A{}^i(x)\, \frac{\partial}{\partial f^{(n)}} \\ &= \omega^{-1i}{}_\alpha F'^{(n)}{}_A{}^\alpha(x)\, \frac{\partial}{\partial f^{(n)}} \,, \end{aligned} \tag{966}$$

or, for short,

$$\begin{aligned} \frac{\delta}{\delta e^a{}_\alpha} &= B'^{(n)}{}_a{}^\alpha\, \frac{\partial}{\partial b^{(n)}} \\ \frac{\delta}{\delta \psi^A{}_\alpha} &= F'^{(n)}{}_a{}^\alpha\, \frac{\partial}{\partial f^{(n)}} \,. \end{aligned} \tag{967}$$

These rather standard conventions for expressing basis changes in index notation have been made explicit here in order to avoid any ambiguities in the following discussion.

First of all consider now the primed supersymmetry generator expressed in terms of the new frame $\boldsymbol{\omega}^\alpha$:

4.44 (Primed supersymmetry generator in arbitrary cotangent frame) Using the form (955) obtained in 4.42 (p.226) one finds

$$\hat{\bar{S}}_{(n)A'} = -\int\limits_\Sigma \hat{\psi}^A{}_\alpha \sigma^a{}_{AA'} \left(\frac{\hbar\kappa^2}{2} C_{(n)} \frac{\delta}{\delta e^a{}_\alpha} + \omega^\alpha{}_i(x)\, \sigma_a{}^{CC'} \left(\delta^B{}_C \partial_j - \omega_j{}^B{}_C\right) C_{(n)} \epsilon^{ijk} \hat{e}_{b\beta} \omega^\beta{}_k(x)\, \sigma^b{}_{BC'} \right) d^3x \,. \tag{968}$$

One could, following convention, just as well have replaced *all* appearances of the indices i, j by α, β. However, that would somewhat hide the crucial coordinate-dependent factors $\omega(x)$ that are introduced into the superpotential-dependent second term, but not into the first term, where the factor $\omega^{\alpha}{}_{i}$ coming from the gravitino field operator cancels with the $\omega^{-1i}{}_{\alpha}$ factor coming from the vielbein functional derivative.

The above considerations now allow to discuss homogeneous modes of the supersymmetry generators, which is the content of the next subsection.

4.3.2 Homogeneous modes

Introduction. For practical calculations in physics it is useful to be able to divide, at least formally, the degrees of freedom of the system under consideration into those that describe its "general", or "global", behavior (in some suitable sense depending on the application in question) and those that describe deviations therefrom. The standard example is a composite body moving through space: Here one usually splits off the center-of-mass motion of the whole body from the relative motion of each of its parts.

A similar split is the starting point of all cosmological models: Some (usually very few) "global" degrees of freedom, most notably the scale factor of the universe, are chosen to describe the overall state of the spatial section Σ. They correspond to the center-of-mass coordinates of a moving body. All other degrees of freedom, for instance gravity waves traveling on Σ, are regarded as describing deviations from the idealized global state of the system and are treated separately – if at all.

The standard approach is to regard the *homogeneous* components of the spatial metric on Σ as constituting the "essential" degrees of freedom of the universe, in the above sense (a method that is justified by the high homogeneity observed in our universe on large scales). This gives rise to the homogeneous (but in general anisotropic) "Bianchi" cosmologies, which contain the (isotropic) Friedmann-Robertson-Walker cosmologies as a special case. In the present framework, such an approach to describe the total of spacetime means that a certain set of the modes (821) of the graviton and gravitino field (*cf.* 4.16 (p.196)) are considered special, namely simply those which are, with respect to a suitable coframe (957) (see 4.43 (p.227)) spatially *constant*. These certainly provide a good general first order approximation to the exact fields, just like the center-of-mass coordinate of a composite body is in general the best simple approximation to the position of any single part of it.

The purpose of the following section is to establish which constraints on the purely homogeneous field modes are induced by the full set of constraints of canonical quantum supergravity. That is, the analog of the Wheeler-DeWitt equation is derived in the context of supergravity. In the above analogy this corresponds to finding the Hamiltonian which describes the center-of-mass motion alone. Just as in this simple case, it is hoped that, at least when the free parameters are chosen suitably, a solution to the homogeneous constraints can, in principle at least, be extended to a solution to full supergravity involving the full infinite set of modes[56].

One motivation behind the following discussions is an attempt to relate the "Lagrangian" and the "Hamiltonian" routes (in the nomenclature of the discussion on p. 9 of the introduction) to supersymmetric quantum cosmology in general and to Bianchi cosmologies in particular.

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[56]It is certainly possible that this hope may fail in special cases. In particular it may well fail in the most interesting cases, as for instance when the universe is near a cosmological singularity with the scale factor tending to zero. Similarly, the center-of-mass of an extended body will, while approaching another extended body, follow the point-particle approximation to its motion only as long as both bodies are not as close as to actually collide, in which case the formerly neglected degrees of freedom of the constituents of both bodies dominate the dynamics.

Literature. A general brief introduction to quantum cosmology is [273]. Standard texts on supersymmetric quantum cosmology are [83] and [197]. The standard "Lagrangian" approach to supersymmetric Bianchi cosmologies is detailed in both of these. The corresponding "Hamiltonian" approach is presented in [25] and references therein.

4.45 (General conventions) In this section we will assume a fixed cotangent basis $\boldsymbol{\omega}^\alpha$ on $T^*(\Sigma)$, as described in 4.43 (p.227). All fields will be given with respect to this basis:

$$\begin{aligned} e^a{}_\alpha(x) &= b^{(n)} B_{(n)}{}^a{}_\alpha(x) \\ \psi^A{}_\alpha(x) &= b^{(n)} F_{(n)}{}^A{}_\alpha(x) \, . \end{aligned} \tag{969}$$

For the following discussion to make sense technically, no special properties of the $\boldsymbol{\omega}^\alpha$ needs to be assumed (except that they form a basis for $T^*_x(\Sigma)$ at each $x \in \Sigma$). However, the main motivation behind the following development are *homogeneous* cosmological models in which case the $\boldsymbol{\omega}^\alpha$ are assumed to be left-invariant with respect to a Lie group of diffeomorphisms acting simply and freely on Σ (*cf.* [269]§7.2). In particular, for homogeneous models of *Bianchi class A*, with which we will be concerned, the $\boldsymbol{\omega}^\alpha$ satisfy the relation

$$\mathbf{d}\boldsymbol{\omega}^\alpha = \frac{1}{2} m^{\beta\gamma} \epsilon_{\beta\gamma\delta} \boldsymbol{\omega}^\gamma \boldsymbol{\omega}^\delta \, . \tag{970}$$

Details of this construction are given in 4.50 (p.237). The vielbein modes which are *constant* with respect to $\boldsymbol{\omega}^\alpha$, i.e. those that satisfy

$$B^{\mathrm{hom}\,a}_{(n)}{}_\alpha(x) = B^{\mathrm{hom}\,a}_{(n)}{}_\alpha(x_0) \, , \tag{971}$$

give rise to the homogeneous spatial metric

$$ds^2 = \eta_{ab} \left(e^{\mathrm{hom}}\right)^a{}_\alpha \left(e^{\mathrm{hom}}\right)^b{}_\beta \boldsymbol{\omega}^\alpha \otimes \boldsymbol{\omega}^\beta \, ,$$

and will therefore be called *homogeneous modes*, for short. Even though, as mentioned above, most of the following considerations are completely independent of the special condition (970) imposed on the basis elements $\boldsymbol{\omega}^\alpha$, we will, for simplicity, refer to constant modes as in (971) generally as *homogeneous modes*. Note that, naturally, whether or not a given mode is constant depends on the chosen basis $\boldsymbol{\omega}^\alpha$.

A useful starting point for the following discussion is the form (968) of the primed supersymmetry generator given in 4.44 (p.228).

It is clear that homogeneous modes of $\hat{\psi}^A{}_\alpha$ may be pulled out of the integral in (968). For that purpose denote by $I^{\mathrm{hom_f}}$ the set of integers that correspond to homogeneous fermionic modes, i.e.:

$$\left(\partial_i F_{(n)}{}^A{}_\alpha(x) = 0, \;\; \forall\, i, \alpha, A\right) \Leftrightarrow n \in I^{\mathrm{hom_f}} \, . \tag{972}$$

4.46 (Splitting off the fermionic homogeneous part of susy generators) With this notation the above expression may be split into contributions from homogeneous and inhomogeneous fermionic modes as follows:

$$
\begin{aligned}
\hat{\bar{S}}_{(n)A'} &= \int_\Sigma \hat{\psi}^A{}_\alpha \sigma^a{}_{AA'} \left(\cdots\right)_a \; d^3x \\
&= \hat{f}^{(n)} \int_\Sigma F_{(n)}{}^A{}_\alpha(x)\, \sigma^a{}_{AA'} \left(\cdots\right)_a{}^\alpha \; d^3x \\
&= \underbrace{\sum_{n\in I^{\mathrm{hom_f}}} \hat{f}^{(n)} F_{(n)}{}^A{}_\alpha \sigma^a{}_{AA'} \int_\Sigma \left(\cdots\right)_a{}^\alpha \; d^3x}_{:=\hat{\bar{S}}^{\mathrm{hom_f}}_{(n)A'}} + \underbrace{\sum_{n\notin I^{\mathrm{hom_f}}} \hat{f}^{(n)} \int_\Sigma F_{(n)}{}^A{}_\alpha(x)\, \sigma^a{}_{AA'} \left(\cdots\right)_a \; d^3x}_{:=\hat{\bar{S}}^{\mathrm{in_f}}_{(n)A'}} \\
&:= \hat{\bar{S}}^{\mathrm{hom_f}}_{(n)A'} + \hat{\bar{S}}^{\mathrm{in_f}}_{(n)A'} \,.
\end{aligned}
\tag{973}
$$

This leaves us with a decomposition of $\hat{\bar{S}}_{(n)A'}$ into two components, $\hat{\bar{S}}^{\mathrm{hom_f}}_{(n)A'}$ and $\hat{\bar{S}}^{\mathrm{in_f}}_{(n)A'}$.

Some important properties of these operators are immediate:

4.47 (Algebra of fermionic-homogeneous component of the susy generator) Consider for instance the identity

$$
\begin{aligned}
0 &= \left(\hat{\bar{S}}_{(n)A'}\right)^2 \\
&= \left(\hat{\bar{S}}^{\mathrm{hom_f}}_{(n)A'} + \hat{\bar{S}}^{\mathrm{in_f}}_{(n)A'}\right)^2 \\
&= \left(\hat{\bar{S}}^{\mathrm{hom_f}}_{(n)A'}\right)^2 + \left(\hat{\bar{S}}^{\mathrm{in_f}}_{(n)A'}\right)^2 + \left\{\hat{\bar{S}}^{\mathrm{hom_f}}_{(n)A'}, \hat{\bar{S}}^{\mathrm{in_f}}_{(n)A'}\right\} \,.
\end{aligned}
\tag{974}
$$

The three terms in the last line contain strictly different bilinear terms in the fermionic mode amplitudes $\hat{f}^{(n)}$: The first contains only terms bilinear in the homogeneous amplitudes, the second only those bilinear in the inhomogeneous amplitudes and the last only products of one homogeneous and one inhomogeneous fermionic mode amplitude operator. It follows that all three must vanish separately:

$$
\begin{aligned}
\left\{\hat{\bar{S}}^{\mathrm{hom_f}}_{(n)A'}, \hat{\bar{S}}^{\mathrm{hom_f}}_{(n)A'}\right\} &= 0 \\
\left\{\hat{\bar{S}}^{\mathrm{in_f}}_{(n)A'}, \hat{\bar{S}}^{\mathrm{in_f}}_{(n)A'}\right\} &= 0 \\
\left\{\hat{\bar{S}}^{\mathrm{hom_f}}_{(n)A'}, \hat{\bar{S}}^{\mathrm{in_f}}_{(n)A'}\right\} &= 0 \,.
\end{aligned}
\tag{975}
$$

Now consider the term $\hat{\bar{S}}^{\mathrm{hom_f}}_{(0)A'}$, i.e. the component of the 0-mode of the primed supersymmetry generator that contains only homogeneous fermionic modes:

4.48 (The fermionic-homogeneous 0-mode of the susy generator) By assumption (819) (see 4.15 (p.194)) the zeroth scalar mode is constant:

$$C_{(0)}(x) = 1/\sqrt{V} \,,$$

so that

$$\hat{\bar{S}}^{\mathrm{hom_f}}_{(0)A'} = \sum_{n\in I^{\mathrm{hom_f}}} \hat{f}^{(n)} F_{(n)}{}^{A}{}_{\alpha}\sigma^{a}{}_{AA'}\,\frac{1}{\sqrt{V}}\int\limits_{\Sigma}\left(-\frac{\hbar\kappa^2}{2}\frac{\delta}{\delta e^{a}{}_{\alpha}} + \sigma_{a}{}^{CC'}\omega_{j}{}^{B}{}_{C}\epsilon^{ijk}\hat{e}_{bk}\sigma^{b}{}_{BC'}\right)\, d^3x\,. \tag{976}$$

The integral over Σ projects out certain modes from its integrand: Let, in analogy to (**??**), $I^{\mathrm{hom_b}}$ be the set of mode indices of the homogeneous bosonic *dual* modes:

$$\left(\partial_i B'^{(n)}{}_{a}{}^{i}(x) = 0,\ \ \forall\, \alpha, i, a\right) \Leftrightarrow n \in I^{\mathrm{hom_b}}\,. \tag{977}$$

The first term in (976), containing the functional derivative, can then be written as follows:

$$\begin{aligned}\int\limits_{\Sigma}\frac{\delta}{\delta e^{a}{}_{\alpha}(x)}\, d^3x &= \int\limits_{\Sigma} B'^{(n)}{}_{a}{}^{\alpha}(x)\ d^3x\; b^{\#}_{(n)} \\ &= V\sum_{n\in I^{\mathrm{hom_b}}} B'^{(n)}{}_{a}{}^{\alpha}\; b^{\#}_{(n)}\,.\end{aligned} \tag{978}$$

(This is because the integral over the inhomogeneous modes $B'^{()n}{}_{a}{}^{\alpha}(x)\,,\ \ n \notin I^{\mathrm{hom_b}}$ vanishes, since these modes are orthogonal to all constant modes and hence each component (for a particular value of a and α) is orthogonal to the constant scalar function 1.)

In 4.24 (p.203), eq. (858), the $\hat{f}^{(n)}$ had generally been identified with differential form creators $\hat{\mathrm{c}}^{\dagger\,(n)}$. These did not carry a spinor index but instead the operators $\frac{\partial}{\partial X^{A'(n)}}$ did. This general identification of course still holds in the present homogeneous context. However, since for the homogeneous modes the terms carrying the spinor index may be factored from the partial derivatives, as has been shown above, there is another identification of differential forms and partial derivatives, obtained from the original one by a linear transformation, which appears naturally:

Introduce the notation:

$$\begin{aligned}\hat{f}^{\mathrm{hom}}{}_{\alpha}{}^{a}{}_{A'} &:= \sum_{n\in I^{\mathrm{hom_f}}} \hat{f}^{(n)} F_{(n)}{}^{A}{}_{\alpha}\sigma^{a}{}_{AA'} \\ \hat{b}^{\mathrm{hom}\,a}{}_{\alpha} &:= -\frac{2}{\kappa^2}\frac{1}{\sqrt{V}}\sum_{n\in I^{\mathrm{hom_b}}} B_{(n)}{}^{a}{}_{\alpha}\,\hat{b}^{(n)} \\ \hat{b}^{\#\mathrm{hom}}{}_{a}{}^{\alpha} &:= -\frac{\kappa^2}{2}\sqrt{V}\sum_{n\in I^{\mathrm{hom_b}}} B'^{(n)}{}_{a}{}^{\alpha}\,\hat{b}^{\#}_{(n)} \\ U^{\mathrm{hom_f}}{}_{a}{}^{\alpha} &:= \frac{1}{2}\int\limits_{\Sigma}\sigma_{a}{}^{CC'}\omega_{j}{}^{B}{}_{C}\epsilon^{ijk}\hat{e}_{bk}\sigma^{b}{}_{BC'}\, d^3x\,,\end{aligned} \tag{979}$$

adapted to the homogeneous objects under consideration. Then (976) may be rewritten in the form

$$\hat{\bar{S}}^{\mathrm{hom_f}}_{(0)A'} = \hat{f}^{\mathrm{hom}}{}_{\alpha}{}^{a}{}_{A'}\left(\hbar\hat{b}^{\#\mathrm{hom}}{}_{a}{}^{\alpha} + U^{\mathrm{hom_f}}{}_{a}{}^{\alpha}\right)\,. \tag{980}$$

It is natural to again reformulate this in terms of a suitable exterior derivative. With the definitions

$$\begin{aligned}
\hat{c}^{\dagger\,\mathrm{hom}}{}_{\alpha}{}^{a}{}_{A'} &:= \hat{f}^{\mathrm{hom}}{}_{\alpha}{}^{a}{}_{A'} \\
\partial^{\mathrm{hom}}_{X^{a}{}_{\alpha}} &:= \hat{b}^{\#\mathrm{hom}}{}_{a}{}^{\alpha} \\
\mathbf{d}^{\mathrm{hom}}_{A'} &:= \hat{f}^{\mathrm{hom}}{}_{\alpha}{}^{a}{}_{A'}\hbar\hat{b}^{\#\mathrm{hom}}{}_{a}{}^{\alpha} \\
&= \hat{c}^{\dagger\,\mathrm{hom}}{}_{\alpha}{}^{a}{}_{A'}\partial^{\mathrm{hom}}_{X^{a}{}_{\alpha}}
\end{aligned} \tag{981}$$

(*cf.* 4.24 (p.203)) one has

$$\hat{\bar{S}}^{\mathrm{hom}_{\mathrm{f}}}_{(0)A'} = \hbar\mathbf{d}^{\mathrm{hom}}_{A'} + \hat{f}^{\mathrm{hom}}{}_{\alpha}{}^{a}{}_{A'}U^{\mathrm{hom}_{\mathrm{f}}}{}_{a}{}^{\alpha}\,. \tag{982}$$

Note, however, that the $\hat{c}^{\dagger\,\mathrm{hom}}{}_{\alpha}{}^{a}{}_{A'}$ are not all linearly independent. This "degeneracy" is however removed when diagonal models are considered. This will be discussed in 4.53 (p.241).

Since all the $\partial^{\mathrm{hom}}_{X^{a}{}_{\alpha}}$ commute among each other and all the $\hat{c}^{\dagger\,\mathrm{hom}}{}_{\alpha}{}^{a}{}_{A'}$ anticommute among each other one has

$$\left\{\mathbf{d}^{\mathrm{hom}}_{A'}, \mathbf{d}^{\mathrm{hom}}_{B'}\right\} = 0, \quad \forall\, A', B'\,. \tag{983}$$

By the same reasoning as in 4.25 (p.205) there must locally be a "superpotential" W^{hom} such that

$$\begin{aligned}
&\left[\mathbf{d}^{\mathrm{hom}}_{A'}, W^{\mathrm{hom}}\right] = \hat{f}^{\mathrm{hom}}{}_{\alpha}{}^{a}{}_{A'}U^{\mathrm{hom}_{\mathrm{f}}}{}_{a}{}^{\alpha} \\
\Rightarrow\quad &\hat{\bar{S}}^{\mathrm{hom}_{\mathrm{f}}}_{(0)A'} = e^{-W^{\mathrm{hom}}/\hbar}\,\hbar\mathbf{d}^{\mathrm{hom}}_{A'}\,e^{W^{\mathrm{hom}}/\hbar}\,.
\end{aligned} \tag{984}$$

In fact, according to (866) and (973) one already knows that

$$\hat{\bar{S}}^{\mathrm{hom}_{\mathrm{f}}}_{(0)A'} = e^{-W_{(0)}/\hbar}\,\hbar\mathbf{d}^{\mathrm{hom}}_{A'}\,e^{W_{(0)}/\hbar}\,. \tag{985}$$

Obviously, W^{hom} may be chosen to be $W_{(0)}$ minus that part of $W_{(0)}$ which commutes with $\mathbf{d}^{\mathrm{hom}}_{A'}$:

$$\begin{aligned}
&W_{(0)} := W^{\mathrm{hom}} + W^{\mathrm{in}} \\
&\left[\mathbf{d}^{\mathrm{hom}}_{A'}, W^{\mathrm{in}}\right] = 0\,.
\end{aligned} \tag{986}$$

The potential term $U^{\mathrm{hom}_{\mathrm{f}}}{}_{a}{}^{\alpha}$ in (979) carries a superscript "hom$_{\mathrm{f}}$" instead of "hom" because it does in general depend on all bosonic amplitudes $b^{(n)}$, not only on the homogeneous ones. Therefore one may want to split it into the sum of two terms

$$U^{\mathrm{hom}_{\mathrm{f}}}{}_{a}{}^{\alpha} := U^{\mathrm{hom}_{\mathrm{f}},\mathrm{hom}_{\mathrm{b}}}{}_{a}{}^{\alpha} + U^{\mathrm{hom}_{\mathrm{f}},\mathrm{in}_{\mathrm{b}}}{}_{a}{}^{\alpha}\,, \tag{987}$$

where the first depends on homogeneous bosonic modes exclusively and the second depends on homogeneous bosonic modes only in so far as they couple to inhomogeneous bosonic modes. More precisely:

$$\left[\hat{b}^{\#}_{(n\notin I^{\mathrm{hom}_{\mathrm{b}}})}, U^{\mathrm{hom}_{\mathrm{f}},\mathrm{hom}_{\mathrm{b}}}{}_{a}{}^{\alpha}\right] = 0 \tag{988}$$

and

$$\left(\left[\hat{b}^{\#}_{(n\in I^{\mathrm{hom_b}})}, U^{\mathrm{hom_f,in_b}}{}_a{}^\alpha\right] \neq 0\right) \Rightarrow \left(\left[\hat{b}^{\#}_{(n\notin I^{\mathrm{hom_b}})}, \left[\hat{b}^{\#}_{(n\in I^{\mathrm{hom_b}})}, U^{\mathrm{hom_f,in_b}}{}_a{}^\alpha\right]\right] \neq 0\right) . \tag{989}$$

This splitting allows to define that part of the zero mode of the primed supersymmetry generator which depends solely on homogeneous modes:

$$\hat{\bar{S}}^{\mathrm{hom_f,hom_b}}_{(0)A'} := \hat{f}^{\mathrm{hom}}{}_\alpha{}^a{}_{A'} \left(\hbar \hat{b}^{\#\mathrm{hom}}{}_a{}^\alpha + U^{\mathrm{hom_f,hom_b}}{}_a{}^\alpha\right) \tag{990}$$

This operator, which depends purely on homogeneous modes, bosonic and fermionic ones, is precisely the primed supersymmetry generator that appears in the literature on homogeneous supersymmetric quantum cosmologies, e.g. in [69], eq. (3). (See [83]§5 and [197]§3 for reviews.) It is one term of the full zero mode of the primed supersymmetry generator

$$\begin{aligned}\hat{\bar{S}}_{(0)A'} &\overset{(973)}{=} \hat{\bar{S}}^{\mathrm{hom_f}}_{(n)A'} + \hat{\bar{S}}^{\mathrm{in_f}}_{(n)A'} \\ &\overset{(990)}{=} \hat{\bar{S}}^{\mathrm{hom_f,hom_b}}_{(0)A'} + \hat{f}^{\mathrm{hom}}{}_\alpha{}^a{}_{A'} U^{\mathrm{hom_f,in_b}}{}_a{}^\alpha + \hat{\bar{S}}^{\mathrm{in_f}}_{(n)A'} .\end{aligned} \tag{991}$$

By (984) the splitting of $U^{\mathrm{hom}}{}_a{}^\alpha$ locally corresponds to a splitting of the superpotential

$$W^{\mathrm{hom}} := W^{\mathrm{hom_f,hom_b}} + W^{\mathrm{hom_f,in_b}} , \tag{992}$$

defined by

$$\begin{aligned}\left[\mathbf{d}^{\mathrm{hom}}_{A'}, W^{\mathrm{hom_f,hom_b}}\right] &= \hat{f}^{\mathrm{hom}}{}_\alpha{}^a{}_{A'} U^{\mathrm{hom_f,hom_b}}{}_a{}^\alpha \\ \left[\mathbf{d}^{\mathrm{hom}}_{A'}, W^{\mathrm{hom_f,in_b}}\right] &= \hat{f}^{\mathrm{hom}}{}_\alpha{}^a{}_{A'} U^{\mathrm{hom_f,in_b}}{}_a{}^\alpha .\end{aligned} \tag{993}$$

Hence (984) may be refined by writing

$$\begin{aligned}\hat{\bar{S}}^{\mathrm{hom_f}}_{(0)A'} &= e^{-W^{\mathrm{hom}}/\hbar}\, \hbar \mathbf{d}^{\mathrm{hom}}_{A'}\, e^{W^{\mathrm{hom}}/\hbar} \\ &= e^{-\left(W^{\mathrm{hom_f,hom_b}} + W^{\mathrm{hom_f,in_b}}\right)/\hbar}\, \hbar \mathbf{d}^{\mathrm{hom}}_{A'}\, e^{\left(W^{\mathrm{hom_f,hom_b}} + W^{\mathrm{hom_f,in_b}}\right)/\hbar} \\ &= e^{-W^{\mathrm{hom_f,in_b}}/\hbar}\, e^{-W^{\mathrm{hom_f,hom_b}}/\hbar}\, \hbar \mathbf{d}^{\mathrm{hom}}_{A'}\, e^{W^{\mathrm{hom_f,hom_b}}/\hbar}\, e^{W^{\mathrm{hom_f,in_b}}/\hbar} \\ &= e^{-W^{\mathrm{hom_f,in_b}}/\hbar} \hat{\bar{S}}^{\mathrm{hom_f,hom_b}}_{(0)A'} e^{W^{\mathrm{hom_f,in_b}}/\hbar} ,\end{aligned} \tag{994}$$

where we have identified

$$\hat{\bar{S}}^{\mathrm{hom_f,hom_b}}_{(0)A'} = e^{-W^{\mathrm{hom_f,hom_b}}/\hbar}\, \hbar \mathbf{d}^{\mathrm{hom}}_{A'}\, e^{W^{\mathrm{hom_f,hom_b}}/\hbar} . \tag{995}$$

This, together with (983), shows in particular that $\hat{\bar{S}}^{\mathrm{hom_f,hom_b}}_{(0)A'}$ is still nilpotent by itself, which is important since it implies that the superalgebra of the homogeneous components of the supersymmetry generators still has the expected form:

$$\begin{aligned}\left\{\hat{\bar{S}}^{\mathrm{hom_f,hom_b}}_{(0)A'}, \hat{\bar{S}}^{\mathrm{hom_f,hom_b}}_{(0)B'}\right\} &= 0 \\ \left\{\hat{S}^{\mathrm{hom_f,hom_b}}_{(0)A}, \hat{S}^{\mathrm{hom_f,hom_b}}_{(0)B}\right\} &= 0 \\ \left\{\hat{\bar{S}}^{\mathrm{hom_f,hom_b}}_{(0)A'}, \hat{S}^{\mathrm{hom_f,hom_b}}_{(0)B}\right\} &= \hat{H}^{\mathrm{hom_f,hom_b}}_{BA'} ,\end{aligned} \tag{996}$$

Here we have introduced the related objects $\hat{S}^{\mathrm{hom_f},\mathrm{hom_b}}_{(0)A}$ and $\hat{H}^{\mathrm{hom_f},\mathrm{hom_b}}_{AA'}$, where the latter is defined by the last line in (996) and the former is, according to 4.32 (p.215), given by

$$\hat{S}^{\mathrm{hom_f},\mathrm{hom_b}}_{(0)A} \quad := \quad \left(\hat{\bar{S}}^{\mathrm{hom_f},\mathrm{hom_b}}_{(0)A'}\right)^{\dagger} . \tag{997}$$

4.49 (Homogeneous constraints) One can now contemplate the constraints

$$\begin{aligned} \hat{S}^{\mathrm{hom_f},\mathrm{hom_b}}_{(0)A} \left|\phi\right\rangle &= 0 \\ \hat{\bar{S}}^{\mathrm{hom_f},\mathrm{hom_b}}_{(0)A'} \left|\phi\right\rangle &= 0 . \end{aligned} \tag{998}$$

These involve only finitely many degrees of freedom, but are nevertheless generally expected to contain some relevant physical information. But these constraints do not directly follow from the full theory: The full theory demands that any physical state satisfies (among other constraints) the zeroth mode of the full supersymmetry constraints, i.e.

$$\begin{aligned} & \hat{\bar{S}}_{(0)A'} \left|\phi\right\rangle = 0 \\ \overset{(991)}{\Leftrightarrow} \quad & \left(\hat{\bar{S}}^{\mathrm{hom_f},\mathrm{hom_b}}_{(0)A'} + \hat{f}^{\mathrm{hom}}{}_{\alpha}{}^{a}{}_{A'} U^{\mathrm{hom_f},\mathrm{in_b}}{}_{a}{}^{\alpha} + \hat{\bar{S}}^{\mathrm{in_f}}_{(n)A'}\right) \left|\phi\right\rangle = 0 , \end{aligned} \tag{999}$$

and similarly for the unprimed supersymmetry generator. Hence the constraints (998) are not necessarily implied by the full theory, but they are not generally excluded, either. Instead, the homogeneous constraints (998) can be regarded as expressing a special case of (999), namely the one in which two terms (a), (b) in

$$(999) \Leftrightarrow \qquad \underbrace{\hat{\bar{S}}^{\mathrm{hom_f},\mathrm{hom_b}}_{(0)A'} \left|\phi\right\rangle}_{\text{(a)}} + \underbrace{\left(\hat{f}^{\mathrm{hom}}{}_{\alpha}{}^{a}{}_{A'} U^{\mathrm{hom_f},\mathrm{in_b}}{}_{a}{}^{\alpha} + \hat{\bar{S}}^{\mathrm{in_f}}_{(n)A'}\right) \left|\phi\right\rangle}_{\text{(b)}} = 0$$

vanish separately.

It is not exactly clear how much the space of solutions is cut down by this additional requirement. In particular one would need to understand if *any* solution of the full theory can satisfy it. But let us in the following assume what is generally assumed in quantum cosmology, namely that (a) = 0 is a valid assumption, maybe at least in some suitable kind of approximation. Since $\hat{\bar{S}}^{\mathrm{hom_f},\mathrm{hom_b}}_{(0)A'}$ and $\hat{S}^{\mathrm{hom_f},\mathrm{hom_b}}_{(0)A}$ only involve bosonic and fermionic amplitudes (and their derivatives) of homogeneous modes, one is led to factor any physical state $\left|\phi\right\rangle$ as

$$\left|\phi\right\rangle \quad := \quad \left|\phi_{\mathrm{hom}}\right\rangle \otimes \left|\phi_{\mathrm{in}}\right\rangle , \tag{1000}$$

in such a way that $\left|\phi_{\mathrm{in}}\right\rangle$ is independent of any homogeneous modes:

$$\hat{b}^{\#}_{(n)} \left|\phi_{\mathrm{in}}\right\rangle = 0 = \hat{f}^{\#}_{(m)} \left|\phi_{\mathrm{in}}\right\rangle , \quad \forall\, n \in I^{\mathrm{hom_b}},\ m \in I^{\mathrm{hom_f}} . \tag{1001}$$

(*cf.* (**??**) (977)). This implies that $\left|\phi_{\mathrm{in}}\right\rangle$ is trivially annihilated by $\mathbf{d}^{\mathrm{hom}}_{A'}$ and $\mathbf{d}^{\dagger}{}^{\mathrm{hom}}_{A}$. Hence (998) is equivalent to

$$\begin{aligned} \hat{S}^{\mathrm{hom_f},\mathrm{hom_b}}_{(0)A} \left|\phi_{\mathrm{hom}}\right\rangle &= 0 \\ \hat{\bar{S}}^{\mathrm{hom_f},\mathrm{hom_b}}_{(0)A'} \left|\phi_{\mathrm{hom}}\right\rangle &= 0 . \end{aligned} \tag{1002}$$

Now $|\phi_{\text{hom}}\rangle$ may still depend on inhomogeneous modes, but these enter (1002) only as arbitrary parameters. Let

$$\begin{aligned} X^{\text{hom}} &= \left\{ b^{(n)|n\in I^{\text{hom}_b}} \right\} \\ X^{\text{in}} &= \left\{ b^{(n)|n\notin I^{\text{hom}_b}} \right\} \end{aligned} \tag{1003}$$

and let the general solution to (1002) be

$$|\phi_{\text{hom}}\rangle = \left|\phi_{\text{hom}}\left(X^{\text{hom}}, p_1, p_2, \cdots\right)\right\rangle , \tag{1004}$$

where p_i are the arbitrary parameters that may be assumed to be constant as far as (1002) is concerned. Given any such solution to the completely homogeneous constraints one can solve for a solution to the full theory by turning the p_i into functions of the inhomogeneous modes

$$p_i = p_i\left(X^{\text{in}}\right)$$

and entering the full set of constraint equations of supergravity with the ansatz

$$|\phi\rangle = \left|\phi_{\text{hom}}\left(X^{\text{hom}}.p_1\left(X^{\text{in}}\right), p_2\left(X^{\text{in}}\right), \cdots\right)\right\rangle \otimes \left|\phi_{\text{in}}\left(X^{\text{in}}\right)\right\rangle . \tag{1005}$$

We will *not* try to investigate here the problem of finding solutions to full supergravity subject to the above ansatz (1005). We just assume that such solutions do exist (at least suitably approximate ones) and instead now focus on the much simpler task of solving the constraints (1002) of the purely homogeneous sector of supergravity.

Before proceeding, the next example briefly presents the most general scenario of homogeneous cosmologies in $3 + 1$ spacetime dimensions:

Example 4.50 (Supersymmetric Bianchi cosmologies) Bianchi models are defined (see [269]§7.2 for an introduction to the classical aspects of Bianchi models) by a spatial metric on Σ with the line element

$$ds^2 = h_{\alpha\beta}\boldsymbol{\omega}^\alpha \otimes \boldsymbol{\omega}^\beta . \tag{1006}$$

Here $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2, \boldsymbol{\omega}^3$ is a basis of 1-forms on Σ which satisfy the defining relation

$$\mathbf{d}\omega^\alpha = \frac{1}{2} C^\alpha{}_{\beta\gamma}\boldsymbol{\omega}^\beta \wedge \boldsymbol{\omega}^\gamma . \tag{1007}$$

In the case of so called *class A* Bianchi models, which are of interest here, the *structure constants* $C^\alpha{}_{\beta\gamma}$ are given by

$$C^\alpha{}_{\beta\gamma} := m^{\alpha\delta}\epsilon_{\delta\beta\gamma} , \tag{1008}$$

with ϵ the completely antisymmetric symbol. The matrix

$$m = \left(m^{\alpha\beta}\right) \tag{1009}$$

is a constant symmetric matrix which determines the *type* of the class A Bianchi model. The metric tensor is supposed to be constant over Σ, i.e. it depends only on coordinate time t:

$$h_{\alpha\beta} = e^a{}_\alpha\, e_{a\beta} = h_{\alpha\beta}(t) . \tag{1010}$$

The bosonic and fermionic degrees of freedom are the homogeneous vielbein $e^a{}_\alpha$ and the homogeneous gravitino field $\psi^A{}_\alpha$. The $e^a{}_\alpha$ parameterize the *homogeneous bosonic configuration space* $\mathcal{M}^{(\mathrm{conf})_{\mathrm{hom}}}$. Their quantum operator versions read in the notation introduced in (979):

$$\begin{aligned}\hat{e}^a{}_\alpha &= -\frac{\kappa^2}{2}\sqrt{V}\,\hat{b}^{\mathrm{hom}\,a}{}_\alpha \\ \hat{\psi}^A{}_\alpha &= \sigma_a{}^{AA'}\,\hat{f}^{\mathrm{hom}}{}_\alpha{}^a{}_{A'}\,.\end{aligned} \tag{1011}$$

The (homogeneous part of the) primed supersymmetry constraint obtained from the data (1006)-(1009) is given in [69],eq. (3) as

$$\hat{\bar{S}}^{\mathrm{hom_f},\mathrm{hom_b}}_{(0)A'} = \hat{\psi}^A{}_\alpha \sigma^a{}_{AA'}\left(-\hbar\frac{\partial}{\partial e^a{}_\alpha} + \frac{1}{2}V m^{pq} e_{aq}\right). \tag{1012}$$

Here V is defined by

$$V := \int_\Sigma \omega^1 \wedge \omega^2 \wedge \omega^3\,. \tag{1013}$$

In the notation introduced in (979) this reads

$$\hat{\bar{S}}^{\mathrm{hom_f},\mathrm{hom_b}}_{(0)A'} = \hat{f}^{\mathrm{hom}}{}_\alpha{}^a{}_{A'}\left(\hbar\hat{b}^{\#\mathrm{hom}}{}_a{}^\alpha - V m^{pq}\hat{b}^{\mathrm{hom}}{}_{aq}\right), \tag{1014}$$

where $\kappa^2\sqrt{V}$ has been set to unity

$$\kappa^2\sqrt{V} := 1\,. \tag{1015}$$

For sake of readability we will from now on drop the "hom" superscripts in this example and simply write

$$\hat{\bar{S}}_{A'} = \hat{f}_\alpha{}^a{}_{A'}\left(\hbar\hat{b}^{\#}{}_a{}^\alpha - V m^{pq}\hat{b}_{aq}\right). \tag{1016}$$

The superpotential W (as in (995)) is readily found to be given by

$$\begin{aligned}W &= -\frac{1}{2}V\,m^{\alpha\beta}\,\hat{b}^a{}_\alpha\,\hat{b}_{a\beta} \\ &= -\frac{1}{2}V\,m^{\alpha\beta}\,h_{\alpha\beta} \\ &= -\frac{1}{2}V\,m^\alpha{}_\alpha \\ &:= -\frac{1}{2}V\,m\end{aligned} \tag{1017}$$

(where $m = m(\hat{b}^\cdot.)$ is the trace of the matrix $m^{\alpha\beta}$ with respect to the metric induced by $e^a{}_\alpha$), so that

$$\begin{aligned}\hat{\bar{S}}_{A'} &= e^{-W/\hbar}\,\hat{f}_\alpha{}^a{}_{A'}\hbar\hat{b}^{\#}{}_a{}^\alpha e^{W/\hbar} \\ &:= e^{-W/\hbar}\,\hat{f}^I_{A'}\hbar\hat{b}^{\#}{}_I e^{W/\hbar}\,,\end{aligned} \tag{1018}$$

where in the last line we have introduced multi-indices $I = (a, \alpha)$. Using these one can conveniently identify the standard objects of exterior calculus on configuration space as in (981):

$$\begin{aligned}
\hat{X}^I &:= \hat{b}^I \\
\partial_I &:= \hat{b}^{\#}{}_I \\
\hat{\mathrm{c}}^{\dagger I}{}_{A'} &:= \hat{f}^I_{A'} \\
\mathbf{d}_{A'} &:= \hat{\mathrm{c}}^{\dagger I}{}_{A'} \partial_I \,.
\end{aligned} \tag{1019}$$

(But note again, as in (982), that the $\hat{\mathrm{c}}^{\dagger I}{}_{A'}$ are in general not linearly independent. They become so only when the model is further restricted. See 4.53 (p.241).) With this notation (1018) can finally be written in the standard form

$$\hat{\bar{S}}_{A'} = e^{-W/\hbar}\, \mathbf{d}_{A'}\, e^{W/\hbar} \tag{1020}$$

of a deformed exterior derivative (*cf.* 4.24 (p.203)) on configuration space $\mathcal{M}^{(\mathrm{conf})}_{\mathrm{hom}}$.

The primed supersymmetry generator of the general class A Bianchi model has been identified in its representation on $\Lambda\left(\mathcal{M}^{(\mathrm{conf})}_{\mathrm{hom}}\right)$, the exterior bundle over the homogeneous bosonic configuration space. Following the general course of argument detailed in 4.32 (p.215) one now has to specify a metric G_{IJ} on $\mathcal{M}^{(\mathrm{conf})}_{\mathrm{hom}}$ in order to find the inner product $\langle\cdot|\cdot\rangle$ for physical states as well as the unprimed supersymmetry generator $\hat{S}_A := \hat{S}^{\mathrm{hom_f},\mathrm{hom_b}}_{(0)A}$ by way of taking the adjoint of $\hat{\bar{S}}$ with respect to this inner product:

$$\hat{S}_A = \left(\hat{\bar{S}}_{A'}\right)^\dagger \,. \tag{1021}$$

(Note that we are still using the convention (1016) and suppress "hom"-superscripts in the context of the present example.)

4.3.3 Supersymmetric quantum cosmology as SQM in configuration space

According to 4.27 (p.208) the metric on configuration space is closely related to the matrix ${D^{AA'}}_{ij}$ defined in 4.12 (p.189). For the present homogeneous Bianchi cosmology this matrix is found in [70], eq. (5) to read:

$$ {D^{AA'}}_{qp} = \frac{1}{V}\left(\frac{i}{\sqrt{h}}h_{pq}n^a - \epsilon_{pqr}e^{ra}\right){\bar\sigma_a}^{A'A} \,. \tag{1022}$$

The adjoint of ${\hat\psi^A}_\alpha$ (1011) is, according to (882) in 4.27 (p.208), given by

$$\begin{aligned} {\bar\psi^{A'}}_\alpha &= \left({\hat\psi^A}_\alpha\right)^\dagger \\ &= -i{D^{AA'}}_{ji}\frac{\partial}{\partial {\psi^A}_j}\,. \end{aligned}\tag{1023}$$

With the relations (1019) and (1011) this gives the metric tensor G_{IJ} on configuration space via

$$\begin{aligned} \hat{c}^I_A &= \left({\hat{c}^{\dagger I}}_{A'}\right)^\dagger \\ &= G^{IJ}\left(\hat{c}_A\right)_J\,. \end{aligned}\tag{1024}$$

However, since G_{IJ} is essentially the truncated DeWitt metric, it may in practice be obtained more directly and more elegantly from the knowledge of the *bosonic*, i.e. the ordinary, Wheeler-DeWitt equation:

4.51 (Determination of G_{IJ} from the ordinary Wheeler-DeWitt equation) The operators $\hat H_{AA'}$ obtained from (996) are linear combinations of the Hamiltonian $\hat H_\perp$ and of the diffeomorphism generators $\hat H_\alpha$. The latter are first order in the partial derivatives $\frac{\partial}{\partial {e^a}_\alpha}$ and hence also first order in the ∂_I. The Hamiltonian generator however contains a bilinear term in ∂_I. This term can be found from

$$\begin{aligned} \hat H_{11'} &= \left\{e^{-W/\hbar}\hbar\,\mathbf{d}_{1'}\,e^{W/\hbar}, e^{W/\hbar}\hbar\,{\mathbf{d}^\dagger}_1\,e^{-W/\hbar}\right\} \\ &= \hbar^2\left\{\mathbf{d}_{1'},{\mathbf{d}^\dagger}_1\right\}+\cdots \\ &= -\hbar^2\underbrace{\left\{{\hat{c}^{\dagger I}}_{1'},\hat{c}^J_1\right\}}_{=G^{IJ}_1}\partial_I\partial_J+\cdots \\ &= -\hbar^2 G^{IJ}_1\partial_I\partial_J+\cdots\,, \end{aligned}\tag{1025}$$

where the ellipsis $\cdots$ indicates terms with derivatives of first or zeroth order. Similarly one has

$$ \hat H_{22'} = -\hbar^2 G^{IJ}_2\partial_I\partial_J+\cdots\,. \tag{1026}$$

Since, by the above argument, the bilinear term can only come from the Hamiltonian generator

$$ \hat H_\perp = -\hbar^2 G^{IJ}_2\partial_I\partial_J+\cdots\,, \tag{1027}$$

it follows that

$$G^{IJ} = G_1^{IJ} = G_2^{IJ} \,. \tag{1028}$$

Hence the inverse metric G^{IJ} on configuration space, which is also involved in the anticommutators

$$\begin{aligned} \left\{\hat{c}^{\dagger}{}^{I}_{1'}, \hat{c}^{J}_{1}\right\} &= G^{IJ} \\ \left\{\hat{c}^{\dagger}{}^{I}_{2'}, \hat{c}^{J}_{2}\right\} &= G^{IJ} \,, \end{aligned} \tag{1029}$$

can be read off from the *symbol* (in the sense of differential operators) of the ordinary (bosonic) Hamiltonian generator: It is the well known DeWitt metric (or rather its projection on the homogeneous sector).

One more crucial part of information can now be obtained from the ordinary Hamiltonian:

4.52 (Determination of the Superpotential from the ordinary WDW equation) When the metric $G_{(n)(m)}$ and the *ordinary* potential V entering the Hamiltonian are known, the superpotential W can be obtained by solving the equation (*cf.* 2.2.2 (p.61))

$$V = G^{(n)(m)} \left(\partial_{(n)} W\right) \left(\partial_{(M)} W\right) \,. \tag{1030}$$

This equation will in general have two independent solutions W_1, W_2.

This implies that given any ordinary, i.e. bosonic, mini-superspace Hamiltonian

$$H = -\hbar^2 G^{(n)(m)} \partial_{(n)} \partial_{(m)} + V + \cdots$$

one finds a supersymmetric extension of the model, given by the homogeneous supercharges (995) (see 4.48 (p.232) for more details) by choosing one of the two possible superpotentials. Since the resulting super-Hamiltonians associated with both choices differ only in their purely fermionic component

$$\left[\hat{c}^{\dagger (n)}, \hat{c}^{(m)}\right] \left(\nabla_{(m)} \partial_{(n)} W\right) \,,$$

the two choices of the superpotential should correspond to two different models for the gravitino field that go along with the chosen model for the graviton field that led to the bosonic Hamiltonian.

This is a very convenient circumstance. It has been employed e.g. in [25] to systematically find the supersymmetric extensions of all "diagonal" Bianchi-cosmologies.

The further simplification that leads to diagonal models is discussed now:

4.53 (Diagonal homogeneous models) One may choose to work not with all homogeneous modes in general but with a *diagonal* vielbein field (*cf.* [25]) where only the homogeneous $e^1{}_1, e^2{}_2, e^3{}_3$ components are taken into account in the ansatz 4.49 (p.236), instead of all homogeneous modes. This gives rise to an interesting effect with respect to the associated fermionic modes: The homogeneous $\psi^A{}_\alpha$ are 6 independent modes (2 values of A times 3 values of α), so that there are exactly two fermionic modes associated with each of the three $e^1{}_1, e^2{}_2, e^3{}_3$. In particular one finds from (981) that

$$\mathbf{d}^{\text{hom}}_{A'} = \hat{f}^{\text{hom}}{}_1{}^1{}_{A'}\hbar\hat{b}^{\text{hom}}{}_1{}^1 + \hat{f}^{\text{hom}}{}_2{}^2{}_{A'}\hbar\hat{b}^{\text{hom}}{}_2{}^2 + \hat{f}^{\text{hom}}{}_2{}^2{}_{A'}\hbar\hat{b}^{\text{hom}}{}_2{}^2 + \cdots\,, \tag{1031}$$

and, because of

$$\hat{f}^{\text{hom}}{}_\alpha{}^a{}_{A'} = \hat{\psi}^{\text{hom}A}{}_\alpha\sigma^a_{AA'}\,, \tag{1032}$$

one has the two distinct sets of fermions

$$\begin{aligned}
\hat{f}^{\text{hom}}{}_1{}^1{}_{1'} &= \frac{1}{\sqrt{2}}\hat{\psi}^{\text{hom}2}{}_1\\
\hat{f}^{\text{hom}}{}_2{}^2{}_{1'} &= \frac{1}{\sqrt{2}}i\hat{\psi}^{\text{hom}2}{}_2\\
\hat{f}^{\text{hom}}{}_3{}^3{}_{1'} &= \frac{1}{\sqrt{2}}\hat{\psi}^{\text{hom}1}{}_3
\end{aligned} \tag{1033}$$

and

$$\begin{aligned}
\hat{f}^{\text{hom}}{}_1{}^1{}_{2'} &= \frac{1}{\sqrt{2}}\hat{\psi}^1{}_1\\
\hat{f}^{\text{hom}}{}_2{}^2{}_{2'} &= -\frac{1}{\sqrt{2}}i\hat{\psi}^1{}_2\\
\hat{f}^{\text{hom}}{}_3{}^3{}_{2'} &= -\frac{1}{\sqrt{2}}\hat{\psi}^2{}_2\,.
\end{aligned} \tag{1034}$$

Under the restriction to diagonal modes we hence drop the multi-index used in (1019) and write for short

$$\begin{aligned}
\hat{c}^{(1)}_{A'} &:= \hat{f}^{\text{hom}}{}_1{}^1{}_{A'}\\
\hat{c}^{(2)}_{A'} &:= \hat{f}^{\text{hom}}{}_2{}^2{}_{A'}\\
\hat{c}^{(3)}_{A'} &:= \hat{f}^{\text{hom}}{}_3{}^3{}_{A'}\,.
\end{aligned} \tag{1035}$$

By the above consideration these fermionic operators, together with their adjoints $\hat{\mathrm{c}}^n_A = (\hat{c}^n_{A'})^\dagger$, constitute two anticommuting copies of the canonical creation and annihilation algebra with the non-vanishing anticommutator being (eg 4.55 (p.244) below for index conventions):

$$\begin{aligned}
\left\{\hat{\mathrm{c}}^{(\mu)}_A, \hat{c}^{(\nu)}_{A'}\right\} &= \delta_{AA'}\,G^{\mu\nu}\\
\Leftrightarrow \left\{\hat{\mathrm{c}}_{A\mu}, \hat{c}^{(\nu)}_{A'}\right\} &= \delta_{AA'}\,\delta^\nu_\mu\,.
\end{aligned} \tag{1036}$$

(This is in agreement with the equation given for these anticommutators in [112], eq.(8).) The expression $\delta_{AA'}$, which may be viewed as

$$\delta_{AA'} = \sqrt{2}\sigma^0{}_{AA'} \tag{1037}$$

if preferred, is of course not Lorentz covariant. It appears here because the restriction to diagonal vielbein elements implicitly *fixes a Lorentz gauge* and all consideration within the diagonally restricted model are to be understood with respect to this gauge.

For the rest of this section we will be concerned exclusively with diagonal homogeneous models. To ease the notation, various now redundant "hom"-labels will be suppressed (there should be no source of ambiguity):

4.54 (Notational convention for diagonal homogeneous models) Let $(\lambda), (\mu), (\nu) \in \{1, 2, 3\}$ be mode indices labeling the homogeneous and diagonal modes. Since we are working in a fixed Lorentz gauge it is convenient to *suppress primes on spinor indices*. This helps to simplify expressions in which $A = A'$ is understood.
Identify

$$\begin{aligned}
\hat{x}^{(1)} &:= e^{\mathrm{hom}1}{}_1 \\
\hat{x}^{(2)} &:= e^{\mathrm{hom}2}{}_2 \\
\hat{x}^{(3)} &:= e^{\mathrm{hom}3}{}_3 \\
\\
\hat{c}_A^{(1)} &:= \hat{f}^{\mathrm{hom}}{}_1{}^1{}_{A'} \\
\hat{c}_A^{(2)} &:= \hat{f}^{\mathrm{hom}}{}_2{}^2{}_{A'} \\
\hat{c}_A^{(3)} &:= \hat{f}^{\mathrm{hom}}{}_3{}^3{}_{A'} \,.
\end{aligned} \tag{1038}$$

Let

$$\left(\mathcal{M}^{(\mathrm{conf})}_{\mathrm{hom,diag}}, G_{(\mu)(\nu)}\right) \tag{1039}$$

be the (3-dimensional) bosonic configuration space ("minisuperspace"), obtained by restriction to homogeneous and diagonal models. On this the exterior derivatives are defined by (see B.13 (p.304)):

$$\begin{aligned}
\mathbf{d}_{A'} &:= \hat{c}^{\dagger(\mu)}_{A'} \hat{\nabla}_{(\mu)} \\
&= \hat{c}^{\dagger(\mu)}_{A} \partial_{(\mu)} \\
\mathbf{d}^{\dagger}{}_A &:= -\hat{c}^{(\mu)}_{A} \hat{\nabla}_{(\mu)} \\
&= -\hat{c}^{(\mu)}_{A} \left(\partial_{(\mu)} - \Gamma_{(\mu)(\lambda)(\nu)} \hat{c}^{\dagger(\nu)}_{A'} \hat{c}^{\lambda}_{A}\right) ,
\end{aligned} \tag{1040}$$

where Γ is the Levi-Civita connection of the metric $G_{(\mu)(\nu)}$ (1039) on minisuperspace[57]. Note that (1036) implies that for $A \neq B$

$$\begin{aligned}
\left\{\mathbf{d}_A, \hat{c}^{\dagger(\mu)}_B\right\} = \left\{\mathbf{d}_A, \hat{c}_{B(\mu)}\right\} &= 0 \\
\left\{\mathbf{d}^{\dagger}{}_A, \hat{c}^{\dagger(\mu)}_B\right\} = \left\{\mathbf{d}^{\dagger}{}_A, \hat{c}_{B(\mu)}\right\} &= 0, \quad A \neq B \,.
\end{aligned} \tag{1041}$$

Denote that part of the homogeneous superpotential (992), which depends solely on homogeneous *and* diagonal modes simply by W:

$$W^{\mathrm{hom_f,hom_b}} = W^{\mathrm{hom_f,hom_b,non-diag}} + W \,. \tag{1042}$$

[57] The fact that $\Gamma_{(\mu)(\lambda)(\nu)} \hat{c}^{\dagger(\nu)}_A \hat{c}^{\lambda}_A$ does not mix the two anticommuting copies, $\hat{c}^{\dagger(\mu)}_1$ and $\hat{c}^{\dagger(\mu)}_2$, of the exterior algebra is consistent with the fact that the metric on full configuration space (see 4.27 (p.208)) should be Kähler. For a Kähler metric the only non-vanishing components of the Levi-Civita connection are $\Gamma_j{}^i{}_k$ and $\Gamma_{\bar{j}}{}^{\bar{i}}{}_{\bar{k}}$, where unbarred and barred indices run over holomorphic and antiholomorphic components, respectively (*cf.* e.g. [52]), which must be identified with the $A, A' = 1$ and $A, A' = 2$ components, respectively, in this simplified model.

In close analogy to (995) the homogeneous and diagonal part of the supersymmetry generator is obtained as

$$\begin{aligned} \bar{S}_A^{\text{hom,diag}} &= \mathbf{d}_A^W \\ S_A^{\text{hom,diag}} &= \mathbf{d}_A^{\dagger W} , \end{aligned} \tag{1043}$$

where

$$\begin{aligned} \mathbf{d}_A^W &:= e^{-W}\,\mathbf{d}_A\,e^{W} \\ &= \mathbf{d}_A + \hat{c}_A^{\dagger(\mu)}\left(\partial_{(\mu)}W\right) \\ \mathbf{d}_A^{\dagger W} &:= e^{W}\,\mathbf{d}_A^{\dagger}\,e^{-W} \\ &= \mathbf{d}_A^{\dagger} + \hat{c}_A^{(\mu)}\left(\partial_{(\mu)}W\right) . \end{aligned} \tag{1044}$$

Here $\hbar = 1$ has been set for convenience.

With these definitions made one may analyse the structure of the theory of homogeneous and diagonal supersymmetric cosmological models:

4.55 (Superalgebra for homogeneous and diagonal models) The operators (1044) constitute two copies, one for each Weyl-spinor index, of the Witten model generators (*cf.* 2.2.2 (p.61)). The supercommutators among operators with the same spin index follow trivially from those where only one copy of the algebra is present (see 2.62 (p.61)). Some "mixed" supercommutators of interest are the following: Let $A \neq B$, then:

$$\begin{aligned} \left\{\mathbf{d}_B^W, \mathbf{d}_A^{\dagger W}\right\} &= \Big\{\hat{c}_B^{\dagger(\mu)}\partial_{(\mu)} + \hat{c}_B^{\dagger(\mu)}\left(\partial_{(\mu)}W\right) , \\ &\qquad -\hat{c}_{A(\lambda)}g^{\lambda\nu}\partial_{(\nu)} + \hat{c}_{A(\lambda)}\hat{c}_A^{\dagger\kappa}\hat{c}_{A\nu}\Gamma^{(\lambda)(\nu)}{}_{(\kappa)} + \hat{c}_A^{(\nu)}\left(\partial_{(\nu)}W\right)\Big\} \\ &\overset{A\neq B}{=} -\hat{c}_B^{\dagger\mu}\hat{c}_{A\lambda}\left(\partial_{(\mu)}g^{(\lambda)(\nu)}\right)\partial_{(\nu)} + \hat{c}_B^{\dagger(\mu)}\hat{c}_{A(\lambda)}\hat{c}_A^{\dagger(\kappa)}\hat{c}_{A(\nu)}\left(\partial_{(\mu)}\Gamma^{(\lambda)(\nu)}{}_{(\kappa)}\right) + \\ &\qquad +2\hat{c}_B^{\dagger\mu}\hat{c}_A^{\nu}\left(\partial_\mu\partial_\nu W\right) \end{aligned} \tag{1045}$$

$$\begin{aligned} \left[\mathbf{d}_A^W, \left\{\mathbf{d}_{B'}^W, \mathbf{d}_A^{\dagger W}\right\}\right] &= \hat{c}_B^{\dagger\mu}\left(\partial_{(\mu)}g^{(\lambda)(\nu)}\right)\partial_{(\nu)}\partial_{(\lambda)} - \hat{c}_A^{\dagger(\rho)}\hat{c}_B^{\dagger\mu}\hat{c}_{A\lambda}\left(\partial_{(\rho)}\partial_{(\mu)}g^{(\lambda)(\nu)}\right)\partial_{(\nu)} - \\ &\quad -\hat{c}_B^{\dagger(\mu)}\hat{c}_A^{\dagger(\kappa)}\hat{c}_{A(\nu)}\left(\partial_{(\mu)}\Gamma^{(\lambda)(\nu)}{}_{(\kappa)}\right)\left(\partial_{(\lambda)} + \left(\partial_{(\lambda)}W\right)\right) - \\ &\quad -\hat{c}_B^{\dagger(\mu)}\hat{c}_{A(\lambda)}\hat{c}_A^{\dagger(\kappa)}\left(\partial_{(\mu)}\Gamma^{(\lambda)(\nu)}{}_{(\kappa)}\right)\left(\partial_{(\nu)} + \left(\partial_{(\nu)}W\right)\right) + \\ &\quad +\hat{c}_A^{\dagger(\rho)}\hat{c}_B^{\dagger(\mu)}\hat{c}_{A(\lambda)}\hat{c}_A^{\dagger(\kappa)}\hat{c}_{A(\nu)}\left(\partial_{(\rho)}\partial_{(\mu)}\Gamma^{(\lambda)(\nu)}{}_{(\kappa)}\right) + \\ &\quad +2\hat{c}_A^{\dagger\lambda}\hat{c}_B^{\dagger\mu}\hat{c}_A^{\nu}\left(\partial_\lambda\partial_\mu\partial_\nu W\right) - 2\hat{c}_{B'}^{\dagger\mu}\left(\partial_\mu\partial^\nu W\right)\left(\partial_\nu + \left(\partial_\nu W\right)\right) \end{aligned} \tag{1046}$$

Literature. Expression (1045) was first given, in the context of supergravity and for a flat minisuperspace metric, in [112], eq. (16). Note, however, when comparing this reference with the present discussion, that the choice of homogeneous constraints used in [112] differ from those used here.

We are now in the position to apply the general construction 4.35 (p.218) to the simple case at hand:

4.56 (Conditions on solutions to the constraints) We are looking for Lorentz invariant solutions $|\phi_{\mathrm{hom}}\rangle$ to the constraints

$$\begin{aligned} \mathbf{d}_{A'} \, |\phi_{\mathrm{hom}}\rangle &= 0 \\ \mathbf{d}^{\dagger}{}_{A} \, |\phi_{\mathrm{hom}}\rangle &= 0, \quad A, A' \in \{1,2\} \, . \end{aligned} \tag{1047}$$

Since $|\phi_{\mathrm{hom}}\rangle$ is hence supposed to be exact with respect to $\mathbf{d}^W_{A'}$ it must

- either be a 0-form
$$\hat{c}^{\mu}_{A} \, |\phi_{\mathrm{hom}}\rangle = 0, \quad \forall \, \mu, A \, ,$$
- or be locally exact with respect to $\mathbf{d}^W_{1'}$ *and* (to ensure Lorentz invariance) $\mathbf{d}^W_{2'}$:
$$\begin{aligned} |\phi_{\mathrm{hom}}\rangle &= 2\mathbf{d}^W_{1'} \mathbf{d}^W_{2'} \, |\phi_{\mathrm{hom},0}\rangle \\ &= \mathbf{d}^W_{A'} \mathbf{d}^{W A'} \, |\phi_{\mathrm{hom},0}\rangle \, , \end{aligned} \tag{1048}$$
where $|\phi_{\mathrm{hom},0}\rangle$ is itself Lorentz invariant.

Since the constraints (1047) hold, when they hold, in every fermion-number sector separately, different cases may be discussed:

- *0-fermion sector:* If $|\phi_{\mathrm{hom}}\rangle = |\phi_{\mathrm{hom},0}\rangle$ is a 0-fermion state, the two unprimed constraints
$$\mathbf{d}^{\dagger}{}^W_A \, |\phi_{\mathrm{hom},0}\rangle = 0$$
are automatically satisfied for any $|\phi_{\mathrm{hom},0}\rangle$ and hence vacuous. Furthermore, the two primed constraints
$$\begin{aligned} \mathbf{d}^W_{A'} \, |\phi_{\mathrm{hom},0}\rangle &= 0 \\ \Leftrightarrow \hat{c}^{\dagger\mu}_{A'} \partial_\mu \, e^W \, |\phi_{\mathrm{hom},0}\rangle &= 0 \end{aligned}$$
are *uniquely* solved by
$$|\phi_{\mathrm{hom},0}\rangle \quad \propto \quad e^{-W} \, |0\rangle \, . \tag{1049}$$
This is the well known semiclassical-like type of state discussed in detail e.g. in [25] and references therein.
- *2-fermion sector:* Let $|\phi_{\mathrm{hom}}\rangle = 2\mathbf{d}^W_{1'} \mathbf{d}^W_{2'} \, |\phi_{\mathrm{hom},0}\rangle$ be a 2-fermion state, i.e.
$$\begin{aligned} \hat{N} \, |\phi_{\mathrm{hom}}\rangle &= 2 \, |\phi_{\mathrm{hom}}\rangle \\ \Leftrightarrow \hat{N} \, |\phi_{\mathrm{hom},0}\rangle &= 0 \, . \end{aligned}$$
The primed supersymmetry constraints are automatically solved by construction of the ansatz (1048). The unprimed supersymmetry constraints yield for $|\phi_{\mathrm{hom},0}\rangle$ the condition
$$\begin{aligned} & \mathbf{d}^{\dagger}{}^W_A \mathbf{d}^W_{A'} \mathbf{d}^{W A'} \, |\phi_{\mathrm{hom},0}\rangle = 0 \\ \Leftrightarrow \quad & e^W \, \mathbf{d}^{\dagger}{}_A \, e^{-2W} \, \mathbf{d}_{A'} \mathbf{d}^{A'} \, e^W \, |\phi_{\mathrm{hom},0}\rangle = 0 \, . \end{aligned} \tag{1050}$$
With the general ansatz
$$|\phi_{\mathrm{hom},0}\rangle := e^{-W} \, |\phi'_{\mathrm{hom},0}\rangle \tag{1051}$$

this becomes

$$\begin{aligned}
&\Leftrightarrow \quad \left(\mathbf{d}^\dagger{}_A + 2\hat{c}^\mu_A \left(\partial_\mu W\right)\right) \mathbf{d}_{A'} \mathbf{d}^{A'} \left|\phi'_{\mathrm{hom},0}\right\rangle = 0 \\
&\Leftrightarrow \quad \left(\frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} - 2\left(\partial_\mu W\right)\right) \partial^\mu \, \partial_\nu \left|\phi'_{\mathrm{hom},0}\right\rangle = 0, \quad \forall\, \nu
\end{aligned} \tag{1052}$$

These relations allow one to solve the supersymmetry constraints for $|\phi_{\mathrm{hom}}\rangle$ by finding a solution $|\phi_{\mathrm{hom},0}\rangle$ to a second order differential equation. While this is conceptually important, in particular because it sheds light on the number of solutions one may principally expect to find (*cf.* [68], [70]), one thereby loses the advantage of working only with first order differential operators. Other methods, that do not involve second order differential equations, are discussed below:

4.57 (Closing $N = 2$ solutions to obtain $N = 4$ solutions) Consider the following ansatz: Let $|\phi_{\mathrm{hom},1}\rangle$ be a one fermion state of the form

$$|\phi_{\mathrm{hom},1}\rangle = \mathbf{d}^W_{1'} \, |\phi_{\mathrm{hom},0}\rangle$$

which solves the $A = A' = 1$ components of the supersymmetry constraints:

$$\begin{aligned}
\mathbf{d}^W_{1'} \, |\phi_{\mathrm{hom},1}\rangle &= 0 \\
\mathbf{d}^\dagger{}^W_1 \, |\phi_{\mathrm{hom},1}\rangle &= 0\,.
\end{aligned} \tag{1053}$$

Under what conditions is the state $|\phi_{\mathrm{hom},2}\rangle$ obtained from $|\phi_{\mathrm{hom},1}\rangle$ by "closing" it with respect to $\mathbf{d}^W_{2'}$

$$|\phi_{\mathrm{hom},2}\rangle \;:=\; \mathbf{d}^W_2 \, |\phi_{\mathrm{hom},1}\rangle \tag{1054}$$

a solution to all constraints $\mathbf{d}^W_{A'} \, |\phi_{\mathrm{hom},2}\rangle = \mathbf{d}^\dagger{}^W_A \, |\phi_{\mathrm{hom},2}\rangle$?

To answer this question the following may be noted: Since

$$\begin{aligned}
0 &= \mathbf{d}^\dagger{}^W_1 \, |\phi_{\mathrm{hom},1}\rangle \\
&= \mathbf{d}^\dagger{}^W_1 \mathbf{d}^W_{1'} \, |\phi_{\mathrm{hom},1}\rangle \\
&= \left(\frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} \partial^\mu + \left(\partial_\mu W\right)\left(\partial^\mu W\right) - \left(\nabla_\mu \partial^\mu W\right)\right) |\phi_{\mathrm{hom},0}\rangle \\
&= \mathbf{d}^\dagger{}^W_2 \mathbf{d}^W_{2'} \, |\phi_{\mathrm{hom},0}\rangle
\end{aligned} \tag{1055}$$

the state $|\phi_{\mathrm{hom},0}\rangle$ solves the $\left\{\mathbf{d}^W_{1'}, \mathbf{d}^W_1\right\} = \hat{\mathbf{H}}_{11'}$ and $\left\{\mathbf{d}^W_{2'}, \mathbf{d}^W_2\right\} = \hat{\mathbf{H}}_{22'}$ homogeneous Hamiltonian constraints automatically. Therefore one has

$$\begin{aligned}
0 \stackrel{!}{=} \mathbf{d}^\dagger{}^W_2 \, |\phi_{\mathrm{hom},2}\rangle &= \mathbf{d}^\dagger{}^W_2 \mathbf{d}^W_{1'} \mathbf{d}^W_{2'} \, |\phi_{\mathrm{hom},0}\rangle \\
&= \left\{\mathbf{d}^\dagger{}^W_2, \mathbf{d}^W_{1'}\right\} \mathbf{d}^W_{2'} \, |\phi_{\mathrm{hom},0}\rangle - \mathbf{d}^W_{1'} \underbrace{\mathbf{d}^\dagger{}^W_2 \mathbf{d}^W_{2'} \, |\phi_{\mathrm{hom},0}\rangle}_{=0} \\
&= \left[\left\{\mathbf{d}^\dagger{}^W_2, \mathbf{d}^W_{1'}\right\}, \mathbf{d}^W_{2'}\right] |\phi_{\mathrm{hom},0}\rangle\ .
\end{aligned} \tag{1056}$$

This expression can be found from (1046). To get a qualitative insight into this constraint consider the special case where the metric is Lorentzian

$$g_{(\mu)(\nu)} = \eta_{(\mu)(\nu)} \,.$$

Then the above is equal to

$$\cdots \stackrel{g_{(\mu)(\nu)}=\eta_{(\mu)(\nu)}}{=} 2\hat{c}^{\dagger\mu}_{1'} \left(\partial_\mu \partial^\nu W\right)\left(\partial_\nu + (\partial_\nu W)\right)|\phi_{\mathrm{hom},0}\rangle \,. \tag{1057}$$

Analogously one finds

$$\mathbf{d}^{\dagger W}_{1} |\phi_{\mathrm{hom},2}\rangle = 2\hat{c}^{\dagger\mu}_{1'} \left(\partial_\mu \partial^\nu W\right)\left(\partial_\nu + (\partial_\nu W)\right)|\phi_{\mathrm{hom},0}\rangle \,. \tag{1058}$$

It follows that

$$\begin{aligned} &\left(\mathbf{d}^{\dagger W}_{A} |\phi_{\mathrm{hom},2}\rangle = 0\right) \\ \Leftrightarrow \quad &\left(\left(\partial_\mu \partial^\nu W\right)\left(\partial_\nu + (\partial_\nu W)\right)|\phi_{\mathrm{hom},0}\rangle = 0, \ \ \forall\, \mu\right) \end{aligned} \tag{1059}$$

is the condition on that has to be imposed in addition to (1053). The matrix $(\partial_\mu \partial_\nu W)$ is symmetric and may hence be diagonalized. Some of the diagonal entries may be zero. The above condition says that $e^{-W}|\phi_{\mathrm{hom},0}\rangle$ may only depend on the coordinates corresponding to these entries. Except when $(\partial_\mu \partial^\nu W)$ is highly degenerate this severely restricts the set of admissible $|\phi_{\mathrm{hom},0}\rangle$.

4.58 (Remark) Maybe this fact seems to be in contradiction with the general argument in 4.35 (p.218), that $|\phi_0\rangle$ should be a solution of the Hamiltonian and diffeomorphism constraints. But note with 4.36 (p.219) that this assumed that the commutator (1046) is proportional to the Lorentz generators. For a certain factor ordering, this is known to be true for the full theory (*cf.* [68]) and for general homogeneous models (*cf.* [70]). But already this latter case does not self-evidently follow from the former, because the restriction to homogeneous models involves a truncation of the supersymmetry generators (as discussed in 4.48 (p.232) and 4.49 (p.236)). After such a truncation there is no a-priori guarantee that the truncated operator algebra still closes up to terms proportional to (truncated) Lorentz generators. And indeed, while it turns out that it still does so in the case of general homogeneous models, the above calculation shows that it no longer does in the case of *diagonal* homogeneous models. (This problem is also discussed in [112].) This may have been expected on the grounds that the diagonal homogeneous model is restricted to a context in which a Lorentz gauge has been fixed (see the discussion at equation (1037) in 4.53 (p.241)). Maybe this indicates that a consistent treatment of diagonal homogeneous models in supergravity should be subjected to a modified, possibly reduced, set of constraints. As has been emphasized in 4.48 (p.232) and 4.49 (p.236), this ambiguity is ultimately due to the fact that the step from the full theory to truncated models involves some (more or less) arbitrary choice of which constraints one should impose on the first factor $|\phi_{\mathrm{hom}}\rangle$ in equation (1000). This choice is not obviously demanded by the theory, but is a part of the model. It needs to be put in by hand. The whole problem amounts to the question, which (more or less arbitrary) choice of factoring (1000) together with which (also more or less arbitrary) choice of constraints (998) gives a good approximation to the full theory. This is a question which deserves further study, but which will not be addressed any further in the present context. (See point (4) in the list of open question in 6.2 (p.291).)

Another way how solutions to only one spin component of the constraints give rise to full solutions is the following:

4.59 (Solving the $A = 1$ constraints on one $A = 2$-component) Consider the following identical transformations true in the present simplified setting with two anticommuting copies of the exterior algebra (1036). Because of (1041) one has:

$$\begin{aligned}
|\phi_{\text{hom},2}\rangle &= \mathbf{d}_1^W \mathbf{d}_2^W \, |\phi_{\text{hom},0}\rangle \\
&= \hat{c}_1^{\dagger\mu} \left(\partial_\mu + (\partial_\mu W)\right) \hat{c}_2^{\dagger\nu} \left(\partial_\nu + (\partial_\nu W)\right) |\phi_{\text{hom},0}\rangle \\
&= -\hat{c}_2^{\dagger\nu} \hat{c}_1^{\dagger\mu} \left(\partial_\mu + (\partial_\mu W)\right) \left(\partial_\nu + (\partial_\nu W)\right) |\phi_{\text{hom},0}\rangle \\
&= -\hat{c}_2^{\dagger\nu} \mathbf{d}_1^W \left(\partial_\nu + (\partial_\nu W)\right) |\phi_{\text{hom},0}\rangle
\end{aligned} \tag{1060}$$

It follows that:

$$\begin{aligned}
& \mathbf{d}^{\dagger}{}_1^W \, |\phi_{\text{hom},2}\rangle = 0 \\
\Leftrightarrow \quad & \hat{c}_2^{\dagger\nu} \underbrace{\mathbf{d}^{\dagger}{}_1^W \mathbf{d}_1^W \left(\partial_\nu + (\partial_\nu W)\right) |\phi_{\text{hom},0}\rangle}_{\text{(a)}} = 0 \,.
\end{aligned} \tag{1061}$$

The term (a) contains no fermions with index 2 so that the above is equivalent to

$$\mathbf{d}^{\dagger}{}_1^W \mathbf{d}_1^W \left(\partial_\nu + (\partial_\nu W)\right) |\phi_{\text{hom},0}\rangle = 0, \quad \forall\, \nu \,. \tag{1062}$$

This means that $\mathbf{d}_1^W \left(\partial_\nu + (\partial_\nu W)\right) |\phi_{\text{hom},0}\rangle$ is a solution to the $A = A = 1$ constraints alone for all ν.

In other words, the solution $|\phi_{\text{hom},2}\rangle$ to the full set of constraints is a linear combination of the $\hat{c}_2^{\dagger\nu}$ times solutions to the $A = 1$ constraints. Hence one may go the other way round:

Find any solution $\left|\phi^1_{\text{hom},1}\right\rangle$ to the $A = 1$ constraints

$$\begin{aligned}
\mathbf{d}_1^W \, |\phi^1_{\text{hom},1}\rangle &= 0 \\
\mathbf{d}_1^{\dagger W} \, |\phi^1_{\text{hom},1}\rangle &= 0 \,.
\end{aligned} \tag{1063}$$

This may be considered as the $\hat{c}_2^{\dagger 1}$ component of an as yet unknown full solution $|\phi_{\text{hom},2}\rangle$ if one can find, in principle, two further $A = 1$ solutions $\left|\phi^2_{\text{hom},1}\right\rangle$ and $\left|\phi^3_{\text{hom},1}\right\rangle$ so that

$$\hat{c}_2^{\dagger 1} \left|\phi^1_{\text{hom},1}\right\rangle + \hat{c}_2^{\dagger 2} \left|\phi^2_{\text{hom},1}\right\rangle + \hat{c}_3^{\dagger 2} \left|\phi^3_{\text{hom},1}\right\rangle$$

is of the form (1060). (Under what conditions this is possible is not investigated here. See item (5) in the list of open question 6.2 (p.291).) But note that even when these are left undetermined one can extract physical information from the state $|\phi_{\text{hom},2}\rangle$: Namely all expectation values which involve the projector on the $\hat{c}_2^{\dagger 1}$ component are already determined by the knowledge of $\left|\phi^1_{\text{hom},1}\right\rangle$ alone: Let $\hat{A}$ be any operator, then obviously

$$\left\langle \hat{c}_{2\,1} \hat{c}_2^{\dagger 1} \hat{A} \hat{c}_2^{\dagger 1} \hat{c}_{2\,1} \right\rangle_{\phi_{\text{hom},2}} = \langle A \rangle_{\phi^1_{\text{hom},1}} \,. \tag{1064}$$

(See §2.3 (p.106) and in particular §2.3.5 (p.140) for more on expectation values in supersymmetric cosmology.)

4.3.4 N-Extended and higher dimensional Canonical Quantum Supergravity

Presently only the $N = 1$, $D = 4$ version of canonical quantum supergravity has been developed in greater detail ([80]). Aspects of this theory have been reviewed and discussed in §4.2 (p.187) and §4.3 (p.192). But of course other supergravity theories exist which involve additional supersymmetry generators, (up to $N = 8$ for $D = 4$) or additional spacetime dimensions (up to $D = 11$ for $N = 1$), or both. (See for instance [252]. A list of relevant introductory literature is given in 5.7 (p.267).) In the present restricted context we of course cannot and will not try to embark on a serious discussion of canonical quantum supergravity for higher N or higher dimensions. But with an eye on applications of the "Hamiltonian" approach (*cf.* p. 9) to supersymmetric quantum cosmology (i.e. by using supersymmetric quantum mechanics in configuration space as discussed in §4.3.3 (p.240)) in the context of 11-dimensional supergravity, we present a conjectural method for tackling the problem of extended supergravity. This method is motivated by and based on the mode-basis representation of $N = 1, D = 4$ supergravity, as discussed in §4.3 (p.192), and on the resulting identification of the supersymmetry generators with generalized exterior derivatives.

As has been shown in §4.3 (p.192), full canonical quantum supergravity in the Schrödinger representation may be regarded as a theory of constrained supersymmetric quantum mechanics (in the sense discussed in detail in §2.2 (p.54)) on infinite dimensional configuration space $\mathcal{M}^{(\mathrm{conf})}$.

But in the context of supersymmetric quantum mechanics the issue of extended supersymmetry is rather well understood (e.g. [101]): As briefly discussed in §2.2.7 (p.90), in particular in 2.99 (p.96), higher N extensions of supersymmetric quantum mechanics on some manifold are in correspondence with higher Kähler symmetries of that manifold. This holds for Riemannian manifolds, but also for semi-Riemannian ones if one appropriately generalizes the notion of complex structure to certain "hidden symmetries" (see §2.2.7 (p.90) for a discussion and in particular 2.97 (p.95) for references to the literature).

Hence from the point of view of the infinite-dimensional SQM perspective on supersymmetric quantum field theory, it seems quite clear what has to happen for a theory to admit higher N supersymmetry: The configuration space has to admit suitable hidden symmetries, namely it has to be Kähler, Hyper-Kähler or even octonionic Hyper-Kähler (or the semi-Riemannian analog thereof). An example of this general fact has already been encountered in §4.3 (p.192) (see 4.27 (p.208) and 4.30 (p.212)), where it was seen that the *four* supersymmetry generators $\bar{S}_{(A'=1)}, \bar{S}_{(A'=2)}, S_{A=1}, S_{A=2}$ went along with a (presumeably) Kähler geometry on configuration space, just as one would expect from the above reasoning.

This general relationship between extended supersymmetry and hidden symmetries on configuration space could allow, this is the conjecture here, to straightforwardly go from $N = 1, D = 4$ canonical supergravity to, say, $N = 2, D = 4$ canonical supergravity along the "Hamiltonian route" (in the nomenclature used in the discussion on p. 9 of the introduction), that is without explicitly considering the respective $N = 2$ field Lagrangian and its canonical quantization, but by instead extending the operator algebra of quantum operators (the quantum supersymmetry and Hamiltonian/diffeomorphism generators): In the extended operator algebra the set of supersymmetry generators must be twice as large as before and must form a representation space of the extended *R-symmetry* under which these supersymmetry generators transform. While "R-symmetry" is a term used in supersymmetric field theory, it is precisely the symmetry group which is generated by the operators associated with complex structures on configuration space [101].

While this is the general picture envisioned here, we will try to make it more concrete only in the very restricted setting of cosmological models. As shown in §4.3.2 (p.230) and §4.3.3 (p.240), after restricting attention to only one mode of the full infinite set of supersymmetry generator modes, the remaining finite number of degrees of freedom are described by supersymmetric quantum mechanics in its usual (i.e. finite, seemingly non-field-theoretic) form. This yields a simple testing ground for the applicability of the above conjectured method for finding higher-N extended supergravities.

In particular, we shall be interested in supersymmetric cosmological models that arise as compactifications of $N = 1, D = 11$ supergravity. This theory contains a single supersymmetry generator with $2^{[11/2]} = 32$ spinor components. As is well known (e.g. [252]), by compactifying the full 11-dimensional theory one obtains lower dimensional supergravity theories with a Kaluza-Klein field content deriving from the components of the higher dimensional fields. In particular, the single 11-dimensional supersymmetry generator spinor breaks up into several, lower component spinors. Since a spinor in 4 spacetime dimensions has $2^{[4/2]} = 4$ components, one can at most obtain $N = 8$ distinct supersymmetry generators after compactifying to 4 dimensions[58]. But whether this maximum number of supersymmetry generators is actually obtained after reducing from 11 to 4 dimensions depends on the precise compactification scheme. Some compactifications *break* supersymmetries, so that one is left with $N < 8$ in 4 dimensions. More precisely, the preservation of higher supersymmetry after compactification requires that the compact dimensions admit *Killing spinors*, i.e. covariantly constant spinor fields. This means that the more symmetric the compact dimensions are, i.e. the more isometries they admit, the more supersymmetry is preserved in the compactified theory.

It seems (as far as I am aware and from what I have learned from personal communications), that it is not known, or even investigated, how this argument carries over to canonical formulations of supergravity and to their canonical quantization. But with respect to the above formulated conjecture and in the context of cosmological models, there seems to be a natural way to approach the situation, which will be discussed now and which is the basis for the supersymmetric quantization of a Bianchi-I model of 11-dimensional supergravity in §5.2 (p.266):

First recall the essential idea of the "Hamiltonian route" to supersymmetric quantum cosmology in the context of $N = 1$, $D = 4$ supergravity as discussed in §4.3.3 (p.240) (*cf.* e.g. [25]). The approach is based on the fact that all one needs to know to construct the *quantum* supersymmetry generator algebra is knowledge of the metric $G_{(m)(n)}$ and of the *bosonic* potential V on configuration space. From this data one finds a superpotential W satisfying the defining

[58]This is the reason why $D = 11$ is the highest dimension in which consistent supergravity is allowed: Namely $N = 8$ is known to be the highest N in $D = 4$ which gives a consistent field theory. Any higher supersymmetry would give rise to particles with spin greater than 2, which is considered to be unphysical.

relation $G^{(n)(m)}W_{,(n)}W_{,(m)} = V$. The quantum operator versions of the supersymmetry generators are constructed by deforming the exterior (co-)derivative on configuration space with this function W. The crucial point here is that from knownledge of the ordinary, bosonic system alone, its supersymmetrically extended quantum dynamics is found in a systematic fashion by looking at the geometry of the bosonic configuration space.

In the light of the above discussion it is immediately clear how this procedure should generalize: For definiteness, consider the bosonic part of the Lagrangian of 11-dimensional supergravity and insert a cosmological ansatz for the 11-dimensional metric (*cf.* §5.2.1 (p.267)). We can consider this ansatz to describe a compactification down to the ordinary 4 spacetime dimensions and hence to yield some $D = 4$ supergravity model, possibly with higher N. After integrating out the spatial coordinates, the reduced action is obtained, which is formally that of a point particle propagating on a semi-Riemannian manifold (mini-superspace), possibly subject to a potential. (In the model considered in §5.2 (p.266) the mini-superspace potential term happens to vanish.) By the above mentioned procedure we may find the supersymmetry generators for this model. But, since compactification of $N = 1$, $D = 11$ supergravity may give higher-N-extended $D = 4$ supergravity, one should check if the dynamics in mini-superspace admits *more* than the usual supercharges, which should be the case when more of the 32 components of the supercharge in 11-dimensions generate symmetries which remain "unbroken" by the particular cosmological ansatz. As has been discussed above, this will be the case exactly if the geometry of mini-superspace, the reduced DeWitt metric, does admit certain "hidden symmetries", namely certain Killing-Yano-tensors. These hidden symmetries may be checked for systematically (see 5.19 (p.284) for an example). If they exists, one can from them construct generators of higher R-symmetry and obtain the further "hidden" supersymmetry generators by looking at the transformation of the known supersymmetry generators under this extended symmetry. This is discussed in 2.98 (p.95) and 2.99 (p.96). For instance, when minisuperspace is a Kähler manifold, the supersymmetry generators are the two holomorphic and two antiholomorphic exterior (co-)derivatives, possibly deformed (*cf.* e.g. [87]).

While this is a well known way to produce supersymmetric quantum mechanics with extended supersymmetry, the conjecture here is that the additionally found supersymmetry generators on mini-superspace are in fact those that correspond to the additional supersymmetry generators of full $N > 1$ supergravity. This seems natural with regard to the insight 4.25 (p.205) that the usual $N = 1$ generators in canonically quantized supergravity can indeed explicitly be shown to be deformed exterior derivatives on configuration space.

Maybe one interesting check on the consistency of our conjecture is the following: As mentioned above, full $D = 4$ supergravity does not exist for $N > 8$. This fact should have a counterpart in the framework of supersymmetric extensions along the "Hamiltonian route". Going from $N = 1$ to $N = 8$ extended supersymmetry in mini-superspace means that enough hidden symmetries have to be present to allow the number of supercharges to increase by a factor of 8. This is possible when mini-superspace is an octonionic-Kähler manifold (e.g. [101]), in which case the DeWitt metric admits 7 covariantly constant Killing-Yano tensors that square to -1 (*cf.* 2.99 (p.96)). But more than 7 such tensors can only be supported by *flat* geometries. This is ultimately related to the fact that there is no division algebra beyond the Octonions. Hence, in generic cases at least, there is also no supersymmetric extension of the mini-superspace dynamics with more than 8 times the standard amount of supersymmetry generators.

While this looks promising, there is a subtlety. By now the reader will have noticed that we have avoided to state what exactly is the “standard amount” of supersymmetry generators on mini-superspace. As discussed in 4.27 (p.208) the configuration space of the full canonical theory of $N = 1$, $D = 4$ supergravity is in fact Kähler and accordingly there are four supersymmetry generators (per mode), $S_1, S_2, \bar{S}_1, \bar{S}_2$. But there is also a redundancy present, in that the Lorentz symmetry has not been modded out. One consequence of this is that, according to 4.39 (p.223), the four supersymmetry generators give rise to only 2 independent supersymmetry constraints. This is notably different from the usual situation in covariant supersymmetric quantum mechanics (§2.2 (p.54)) where, when several supersymmetry generators are present, these in general do induce independent constraints on physical states. Hence it seems reasonable to expected that, once the Lorentz symmetry is dealt with (either by restricting to the Lorentz invariant subspace of states or by fixing a Lorentz gauge) the restricted configuration space is no longer Kähler and the number of supersymmetry generators is halved. How exactly this may come about has been discussed in highly simplied situations in 4.57 (p.246) and 4.59 (p.248) in the context of diagonal homogenous models of $N = 1, D = 4$ supergravity, where mini-superspace is 3-dimensional and hence certainly not Kähler (because a Kähler manifold is necessarily even-dimensional). Of course, as has been stressed there, diagonal homogeneous models involve a lot of simplifying assumptions and hence cannot well serve to prove much about the full theory. But a number of circumstances seem to point in the direction that the true superspace (i.e. configuration space) obtained from $N = 1, D = 4$ canonical supergravity, i.e. that space obtained by dividing configuration space by the Lorentz and the diffeomorphism group, is a Riemannian manifold without special Kähler symmetry. According to our conjecture, any Kähler symmetry of this superspace should correspond to $N > 1$, up to $N = 8$ for octonionic-Kähler symmetry.

Some elements of this discussion are assembled in the following table:

full theory	dimension D of $N = 1$ sugra	$2, 3$	$4, 5$	$6, 7$	$8, 9$	$10, 11$
compactification to $D = 4$	maximal N	–	1	2	4	8
	susy automorphism group	–	$U(1)$	$U(2)$	$U(4)$	$U(8)$
	number of susy generators	(2)	4	8	16	32
	non-redundant susy constraints	(1)	2	4	8	16
Lorentz invariant config. manifold $\mathcal{M}^{(\text{conf})}/L$	geometry of $\mathcal{M}^{(\text{conf})}/L$	S	R	K	HK	OK
	complex structures on $T\mathcal{M}^{(\text{conf})}/L$	0	0	1	3	7
	associated division algebra	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
susy on $\mathcal{M}^{(\text{conf})}/L$	Dirac operators on $\mathcal{M}^{(\text{conf})}$	1	2	4	8	16
	automorphism group	–	U(1)	U(2)	?	?

S = spin geometry, R = Riemannian geometry, K = Kähler geometry, HK = hyper-Kähler, OK = octonionic Kähler

An example where these considerations might apply is the homogeneous model in $N = 1, D = 11$ supergravity, which is discussed in 5.2 (p.266). Here the higher dimensions are compactified on $T^6 \otimes S^1$, and the reduced system may be regarded as a model for $D = 4$ supergravity with additional fields due to the Kaluza-Klein reduction and due to the 3-form field present in 11 dimensions. It is found in 5.19 (p.284) that, indeed, the mini-superspace metric admits "hidden" symmetries in the form of two covariantly constant Killing-Yano tensors which square to minus the identity. The associated hidden supercharges can be calculated and their superalgebra turns out to contain central charges (i.e. even-graded generators that commute with all other generators, *cf.* 2.38 (p.47) and 4.13 (p.193)).

Due to the simplifying assumptions that go into a cosmological model like this, it is, without further investigation, hard to say if the hidden symmetries thus found really have any direct relation to some of the 32 supersymmetry generators in full $N = 1$, $D = 11$ supergravity, as has been conjectured above, or if they are just artifacts of an overly simplistic ansatz, which is very well possible. But at least these findings are consistent with the above idea that dimensional reduction of higher dimensional supergravity gives rise to supersymmetric quantum mechanics in mini-superspace with additional hidden supersymmetries. Further investigations in this direction might be worthwhile. (See point 6 of 6.2 (p.291).)

5 Supersymmetric Quantum Cosmological Models.

Outline. This section applies the theory presented in §2 (p.14) and §4 (p.181) to the quantization of supersymmetric homogeneous cosmological models. First, in §5.1 (p.255), some rather well known examples in $N = 1$, $D = 4$ supergravity are reconsidered. Then in §5.2 (p.266) a more recent model deriving from $N = 1$, $D = 11$ supergravity is analyzed in some detail. Numerical simulations of classical and quantum dynamics provide insight into the nature of possible solutions. The dominating feature of all models, most notably of the eleven-dimensional one, which is found to classically exhibit "Mixmaster"-type behavior, is the presence of exponential (effective) potential walls in mini-superspace at which the "universe point" may scatter. The numerical simulations concentrate on these scattering events and graphical representations of probability amplitudes and probability currents (*cf.* 2.79 (p.78)) are given.

5.1 Cosmological models in $N = 1$, $D = 4$ supergravity.

Outline. The purpose of the following subsection is to give examples for the general considerations presented in §4.3.2 (p.230) and §4.3.3 (p.240) on homogeneous cosmological models in $N = 1$, $D = 4$ supergravity. The discussion is based on some of the mini-superspace Hamiltonians for homogeneous models that are derived and assembled in [25]. By using methods discussed in §2.2.8 (p.100), non-trivial solutions to these models are obtained and approximated numerically. Graphical representations of the associated probability amplitudes and probability currents (see §2.2.4 (p.78)) show interesting behavior of these solutions near reflection points at potential walls in mini-superspace. These investigations serve as a testing ground and preparation for a similar analysis of a model deriving from $D = 11$-supergravity that is given in 5.2 (p.266).

5.1 (Basics of diagonal Bianchi-type cosmology) The models considered here are examples of the homogeneous Bianchi-type models with *diagonal* metric that are discussed in [25]. (See 4.50 (p.237) and 4.53 (p.241) for more information on general and diagonal supersymmetric Bianchi cosmologies.) General Bianchi models are described by spacetime metrics of the form

$$ds^2 \quad = \quad -N^2(t)\, dt^2 + h_{\alpha\beta}(t)\, \boldsymbol{\omega}^\alpha \boldsymbol{\omega}^\beta \,, \tag{1065}$$

where $\boldsymbol{\omega}^i$ are 1-forms invariant under some Lie group of diffeomorphism (*cf.* [269]§7.2 for a detailed discussion). Restriction to such cases where $h = (h_{\alpha\beta})$ can be chosen diagonal gives rise to a 3-dimensional configuration space ('mini-superspace'). Traditionally this is parameterized either by coordinates $\beta^1, \beta^2, \beta^3$ or by α, β_+, β_- as follows:

$$\begin{aligned} h &:= \frac{1}{6\pi}\mathrm{diag}\Big(e^{2\beta^1(t)}, e^{2\beta^2(t)}, e^{2\beta^3(t)}\Big) \\ &:= \frac{1}{6\pi}\mathrm{diag}\Big(e^{2\left(\alpha(t)+\beta_+(t)+\sqrt{3}\beta_-(t)\right)}, e^{2\left(\alpha(t)+\beta_+(t)-\sqrt{3}\beta_-(t)\right)}, e^{2(\alpha(t)-\beta_-(t))}\Big) \,. \end{aligned} \tag{1066}$$

(Still another, slightly different, parameterization is used in example 5.3 (p.257) below.) Entering this ansatz into the *bosonic* part of the action i.e. the Einstein-Hilbert action, yields the following Hamiltonian:

$$\begin{aligned} H &= \frac{1}{2}\left(-p_\alpha^2 + p_+^2 + p_-^2\right) + V^{(0)}(\alpha, \beta_+, \beta_-) \\ &= \frac{3}{2}\left[p_1^2 + p_2^2 + p_3^2 - 2p_1p_2 - 2p_1p_3 - 2p_2p_3\right] + V^{(0)}\left(\beta^1, \beta^2, \beta^3\right) \end{aligned} \tag{1067}$$

Here the superspace potential

$$V^{(0)} = -12\pi^2\sqrt{g}^3 R \tag{1068}$$

(where R is the scalar curvature of the 3-metric) depends on the exact nature of the invariant forms ω and thus on the particular model under consideration.

With the bosonic Hamiltonian known, the associated supersymmetry generators (995), p. 235, of $N = 1$, $D = 4$ supergravity, in their truncated mini-superspace form (see 4.48 (p.232)) are obtained, according to the methods described in 4.51 (p.240) and 4.52 (p.241), by choosing a superpotential W such that

$$V = G^{(n)(m)}\left(\partial_{(n)}W\right)\left(\partial_{(M)}W\right)$$

and setting

$$\begin{aligned} \hat{\bar{S}} &= \exp(-W)\,\mathbf{d}\,\exp(W) \\ &:= \mathbf{d}^W \\ \hat{S} &= \exp(W)\,\mathbf{d}^\dagger\,\exp(-W) \\ &:= {\mathbf{d}^\dagger}^W \end{aligned} \tag{1069}$$

for the completely homogeneous supersymmetry generators $\hat{\bar{S}}$, $\hat{S}$. Here $\mathbf{d}$ and $\mathbf{d}^\dagger$ are the exterior derivative and the exterior co-derivative on mini-superspace (*cf.* 2.2 (p.16)) and differential forms are identified with gravitino mode amplitudes (*cf.* 4.24 (p.203)). But note the following:

5.2 (Spinor components of the supersymmetry generators) Since we are concerned here with *diagonal* homogeneous models there are, according to the considerations in 4.53 (p.241), two anticommuting copies of fermionic creation and annihilation operators. The two Weyl-spinor components of the supersymmetry generators give two copies of the deformed exterior (co-)derivative, one for each copy of the fermionic algebra. Following 4.59 (p.248) it is sufficient to know solutions to only one of the spinor components of the supersymmetry constraints. Therefore we will concentrate in the following on only one spin component of the supersymmetry generators and suppress their spinor index.

The first example is a model of Bianchi-I type, i.e. the spatial section is a flat 3-torus (we consider compact spatial sections only), the dimensions of which vary in time. To make this simple model a little more interesting, a small *inhomogeneous* perturbation is added. This gives rise to a potential term (which otherwise vanishes due to the flatness of space), for which the superpotential is found.

Example 5.3 (Inhomogeneously perturbed Bianchi I model) Consider a cosmological model on the 3-torus

$$\mathcal{M} = T^{(3)} \otimes \mathbb{R}\,.$$

Choose standard coordinates on $T^{(3)}$ with

$$x^{1,2,3} \in [0,1]\ .$$

A natural complete set of scalar modes on $T^{(3)}$, with respect to this coordinate patch, is the usual set of Fourier modes:

$$\begin{aligned}
\phi_{(+n^1,+n2,+n3)}\left(x^1,x^2,x^3\right) &:= N\sin\left(2\pi n^1x^1\right)\sin\left(2\pi n^2x^2\right)\sin\left(2\pi n^3x^3\right)\\
\phi_{(-n^1,+n2,+n3)}\left(x^1,x^2,x^3\right) &:= N\cos\left(2\pi n^1x^1\right)\sin\left(2\pi n^2x^2\right)\sin\left(2\pi n^3x^3\right)\\
\cdots &
\end{aligned} \tag{1070}$$

With t the time coordinate, the Bianchi I model on the torus is defined by the following metric (in Misner parameterization):

$$g^{(\mathrm{BI})}_{\mu\nu}\left(t,x^i\right) = e^{-\Omega(t)}\mathrm{diag}\Big(-N^2(t)\,, e^{\beta_+(t)+\sqrt{3}\beta_-(t)}, e^{\beta_+(t)-\sqrt{3}\beta_-(t)}, e^{-2\beta_+(t)}\Big)\ . \tag{1071}$$

The functions Ω, β_+, β_- may be regarded as (logarithms of) amplitudes of the constant ϕ_0-mode of the gravitational field. N is the lapse function, as usual. In order to perturb this model a little, further amplitudes of higher modes may be taken to be nonvanishing. For instance one could set

$$\begin{aligned}
g^{(\mathrm{pBI})}_{00}\left(t,x^i\right) &= N(t)\exp(-\Omega(t))\\
g^{(\mathrm{pBI})}_{11}\left(t,x^i\right) &= \exp\Big(-\Omega(t)+\beta_+(t)+\sqrt{3}\beta_-(t)+s(t)\sin\left(2\pi x^3\right)+c(t)\cos\left(2\pi x^3\right)\Big)\\
g^{(\mathrm{pBI})}_{22}\left(t,x^i\right) &= \exp\Big(-\Omega(t)+\beta_+(t)-\sqrt{3}\beta_-(t)-s(t)\sin\left(2\pi x^3\right)-c(t)\cos\left(2\pi x^3\right)\Big)\\
g^{(\mathrm{pBI})}_{33}\left(t,x^i\right) &= \exp(-\Omega(t)-2\beta_+(t))
\end{aligned} \tag{1072}$$

(all other components of $g^{(\mathrm{pBI})}$ being zero) with two additional time dependent amplitudes s,c describing an inhomogeneous (x^3-dependent) anisotropy. This ansatz may now be inserted into the Einstein-Hilbert action and then dimensionally reduced. One finds

$$\begin{aligned}
&\int_{\mathcal{M}} \sqrt{\det\left(g^{(\mathrm{pBI})}\right)}R^{(\mathrm{pBI})}\ dx^1\,dx^2\,d^3x\,dt\\
&= \int_{\mathbb{R}} L\left[N,\Omega,\beta_+,\beta_-,s,c,\dot N,\dot\Omega,\dot\beta_+,\dot\beta_-,\dot s,\dot c\right]\ dt + (\text{boundary term})
\end{aligned} \tag{1073}$$

with the Lagrangian

$$L = \frac{3}{2N}e^{-\frac{3}{2}\Omega}\left(-\dot\Omega^2+\dot\beta_+^2+\dot\beta_-^2+\frac{1}{6}\dot s^2+\frac{1}{6}\dot c^2\right) - N\pi^2e^{-\frac{1}{2}\Omega+2\beta_+}\left(s^2+c^2\right)\ . \tag{1074}$$

Except for its N-dependence this Lagrangian is non-singular.
The Hamiltonian associated with (1074) is

$$\begin{aligned} H &= NH_0 \\ &= N\frac{1}{6}e^{\frac{3}{2}\Omega}\left(-p_\Omega^2 + p_{\beta_+}^2 + p_{\beta_-}^2 + 6p_{s^2} + 6p_{c^2} + 6\pi^2 e^{-2\Omega+2\beta_+}\left(s^2+c^2\right)\right) . \end{aligned} \tag{1075}$$

For $s = 0$, $c = 0$ the above constraint H_0 is the usual Hamiltonian constraint of the Bianchi I model, which induces free motion in the reduced configuration space. For nonvanishing inhomogeneity (non-vanishing s, c) configuration space has two more dimensions, coordinatized by s and c, and the motion along these dimensions is subject to an oscillator potential

$$V = 6\pi^2 e^{-2\Omega+2\beta_+}\left(s^2+c^2\right) .$$

Hence the model is stable with respect to perturbations in s and c. Classically, a displacement of s, c away from $s = c = 0$ leads to an oscillation of these parameters (while the universe point traces out its trajectory in configuration space). Furthermore, due to the factor $e^{-2\Omega+2\beta_+}$, the amplitude of these oscillations is smaller for higher anisotropy and for greater scale factor of the universe. In this model an initial inhomogeneity should decrease while the universe is expanding.

Now the above model of ordinary $D = 4$ gravity shall be extended to a model in $D = 4$, $N = 1$ supergravity. By 4.51 (p.240) and 4.52 (p.241), to do so it is sufficient to identify a suitable metric G_{mn} on configuration space as well as a superpotential W. Since one has the freedom to conformally rescale the Hamiltonian constraint, one may choose as a convenient metric, compatible with (1075), the flat Lorentzian metric

$$G(\Omega, \beta_+\beta_-, s, c)_{mn} := \mathrm{diag}\left(-1, 1, 1, \frac{1}{6}, \frac{1}{6}\right) .$$

Hence the (bosonic) Wheeler-deWit constraint operator reads

$$\hat{H}_0^{(\mathrm{bosonic})} = -\hbar^2 G^{mn}\partial_{X^m}\partial_{X^n} + V + \mathcal{O}(\hbar) , \tag{1076}$$

where, of course, $X^m \in \{\Omega, \beta_+\beta_-, s, c\}$. Terms of order $\hbar$ will be fixed by the supersymmetry algebra: The superpotential W has to satisfy the equation

$$G^{mn}\left(\partial_{X^m}W\right)\left(\partial_{X^n}W\right) = V \tag{1077}$$

(*cf.* 2.62 (p.61)). The two solutions are

$$W_\pm = \pm\frac{\pi}{2}e^{-\Omega+\beta_+}\left(s^2+c^2\right) . \tag{1078}$$

From these one finds two different supersymmetric extensions of the above ordinary quantum cosmological model by choosing supersymmetry constraints (*cf.* (866)):

$$\begin{aligned} \bar{S}_\pm &= e^{-W_\pm/\hbar}\,\hbar\mathbf{d}_{\mathcal{M}^{(\mathrm{conf})}}\,e^{W_\pm/\hbar} \\ S_\pm &= e^{W_\pm/\hbar}\,\hbar\mathbf{d}^\dagger{}_{\mathcal{M}^{(\mathrm{conf})}}\,e^{-W_\pm/\hbar} . \end{aligned} \tag{1079}$$

Their anticommutator gives the supersymmetrically extended version of (1076):

$$\hat{H}_{0\pm} = -\hbar^2 G^{mn}\partial_{X^m}\partial_{X^n} + V + \hbar \left[\hat{c}^{\dagger m}, \hat{c}^n\right] (\partial_{X^m}\partial_{X^n} W_\pm) \, . \tag{1080}$$

The two choices ($\pm$) differ by a term of order $\hbar$. From the discussion in the above outline one knows that they correspond to two different models for the gravitino field.

Next we turn to a special case of a diagonal Bianchi cosmology in which mini-superspace is only 1+1 dimensional, namely the Kantowski-Sachs model. Because of the low dimension of mini-superspace the dynamics of these models is directly comparable to the supersymmetric checkerboard models that have been discussed in 2.85 (p.83).

Example 5.4 (Kantowski-Sachs model) This model has only 2 independent parameters, say α, β_+, which represent the scale factor and an anisotropy parameter, respectively. The ordinary Hamiltonian given in [25], eq. (1.14), is

$$H'_{\mathrm{KS}} := \frac{1}{2}\left(-p_\alpha^2 + p_{\beta_+}^2\right) - \frac{3}{2} e^{4\alpha} e^{-2\beta_+} \, . \tag{1081}$$

The supersymmetric extension according to 4.52 (p.241) is accomplished with the choice

$$W_{\mathrm{KS}} \quad := \quad \frac{1}{3} e^{2\alpha - \beta_+} \tag{1082}$$

(*cf.* [25](2.16)).

This superpotential does depend on the time-like configuration space coordinate α, so that, according to §2.2.4 (p.78), 2.79 (p.78), 2.80 (p.78), and 2.81 (p.79), it will not yield a conserved probability current. This is in fact a generic feature of the superpotentials listed in [25], they are all proportional to $e^{2\alpha}$, which is to be expected from 4.5 (p.184). A conformal transformation, as discussed in remark 4.9 (p.186), does not affect the superpotential and thus cannot be used to make it α-independent on the quantum level. (See also the discussion in [203], summarized on page 79, about the failure of certain scalar products in FRW quantum cosmology to be conserved due to α-dependence of the minisuperspace potential.)

In the following we find it helpful, in order to get insight into the model and into the theory, to proceed in two steps: First we *neglect* the α-dependence of the superpotential, investigate the resulting dynamics and then, in a second step where the α-dependence is turned on again, compare it to the "true" dynamics. Neglecting the α-dependence of the potential can physically be justified when the analysis is restricted to a small interval $-\epsilon < \alpha < +\epsilon$ around $\alpha = 0$. The simulations shown in figures 4 (p.262) and 5 (p.263) below involve an α-interval which is probably too large to be a good approximation, but one may regard a small strip of these diagrams along the $\alpha = 0$ axis as giving physically meaningful information.

But a stronger motivation for suppressing the α-dependence for a while is that it gives a useful insight into the general theory. Most notably, it is obvious from the lower diagram in figure 4 (p.262) (the generation of this diagram and the following ones is discussed in the next paragraph 5.5 (p.260)) that the probability current in this case is indeed conserved and the difference between this conserved current and the "true" current for proper α-dependent potential, which is shown in figure 7 (p.265) and which is definitely not conserved, is obvious. While the non-conserved current shows an increase in current *amplitude* as soon as the "time"-dependent increase of the potential becomes noticeable, the current *direction* in both cases shows the same underlying reflection phenomenon.

So consider first the approximate potential

$$W_{\mathrm{KS}} \stackrel{\alpha\to 0}{\longrightarrow} \frac{1}{3}e^{-\beta_+} \,. \tag{1083}$$

The Witten-Dirac operator associated with this potential reads

$$\mathbf{D} \;=\; \hat{\gamma}^0_-\partial_0 + \hat{\gamma}^1_-\partial_1 - \hat{\gamma}^1_+ c e^{-x^1} \,, \tag{1084}$$

where the identification

$$\begin{array}{ccc} \alpha & \leftrightarrow & x^0 \\ \beta & \leftrightarrow & x^1 \end{array}$$

is used and where c is some constant. The respective 'time' evolution equation is (*cf.* 2.101 (p.100) and 2.102 (p.101))

$$\partial_0 \left|\phi\right\rangle \;=\; \left(-\hat{\gamma}^0_-\hat{\gamma}^1_-\partial_1 + \hat{\gamma}^0_-\hat{\gamma}^1_+ c e^{-x^1}\right)\left|\phi\right\rangle \,. \tag{1085}$$

5.5 (Kantowski-Sachs scattering event) In the Kantowski-Sachs model (example 5.4 (p.259)) The potential rises in the $-x^1$ direction. In order to see scattering off the exponential potential wall we choose as initial state a Gaussian distribution in the "left going" component (*cf.* 2.83 (p.81) and 2.85 (p.83)):

$$\left|\phi(x^0=0)\right\rangle = e^{-\left(x^1\right)^2}\frac{1}{2}\left(1+\hat{\gamma}^0_-\hat{\gamma}^1_-\right)\frac{1}{2}\left(1+\hat{\gamma}^0_-\right)\left|0\right\rangle \,.$$

The result of a numerical propagation of this $|\phi_0\rangle$ is shown in figures 4 (p.262) and 5 (p.263).

These have been obtained, following the methods described in §2.2.8 (p.100) (see in particular 2.104 (p.103)) by expanding the exponential in the formal solution

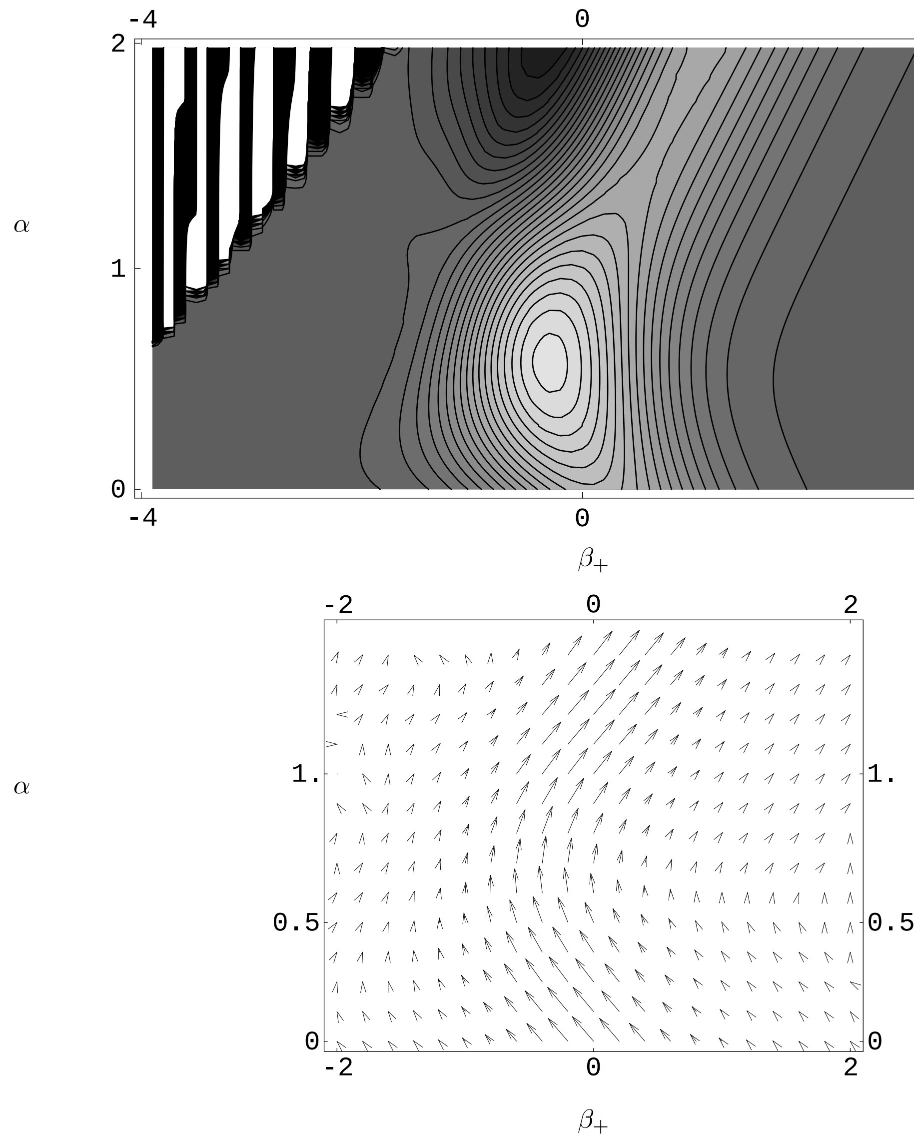
$$\left|\phi(x^0)\right\rangle \;=\; \exp\!\left(x^0\left(-\hat{\gamma}^0_-\hat{\gamma}^1_-\partial_1 - \hat{\gamma}^0_-\hat{\gamma}^1_+ c e^{-x^1}\right)\right)\left|\phi_0\right\rangle$$

to 64-th order. As can be seen in the figures, this approximation suffices to display smooth evolution of the wave packet but cannot properly resolve the heavily oscillating parts of the reflected wave packet near the potential wall on the left.

The figures illustrate how the initial Gaussian wave packet moves towards the potential wall where it is reflected. In the course of this process the amplitude of the “left moving” component of the wave function diminishes and becomes negative later on, while the amplitude of the “right moving” component, which is zero at the beginning, increases to form a Gauss-like wave packet of its own, propagating uniformly to the right. (Compare this with the discussion of the 1+1 dimensional supersymmetric checkerboard models in §2.2.5 (p.81).)

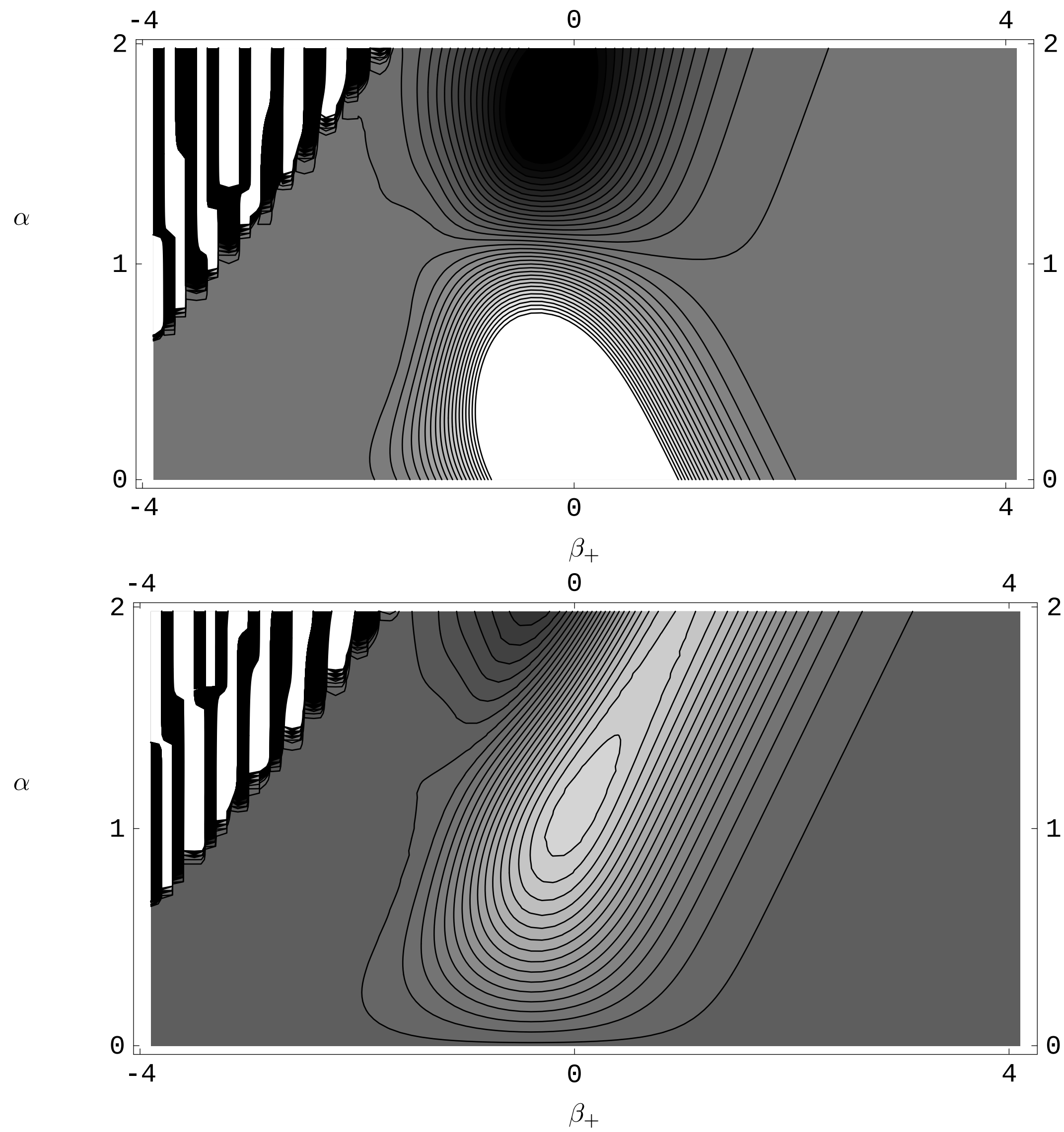
The corresponding probability current (*cf.* §2.2.4 (p.78), 2.79 (p.78)) is displayed in the lower figure 4 (p.262) . (Since it is quadratic in the wave function and thus numerically more demanding it has been calculated to only 20-th order in the generator (1086).) As long as the superpotential (1082) is approximated by the α-independent term (1083) this current is *conserved*, as follows from 2.81 (p.79).

Figure 4



Scattering event in minisuperspace for a Kantowski-Sachs model - total amplitude This figure displays the total amplitude ϕ_{tot}, i.e. the sum of left and right moving amplitudes $|\phi\rangle_{\mathrm{tot}} = \phi_{\mathrm{tot}}\frac{1}{2}\left(1+\hat{\gamma}^0_-\right)|0\rangle$ of the supersymmetric Kantowski-Sachs model (see example 5.4 (p.259)) for the 'initial' state a Gaussian in the *left* moving component: $|\psi_0\rangle = e^{-\left(x^1\right)^2}\frac{1}{2}\left(1+\hat{\gamma}^1_-\hat{\gamma}^0_-\right)\frac{1}{2}\left(1+\hat{\gamma}^0_-\right)|0\rangle$. Bright shading indicates high values and dark shading low (and negative) values of the amplitude. Each arrow represents the local probability current (334), p. 78. The vertical component of an arrow is given by $J^{(\alpha)} = \langle\phi|\,\hat{\gamma}^{(\alpha)}_-\hat{\gamma}^{(\alpha)}_+\,|\phi\rangle_{\mathrm{loc}}$ and the horizontal component by $J^{(\beta_+)} = \langle\phi|\,\hat{\gamma}^{(\beta_+)}_-\hat{\gamma}^{(\alpha)}_+\,|\phi\rangle_{\mathrm{loc}}$. Note that because of the approximation (1083) it follows from 2.81 (p.79) that this current is *conserved*. This very plausibly agrees with its visual impression.

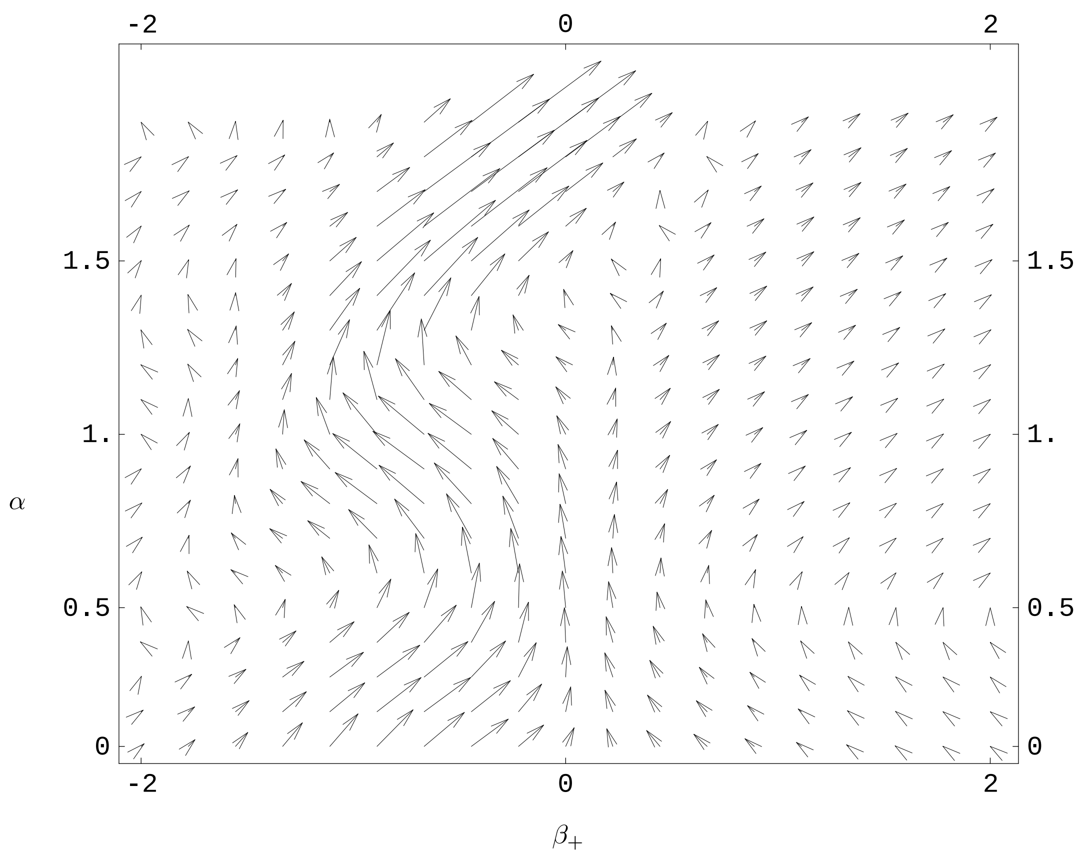
Figure 5



Scattering event in minisuperspace for a Kantowski-Sachs model - left and right going amplitude Shown is the amplitude ϕ of the left and right going component $\phi\frac{1}{2}\left(1 \pm \hat{\gamma}_-^0\hat{\gamma}_-^1\right)\frac{1}{2}\left(1+\hat{\gamma}_-^0\right)|0\rangle$, respectively, for the numerical propagation discussed in example 5.4 (p.259). See the caption of figure 4 (p.262) for details.

5.6 ($N = 2$ supersymmetric solution) So far we have only calculated the $N = 1$ supersymmetric solution that is annihilated by the operator (1084), p. 260. According to 2.105 (p.103), in order to find an $N = 2$-supersymmetric solutions that is annihilated by $\mathbf{D} = \mathbf{d}^W + \mathbf{d}^{\dagger^W}$ *and* by $\tilde{\mathbf{D}} = \mathbf{d}^W - \mathbf{d}^{\dagger^W}$, and hence by both supersymmetry generators $\hat{S} = \mathbf{d}^{\dagger^W}$, $\tilde{\hat{S}} = \mathbf{d}^W$ (see (1069), p. 256), we have to apply $\tilde{\mathbf{D}}$ to $|\phi\rangle$ (1086). The probability current of the resulting state is displayed in figure 6.

Figure 6

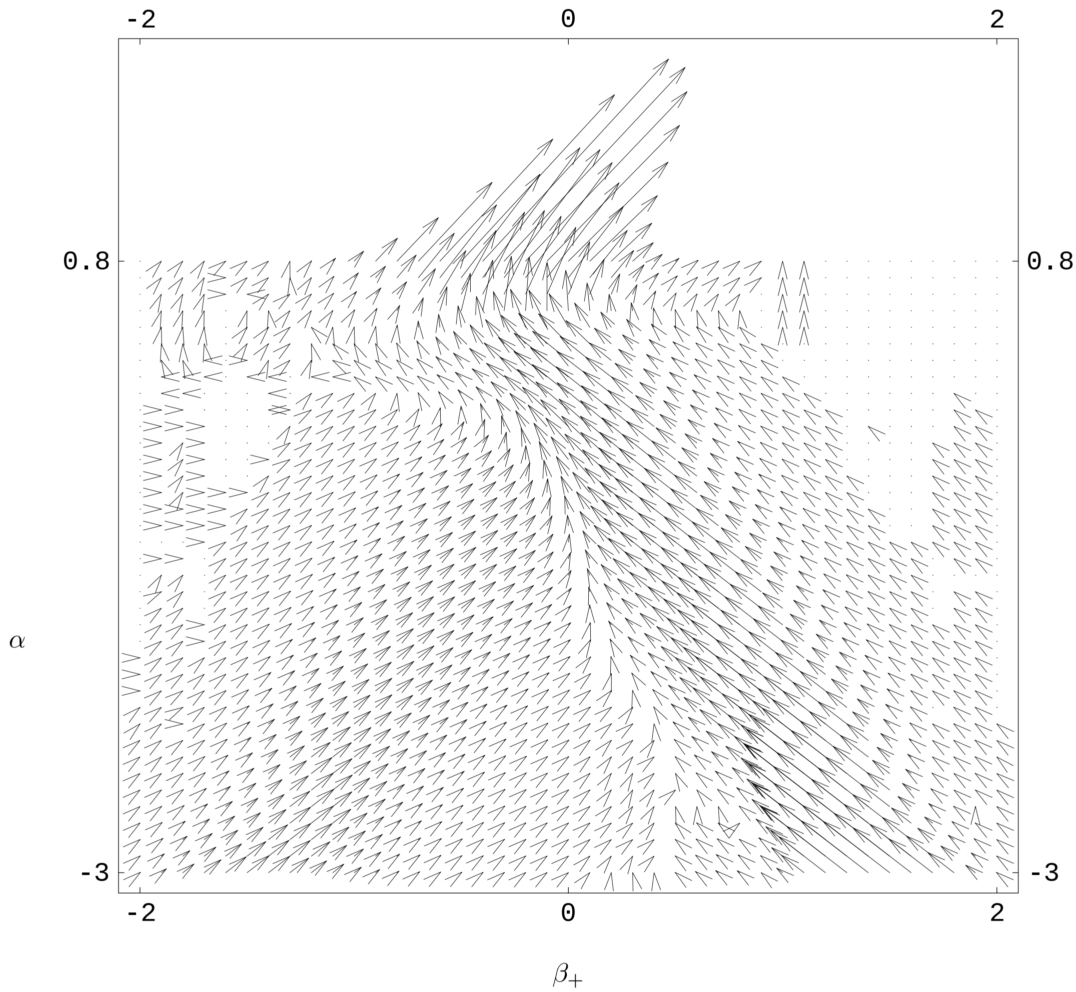


Scattering event in minisuperspace for a Kantowski-Sachs model - $N = 2$-supersymmetric current This is the current of the state $|\psi'\rangle$ obtained from the solution displayed in figure 4 (p.262) by acting on it with the second supercharge $\bar{\mathbf{D}} = i\left(\hat{\gamma}_+^\mu \partial_\mu + \hat{\gamma}_-^\mu \left(\partial_\mu W\right)\right)$: $|\psi'\rangle = \bar{\mathbf{D}}\,|\psi\rangle$. This makes $|\psi'\rangle$ invariant under the action of both supercharges: $\mathbf{D}\,|\psi'\rangle = 0$, $\bar{\mathbf{D}}\,|\psi'\rangle = 0$.

As opposed to the associated $N = 1$-current (lower part of figure 4 (p.262)), the above current is “S”-shaped. The same is found again for the $N = 2$-current of the $D = 11$-supergravity model in §5.2 (p.266), which is displayed in figure 10 (p.283). It may be a sign of generic *zitterbewegung* in supersymmetric quantum mechanics, which has to be expected due to the close relationship with the dynamics of the Dirac particle. See the discussion in §2.2.5 (p.81) for more details on zitterbewegung in (relativistic) supersymmetric quantum mechanics and in particular see the current displayed in figure 2 (p.86).

Figures 4, 5, and 6 give an impression of the dynamics of a simple system with "time"-independent potential. When the current is calculated instead for the "true" mini-superspace potential (1082), p. 259, one obtains the result displayed in figure 7 (*cf.* the discussion leading to the ansatz (1083), .p 260).

Figure 7



Scattering event in minisuperspace for a Kantowski-Sachs model - Current obtained for α-dependent superpotential The figure shows part of the would-be probability current of the Kantowski-Sachs model with the same initial conditions as given in the caption of figure 4 (p.262), but now with the exact α-dependent superpotential $e^{\alpha-\beta_+}$ (*cf.* 5.5 (p.260)). Clearly, now the probability current is no longer conserved since (formal) 'energy' is fed into the system at an exponential rate (*cf.* §2.2.4 (p.78), example 2.81 (p.79)). To better illustrate this effect, the current is here shown also for negative values of α. As long as $-\alpha$ is large, wave-packet dynamics is almost free. The Gaussian travels in good approximation uniformly at unit speed ('lightlike') from large positive β_+ to $\beta_+ \approx 0$, where it scatters and reverses its direction of motion. Then the potential increases appreciably with α and the overall current density builds up.

5.2 Bianchi-I model of $N = 1$, $D = 11$ supergravity

Introduction. Supergravity theories can be formulated in $D = 4$ up to $D = 11$ spacetime dimensions (*cf.* 4.3.4 (p.250)). 11 dimensional supergravity is conjectured to be a low-energy limit of M-theory, the hypothetical non-perturbative version of string theory, which is currently widely considered a promising candidate for what is expected to be a so-called "theory of everything". Hence there is some interest in studying cosmological models derived from 11-dimensional supergravity. The currently most fashionable sort are "brane-world" models in which the 7 additional spatial dimensions are assumed to have the topology $S^1/Z^2 \otimes \mathcal{X}$, where S^1/Z^2 is the circle with two half-arcs identified and where $\mathcal{X}$ is some Calabi-Yau space. The topology S^1/Z^2 of the 10-th spatial dimension is of interest, because with it the eleven dimensional theory reduces to the heterotic $E8 \times E8$ string theory in 10 dimensions, which seems to be the most promising of the various string theories with respect to reproducing standard model phenomenology.

In the following, however, we will study a cosmological model of $N = 1$, $D = 11$ supergravity which is compactified not on $S^1/Z^2 \otimes \mathcal{X}$ but simply on $S^1 \otimes \mathcal{X}$, where furthermore $\mathcal{X}$ is taken to be the flat 6-torus T^6. We assume the remaining ordinary 4 spacetime dimensions to be spatially homogeneous with Bianchi-I symmetry and to furthermore have spatially the topology of the 3-torus T^3. The purely bosonic version of this model (with a slightly different ansatz for 4-dimensional spacetime) has been constructed and investigated in [34], [46], and [47]. Even though the phenomenological value of such a model is, without further enhancement, probably rather small, it will serve us here as an interesting testing ground for the various constructions and techniques of supersymmetric quantum cosmology that have been discussed in §2 (p.14) and §4 (p.181).

In particular, we will follow the "Hamiltonian route" (in the terminology of the discussion on p. 9 in the introduction, see also §4.3.2 (p.230) and §4.3.3 (p.240)) to find a supersymmetric extension of the bosonic model presented in [34], i.e. to find an ansatz for the gravitino field compatible under supersymmetry with the given ansatz for the gravitational field. It is the remarkable strength of the "Hamiltonian route", that such an extension can be straightforwardly and systematically achieved without considering the full action of $N = 1$, $D = 11$ supergravity: Instead we follow §2.2.1 (p.55) and obtain the supersymmetric extension of the model by finding a supersymmetric extension of its algebra of quantum gauge generators, namely by accompanying the Hamiltonian operator on mini-superspace with its Dirac "square roots". Due to 4.51 (p.240) this can be done after merely identifying the DeWitt metric on mini-superspace. (Ordinarily the second step would be to find the superpotential on mini-superspace, as in 4.52 (p.241), but it turns out that in the present model no potential term appears, so that the superpotential also vanishes.) The supersymmetry generators are obtained as deformed exterior derivatives on mini-superspace, which is thus extended to "super-mini-superspace", namely the 1-form bundle over mini-superspace. These 1-forms in turn are the Grassmann-valued amplitudes of the gravitino field, as has been shown in 4.24 (p.203), and a state is a superfield over this space, namely a section of the form bundle.

The ordinary 4-dimensional Bianchi-I model has, due to the vanishing of its spatial curvature, vanishing mini-superspace potential, so that its dynamics is trivial. That the latter is not true for the above model is due to the presence of the 3-form field. While (for a homogenous form field) there is still no potential term, the "*kinetic*" energy of this field can be seen to act as an *effective* potential for the dynamics of the gravitational degrees of freedom. In accord with general considerations on cosmologies with form-field contributions (see e.g. [71]), it is found that this effective potential constitutes a well with exponential "walls", which, classically, gives rise to a Mixmaster-like behavior of the internal and external spatial dimensions.

In close analogy to the scattering event of the $D = 4$ Kantowski-Sachs model of §5.1 (p.255), we will construct a wave packet incident on one of these walls and numerically investigate its reflection.

A special feature of the quantum mechanics of the present model is that the mini-superspace metric has certain "hidden" symmetries (see §2.2.7 (p.90) and in particular 2.96 (p.94), 2.98 (p.95), and 2.99 (p.96)) due to which the original supersymmetry operators are accompanied by further supercharges, which together satisfy a superalgebra with central charges (*cf.* 2.38 (p.47)). As is discussed in 4.3.4 (p.250), this feature might be related to unbroken supersymmetries stemming from the higher dimensional theory.

5.7 (Literature.) A brief introduction to supergravities in various dimensions is given in [252]. The supersymmetric extension of 11-dimensional gravity was found by Cremmer, Julia, and Scherk in 1978 ([67][65][66]). Cosmologies from 11-dimensional supergravity are discussed for instance in [99][34][46][47]. A general theme in higher dimensional theories is the question as to why we observe exactly three *large* spatial dimensions, why the other spatial dimensions are compactified and which fields in the 'large' dimensions arise from metric field components associated to 'small' dimensions. Investigations in this direction started with the advent of Kaluza-Klein theories, which try to model non-gravitational interaction (electromagnetism, weak force) by higher dimensional gravitation (see [90] for a review of supersymmetric Kaluza-Klein theories), and were revived in a somewhat generalized fashion when it was realized that superstring theory requires higher dimensions for consistence. (See [270].) Cosmological models have the potential to give a dynamical description of compactification and decompactification of dimensions (*cf.* [242] [236] [155] [3] [1], and see §5.2.2 (p.275)). [99] discusses the possibility that the degree of the 3-form field $\mathcal{A}$ of supergravity might single out 3 spatial dimensions.
Technical details of supergravity compactification with emphasis on supersymmetry breaking, effective superpotentials, and the role of the form fields are given in [24] [116] [276].

5.2.1 The model.

First consider the bosonic sector of full $N = 1$, $D = 11$ supergravity:

5.8 (The action of 11-D supergravity) The bosonic sector of the action of 11-D supergravity (i.e. the part that remains when the gravitino field vanishes) reads ([67][65][66]):

$$S = \int (*R - \mathcal{F} \wedge *\mathcal{F} - k\mathcal{A} \wedge \mathcal{F} \wedge \mathcal{F}) \ , \tag{1086}$$

where integration is over physical spacetime represented by a pseudo-Riemannian 11-manifold (M, g), and where

$$*R = R \,\mathrm{vol}$$

is the Hodge dual of the *Ricci curvature scalar*

$$R = R(g) \ ,$$

and

$$\mathrm{vol} = \sqrt{-g} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{10}$$

is, as in 2.2 (p.16), the volume pseudo-form on (M, g). The 3-*form field*

$$\mathcal{A} := \mathcal{A}_{\lambda\mu\nu} dx^\lambda \wedge dx^\mu \wedge dx^\nu$$

is a 3-form section over M and

$$\mathcal{F} := \mathbf{d}\mathcal{A}$$

the corresponding *field strength*

$$\mathcal{F} = \mathcal{F}_{\kappa\lambda\mu\nu} dx^\kappa \wedge dx^\lambda \wedge dx^\mu \wedge dx^\nu \ .$$

We have absorbed a normalization constant into $\mathcal{F}$. The constant k will not be of interest since the term $\mathcal{A} \wedge \mathcal{F} \wedge \mathcal{F}$ makes no contribution to the action in the homogeneous model considered below.

Now we state the ansatz for the metric and the 3-form field with which we will enter the above action:

5.9 (The metric) As already mentioned in the introduction, we assume spacetime to be given by

$$\mathcal{M}^{(\text{spacetime})} = \underbrace{R^1}_{\text{time}} \otimes \underbrace{S^1 \otimes T^6}_{\substack{\text{internal}\\ \text{space}}} \otimes \underbrace{T^3}_{\substack{\text{external}\\ \text{space}}} \ . \tag{1087}$$

We make a *homogeneous* ansatz with translational (Bianchi-I-like) symmetry for the 11-dimensional metric. To facilitate comparison with the existing literature we will consider (a slight generalization of) the ansatz for dimensional reduction that is used in [34] [46] [47]:

The metric of 4-dimensional 'external' spacetime is chosen to be

$$g^{(\text{ext})} := \begin{bmatrix} -N^2 & 0 & 0 & 0 \\ 0 & e^{2\alpha_1} & 0 & 0 \\ 0 & 0 & e^{2\alpha_2} & 0 \\ 0 & 0 & 0 & e^{2\alpha_3} \end{bmatrix} , \tag{1088}$$

$$\begin{aligned} x^0 &= -\infty \ldots \infty \\ x^i &= 0 \ldots 1, \qquad i \in \{1,2,3\} \\ N &= N(x^0) \\ \alpha_i &= \alpha_i(x^0) \ , \end{aligned}$$

which is the usual Bianchi-I form of flat 3-torus space plus time. N is the *lapse* function that measures the amount of proper time per coordinate time x^0 and e^{α_i} is the circumference of the 3-torus in x^i-direction.

From superstring theory one knows that the internal space needs to be Calabi-Yau, i.e. compact Kähler and Ricci flat. This is enforced by choosing T^6 with a trivial flat metric:

$$g^{(\mathrm{int})} := e^{2\beta}\mathrm{diag}(1,1,1,1,1,1) \tag{1089}$$

$$\begin{aligned} x^\mu &= 0\ldots 1, \qquad \mu \in \{4,5,6,7,8,9\} \\ \beta &= \beta(x^0)\ . \end{aligned}$$

Finally, the tenth spatial dimension is assumed to be a circle of radius $e^{\Phi/2}$:

$$g^{(\mathrm{dil})} := e^{\Phi} \tag{1090}$$

$$\begin{aligned} x^{10} &= 0\ldots 1 \\ \Phi &= \Phi(x^0) \end{aligned}$$

It proves convenient (see [34]) to conformally scale the resulting metric by a factor $e^{-\frac{1}{3}\Phi}$, which finally gives the full 11 dimensional line element of our model:

$$\left(ds^{(11)}\right)^2 := e^{-\frac{1}{3}\Phi}\left(-N^2 dx^0 dx^0 + \sum_{i=1}^{3} e^{2\alpha_i} dx^i dx^i + e^{2\beta}\sum_{j=4}^{9} dx^j dx^j + e^{\Phi} dx^{10} dx^{10}\right). \tag{1091}$$

With all the assumptions of the model presented, the gravitational part of the action (1086) can now be dimensionally reduced by integrating over all spatial variables. This is, by construction of the homogeneous model, totally trivial:

$$\begin{aligned} S &= \int *R^{(11)} \\ &= \int \sqrt{g^{(11)}(x^0)}R^{(11)}\left(x^0\right) dx^0 \cdots dx^{10} \\ &= \int \sqrt{g^{(11)}(x^0)}R^{(11)}\left(x^0\right) dx^0\,. \end{aligned} \tag{1092}$$

The determinant of the metric is also easily read off:

$$\sqrt{-\det\left(g^{(11)}\right)} = N e^{-\frac{4}{3}\Phi+\alpha_{(1)}+\alpha_{(2)}+\alpha_{(3)}+6\beta}\,. \tag{1093}$$

The somewhat more tedious part is to calculate the curvature density $\sqrt{g^{(11)}(x^0)}R^{(11)}\left(x^0\right)$. It turns out to be decomposable as

$$\sqrt{g^{(11)}(x^0)}R^{(11)}\left(x^0\right) = L(N, N', \alpha_i, {\alpha_i}', \beta, \beta', \Phi, \Phi') + F'$$

(where a prime indicates the differential with respect to x^0), i.e. as a functional of the physical fields and their first derivatives plus a total derivative. As usual, the latter can be ignored, since $\int F' = \text{const}$ does not affect the physics described by the action S, leaving us with $L_{(*R)}$, which is the gravitational Lagrangian of our model. It evaluates to

$$L_{(*R)} = \frac{-e^{\left(6\beta-\Phi+\sum_{i=1}^{3}\alpha_i\right)}}{N}\left(30\beta'^2+\Phi'^2-12\beta'\left(\Phi'-\sum_{i=1}^{3}\alpha_i{}'\right)-2\Phi'\sum_{i=1}^{3}\alpha_i{}'+\sum_{i\neq j=1}^{3}\alpha_i{}'\alpha_j{}'\right). \tag{1094}$$

By a linear transformation of coordinates in mini-superspace, replacing Φ by

$$\phi \;:=\; \Phi-\sum_{i=1}^{3}\alpha_i-6\beta\,, \tag{1095}$$

this can be simplified to finally give (*cf.* [46], eq. (12))

$$L_{(*R)} = \frac{1}{Ne^{\phi}}\left(-\phi'^2+\sum_{i=1}^{3}\alpha_i'^2+6\beta'^2\right). \tag{1096}$$

The volume density in the new coordinates (1095) reads

$$\sqrt{-\det\bigl(g^{(11)}\bigr)} \;=\; Ne^{\phi-\frac{1}{3}\left(\alpha_{(1)}+\alpha_{(2)}+\alpha_{(3)}+6\beta\right)}. \tag{1097}$$

5.10 (The form field) The dynamics (1096) of the metric field (1091) alone is quite uninteresting. One can perturb it by considering a non-vanishing, but still homogeneous, 3-form field $\mathcal{A}$.

A 3-form in 11 dimensions has in general $\binom{11}{3}=165$ independent components. However, the number of dynamically distinguishable components reduces drastically in the simple model considered here, for two reasons:

1. Since $\mathcal{A}$ is assumed to be homogeneous, i.e. $\mathcal{A}=\mathcal{A}\bigl(x^0\bigr)$, the field strength

$$\begin{aligned}\mathcal{F} &= \mathbf{d}\mathcal{A}\\ &= \mathbf{d}\mathcal{A}_{\lambda\mu\nu}dx^\lambda\wedge dx^\mu\wedge dx^\nu\\ &= \left(\partial_0\mathcal{A}\right)_{\lambda\mu\nu}dx^0\wedge dx^\lambda\wedge dx^\mu\wedge dx^\nu\end{aligned} \tag{1098}$$

   will always be proportional to dx^0. This implies that

$$\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F} \;=\; 0 \tag{1099}$$

   vanishes identically. The only remaining contribution of the 3-form field to the action (1086) is via the term

$$\begin{aligned}\mathcal{F}\wedge *\mathcal{F} &= 4!\sqrt{g_{(11)}}\mathcal{F}_{\kappa\lambda\mu\nu}\mathcal{F}^{\kappa\lambda\mu\nu}\\ &= 4\cdot 4!\sqrt{g_{(11)}}\,\mathcal{F}_{0\lambda\mu\nu}\mathcal{F}^{0\lambda\mu\nu}\,.\end{aligned} \tag{1100}$$

(We will in the following absorb the factor $4 \cdot 4!$ into the normalization of $\mathcal{F}$ and hence of $\mathcal{A}$.) But this means that any component of $\mathcal{A}$ proportional to dx^0 does not contribute to the action. The corresponding canonical coordinate and canonical momenta (see below) are both cyclic and can hence be ignored. This reduces the 165 independent components of $\mathcal{A}$ to $\binom{10}{3} = 120$ components that may actually appear in the Lagrangian.

2. The Lagrangian of the 3-form field strength (1100) can be written in components as:

$$\sqrt{g_{(11)}}\mathcal{F}_{0\lambda\mu\nu}\mathcal{F}^{0\lambda\mu\nu} = \sqrt{g_{(11)}}\,(\partial_0\mathcal{A})_{\lambda\mu\nu}\,(\partial_0\mathcal{A})_{\lambda'\mu'\nu'}\,g^{00}_{(11)}g^{\lambda\lambda'}_{(11)}g^{\mu\mu'}_{(11)}g^{\nu\nu'}_{(11)}\,. \tag{1101}$$

The terms $(\partial_0\mathcal{A})_{\lambda\mu\nu}$ are essentially canonical momenta of the dimensionally reduced Lagrangian (see below) and their dynamics will be determined by the DeWitt metric of this model, which, for the 3-form components, is seen to be

$$G^{(\lambda\mu\nu)(\lambda'\mu'\nu')} := \sqrt{g_{(11)}}g^{00}_{(11)}g^{\lambda\lambda'}_{(11)}g^{\mu\mu'}_{(11)}g^{\nu\nu'}_{(11)}\,. \tag{1102}$$

Whenever n of the diagonal elements of G are identically equal, they will belong to a trivial n-dimensional subspace of configuration space. All such subspaces can be collapsed to 1 dimension without losing information about the dynamics (as long as there are no potential terms varying in these subspaces). Hence, with respect to the dynamics, all components $\mathcal{A}_{\lambda\mu\nu}$ of $\mathcal{A}$ with identical configuration space metric $G^{(\lambda\mu\nu)(\lambda\mu\nu)}$ can be identified.

The 11-dimensional spacetime metric $g^{(11)}$ has 6 identical entries corresponding to the internal 6-torus T^6. By the above argument, all components $\mathcal{A}_{\lambda\mu\nu}$ associated to the same number of indices with values on this 6-torus, and otherwise identical indices, have the same kinetic metric components $G^{(\lambda\mu\nu)(\lambda\mu\nu)}$ and qualitatively give the same contribution to the action. Therefore, without loss of generality, of all the indistinguishable components only one representative is included in the following. This reduces the number of components to a mere 15, which constitute the following ansatz for the 3-form field (recall that indices $1, 2, 3$ correspond to external space, $4, 5, 6, 7, 8, 9$ to the internal Calabi-Yau space, and 10 to the 'internal circle'):

$$\begin{array}{rcccccc}
\mathcal{A}(x^0) := & & & & \mathcal{A}_{1,2,3}dx^1dx^2dx^3 & & \\
& + & \mathcal{A}_{1,2,4}dx^1dx^2dx^4 & + & \mathcal{A}_{1,3,4}dx^1dx^3dx^4 & + & \mathcal{A}_{2,3,4}dx^2dx^3dx^4 \\
& + & \mathcal{A}_{1,4,5}dx^1dx^4dx^5 & + & \mathcal{A}_{2,4,5}dx^2dx^4dx^5 & + & \mathcal{A}_{3,4,5}dx^3dx^4dx^5 \\
& + & \mathcal{A}_{1,2,10}dx^1dx^2dx^{10} & + & \mathcal{A}_{1,3,10}dx^1dx^3dx^{10} & + & \mathcal{A}_{2,3,10}dx^2dx^3dx^{10} \\
& + & \mathcal{A}_{1,4,10}dx^1dx^4dx^{10} & + & \mathcal{A}_{2,4,10}dx^2dx^4dx^{10} & + & \mathcal{A}_{3,4,10}dx^3dx^4dx^{10} \\
& & & + & \mathcal{A}_{4,5,6}dx^4dx^5dx^6 & & \\
& & & + & \mathcal{A}_{4,5,10}dx^4dx^5dx^{10} & &
\end{array} \tag{1103}$$

With this ansatz for $\mathcal{A}$ the form field Lagrangian becomes

$$
\begin{aligned}
-\mathcal{F} \wedge *\mathcal{F} &= L_{(\mathcal{F}\wedge*\mathcal{F})} dx^0 \wedge dx^2 \wedge \cdots dx^{10} \\
N e^{\phi} L_{(\mathcal{F}\wedge*\mathcal{F})} &= e^{(\phi+6\beta-\alpha_{(1)}-\alpha_{(2)}-\alpha_{(3)})}\left(\dot{\mathcal{A}}_{1,2,3}\right)^2 \\
&+ e^{(-2\alpha_{(1)}-2\alpha_{(2)})}\left(\dot{\mathcal{A}}_{1,2,10}\right)^2 \\
&+ e^{(-2\alpha_{(1)}-2\alpha_{(3)})}\left(\dot{\mathcal{A}}_{1,3,10}\right)^2 \\
&+ e^{(-2\alpha_{(2)}-2\alpha_{(3)})}\left(\dot{\mathcal{A}}_{2,3,10}\right)^2 \\
&+ e^{(\phi+4\beta-\alpha_{(1)}-\alpha_{(2)}+\alpha_{(3)})}\left(\dot{\mathcal{A}}_{1,2,4}\right)^2 \\
&+ e^{(\phi+4\beta-\alpha_{(1)}+\alpha_{(2)}-\alpha_{(3)})}\left(\dot{\mathcal{A}}_{1,3,4}\right)^2 \\
&+ e^{(\phi+4\beta+\alpha_{(1)}-\alpha_{(2)}-\alpha_{(3)})}\left(\dot{\mathcal{A}}_{2,3,4}\right)^2 \\
&+ e^{(-2\beta-2\alpha_{(1)})}\left(\dot{\mathcal{A}}_{1,4,10}\right)^2 \\
&+ e^{(-2\beta-2\alpha_{(2)})}\left(\dot{\mathcal{A}}_{2,4,10}\right)^2 \\
&+ e^{(-2\beta-2\alpha_{(3)})}\left(\dot{\mathcal{A}}_{3,4,10}\right)^2 \\
&+ e^{(\phi+2\beta-\alpha_{(1)}+\alpha_{(2)}+\alpha_{(3)})}\left(\dot{\mathcal{A}}_{1,4,5}\right)^2 \\
&+ e^{(\phi+2\beta+\alpha_{(1)}-\alpha_{(2)}+\alpha_{(3)})}\left(\dot{\mathcal{A}}_{2,4,5}\right)^2 \\
&+ e^{(\phi+2\beta+\alpha_{(1)}+\alpha_{(2)}-\alpha_{(3)})}\left(\dot{\mathcal{A}}_{3,4,5}\right)^2 \\
&+ e^{(\phi+\alpha_{(1)}+\alpha_{(2)}+\alpha_{(3)})}\left(\dot{\mathcal{A}}_{4,5,6}\right)^2 \\
&+ e^{(-4\beta)}\left(\dot{\mathcal{A}}_{4,5,10}\right)^2
\end{aligned}
\tag{1104}
$$

5.11 (Mini-Superspace metric) We have now finished the dimensional reduction of the homogeneous supergravity cosmology (1091) with general homogeneous form field (1103). The reduced Lagrangian

$$
L := L_{(*R)} + L_{(\mathcal{F}\wedge*\mathcal{F})} + \underbrace{L_{(\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F})}}_{=0} \tag{1105}
$$

has been derived, which describes free relativistic dynamics of a point in 20-dimensional pseudo-Riemannian mini-superspace $\left(\mathcal{M}^{(\mathrm{conf})}, G^{(\mathrm{conf})}\right)$, coordinatized by $\phi, \alpha_{(1)}, \alpha_{(2)}, \alpha_{(3)}, \beta, \mathcal{A}_{(1)}, \ldots, \mathcal{A}_{(15)}$ and equipped with the following metric:

$$
\begin{aligned}
N e^{\phi} G^{(\mathrm{conf})} := G &= \mathrm{diag}\Big(G_{(\phi,\phi)}, G_{(\alpha_{(1)},\alpha_{(1)})}, \ldots, G_{(\mathcal{A}_{4,5,6},\mathcal{A}_{4,5,6})}, G_{(\mathcal{A}_{4,5,10},\mathcal{A}_{4,5,10})}\Big) \\
G_{(\phi,\phi)} &= -1 \\
G_{(\alpha_{(i)},\alpha_{(i)})} &= 1
\end{aligned}
$$

$$
\begin{aligned}
G_{(\beta,\beta)} &= 6 \\
G_{(\mathcal{A}_{1,2,3},\mathcal{A}_{1,2,3})} &= e^{(\phi-\alpha_{(1)}-\alpha_{(2)}-\alpha_{(3)}+6\beta)} \\
G_{(\mathcal{A}_{1,2,10},\mathcal{A}_{1,2,10})} &= e^{(-2\alpha_{(1)}-2\alpha_{(2)})} \\
G_{(\mathcal{A}_{1,3,10},\mathcal{A}_{1,3,10})} &= e^{(-2\alpha_{(1)}-2\alpha_{(3)})} \\
G_{(\mathcal{A}_{2,3,10},\mathcal{A}_{2,3,10})} &= e^{(-2\alpha_{(2)}-2\alpha_{(3)})} \\
G_{(\mathcal{A}_{1,2,4},\mathcal{A}_{1,2,4})} &= e^{(\phi+4\beta-\alpha_{(1)}-\alpha_{(2)}+\alpha_{(3)})} \\
G_{(\mathcal{A}_{1,3,4},\mathcal{A}_{1,3,4})} &= e^{(\phi+4\beta-\alpha_{(1)}+\alpha_{(2)}-\alpha_{(3)})} \\
G_{(\mathcal{A}_{2,3,4},\mathcal{A}_{2,3,4})} &= e^{(\phi+4\beta+\alpha_{(1)}-\alpha_{(2)}-\alpha_{(3)})} \\
G_{(\mathcal{A}_{1,4,10},\mathcal{A}_{1,4,10})} &= e^{(-2\beta-2\alpha_{(1)})} \\
G_{(\mathcal{A}_{2,4,10},\mathcal{A}_{2,4,10})} &= e^{(-2\beta-2\alpha_{(2)})} \\
G_{(\mathcal{A}_{3,4,10},\mathcal{A}_{3,4,10})} &= e^{(-2\beta-2\alpha_{(3)})} \\
G_{(\mathcal{A}_{1,4,5},\mathcal{A}_{1,4,5})} &= e^{(\phi+2\beta-\alpha_{(1)}+\alpha_{(2)}+\alpha_{(3)})} \\
G_{(\mathcal{A}_{2,4,5},\mathcal{A}_{2,4,5})} &= e^{(\phi+2\beta+\alpha_{(1)}-\alpha_{(2)}+\alpha_{(3)})} \\
G_{(\mathcal{A}_{3,4,5},\mathcal{A}_{3,4,5})} &= e^{(\phi+2\beta+\alpha_{(1)}+\alpha_{(2)}-\alpha_{(3)})} \\
G_{(\mathcal{A}_{4,5,6},\mathcal{A}_{4,5,6})} &= e^{(\phi+\alpha_{(1)}+\alpha_{(2)}+\alpha_{(3)})} \\
G_{(\mathcal{A}_{4,5,10},\mathcal{A}_{4,5,10})} &= e^{(-4\beta)} \,.
\end{aligned}
\tag{1106}
$$

With $\left(\mathcal{M}^{(11)}, G^{(\mathrm{conf})}\right)$ known, one can leave the details of cosmology in general and supergravity in particular behind and concentrate on the task of quantizing the relativistic point propagating on a curved manifold. In order to do so one should switch from the Lagrangian to the Hamiltonian description:

5.12 (The Hamiltonian) Let

$$
X := X(t) = \left(X^{(n)}(t)\right) = \left[\phi, \alpha_{(i)}, \beta, \mathcal{A}_{1,2,3}, \ldots, \mathcal{A}_{5,4,10}\right]^{\mathrm{T}}(t) \tag{1107}
$$

be a curve in $\left(\mathcal{M}^{(\mathrm{conf})}, G^{(\mathrm{conf})}\right)$ parameterized by t and let

$$
\dot{X} = \left(\dot{X}^{(n)}\right) = \left[\dot{\phi}, \dot{\alpha}_{(i)}, \dot{\beta}, \dot{\mathcal{A}}_{1,2,3}, \ldots, \dot{\mathcal{A}}_{5,4,10}\right]^{\mathrm{T}}(t)
$$

be the respective tangent vector. The Hamiltonian H is obtained as usual (see §A (p.293) for details) by means of a Legendre transformation of the Lagrangian density:

$$
\begin{aligned}
L\left(X, \dot{X}\right) &= \frac{1}{Ne^{\phi}}\, G_{(m)(n)} \dot{X}^{(m)} \dot{X}^{(n)} \\
\Rightarrow P_{(n)} &= \frac{\partial L}{\partial \dot{X}^{(n)}} \\
&= 2\frac{1}{Ne^{\phi}} G_{(m)(n)} \dot{X}^{(n)} \\
\tilde{H}(X,P) &= P_{(n)} \dot{X}^{(n)} - L \\
&= \frac{1}{4} Ne^{\phi}\, G^{(m)(n)} P_{(m)} P_{(n)} \\
&:= \frac{1}{4} Ne^{\phi}\, H \,.
\end{aligned}
\tag{1108}
$$

The classical *Hamiltonian constraint* is[59]

$$H := G^{(m)(n)} P_{(m)} P_{(n)} \stackrel{!}{=} 0 \,, \tag{1109}$$

up to classically irrelevant conformal transformations. As discussed in 4.9 (p.186), one has to fix a conformal scaling in order to obtain a unique quantum constraint operator. Following [34], [46], and [47] we choose H as given above.

[59] If this were the Hamiltonian constraint of a real physical particle it would describe a *massless* relativistic particle in curved spacetime.

5.2.2 Solutions.

5.13 (The classical equations of motion) The classical equations of motion

$$\begin{aligned}\dot{X}^{(n)} &= \frac{\partial H}{\partial P_{(n)}}\\ \dot{P}_{(n)} &= -\frac{\partial H}{\partial X^{(n)}}\end{aligned}$$

for

$$H(x,p) = G^{\mu\nu} P_{(m)} P_{(n)}$$

with G given by (1106) are as follows:

$$\begin{aligned}
\dot{\phi} &= -2p_\phi\\
\dot{\alpha}_{(i)} &= 2p_{\alpha_{(i)}}\\
\dot{\beta} &= \frac{1}{3}p_\beta\\
\dot{\mathcal{A}}_{1,2,3} &= 2e^{(-\phi-6\beta+\alpha_{(1)}+\alpha_{(2)}+\alpha_{(3)})}\, p_{\mathcal{A}_{1,2,3}}\\
\dot{\mathcal{A}}_{1,2,10} &= 2e^{2(+\alpha_{(1)}+\alpha_{(2)})}\, p_{\mathcal{A}_{1,2,10}}\\
\dot{\mathcal{A}}_{1,3,10} &= 2e^{2(+\alpha_{(1)}+\alpha_{(3)})}\, p_{\mathcal{A}_{1,3,10}}\\
\dot{\mathcal{A}}_{2,3,10} &= 2e^{2(+\alpha_{(2)}+\alpha_{(3)})}\, p_{\mathcal{A}_{2,3,10}}\\
\dot{\mathcal{A}}_{1,2,4} &= 2e^{(-\phi-4\beta+\alpha_{(1)}+\alpha_{(2)}-\alpha(3))}\, p_{\mathcal{A}_{1,2,4}}\\
\dot{\mathcal{A}}_{1,3,4} &= 2e^{(-\phi-4\beta+\alpha_{(1)}-\alpha_{(2)}+\alpha(3))}\, p_{\mathcal{A}_{1,3,4}}\\
\dot{\mathcal{A}}_{2,3,4} &= 2e^{(-\phi-4\beta-\alpha_{(1)}+\alpha_{(2)}+\alpha(3))}\, p_{\mathcal{A}_{2,3,4}}\\
\dot{\mathcal{A}}_{1,4,10} &= 2e^{2(\beta+\alpha_{(1)})}\, p_{\mathcal{A}_{1,4,10}}\\
\dot{\mathcal{A}}_{2,4,10} &= 2e^{2(\beta+\alpha_{(2)})}\, p_{\mathcal{A}_{2,4,10}}\\
\dot{\mathcal{A}}_{3,4,10} &= 2e^{2(\beta+\alpha_{(3)})}\, p_{\mathcal{A}_{3,4,10}}\\
\dot{\mathcal{A}}_{1,4,5} &= 2e^{(-\phi-2\beta+\alpha_{(1)}-\alpha_{(2)}-\alpha_{(3)})}\, p_{\mathcal{A}_{1,4,5}}\\
\dot{\mathcal{A}}_{2,4,5} &= 2e^{(-\phi-2\beta-\alpha_{(1)}+\alpha_{(2)}-\alpha_{(3)})}\, p_{\mathcal{A}_{2,4,5}}\\
\dot{\mathcal{A}}_{3,4,5} &= 2e^{(-\phi-2\beta-\alpha_{(1)}-\alpha_{(2)}+\alpha_{(3)})}\, p_{\mathcal{A}_{3,4,5}}\\
\dot{\mathcal{A}}_{4,5,6} &= 2e^{(-\phi-\alpha_{(1)}-\alpha_{(2)}-\alpha_{(3)})}\, p_{\mathcal{A}_{4,5,6}}\\
\dot{\mathcal{A}}_{4,5,10} &= 2e^{(4\beta)}\, p_{\mathcal{A}_{4,5,10}}
\end{aligned}$$

$$\begin{aligned}
\dot{p}_\phi &= e^{(-\phi-\alpha_{(1)}-\alpha_{(2)}-\alpha_{(3)}-6\beta)}\Big(e^{2(\alpha_{(1)}+\alpha_{(2)}+\alpha_{(3)})}p^2_{\mathcal{A}_{1,2,3}} + e^{2\beta}\Big(e^{2(\alpha_{(1)}+\alpha_{(2)})}p^2_{\mathcal{A}_{1,2,4}} +\\
&\quad + e^{2(\alpha_{(1)}+\alpha_{(3)})}p^2_{\mathcal{A}_{1,3,4}} + e^{2(\alpha_{(2)}+\alpha_{(3)})}p^2_{\mathcal{A}_{2,3,4}} + e^{2(\beta+\alpha_{(1)})}p^2_{\mathcal{A}_{1,4,5}} + e^{2(\beta+\alpha_{(2)})}p^2_{\mathcal{A}_{2,4,5}}\\
&\quad + e^{2(\beta+\alpha_{(3)})}p^2_{\mathcal{A}_{3,4,5}} + e^{4\beta}p^2_{\mathcal{A}_{4,5,6}}\Big)\Big)\\
\dot{p}_{\alpha_{(1)}} &= -e^{(-\phi-6\beta-\alpha_{(1)}-\alpha_{(2)}-\alpha_{(3)})}\Big(e^{2(\alpha_{(1)}+\alpha_{(2)}+\alpha_{(3)})}p^2_{\mathcal{A}_{1,2,3}} +\\
&\quad + e^{2\beta}\Big(2e^{(\phi+3\alpha_{(1)}+3\alpha_{(2)}+\alpha_{(3)}+4\beta)}p^2_{\mathcal{A}_{1,2,10}} + 2e^{(\phi+3\alpha_{(1)}+\alpha_{(2)}+3\alpha_{(3)}+4\beta)}p^2_{\mathcal{A}_{1,3,10}} +\\
&\quad + e^{2(\alpha_{(1)}+\alpha_{(2)})}p^2_{\mathcal{A}_{1,2,4}} + e^{2(\alpha_{(1)}+\alpha_{(3)})}p^2_{\mathcal{A}_{1,3,4}} - e^{2(\alpha_{(2)}+\alpha_{(3)})}p^2_{\mathcal{A}_{2,3,4}} +
\end{aligned}$$

$$
\begin{aligned}
& +2e^{(\phi+3\alpha_{(1)}+\alpha_{(2)}+\alpha_{(3)}+6\beta)}p^2_{\mathcal{A}_{1,4,10}} + e^{2(\alpha_{(1)}+\beta)}p^2_{\mathcal{A}_{1,4,5}} - e^{2(\alpha_{(2)}+\beta)}p^2_{\mathcal{A}_{2,4,5}} - \\
& -e^{2(\alpha_{(3)}+\beta)}p^2_{\mathcal{A}_{3,4,5}} - e^{4\beta}p^2_{\mathcal{A}_{4,5,6}}\Big)\Big) \\
\dot{p}_{\alpha_{(2)}} = & -e^{(-\phi-6\beta-\alpha_{(1)}-\alpha_{(2)}-\alpha_{(3)})}\Big(e^{2(\alpha_{(1)}+\alpha_{(2)}+\alpha_{(3)})}p^2_{\mathcal{A}_{1,2,3}} + \\
& +e^{2\beta}\Big(2e^{(\phi+3\alpha_{(1)}+3\alpha_{(2)}+\alpha_{(3)}+4\beta)}p^2_{\mathcal{A}_{1,2,10}} + 2e^{(\phi+\alpha_{(1)}+3\alpha_{(2)}+3\alpha_{(3)}+4\beta)}p^2_{\mathcal{A}_{2,3,10}} + \\
& +e^{2(\alpha_{(1)}+\alpha_{(2)})}p^2_{\mathcal{A}_{1,2,4}} + e^{2(\alpha_{(1)}+\alpha_{(3)})}p^2_{\mathcal{A}_{1,3,4}} - e^{2(\alpha_{(2)}+\alpha_{(3)})}p^2_{\mathcal{A}_{2,3,4}} + \\
& +2e^{(\phi+\alpha_{(1)}+3\alpha_{(2)}+\alpha_{(3)}+6\beta)}p^2_{\mathcal{A}_{2,4,10}} - e^{2(\alpha_{(1)}+\beta)}p^2_{\mathcal{A}_{1,4,5}} + e^{2(\alpha_{(2)}+\beta)}p^2_{\mathcal{A}_{2,4,5}} - \\
& -e^{2(\alpha_{(3)}+\beta)}p^2_{\mathcal{A}_{3,4,5}} - e^{4\beta}p^2_{\mathcal{A}_{4,5,6}}\Big)\Big) \\
\dot{p}_{\alpha_{(3)}} = & -e^{(-\phi-6\beta-\alpha_{(1)}-\alpha_{(2)}-\alpha_{(3)})}\Big(e^{2(\alpha_{(1)}+\alpha_{(2)}+\alpha_{(3)})}p^2_{\mathcal{A}_{1,2,3}} + \\
& +e^{2\beta}\Big(2e^{(\phi+3\alpha_{(1)}+\alpha_{(2)}+3\alpha_{(3)}+4\beta)}p^2_{\mathcal{A}_{1,3,10}} + 2e^{(\phi+\alpha_{(1)}+3\alpha_{(2)}+3\alpha_{(3)}+4\beta)}p^2_{\mathcal{A}_{2,3,10}} + \\
& +e^{2(\alpha_{(1)}+\alpha_{(2)})}p^2_{\mathcal{A}_{1,2,4}} + e^{2(\alpha_{(1)}+\alpha_{(3)})}p^2_{\mathcal{A}_{1,3,4}} - e^{2(\alpha_{(2)}+\alpha_{(3)})}p^2_{\mathcal{A}_{2,3,4}} + \\
& +2e^{(\phi+\alpha_{(1)}+\alpha_{(2)}+3\alpha_{(3)}+6\beta)}p^2_{\mathcal{A}_{3,4,10}} - e^{2(\alpha_{(1)}+\beta)}p^2_{\mathcal{A}_{1,4,5}} - e^{2(\alpha_{(2)}+\beta)}p^2_{\mathcal{A}_{2,4,5}} + \\
& +e^{2(\alpha_{(3)}+\beta)}p^2_{\mathcal{A}_{3,4,5}} - e^{4\beta}p^2_{\mathcal{A}_{4,5,6}}\Big)\Big) \\
\dot{p}_{\beta} = & \; 2e^{(-\phi-\alpha_{(1)}-\alpha_{(2)}-\alpha_{(3)}-6\beta)}\Big(3e^{2(\alpha_{(1)}+\alpha_{(2)}+\alpha_{(3)})}p^2_{\mathcal{A}_{1,2,3}} + \\
& +e^{2\beta}\Big(2e^{2(\alpha_{(1)}+\alpha_{(2)})}p^2_{\mathcal{A}_{1,2,4}} + 2e^{2(\alpha_{(1)}+\alpha_{(3)})}p^2_{\mathcal{A}_{1,3,4}} + 2e^{2(\alpha_{(2)}+\alpha_{(3)})}p^2_{\mathcal{A}_{2,3,4}} - \\
& -e^{(\phi+3\alpha_{(1)}+\alpha_{(2)}+\alpha_{(3)}+6\beta)}p^2_{\mathcal{A}_{1,4,10}} - e^{(\phi+\alpha_{(1)}+3\alpha_{(2)}+\alpha_{(3)}+6\beta)}p^2_{\mathcal{A}_{2,4,10}} - \\
& -e^{(\phi+\alpha_{(1)}+\alpha_{(2)}+3\alpha_{(3)}+6\beta)}p^2_{\mathcal{A}_{3,4,10}} + e^{2(\alpha_{(1)}+\beta)}p^2_{\mathcal{A}_{1,4,5}} + e^{2(\alpha_{(2)}+\beta)}p^2_{\mathcal{A}_{2,4,5}} + e^{2(\alpha_{(3)}+\beta)}p^2_{\mathcal{A}_{3,4,5}}\Big)\Big) \\
\dot{p}_{\mathcal{A}_{\lambda\mu\nu}} = & \; 0\,. \qquad (1110)
\end{aligned}
$$

5.14 (Discussion) Since the form field is a *cyclic* coordinate in configuration space, the form field momenta are conserved in the present model. The contribution (1104) of the form field Lagrangian, though a kinetic term, represents an *effective* potential for the motion of the metric degrees of freedom (the moduli fields). The respective classical 'forces' are displayed above. Note that in the form field Lagrangian (1104) the moduli of the external and internal dimensions, $\alpha_{(i)}$ and β, appear with *both* signs in the exponents, hence

$$
\begin{aligned}
\lim_{\alpha_{(i)}\to\pm\infty} L_{(\mathcal{F}\wedge *\mathcal{F})} &= -\infty \\
\lim_{\beta\to\pm\infty} L_{(\mathcal{F}\wedge *\mathcal{F})} &= -\infty\,.
\end{aligned}
$$

But since $L_{(\mathcal{F}\wedge *\mathcal{F})} = -V_{\text{effective}}$, this means that there is a *potential well* with respect to $\alpha_{(i)}$ and β. (The walls that make up this potential well go under the name *electric p-form walls*, see e.g. [71] and references therein, because they are due to field strengths $\mathcal{F}$ proportional to dx^0. Potential walls arising due to field strengths with no x^0 component are accordingly called *magnetic*. These cannot arise in a homogeneous model.) Furthermore, since some of the terms in $L_{(\mathcal{F}\wedge *\mathcal{F})}$ have a ϕ dependence of $e^{-\phi}$ (1095), some of the infinitely high 'walls' of this potential well are receding with increasing ϕ (which is the time-like coordinate in configuration space).

Exactly such a situation, a potential well with exponentially increasing and receding walls, is what characterizes the well known *Mixmaster* scenario, which, in ordinary 4-dimensional homogeneous cosmology, is known from the dynamics of the Bianchi-IX model (*cf.* [134][136][135]).

In this scenario the 'universe point' in configuration space undergoes essentially free propagation until it hits one of the receding walls, whereupon it is reflected almost like in a billiard. The name 'Mixmaster' derives from the fact that the billiard-like motion in configuration space physically corresponds to a succession of epochs, in which some of the scales of the universe expand uniformly, while others contract, possibly changing roles in the next epoch.

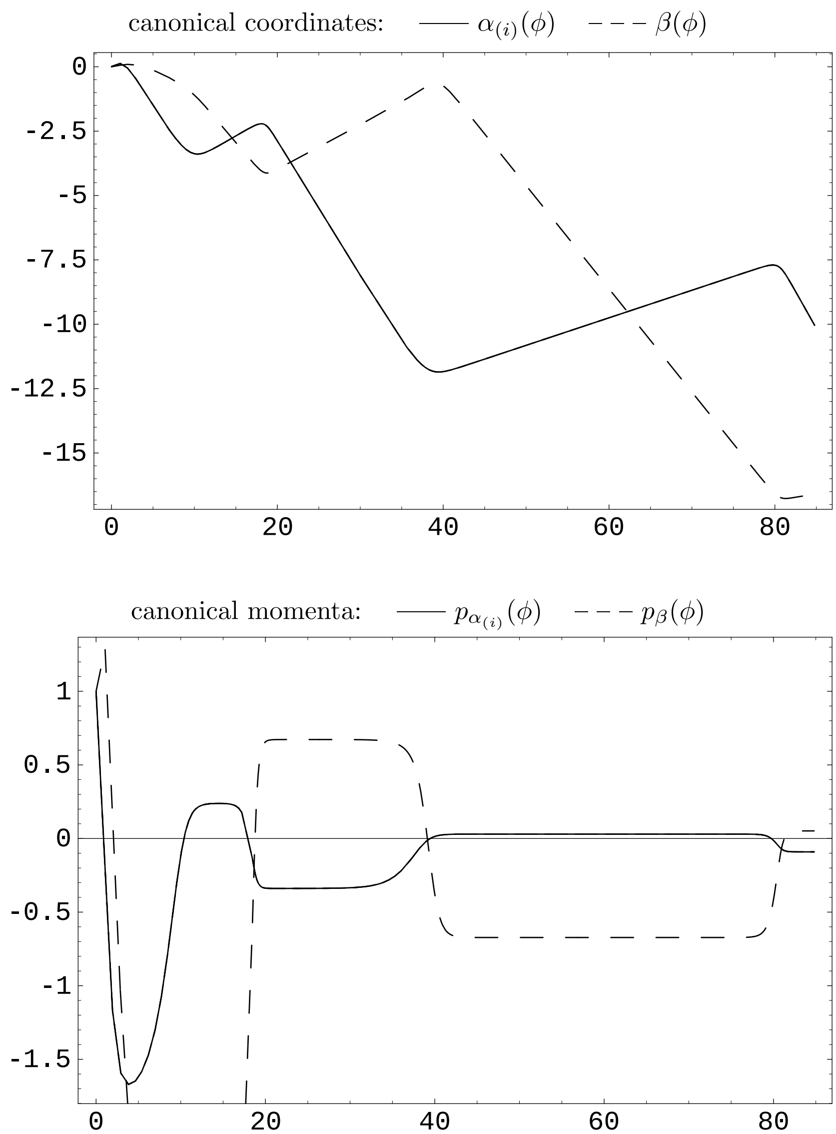
This behavior arises here in 11-D supergravity already in a simple Bianchi-I model (which has trivial free dynamics in the ordinary case) because of the presence of the 3-form field. In fact, from general considerations (*cf.* [71] [73][72] [74]) it is known that higher dimensional string and supergravity cosmologies will *generically* exhibit chaotic behavior. The present model realizes a special case of the chaos intrinsic to dynamics derived from string theory Lagrangians with p-form contributions.

5.15 (Numerical solution to the classical dynamics) In order to get some intuitive insight into the classical dynamics of our model, the equations of motion (1110) can be integrated numerically. Doing so requires the specification of initial values for the canonical coordinates (1107) and their associated momenta. Due to lack of any reason to prefer one such set over another, we set all initial coordinates, as well as all the moduli momenta (except for p_ϕ) to zero, and set all the 3-form field momenta to one. p_ϕ is then determined by solving the classical constraint $H = 0$:

$$\begin{aligned}
\phi(0)\,,\alpha_{(i)}(0)\,,\beta(0)\,,\mathcal{A}_{1,2,3}(0)\,,\ldots,\mathcal{A}_{5,4,10}(0) &:= 0\\
p_\phi(0)\,,p_{\alpha_{(i)}}(0)\,,p_\beta(0) &:= 0\\
p_{\mathcal{A}_{1,2,3}}(0)\,,\ldots,p_{\mathcal{A}_{5,4,10}}(0) &:= 1\\
p_\phi(0) &:= -\sqrt{\frac{109}{6}}
\end{aligned} \tag{1111}$$

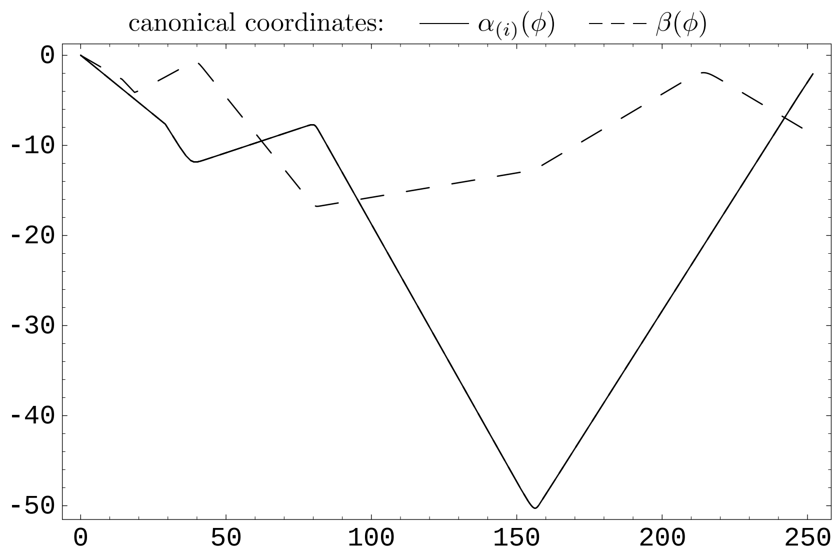
The result of the numerical solution of (1110) with initial values (1111) is displayed graphically in figures 8 (p.278), 9 (p.279).

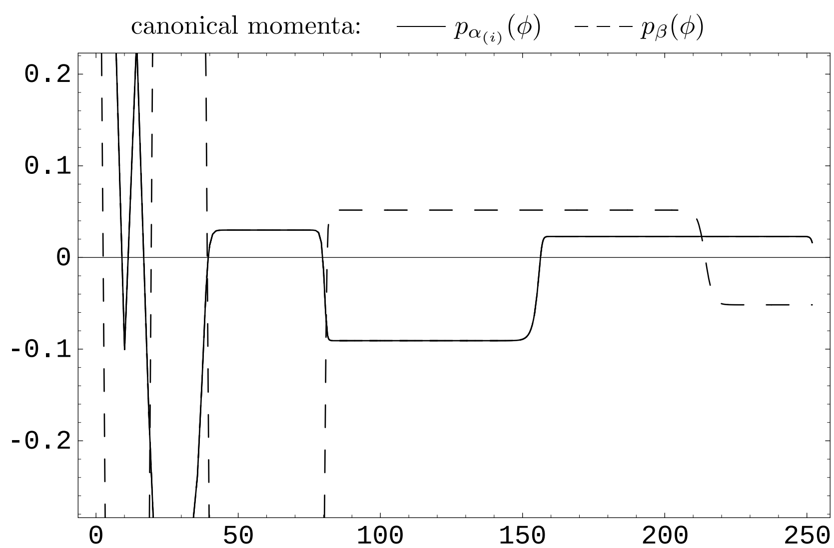
Figure 8



'Mixmaster' behavior of external and internal dimensions in 11-D super Bianchi-I. Plotted is the numerical solution of the classical equations of motion (1110) for initial values (1111) of the 11-D supergravity Bianchi-I model (1091) with general homogeneous 3-form field (1103). In the upper diagram the solid and dashed lines indicate, respectively, the values of the moduli $\alpha_{(i)}$ (size of external space dimensions) and β (size of internal Calabi-Yau dimensions) in dependence of the value of ϕ (playing the role of 'cosmological time'), which varies along the horizontal axis. The lower diagram shows the corresponding canonical momenta $p_{\alpha_{(i)}}$ and p_β. The same plot for a greater range of ϕ is shown in figure 9 (p.279)

Figure 9





'Mixmaster' behavior of external and internal dimensions in 11-D super Bianchi-I. This plot shows the same numerical result as figure 8 (p.278) (see there), but for a greater range of 'cosmological time' ϕ.

5.16 (The quantum Hamiltonian and Supersymmetry generators) Quantization of the model can follow the general prescriptions for homogeneous and diagonal models of supergravity given in §4.3.2 (p.230) and §4.3.3 (p.240). The supersymmetry generators (see note on spinor indices in 5.2 (p.256)) are

$$\begin{aligned} \hat{\bar{S}} &= \mathbf{d}_{\mathcal{M}^{(\mathrm{conf})}} \\ \hat{S} &= \mathbf{d}^{\dagger}{}_{\mathcal{M}^{(\mathrm{conf})}} \,. \end{aligned} \tag{1112}$$

The supersymmetrically extended Hamiltonian is, due to the vanishing of the superpotential, simply the Laplace-Beltrami-operator on $\mathcal{M}^{(\mathrm{conf})}$:

$$\hat{\mathbf{H}} = \left\{\mathbf{d}, \mathbf{d}^{\dagger}\right\} \,. \tag{1113}$$

We denote in the following the associated Dirac operators by

$$\begin{aligned} \mathbf{D} = \mathbf{D}_1 &:= \mathbf{d}_{\mathcal{M}^{(\mathrm{conf})}} + \mathbf{d}^{\dagger}{}_{\mathcal{M}^{(\mathrm{conf})}} \\ \mathbf{D}_2 &:= i\left(\mathbf{d}_{\mathcal{M}^{(\mathrm{conf})}} - \mathbf{d}^{\dagger}{}_{\mathcal{M}^{(\mathrm{conf})}}\right) \,. \end{aligned} \tag{1114}$$

The investigation of the quantum mechanics of the 11-D super-Bianchi-I model reduces to the study of the common space of zeros

$$\mathbf{D}_i \left|\phi\right\rangle = 0 \tag{1115}$$

of $\mathbf{D}_1$ and $\mathbf{D}_2$ on $\left(\mathcal{M}^{(\mathrm{conf})}, G\right)$.

(It is noteworthy that because the superpotential vanishes, so that the supersymmetry generators are ordinary (non-deformed) exterior (co-)derivatives, this system of constraints is exactly that of source free classical electromagnetism on $\mathcal{M}^{(\mathrm{conf})}$ (*cf.* 2.2.3 (p.70)), or rather, since $\mathcal{M}^{(\mathrm{conf})}$ is 20-dimensional, that of *generalized* source-free electromagnetism, as defined in 2.73 (p.73), 2.74 (p.73).)

Due to the relatively high dimensionality of $\mathcal{M}^{(\mathrm{conf})}$, solving (1115) is, while conceptually straightforward, a practically rather demanding task. For this reason some simple special cases of (1115) will be studied in order to gain qualitative insight into the system. The following definition gives a possible family of scenarios that arise by setting various components of the 3-form field $\mathcal{A}$ to zero:

5.17 (Configuration space scenarios)
By neglecting all components of $\mathcal{A}$ except for either of $\mathcal{A}_{(1,2,3)}$,$\mathcal{A}_{(4,5,6)}$, or $\mathcal{A}_{(4,5,10)}$ (see (1106)) one arrives at an effectively 4-dimensional configuration space with a metric of the form

$$G = \left(G_{(m)(n)}\right) = \mathrm{diag}\Big(-1, 3, 6, e^{2ax^0+2bx^1+2cx^2}\Big) \tag{1116}$$

for some real constants a, b, c.

This highly simplified configuration space does not exhibit an effective potential well as found in the full model (*cf.* note 5.14 (p.276)) but still features a single (effective) potential wall. This, together with its simplicity, makes the metric (1116) well suited for studying reflections of wave packets at such walls (*cf.* simulation 5.18 (p.281)). Note that, for constant form field momenta, the qualitative dynamics induced by (1116) is very similar to that of the

Kantowski-Sachs model of ordinary 4-dimensional gravity[60] (*cf.* example 5.4 (p.259)). Further note that this scenario corresponds to those studied in [46] (where, however, the kinetic terms that act as effective potential energies are treated like true potentials).

From the classical dynamics (*cf.* simulation 5.15 (p.277)) we know that localized wave packets will undergo scattering at the exponential effective potential walls induced by the configuration space metric associated with the form field (1106). In order to get an impression of this effect, the following simulation numerically propagates a certain wave packet of the form of that studied in simulation 5.5 (p.260), along the 'time-like' coordinate of configuration space.

5.18 (Kantowski-Sachs-like scenario in 11-D super-Bianchi-I) A dynamical situation similar to that of the Kantowski-Sachs scenario (simulation 5.5 (p.260)) can be found in the 11-D super-Bianchi-I model by concentrating on the effective potential induced by a single component of the three form field which increases exponentially with one of the α or β coordinates. There are several possibilities to reduce the configuration space to merely 3 dimensions $\left(\mathcal{M}^{(\mathrm{rconf})}, g^{(\mathrm{rconf})}\right)$, $D\left(\mathcal{M}^{(\mathrm{rconf})}\right) = 3$, such that the metric is:

$$G^{(\mathrm{rconf})} := \mathrm{diag}\Big(-1, 1, e^{(x^1)}\Big) \tag{1117}$$

$$\begin{aligned} x^0 &= \phi \in \{-\infty, \infty\} \\ x^1 &= \left\{\alpha_{(i)}, \beta\right\} \in \{-\infty, \infty\} \\ x^2 &= \mathcal{A}_{\dots} \in \{-\infty, \infty\} \ . \end{aligned} \tag{1118}$$

The exterior Dirac operator on this reduced configuration space locally reads:

$$\begin{aligned} \mathbf{D} &= \left(\mathbf{d} + \mathbf{d}^\dagger\right)_{G^{(\mathrm{rconf})}} \\ &= \hat{\gamma}_-^0 \partial_0 + \hat{\gamma}_-^1 \partial_1 + e^{-(x^1)/2} \hat{\gamma}_-^3 \partial_3 + \frac{1}{2} \hat{\gamma}_-^2 \,\hat{\mathrm{e}}^{\dagger^3} \hat{\mathrm{e}}^2 \,, \end{aligned}$$

and accordingly the generator of x^0 evolution is: (*cf.* 2.101 (p.100) and 2.102 (p.101))

$$\hat{A} \;=\; -\hat{\gamma}_-^0 \hat{\gamma}_-^1 \partial_1 - e^{-(x^1)/2} \hat{\gamma}_-^0 \hat{\gamma}_-^2 \partial_2 - \frac{1}{2} \hat{\gamma}_-^0 \hat{\gamma}_-^2 \,\hat{\mathrm{e}}^{\dagger^2} \hat{\mathrm{e}}^1 \,. \tag{1119}$$

In the tradition of the other simulation (5.5 (p.260)) the initial state is chosen to be a Gaussian with respect to the x^1-coordinate in the 'left going' component (*cf.* §2.2.5 (p.81)):

$$|\phi_0\rangle := e^{-(x^1)} \sin\!\left(x^2\right) \frac{1}{2} \left(1 + \hat{\gamma}_-^0 \hat{\gamma}_-^1\right) \frac{1}{2} \left(1 + \hat{\gamma}_-^0\right) |0\rangle \ . \tag{1120}$$

Since the kinetic energy of the form field constitutes the effective potential (see note 5.14 (p.276)) the additional x^2-dependence here is chosen to be a simple sine-wave. This roughly corresponds classically to the initial values chosen in simulation 5.15 (p.277).

[60] The dynamics in mini-superspace, that is. The corresponding dynamics of the physical universe is of course radically different in both cases.

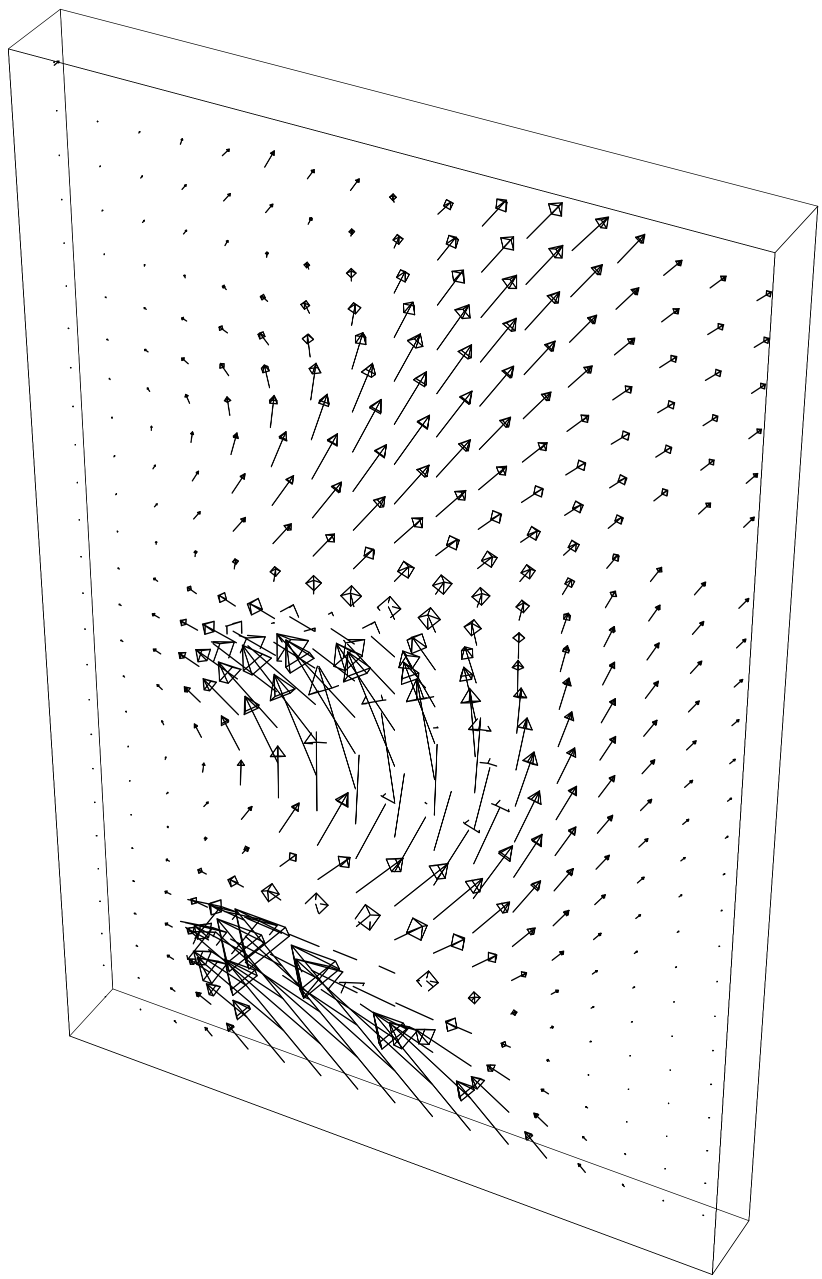
For numerical purposes, the expression

$$\left|\phi\left(x^0\right)\right\rangle = \exp\left(x^0 \hat{A}\right) \left|\phi_0\right\rangle$$

has been expanded to 70-th order in $\hat{A}$. It turns out, surprisingly, that this $|\phi\rangle$, which is by construction a solution to one of the supersymmetry constraints, $\mathbf{D}\,|\phi\rangle = 0$, is also apparently annihilated by the other supercharge, $\mathbf{D}_2\,|\phi\rangle \approx 0$ (namely within the precision of the numerics). This means that one need not follow the prescription 2.105 (p.103) to turn $|\phi\rangle$ from an $N = 1$ to an $N = 2$ supersymmetric state by acting with $\mathbf{D}_2$ on it. Compare this with the opposite case 5.6 (p.264) in the similar situation 5.5 (p.260) arising in the Kantowski-Sachs model of $N = 1$, $D = 4$ supergravity, 5.4 (p.259).

Figure 10 (p.283) shows a 3-dimensional view of the (conserved) probability current (see corollary 2.80 (p.78)) for fixed $x^2 = \pi/3$. (The fact that in the graphic representation the current does not *appear* to be conserved is due to the diagram showing only a single slice through configuration space at fixed $x^2 = \pi/3$.) As expected, the wave packet is reflected at the effective potential wall to the left (compare with 5.5 (p.260)). But, interestingly, part of the packet returns to scatter a second time. This is reminiscent of the same behavior of the $N = 2$ supersymmetric current of the Kantowski-Sachs model in figure 6 (p.264).

Figure 10



Probability current of a scattering event in 11-D super Bianchi-I: Shown is the probability current (*cf.* §2.2.4 (p.78)) on a slice through 3-dimensional mini-superspace of a simple scenario (5.18 (p.281)) of the homogeneous model of $N = 1$, $D = 11$ supergravity introduced in §5.2 (p.266). The spacetime moduli field $x^1 = \alpha$ (scale factor of the external spatial dimensions) varies along the horizontal axis, the (modified) dilaton $x^0 = \phi$ (being the time-like parameter) along the vertical axis, and the third dimension indicates the range of the sole component of the 3-form field amplitude $x^3 = \mathcal{A}$.

Remarkably, the current shows again the *zitterbewegung*-type form already noticed in the caption of figure 6 (p.264). See there for more details.

Finally we turn to an investigation of possible hidden symmetries on configuration space (*cf.* §2.2.7 (p.90) and §4.3.4 (p.250)):

5.19 (Algebra and symmetries of scenario 5.17 (p.280)) The supersymmetric quantum mechanics associated with scenario 5.17 (p.280) (1116) of the 11-dimensional super-Bianchi-I model (1091) is governed by the supersymmetry constraint

$$\mathbf{D}\,|\phi\rangle = 0\,,$$

where $\mathbf{D}$ is the exterior Dirac operator associated with G:

$$\mathbf{D} = \mathbf{d}_G + \mathbf{d}^\dagger{}_G\,.$$

The underlying superalgebra is (*cf.* §2.1.3 (p.43))

$$\left\{\mathbf{\Delta}_G = \mathbf{D}^2, \mathbf{D}, \iota = (-1)^{\hat{N}}\right\}\,,$$

where the even generator $\mathbf{\Delta}_G$ is the Laplace-Beltrami operator on G representing the supersymmetrically extended Wheeler-deWitt constraint:

$$\mathbf{\Delta}\,|\phi\rangle = 0\,.$$

This superalgebra can be further investigated by systematically examining the geometric properties of the underlying manifold, most importantly among which are symmetries associated with generalized Killing tensors of the metric (*cf.* 2.2.7 (p.90)). It turns out that G admits two complex structures which give rise to two *hidden* supersymmetries and two associated central charges. **Note** that in the following coordinates are labeled x^1, x^2, x^3 instead of x^0, x^1, x^2.

1. *Basic geometric quantities:*

$$\begin{aligned}
\sqrt{\det(g)} &= 3\sqrt{2}e^{ax^1+bx^2+cx^3}\\
R &= \frac{1}{3}\left(6a^2 - 2b^2 - c^2\right)\\
e = (e^\mu{}_a) &= \mathrm{diag}\left(1, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, e^{-ax^1-bx^2-cx^3}\right)\\
\tilde{e} = (\tilde{e}^\mu{}_a) &= \mathrm{diag}\left(1, \sqrt{3}, \sqrt{6}, e^{ax^1+bx^2+cx^3}\right)
\end{aligned}$$

$$\begin{aligned}
\omega_{(\mu=1)} = \left(\omega_{(\mu=1)}{}^a{}_b\right) &= 0\\
\omega_{(\mu=2)} = \left(\omega_{(\mu=2)}{}^a{}_b\right) &= 0\\
\omega_{(\mu=3)} = \left(\omega_{(\mu=3)}{}^a{}_b\right) &= 0\\
\omega_{(\mu=4)} = \left(\omega_{(\mu=4)}{}^a{}_b\right) &= e^{ax^1+bx^2+cx^3}\begin{bmatrix} 0 & 0 & 0 & a\\ 0 & 0 & 0 & -b/\sqrt{3}\\ 0 & 0 & 0 & -c/\sqrt{6}\\ a & b/\sqrt{3} & c/\sqrt{6} & 0\end{bmatrix}.
\end{aligned}$$

2. *Killing-Yano tensor of valence 2:* One finds the following general second rank Killing-Yano tensor $f^{(2)}$:

$$f^{(2)}(k_1,k_2)=\left(f^{(2)}_{\mu\nu}\right)=$$
$$\begin{bmatrix} 0 & -k_2\frac{1}{6}c & k_1\frac{1}{3}b & k_2ae^{ax^1+bx^2+c^3} \\ k_2\frac{1}{6}c & 0 & k_1a & k_2be^{ax^1+bx^2+c^3} \\ -k_1\frac{1}{3}b & -k_1a & 0 & k_2ce^{ax^1+bx^2+c^3} \\ -k_2ae^{ax^1+bx^2+c^3} & -k_2be^{ax^1+bx^2+c^3} & -k_2ce^{ax^1+bx^2+c^3} & 0 \end{bmatrix}, \tag{1121}$$

which turns out to be covariantly constant:

$$\nabla_\lambda f^{(2)}_{\mu\nu}=0\,. \tag{1122}$$

3. *Killing-Yano tensor of valence 3:* The metric also allows the following general third rank Killing-Yano tensor:

$$\begin{aligned} f^{(3)}(k_3,k_4) &= \left(f^{(3)}_{\mu\nu\lambda}\right) \\ f^{(3)}_{0,1,2} &= 0 \\ f^{(3)}_{0,1,3} &= k_3be^{ax^1+bx^2+cx^3} \\ f^{(3)}_{0,2,3} &= (k_3c+k_4a)\,e^{ax^1+bx^2+cx^3} \\ f^{(3)}_{1,2,3} &= k_4be^{ax^1+bx^2+cx^3}\,. \end{aligned} \tag{1123}$$

which is also covariantly constant:

$$\nabla_\kappa f^{(3)}_{\mu\nu\lambda}=0\,. \tag{1124}$$

4. *Stäckel-Killing tensor of valence 2:* From the square of the valence 2 Killing-Yano tensor $f^{(2)}$ one obtains the symmetric Stäckel-Killing tensor $K^{(2)}$:

$$K^{(2)}(k_1,k_2)=\left(K^{(2)}_{\mu\nu}\right)=f^{(2)}_{\mu\kappa}g^{\kappa\lambda}f^{(2)}_{\lambda\nu}=$$
$$\begin{bmatrix} -k_2^2a^2-k_1^2\frac{1}{108}\left(2b^2+c^2\right) & -\left(k_2^2+\frac{1}{18}k_1^2\right)ab & -\left(k_2^2+\frac{1}{18}k_1^2\right)ac & 0 \\ -\left(k_2^2+\frac{1}{18}k_1^2\right)ab & -k_2^2b^2+k_1^2\frac{1}{36}\left(c^2-6a^2\right) & -\left(k_2^2+\frac{1}{18}k_2^2\right)bc & 0 \\ -\left(k_2^2+\frac{1}{18}k_1^2\right)ac & -\left(k_2^2+\frac{1}{18}k_1^2\right)bc & -k_2^2c^2+k_1^2\frac{1}{9}\left(b^2-3a^2\right) & 0 \\ 0 & 0 & 0 & K_{4,4} \end{bmatrix}$$
$$K_{4,4}=k_2^2\frac{1}{6}\left(6a^2-2b^2-c^2\right)e^{2ax^1+2bx^2+2cx^3} \tag{1125}$$

For a certain choice of constants this is equal to minus the metric (1116):

$$K^{(2)}\left(k_1=\pm\sqrt{\frac{108}{6a^2-2b^2-c^2}},\,k_2=\pm\sqrt{\frac{-6}{6a^2-2b^2-c^2}}\right)=-g\,. \tag{1126}$$

Note that for given real constants a,b,c one of k_1,k_2 in (1126) will be real, the other imaginary. This is a consequence of the indefiniteness of the metric (1116).

5. *Complex structures:* According to (1126) one finds two complex structures[61] on the tangent bundle:

$$
\begin{aligned}
J_+ = \left(J^{\mu}_{+\nu}\right) &:= \left(g^{\mu\kappa} f^{(2)}_{\kappa\nu}\left(k_1 = +\sqrt{\frac{108}{6a^2-2b^2-c^2}}, k_2 = +\sqrt{\frac{-6}{6a^2-2b^2-c^2}}\right)\right) \\
J_- = \left(J^{\mu}_{-\nu}\right) &:= \left(g^{\mu\kappa} f^{(2)}_{\kappa\nu}\left(k_1 = +\sqrt{\frac{108}{6a^2-2b^2-c^2}}, k_2 = -\sqrt{\frac{-6}{6a^2-2b^2-c^2}}\right)\right)
\end{aligned}
\tag{1127}
$$

By construction, J_+, J_- both square to minus the identity

$$
J^2_{\pm} \;=\; -1\,. \tag{1128}
$$

They commute and their product is a symmetric (and covariantly constant) Stäckel-Killing tensor:

$$
J_+ J_- = J_- J_+ \;:=\; (g^{\mu\kappa}\tilde{g}_{\kappa\nu}) \tag{1129}
$$

that plays the role of a *dual* metric $\tilde{g}$ (in the sense of [225]):

$$
\tilde{g} = (\tilde{g}_{\mu\nu}) \;=\; \begin{bmatrix}
\frac{6a^2+2b^2+c^2}{-6a^2+2b^2+c^2} & \frac{12ab}{-6a^2+2b^2+c^2} & \frac{12ac}{-6a^2+2b^2+c^2} & 0 \\
\frac{12ab}{-6a^2+2b^2+c^2} & \frac{18a^2+6b^2-3c^2}{-6a^2+2b^2+c^2} & \frac{12bc}{-6a^2+2b^2+c^2} & 0 \\
\frac{12ac}{-6a^2+2b^2+c^2} & \frac{12bc}{-6a^2+2b^2+c^2} & \frac{36a^2-12b^2+6c^2}{-6a^2+2b^2+c^2} & 0 \\
0 & 0 & 0 & e^{2ax^1+2bx^2+2cx^3}
\end{bmatrix} \tag{1130}
$$

6. *Clifford-2-vectors associated with the Killing-Yano tensor:* It will prove convenient to introduce the following Clifford-2-vectors associated with the Killing-Yano tensor $f^{(2)}$:

$$
\begin{aligned}
\mathbf{f}_1 &:= \frac{1}{2} f^{(2)}_{ab}\left(k_1 = \sqrt{\frac{108}{6a^2-2b^2-c^2}}, k_2 = 0\right)\hat{\gamma}^a_-\hat{\gamma}^b_- \\
\mathbf{f}_2 &:= \frac{1}{2} f^{(2)}_{ab}\left(k_1 = 0, k_2 = \sqrt{\frac{6}{6a^2-2b^2-c^2}}\right)\hat{\gamma}^a_-\hat{\gamma}^b_- \\
\mathbf{J}_+ &:= \frac{1}{2} J_{+ab}\hat{\gamma}^a_-\hat{\gamma}^b_- \\
&= \mathbf{f}_1 + i\mathbf{f}_2 \\
\mathbf{J}_- &:= \frac{1}{2} J_{-ab}\hat{\gamma}^a_-\hat{\gamma}^b_- \\
&= \mathbf{f}_1 - i\mathbf{f}_2\,.
\end{aligned}
\tag{1131}
$$

[61] Here the term “complex structure”, though convenient, is abuse of terminology, since $J\pm$ involve the imaginary unit and hence are linear operators on the already complexified tangent bundle. The point is that these tensors do square to minus the identity tensor when contracted with the semi-Riemannian metric.

These can be checked to satisfy:

$$\begin{aligned}
(\mathbf{f}_{1,2})^\dagger &= -\mathbf{f}_{1,2} \\
(\mathbf{f}_1)^2 &= -1 \\
(\mathbf{f}_2)^2 &= 1 \\
[\mathbf{f}_1, \mathbf{f}_2] &= 0 \\
\{\mathbf{f}_1, \mathbf{f}_2\} &= 2\mathrm{I}_- \,, \qquad (1132)
\end{aligned}$$

which implies:

$$\begin{aligned}
(\mathbf{J}_\pm)^\dagger &= -\mathbf{J}_\mp \qquad (1133) \\
(\mathbf{J}_+)^2 &= -2\,(1 - i\mathrm{I}_-) \\
&= -2\,(-1 + \bar{\gamma}_-) \\
(\mathbf{J}_-)^2 &= -2\,(1 + i\mathrm{I}_-) \\
&= -2\,(1 + \bar{\gamma}_-) \\
\mathbf{J}_\pm \mathbf{J}_\mp &= 0\,. \qquad (1134)
\end{aligned}$$

(For definitions and properties of the pseudoscalar I and the chirality operator $\bar{\gamma}$ see §2.1.1 (p.15) and in particular B.14 (p.305) and B.16 (p.307).)

7. **D**-*harmonic operators:* The following operators can be checked to be harmonic, i.e. they commute with the Laplace operator $\mathbf{D}^2$ (*cf.* definition 2.87 (p.91)):

   - *Killing-Yano Clifford-2-vector:*

   $$\frac{1}{2} f^{(2)}_{ab} \hat{\gamma}^a_- \hat{\gamma}^b_- \,. \qquad (1135)$$

   - *Fundamental symplectic operators:*

   $$\begin{aligned}
   &\frac{1}{2} f^{(2)}_{ab} \hat{\mathrm{e}}^{\dagger a} \hat{\mathrm{e}}^{\dagger b} \\
   &\frac{1}{2} f^{(2)}_{ab} \hat{\mathrm{e}}^{a} \hat{\mathrm{e}}^{b} \qquad (1136)
   \end{aligned}$$

   - *Killing-Yano Clifford-3-vector:*

   $$\frac{1}{6} f^{(3)}_{abc} \hat{\gamma}^b_- \hat{\gamma}^b_- \hat{\gamma}^c_- \qquad (1137)$$

8. *Extended supersymmetry algebra:* According to consequence 2.89 (p.92), every harmonic operator can be "closed" to yield an operator that super-commutes with **D**. Associated with the harmonic operators (1135) are two *hidden supercharges* $\tilde{\mathbf{D}}_\pm$:

$$\begin{aligned}
\tilde{\mathbf{D}}_\pm &:= \frac{1}{2}\left[\mathbf{D}, \mathbf{J}_\pm\right] \\
&= \frac{1}{2}\left[\hat{\gamma}^a_- \hat{\nabla}_a, \frac{1}{2} J_{\pm ab} \hat{\gamma}^a_- \hat{\gamma}^b_-\right] \\
&= \frac{1}{2}\left[\hat{\gamma}^a_-, \frac{1}{2} J_{\pm ab} \hat{\gamma}^a_- \hat{\gamma}^b_-\right] \hat{\nabla}_a + \hat{\gamma}^a_- \left[\hat{\nabla}_a, \frac{1}{2} J_{\pm ab} \hat{\gamma}^a_- \hat{\gamma}^b_-\right] \\
&= J^a_{\pm b} \hat{\gamma}^b_- \hat{\nabla}_a + \hat{\gamma}^a_- \frac{1}{2} \left(\nabla_a J_{\pm bc}\right) \hat{\gamma}^b_- \hat{\gamma}^c_- \\
&= J^a_{\pm b} \hat{\gamma}^b_- \hat{\nabla}_a \,, \qquad (1138)
\end{aligned}$$

where the last line follows because $J_\pm$ is covariantly constant. These hidden charges satisfy:

$$\begin{aligned}
\left(\tilde{\mathbf{D}}_\pm\right)^\dagger &= \tilde{\mathbf{D}}_\mp \\
\left\{\mathbf{D}, \tilde{\mathbf{D}}_\pm\right\} &= 0 \\
\left\{\tilde{\mathbf{D}}_\pm, \tilde{\mathbf{D}}_\pm\right\} &= \{\mathbf{D}, \mathbf{D}\} \\
&= 2\mathbf{\Delta}_G \\
\left\{\tilde{\mathbf{D}}_+, \tilde{\mathbf{D}}_-\right\} &= 2\mathbf{\Delta}_{\tilde{G}} \\
[\mathbf{\Delta}_G, \mathbf{\Delta}_{\tilde{G}}] &= 0\,,
\end{aligned} \tag{1139}$$

where $\mathbf{\Delta}_{\tilde{G}}$ is the exterior Laplace operator associated with the 'dual' metric $\tilde{g}$ (1130). (The first equation follows from (1133) together with the definition (1138). The second is the usual consequence of $\mathbf{J}_\pm$ being $\mathbf{D}$-harmonic.)
In order to diagonalize these anti-commutation relations introduce the following operators:

$$\begin{aligned}
\mathbf{D}_0 &:= \mathbf{D} \\
\mathbf{D}_1 &:= \frac{1}{\sqrt{2}}\left(\tilde{\mathbf{D}}_+ + \tilde{\mathbf{D}}_-\right) \\
&= \frac{1}{\sqrt{2}}[\mathbf{D}, \mathbf{f}_1] \\
\mathbf{D}_2 &:= \frac{-i}{\sqrt{2}}\left(\tilde{\mathbf{D}}_+ - \tilde{\mathbf{D}}_-\right) \\
&= \frac{1}{\sqrt{2}}[\mathbf{D}, \mathbf{f}_2] \\
\mathbf{Z}_0 &:= \mathbf{\Delta}_G \\
\mathbf{Z}_1 &= \mathbf{\Delta}_G + \mathbf{\Delta}_{\tilde{G}} \\
\mathbf{Z}_2 &= -\mathbf{\Delta}_G + \mathbf{\Delta}_{\tilde{G}}\,.
\end{aligned} \tag{1140}$$

These then satisfy the relations:

$$\begin{aligned}
(\mathbf{D}_i)^\dagger &= \mathbf{D}_i \\
\{\mathbf{D}_i, \mathbf{D}_j\} &= 2\delta_{ij}\mathbf{Z}_i
\end{aligned} \tag{1141}$$

for $i, j \in \{0, 1, 2\}$ (no sum over repeated indices). This establishes an extended superalgebra with central charges $\mathbf{Z}_i$.
All these supercharges can be decomposed into their nilpotent components

$$\mathbf{D}_i = \mathbf{d}_i + \mathbf{d}^\dagger{}_i \tag{1142}$$

satisfying the canonical creation and annihilation algebra

$$\begin{aligned}
\{\mathbf{d}_i, \mathbf{d}_j\} &= 0 \\
\left\{\mathbf{d}^\dagger{}_i, \mathbf{d}^\dagger{}_j\right\} &= 0 \\
\left\{\mathbf{d}_i, \mathbf{d}^\dagger{}_j\right\} &= 2\delta_{ij}\mathbf{Z}_i\,.
\end{aligned} \tag{1143}$$

Representations of this algebra are found as usual by considering appropriate 'vacuum' states $\left|E, \tilde{E}\right\rangle$ that are eigenstates of the Casimir operators and annihilated by the $\mathbf{d}^\dagger{}_i$:

$$\begin{aligned} \mathbf{\Delta}_G \left|E, \tilde{E}\right\rangle &= E \\ \mathbf{\Delta}_{\tilde{G}} \left|E, \tilde{E}\right\rangle &= \tilde{E} \\ \mathbf{d}^\dagger{}_i \left|E, \tilde{E}\right\rangle &= 0\,. \end{aligned} \tag{1144}$$

The general supermultiplet then consists of the $2^3 = 8$ states obtained from $\left|E, \tilde{E}\right\rangle$ by acting on it with with the 'creators' $\mathbf{d}_i$. Of these are physical only those annihilated by $\mathbf{d}_0$:

$$\mathbf{d}_0 \left|E, \tilde{E}\right\rangle_{\mathrm{ph}} = 0\,. \tag{1145}$$

This gives physical supermultiplets with $2^2 = 4$ states:

$$\left\{ \mathbf{d}_1\mathbf{d}_2 \left|E, \tilde{E}\right\rangle_{\mathrm{ph}}, \mathbf{d}_1 \left|E, \tilde{E}\right\rangle_{\mathrm{ph}}, \mathbf{d}_2 \left|E, \tilde{E}\right\rangle_{\mathrm{ph}}, \left|E, \tilde{E}\right\rangle_{\mathrm{ph}} \right\}\,. \tag{1146}$$

In particular, states with $\bar{E} = 0$, are states of higher supersymmetry.

In summary, the above calculation shows that scenario 5.17 (p.280) of the general homogeneous super-Bianchi-I model features higher supersymmetry. This symmetry can be expressed in terms of geometric operators of a 'dual' metric $\tilde{G}$ on configuration space. If this higher symmetry has any deeper relevance remains to be investigated (see point 6 of 6.2 (p.291)).

Example 5.20 The simplest non-trivial example of the above considerations arises when the only excited 3-form field component is $\mathcal{A}_{(4,5,10)}$. According to (1106) one has

$$g\big(\mathcal{A}_{(4,5,10)}, \mathcal{A}_{(4,5,10)}\big) = e^{(-4\beta)}$$

and so this situation is described by scenario 1 (1116) with parameters

$$\begin{aligned} a &= 0 \\ b &= -2 \\ c &= 0\,. \end{aligned} \tag{1147}$$

The metric on configuration space is

$$g = \mathrm{diag}\Big(-1, 3, 6, e^{(-4\beta)}\Big)\,. \tag{1148}$$

Inserting (1147) into the results of calculation 5.19 (p.284) yields:

$$\begin{aligned} \sqrt{\det(g)} &= 3\sqrt{2}e^{-4\beta} \\ R &= -\frac{8}{3} \end{aligned}$$

$$\begin{aligned}
f^{(2)} = \left(f^{(2)}_{\mu\nu}\right) &= \begin{bmatrix} 0 & 0 & -\frac{2}{3}k_1 & 0 \\ 0 & 0 & 0 & k_2 e^{-2x^2} \\ \frac{2}{3}k_1 & 0 & 0 & 0 \\ 0 & k_2 2e^{-2x^2} & 0 & 0 \end{bmatrix} \\
f^{(3)} = \left(f^{(3)}_{\mu\nu\lambda}\right) &= \left(f^{(3)}_{0,1,2} = 0\,,\ f^{(3)}_{0,1,3} = -2k_3 e^{-2x^2}\,,\ f^{(3)}_{0,2,3} = 0\,,\ f^{(3)}_{1,2,3} = -2k_4 e^{-2x^2}\,,\ \right) \\
K^{(2)} = \left(K^{(2)}_{\mu\nu}\right) &= \mathrm{diag}\left(-\frac{2}{27}k_1^2,\ -3\frac{4}{3}k_2^2,\ 6\frac{2}{27}k_1^2,\ -\frac{4}{3}k_2^2 e^{-4x^2}\right) \qquad (1149)
\end{aligned}$$

.

6 Discussion

6.1 (Summary and conclusion) It has been shown that canonical quantum supergravity is governed by an infinite dimensional version of the constraints of covariant supersymmetric quantum mechanics, and, at least when reduced to a small finite number of degrees of freedom, the latter has been found to exhibit a rich formal structure and to admit constructions that may proof useful in the study of supersymmetric quantum cosmology. The models that have been discussed testify to the wealth of interesting phenomena to be discovered in this field.

If nothing else, it seems one can conclude that the "Hamiltonian route" (*cf.* p. 9) to supersymmetric quantum cosmology is a promising way to approach the physical and technical issues of this subject. It is true to the geometrical character inherent to supersymmetry and thus makes transparent some otherwise not as clearly visible structures underlying the formalism. While this text has concentrated on simplified models and the associated finite-dimensional quantum mechanics, the discussion of §4.3 (p.192), which in particular shows that the supersymmetry generators of canonical supergravity are deformed exterior derivatives, indicates that it is maybe not completely unreasonable to expect that some techniques based on the differential geometric nature of covariant SQM, which governs supersymmetric quantum cosmology, can be lifted without too much effort to the full theory of supergravity. This certainly requires and motivates further study.

6.2 (Open questions) The whole field of supersymmetric quantum cosmology is rich in very deep and very general open questions, some of which have been briefly mentioned in §4.8 (p.185). These shall not concern us here. Instead, there are a couple of questions that immediately pose themselves in the restricted context of this text, and which I have not managed to address, not to mention answer, under the given constraints on ability and time. Some of these questions should require nothing but a little straightfroward calculation, others may require new principal ideas:

1. **Description of higher-dimensional SQM by means of random walks as in the Feynman checkerboard model.** In §2.2.5 (p.81) it is shown how the description of the Dirac particle in 1+1 dimensions ("Feynman checkerboard model") generalizes to supersymmetric quantum mechanics. In more than 2-dimensions a similar approach should be possible, but encounters technical problems that are absent in the simplistic 1+1 dimensional setting. *What is the correct description of SQM in arbitrary dimensions (and arbitrary fermion sectors) by means of checkerboard-like random walks?*

2. **Statistical ensembles for supersymmetric cosmological models.** In 2.107 (p.105) and 2.108 (p.105) a formal method for studying canonical ensembles of gauge covariant systems in general and systems governed by a constraint of the form of a generalized Dirac operator in particular has been discussed. *Can this method be fruitfully applied in supersymmetric quantum cosmology?*

3. **Complex geometry of $\mathcal{M}^{(\mathrm{conf})}$.** According to 4.24 (p.203), 4.27 (p.208), 4.28 (p.210), and 4.29 (p.210) the configuration space $\mathcal{M}^{(\mathrm{conf})}$ of $N =$

1, $D = 4$ supergravity should be a Kähler manifold. *What exactly is the complex structure, what are the holomorphic and the antiholomorphic coordinates and how do the metric, the connection and other geometrical quantities read in the holomorphic/antiholomorphic basis?*

4. **Choosing a factoring for states and associated constraints for truncated models.** The "Born-Oppenheimer"-like approximation to full supergravity by means of truncated cosmological models requires a somewhat arbitrary choice of factorization of the state vector and of the truncated constraints (*cf.* 4.58 (p.247)). *Which truncated constraints should be imposed on which factoring to produce a good approximation to the full dynamics?*

5. **Solving $N = 4$ SQM in configuration space by finding solutions to $N = 2$ constraints.** The two spin components of the two supersymmetry generators give 4 supersymmetry generators per mode. This implies in particular 4 supersymmetry generators on mini-superspace of homogeneous models. But only 2 of these are independent as constraints on Lorentz invariant states (*cf.* 4.39 (p.223)). Hence it is in principle sufficient to solve only one spin component of the supersymmetry constraint and automatically obtain a solution to the full set of constraints. For simple examples this is easily demonstrated, but (*cf.* 4.59 (p.248)): *How exactly does an $N = 2$ solution (in one spin component) give rise to an $N = 4$ solution (in both spin components) in general and in particular?*

6. **N-extended supergravity and hidden supercharges on configuration space.** The cosmological model of $N = 1$, $D = 11$ supergravity discussed in §5.2 (p.266) is found to give rise to a mini-superspace whose geometry admits hidden supersymmetries and central charges (see 5.19 (p.284)). According to the conjecture presented in §4.3.4 (p.250) these might be related to spacetime supersymmetries in 11 dimensions which remain unbroken after dimensional reduction. *Can this connection be substantiated? Or, else, can it be ruled out?*

A Mechanics

Outline. In this section some aspects of classical and quantum mechanics are reviewed, with an emphasis on the covariant relativistic case, insofar as they are needed in the main text.

Consider the relativistic point propagarting on the pseudo-Riemannian manifold $(\mathcal{M}, g_{\mu\nu})$ (see 2.2 (p.16) for notational conventions). Classically, there are several equivalent action functionals describing the dynamics. One, applying to the free case, is

$$\begin{aligned} S &= -m \int ds \\ &= -m \int \sqrt{-g_{\mu\nu}(x)\,\dot{x}^\mu \dot{x}^\nu}\; d\tau\,. \end{aligned} \tag{1150}$$

Here an overdot indicates the derivative with respect to the parameter τ along the point's worldline:

$$\dot{x}^\mu := \partial_\tau x^\mu\,.$$

Hence the respective Lagrangian is

$$L = -m\sqrt{-g_{\mu\nu}(x)\,\dot{x}^\mu \dot{x}^\nu}\,, \tag{1151}$$

from which the canonical momenta are obtained as:

$$\begin{aligned} p_\mu &= \frac{\delta L}{\delta \dot{x}^\mu} \\ &= -m \frac{g_{\mu\nu}\dot{x}^\nu}{\sqrt{-\dot{x}^\mu \dot{x}^\nu}}\,. \end{aligned} \tag{1152}$$

It follows that

$$\Rightarrow\; p_\mu p^\mu \;=\; -m^2\,, \tag{1153}$$

which expresses the *Hamiltonian constraint* to which the system is subjected. To make this more explicit one can use the equivalent action

$$\begin{aligned} S' &= \int L'\; d\tau \\ &= \int \left(p_\mu \dot{x}^\mu - H'\right)\; d\tau \\ &= \int \left(p_\mu \dot{x}^\mu - N\frac{1}{2}\left(\frac{1}{m} g^{\mu\nu}(x)\, p_\mu p_\nu + m\right)\right)\; d\tau\,, \end{aligned} \tag{1154}$$

which leads to

$$\begin{aligned} \dot{x}^\mu = \frac{\partial H'}{\partial p_\mu} &= \frac{N}{m} g^{\mu\nu} p_\nu \\ \dot{p}_\mu = -\frac{\partial H'}{\partial x^\mu} &= -\frac{N}{m}\frac{1}{2}\partial_\mu g^{\nu\lambda} p_\nu p_\lambda \\ &= -\frac{m}{N}\left(\frac{1}{2}\partial_\mu g^{\nu\lambda}\right)\dot{x}_\nu \dot{x}_\lambda \end{aligned} \tag{1155}$$

and (choosing the gauge $N = m$ for convenience)

$$\begin{aligned}\ddot{x}^\mu &= \dot{g}^{\mu\nu} p_\nu + g^{\mu\nu} \dot{p}_\nu \\ &= \partial^\lambda g^{\mu\nu} \dot{x}_\lambda \dot{x}_\nu - \frac{1}{2} g^{\mu\nu} \partial_\mu g^{\nu\lambda} \dot{x}_\nu \dot{x}_\lambda \\ &= -\Gamma_\nu{}^\mu{}_\lambda \dot{x}^\nu \dot{x}^\lambda \,. \end{aligned} \tag{1156}$$

Variation with respect to N reproduces the mass shell condition (1153). The Lagrangian associated with the above Hamiltonian is

$$L = -\frac{m}{2} \left(\frac{1}{N} \dot{x}_\mu \dot{x}^\mu - N \right) \,. \tag{1157}$$

This can be reformulated as

$$L = -\frac{m}{2} \sqrt{|\gamma|} \left(\gamma^{\tau\tau} \left(\partial_\tau x_\mu \right) \left(\partial_\tau x^\mu \right) + 1 \right) \,, \tag{1158}$$

where $\gamma_{\tau\tau} = -N^2$ is the (arbitrary) metric on the world line of the particle. Its inverse is $\gamma^{\tau\tau} = N^{-2}$. Written this way the action is the 1-dimensional analog of the Polyakov action for the relativistic string (*cf.* 3.1 (p.153)).

The Hamiltonian in (1154) is readily generalized to include a scalar potential V and a vector potential A:

$$H = N \left(\frac{1}{2} g^{\mu\nu} \left(p_\mu - A_\mu \right) \left(p_\nu - A_\nu \right) + V(x) \right) \,, \tag{1159}$$

which leads to

$$\begin{aligned}\dot{x}^\mu &= N g^{\mu\nu} \left(p_\nu - A_\nu \right) \\ \dot{p}_\mu &= N \left(- \left(\frac{1}{2} \partial_\mu g^{\nu\lambda} \right) \dot{x}_\nu \dot{x}_\lambda + g^{\nu\lambda} \partial_\mu A_\nu \dot{x}_\lambda - \partial_\mu V \right) \end{aligned} \tag{1160}$$

and

$$\begin{aligned} L &= p_\mu \dot{x}^\mu - H \\ &= \left(\frac{1}{N} \dot{x}_\mu + A_\mu \right) \dot{x}^\mu - \frac{1}{2N} \dot{x}_\mu \dot{x}^\mu - NV \\ &= \frac{1}{2N} \dot{x}_\mu \dot{x}^\mu + A_\mu \dot{x}^\mu - NV \,. \end{aligned} \tag{1161}$$

Fixing again the gauge to $N = 1$, the equations of motion now read:

$$\begin{aligned} \ddot{x}^\mu &= \dot{g}^{\mu\nu} \dot{x}_\nu + g^{\mu\nu} \dot{p}_\nu - g^{\mu\nu} \dot{A}_\nu \\ &= -\Gamma_\nu{}^\mu{}_\lambda \dot{x}^\nu \dot{x}^\lambda + g^{\nu\lambda} \partial_\mu A_\nu \dot{x}_\lambda - g^{\mu\nu} \partial_\lambda A_\nu \dot{x}^\lambda \partial_\mu V \\ &= -\Gamma_\nu{}^\mu{}_\lambda \dot{x}^\nu \dot{x}^\lambda + F^\mu{}_\nu \dot{x}^\nu - \partial^\mu V \,. \end{aligned} \tag{1162}$$

On the right hand side one has

- "gravitational forces" bilinear in the proper velocity,
- electromagnetic forces linear in the proper velocity,
- and forces deriving from a scalar potential .

The mini-superspace actions that are of interest in the main text (*cf.* 5.12 (p.273)) only feature the latter, therefore in the following consider

$$
\begin{aligned}
L &= \frac{1}{2N}\dot{x}_\mu \dot{x}^\mu - NV \\
H &= N\frac{1}{2}\left(p_\mu p^\mu + V\right) .
\end{aligned}
\tag{1163}
$$

(Here we have substituted $V \to \frac{1}{2}V$ for later convenience.)

The associated relativistic Hamilton-Jacobi equation for the classical action S is

$$
\left(\nabla_\mu S\right)\left(\nabla^\mu S\right) + V = 0 \,,
\tag{1164}
$$

with

$$
\partial_\tau S = 0 \,.
$$

For instance for $V = m^2$ equation (1164) again gives the mass-shell constraint $p_\mu p^\mu = -m^2$.

A.1 (The quantum fluid analogy) Now quantize the theory. The Hamiltonian operator reads

$$
\begin{aligned}
\hat{H} &= -\hbar^2 \nabla_\mu \nabla^\mu + V \\
&= \hbar^2 \mathbf{d}^\dagger \mathbf{d}_{|0} + V \,,
\end{aligned}
\tag{1165}
$$

where we have suppressed N for notational convenience. $\mathbf{d}^\dagger \mathbf{d}_{|0}$ is the Laplace-Beltrami operator $\mathbf{d}^\dagger\mathbf{d} + \mathbf{d}\mathbf{d}^\dagger$ on $\mathcal{M}$ restricted to 0-forms, i.e. to function $\mathcal{M} \to \mathbb{C}$.

There is a well known relation between quantum mechanics and the classical Hamilton-Jacobi equation:

Define two real functions

$$
R, S : \mathcal{M} \to \mathbb{R}
\tag{1166}
$$

and set

$$
\sqrt{\rho} := e^{R/\hbar} \,.
\tag{1167}
$$

Any kinematical state $|\psi\rangle$ of the quantum theory may be written as

$$
\begin{aligned}
|\psi\rangle &= \sqrt{\rho} e^{S/i\hbar} \\
&= e^{(R-iS)/\hbar} \,.
\end{aligned}
\tag{1168}
$$

In this parameterization of $|\psi\rangle$ the Hamiltonian constraint is equivalent to

$$
\begin{aligned}
& \hat{H}\,|\psi\rangle = 0 \\
\Leftrightarrow\quad & \left(-\nabla_\mu \nabla^\mu + V\right)|\psi\rangle = 0 \\
\Leftrightarrow\quad & -\hbar^2 \frac{\nabla_\mu \nabla^\mu \sqrt{\rho}}{\sqrt{\rho}} + \frac{2i\hbar\left(\nabla_\mu \sqrt{\rho}\right)\left(\nabla^\mu S\right)}{\sqrt{\rho}} + \left(\nabla_\mu S\right)\left(\nabla^\mu S\right) + i\hbar \nabla_\mu \nabla^\mu S + V = 0 \\
\Leftrightarrow\quad & \left\{ \begin{array}{l} -\hbar^2 \left(\nabla_\mu \nabla^\mu \sqrt{\rho}\right)/\sqrt{\rho} + \left(\nabla_\mu S\right)\left(\nabla^\mu S\right) + V = 0 \\ 2\left(\nabla_\mu \sqrt{\rho}\right)\left(\nabla^\mu S\right) + \sqrt{\rho}\nabla_\mu \nabla^\mu S = 0 \end{array} \right. \\
\Leftrightarrow\quad & \left\{ \begin{array}{l} -\hbar^2 \left(\nabla_\mu \nabla^\mu \sqrt{\rho}\right)/\sqrt{\rho} + \left(\nabla_\mu S\right)\left(\nabla^\mu S\right) + V = 0 \\ \nabla_\mu \left(\rho \left(\nabla^\mu S\right)\right) = 0 \end{array} \right. .
\end{aligned}
\tag{1169}
$$

This is a coupled system of differential equations for R and S. One is the classical Hamilton-Jacobi equation modified by the so-called *quantum potential*

$$V_{\mathrm{QM}} \quad := \quad -\hbar^2 \left(\nabla_\mu \nabla^\mu \sqrt{\rho}\right) / \sqrt{\rho}\,, \tag{1170}$$

The other is the continuity equation (expressing conservation of energy-momentum) for a current with velocity

$$U^\mu \quad = \quad \nabla^\mu S \tag{1171}$$

and density ρ. This equivalent reformulation of quantum mechanics lends itself to a treatment by means of the theory of stochastic processes. (See for instance [240] for a review.) This may be relevant in the context of supersymmetric quantum mechanics, since here one can find close connections to stochastic theory. For instance, note that due to the identity

$$\frac{\nabla^2 f}{f} \quad = \quad |\nabla \ln f|^2 + \nabla^2 \ln f \tag{1172}$$

the *quantum potential* (1170) is equal to

$$\begin{aligned} V_{\mathrm{QM}} \quad &= \quad -\hbar^2 \left(|\nabla \ln \sqrt{\rho}|^2 + \nabla^2 \ln \rho \right) \\ &\overset{(1167)}{=} \quad -\left((\nabla_\mu R)\,(\nabla^\mu R) + \hbar \nabla_\mu \nabla^\mu R \right) . \end{aligned} \tag{1173}$$

But this is immediately recognized as the form of the Witten-model superpotential in the 0-fermion sector (see 2.2.2 (p.61)):

$$\left(\mathbf{D}^W\right)^2 |\psi\rangle \quad = \quad \left(-\hbar^2 \nabla_\mu \nabla^\mu + (\nabla_\mu W)\,(\nabla^\mu W) - \hbar \nabla^\mu \nabla_\mu W \right) |\psi\rangle \;. \tag{1174}$$

This motivates looking at the supersymmetric extension of (1165):

A.2 (The superparticle) We had already suggestively written the wave operator in (1165) as

$$\begin{aligned} -\nabla_\mu \nabla^\mu |\psi\rangle \quad &= \quad -\frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu\nu} \partial_\nu |\psi\rangle \\ &= \quad \mathbf{d}^\dagger \mathbf{d} |\psi\rangle \\ &= \quad \left(\mathbf{d} + \mathbf{d}^\dagger\right)^2 |\psi\rangle \;, \end{aligned} \tag{1175}$$

where $|\psi\rangle$ is an ordinary function $\mathcal{M} \to \mathbb{C}$. There is nothing more natural then letting be $|\psi\rangle$ take values in the full domain of $\left(\mathbf{d} + \mathbf{d}^\dagger\right)^2$, that is let $|\psi\rangle$ be an element of the exterior bundle $\Lambda(\mathcal{M})$ (see (7), p. 17 and 2.2 (p.16) in general).

The system described by this Hamiltonian is called the *superparticle*. It can be regarded as the point particle limit of the superstring (*cf.* 3.14 (p.166)).

B More on exterior and Clifford algebra

Introduction. In the following a couple of definitions and results in the context of §2.1.1 (p.15) are given for reference.

B.1 Clifford and exterior algebra

The graded vector spaces of central importance for supersymmetric quantum mechanics are those spanned by Clifford algebras and, as an important special case, Grassmann algebras. In a sense, the very nature of the complex of ideas that is generated by the notions geometry, spin, and supersymmetry (*cf.* e.g. [165]) is already present and naturally captured in the structure of Clifford algebras. In particular, the idea of supersymmetry to replace second-order differential operators by their first-order 'square roots' directly leads to Clifford algebras (or vice versa), which arise as 'square roots' of quadratic forms:

B.1 (Clifford algebra) The real/complex Clifford algebra $\mathrm{Cl}(Q(\cdot))$ associated to a 'quadratic' vector space $(V, Q(\cdot))$, which comes with a quadratic form ('norm square')

$$Q(\cdot) : V \ni v \mapsto Q(v) \in \mathbb{R}$$

is the algebra over $\mathbb{R}/\mathbb{C}$ generated by the elements of $V \cup \mathbb{R}$ or $V \cup \mathbb{C}$ and equipped with an algebra product such that the square of a vector in the algebra gives its norm squared:

$$V \ni v \to \hat{\gamma}_v \in \mathrm{Cl} \Rightarrow (\hat{\gamma}_v)^2 = Q(v)\,.$$

It follows that in an orthonormal basis

$$e_j \in V;\; Q(e_j) = \pm 1;\; Q(e_j + e_k) = Q(e_j) + Q(e_k)$$

one has the anticommutation relation familiar from the theory of the Dirac electron:

$$\left\{\hat{\gamma}_{e_j}, \hat{\gamma}_{e_k}\right\} = 2Q(e_j)\delta_{j,k}\,.$$

B.2 (Grassmann algebra) An important special case of a Clifford algebra, called a *Grassmann algebra*, is that where the defining quadratic form vanishes identically

$$Q(v) = 0,\, \forall v \in V\,.$$

This implies that all $v \in V$ are, as elements of the algebra, nilpotent and mutually anticommuting.

The standard example of a Grassmann algebra, which is be of central importance here, is the exterior algebra of differential forms.

B.3 (Relation Exterior/Clifford algebra) *The algebra of linear operators on the exterior algebra* $\mathbf{\Lambda}$ *is isomorphic to the Clifford algebra* $\mathrm{Cl}(D, D)$.

Proof: The algebra of linear operators on $\mathbf{\Lambda}$ is generated by the creation and annihilation operators $\hat{\mathrm{e}}^{\dagger a}$, $\hat{\mathrm{e}}^{a}$, having the non-vanishing bracket

$$\left\{\hat{\mathrm{e}}^{\dagger a}, \hat{\mathrm{e}}^{b}\right\} = \eta^{ab}.$$

The Clifford algebra $\mathrm{Cl}(D,D)$ arises by a simple change of basis:

$$\begin{aligned}\hat\gamma^a_\pm &:= \hat{\mathrm{e}}^{\dagger a} \pm \hat{\mathrm{e}}^a \\ \Rightarrow \hat{\mathrm{e}}^{\dagger a} &= \frac{1}{2}\left(\hat\gamma^a_+ + \hat\gamma^a_-\right) \\ \hat{\mathrm{e}}^a &= \frac{1}{2}\left(\hat\gamma^a_+ - \hat\gamma^a_-\right) ,\end{aligned} \tag{1176}$$

with

$$\{\hat\gamma^a_s, \hat\gamma^a_{s'}\} = 2s\delta_{ss'}\eta^{ab} . \tag{1177}$$

Alternatively, this may be regarded as two anti-commuting Clifford algebras $\mathrm{Cl}(D,D) \simeq \mathrm{Cl}(s, D-s) \otimes \mathrm{Cl}(D-s, s)$:

$$\begin{aligned}\left\{\hat\gamma^a_\pm, \hat\gamma^b_\pm\right\} &= \pm 2\eta^{ab} \\ \left\{\hat\gamma^a_\mp, \hat\gamma^b_\pm\right\} &= 0 .\end{aligned} \tag{1178}$$

In a coordinate basis this reads:

$$\begin{aligned}\hat\gamma^\mu_{g\pm} &:= \tilde{\mathrm{e}}^\mu{}_a \hat\gamma^a_\pm \\ &= \left(\hat{\mathrm{c}}^{\dagger\mu} \pm \hat{\mathrm{c}}^\mu\right)\end{aligned} \tag{1179}$$

$$\left\{\hat\gamma^\mu_{g\pm}, \hat\gamma^\nu_{g\pm}\right\} = \pm 2 g^{\mu\nu} . \tag{1180}$$

B.4 (Projection on Clifford scalars) A central operation in Clifford algebra is the projection on Clifford-0-vectors, i.e. scalars (*cf.* e.g. [129][126][127]):

$$\left\langle \omega^0 + \omega^1_\mu \hat\gamma^\mu + \omega^2_{[\mu\nu]}\hat\gamma^\mu\hat\gamma^\nu + \cdots \right\rangle_0 := \omega^0 . \tag{1181}$$

Due to (40) and (1176) this operation coincides with the local inner product on $\Lambda(\mathcal{M})$ in the sense that:

$$\left\langle \hat A \right\rangle_0 \mathrm{vol} = \langle 0|\, \hat A \,|0\rangle_{\mathrm{loc}} \tag{1182}$$

B.5 (Symbol map) (*cf.* [31]) The exterior bundle $\Lambda(\mathcal{M})$ and any of the Clifford bundles $\mathrm{Cl}(\mathcal{M})_\pm$ are isomorphic as vector spaces. This relation is made explicit by the *symbol map* σ that maps Clifford elements to forms by applying them on the *vacuum state*:

$$\mathrm{Cl}(\mathcal{M})_\pm \ni \hat x \stackrel{\sigma}{\mapsto} \hat x \,|0\rangle \in \mathrm{Cl}(\mathcal{M})_\pm |0\rangle = \Lambda(\mathcal{M}) . \tag{1183}$$

Because of

$$\hat{\mathrm{e}}^a |0\rangle = 0$$

one has

$$\hat\gamma^a_+ |0\rangle = \hat\gamma^a_- |0\rangle = \hat{\mathrm{e}}^{\dagger a} |0\rangle . \tag{1184}$$

This simple and well known observation leads to the following important fact:

B.6 (Right action of $\mathrm{Cl}(\mathcal{M})^{\mathrm{even}}$**)** *The elements of* $\mathrm{Cl}(\mathcal{M})^{\mathrm{even}}_{\mp}$ *are represented on* $\mathrm{Cl}(\mathcal{M})_{\pm}\,|0\rangle = \Lambda(\mathcal{M})$ *by the right action of* $\mathrm{Cl}(\mathcal{M})^{\mathrm{even}}_{\pm}$*.*

Proof: Let

$$\begin{aligned} \hat{\gamma}^{ij}_{\mp} &\in \mathrm{Cl}(\mathcal{M})^{\mathrm{even}}_{\mp} \\ \hat{x}_{\pm}\,|0\rangle &\in \mathrm{Cl}(\mathcal{M})_{\pm}\,|0\rangle \end{aligned} \tag{1185}$$

then, by (1184),

$$\begin{aligned} \hat{\gamma}^{ij}_{\mp}\,\hat{x}_{\pm}\,|0\rangle &= \hat{x}_{\pm}\,\hat{\gamma}^{ij}_{\mp}\,|0\rangle \\ &= \hat{x}_{\pm}\,\hat{\gamma}^{ij}_{\pm}\,|0\rangle\ . \end{aligned} \tag{1186}$$

□

B.7 *A multiplicative derivation*

$$\hat{A} = A_{ij}\hat{\mathrm{e}}^{\dagger i}\hat{\mathrm{e}}^{j}$$

on $\Lambda(\mathcal{M})$ *is represented on* $\mathrm{Cl}(\mathcal{M})_{\pm}\,|0\rangle$ *by the adjoint action of* $\mathrm{Cl}(\mathcal{M})^{\mathrm{even}}_{\pm}$*:*

$$\begin{aligned} \hat{A}\,\hat{x}_{\pm}\,|0\rangle &= A_{ij}\hat{\mathrm{e}}^{\dagger i}\hat{\mathrm{e}}^{j}\,\hat{x}_{\pm}\,|0\rangle \\ &= \pm\frac{1}{2}A_{ij}\left[\hat{\gamma}^{i}_{\pm}\hat{\gamma}^{j}_{\pm}\,,\,\hat{x}_{\pm}\right]|0\rangle\ . \end{aligned} \tag{1187}$$

Proof: Observe that

$$\hat{\mathrm{e}}^{\dagger i}\hat{\mathrm{e}}^{j} = \frac{1}{2}\left(\hat{\gamma}^{i}_{+}\hat{\gamma}^{j}_{+} - \hat{\gamma}^{i}_{-}\hat{\gamma}^{j}_{-}\right) \tag{1188}$$

and that (with $\hat{x}_{\pm} \in \mathrm{Cl}(\mathcal{M})_{\pm}$)

$$\left(\hat{\gamma}^{i}_{+}\hat{\gamma}^{j}_{+} - \hat{\gamma}^{i}_{-}\hat{\gamma}^{j}_{-}\right)\hat{x}_{\pm}\,|0\rangle = \pm\left[\hat{\gamma}^{i}_{\pm}\hat{\gamma}^{j}_{\pm}\,,\,\hat{x}_{\pm}\right]|0\rangle\ . \tag{1189}$$

□

These simple results will make transparent the relation between Dirac operators on the exterior bundle and on the spin bundle in §B.2 (p.311).

B.8 (Automorphsims of the Clifford algebra) $\tilde{A}$ denotes the Clifford reverse *and* complex conjugate of A: Let $k = \mathrm{grade}\left(\hat{x}_{(k)}\right)$ be the grade of a homogeneous element of $\hat{x}_{(k)} \in \mathrm{Cl}(\mathcal{M})_{\pm}$, then

$$\widetilde{\hat{x}_{(k)}} = (-1)^{k(k-1)/2}\hat{x}^{*}_{(k)}\,. \tag{1190}$$

B.9 (Spin) The spin bundle $\mathrm{Spin}(\mathcal{M})_{\pm}$ and its Lie algebra bundle $\mathrm{spin}(\mathcal{M})_{\pm}$ are defined by

$$\begin{aligned} \mathrm{Spin}(\mathcal{M})_{\pm} &:= \left\{\hat{s} \in \mathrm{Cl}(\mathcal{M})^{\mathrm{even}}_{\pm}\,|\hat{s}\tilde{\hat{s}} = 1, \mathrm{grade}\left(\hat{s}\hat{x}_{(1)}\tilde{\hat{s}}\right) = 1\right\} \\ \mathrm{spin}(\mathcal{M})_{\pm} &= \mathrm{Cl}(\mathcal{M})^{(2)}_{\pm}\ . \end{aligned} \tag{1191}$$

Theorem B.10 (Basic Relations from Exterior Calculus)

1. Let I and J be disjoint multi-indices $I = (i_1, i_2, \ldots, i_p)$ and $J = (j_1, j_2, \ldots, j_{D-p})$ and let $\sigma(J, I)$ be the signature of the permutation $(J, I) \to (1, 2, 3, \ldots, D)$.

   The action of the Hodge-$*$ operator on spin-frame basis states is defined by

$$* \, \hat{e}^{\dagger I} \, |0\rangle = (-1)^{s(I)} \sigma(J, I) \; \hat{e}^{\dagger J} \, |0\rangle \ ,$$

   where $s(I)$ denotes the number of indices in I that correspond to negative eigenvalues of η^{ab}.

2. Let

$$\begin{aligned} |\alpha\rangle &= \alpha_{a_1, a_2, \cdots, a_p} \hat{\mathrm{e}}^{\dagger a_1} \hat{\mathrm{e}}^{\dagger a_2} \cdots \hat{\mathrm{e}}^{\dagger a_p} \, |0\rangle \\ |\beta\rangle &= \beta_{a_1, a_2, \cdots, a_p} \hat{\mathrm{e}}^{\dagger a_1} \hat{\mathrm{e}}^{\dagger a_2} \cdots \hat{\mathrm{e}}^{\dagger a_p} \, |0\rangle \end{aligned}$$

   The Hodge inner product

$$\langle \alpha | \beta \rangle_{\mathrm{loc}} = \alpha \wedge * \beta \tag{1192}$$

   reads in components

$$\langle \alpha | \beta \rangle_{\mathrm{loc}} = p! \, \alpha_{a_1, a_2, \cdots, a_p} \beta^{a_1, a_2, \cdots, a_p} \, |\mathrm{vol}\rangle \ . \tag{1193}$$

   Often it is desirable to avoid overcounting of index permutations. Because of

$$\begin{aligned} |\alpha\rangle &= \sum_{0 \leq a_1 < a_2 < a_p} p! \, \alpha_{a_1, a_2, \cdots, a_p} \hat{\mathrm{e}}^{\dagger a_1} \hat{\mathrm{e}}^{\dagger a_2} \cdots \hat{\mathrm{e}}^{\dagger a_p} \, |0\rangle \\ |\beta\rangle &= \sum_{0 \leq a_1 < a_2 < a_p} p! \, \beta_{a_1, a_2, \cdots, a_p} \hat{\mathrm{e}}^{\dagger a_1} \hat{\mathrm{e}}^{\dagger a_2} \cdots \hat{\mathrm{e}}^{\dagger a_p} \, |0\rangle \end{aligned}$$

   it is natural to define

$$\begin{aligned} \alpha'_{a_1, a_2, \cdots, a_p} &:= p! \, \alpha_{a_1, a_2, \cdots, a_p} \\ \beta'_{a_1, a_2, \cdots, a_p} &:= p! \, \beta_{a_1, a_2, \cdots, a_p} \, . \end{aligned} \tag{1194}$$

   With this definition one has the following, sometimes preferable, expression for the local Hodge inner product.

$$\langle \alpha | \beta \rangle_{\mathrm{loc}} = \sum_{0 \leq a_1 < a_2 < a_p} \alpha'_{a_1, a_2, \cdots, a_p} \beta'^{a_1, a_2, \cdots, a_p} \, |\mathrm{vol}\rangle \ . \tag{1195}$$

3. *The square of the* Hodge-$*$ *operator is:*

$$*^2 = (-1)^{\hat{N}(D - \hat{N}) + s} \ . \tag{1196}$$

Proof: Assume first that $s = 0$: In this case the HODGE-$*$ acts by definition on any orthonormal basis form $\hat{e}^{\dagger I} |0\rangle$ as

$$* \, \hat{e}^{\dagger I} |0\rangle = \sigma(J, I) \, \hat{e}^{\dagger J} |0\rangle \ .$$

Applying it again yields

$$\begin{aligned} * * \hat{e}^{\dagger I} |0\rangle &= \sigma(I,J)\sigma(J,I)\hat{e}^{\dagger I} |0\rangle \\ &= (-1)^{p(D-p)}\hat{e}^{\dagger I} |0\rangle \ . \end{aligned}$$

For $s > 0$ a factor of -1 is picked up for every occurrence of a $\hat{e}^{\dagger \mu}$ corresponding to a negative eigenvalue. Since every $\hat{e}^{\dagger \mu}$ appears exactly once there is a total factor of $(-1)^s$. □

4. *The* HODGE-$*$ *operator relates the exterior derivative* $\mathbf{d}$ *with its adjoint* $\mathbf{d}^\dagger$ *via*

$$\mathbf{d}^\dagger \;=\; * \, \mathbf{d} * (-1)^{D(\hat{N}-1)+1+s} \ . \tag{1197}$$

Proof: Let α *be a* $(p-1)$*-form and* β *a* p*-form, then:*

$$\begin{aligned} 0 &= \int \mathbf{d}\left(\alpha \wedge *\beta\right) \\ &= \int (\mathbf{d}\alpha) \wedge *\beta + \int \alpha \wedge \left(\mathbf{d}*(-1)^{\hat{N}-1}\beta\right) \\ &\overset{(1196)}{=} \int (\mathbf{d}\alpha) \wedge *\beta + \int \alpha \wedge \left((-1)^{\hat{N}(D-\hat{N})+s} **\mathbf{d}*(-1)^{\hat{N}-1}\beta\right) \\ &= \int (\mathbf{d}\alpha) \wedge *\beta + \int \alpha \wedge \left(**\mathbf{d}*(-1)^{(D-\hat{N}+1)(\hat{N}-1)+\hat{N}-1+s}\beta\right) \\ &= \langle \mathbf{d}\alpha|\beta\rangle + \left\langle \alpha | * \mathbf{d} * (-1)^{D(\hat{N}-1)+s}\beta \right\rangle \\ \Rightarrow \langle \mathbf{d}\alpha|\beta\rangle &= \left\langle \alpha|\mathbf{d}^\dagger \beta\right\rangle = \left\langle \alpha | * \mathbf{d} * (-1)^{D(\hat{N}-1)+1+s}\beta \right\rangle \end{aligned}$$

□

(This relation simplifies when the Hodge-$*$ is replace by Clifford chirality operators, see (1228).)

B.11 (Local representations of various operators) The following list summarizes the explicit local representation of various important operators. (Compare for instance [165]. For a nice physically motivated discussion see [53].)

1. *spin connection on the exterior bundle*

$$\Omega_\mu \;:=\; \omega_{\mu a b}\hat{e}^{\dagger b}\hat{e}^a \tag{1198}$$

2. *Covariant derivative on the exterior bundle.*

$$\begin{aligned} \hat{\nabla}_\mu &:= \partial_\mu - \omega_{\mu a b}\hat{e}^{\dagger b}\hat{e}^a \\ &= \partial_\mu + \omega_{\mu a b}\hat{e}^{\dagger a}\hat{e}^b \\ &= \partial_\mu + \frac{1}{4}\omega_{\mu a b}\left(\hat{\gamma}_+^a\hat{\gamma}_+^b - \hat{\gamma}_-^a\hat{\gamma}_-^b\right) \end{aligned} \tag{1199}$$

(where $\omega_{\mu ab} = \omega_{\mu[ab]}$ are the components of the spin connection ω, i.e. the Levi-Civita connection with respect to a (pseudo-) orthonormal frame $\{\partial_a\}_a$).

Proof: $\hat{\nabla}$ is obviously a derivation, so it suffices to observe that it has the correct action on 0-forms, which is trivially true, and on 1-forms:

$$\begin{aligned}\hat{\nabla}_\mu \,\hat{\mathrm{e}}^{\dagger i}\,|0\rangle &= -\omega_{\mu ab}\hat{\mathrm{e}}^{\dagger b}\hat{\mathrm{e}}^a\,\hat{\mathrm{e}}^{\dagger i}\,|0\rangle \\ &= -\omega_\mu{}^i{}_b\hat{\mathrm{e}}^{\dagger b}\,|0\rangle\ .\end{aligned}$$

3. *Exterior derivative.*

$$\mathbf{d} = \hat{\mathrm{c}}^{\dagger\mu}\hat{\nabla}_\mu \tag{1200}$$

Proof: From the requirement

$$\mathbf{d}\,\phi\,|0\rangle \stackrel{!}{=} \hat{\mathrm{c}}^{\dagger\mu}\phi_{,\mu}\,|0\rangle$$

one has

$$\mathbf{d} = \hat{\mathrm{c}}^{\dagger\mu}\left(\partial_\mu + \hat{A}\right)\ ,$$

where $\hat{A}$ is some operator that annihilates the vacuum and that, in order to ensure that $\mathbf{d}$ acts as an anti-derivation, commutes with the number operator. $\hat{A}$ can be determined by imposing Cartan's first structure equation for vanishing torsion:

$$\begin{aligned}\mathbf{d}\,\hat{\mathrm{e}}^{\dagger a}\,|0\rangle + \hat{\omega}^a{}_b\hat{\mathrm{e}}^{\dagger b}\,|0\rangle &= 0 \\ \Rightarrow \hat{A}\,\hat{\mathrm{e}}^{\dagger a}\,|0\rangle + \hat{\omega}^a{}_b\hat{\mathrm{e}}^{\dagger b}\,|0\rangle &= 0\end{aligned} \tag{1201}$$

It follows that

$$\hat{A} = -\omega_{\mu ab}\hat{\mathrm{e}}^{\dagger b}\hat{\mathrm{e}}^{\dagger a}\ .$$

□

Note that these formulas hold for the Levi-Civita connection and in particular assume vanishing torsion.

4. *Exterior coderivative.*

$$\mathbf{d}^\dagger = -\hat{\mathrm{c}}^\mu\,\hat{\nabla}_\mu \tag{1202}$$

Proof: Since $\mathbf{d}^\dagger$ is apparently an anti-derivation, it suffices to observe that on 1-forms

$$\mathbf{d}^\dagger\,\hat{\mathrm{c}}^{\dagger\mu}\alpha_\mu\,|0\rangle = -\nabla^\mu\alpha_\mu\,|0\rangle\ ,$$

as it should be. □

5. *Lie derivative $\mathcal{L}$ with respect to a vector field $v^\mu\partial_\mu$.:*

$$\hat{\mathcal{L}}_v = \{\mathbf{d}, v^\mu\hat{\mathrm{c}}_\mu\} \tag{1203}$$

Proof: This is Cartan's homotopy formula. (e.g. [98] p.135)
The Lie derivative operator is related to the Lie bracket of vector fields by:

$$\left[\hat{\mathcal{L}}_v, w^\mu \hat{\mathrm{c}}_\mu\right] = [v,w]^\mu \hat{\mathrm{c}}_\mu \tag{1204}$$

6. *Curvature.*

$$\left[\hat{\nabla}_\mu, \hat{\nabla}_\nu\right] = R_{\mu\nu ab} \hat{\mathrm{e}}^{\dagger a} \hat{\mathrm{e}}^b \tag{1205}$$

Proof:

$$\begin{aligned}\left[\hat{\nabla}_\mu, \hat{\nabla}_\nu\right] &\overset{(1199)}{=} \left[\partial_\mu - \omega_{\mu ab}\hat{\mathrm{e}}^{\dagger b}\hat{\mathrm{e}}^a, \partial_\nu - \omega_{\nu ab}\hat{\mathrm{e}}^{\dagger b}\hat{\mathrm{e}}^a\right] \\ &= \left(\partial_{[\mu}\omega_{\nu]ab} + \omega_{\mu ac}\omega_\nu{}^c{}_b - \omega_{\nu ac}\omega_\mu{}^c{}_b\right)\hat{\mathrm{e}}^{\dagger a}\hat{\mathrm{e}}^b\end{aligned}$$

□

7. *Ordinary exterior Dirac operator.*[62]

$$\begin{aligned}\mathbf{D} &:= \mathbf{d} + \mathbf{d}^\dagger \\ &= \hat{\gamma}_-^\mu \hat{\nabla}_\mu\end{aligned} \tag{1206}$$

Proof: By (1200) and (1202). □

8. *Exterior Laplace operator (Laplace-Beltrami operator).*

$$\begin{aligned}\mathbf{\Delta} &:= \left(\mathbf{d} + \mathbf{d}^\dagger\right)^2 = \left(\hat{\gamma}_-^\mu \hat{\nabla}_\mu\right)^2 \\ &= -g^{\mu\nu}\left(\nabla_\mu\nabla_\nu - \Gamma_\mu{}^\kappa{}_\nu\nabla_\kappa\right) - R_{\mu\nu cd}\hat{\mathrm{c}}^{\dagger\mu}\hat{\mathrm{c}}^\nu\hat{\mathrm{e}}^{\dagger c}\hat{\mathrm{e}}^d \\ &= -g^{\mu\nu}\left(\nabla_\mu\nabla_\nu - \Gamma_\mu{}^\kappa{}_\nu\nabla_\kappa\right) + \frac{R}{4} + \frac{1}{8}R_{\mu\nu cd}\hat{\gamma}_{g_+}^{\ \mu}\hat{\gamma}_{g_+}^{\ \nu}\hat{\gamma}_-^c\hat{\gamma}_-^d\end{aligned}$$

Proof: (This is the Weitzenböck formula.)

The following relation is sometimes useful:

B.12

$$\left\{\hat{\gamma}_-^a \hat{\nabla}_a, \hat{\gamma}_+^b f_b\right\} = \hat{\gamma}_-^a \hat{\gamma}_+^b \left(\nabla_a f_b\right) \tag{1207}$$

Proof: By (1199) one has

$$\hat{\nabla}_a = \partial_a - \omega_{abc}\hat{\mathrm{e}}^{\dagger c}\hat{\mathrm{e}}^b\,.$$

The different contributions to the commutator are:

$$\left\{\hat{\gamma}_-^a \partial_a, \hat{\gamma}_+^d f_d\right\} = \hat{\gamma}_-^a \hat{\gamma}_+^d \left[\partial_a, f_d\right]$$

[62] Sometimes called the *vector derivative* in Clifford formalism, *cf.* [130].

and

$$
\begin{aligned}
\left\{\hat{\gamma}_-^a \omega_{abc} \hat{\mathrm{e}}^{\dagger c} \hat{\mathrm{e}}^b, \hat{\gamma}_+^d f_d\right\} &\overset{(1188)}{=} \left\{\hat{\gamma}_-^a \frac{1}{2}\omega_{abc}\left(\hat{\gamma}_+^c \hat{\gamma}_+^b - \hat{\gamma}_-^c \hat{\gamma}_-^b\right), \hat{\gamma}_+^d f_d\right\} \\
&= \frac{1}{2}\hat{\gamma}_-^a \omega_{abc}\left[\hat{\gamma}_+^c \hat{\gamma}_+^b, \hat{\gamma}_+^d\right] f_d \\
&= \hat{\gamma}_-^a \hat{\gamma}_+^c \omega_a{}^b{}_c f_b \,.
\end{aligned}
\tag{1208}
$$

□

B.13 (Coordinate dependence of 1-form basis) It makes sense to consider different conventions with respect to the commutator of the coordinate derivative with basis 1-forms. As defined above, denote by $\hat{\mathrm{e}}^{\dagger a}$ a (pseudo-)orthonormal basis of form creators,

$$
\left\{\hat{\mathrm{e}}^{\dagger a}, \hat{\mathrm{e}}^b\right\} = \eta^{ab}\,,
$$

and by $\hat{\mathrm{c}}^{\dagger \mu}$ a holonomic basis of form creators (i.e. one associated to a set of coordinates x^μ by $\hat{\mathrm{e}}^{\dagger \mu}\,|0\rangle = dx^\mu$), so that

$$
\left\{\hat{\mathrm{c}}^{\dagger \mu}, \hat{\mathrm{c}}^\mu\right\} = g^{\mu\nu}\,.
$$

Both are related by a vielbein field $e^a{}_\mu(x)$:

$$
\begin{aligned}
\hat{\mathrm{e}}^{\dagger a} &= e^a{}_\mu \hat{\mathrm{c}}^{\dagger \mu} \\
\hat{\mathrm{c}}^{\dagger \mu} &= \tilde{e}^\mu{}_a \hat{\mathrm{e}}^{\dagger \mu} \\
e^a{}_\mu \, \tilde{e}^\mu{}_b &= \delta^a_b \\
\tilde{e}^\mu{}_a e^a{}_\nu &= \delta^\mu_\nu \,.
\end{aligned}
\tag{1209}
$$

Now, since the coordinate derivative ∂_μ is not a covariant object, the commutators

$$
\left[\partial_\mu, \hat{\mathrm{e}}^{\dagger a}\right],\ \left[\partial_\mu, \hat{\mathrm{c}}^{\dagger \nu}\right],
$$

have no invariant meaning. What does have invariant meaning is the commutator with the covariant derivative:

$$
\begin{aligned}
\left[\hat{\nabla}_\mu, \hat{\mathrm{e}}^{\dagger a}\right] &= -\omega_\mu{}^a{}_b\, \hat{\mathrm{e}}^{\dagger b} \\
\left[\hat{\nabla}_\mu, \hat{\mathrm{c}}^{\dagger \nu}\right] &= -\Gamma_\mu{}^\nu{}_\kappa\, \hat{\mathrm{c}}^{\dagger \kappa}\,,
\end{aligned}
\tag{1210}
$$

so that

$$
\begin{aligned}
\hat{\nabla}_\mu\, f_a \hat{\mathrm{e}}^{\dagger a}\,|0\rangle &= (\nabla_\mu\, f_a)\, \hat{\mathrm{e}}^{\dagger a}\,|0\rangle \\
\hat{\nabla}_\mu\, f_\nu \hat{\mathrm{c}}^{\dagger \nu}\,|0\rangle &= (\nabla_\mu\, f_\nu)\, \hat{\mathrm{c}}^{\dagger \nu}\,|0\rangle\ ,
\end{aligned}
\tag{1211}
$$

as it should be. (Here Γ is the Christoffel symbol of the Levi-Civita connection and ω its spin frame version.) In order to give a local representation of the covariant derivative operator $\hat{\nabla}_\mu$ one must decide which of the basis forms is supposed to be independent of the coordinates in the sense that its commutator with the derivative ∂_μ vanishes. Two useful choices are:

$$
\begin{aligned}
&\left[\partial_\mu, \hat{\mathrm{e}}^{\dagger a}\right] := 0 \\
\Rightarrow &\left\{
\begin{array}{l}
[\partial_\mu, \hat{\mathrm{e}}_a] = [\partial_\mu, \hat{\mathrm{e}}^a] = 0 \\
\left[\partial_\mu, \hat{\mathrm{c}}^{\dagger\nu}\right] = (\partial_\mu \tilde{e}^\nu{}_a)\, \hat{\mathrm{e}}^{\dagger a} = e^a{}_\kappa\, (\partial_\mu \tilde{e}^\nu{}_a)\, \hat{\mathrm{c}}^{\dagger\kappa} \\
\hat{\nabla}_\mu = \partial_\mu - \omega_{\mu a b} \hat{\mathrm{e}}^{\dagger b} \hat{\mathrm{e}}^a
\end{array}
\right.
\end{aligned}
\tag{1212}
$$

and

$$
\begin{aligned}
&\left[\partial_\mu, \hat{\mathrm{c}}^{\dagger\mu}\right] := 0 \\
\Rightarrow &\left\{
\begin{array}{l}
[\partial_\mu, \hat{\mathrm{c}}_\nu] = 0 \\
\left[\partial_\mu, \hat{\mathrm{e}}^{\dagger a}\right] = (\partial_\mu e^a{}_\nu)\, \hat{\mathrm{c}}^{\dagger\nu} = \tilde{e}^\nu{}_b\, (\partial_\mu e^a{}_\nu)\, \hat{\mathrm{e}}^{\dagger b} \\
\hat{\nabla}_\mu = \partial_\mu - \Gamma_{\mu\alpha\beta} \hat{\mathrm{c}}^{\dagger\beta} \hat{\mathrm{c}}^\alpha
\end{array}
\right.
\end{aligned}
\tag{1213}
$$

The second convention (1213) arises naturally for instance when supergravity is formulated in a mode-amplitude basis as described in §4.3 (p.192). It has the advantage that, when adopted, the exterior derivative looks particularly simple:

$$
\begin{aligned}
(1213) \Rightarrow \mathbf{d} &\overset{(1200)}{=} \hat{\mathrm{c}}^{\dagger\mu} \hat{\nabla}_\mu \\
&\overset{(1213)}{=} \hat{\mathrm{c}}^{\dagger\mu} \left(\partial_\mu - \Gamma_{\mu\alpha\beta} \hat{\mathrm{c}}^{\dagger\beta} \hat{\mathrm{c}}^\alpha\right) \\
&= \hat{\mathrm{c}}^{\dagger\mu} \partial_\mu - \hat{\mathrm{c}}^{\dagger} \underbrace{\Gamma_{[\mu|\alpha|\beta]}}_{=0} \hat{\mathrm{c}}^{\dagger\beta} \hat{\mathrm{c}}^\alpha \\
&= \hat{\mathrm{c}}^{\dagger\mu} \partial_\mu \,.
\end{aligned}
\tag{1214}
$$

Also, its Hodge adjoint, the exterior coderivative, has a simple form on the space Λ^1 of 1-forms:

$$
\begin{aligned}
\mathbf{d}^\dagger &\overset{(1202)}{=} -\hat{\mathrm{c}}^\mu\, \hat{\nabla}_\mu \\
&= -\hat{\mathrm{c}}^\mu \left(\partial_\mu - \Gamma_{\mu\alpha\beta} \hat{\mathrm{c}}^{\dagger\beta} \hat{\mathrm{c}}^\alpha\right) \\
\Rightarrow \mathbf{d}^\dagger{}_{|\Lambda^1} &= -\hat{\mathrm{c}}^\mu \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} \,.
\end{aligned}
\tag{1215}
$$

Helicity The Hodge-$*$ is a duality operation on $\Lambda(\mathcal{M})$, but its sign is adjusted so as to give the proper scalar product (1193). A slight modification of the Hodge-$*$ operator gives the usual *chirality* operator $\bar{\gamma}_\pm$ of the Clifford algebra of sign $\pm$. The starting point is the Clifford *pseudo-scalar* operator:

B.14 (Volume pseudoscalar) *The volume element is represented by the pseudoscalar* $\mathrm{I}_\pm$ *in the Clifford bundle* $\mathrm{Cl}(\mathcal{M})_\pm$*:*

$$
\mathrm{I}_\pm := \hat{\gamma}^1_\pm \hat{\gamma}^2_\pm \cdots \hat{\gamma}^D_\pm \,.
\tag{1216}
$$

B.15 (Basic relations concerning the pseudoscalar) *One has the following basic relations:*

1. *The volume pseudo-scalar* $\mathrm{I}_\pm$ *is, up to a sign which depends on the form degree, the Hodge-$*$ operator:*

$$\mathrm{I} = *\,(-1)^{\hat{N}(\hat{N}\mp 1)/2}\,. \tag{1217}$$

Proof: It is sufficient to verify this for (pseudo-)orthonormal basis p-forms

$$|\alpha\rangle_p = \hat{e}^{\dagger i_1}\hat{e}^{\dagger i_2}\cdots\hat{e}^{\dagger i_p}\,|0\rangle\ .$$

Their Hodge adjoint is, by definition:

$$\begin{aligned} *\alpha_p &= *\,\hat{e}^{\dagger i_1}\hat{e}^{\dagger i_2}\cdots\hat{e}^{\dagger i_p}\,|0\rangle \\ &= \sigma(J,I)(-1)^{s(I)}\hat{e}^{\dagger j_1}\hat{e}^{\dagger j_2}\cdots\hat{e}^{\dagger j_{D-p}}\,|0\rangle\ , \end{aligned}$$

where $s(I)$ is the number of i_n that correspond to negative eigenvalues of η. On the other hand, the action of $\mathrm{I}_\pm$ on α_p is

$$\begin{aligned} \mathrm{I}_\pm\,\alpha_p &= \left(\hat{\gamma}^1_\pm\hat{\gamma}^2_\pm\cdots\hat{\gamma}^D_\pm\right)\left(\hat{e}^{\dagger i_1}\hat{e}^{\dagger i_2}\cdots\hat{e}^{\dagger i_p}\right)|0\rangle \\ &= \sigma(J,I)\,\left(\hat{\gamma}^{j_1}_\pm\hat{\gamma}^{j_2}_\pm\cdots\hat{\gamma}^{j_{D-p}}_\pm\right)\left(\hat{\gamma}^{i_1}_\pm\hat{\gamma}^{i_2}_\pm\cdots\hat{\gamma}^{i_p}_\pm\right)\left(\hat{e}^{\dagger i_1}\hat{e}^{\dagger i_2}\cdots\hat{e}^{\dagger i_p}\right)|0\rangle \\ &= (-1)^{p(p-1)/2}\,\sigma(J,I)\left(\hat{\gamma}^{j_1}\hat{\gamma}^{j_2}\cdots\hat{\gamma}^{j_{D-p}}\right)\left(\hat{\gamma}^{i_1}\hat{\gamma}^{i_2}\cdots\hat{\gamma}^{i_p}\right)\left(\hat{e}^{\dagger i_p}\hat{e}^{\dagger i_{p-1}}\cdots\hat{e}^{\dagger i_1}\right)|0\rangle \\ &= (-1)^{p(p\mp1)/2}\,\sigma(J,I)(-1)^{s(I)}\hat{\gamma}^{j_1}\hat{\gamma}^{j_2}\cdots\hat{\gamma}^{j_{D-p}}\,|0\rangle \\ &= (-1)^{p(p\mp1)/2}\,\sigma(J,I)(-1)^{s(I)}\hat{e}^{\dagger j_1}\hat{e}^{\dagger j_2}\cdots\hat{e}^{\dagger j_{D-p}}\,|0\rangle \\ &= (-1)^{p(p\mp1)/2}\,*\alpha_p \end{aligned}$$

□

2. *The square of* $\mathrm{I}_\pm$ *is*

$$\mathrm{I}^2_\pm = (-1)^{D(D\mp1)/2+s} \tag{1218}$$

Proof:

$$\begin{aligned} \mathrm{I}^2_\pm &= \left(\hat{\gamma}^1_\pm\hat{\gamma}^2_\pm\cdots\hat{\gamma}^D_\pm\right)\left(\hat{\gamma}^1_\pm\hat{\gamma}^2_\pm\cdots\hat{\gamma}^D_\pm\right) \\ &= (-1)^{D(D-1)/2}\,\left(\hat{\gamma}^1_\pm\hat{\gamma}^2_\pm\cdots\hat{\gamma}^D_\pm\right)\left(\hat{\gamma}^D_\pm\hat{\gamma}^{D-1}_\pm\cdots\hat{\gamma}^1_\pm\right) \\ &= (-1)^{D(D-1)/2}(-1)^{D/2\pm(s-D/2)} \\ &= (-1)^{D(D\mp1)/2+s} \end{aligned} \tag{1219}$$

3. $\mathrm{I}_\pm$ is constant and covariantly constant:

$$\begin{aligned} \left[\partial_\mu, \mathrm{I}_\pm\right] &= 0 \\ \left[\hat{\nabla}_\mu, \mathrm{I}_\pm\right] &= 0\,. \end{aligned} \tag{1220}$$

Proof: Constancy follows because the $\hat{\gamma}^a_\pm$ are defined to commute with ∂_μ. Covariant constancy follows by (1199) and the fact that every Clifford 2-vector commutes with the pseudoscalar (of either sign).

Normalizing this to unity yields the idempotent *chirality* operators:

B.16 (Chirality) The involutions

$$\begin{aligned} \bar{\gamma}_{\pm} &:= i^{D(D\mp 1)/2+s}\, \mathrm{I}_{\pm} \\ &\overset{(1217)}{=} * \, i^{D(D\mp 1)/2+s}(-1)^{\hat{N}(\hat{N}-1)/2} && (1221) \\ \bar{\gamma}_{\pm}^2 &= 1\,. && (1222) \end{aligned}$$

are called *chirality* operators.

Theorem B.17 (Basic relations concerning chirality)

1. *The chirality operator is covariantly constant:*

$$\left[\hat{\nabla}_\mu, \bar{\gamma}_{\pm}\right] = 0\,. \qquad (1223)$$

Proof: From definition 1199 (p.301) one has

$$\hat{\nabla}_\mu = \partial_\mu + \omega_{\mu ab}\hat{\mathrm{e}}^{\dagger a}\hat{\mathrm{e}}^b = \partial_\mu + \frac{1}{4}\omega_{\mu ab}\left(\hat{\gamma}_+^a\hat{\gamma}_+^b - \hat{\gamma}_-^a\hat{\gamma}_-^b\right)\,.$$

By assumption the spin-frame Clifford elements are constant

$$\left[\partial_\mu, \hat{\gamma}_{\pm}^a\right] = 0$$

and hence so is their pseudoscalar:

$$\left[\partial_\mu, \bar{\gamma}_{\pm}\right] = 0\,.$$

Furthermore

$$\left[\omega_{\mu ab}\left(\hat{\gamma}_-^a\hat{\gamma}_-^b - \hat{\gamma}_+^a\hat{\gamma}_+^b\right), \bar{\gamma}_{\pm}\right] = 0$$

because $\hat{\gamma}_{\pm}$ either commute or anticommute with the pseudoscalar.

2. *The chirality operators of both Clifford algebras* (1178) *commute (anti-commute) in even (odd) dimensions:*

$$\left[\bar{\gamma}_+, \bar{\gamma}_-\right]_{(-)^{D+1}} = 0 \qquad (1224)$$

3. *The product of both chirality operators is proportional to the Witten operator* $(-1)^{\hat{N}}$*:*

$$\bar{\gamma}_-\bar{\gamma}_+ = i^{D^2}(-1)^{D(D+1)/2}\,(-1)^{\hat{N}} \qquad (1225)$$

Proof: Observe that

$$\begin{aligned} \hat{\gamma}_-^i\hat{\gamma}_+^i &= \left[\hat{\mathrm{e}}^{\dagger i}, \hat{\mathrm{e}}^i\right] \\ &= 2\hat{\mathrm{e}}^{\dagger i}\hat{\mathrm{e}}^i - \eta^{ii} \\ &= \mathrm{sign}\left(\eta^{ii}\right)(-1)^{\hat{N}^{(i)}}\,. && (1226) \end{aligned}$$

It follows that:

$$\begin{aligned}
\bar\gamma_-\bar\gamma_+ &= i^{D(D+1)/2+s} i^{D(D-1)/2+s}\mathrm{I}_-\mathrm{I}_+ \\
&= i^{D(D+1)/2+s} i^{D(D-1)/2+s}(-1)^{D(D-1)/2+s}(-1)^{\hat N} \\
&= i^{D^2}(-1)^{D(D-1)/2}(-1)^{\hat N}
\end{aligned} \tag{1227}$$

□

4. **Theorem B.18 (Chirality and adjoint)** *The chirality operator $\bar\gamma_\pm$ relates creation and annihilation operators as well as the exterior derivatives to their respective adjoints via:*

$$\begin{aligned}
\hat{\mathrm{e}}^a &= \mp(-1)^D\bar\gamma_\pm\,\hat{\mathrm{e}}^\dagger\,\bar\gamma_\pm \\
\mathbf{d}^\dagger &= \pm(-1)^D\,\bar\gamma_\pm\mathbf{d}\bar\gamma_\pm\,.
\end{aligned} \tag{1228}$$

(*cf.* (1197))

Proof:

$$\begin{aligned}
\bar\gamma_\pm\,\hat{\mathrm{e}}^\dagger\,\bar\gamma_\pm &= i^{D(D\mp1)/2+s}\mathrm{I}\,\hat{\mathrm{e}}^\dagger\,i^{D(D\mp1)/2+s}\mathrm{I} \\
&= i^{D(D\mp1)/2+s}\hat\gamma_\pm^1\hat\gamma_\pm^2\cdots\hat\gamma_\pm^D\,\hat{\mathrm{e}}^\dagger\,i^{D(D\mp1)/2+s}\mathrm{I} \\
&= \pm(-1)^{D+1}i^{D(D\mp1)/2+s}\,\hat{\mathrm{e}}\,\hat\gamma_\pm^1\hat\gamma_\pm^2\cdots\hat\gamma_\pm^D i^{D(D\mp1)/2+s}\mathrm{I} \\
&= \pm(-1)^{D+1}\hat{\mathrm{e}}\bar\gamma_\pm\bar\gamma_\pm \\
&= \pm(-1)^{D+1}\hat{\mathrm{e}}
\end{aligned}$$

$$\begin{aligned}
\bar\gamma_\pm\mathbf{d}\bar\gamma_\pm &\overset{(1221)}{=} i^{D(D\mp1)/2+s}\mathrm{I}\,\mathbf{d}\,i^{D(D\mp1)/2+s}\mathrm{I} \\
&\overset{(1217)}{=} (-1)^{D(D\mp1)/2+s}\,*(-1)^{\hat N(\hat N\mp1)/2}\,\mathbf{d}\,*(-1)^{\hat N(\hat N\mp1)/2} \\
&= (-1)^{D(D\mp1)/2+s}\,*\,\mathbf{d}\,*(-1)^{(D-\hat N+1)(D-\hat N+1\mp1)/2}(-1)^{\hat N(\hat N\mp1)/2} \\
&\overset{(1197)}{=} (-1)^{D(D\mp1)/2+s}\,\mathbf{d}^\dagger(-1)^{D(\hat N-1)+1+s}(-1)^{(D-\hat N+1)(D-\hat N+1\mp1)/2}(-1)^{\hat N(\hat N\mp1)/2} \\
&= \pm(-1)^D\,\mathbf{d}^\dagger
\end{aligned}$$

□

5. From (1216 (p.305)) and (1221 (p.307)) one has

$$\begin{aligned}
\mathrm{I}_\pm^\dagger &= (-1)^{D(D\mp1)/2}\mathrm{I}_\pm \\
\overline{\bar\gamma_\pm} &= (-1)^{D(D\mp1)/2+s}\bar\gamma_\pm \\
\bar\gamma_\pm^\dagger &= (-1)^s\,\bar\gamma_\pm\,.
\end{aligned} \tag{1229}$$

B.19 (Helicity) *The projectors on the chirality eigenspaces will be called helicity projectors:*

$$\begin{aligned}
\hat{\mathrm{h}}_\pm &:= \frac{1}{2}(1\pm\bar\gamma_+) \\
\hat{\mathrm{h}}'_\pm &:= \frac{1}{2}(1\pm\bar\gamma_-)
\end{aligned} \tag{1230}$$

$$\begin{aligned}
\Rightarrow \hat{\mathrm{h}}_\pm^{(\prime)}\hat{\mathrm{h}}_\pm^{(\prime)} &= \hat{\mathrm{h}}_\pm^{(\prime)} \\
\hat{\mathrm{h}}_\pm^{(\prime)}\hat{\mathrm{h}}_\mp^{(\prime)} &= 0\,.
\end{aligned} \tag{1231}$$

Theorem B.20 (Basic relations concerning helicity)

1. *The helicity projectors are self-adjoint on euclidian manifolds and mutually adjoint on lorentzian manifolds:*

$$\left(\hat{\mathrm{h}}_{\pm}^{(\prime)}\right)^{\dagger} = \hat{\mathrm{h}}_{\pm(-)^s}^{(\prime)} \,, \tag{1232}$$

2. As a corollary one has:

   States of opposite (equal) helicity have vanishing scalar product for Euclidean (lorentzian) metrics:

$$\left\langle \hat{\mathrm{h}}_{\pm}^{(\prime)}\alpha | \hat{\mathrm{h}}_{\mp(-)^s}^{(\prime)}\alpha \right\rangle_{\mathrm{loc}} = 0 \tag{1233}$$

3. *The helicity projectors $\hat{\mathrm{h}}_{\pm}$ relate the canonical algebra $\hat{\mathrm{e}}^{\dagger}, \hat{\mathrm{e}}$ with the Clifford algebra $\hat{\gamma}_{\pm}$ via the following relations:*

$$\begin{aligned}
\hat{\mathrm{h}}_{\pm}\,\hat{\mathrm{e}}^{\dagger a}\,\hat{\mathrm{h}}_{\pm} &= \frac{1}{2}\,\hat{\mathrm{h}}_{\pm}\,\hat{\gamma}^{a}_{(-)^{D+1}} \\
&= \frac{1}{2}\,\hat{\gamma}^{a}_{(-)^{D+1}}\hat{\mathrm{h}}_{\pm} \\
\hat{\mathrm{h}}_{\pm}\,\hat{\mathrm{e}}^{\dagger a}\,\hat{\mathrm{h}}_{\mp} &= \frac{1}{2}\,\hat{\mathrm{h}}_{\pm}\,\hat{\gamma}^{a}_{(-)^{D}} \\
&= \frac{1}{2}\,\hat{\gamma}^{a}_{(-)^{D}}\hat{\mathrm{h}}_{\mp} \\
\hat{\mathrm{h}}_{\pm}\,\hat{\mathrm{e}}^{a}\,\hat{\mathrm{h}}_{\pm} &= \frac{1}{2}(-1)^{D+1}\,\hat{\mathrm{h}}_{\pm}\,\hat{\gamma}^{a}_{(-)^{D+1}} \\
&= \frac{1}{2}(-1)^{D+1}\,\hat{\gamma}^{a}_{(-)^{D+1}}\hat{\mathrm{h}}_{\pm} \\
\hat{\mathrm{h}}_{\pm}\,\hat{\mathrm{e}}^{a}\,\hat{\mathrm{h}}_{\mp} &= \frac{1}{2}(-1)^{D}\,\hat{\mathrm{h}}_{\pm}\,\hat{\gamma}^{a}_{(-)^{D}} \\
&= \frac{1}{2}(-1)^{D}\,\hat{\gamma}^{a}_{(-)^{D}}\hat{\mathrm{h}}_{\mp}
\end{aligned} \tag{1234}$$

The relations for $\hat{\mathrm{h}}'_{\pm}$ are the same except for the substitution $D \to D+1$. *Proof:* The first equation follows from:

$$\begin{aligned}
\hat{\mathrm{h}}_{\pm}\,\hat{\mathrm{e}}^{\dagger a}\,\hat{\mathrm{h}}_{\pm} &= \frac{1}{4}\left((1\pm\bar{\gamma})\,\hat{\mathrm{e}}^{\dagger a}\,(1\pm\bar{\gamma})\right) \\
&= \frac{1}{4}\left(\hat{\mathrm{e}}^{\dagger a} + \bar{\gamma}\hat{\mathrm{e}}^{\dagger a}\bar{\gamma} \pm \bar{\gamma}\hat{\mathrm{e}}^{\dagger a} \pm \hat{\mathrm{e}}^{\dagger a}\bar{\gamma}\right) \\
&= \frac{1}{4}\left(\hat{\mathrm{e}}^{\dagger a} + (-1)^{D+1}\hat{\mathrm{e}}^{a} \pm \bar{\gamma}\hat{\mathrm{e}}^{\dagger a} \pm \bar{\gamma}(-1)^{D+1}\hat{\mathrm{e}}^{a}\right) \\
&= \frac{1}{4}\,(1\pm\bar{\gamma})\left(\hat{\mathrm{e}}^{\dagger a} + (-1)^{D+1}\hat{\mathrm{e}}^{a}\right) \\
&= \frac{1}{2}\,\hat{\mathrm{h}}_{\pm}\,\hat{\gamma}^{a}_{(-)^{D+1}}\,.
\end{aligned}$$

Similarly for the other equations. □

4. *The exterior Dirac operators* $\mathbf{D}_\pm = \mathbf{d} \pm \mathbf{d}^\dagger = \hat{\gamma}_{g\pm}{}^\mu \hat{\nabla}_\mu$ *can be identically decomposed as*

$$\begin{aligned} \mathbf{D}_\pm &= 2\left(\hat{\mathrm{h}}_+ \mathbf{d}\, \hat{\mathrm{h}}_{\pm(-)^D} + \hat{\mathrm{h}}_- \mathbf{d}\, \hat{\mathrm{h}}_{\mp(-)^D}\right) \\ &= \pm 2\left(\hat{\mathrm{h}}_+ \mathbf{d}^\dagger\, \hat{\mathrm{h}}_{\pm(-)^D} + \hat{\mathrm{h}}_- \mathbf{d}^\dagger\, \hat{\mathrm{h}}_{\mp(-)^D}\right) \\ &= \hat{\mathrm{h}}_+ \left(\mathbf{d} \pm \mathbf{d}^\dagger\right) \hat{\mathrm{h}}_{\pm(-)^D} + \hat{\mathrm{h}}_- \left(\mathbf{d} \pm \mathbf{d}^\dagger\right) \hat{\mathrm{h}}_{\mp(-)^D} \,. \end{aligned} \tag{1235}$$

Proof:

$$\begin{aligned} & 2\left(\hat{\mathrm{h}}_+ \mathbf{d}\, \hat{\mathrm{h}}_{\pm(-)^D} + \hat{\mathrm{h}}_- \mathbf{d}\, \hat{\mathrm{h}}_{\mp(-)^D}\right) \\ =\;& \frac{1}{2}\left((1+\bar{\gamma})\,\mathbf{d}\left(1 \pm (-1)^D \bar{\gamma}\right) + (1-\bar{\gamma})\,\mathbf{d}\left(1 \mp (-1)^D \bar{\gamma}\right)\right) \\ =\;& \mathbf{d} \pm (-1)^D \bar{\gamma}\mathbf{d}\bar{\gamma} \\ \overset{(1228)}{=}\;& \mathbf{d} \pm \mathbf{d}^\dagger \end{aligned}$$

□

5. Accordingly, the *exterior Laplace operator*

$$\mathbf{\Delta} := \pm(\mathbf{D}_\pm)^2 \tag{1236}$$

can be written as

$$\begin{aligned} \mathbf{\Delta} &= 4\left(\hat{\mathrm{h}}_+ \mathbf{d}\, \hat{\mathrm{h}}_\pm \mathbf{d}\, \hat{\mathrm{h}}_+ + \hat{\mathrm{h}}_- \mathbf{d}\, \hat{\mathrm{h}}_\mp \mathbf{d}\, \hat{\mathrm{h}}_-\right) \\ &= 4\left(\hat{\mathrm{h}}_+ \mathbf{d}^\dagger\, \hat{\mathrm{h}}_\pm \mathbf{d}^\dagger\, \hat{\mathrm{h}}_+ + \hat{\mathrm{h}}_- \mathbf{d}^\dagger\, \hat{\mathrm{h}}_\mp \mathbf{d}^\dagger\, \hat{\mathrm{h}}_-\right) \,. \end{aligned} \tag{1237}$$

B.2 Dirac operators on Clifford and exterior bundles

Introduction. The Dirac operators on the exterior bundle and on the spin bundle are intimately related. The exterior bundle $\Lambda(\mathcal{M})$, on which the $(N=2)$-Dirac operator acts, is isomorphic to the tensor product of two spin bundles:

$$\Lambda(\mathcal{M}) \simeq \mathrm{Spin}(\mathcal{M})_+ \otimes \mathrm{Spin}(\mathcal{M})_- \,. \tag{1238}$$

This isomorphism can be made explicit by choosing a basis of $\Lambda(\mathcal{M})$ consisting of elements that factor into two *algebraic spinors* ([171][268]):

$$\Lambda(\mathcal{M}) \ni \lambda := s \otimes \tilde{s} \in \mathrm{Spin}(\mathcal{M})_+ \otimes \mathrm{Spin}(\mathcal{M})_- \,. \tag{1239}$$

Using this basis it is shown that the $(N=2)$-Dirac operator is the sum of two $(N=1)$-Dirac operators which act on $\mathrm{Spin}(\mathcal{M})_\pm$. The $(N=1)$-Dirac operator is recognized as the ordinary Dirac operator of the theory of the relativistic electron.

The following demonstration, which is based on the previous section §B.1 (p.297) (in particular on B.6-B.9) complies with the material in [268][267], though in spirit it is somewhat more indebted to [101]. A closely related discussion can be found in section 2.1 of [33].

B.21 The Clifford bundle $\mathrm{Cl}(\mathcal{M})_\pm$ can be spanned by elements of the form

$$\psi_\pm \, \hat{O} \, \tilde{\psi_\pm} \tag{1240}$$

where $\psi_\pm$ is a Dirac-Hestenes state

$$\begin{aligned} &\psi_\pm = \rho R_\pm \\ &R_\pm \in \mathrm{Spin}(\mathcal{M})_\pm \end{aligned} \tag{1241}$$

and $\hat{O}$ is any constant element of $\mathrm{Cl}(\mathcal{M})_\pm$, i.e.

$$\begin{aligned} &\hat{O} \in \mathrm{Cl}(\mathcal{M})_\pm \\ &\left[\partial_\mu, \hat{O}\right] = 0 \,. \end{aligned} \tag{1242}$$

Accordingly, $\mathrm{Cl}(\mathcal{M})_\pm \left|0\right\rangle$ can be spanned by elements of the form

$$\psi_\pm \, \hat{O} \, \tilde{\psi_\pm} \left|0\right\rangle \,. \tag{1243}$$

B.22 (Levi-Civita connection and Clifford connection) The Levi-Civita connection on the exterior bundle, with local representation

$$\begin{aligned} \hat{\nabla}_\mu &= \partial_\mu - \omega_{\mu ab} \hat{\mathrm{e}}^{\dagger b} \hat{\mathrm{e}}^a \\ &= \partial_\mu + \omega_{\mu ab} \hat{\mathrm{e}}^{\dagger a} \hat{\mathrm{e}}^b \\ &= \partial_\mu + \frac{1}{4} \omega_{\mu ab} \left(\hat{\gamma}_+^a \hat{\gamma}_+^b - \hat{\gamma}_-^a \hat{\gamma}_-^b \right) \\ &= \partial_\mu \pm \frac{1}{4} \omega_{\mu ab} \left[\hat{\gamma}_\pm^a \hat{\gamma}_\pm^b, \cdot \right] \,, \end{aligned} \tag{1244}$$

acts on these elements as follows:

$$\begin{aligned}\hat{\nabla}_\mu \psi_\pm \, \hat{O} \, \tilde{\psi_\pm} \left|0\right\rangle &= \left(\left(\partial_\mu \pm \frac{1}{4} \omega_{\mu a b} \hat{\gamma}^a_\pm \hat{\gamma}^b_\pm \right) \psi \right) \hat{O} \tilde{\psi} \left|0\right\rangle \\ &\quad + \psi \hat{O} \left(\left(\partial_\mu \pm \frac{1}{4} \omega_{\mu a b} \hat{\gamma}^a_\pm \hat{\gamma}^b_\pm \right) \psi \right)^\sim \left|0\right\rangle \, . \end{aligned} \tag{1245}$$

Since the operation $\tilde{\cdot}$ includes complex conjugation, this is equivalent to

$$\begin{aligned}\hat{\nabla}_\mu \psi_\pm \, \hat{O} \, \tilde{\psi_\pm} \left|0\right\rangle &= \left(\left(\partial_\mu \pm \frac{1}{4} \omega_{\mu a b} \hat{\gamma}^a_\pm \hat{\gamma}^b_\pm + i A_\mu \right) \psi \right) \hat{O} \tilde{\psi} \left|0\right\rangle \\ &\quad + \psi \hat{O} \left(\left(\partial_\mu \pm \frac{1}{4} \omega_{\mu a b} \hat{\gamma}^a_\pm \hat{\gamma}^b_\pm + i A_\mu \right) \psi \right)^\sim \left|0\right\rangle \end{aligned} \tag{1246}$$

for all covariant scalar sections A_μ. One recognizes the operator

$$\hat{\nabla}^S := \partial_\mu \pm \frac{1}{4} \omega_{\mu a b} \hat{\gamma}^a_\pm \hat{\gamma}^b_\pm + i A_\mu \tag{1247}$$

as the local representation of a Clifford connection on Spin($\mathcal{M}$) (i.e. a Spin-connection compatible with the Levi-Civita connection, see [101] (3.4)).

Thus elements of the form $\psi \hat{O} \tilde{\psi} \left|0\right\rangle$ form a basis of the exterior bundle which explicitly exhibits the isomorphism with the twisted spin bundle (see [101] pp. 34):

$$\begin{aligned}\Lambda(\mathcal{M}) &= \mathrm{Spin}(\mathcal{M})_+ \otimes \mathrm{Spin}(\mathcal{M})_- \\ \hat{\nabla} &= \hat{\nabla}^{S_+} + \hat{\nabla}^{S_-} \, . \end{aligned} \tag{1248}$$

The above construction rests on the fact that the $N = 2$ connection transforms vectors, while the $N = 1$ connection transforms spinors, which may be regarded as square roots of vectors in the sense of the factorization (1248). Similar constructions arise in the study of the relation between Maxwell's theory and Dirac's electron. Note that the homogeneous exterior Dirac equation $\mathbf{D} \left|\psi\right\rangle = 0$ reduces to the free Maxwell equations when restricted to 2-forms. (*cf.* [264], where, however, no curvature is considered).

Literature.

1. The relation between the Dirac operator on the exterior bundle and that on the spin bundle has been the subject of various investigations. A brief account of their history is given in the introduction of [267]. In the context of relating Maxwell's equations to the Dirac electron this article also presents the method (1240) (1243) to constructively show the relation between both Dirac operators (*cf.* §2.2.3 (p.70)), which apparently originates in [264]. Even though these papers do not consider curved spacetime, the inclusion of curvature in (1246) is, of course, straightforward.

   In the same spirit, the articles [263] and [226] show how the 'classical' Dirac-Hestenes equation is obtained from the operator

$$\begin{aligned} v_a \hat{\gamma}^a_- \cdot \mathbf{D} &= v_a \hat{\gamma}^a_- \cdot \hat{\gamma}^\mu_- \left(\partial_\mu - \Omega_\mu \right) \\ &= \partial_v - \Omega_v \end{aligned}$$

where v is the tangent vector to some trajectory. Interestingly, $\partial_v - \Omega_v$ applied on the accompanying vielbein of that trajectory gives Frenet's equations. Also, for $\mathbf{D} = \mathbf{d} + \mathbf{d}^\dagger$ the ordinary exterior Dirac operator

$$\begin{aligned} v_a \hat{\gamma}_-^a \cdot \mathbf{D} &= v_a \hat{\gamma}_-^a \cdot \hat{\gamma}_-^\mu \hat{\nabla}_\mu \\ &= \hat{\nabla}_v \end{aligned} \tag{1249}$$

shows that the classical solution to $\mathbf{D}\,|\psi\rangle = 0$ is a frame parallel translated along its trajectory.

On the other hand, somewhat contrary to these ideas, there are several authors who consider the action of the *exterior* Dirac operator on *spinors* ψP (instead of on bispinors $\psi \hat{O} \tilde{\psi}$ as above), i.e. on minimal left ideals of the projector P. But since a spinor transforms under the *left* action spin(n), while the exterior Dirac operator includes the *adjoint* action of spin(n), such an investigation more or less leave ordinary conceptual frameworks. E.g. [51] interpret the right contribution of the adjoint action on spinors as gauge transformations in some isospin space (with respect to such 'isospin' transformations also compare [220]). What is somewhat unexpected here is that the usual *spin connection* should act from both sides on spinors (*cf.* [194]). But of course it is possible and meaningful to transform a spinor from the right. In fact, investigation of this possibility naturally lead to the notion of *families* of particles, which might well shed light on the standard model (see [258]).

2. The expression $\hat{\gamma}_-^\mu \partial_\mu$ is, without specifying a representation for $\hat{\gamma}_-$ ambiguous with respect to extensions to curved manifolds, since one could choose either of

$$\begin{aligned} \mathbf{D}_{\mathrm{Spin}(\mathcal{M})} &= \hat{\gamma}_-^\mu \left(\partial_\mu + \frac{1}{2} \omega_{\mu ab} \hat{\gamma}_-^a \hat{\gamma}_-^b \right) \\ \mathbf{D}_{\Lambda(\mathcal{M})} &= \hat{\gamma}_-^\mu \left(\partial_\mu + \frac{1}{2} \omega_{\mu ab} \left(\hat{\gamma}_-^a \hat{\gamma}_-^b - \hat{\gamma}_+^a \hat{\gamma}_+^b \right) \right) . \end{aligned} \tag{1250}$$

Sometimes in the school of *Geometric Algebra* the convention is used to write ∂_μ for what is denoted by $\hat{\nabla}_\mu$ in (1199).

Using this convention [217] remarks that choosing

$$H \sim \left(\hat{\gamma}_-^\mu \partial_\mu \right)^2 , \tag{1251}$$

restricted to scalar fields, as the Hamiltonian of a free quantum particle removes the usual ordering ambiguity of quantum mechanics and that the 'correspondence rule' $p \to \hat{\gamma}_-^\mu \partial_\mu$ gives a well behaved quantum observable. In the context of the discussion in §2.2.1 (p.55) this operator has been called the 'supercharge', as, in fact, it is commonly known (*cf.* references in §2.2 (p.54)). This is an indication of the fact that 'supersymmetric quantum mechanics' and 'Geometric Calculus' are partly about the same topic only under different headings.

B.3 Superanalysis

Outline. Selected basics of rudimentary superanalysis are discussed. Definitions and results are given which are needed in §4 (p.181) to translate between the language of superanalysis and the language of forms (see in particular 4.31 (p.213)).

Literature. Standard textbooks on superanalysis are [30] and [79]. The material of interest in the present context is mainly that presented in [95]§2.4. In the context of "geometric algebra" there has been some effort to relate superanalysis with Clifford algebra, see for instance [263].

B.23 (Basic definitions) The objects

$$c^n,\ n \in \{1, \dots, D\}$$

are taken to be anticommuting and nilpotent generators of a Grassmann algebra:

$$c^n c^m + c^m c^n \quad = \quad 0, \quad \forall\, n, m \in \{1, \dots, D\}\ . \tag{1252}$$

Together with a second copy, $c^{*n},\ n \in \{1, \dots, D\}$, of this algebra and an involutive operation $*$, this is said to generate a complex Grassmann algebra, defined by the relations

$$\begin{aligned} & c^n c^m + c^m c^n = c^n c^{*m} + c^{*m} c^n = c^{*n} c^{*m} + c^{*m} c^{*n} = 0 \\ & (c^n)^* = c^{*n},\ (c^{*n})^* = c^n \\ & (AB)^* = B^* A^*\ . \end{aligned} \tag{1253}$$

Grassmann analytic functions f are polynomials in c^n, c^{*n}:

$$f(c^{*\cdot} c^{\cdot}) \quad = \quad \sum_{p+q=0}^{D} f_{n_1,n_2,\dots,n_p,m_1,m_2,\cdots,m_q} c^{*n_1} c^{*n_2} \cdots c^{*n_p} c^{m_1} c^{m_2} \cdots c^{m_q}\ .$$

The terms in this sum for various p,q are said to be of degree deg $= p + q$. One introduces Grassmannian partial derivative operators

$$\frac{\partial}{\partial c^n},\ \frac{\partial}{\partial c^{*n}}$$

by the rule (all terms assumed to be non-vanishing)

$$\begin{aligned} \frac{\partial}{\partial c^n} A c^n B \quad &:= \quad (-1)^{\deg(A)} AB \\ \frac{\partial}{\partial c^{*n}} A c^{*n} B \quad &:= \quad (-1)^{\deg(A)} AB\,. \end{aligned} \tag{1254}$$

Functions f that only depend on the c^{*n}

$$\frac{\partial}{\partial c^n} f = 0$$

are called *holomorphic* and those that depend only on the c^n

$$\frac{\partial}{\partial c^{*n}} f = 0$$

are called *antiholomorphic*.
In the context of superanalysis one tries to write operations on the Grassmann algebra as analogs of operations in ordinary analysis. A formal definite integration, known as *Berezin integration*,

$$A \mapsto \int A \prod_{\substack{n=1\ldots D \\ m=D\ldots 1}}^{D} dc^{*n} dc_m$$

is defined by demanding

$$\begin{aligned} \deg(A) < 2D \;\Rightarrow\; \int A \prod_{\substack{n=1\ldots D \\ m=D\ldots 1}}^{D} dc^{*n} dc^m = 0 \\ \int c^1 c^2 \cdots c^D c^{*D} c^{*D-1} \cdots c^{*1} \prod_{\substack{n=1\ldots D \\ m=D\ldots 1}} dc^{*n} dc^m = 1\,, \end{aligned} \tag{1255}$$

and extending this linearly to the whole algebra.

Conventions. Throughout this section the following conventions are used: ϕ and ψ are a p-form and a q-form, respectively, on the D-dimensional (semi-) Riemannian manifold $(\mathcal{M}, g)$ with local coordinates $\{x^1, x^2, \ldots, x^D\}$:

$$\begin{aligned} \phi &:= f_{m_1,m_2,\ldots,m_p} dx^{m_1} \wedge dx^{m_2} \wedge \cdots \wedge dx^{m_p} \\ &= f_{m_1,m_2,\ldots,m_p} \hat{c}^{\dagger m_1} \hat{c}^{\dagger m_2} \cdots \hat{c}^{\dagger m_p} |0\rangle \\ \psi &:= g_{n_1,n_2,\ldots,n_q} dx^{n_1} \wedge dx^{n_2} \wedge \cdots \wedge dx^{n_q} \\ &= g_{n_1,n_2,\ldots,n_q} \hat{c}^{\dagger n_1} \hat{c}^{\dagger n_2} \cdots \hat{c}^{\dagger n_q} |0\rangle \,. \end{aligned} \tag{1256}$$

f and g denote the associated Grassmannian holomorphic objects:

$$\begin{aligned} f &:= f_{m_1,m_2,\ldots,m_p} c^{*m_1} c^{*m_2} \cdots c^{*m_p} \\ g &:= g_{n_1,n_2,\ldots,n_q} c^{*n_1} c^{*n_2} \cdots c^{*n_q}\,, \end{aligned}$$

and the operation of switching between the two notations is indicated by S, so that, for example:

$$\begin{aligned} S(\phi) &= f \\ S^{-1}(f) &= \phi\,. \end{aligned} \tag{1257}$$

The metric g is used to shift indices carried by Grassmann elements:

$$\begin{aligned} c_n &:= g_{nm} c^m \\ c^*_n &:= g_{nm} c^{*m}\,. \end{aligned} \tag{1258}$$

Finally, let P_0 be the projector on the 0-form sector:

$$P_0 \phi = \begin{cases} \phi & \text{if } \hat{N}\phi = 0 \\ 0 & \text{otherwise} \end{cases} \,. \tag{1259}$$

B.24 (Change of fermionic variables) Under a change of variables in an integral over Grassmann numbers the integrand picks up the *inverse* of the Jacobian of the transformation. (*cf.* [30], pp. 78) Let

$$F^i{}_j := \frac{\partial c'^i}{\partial c^j}\,,$$

then

$$\int f\, dc^1\, dc^2 \cdots dc^n \;=\; \det(F) \int f\, dc'^1\, dc'^2 \cdots dc'^n\,. \tag{1260}$$

Proof: Let

$$dc'^1\, dc'^2 \cdots dc'^n \;=\; K dc^1\, dc^2 \cdots dc^n \tag{1261}$$

and recall

$$\begin{aligned}\int f\left(c^i\right) dc^1\, dc^2 \cdots dc^n &= \int f_{12\cdots n} c^1 c^2 \cdots c^n\, dc^1\, dc^2 \cdots dc^n \\ &= f_{12\cdots n}\,.\end{aligned} \tag{1262}$$

It follows that

$$\begin{aligned}\int f_{12\cdots n} c^1 c^2 \cdots c^n\, dc^1\, dc^2 \cdots dc^n &= \frac{1}{\det(F)} \int f_{12\cdots n} c'^1 c'^2 \cdots c'^n\, dc^1\, dc^2 \cdots dc^n \\ &= \frac{K}{\det(F)} \int f_{12\cdots n} c'^1 c'^2 \cdots c'^n dc^1\, dc^2 \cdots dc^n \\ &= \frac{K}{\det(F)} f_{12\cdots n}\,,\end{aligned} \tag{1263}$$

and hence $K = \det(F)$.

B.25 The Hodge dual $*\phi$ of the form ϕ is related related to the Grassmannian Fourier transformation of the conjugate f^* of the associated holomorphic Grassmann function f by:

$$\int f^* \exp(-c^{*n} c_n) \prod_{m=D\ldots 1}^{D} dc_m \;=\; (-1)^{D(D+1)/2+p} \frac{1}{\sqrt{g}} S(*\phi)\,. \tag{1264}$$

Proof:

$$\begin{aligned}&\int f^* \exp(-c^{*n} c_n) \prod_{m=D\ldots 1}^{D} dc_m \\ =\;& (-1)^{p(p-1)/2} \int f^{m_1,m_2,\ldots,m_p} c_{m_1} c_{m_2} \cdots c_{m_p} \exp(-c^{*n} c_n) \prod_{m=D\ldots 1}^{D} dc_m \\ =\;& (-1)^{p(p-1)/2+D-p} f^{m_1,m_2,\ldots,m_p} \\ &\qquad \int c_{m_1} c_{m_2} \cdots c_{m_p} \sum_{1\leq r_1 < r_2 < \cdots < r_{D-p} \leq D} c^{*r_1} c_{r_1} c^{*r_2} c_{r_2} \cdots c^{*r_{D-p}} c_{r_{D-p}} \prod_{m=D\ldots 1}^{D} dc_m \\ =\;& (-1)^{p(p-1)/2+D-p+(D-p)(D-p-1)/2+(D-p)p}\, f^{m_1,m_2,\ldots,m_p}\end{aligned}$$

$$
\begin{aligned}
& \sum_{1\leq r_1<r_2<\cdots<r_{D-p}\leq D} c^{*r_1}c^{*r_2}\cdots c^{*r_{D-p}} \underbrace{\int c_{m_1}c_{m_2}\cdots c_{m_p}c_{r_1}c_{r_2}\cdots c_{r_{D-p}} \prod_{m=D\ldots 1}^{D} dc_m}_{=\epsilon_{m_1,m_2,\cdots,m_p,r_1,r_2,\cdots,r_{D-p}}} \\
&= (-1)^{D(D+1)/2+p}\frac{1}{(D-p)!}f^{m_1,m_2,\ldots,m_p}\epsilon_{m_1,m_2,\cdots,m_p,r_1,r_2,\cdots,r_{D-p}}c^{*r_1}c^{*r_2}\cdots c^{*r_{D-p}} \\
&= (-1)^{D(D+1)/2+p}\frac{1}{\sqrt{g}}S(*\phi)\;. \qquad (1265)
\end{aligned}
$$

B.26 (Fermionic inner product) *The local inner product in terms of differential forms coincides with that for Grassmann analytic functions in the following sense:*

$$
\begin{aligned}
& P_0 * (\phi\wedge *\psi) \\
&= \int f^*\, g\, \exp(-c^{*n}c_n) \prod_{\substack{n=1\ldots D\\ m=D\ldots 1}} dc^{*n}dc_m \\
&= \begin{cases} p!\, f^{m_1,m_2,\ldots,m_p}g_{m_1,m_2,\ldots,m_p} & \text{if } p=q \\ 0 & \text{if } p\neq q \end{cases}\;. \qquad (1266)
\end{aligned}
$$

Proof: For differential forms this is stated in (1193). The following gives the proof for the Grassmann inner product:

$$
\begin{aligned}
& \int f^*\, g\, \exp(-c^{*n}c_n) \prod_{\substack{n=1\ldots D\\ m=D\ldots 1}} dc^{*n}dc_m \\
&= \int f^*_{m_1,m_2,\ldots,m_p}c^{m_1}c^{m_2}\cdots c^{m_p}\, g_{n_1,n_2,\ldots,n_q}c^{*n_1}c^{*n_2}\cdots c^{*n_q}\, \exp(-c^{*n}c_n)\, dc^{*1}\cdots dc^{*D}dc_D\cdots dc_1 \\
&= (-1)^{p(p-1)/2}\int f^{*m_1,m_2,\ldots,m_p}c_{m_1}c_{m_2}\cdots c_{m_p}\, g_{n_1,n_2,\ldots,n_q}c^{*n_1}c^{*n_2}\cdots c^{*n_q} \\
&\qquad \sum_r (-1)^r \sum_{0\leq l_1<l_2<\cdots<l_r\leq D} c^{*l_1}c_{l_1}c^{*l_2}c_{l_2}\cdots c^{*l_r}c_{l_r} \prod_{\substack{n=1\ldots D\\ m=D\ldots 1}} dc^{*n}dc_m \\
&= (-1)^{p(p-1)/2}\int f^{*m_1,m_2,\ldots,m_p}c_{m_1}c_{m_2}\cdots c_{m_p}\, g_{n_1,n_2,\ldots,n_q}c^{*n_1}c^{*n_2}\cdots c^{*n_q} \\
&\qquad \sum_r (-1)^r(-1)^{r(r+1)/2} \sum_{0\leq l_1<l_2<\cdots<l_r\leq D} c_{l_1}c_{l_2}\cdots c_{l_r}\, c^{*l_1}c^{*l_2}\cdots c^{*l_r} \prod_{\substack{n=1\ldots D\\ m=D\ldots 1}} dc^{*n}dc_m \\
&= (-1)^{p(p-1)/2}f^{*m_1,m_2,\ldots,m_p}g_{n_1,n_2,\ldots,n_q}\int \sum_r (-1)^{r+r(r+1)/2+rq} \sum_{0\leq l_1<l_2<\cdots<l_r\leq D} \\
&\qquad c_{m_1}c_{m_2}\cdots c_{m_p}\, c_{l_1}c_{l_2}\cdots c_{l_r}\, c^{*n_1}c^{*n_2}\cdots c^{*n_q}\, c^{*l_1}c^{*l_2}\cdots c^{*l_r} \prod_{\substack{n=1\ldots D\\ m=D\ldots 1}} dc^{*n}dc_m\,.
\end{aligned}
$$

At this point it is obvious that the Berezin integral vanishes if $p\neq q$. Hence assume $p=q$ in the following. Then the only contribution from the sum over r is that for $r=D-p$:

$$
= (-1)^{p(p-1)/2+D-p+(D-p)(D-p+1)/2+(D-p)p}\, f^{*m_1,m_2,\ldots,m_p}g_{n_1,n_2,\ldots,n_p}
$$

$$
\begin{aligned}
& \int \sum_{0\le l_1<l_2<\cdots<l_{D-p}\le D} c_{m_1}c_{m_2}\cdots c_{m_p}\, c_{l_1}c_{l_2}\cdots c_{l_{D-p}}\, c^{*n_1}c^{*n_2}\cdots c^{*n_p}\, c^{*l_1}c^{*l_2}\cdots c^{*l_{D-p}} \prod_{\substack{n=1\ldots D\\ m=D\ldots 1}} dc^{*n}dc_m \\
= \; & (-1)^{D(D-1)/2}\,(p!)^2 \sum_{\substack{1\le m_1<\cdots<m_p\le D\\ 1\le n_1<\cdots<n_p\le D}} f^{*m_1,m_2,\ldots,m_p} g_{n_1,n_2,\ldots,n_p} \\
& \int \sum_{0\le l_1<l_2<\cdots<l_{D-p}\le D} c_{m_1}c_{m_2}\cdots c_{m_p}\, c_{l_1}c_{l_2}\cdots c_{l_{D-p}}\, c^{*n_1}c^{*n_2}\cdots c^{*n_p}\, c^{*l_1}c^{*l_2}\cdots c^{*l_{D-p}} \prod_{\substack{n=1\ldots D\\ m=D\ldots 1}} dc^{*n}dc_m
\end{aligned}
$$

In the last line the implicit sum over the indices m_i, n_i has been made explicit and ordered. This makes it obvious that the only contribution to the Berezin integral comes from terms where $m_i = n_i$:

$$
\begin{aligned}
&= (p!)^2 \sum_{\substack{1\le m_1<\cdots<m_p\le D\\ 1\le n_1<\cdots<n_p\le D}} f^{*m_1,m_2,\ldots,m_p} g_{n_1,n_2,\ldots,n_p} \delta^{m_1}_{n_1}\delta^{m_2}_{n_2}\cdots\delta^{m_p}_{n_p} \\
&= (p!)^2 \sum_{1\le m_1<\cdots<m_p\le D} f^{*m_1,m_2,\ldots,m_p} g_{m_1,m_2,\ldots,m_p} \\
&= p!\, f^{*m_1,m_2,\ldots,m_p} g_{m_1,m_2,\ldots,m_p}\,.
\end{aligned}
$$

□

C More on symmetries

Introduction. The results of §2.2.7 (p.90) can be used to systematically list the Dirac operators that can be constructed by closing the generic $\hat{N}$ and $*$ symmetries, as well as using symmetries of covariantly constant Killing-Yano tensors. The main result is (C.4 (p.326)) that the introduction of a superpotential by means of the Witten model (*cf.* 2.2.2 (p.61)) breaks the symmetry induced by $*$ and thus reduces the number of independent Dirac operators that can otherwise be constructed by one half.

C.1 (Number of supercharges on the exterior bundle) Let $\hat{N}$ be the number operator on $\Lambda(\mathcal{M})$ counting the degree of forms and let $*$ be the Hodge duality operator. Let

$$\mathbf{H} := \left(\mathbf{d} + \mathbf{d}^\dagger\right)^2$$

be the exterior Laplace operator with the generic supercharge

$$\begin{aligned} \mathbf{D} &:= \mathbf{d} + \mathbf{d}^\dagger \\ \mathbf{D}^2 &= \mathbf{H}\,. \end{aligned} \tag{1267}$$

According to §2.2.7 (p.90) every symmetry of $\mathbf{H}$ can be 'closed' to give a symmetry of $\mathbf{D}$. Furthermore, $\mathbf{H}$ always has the symmetries $\hat{N}$ and $*$:

$$\begin{aligned} \left[\hat{N}, \mathbf{H}\right] &= 0 \\ [*, \mathbf{H}] &= 0\,. \end{aligned} \tag{1268}$$

(Instead of using the Hodge operator it is more convenient to look at the chirality operators $\bar{\gamma}_\pm$, which are related to $*$ and $\hat{N}$ by (*cf.* definition B.15 (p.305), B.16 (p.307))

$$\bar{\gamma}_\pm = *(-1)^{\hat{N}\left(\hat{N}\mp 1\right)/2} i^{D(D\mp 1)/2+s}\,.) \tag{1269}$$

By closing these symmetries one finds the following four supercharges

$$\begin{aligned} \mathbf{D}^{(1)} &:= \mathbf{D} = \hat{\gamma}_-^\mu \hat{\nabla}_\mu \\ \mathbf{D}^{(2)} &:= i\left[\hat{N}, \mathbf{D}\right] = i\hat{\gamma}_+^\mu \hat{\nabla}_\mu \\ \mathbf{D}^{(3)} &:= \frac{i}{2}\left[\bar{\gamma}_{-(-1)^D}, \mathbf{D}^{(1)}\right] = i\bar{\gamma}_{-(-1)^D}\mathbf{D}^{(1)} \\ \mathbf{D}^{(4)} &:= \frac{-i}{2}\left[\bar{\gamma}_{(-1)^D}, \mathbf{D}^{(2)}\right] = \bar{\gamma}_{(-1)^D}\mathbf{D}^{(2)}\,, \end{aligned} \tag{1270}$$

which, because of the relations

$$\begin{aligned} \left\{\mathbf{D}, \bar{\gamma}_{(-1)^D}\right\} &= 0 \\ \left[\mathbf{D}, \bar{\gamma}_{-(-1)^D}\right] &= 0 \\ \left[\hat{\gamma}_-^a, \hat{N}\right] &= \hat{\gamma}_+^a\,, \end{aligned} \tag{1271}$$

indeed satisfy

$$\left\{\mathbf{D}^{(i)}, \mathbf{D}^{(j)}\right\} = 2\delta^{ij}\mathbf{H}\,.$$

To find more than 4 square roots of the exterior Laplace operator the metric must admit complex structures on the tangent bundle: Extended supersymmetry on Riemannian manifolds goes along with Kähler, Hyperkähler, and octonionic geometry, which, algebraically, is related to the existence of certain Killing-Yano tensors and associated operators commuting with $\mathbf{H}$. (See §2.2.7 (p.90) for details.) A covariantly constant Killing-Yano tensor J, squaring to minus the identity (i.e. an almost complex structure)

$$\begin{aligned} &J = (J^{\mu}{}_{\nu}) \\ &J^2 = -1 \\ &\mathbf{d}J_{\mu\nu}dx^{\mu}dx^{\nu} = 0 \end{aligned} \tag{1272}$$

induces an operator

$$\begin{aligned} &\mathbf{J} := \frac{1}{2}J_{\mu\nu}\hat{\gamma}_-^{\mu}\hat{\gamma}_-^{\nu} \\ &[\mathbf{J},\mathbf{H}] = 0 \end{aligned} \tag{1273}$$

which commutes with the exterior Laplace operator and gives, by 'closing' it, rise to a '$J-$holomorphic' Dirac operator

$$\begin{aligned} \mathbf{D}_{(J)} &:= \frac{1}{2}[\mathbf{J},\mathbf{D}] \\ &= \frac{1}{2}\left[\mathbf{J}, e_a^{\mu}\hat{\gamma}_-^a\,\hat{\nabla}_{\mu}\right] \\ &= J_a^{\mu}\hat{\gamma}_-^a\,\hat{\nabla}_{\mu}\,. \end{aligned} \tag{1274}$$

One such J indicates Kähler geometry. Three almost complex structures $J^{(i)}$, which furthermore satisfy the quaternion algebra, make the geometry Hyper-Kähler and give rise to three new Dirac operators

$$\mathbf{D}_{J^{(i)}} := \frac{1}{2}\left[\mathbf{J^{(i)}},\mathbf{D}\right]\,.$$

Finally, seven complex structures, satisfying suitable conditions, give rise to the highest possible symmetry, governed by the octonions. Since these are the largest normed division algebra the sequence ends here, so that at most seven additional Dirac operators are obtained this way (*cf.* [101]). However, from every Dirac operator $J^{(i)}$ obtained by 'closing' a symmetry due to an almost complex structure, one can again, just as in (1270), construct the four Dirac operators

$$\begin{aligned} \mathbf{D}^{(1)}_{J^{(i)}} &:= \mathbf{D} = J^{\mu}{}_a\hat{\gamma}_-^a\hat{\nabla}_{\mu} \\ \mathbf{D}^{(2)}_{J^{(i)}} &:= i\left[\hat{N},\mathbf{D}\right] = iJ^{\mu}{}_a\hat{\gamma}_+^a\hat{\nabla}_{\mu} \\ \mathbf{D}^{(3)}_{J^{(i)}} &:= \frac{i}{2}\left[\bar{\gamma}_{-(-1)^D},\mathbf{D}^{(1)}\right] = i\bar{\gamma}_{-(-1)^D}\mathbf{D}^{(1)}_{J^{(i)}} \\ \mathbf{D}^{(4)}_{J^{(i)}} &:= \frac{-i}{2}\left[\bar{\gamma}_{(-1)^D},\mathbf{D}^{(2)}\right] = \bar{\gamma}_{(-1)^D}\mathbf{D}^{(2)}_{J^{(i)}}\,. \end{aligned} \tag{1275}$$

C.2 (List of supercharges of $\Lambda(\mathcal{M})$)

1. *Spin geometry:* Consider a Dirac operator

$$\mathbf{D}^{(1)} = \hat{\gamma}^\mu \hat{\nabla}^{(S)}_\mu$$

on the spin bundle and and any grading inducing involution ι, $\{\mathbf{D}, \iota\} = 0$. The associated generalized Laplace operator

$$\mathbf{H} = \left(\mathbf{D}^{(2)}\right)^2$$

is invariant under ι

$$[\iota, \mathbf{H}] = 0\,.$$

Closing this symmetry (as described in §2.2.7 (p.90)) yields the associated *dual* Dirac operator

$$\frac{i}{2}[\iota, \mathbf{D}] \;=\; i\iota\mathbf{D}\,, \qquad (1276)$$

which is easily seen to satisfy

$$\left\{\mathbf{D}^{(i)}, \mathbf{D}^{(j)}\right\} = 2\delta^{ij}\mathbf{H}, \qquad i,j \in \{1,2\}\;.$$

2. *Riemannian geometry, even dimension:* The even graded generator

$$\mathbf{H} = \mathbf{d}\mathbf{d}^\dagger + \mathbf{d}^\dagger\mathbf{d}$$

has two generic symmetries. It preserves fermion number (i.e. form degree) and chirality:

$$\begin{aligned}\left[\hat{N}, \mathbf{H}\right] &= 0 \\ [\bar{\gamma}_\pm, \mathbf{H}] &= 0\,.\end{aligned} \qquad (1277)$$

From the standard Dirac operator

$$\mathbf{D} = \mathbf{d} + \mathbf{d}^\dagger$$

one obtains three further Dirac operators by '**D**-closing' the above symmetries (*cf.* §2.2.7 (p.90)):

$$\begin{aligned}
\mathbf{D}^{(1)} &:= \mathbf{D} \\
&= \mathbf{d} + \mathbf{d}^\dagger \\
&= \hat{\gamma}^\mu_- \hat{\nabla}_\mu \\
\mathbf{D}^{(2)} &:= i\left[\hat{N}, \mathbf{D}^{(1)}\right] \\
&= i\left[\hat{N}, \mathbf{d} + \mathbf{d}^\dagger\right] \\
&= i\left(\mathbf{d} - \mathbf{d}^\dagger\right) \\
&= i\hat{\gamma}^\mu_+ \hat{\nabla}_\mu \\
\mathbf{D}^{(3)} &:= \frac{i}{2}\left[\bar{\gamma}_-, \mathbf{D}^{(1)}\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{i}{2}\left[\bar{\gamma}_-, \hat{\gamma}_-^\mu \hat{\nabla}_\mu\right] \\
&= i\bar{\gamma}_- \mathbf{D}^{(1)} \\
\mathbf{D}^{(4)} &:= \frac{-i}{2}\left[\bar{\gamma}_+, \mathbf{D}^{(2)}\right] \\
&= \frac{1}{2}\left[\bar{\gamma}_+, \hat{\gamma}_+^\mu \hat{\nabla}_\mu\right] \\
&= \bar{\gamma}_+ \mathbf{D}^{(2)} .
\end{aligned} \tag{1278}$$

These four operators satisfy the superalgebra

$$\left\{\mathbf{D}^{(i)}, \mathbf{D}^{(j)}\right\} = 2\delta^{ij}\mathbf{H} . \tag{1279}$$

Proof of the superalgebra relations: For $i, j \in \{1, 2\}$ one has as usual

$$\left\{\mathbf{D}^{(i)}, \mathbf{D}^{(j)}\right\} = 2\delta^{ij}\left(\mathbf{d}\mathbf{d}^\dagger + \mathbf{d}^\dagger\mathbf{d}\right) = 2\delta^{ij}\mathbf{H} , \qquad i, j \in \{1, 2\} .$$

Furthermore, the following relations hold (for $D = 2n$):

$$\begin{aligned}
\left\{\bar{\gamma}_-, \mathbf{D}^{(1)}\right\} &= 0 \\
\left\{\bar{\gamma}_+, \mathbf{D}^{(2)}\right\} &= 0 \\
\left[\bar{\gamma}_+, \bar{\gamma}_-\right] &= 0 .
\end{aligned} \tag{1280}$$

(This follows immediately from

$$\begin{aligned}
\left[\hat{\nabla}_\mu, \bar{\gamma}_\pm\right] &= 0 \\
\left\{\hat{\gamma}_\pm^a, \bar{\gamma}_{\pm(-1)^D}\right\} &= 0 \\
\left[\hat{\gamma}_\pm^a, \bar{\gamma}_{\mp(-1)^D}\right] &= 0 ,
\end{aligned}$$

cf. theorem B.17 (p.307)). Using (1280) the superalgebra can easily be verified.

3. *Riemannian geometry, odd dimension:* In odd dimensions, $D = 2n + 1$, the construction of the superalgebra is similar as in even dimensions, but now there is a central charge in the algebra: Define $\mathbf{D}^{(1)}$, $\mathbf{D}^{(2)}$ as in (1278). The analog of relations (1280) now reads:

$$\begin{aligned}
\left\{\bar{\gamma}_+, \mathbf{D}^{(1)}\right\} &= 0 \\
\left\{\bar{\gamma}_-, \mathbf{D}^{(2)}\right\} &= 0 \\
\left\{\bar{\gamma}_+, \bar{\gamma}_-\right\} &= 0 .
\end{aligned} \tag{1281}$$

Hence one defines

$$\begin{aligned}
\mathbf{D}^{(1)} &= \mathbf{d} + \mathbf{d}^\dagger \\
\mathbf{D}^{(2)} &= i\left(\mathbf{d} - \mathbf{d}^\dagger\right) \\
\mathbf{D}^{(3)} &= i\bar{\gamma}_+ \mathbf{D}^{(1)} \\
\mathbf{D}^{(4)} &= i\bar{\gamma}_- \mathbf{D}^{(2)}
\end{aligned} \tag{1282}$$

and finds the superalgebra:

$$\left\{\mathbf{D}^{(i)}, \mathbf{D}^{(j)}\right\} = 2\delta^{ij}\mathbf{H} + 2\mathbf{Z}^{ij} \tag{1283}$$

with central charge

$$\begin{aligned}
\mathbf{Z}^{ij} &= \mathbf{Z}^{ji} \\
\mathbf{Z}^{3,4} &= -\bar{\gamma}_+\bar{\gamma}_-\mathbf{D}^{(1)}\mathbf{D}^{(2)} \\
\mathbf{Z}^{ij} &= 0\,, \quad \text{otherwise}\,.
\end{aligned} \tag{1284}$$

4. *Kähler geometry:* (*cf.* [101] §3.4) The manifold $(\mathcal{M}, g)$ is Kähler iff there exists a complex structure J

$$\begin{aligned}
J : T\mathcal{M} &\to T\mathcal{M} \\
v^\mu \partial_\mu &\mapsto J^\mu{}_\nu v^\nu \partial_\mu
\end{aligned}$$

on the tangent bundle $T\mathcal{M}$ such that

$$\begin{aligned}
J^2 &= -1 \\
\partial_{[\lambda} J_{\mu\nu]} &= 0\,.
\end{aligned} \tag{1285}$$

In particular, this implies an even number of dimensions: $D = 2n$. By construction, the operator

$$\mathbf{J} := J_{\mu\nu}\hat{c}^{\dagger\mu}\hat{c}^\nu \tag{1286}$$

has eigenvalues $\pm i$ on holomorphic/antiholomorphic 1-forms and the maps

$$\begin{aligned}
P^{J\pm} : T\mathcal{M} &\to T\mathcal{M} \\
v^\mu \partial_\mu &\mapsto \frac{1}{2}\left(\delta^\mu{}_\mu \mp iJ^\mu{}_\nu\right) v^\nu \partial_\mu
\end{aligned} \tag{1287}$$

act as projectors on the respective eigenspaces. Hence the operators

$$\begin{aligned}
\hat{N}^{J\pm} &:= \left(P^{J\pm}\right)_{ab} \hat{e}^{\dagger a}\hat{e}^b \\
&= \frac{1}{2}\left(\hat{N} \mp i\mathbf{J}\right)
\end{aligned} \tag{1288}$$

count the holomorphic (+) and antiholomorphic (−) degree of a form. The (anti-)holomorphic exterior derivatives are then[63]

$$\mathbf{d}^{J\pm} := \left[\hat{N}^{J\pm}, \mathbf{d}\right] \tag{1290}$$

[63] The usual notation is

$$\begin{aligned}
\mathbf{d}^{J+} &= \partial \\
\mathbf{d}^{\dagger J+} &= \partial^* \\
\mathbf{d}^{J-} &= \bar{\partial} \\
\mathbf{d}^{\dagger J-} &= \bar{\partial}^*\,.
\end{aligned} \tag{1289}$$

satisfying

$$
\begin{aligned}
&\mathbf{d} = \mathbf{d}^{J+} + \mathbf{d}^{J-} \\
&\left\{\mathbf{d}^{J\pm}, \mathbf{d}\right\} = 0 \\
&\left\{\mathbf{d}^{J\pm}, \mathbf{d}^{J\pm}\right\} = 0 \\
&\left\{\mathbf{d}^{J\pm}, \mathbf{d}^{J\mp}\right\} = 0 \\
&\left\{\mathbf{d}^{J\pm}, \mathbf{d}^{\dagger\, J\mp}\right\} = 0 \\
&\left\{\mathbf{d}^{J\pm}, \mathbf{d}^{\dagger\, J\pm}\right\} = \left(\mathbf{d} + \mathbf{d}^\dagger\right)^2 = \mathbf{H}\,.
\end{aligned}
\tag{1291}
$$

Diagonalizing the above polar superalgebra gives the usual four Dirac operators

$$
\begin{aligned}
\mathbf{D}^{(1)} &= \mathbf{d}^{J+} + \mathbf{d}^{\dagger\, J+} \\
\mathbf{D}^{(2)} &= i\left(\mathbf{d}^{J+} - \mathbf{d}^{\dagger\, J+}\right) \\
\mathbf{D}^{(3)} &= \mathbf{d}^{J-} + \mathbf{d}^{\dagger\, J-} \\
\mathbf{D}^{(4)} &= i\left(\mathbf{d}^{J-} - \mathbf{d}^{\dagger\, J-}\right)
\end{aligned}
\tag{1292}
$$

and their anticommutators:

$$
\left\{\mathbf{D}^{(i)}, \mathbf{D}^{j}\right\} = 2\delta^{ij}\mathbf{H}\,, \qquad i,j \in \{1,2,3,4\}\,.
\tag{1293}
$$

This is the superalgebra associated with the symmetries $\hat{N}$ and $\hat{N}^{J\pm}$ of $\mathbf{H}$. As in the Riemannian case, further supercharges are generated by duality/chirality symmetry: One can construct holomorphic and antiholomorphic chirality operators $\bar{\gamma}^{J\pm}$ with the property

$$
\begin{aligned}
&\left(\bar{\gamma}^{J\pm}\right)^2 = 0 \\
&\left[\bar{\gamma}^{J+}, \bar{\gamma}^{J-}\right] = 0\,,
\end{aligned}
\tag{1294}
$$

so that

$$
\begin{aligned}
\left\{\bar{\gamma}^{J\pm}, \mathbf{d}^{J\pm} + s\mathbf{d}^{\dagger\, J\pm}\right\} &= 0\,, \quad s \in \{-1,+1\} \\
\left[\bar{\gamma}^{J\mp}, \mathbf{d}^{J\pm} + s\mathbf{d}^{\dagger\, J\pm}\right] &= 0\,, \quad s \in \{-1,+1\}\,.
\end{aligned}
\tag{1295}
$$

Closing this symmetry analogously to the Riemannian case again doubles the number of supercharges (1292):

$$
\begin{aligned}
\mathbf{D}^{(1)} &= \mathbf{d}^{J+} + \mathbf{d}^{\dagger\, J+} \\
\mathbf{D}^{(2)} &= i\left(\mathbf{d}^{J+} - \mathbf{d}^{\dagger\, J+}\right) \\
\mathbf{D}^{(3)} &= \mathbf{d}^{J-} + \mathbf{d}^{\dagger\, J-} \\
\mathbf{D}^{(4)} &= i\left(\mathbf{d}^{J-} - \mathbf{d}^{\dagger\, J-}\right) \\
\mathbf{D}^{(5)} &= i\bar{\gamma}^{J+}\left(\mathbf{d}^{J+} + \mathbf{d}^{\dagger\, J+}\right) \\
\mathbf{D}^{(6)} &= \bar{\gamma}^{J+}\left(\mathbf{d}^{J+} - \mathbf{d}^{\dagger\, J+}\right)
\end{aligned}
$$

$$\begin{aligned}
\mathbf{D}^{(7)} &= i\bar{\gamma}^{J-}\left(\mathbf{d}^{J-} + \mathbf{d}^{\dagger\,J-}\right) \\
\mathbf{D}^{(8)} &= \bar{\gamma}^{J-}\left(\mathbf{d}^{J-} - \mathbf{d}^{\dagger\,J-}\right) .
\end{aligned} \tag{1296}$$

From (1295) and (1296) the following superalgebra can be read off:

$$\left\{\mathbf{D}^{(i)}, \mathbf{D}^{(j)}\right\} = 2\delta^{ij}\mathbf{H} + 2\mathbf{Z}^{ij} , \tag{1297}$$

where $\mathbf{Z}^{ij}$ are the following *central charges*:

$$\begin{aligned}
\mathbf{Z}^{ij} &= \mathbf{Z}^{ji} \\
\mathbf{Z}^{1,6} &= \mathbf{D}^{(1)}\mathbf{D}^{(6)} \\
&= \bar{\gamma}^{J+}\left(\mathbf{d}^{J+} - \mathbf{d}^{\dagger\,J+}\right)\left(\mathbf{d}^{J+} + \mathbf{d}^{\dagger\,J+}\right) \\
\mathbf{Z}^{2,5} &= \mathbf{Z}^{1,6} \\
\mathbf{Z}^{3,8} &= \mathbf{D}^{(3)}\mathbf{D}^{(8)} \\
&= \bar{\gamma}^{J-}\left(\mathbf{d}^{J-} - \mathbf{d}^{\dagger\,J-}\right)\left(\mathbf{d}^{J-} + \mathbf{d}^{\dagger\,J-}\right) \\
\mathbf{Z}^{4,7} &= \mathbf{Z}^{3,8} \\
\mathbf{Z}^{ij} &= 0 , \quad \text{otherwise}
\end{aligned} \tag{1298}$$

Example C.3 The simplest example for the Kähler case is the flat Euclidean 2-dimensional plane. One finds

$$\begin{aligned}
\mathbf{d} &= \hat{\mathrm{e}}^{\dagger\,1}\partial_1 + \hat{\mathrm{e}}^{\dagger\,2}\partial_2 \\
\mathbf{d}^{\dagger} &= -\hat{\mathrm{e}}^{1}\partial_1 + \hat{\mathrm{e}}^{2}\partial_2 \\
\mathbf{d}^{J\pm} &= \left(\hat{\mathrm{e}}^{\dagger\,1} \pm i\hat{\mathrm{e}}^{\dagger\,2}\right)(\partial_1 \mp i\partial_2) \\
\mathbf{d}^{\dagger\,J\pm} &= -\left(\hat{\mathrm{e}}^{1} \mp i\hat{\mathrm{e}}^{2}\right)(\partial_1 \pm i\partial_2) .
\end{aligned} \tag{1299}$$

By introducing the holomorphic and antiholomorphic Clifford generators

$$\begin{aligned}
\hat{\gamma}_{+}^{J\pm} &:= \frac{1}{\sqrt{2}}\left(\hat{\mathrm{e}}^{\dagger\,1} \pm i\hat{\mathrm{e}}^{\dagger\,2}\right) + \frac{1}{\sqrt{2}}\left(\hat{\mathrm{e}}^{1} \mp i\hat{\mathrm{e}}^{2}\right) \\
\hat{\gamma}_{-}^{J\pm} &:= \frac{1}{\sqrt{2}}\left(\hat{\mathrm{e}}^{\dagger\,1} \pm i\hat{\mathrm{e}}^{\dagger\,2}\right) - \frac{1}{\sqrt{2}}\left(\hat{\mathrm{e}}^{1} \mp i\hat{\mathrm{e}}^{2}\right)
\end{aligned} \tag{1300}$$

which satisfy

$$\left\{\hat{\gamma}_{s_1}^{Js_2}, \hat{\gamma}_{s_1'}^{Js_2'}\right\} = 2s_1\delta_{(s_1,s_1')}\delta_{(s_2,s_2')} \tag{1301}$$

(for $s_1, s_2, s_1', s_2' \in \{+,-\}$), the above exterior derivatives give rise to the following four Dirac operators:

$$\begin{aligned}
\mathbf{d}^{J\pm} + \mathbf{d}^{\dagger\,J\pm} &:= \hat{\gamma}_{-}^{J\pm}\partial_1 \mp i\hat{\gamma}_{+}^{J\pm}\partial_2 \\
i\left(\mathbf{d}^{J\pm} - \mathbf{d}^{\dagger\,J\pm}\right) &:= i\hat{\gamma}_{+}^{J\pm}\partial_1 \pm \hat{\gamma}_{-}^{J\pm}\partial_2 .
\end{aligned} \tag{1302}$$

The holomorphic and antiholomorphic chirality operators are

$$\begin{aligned}
\bar{\gamma}^{J+} &= \hat{\gamma}_{-}^{J+}\hat{\gamma}_{+}^{J+} \\
\bar{\gamma}^{J-} &= \hat{\gamma}_{-}^{J-}\hat{\gamma}_{+}^{J-} .
\end{aligned} \tag{1303}$$

According to (1296) one has:

$$\begin{aligned}
\bar{\gamma}^{J\pm}\left(\mathbf{d}^{J\pm}+\mathbf{d}^{\dagger\,J\pm}\right) &= \hat{\gamma}_{+}^{J\pm}\partial_1 \mp i\hat{\gamma}_{-}^{J\pm}\partial_2 \\
&= \left(\mathbf{d}^{J\pm}-\mathbf{d}^{\dagger\,J\pm}\right) \\
\bar{\gamma}^{J\pm}i\left(\mathbf{d}^{J\pm}-\mathbf{d}^{\dagger\,J\pm}\right) &= i\hat{\gamma}_{-}^{J\pm}\partial_1 \pm \hat{\gamma}_{+}^{J\pm}\partial_2 \\
&= i\left(\mathbf{d}^{J\pm}+\mathbf{d}^{\dagger\,J\pm}\right),
\end{aligned} \tag{1304}$$

and *no* new supercharges are found in this case. This shows that in $D=2$ the algebra (1296), (1297), (1298) degenerates into two identical copies of the usual four Kähler supercharges with all central charges proportional to $\mathbf{H}$. This can only happen in $D=2$.

C.4 (Witten model breaks the duality symmetry) One way to find square roots of configuration space Hamiltonians

$$H_{\text{bsosonic}} = g^{\mu\nu}\partial_\mu\partial_\nu + U + \cdots \tag{1305}$$

with given bosonic potentials U is by employing the Witten model (*cf.* definition 2.2.2 (p.61)). But, as for instance observed in [25] (p. 802), the Witten model breaks the duality symmetry that was discussed in observation C.1 (p.319), since for nonvanishing superpotential the Witten Hamiltonian

$$\mathbf{H}_W = \left(\mathbf{d}+\mathbf{d}^\dagger\right)^2 + \left(\partial_\mu W\right)\left(\partial^\mu W\right) + \hat{\gamma}_{g_-}^{\ \mu}\hat{\gamma}_{g_+}^{\ \nu}\left(\nabla_\mu\nabla_\nu W\right)$$

with

$$U = \left(\partial_\mu W\right)\left(\partial^\mu W\right)$$

(*cf.* theorem 2.62 (p.61)) is no longer invariant under $*$:

$$\begin{aligned}
\left[*,\mathbf{H}_W\right] &= \underbrace{\left[*,\left(\mathbf{d}+\mathbf{d}^\dagger\right)^2\right]}_{=0} + \underbrace{\left[*,\left(\partial_\mu W\right)\left(\partial^\mu W\right)\right]}_{=0} + \left[*,\hat{\gamma}_{g_-}^{\ \mu}\hat{\gamma}_{g_+}^{\ \nu}\left(\nabla_\mu\nabla_\nu W\right)\right] \\
&= 2*\hat{\gamma}_{g_-}^{\ \mu}\hat{\gamma}_{g_+}^{\ \nu}\left(\nabla_\mu\nabla_\nu W\right).
\end{aligned} \tag{1306}$$

One way to restore the full supersymmetry has been given in [112]. Another way is to modify the Witten deformation: The original Witten model is defined by the Dirac operator

$$\mathbf{D}_W \ := \ e^{-W}\mathbf{d}e^{W} + e^{W}\mathbf{d}^\dagger e^{-W}. \tag{1307}$$

If one instead uses

$$\mathbf{D}'_{W'} \ := \ e^{-W'\hat{N}}\mathbf{d}e^{W'\hat{N}} + e^{W'\hat{N}}\mathbf{d}^\dagger e^{-W'\hat{N}} \tag{1308}$$

(with $W' = W/\left(D^2-D\right)$) one finds another admissible extension of the original bosonic Hamiltonian (**??**)

$$H_{\text{bosonic}} \to \left(\mathbf{D}'_{W'}\right)^2 .$$

This is shown in detail in theorem **??** (p.**??**). But it can already easily be seen by comparing the action of $(\mathbf{D}'_{W'})^2$ on the full form sector (which is identified with the bosonic sector):

$$\begin{aligned} (\mathbf{D}'_{W'})^2 f\,|\mathrm{vol}\rangle &= e^{-W'\hat{N}}\,\mathbf{d}\,e^{2W'\hat{N}}\,\mathbf{d}^\dagger\,e^{-W'\hat{N}} f\,|\mathrm{vol}\rangle \\ &= e^{-W'D}\,\mathbf{d}\,e^{2W'(D-1)}\,\mathbf{d}^\dagger\,e^{-W'D} f\,|\mathrm{vol}\rangle \end{aligned} \tag{1309}$$

with that of the Witten Hamiltonian

$$\begin{aligned} (\mathbf{D}_W)^2 f\,|\mathrm{vol}\rangle &= e^{-W}\,\mathbf{d}\,e^{2W}\,\mathbf{d}^\dagger\,e^{-W} f\,|\mathrm{vol}\rangle \\ &= e^{-W}\,\mathbf{d}\,e^{2W}\,\mathbf{d}^\dagger\,e^{-W} f\,|\mathrm{vol}\rangle\,. \end{aligned} \tag{1310}$$

Hidden symmetries of the 'free' Laplace operator, which does not feature a superpotential, are determined by the metric tensor alone. The presence of a superpotential puts further restrictions on would-be symmetries.

C.5 (Symmetries of the Witten model) A Killing-Yano tensor $f_{\mu\nu}$ gives rise to a symmetry

$$J \;:=\; \frac{1}{2} f_{\mu\nu}\hat{\gamma}_-^\mu \hat{\gamma}_-^\nu \tag{1311}$$

of the Witten model Laplacian (*cf.* definition 2.2.2 (p.61))

$$\mathbf{H} \;=\; \left(\mathbf{d}+\mathbf{d}^\dagger\right)^2 + (\partial^\mu W)\,(\partial_\mu W) + \hat{\gamma}_-^\mu \hat{\gamma}_+^\nu\,(\nabla_\mu \nabla_\nu W)\,, \tag{1312}$$

i.e.

$$[J, \mathbf{H}] = 0\,,$$

if and only if f satisfies

$$f_{\mu\nu}\nabla^\nu \nabla_\kappa W = 0\,. \tag{1313}$$

(*cf.* 2.95 (p.93), 2.96 (p.94), and 2.98 (p.95)).

Proof:

$$\begin{aligned} & \left[\frac{1}{2} f_{\mu\nu}\hat{\gamma}_-^\mu \hat{\gamma}_-^\nu,\ \mathbf{H}\right] = 0 \\ \Leftrightarrow\quad & \left[\frac{1}{2} f_{\mu\nu}\hat{\gamma}_-^\mu \hat{\gamma}_-^\nu,\ \hat{\gamma}_-^{\mu'} \hat{\gamma}_+^{\nu'}\,(\nabla_{\mu'}\nabla_{\nu'} W)\right] = 0 \\ \Leftrightarrow\quad & 2\hat{\gamma}_-^\mu \hat{\gamma}_+^{\nu'} f_{\mu\nu}\,(\nabla^\nu \nabla_{\nu'} W) = 0 \\ \Leftrightarrow\quad & f_{\mu\nu}\,(\nabla^\nu \nabla_{\nu'} W) = 0\,. \end{aligned} \tag{1314}$$

D Proofs and calculations

In this section some proofs and calculations are given which were omitted in the main text to improve readability:

1. **Proof of 2.28 (p.41)**:

   Let (A, V, ι) be a graded algebra A of linear operators on the graded vector space V which contains a graded nilpotent operator $\mathbf{q} \in A$, $\{\mathbf{q}, \iota\} = 0$, $\mathbf{q}^2 = 0$. Let $\mathbf{q}^{\dagger_{\hat{\eta}}}$ be the adjoint of $\mathbf{q}$ with respect to some scalar product on V.

   The alternating trace regulated by $e^{-\left(\mathbf{q}+\mathbf{q}^{\dagger_{\hat{\eta}}}\right)^2}$ is equal to the alternating trace over $\mathrm{Ker}\,(q) \cap \mathrm{Ker}\,\left(q^{\dagger_{\hat{\eta}}}\right)$:

   $$\mathrm{Tr}\left(e^{-\left(\mathbf{q}+\mathbf{q}^{\dagger_{\hat{\eta}}}\right)^2}\,\iota\, a\right) = \mathrm{Tr}(\iota\, a)_{\mathrm{Ker}\left(\mathbf{q}+\mathbf{q}^{\dagger_{\hat{\eta}}}\right)}\,. \tag{1315}$$

   Proof:
   First consider the following, purely formal argument:
   Ignoring issues of convergence, the alternating trace $\mathrm{sTr}(a)_\iota$ *on* V *over any* $\mathbf{q}$-closed *even graded operator* $a \in A$, $[\mathbf{q}, a]_\iota = 0$, $[a, \iota] = 0$ *reduces to the alternating trace on the cohomology* $\mathrm{H}_c(\mathbf{q})$

   $$\mathrm{sTr}(a)_\iota = \mathrm{sTr}\left(a\,\mathrm{P}_{\mathrm{H}_c(\mathbf{q})}\right)_\iota\,, \tag{1316}$$

   where $\mathrm{P}_{\mathrm{H}_c(\mathbf{q})}$ *is the projector onto* $\mathrm{H}_c(\mathbf{q})$.
   Proof of formal argument: By the Hodge decomposition, the trace runs over the three disjoint subspaces $\overline{\mathrm{Im}(\mathbf{q})}$, $\overline{\mathrm{Im}\left(\mathbf{q}^{\dagger_{\hat{\eta}}}\right)}$, and $\mathrm{Ker}\,(\mathrm{Im}(\mathbf{q}))\cap\mathrm{Ker}\,\left(\mathbf{q}^{\dagger_{\hat{\eta}}}\right)$ for some co-operator $\mathbf{q}^{\dagger_{\hat{\eta}}}$. But for every eigenvector $|\alpha\rangle \in \mathrm{Im}\left(\mathbf{q}^{\dagger_{\hat{\eta}}}\right)$ of a there is, because a commutes with $\mathbf{q}$, an eigenvector $|\alpha'\rangle = \mathbf{q}\,|\alpha\rangle \in \mathrm{Im}(\mathbf{q})$ of the same eigenvalue but with opposite grade. It follows that the traces over $\mathrm{Im}(\mathbf{q})$ and $\mathrm{Im}\left(\mathbf{q}^{\dagger_{\hat{\eta}}}\right)$ mutually cancel.□

   To make use of this formal argument, the trace needs to be regulated, i.e. the sum over the canceling subspaces needs to be damped in order to yield an absolutely convergent series. There are several (generally inequivalent) ways of regulating and they correspond to different co-operators $\mathbf{q}^{\dagger_{\hat{\eta}}}$ in the above proof, i.e. to choosing different representatives $|\alpha\rangle \in [|\alpha\rangle]$ in the cohomology equivalence classes $[|\alpha\rangle] \in \mathrm{H}_c(\mathbf{q})$:

   The regulated supertrace

   $$\mathrm{sTr}(a)_{\iota,\mathbf{q},\hat{\eta}} := \mathrm{sTr}\left(e^{-\left\{\mathbf{q},\mathbf{q}^{\dagger_{\hat{\eta}}}\right\}}\,a\right)_\iota \tag{1317}$$

   reduces to the alternating trace over the kernel of $\left\{\mathbf{q}, \mathbf{q}^{\dagger_{\hat{\eta}}}\right\}$*:*

   $$\begin{aligned} \mathrm{sTr}\left(e^{-\left\{\mathbf{q},\mathbf{q}^{\dagger_{\hat{\eta}}}\right\}}\,a\right)_\iota &= \mathrm{Tr}(\iota\, a)_{\mathrm{Ker}\left(\left\{\mathbf{q},\mathbf{q}^{\dagger_{\hat{\eta}}}\right\}\right)} \\ &= \mathrm{Tr}(a)_+ - \mathrm{Tr}(a)_-\,, \end{aligned} \tag{1318}$$

   where $\mathrm{Tr}(\cdot)_\pm$ *is the trace over the* ± 1 *eigensubspace of* ι *in* $\mathrm{Ker}\,\left(\left\{\mathbf{q}, \mathbf{q}^{\dagger_{\hat{\eta}}}\right\}\right)$.
   Proof: The operator $\left\{\mathbf{q}, \mathbf{q}^{\dagger_{\hat{\eta}}}\right\}$ has, since it is by construction a *positive operator*, positive eigenvalues $\lambda \to \infty$ on $\mathrm{Im}(\mathbf{q})$ and $\mathrm{Im}\left(\mathbf{q}^{\dagger_{\hat{\eta}}}\right)$. Hence the

operator $e^{-\{\mathbf{q},\mathbf{q}^{\dagger_{\hat\eta}}\}}$ restricts to the identity on Ker $\left(\{\mathbf{q},\mathbf{q}^{\dagger_{\hat\eta}}\}\right)$ and vanishes monotonically with λ, thereby forcing the supertrace over $\mathrm{Im}(\mathbf{q})\cup\mathrm{Im}\left(\mathbf{q}^{\dagger_{\hat\eta}}\right)$ to converge to zero by the above formal argument. □

Note: It is worth noting that (175), p. 41 tacitly assumes that the trace over the cohomology itself is finite (*cf.* [228]). This is a requirement on $\mathbf{q}^{\dagger_{\hat\eta}}$, i.e. on the scalar product $\langle\cdot|\cdot\rangle_{\hat\eta}$ with respect to which the adjoint is taken.

2. **Details for ?? (p.??):** First consider the obstruction to invertability. The (real) vielbein Lorentz 4-vector $e^a{}_i$, when contracted with $\sigma_a{}^{A1'}$, yields the 2-component spinor (for fixed i):

$$\sigma_a{}^{A1'} e^a{}_i = \frac{1}{\sqrt{2}} \begin{bmatrix} -e^0{}_i + e^3{}_i \\ e^1{}_i + ie^2{}_i \end{bmatrix}^A .$$

Since $e^a{}_i$ is real, the lower component, $A = 2$, of this spinor contains the full information of $e^1{}_i$ and $e^2{}_i$. But one cannot in general extract $e^0{}_i$ and $e^3{}_i$ separately from the first component, $A = 1$ – except when one of them vanishes identically. Hence assume that a Lorentz frame on spacetime is chosen such that $e^0{}_i = 0$, $\forall i \in \{1, 2, 3\}$, identically. This is called the *time gauge* (*cf.* [204], [83] (2.9.2.28)). In this case

$$e_i = (e^a{}_i) = \sqrt{2} \begin{bmatrix} 0 \\ \mathrm{Re}\left(\sigma_a{}^{21'} e^a{}_i\right) \\ \mathrm{Im}\left(\sigma_a{}^{21'} e^a{}_i\right) \\ \sigma_a{}^{11'} e^a{}_i \end{bmatrix} . \tag{1319}$$

According to (1319) the inversion of $\tilde{E}_{(0)1'}$ in the time gauge reads:

$$\begin{aligned} &\mathrm{Re}\left(\tilde{E}_{(0)1'(m)}{}^{(n)} \int\limits_\Sigma F_{(n)}{}^{A=2}{}_i(x)\, B'^{(l)}{}_{a=1}{}^i(x)\ d^3x \right) + \\ &+\mathrm{Im}\left(\tilde{E}_{(0)1'(m)}{}^{(n)} \int\limits_\Sigma F_{(n)}{}^{A=2}{}_i(x)\, B'^{(l)}{}_{a=2}{}^i(x)\ d^3x \right) + \\ &+\tilde{E}_{(0)1'(m)}{}^{(n)} \int\limits_\Sigma F_{(n)}{}^{A=1}{}_i(x)\, B'^{(l)}{}_{a=3}{}^i(x)\ d^3x \\ = \;& \frac{2}{\kappa^2}\, \delta^l_m\, . \end{aligned} \tag{1320}$$

E Ghost algebras

The following gives some technical details related to §2.3 (p.106).

Outline. Several possible realizations of the ghost algebra (462)-(477), p. 119 will be constructed in terms of Clifford elements (e.g. $\hat{\gamma}_{\pm}^{(\lambda)}$) of the graded superalgebra, as described in §2.3.4 (p.134).

All of them will make use of the following definitions:

E.1 (Quantization of $\hat{\gamma}_{\pm}^{(\lambda)}$) The only non-vanishing supercommutator of the fermionic operators associated with the lagrange multiplier λ are

$$\begin{aligned} \left\{\hat{\mathrm{e}}^{\dagger(\lambda)}, \hat{\mathrm{e}}^{(\lambda)}\right\} &:= -1 \\ \Leftrightarrow \left\{\hat{\gamma}_{\pm}^{(\lambda)}, \hat{\gamma}_{\pm}^{(\lambda)}\right\} &:= \mp 2\,. \end{aligned} \tag{1321}$$

It follows that the hermitian metric operator for the local inner product is:

E.2 (Local hermitian metric operator) By the above definition of $\hat{\gamma}_{\pm}^{(\hat{\eta})}$ (1321) the hermitian metric operator

$$\hat{\eta}^{(0)} := \hat{\gamma}_{-}^{0}\hat{\gamma}_{+}^{0}\hat{\gamma}_{-}^{(\lambda)}\hat{\gamma}_{+}^{(\lambda)}\,, \tag{1322}$$

which is a self-adjoint involution

$$\begin{aligned} \hat{\eta}^{(0)\,\dagger} &= \hat{\eta}^{(0)} \\ \hat{\eta}^{(0)\,2} &= 1\,, \end{aligned} \tag{1323}$$

gives a positive definite local inner product:

$$\langle\cdot|\cdot\rangle_{\mathrm{loc},\hat{\eta}^{(0)}} := \langle\cdot|\hat{\eta}\,\cdot\rangle_{\mathrm{loc}} \tag{1324}$$

$$\Rightarrow \langle\alpha|\alpha\rangle_{\mathrm{loc},\hat{\eta}^{(0)}} \geq 0\,. \tag{1325}$$

E.3 (Global hermitian metric operator) In order to get a finite trace over physical states the local hermitian metric operator (1322) needs to be multiplied by a λ-dependent factor, e.g.:

$$\hat{\eta} := \hat{\eta}^{(0)} e^{\lambda x^0}\,, \tag{1326}$$

where X^0 is the coordinate along the integral lines of e_0, i.e.

$$\left[\mathbf{D}, x^0\right] := \hat{\gamma}_{-}^{0}\,. \tag{1327}$$

We now give two possible realizations of the ghost algebra:

E.4 (Ghost algebra based on involution $\iota = \hat{\gamma}_{+}$ in odd dimensions with no superpotential) In *odd dimensions* the exterior bundle can be decomposed as follows

$$\Lambda(\mathcal{M})^{(2n+1)} \simeq \mathrm{Cl}(\mathcal{M})_{+}\,\frac{1}{2}\left(1+\bar{\gamma}_{+}\right)|0\rangle \,\oplus\, \mathrm{Cl}(\mathcal{M})_{+}\,\frac{1}{2}\left(1-\bar{\gamma}_{+}\right)|0\rangle\,,$$

since $\left[\hat{\gamma}^a_+, \bar{\gamma}_+, =\right] 0$. Using the involution $\iota := \bar{\gamma}_+$ to decompose the Dirac operator

$$
\begin{aligned}
\mathbf{D} &= \hat{\gamma}^a_- \hat{\nabla}_a \\
&:= \mathbf{D}_{+\bar{\gamma}_+} + \mathbf{D}_{-\bar{\gamma}_+} \\
&:= \frac{1}{2}\left(1-\bar{\gamma}_+\right)\mathbf{D}\,\frac{1}{2}\left(1+\bar{\gamma}_+\right) + \frac{1}{2}\left(1+\bar{\gamma}_+\right)\mathbf{D}\,\frac{1}{2}\left(1-\bar{\gamma}_+\right) ,
\end{aligned}
$$

one sees that the operator $\mathbf{D}_{+\bar{\gamma}_+}$, which can be used as a BRST operator according to §2.3.4 (p.134), operates between two isomorphic copies of state spaces. This circumstance explains why one naturally finds the usual BRST ghost algebra in this setup, as will be shown in the following, by interpreting the image of $\frac{1}{2}\left(1-\bar{\gamma}_+\right)$ as the 'ghost sector'.

Realization of the ghost algebra: The operators $\mathbf{Q}$, $\mathcal{C}$, $\mathcal{P}$, $\mathbf{p}$ are defined in terms of the exterior algebra and it is shown that they satisfy the expected relations:

- *BRST operator:*

$$
\begin{aligned}
\mathbf{Q} &:= \mathbf{D}_{+\bar{\gamma}_+} \\
&= \hat{\gamma}^a_- \hat{\nabla}_a \hat{\mathrm{h}}_+
\end{aligned}
\tag{1328}
$$

- *ghost creator:*

$$
\mathcal{C} := \bar{\gamma}_- \hat{\mathrm{h}}_+ \tag{1329}
$$

- *ghost annihilator:*

$$
\mathcal{P} := \bar{\gamma}_- \hat{\mathrm{h}}_- \tag{1330}
$$

- *gauge generator*

$$
\begin{aligned}
\mathbf{p} &:= \{\mathbf{Q}, \mathcal{P}\} \\
&\overset{(1337)}{=} \bar{\gamma}_- \frac{1}{2}\mathbf{D}_+ \\
&= \bar{\gamma}_- \hat{\gamma}^a_- \hat{\nabla}_a
\end{aligned}
\tag{1331}
$$

- *ghost number operator:*

$$
\hat{N}_G := \hat{\mathrm{h}}_- \tag{1332}
$$

They satisfy the following relations:

1.

$$
\mathbf{Q} = \mathcal{C}p \tag{1333}
$$

Proof:

$$
\begin{aligned}
\mathcal{C}p &= \bar{\gamma}_- \hat{\mathrm{h}}_+ \bar{\gamma}_- \left(\hat{\mathrm{h}}_- \mathbf{d}\hat{\mathrm{h}}_+ + \hat{\mathrm{h}}_+ \mathbf{d}\hat{\mathrm{h}}_-\right) \\
&= \hat{\mathrm{h}}_- \left(\hat{\mathrm{h}}_- \mathbf{d}\hat{\mathrm{h}}_+ + \hat{\mathrm{h}}_+ \mathbf{d}\hat{\mathrm{h}}_-\right) \\
&= \hat{\mathrm{h}}_- \mathbf{d}\hat{\mathrm{h}}_+ \\
&= \mathbf{D}_B
\end{aligned}
\tag{1334}
$$

□

2.

$$\begin{aligned}\{\mathcal{C},\mathcal{P}\} &= 1\\ \left[\hat{N}_G,\mathcal{C}\right] &= \mathcal{C}\\ \left[\hat{N}_G,\mathcal{P}\right] &= -\mathcal{P}\end{aligned} \tag{1335}$$

Proof:

$$\begin{aligned}\{\mathcal{C},\mathcal{P}\} &= \bar{\gamma}_-\,\hat{\mathrm{h}}_+\,\bar{\gamma}_-\,\hat{\mathrm{h}}_- + \bar{\gamma}_-\,\hat{\mathrm{h}}_-\,\bar{\gamma}_-\,\hat{\mathrm{h}}_+\\ &= \hat{\mathrm{h}}_- + \hat{\mathrm{h}}_+\\ &= 1\\ \left[\hat{N}_G,\mathcal{C}\right] &= \hat{\mathrm{h}}_-\,\bar{\gamma}_-\,\hat{\mathrm{h}}_+ - \bar{\gamma}_-\,\hat{\mathrm{h}}_+\,\hat{\mathrm{h}}_-\\ &= \bar{\gamma}_-\,\hat{\mathrm{h}}_+\\ &= \mathcal{C}\\ \left[\hat{N}_G,\mathcal{P}\right] &= \hat{\mathrm{h}}_-\,\bar{\gamma}_-\,\hat{\mathrm{h}}_- - \bar{\gamma}_-\,\hat{\mathrm{h}}_-\,\hat{\mathrm{h}}_-\\ &= -\bar{\gamma}_-\,\hat{\mathrm{h}}_-\\ &= -\mathcal{P}\end{aligned}$$

□

3.

$$\{\mathcal{C},\mathbf{D}_B\} = 0 \tag{1336}$$

$$\{\mathcal{P},\mathbf{D}_B\} = \mathbf{p} \tag{1337}$$

$$[\mathcal{C},\mathbf{p}] = 0 \tag{1338}$$

$$[\mathcal{P},\mathbf{p}] = 0 \tag{1339}$$

Proof:

$$\begin{aligned}\{\mathcal{C},\mathbf{D}_B\} &= \bar{\gamma}_-\,\hat{\mathrm{h}}_+\,\hat{\mathrm{h}}_-\mathbf{d}\hat{\mathrm{h}}_+ + \hat{\mathrm{h}}_-\mathbf{d}\hat{\mathrm{h}}_+\,\bar{\gamma}_-\,\hat{\mathrm{h}}_+\\ &= \bar{\gamma}_-\,\hat{\mathrm{h}}_+\,\hat{\mathrm{h}}_-\mathbf{d}\hat{\mathrm{h}}_+ + \hat{\mathrm{h}}_-\mathbf{d}\hat{\mathrm{h}}_+\,\hat{\mathrm{h}}_-\bar{\gamma}_-\\ &= 0\\ \{\mathcal{P},\mathbf{D}_B\} &= \bar{\gamma}_-\,\hat{\mathrm{h}}_-\,\hat{\mathrm{h}}_-\mathbf{d}\hat{\mathrm{h}}_+ + \hat{\mathrm{h}}_-\mathbf{d}\hat{\mathrm{h}}_+\,\bar{\gamma}_-\,\hat{\mathrm{h}}_-\\ &= \bar{\gamma}_-\left(\hat{\mathrm{h}}_-\mathbf{d}\hat{\mathrm{h}}_+ + \hat{\mathrm{h}}_+\mathbf{d}\hat{\mathrm{h}}_-\right)\\ &= \mathbf{p}\\ [\mathcal{C},\mathbf{p}] &= \bar{\gamma}_-\hat{\mathrm{h}}_+\,\bar{\gamma}_-\left(\hat{\mathrm{h}}_-\mathbf{d}\hat{\mathrm{h}}_+ + \hat{\mathrm{h}}_+\mathbf{d}\hat{\mathrm{h}}_-\right) - \bar{\gamma}_-\left(\hat{\mathrm{h}}_-\mathbf{d}\hat{\mathrm{h}}_+ + \hat{\mathrm{h}}_+\mathbf{d}\hat{\mathrm{h}}_-\right)\bar{\gamma}_-\hat{\mathrm{h}}_+\\ &= \hat{\mathrm{h}}_-\left(\hat{\mathrm{h}}_-\mathbf{d}\hat{\mathrm{h}}_+ + \hat{\mathrm{h}}_+\mathbf{d}\hat{\mathrm{h}}_-\right) - \left(\hat{\mathrm{h}}_-\mathbf{d}\hat{\mathrm{h}}_+ + \hat{\mathrm{h}}_+\mathbf{d}\hat{\mathrm{h}}_-\right)\hat{\mathrm{h}}_+\\ &= 0\\ [\mathcal{P},\mathbf{p}] &\overset{(1337)}{=} [\mathcal{P},\{\mathcal{P},\mathbf{D}_B\}]\\ &\overset{\mathcal{P}^2=0}{=} 0\end{aligned} \tag{1340}$$

□

4.

$$\hat{N}_G = \mathcal{C}\mathcal{P} \tag{1341}$$

Proof:

$$\begin{aligned}\mathcal{C}\mathcal{P} &= \bar{\gamma}_- \,\hat{\mathrm{h}}_+ \,\bar{\gamma}_- \,\hat{\mathrm{h}}_- \\ &= \hat{\mathrm{h}}_-\end{aligned} \tag{1342}$$

□

5.

$$\begin{aligned}\mathcal{C}^\dagger &= -\mathcal{C} \\ \mathcal{P}^\dagger &= -\mathcal{P} \\ \hat{N}_G^\dagger &= 1 - \hat{N}_G \\ \mathbf{p}^\dagger &= -\mathbf{p}\end{aligned} \tag{1343}$$

Proof: by (1233) for odd s

$$\begin{aligned}\mathbf{p}^\dagger &= \left(\bar{\gamma}_- \frac{1}{2}\mathbf{D}_+\right)^\dagger \\ &= -\frac{1}{2}\mathbf{D}_+ \bar{\gamma}_- \\ &= -\bar{\gamma}_- \frac{1}{2}\mathbf{D}_+\end{aligned} \tag{1344}$$

□

6. Under the isomorphism $\bar{\gamma}_-$ the ghost algebra transforms as:

$$\begin{aligned}\bar{\gamma}_- \,\mathcal{C}\, \bar{\gamma}_- &= \mathcal{P} \\ \bar{\gamma}_- \,\mathcal{P}\, \bar{\gamma}_- &= \mathcal{C} \\ \bar{\gamma}_- \,\hat{N}_G\, \bar{\gamma}_- &= 1 - \hat{N}_G\end{aligned}$$

The fermionic operators associated with the Lagrange multiplier λ can be chosen as follows:

$$\left\{ \hat{\mathrm{e}}^{\dagger\, D+1}, \hat{\mathrm{e}}^{D+1} \right\} = -1 \tag{1345}$$

$$\begin{aligned}\bar{\mathcal{C}} &:= \bar{\gamma}_- \bar{\gamma}_-^D \frac{1}{2}\left(1 + i\bar{\gamma}_+ \bar{\gamma}_- \bar{\gamma}_+^D\right) \\ \bar{\mathcal{P}} &:= \bar{\gamma}_- \bar{\gamma}_-^D \frac{1}{2}\left(1 - i\bar{\gamma}_+ \bar{\gamma}_- \bar{\gamma}_+^D\right) \;,\end{aligned} \tag{1346}$$

where $\bar{\gamma}_\pm^D$ are the chirality operators associated with the single configuration variable λ:

$$\begin{aligned} \bar{\gamma}_+^D &:= i^{1(1-1)/2+1}\hat{\gamma}_+^D \\ &= i\hat{\gamma}_+^D \\ \bar{\gamma}_-^D &:= i^{1(1+1)/2+1}\hat{\gamma}_-^D \\ &= \hat{\gamma}_-^D \\ \left(\bar{\gamma}_\pm^D\right)^2 &= 1 \\ \left(\bar{\gamma}_\pm^D\right)^\dagger &= -\bar{\gamma}_\pm^D \,. \end{aligned} \tag{1347}$$

This gives the relations:

$$\begin{aligned} \bar{\mathcal{C}}^\dagger &= -\bar{\mathcal{C}} \\ \bar{\mathcal{P}}^\dagger &= -\bar{\mathcal{P}} \end{aligned} \tag{1348}$$

$$\left\{\bar{\mathcal{P}},\bar{\mathcal{C}}\right\} = -1 \tag{1349}$$

$$\begin{aligned} \left\{\bar{\mathcal{P}},\bar{\mathcal{P}}\right\} &= 0 \\ \left\{\bar{\mathcal{C}},\bar{\mathcal{C}}\right\} &= 0 \\ \left\{\mathcal{C},\bar{\mathcal{P}}\right\} &= 0 \\ \left\{\mathcal{C},\bar{\mathcal{C}}\right\} &= 0 \\ \left\{\mathcal{P},\bar{\mathcal{P}}\right\} &= 0 \\ \left\{\mathcal{P},\bar{\mathcal{C}}\right\} &= 0 \end{aligned} \tag{1350}$$

$$\begin{aligned} \left[\bar{\mathcal{P}},\mathbf{p}\right] &= 0 \\ \left[\bar{\mathcal{C}},\mathbf{p}\right] &= 0 \,. \end{aligned} \tag{1351}$$

From relation (1349) it follows that the anti-ghost number operator

$$\begin{aligned} \hat{N}_{\bar{G}} &:= \bar{\mathcal{C}}\bar{\mathcal{P}} \\ &= \frac{1}{2}\left(1 - i\bar{\gamma}_+\bar{\gamma}_-\bar{\gamma}_+^D\right) \,, \end{aligned} \tag{1352}$$

which satisfies

$$\hat{N}_{\bar{G}}^\dagger = -1 - \hat{N}_{\bar{G}} \,, \tag{1353}$$

has negative eigenvalues where the ordinary ghost operator has positive ones:

$$\begin{aligned} \hat{N}_{\bar{G}} \left|\bar{\mathcal{P}} = 0\right\rangle &= 0 \\ \hat{N}_{\bar{G}}\,\bar{\mathcal{C}}\left|\bar{\mathcal{P}} = 0\right\rangle &= -\bar{\mathcal{C}}\left|\bar{\mathcal{P}} = 0\right\rangle \,, \end{aligned} \tag{1354}$$

$$\begin{aligned} \left[\hat{N}_{\bar{G}},\bar{\mathcal{C}}\right] &= -\bar{\mathcal{C}} \\ \left[\hat{N}_{\bar{G}},\bar{\mathcal{P}}\right] &= \bar{\mathcal{P}} \,. \end{aligned} \tag{1355}$$

The operator of total ghost number

$$\hat{N}_{\mathcal{G}} = \hat{N}_G + \hat{N}_{\bar{G}} \tag{1356}$$

is anti-hermitian

$$\hat{N}_{\mathcal{G}}^{\dagger} = -\hat{N}_{\mathcal{G}} \qquad (1357)$$

(by (1343) and (1353)). $\hat{N}_{\mathcal{G}}$ gives a grading on $\Lambda(\mathcal{M})$ different from that given by the exterior number operator $\hat{N}$. For example, $\mathcal{C}$ is odd with respect to the $\hat{N}_{\mathcal{G}}$-grading but has no definite grade with respect to the $\hat{N}$-grading.

Under the usual HODGE *scalar product the ghosts and anti-ghosts are automatically subject to the proper* BEREZIN *integration:* Let $|\phi_i\rangle$ be a no-ghost, no-anti-ghost state, e.g. constructed as follows from a no-ghost and no-$\hat{\mathrm{e}}^{\dagger D}$ state $|v_i\rangle$:

$$\begin{aligned}
\hat{\mathrm{e}}^{D}\,|v_i\rangle &:= 0\\
\bar{\gamma}_{+}\,|v_1\rangle &:= -\,|v_1\rangle\\
\bar{\gamma}_{+}\,|v_2\rangle &:= |v_2\rangle\\
|\phi\rangle_i &:= \left(1-\hat{N}_G\right)|v_i\rangle\\
&= \frac{1}{2}\left(1+i\bar{\gamma}_{+}\bar{\gamma}_{-}\bar{\gamma}_{+}^{D}\right)|v_i\rangle\ .
\end{aligned}$$

Then:

$$\begin{aligned}
\left\langle\phi_1|\bar{\mathcal{C}}\,\phi_2\right\rangle_D &= \left\langle\phi_1|\bar{\gamma}_{-}\bar{\gamma}_{-}^{D}\frac{1}{2}\left(1+i\bar{\gamma}_{+}\bar{\gamma}_{-}\bar{\gamma}_{+}^{D}\right)\phi_2\right\rangle_D\\
&= \left\langle v_i|\bar{\gamma}_{-}\bar{\gamma}_{-}^{D}\frac{1}{2}\left(1+i\bar{\gamma}_{+}\bar{\gamma}_{-}\bar{\gamma}_{+}^{D}\right)v_j\right\rangle_D\\
&= \underbrace{\frac{1}{2}\left\langle v_i|\bar{\gamma}_{-}\hat{\mathrm{e}}^{\dagger D}v_j\right\rangle_D}_{=0} - \frac{1}{2}\left\langle v_i|i\bar{\gamma}_{+}\bar{\gamma}_{-}^{D}\bar{\gamma}_{+}^{D}v_j\right\rangle_D\\
&= -\frac{1}{2}\left\langle v_i|\hat{\mathrm{e}}^{D}\hat{\mathrm{e}}^{\dagger D}v_j\right\rangle_D\\
&= \frac{1}{2}\left\langle v_i|v_j\right\rangle\ .
\end{aligned} \qquad (1358)$$

Analogously for the reverse situation:

$$\begin{aligned}
\hat{\mathrm{e}}^{D}\,|v_i\rangle &:= 0\\
\bar{\gamma}_{+}\,|v_1\rangle &:= |v_1\rangle\\
\bar{\gamma}_{+}\,|v_2\rangle &:= |v_2\rangle\\
|\phi\rangle_i &:= \hat{N}_G\,|v_i\rangle\\
&= \frac{1}{2}\left(1-i\bar{\gamma}_{+}\bar{\gamma}_{-}\bar{\gamma}_{+}^{D}\right)|v_i\rangle\ .
\end{aligned}$$

Then:

$$\begin{aligned}
\left\langle\phi_1|\bar{\mathcal{P}}\,\phi_2\right\rangle_D &= \left\langle\phi_1|\bar{\gamma}_{-}\bar{\gamma}_{-}^{D}\frac{1}{2}\left(1-i\bar{\gamma}_{+}\bar{\gamma}_{-}\bar{\gamma}_{+}^{D}\right)\phi_2\right\rangle_D\\
&= \left\langle v_i|\bar{\gamma}_{-}\bar{\gamma}_{-}^{D}\frac{1}{2}\left(1-i\bar{\gamma}_{+}\bar{\gamma}_{-}\bar{\gamma}_{+}^{D}\right)v_j\right\rangle_D
\end{aligned}$$

$$
\begin{aligned}
&= \underbrace{\frac{1}{2}\left\langle v_i|\bar{\gamma}_- \hat{\mathrm{e}}^{\dagger D} v_j\right\rangle_D}_{=0} + \frac{1}{2}\left\langle v_i|i\bar{\gamma}_+\bar{\gamma}_-^D\bar{\gamma}_+^D v_j\right\rangle_D \\
&= +\frac{1}{2}\left\langle v_i|\hat{\mathrm{e}}^D\hat{\mathrm{e}}^{\dagger D} v_j\right\rangle_D \\
&= -\frac{1}{2}\left\langle v_i|v_j\right\rangle \,.
\end{aligned}
\tag{1359}
$$

Under $\hat{\eta}$-*conjugation* one finds the following behavior of various operators:

$$
\begin{aligned}
\hat{\eta}\,\hat{\mathrm{e}}^{(\dagger)^0}\,\hat{\eta} &= -\hat{\mathrm{e}}^{(\dagger)^0} \\
\hat{\eta}\,\hat{\mathrm{e}}^{(\dagger)^i}\,\hat{\eta} &= \hat{\mathrm{e}}^{(\dagger)^i} \quad (i \neq 0) \\
\hat{\eta}\,\bar{\gamma}_\pm\,\hat{\eta} &= -\bar{\gamma}_\pm = \bar{\gamma}_\pm^\dagger \\
\hat{\eta}\,(-1)^{\hat{N}}\hat{\eta} &= (-1)^{\hat{N}} \\
&= \left((-1)^{\hat{N}}\right)^\dagger \\
\hat{\eta}\,\hat{\mathrm{h}}_\pm\,\hat{\eta} &= \hat{\mathrm{h}}_\mp \\
&= \hat{\mathrm{h}}_\pm^\dagger \\
\hat{\eta}\,\mathcal{C}\hat{\eta} &= -\mathcal{P} \\
\hat{\eta}\,\mathcal{P}\hat{\eta} &= -\mathcal{C} \\
\hat{\eta}\,\bar{\mathcal{P}}\hat{\eta} &= \bar{\mathcal{C}} \\
\hat{\eta}\,\bar{\mathcal{C}}\hat{\eta} &= \bar{\mathcal{P}} \\
\hat{\eta}\,\hat{N}_G\,\hat{\eta} &= 1 - \hat{N}_G \\
&= \hat{N}_{\bar{G}}^\dagger \\
\hat{\eta}\,\hat{N}_{\bar{G}}\,\hat{\eta} &= -1 - \hat{N}_G \\
&= \hat{N}_{\bar{G}}^\dagger \\
\hat{\eta}\,\hat{N}_{\mathcal{G}}\,\hat{\eta} &= -\hat{N}_{\mathcal{G}} \\
&= \hat{N}_{\mathcal{G}}^\dagger \,.
\end{aligned}
\tag{1360}
$$

The *coBRST operator* is the $\hat{\eta}$-adjoint of the BRST charge [145][104][246]:

$$
\mathbf{Q}^{\dagger_{\hat{\eta}}} := \hat{\eta}\,\mathbf{Q}\,\hat{\eta}\,. \tag{1361}
$$

This yields the algebra

$$
\begin{aligned}
\{\mathbf{Q},\mathbf{Q}\} &= 0 \\
\left\{\mathbf{Q}^{\dagger_{\hat{\eta}}},\mathbf{Q}^{\dagger_{\hat{\eta}}}\right\} &= 0 \\
\left\{\mathbf{Q},\mathbf{Q}^{\dagger_{\hat{\eta}}}\right\} &= \mathbf{\Delta_Q}\,.
\end{aligned}
\tag{1362}
$$

Since $\langle\cdot|\cdot\rangle_{\hat{\eta}}$ is a proper scalar product (positive definite, non-degenerate), this gives relations analogous to those of $\mathbf{d}$ and $\mathbf{d}^\dagger$ in the Riemannian case:

$$
\begin{aligned}
&\mathbf{\Delta_Q}\,|\alpha\rangle = 0 \\
\Leftrightarrow\quad &\mathbf{Q}\,|\alpha\rangle = 0,\; \mathbf{Q}^{\dagger_{\hat{\eta}}}\,|\alpha\rangle = 0\,.
\end{aligned}
\tag{1363}
$$

E.5 (Ghost algebra based on involution $\iota = i\hat{\gamma}_+^0$ in arbitrary dimensions with time-independent superpotential and $\left\{\mathbf{d} + \mathbf{d}^\dagger, \hat{\gamma}_+^0\right\} = 0$) Let

$$\begin{aligned} \partial_0 W &= 0 \\ \mathbf{D} &= \hat{\gamma}_-^a \hat{\nabla}_a + \hat{\gamma}_+^b \left(\partial_b W\right) . \end{aligned} \tag{1364}$$

The ghost degrees of freedom (those which do not appear in the Dirac operator) are

$$\hat{\gamma}_+^{(\lambda)},\, \hat{\gamma}_-^{(\lambda)},\, \hat{\gamma}_+^0 .$$

An admissible ghost representation is:[64]

$$\begin{aligned} \mathcal{C} &:= (-1)^{\hat{N}} \hat{\gamma}_+^0 \frac{1}{2}\left(1 + i\hat{\gamma}_+^0\right) \\ \mathcal{P} &:= (-1)^{\hat{N}} \hat{\gamma}_+^0 \frac{1}{2}\left(1 - i\hat{\gamma}_+^0\right) \\ \bar{\mathcal{C}} &:= (-1)^{\hat{N}_{(\lambda)}} \hat{\gamma}_+^0 \frac{1}{2}\left(1 + (-1)^{\hat{N}} \hat{\gamma}_-^{(\lambda)}\right) \\ \bar{\mathcal{P}} &:= (-1)^{\hat{N}_{(\lambda)}} \hat{\gamma}_+^0 \frac{1}{2}\left(1 - (-1)^{\hat{N}} \hat{\gamma}_-^{(\lambda)}\right) , \end{aligned} \tag{1365}$$

with gauge generator and BRST operator given by:

$$\begin{aligned} \mathbf{p} &:= (-1)^{\hat{N}} \hat{\gamma}_+^0 \mathbf{D} \\ \mathbf{Q} &:= \mathcal{C}\mathbf{p} + \bar{\mathcal{P}} i\partial_{(\lambda)} \\ &= \mathbf{D}\,\frac{1}{2}\left(1 + i\hat{\gamma}_+^0\right) + \bar{\mathcal{P}} i\partial_{(\lambda)} . \end{aligned} \tag{1366}$$

Together with

$$\begin{aligned} \hat{N}_G &:= \mathcal{C}\mathcal{P} \\ &= \frac{1}{2}\left(1 - i\hat{\gamma}_+^0\right) \\ \hat{N}_{\bar{G}} &:= \bar{\mathcal{C}}\bar{\mathcal{P}} \\ &= -\frac{1}{2}\left(1 - (-1)^{\hat{N}} \hat{\gamma}_-^{(\lambda)}\right) \\ \hat{N}_{\mathcal{G}} &:= \hat{N}_G + \hat{N}_{\bar{G}} \end{aligned} \tag{1367}$$

this reproduces the entire ghost algebra:

---

[64]This has been found by the following reasoning: We need projectors on eigenspaces of two of the ghost degrees of freedom, so choose $i\hat{\gamma}_+^0$ and $\hat{\gamma}_-^{(\lambda)}$ as involutions (i.e. as (-1) to the power of ghost number operator). These projectors must commute, but $\left\{i\hat{\gamma}_+^0, \hat{\gamma}_-^{(\lambda)}\right\} = 0$ so add in an involution that anti-commutes with one of them, e.g. $(-1)^{\hat{N}}$. Next, to construct creators and annihilators, one needs two operators that mutually anticommute and anticommute with one of the involutions, while commuting with the other. One of these operators, the one appearing in the ghost, also needs to resemble the fermionic action of $\mathbf{D}$. But the latter acts with the operators $\hat{\gamma}_-^a$, $\hat{\gamma}_+^{a\neq 0}$. Multiplying these all together gives $\sim \bar{\gamma}_- \bar{\gamma}_+ \hat{\gamma}_+^0 \sim (-1)^{\hat{N}} \hat{\gamma}_+^0$. Using this in the ghost creator, there remains only an appropriate operator for the anti-ghost to be found (by trial and error).

1. The only non-vanishing brackets between the ghosts and anti-ghosts are

$$\begin{aligned} \{\mathcal{C}, \mathcal{P}\} &= 1 \\ \{\bar{\mathcal{C}}, \bar{\mathcal{P}}\} &= -1\,. \end{aligned} \tag{1368}$$

Proof: This follows from the relations

$$\begin{aligned} \left\{(-1)^{\hat{N}}, \hat{\gamma}^0_+\right\} &= 0 \\ \left\{(-1)^{\hat{N}_{(\lambda)}}\hat{\gamma}^0_+, (-1)^{\hat{N}}\hat{\gamma}^{(\lambda)}_-\right\} &= 0 \\ \left((-1)^{\hat{N}}\hat{\gamma}^0_+\right)^2 &= 1 \\ \left((-1)^{\hat{N}_{(\lambda)}}\hat{\gamma}^0_+\right)^2 &= -1 \end{aligned}$$

and the fact that $\frac{1}{2}\left(1 \pm i\hat{\gamma}^0_+\right)$ and $\frac{1}{2}\left(1 \pm (-1)^{\hat{N}}\hat{\gamma}^{(\lambda)}_-\right)$ are mutually orthogonal projectors. $\square$

2. Ghosts and anti-ghosts are (anti-) self-adjoint:

$$\begin{aligned} \mathcal{C}^\dagger &= -\mathcal{C} \\ \mathcal{P}^\dagger &= -\mathcal{P} \\ \bar{\mathcal{C}}^\dagger &= \bar{\mathcal{C}} \\ \bar{\mathcal{P}}^\dagger &= \bar{\mathcal{P}}\,. \end{aligned} \tag{1369}$$

Proof: Because of

$$\begin{aligned} \left((-1)^{\hat{N}}\right)^\dagger &= (-1)^{\hat{N}} \\ \left((-1)^{\hat{N}_{(\lambda)}}\right)^\dagger &= (-1)^{\hat{N}_{(\lambda)}} \\ \left(\hat{\gamma}^0_+\right)^\dagger &= \hat{\gamma}^0_+ \\ \left(\hat{\gamma}^{(\lambda)}_+\right)^\dagger &= \hat{\gamma}^{(\lambda)}_+ \end{aligned} \tag{1370}$$

and

$$\begin{aligned} \left\{(-1)^{\hat{N}}, \hat{\gamma}^a_\pm\right\} &= 0 \\ \left[(-1)^{\hat{N}}, \hat{\gamma}^{(\lambda)}_\pm\right] &= 0 \\ \left\{(-1)^{\hat{N}_{(\lambda)}}, \hat{\gamma}^{(\lambda)}_\pm\right\} &= 0 \\ \left[(-1)^{\hat{N}_{(\lambda)}}, \hat{\gamma}^a_\pm\right] &= 0 \\ \left\{\hat{\gamma}^a_\pm, \hat{\gamma}^{(\lambda)}_\pm\right\} &= 0 \end{aligned} \tag{1371}$$

one has

$$\left(-(-1)^{\hat{N}}\hat{\gamma}^0_+\frac{1}{2}\left(1 \pm i\hat{\gamma}^0_+\right)\right)^\dagger = -\frac{1}{2}\left(1 \mp i\hat{\gamma}^0_+\right)(-1)^{\hat{N}}\hat{\gamma}^0_+$$

$$
\begin{aligned}
&= -(-1)^{\hat N}\hat\gamma^0_+\frac{1}{2}\left(1 \pm i\hat\gamma^0_+\right) \\
\left((-1)^{\hat N_{(\lambda)}}\hat\gamma^0_+\frac{1}{2}\left(1 \pm (-1)^{\hat N}\hat\gamma^{(\lambda)}_-\right)\right)^\dagger &= \frac{1}{2}\left(1 \mp (-1)^{\hat N}\hat\gamma^{(\lambda)}_-\right)(-1)^{\hat N_{(\lambda)}}\hat\gamma^0_+ \\
&= (-1)^{\hat N_{(\lambda)}}\hat\gamma^0_+\frac{1}{2}\left(1 \pm (-1)^{\hat N}\hat\gamma^{(\lambda)}_-\right).
\end{aligned}
$$

3. The total ghost number operator is anti-self-adjoint:

$$
\begin{aligned}
\left(\hat N_G\right)^\dagger &= 1 - \hat N_G \\
\left(\hat N_{\bar G}\right)^\dagger &= -1 - \hat N_G \\
\hat N_{\mathcal G}^\dagger &= -\hat N_{\mathcal G}\,.
\end{aligned}
\tag{1372}
$$

Proof: By the definition (1367) and the relations (1368) and (1369). □

4. The gauge generator commutes with all ghosts and the BRST operator is nilpotent and self-adjoint:

$$
\begin{aligned}
[\mathbf{p}, \mathcal X] &= 0 \qquad \mathcal X \in \left\{\mathcal C, \mathcal P, \bar{\mathcal C}, \bar{\mathcal P}\right\} \\
\mathbf{Q}^\dagger &= \mathbf{Q} \\
\mathbf{Q}^2 &= 0
\end{aligned}
\tag{1373}
$$

Proof: $\mathbf{D}$ has the following brackets with the operators that the ghosts are constructed from in (1365):

$$
\begin{aligned}
\left\{\mathbf{D}, (-1)^{\hat N}\right\} = \left\{\mathbf{D}, \hat\gamma^0_+\right\} = \left\{\mathbf{D}, \hat\gamma^{(\lambda)}_\pm\right\} &= 0 \\
\left[\mathbf{D}, (-1)^{\hat N_\lambda}\right] &= 0\,.
\end{aligned}
$$

This, together with the brackets (1371) shows that

$$
\begin{aligned}
(\mathcal C\mathbf{p})^\dagger &= \mathcal C\mathbf{p} \\
(\mathcal C\mathbf{p})^2 &= 0 \\
\left(\bar{\mathcal P} i\partial_{(\lambda)}\right)^\dagger &= \bar{\mathcal P} i\partial_{(\lambda)} \\
\left(\bar{\mathcal P} i\partial_{(\lambda)}\right)^2 &= 0 \\
\left\{\mathcal C\mathbf{p}, \bar{\mathcal P} i\partial_{(\lambda)}\right\} &= 0\,.
\end{aligned}
$$

□

F Further literature

This section discusses some related literature in more detail.

F.1 In [89] the σ-model approach ("Hamiltonian route" in the terminology of the discussion on p. 9) to supersymmetric quantum mechanics is applied to cosmological models obtained from Einstein gravity coupled to a Yang-Mills field. While the main result is the construction of appropriate superpotentials, among other things, the paper also discusses adjointness relations of the mini-superspace supercharges and the existence of solutions in intermediate fermion sectors. These two issues concern questions discussed in this text. Since there is a certain discrepancy in the respective discussions the following tries to address the question how these arise:

- *Self-adjointness of the supercharges.* On page 1 of [89] it is pointed out, that in [25] (where the σ-model method is presented) it says that the nilpotent supercharges Q and $\bar{Q}$ (*cf.* footnote 9 (p.14)) are *not* mutually adjoint. The authors of [89] then write: "*In this paper we use another construction of the corresponding Hamiltonian, which* [...] *is Hermitian self-adjoint for any type of signature of the metric in minisuperspace.*" This is curious, because a comparison of the supercharges presented on page 5 of [89] with those given in [25] shows that both are in fact identical. This is to be expected, since both papers make (more or less explicitly) use of the Witten model (*cf.* 2.2.2 (p.61)) by introducing superpotentials by means of the deformation $Q = e^{-W} Q_0 e^{W}$. Hence the conclusion on page 5 of [89], that the supercharges "are *mutually Hermitian adjoint with respect to the measure* $\sqrt{|-g|} d^n q$ *and therefore, the energy operator* H *is* self-adjoint *for any signature of the metric* g_{ij}" must be true for [25], too, as indeed it is, essentially by construction. The apparent contradiction can be resolved by noting that one is dealing here with two types of inner products which give rise to two different notions of adjointness: The nilpotent supercharges of the Witten model are indeed self-adjoint with respect to the usual Hodge inner product, (38), p. 21 (with respect to the indefinite metric), alluded to in the quote above. But, for indefinite metrics, the Hodge inner product on sections of the exterior bundle is not positive definite and hence not a scalar product. This has important consequences, most notably of them the fact that Hodge's theorem (2.24 (p.40)) does *not* apply as it does in the case of positive definite metrics, the context in which it is usually discussed. Aware of this problem, the authors of [25] consider instead a modification of the usual Hodge inner product, one in which the time-like components of differential forms do not give rise to a negative sign. With respect to this modified inner product the supercharges are in fact no longer mutually adjoint. This issue of dealing with inner products in a theory with indefinite metric is considered in detail in §2.3 (p.106).

- *Intermediate solutions.* Further below on page 5 of [89] it says that, for vanishing superpotential, the nilpotent σ-model supercharges act as exterior derivative and coderivative, respectively, and that therefore results of deRahm cohomology theory apply, namely: "*[...] the solution of equation* $\bar{Q}_0 |\rho\rangle = 0$ *cannot be written as* $|\rho_p\rangle = \bar{Q}_0 |\sigma_{p-1}\rangle$ *only if* *the corresponding p-th cohomology group* $H^p(M)$ *of the manifold* $M(g_{ij})$ *is nontrivial*", and (page 6): "*Therefore the possible existence of* [solutions $|\rho\rangle$ to $Q\,|\rho\rangle = 0 = \bar{Q}\,|\rho\rangle$] *is directly related with the topology of the considered manifold* $M(g_{ij})$*, since all states except those in purely bosonic and filled fermion sectors can be excluded even without solving* [$Q\,|\rho\rangle = 0 = \bar{Q}\,|\rho\rangle$], *if the topology of the manifold* $M(g_{ij})$ *is trivial.*" It must be noted here that Hodge's theorem, which gives rise to deRahm cohomology, applies to *compact Riemannian* manifolds only. In the present context of supersymmetric quantum cosmology, the underlying manifold (namely mini-superspace) is neither compact nor is it Riemannian. Hodge's theorem does not apply here as it does in the cases for which it is formulated. This can be demonstrated by a simple counter-example to the above quoted claim: Consider flat 2D Minkowski space M^2 with metric $g = \mathrm{diag}(-1, 1)$ and the usual topology of $\mathbb{R}^2$. We have (in the notation introduced in 2.2 (p.16)) $\mathbf{d} = \hat{\mathrm{e}}^{\dagger 0}\partial_0 + \hat{\mathrm{e}}^{\dagger 1}\partial_1$, $\mathbf{d}^\dagger = -\hat{\mathrm{e}}^0\partial_0 - \hat{\mathrm{e}}^1\partial_1$, $\left(\mathbf{d} + \mathbf{d}^\dagger\right)^2 = \partial^\mu\partial_\mu = -\partial_0\partial_0 + \partial_1\partial_1$. A state in the intermediate fermion sector is immediately obtained via Graham's method (*cf.* §2.2.7 (p.90)) by choosing a solution $|\tilde{\rho}\rangle$ of $\partial^\mu\partial_\mu\,|\tilde{\rho}\rangle$, e.g. $|\tilde{\rho}\rangle = e^{ik(x^0 \pm x^1)}\,|0\rangle$ and applying $\mathbf{d}$ to it: $|\rho\rangle = \mathbf{d}\,|\tilde{\rho}\rangle = ik\left(dx^0 \pm dx^1\right)e^{ik(x^0 \pm x^1)}$. It is readily checked that $\mathbf{d}\,|\rho\rangle = 0 = \mathbf{d}^\dagger\,|\rho\rangle$. Superposing such plane wave solutions so as to form a normalizable wave packet gives perfectly admissible solutions in the 1-fermion sector, even though the topology of M^2 is trivial.

G Spinor representations

G.1 (Spinor conventions) The most widely followed convention concerning two component Weyl spinor notation seems to be that every author has his own. Apart from that, possibly the conventions used, for instance, in [36] are most popular in the context of supergravity. But here we follow the notation used in [80] for deriving canonically quantized supergravity in the vielbein formalism. This is summarized in [83],§2.9, pp. 63:

The σ-matrices are taken to be

$$\sigma_0 = -\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1374}$$

and their spinor indices are assumed to be upstairs

$$\sigma_a = \left(\sigma_a{}^{AA'}\right) .$$

Lorentz indices are raised and lowered with the Minkowski metric of signature +2:

$$\eta_{ab} = (-,+,+,+) , \tag{1375}$$

as (currently) usual in gravitation theory. The 2-component spinor metric ϵ is defined by

$$\left(\epsilon^{AB}\right) = \left(\epsilon_{AB}\right) = \left(\epsilon^{A'B'}\right) = \left(\epsilon_{A'B'}\right) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} . \tag{1376}$$

Spinor indices are raised and lowered by means of these objects following the "NW-SE" convention:

$$\begin{aligned} \rho^A &= \epsilon^{AB}\,\rho_B \\ \rho_A &= \rho^B\,\epsilon_{BA} \\ \rho^{A'} &= \epsilon^{A'B'}\,\rho_{B'} \\ \rho_{A'} &= \rho^{B'}\,\epsilon_{B'A'} . \end{aligned} \tag{1377}$$

This implies in particular that, for instance,

$$\begin{aligned} \sigma_{aAA'} &= \sigma_a{}^{BB'}\epsilon_{BA}\epsilon_{B'A'} \\ &= \left(\epsilon^{\mathrm{T}}\sigma_a\epsilon\right)_{AA'} \\ &= -\left(\epsilon\sigma_a\epsilon\right)_{AA'} . \end{aligned} \tag{1378}$$

Barred σ-matrices are always defined by

$$\bar\sigma = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3) . \tag{1379}$$

Their index structure is

$$\bar\sigma_a = \left(\bar\sigma_{aA'A}\right) .$$

Note that also

$$\sigma^{\mathrm{T}}_{aA'A} = -\left(\epsilon\sigma_a^{\mathrm{T}}\epsilon\right)_{A'A} = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3)_{aA'A}$$

so that barred and unbarred σ-matrices are related by

$$\bar{\sigma}_{aA'A} = \sigma^{\mathrm{T}}_{aA'A} \,. \tag{1380}$$

The normalization factor $\frac{1}{\sqrt{2}}$ in (1374) is there so that no factor of 2 appears in the following orthonormality relations:

$$\begin{aligned} {\sigma_a}^{AA'} {\sigma^b}_{AA'} &= -\delta_{ab} & (1381)\\ {\sigma_a}^{AA'} {\sigma^a}_{BB'} &= -\delta^A_B \, \delta^{B'}_{A'} \,. & (1382) \end{aligned}$$

Lorentz vector indices and spinor indices are related by, for instance:

$$\begin{aligned} {e^{AA'}}_i &= {\sigma_a}^{AA'} {e^a}_i \\ \Leftrightarrow {e^a}_i &= -{\sigma^a}_{AA'} {e^{AA'}}_i \,. \end{aligned} \tag{1383}$$

Because of (1381) and (1382) the respective derivatives are related by

$$\begin{aligned} \frac{\delta}{\delta {e^{AA'}}_i} &= -{\sigma^a}_{AA'} \frac{\delta}{\delta {e^a}_i} \\ \frac{\delta}{\delta {e^a}_i} &= -{\sigma_a}^{AA'} \frac{\delta}{\delta {e^{AA'}}_i} \,. \end{aligned} \tag{1384}$$

G.2 (Weyl representations of Clifford algebra in higher dimensions) In even dimensions, $D = 2n$, there is always a representation of the Clifford algebra $\mathrm{Cl}(1, D-1)$ with a diagonal Clifford pseudoscalar and $\hat{\gamma}^0$ block anti-diagonal. More precisely, there is a representation such that:

$$\begin{aligned} \mathrm{I} &= -(-i)^{n-1} \left[\begin{array}{cc} 1_{(2^{n-1})} & 0_{(2^{n-1})} \\ 0_{(2^{n-1})} & -1_{(2^{n-1})} \end{array} \right] = -(-i)^{n-1} \sigma^3 \otimes \bigotimes_{n-1} \sigma^0 \\ \hat{\gamma}^0 &= \left[\begin{array}{cc} 0_{(2^{n-1})} & 1_{(2^{n-1})} \\ 1_{(2^{n-1})} & 0_{(2^{n-1})} \end{array} \right] = \sigma^1 \otimes \bigotimes_{n-1} \sigma^0 \,. \end{aligned} \tag{1385}$$

Here σ is the usual representation of the Pauli algebra:

$$\sigma^0 = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right], \quad \sigma^1 = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \quad \sigma^2 = \left[\begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right], \quad \sigma^3 = \left[\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right] \tag{1386}$$

and matrices are understood to be obtained from tensor products of the Pauli algebra by inserting right factors into left factors.

Proof: One possibility to construct such a representation is by the following recursive definition:

1. For $D = 2$ choose

$$\hat{\gamma}^0 := \sigma^1, \quad \hat{\gamma}^1 := i\sigma^2 \quad . \tag{1387}$$

Then the pseudoscalar reads

$$I = \hat{\gamma}^0 \hat{\gamma}^1 = -\sigma^3 \,. \tag{1388}$$

2. Let $\hat{\gamma}'$ be the representation obtained for $D' = 2n'$. Then for $D = 2n' + 2$ choose

$$\begin{aligned} \hat{\gamma}^0 &:= \hat{\gamma}'^0 \otimes \sigma^0 \\ \hat{\gamma}^i &:= \hat{\gamma}'^i \otimes \sigma^1, \qquad 0 < i < D' \\ \hat{\gamma}^{D-2} &:= \tilde{\hat{\gamma}} \otimes \sigma^2 \\ \hat{\gamma}^{D-1} &:= \tilde{\hat{\gamma}} \otimes \sigma^3 \end{aligned} \tag{1389}$$

with

$$\tilde{\hat{\gamma}} := i\sigma^2 \otimes \bigotimes_{n'-1} \sigma^0 \,. \tag{1390}$$

The corresponding pseudoscalar is represented by

$$I = \; = \; -i\, I' \otimes \sigma^0 \,. \tag{1391}$$

For $D = 4$ this reproduces the usual 4-dimensional Weyl representation of the Dirac algebra:

$$\hat{\gamma}^0 = \begin{bmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{bmatrix}, \quad \hat{\gamma}^1 = i \begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix}, \quad \hat{\gamma}^2 = i \begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix}, \quad \hat{\gamma}^3 = i \begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix} \tag{1392}$$

with

$$I = i \begin{bmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{bmatrix} . \tag{1393}$$

For $D = 6$ one finds:

$$\begin{aligned} \hat{\gamma}^0 &= \sigma^1 \otimes \sigma^0 \otimes \sigma^0 \\ \hat{\gamma}^1 &= i\sigma^2 \otimes \sigma^1 \otimes \sigma^1 \\ \hat{\gamma}^2 &= i\sigma^2 \otimes \sigma^2 \otimes \sigma^1 \\ \hat{\gamma}^3 &= i\sigma^2 \otimes \sigma^3 \otimes \sigma^1 \\ \hat{\gamma}^4 &= i\sigma^2 \otimes \sigma^0 \otimes \sigma^2 \\ \hat{\gamma}^5 &= i\sigma^2 \otimes \sigma^0 \otimes \sigma^3 \end{aligned} \tag{1394}$$

and

$$I = \sigma^3 \otimes \sigma^0 \otimes \sigma^0 \,. \tag{1395}$$

References

[1] R. Abbott, S. Barr, and S. Ellis. Kaluza-Klein cosmologies and inflation. *Phys. Rev. D*, 30:720–727, 1984.

[2] E. Abreu and N. Braga. Superspace formulation for the BRST quantization of the chiral Schwinger model. `hep-th/9609119`, 1996.

[3] E. Alvarez and M. Gavela. Entropy from extra dimensions. *Phys. Rev. Lett.*, 51:931–934, 1983.

[4] L. Alvarez-Gaumé. Supersymmetry and the Atiyah-Singer index theorem. *Comm. Math. Phys.*, :161–173, 1983.

[5] S. Ansoldi, A. Aurilia, and E. Spalluci. String propagator: A loop space representation. 1995. `hep-th/9510133`.

[6] D. Armand-Ugon, R. Gambini, O. Obregón, and J. Pullin. Towards a loop representation for quantum canonical supergravity. *Nucl. Phys. B*, 460:615–631, 1996.

[7] M. Asano, M. Tanimoto, and N. Yoshino. Supersymmetric Bianchi class A models. *Phys. Lett. B*, 314:303, 1993.

[8] K. Aschan, E. Bayro-Corrochano, and P. Lounesto. Covariances of the Maxwell and Dirac equation. *Electromagnetics Laboratory Report Series, Helsinki University of Technology*, 388:1–18, 2001.

[9] F. Assaoui and T. Lhallabi. Supersymmetry and quantum cosmology. `hep-th/0009031`, 200.

[10] J. Azcárraga, J. Izquierdo, and A. Macfarlane. Hidden supersymmetries in supersymmetric quantum mechanics. `hep-th/0101053`, 2001.

[11] J. Baez, I. Segal, and Z. Zhou. *Introduction to Algebraic and Constructive Quantum Field Theory.* Princeton University Press, 1992.

[12] D. Băleanu, V. Dubovnik, and S. Misicu. Dual Killing-Yano symmetry and multipole moments in electromagnetism and mechanics of continua. `math-ph/9806011`, 1999.

[13] J. B. Barbour. Time and complex numbers in canonical quantum gravity. *Phys. Rev. D*, 47(12):5422–5429, 1993.

[14] A. Barut and M. Pavsic. Classical model of the Dirac electron in curved space. *Class. Quant. Grav.*, 4:L41, 1987.

[15] A. Barut and M. Pavsic. Quantization of the classical relativistic Zitterbewegung in the Schrödinger picture. *Class. Quant. Grav.*, 4:L131, 1987.

[16] A. Barut and M. Pavsic. Kaluza-Klein approach to the classical model of the Dirac electron. *Class. Quant. Grav.*, 5:707, 1988.

[17] A. Barut and M. Pavšič. Equivalence of the spinning superparticle descriptions with Grassmann variables or with c-number spinors. *Phys. Lett. B*, 216(3,4):297, 1989.

[18] A. Barvinsky, A. Kamenshchik, and C. Kiefer. Effective action and decoherence by fermions in quantum cosmology. `gr-qc/9901055`, 1999.

[19] A. Barvinsky, A. Kamenshchik, and C. Kiefer. Origin of the inflationary universe. *Mod. Phys. Let. A*, 14:1083–1088, 1999.

[20] A. Barvinsky, A. Kamenshchik, C. Kiefer, and I. Mishakov. Decoherence in quantum cosmology at the onset of inflation. `gr-qc/9812043`, 1998.

[21] I. Batalin and R. Marnelius. Solving general gauge theories on inner product spaces. *Nucl. Phys. B*, 442:669–695, 1995.

[22] W. Baylis. *Clifford (geometric) Algebras.* Birkhäuser Boston, 1996.

[23] W. Baylis. *Electrodynamics: A modern geometrical Approach.* Birkhäuser Boston, 1999.

[24] C. Beasly and E. Witten. A note on fluxes and superpotentials in $\mathcal{M}$-theory compactifications on manifolds of G_2 holonomy. `hep-th/0203061`, 2002.

[25] J. Bene and R. Graham. Supersymmetric homogeneous quantum cosmologies coupled to a scalar field. *Phys. Rev. D*, 1994.

[26] E. Benedict, R. Jackiw, and H. Lee. Functional scrödinger and BRST quantization of $(1+1)$-dimensional gravity. `hep-th/9607062`, 1996.

[27] I. Benn and P. Charlton. Dirac symmetry operators from conformal Killing-Yano tensors. `gr-qc/9612011`, 1996.

[28] I. Benn, P. Charlton, and J. Kress. Debye potentials for Maxwell and Dirac fields from a generalisation of the Killing-Yano equation. `gr-qc/9610037`, 1996.

[29] I. Benn and R. Tucker. *An introduction to Spinors and Geometry.* Adam Hilger, 1987.

[30] F. Berezin. *Introduction to superanalysis.* D. Reidel, 1987.

[31] N. Berline, E. Getzler, and M. Vergne. *Heat Kernels and Dirac Operators.* Springer, 1992.

[32] D. Bigatti and L. Susskind. Review of matrix theory. `hep-th/9712072`, 1997.

[33] A. Bilal, J.P. Derendinger, and K. Sfetsos. (Weak) G_2 holonomy from self-duality, flux and supersymmetry. `hep-th/0111274`, 2002.

[34] A. Billyard, A. Coley, J. Lidsey, and U. Nilsson. Dynamics of M-theory cosmology. `hep-th/9908102`, 1999.

[35] N. Borizov, K. Ilinski, and V. Uzdin. Generalized algebras of supersymmetric quantum mechanics. *Teoreticheskaya i Matematicheskaya Fizika*, 94:418–425, 1993.

[36] F. Brandt. Lectures on supergravity. `hep-th/0204035`, 2002.

[37] L. Brink. Lectures on superstrings. In L. Brink and C., editors, *Principles of String Theory*, pages 3–91. Plenum Press, 1988.

[38] R. Britto-Pacumino, J. Michelson, A. Strominger, and A. Volovich. Lectures on superconformal quantum mechanics and multi-black hole moduli spaces. `hep-th/9911066`, 2000.

[39] R. Capovilla and J. Guven. . *Class. Quant. Grav.*, 11:1961, 1994.

[40] R. Capovilla and O. Obregón. No quantum superminisuperspace with $\Lambda \neq 0$. *Phys. Rev. D*, 49(12):6562, 1994.

[41] A. Carlini and J. Greensite. The mass shell of the universe. `gr-qc/9610020`, 1996.

[42] S. Carlip. Quantum gravity: a progress report. *Rept. Prog. Phys.*, 64:885, also `gr--qc/0108040`, 1996.

[43] S. Carroll, D. Freedman, M. Ortiz, and D. Page. Physical states in canonically quantized supergravity. `hep-th/9401155`, 1994.

[44] L. Castellani, R. D'Auria, and P. Fré. *Supergravity and superstrings.* World Scientific, 1991.

[45] M. Cavaglià and V. Alfaro. Time gauge fixing and hilbert space in quantum string cosmology. `gr-qc/9605020`, 1996.

[46] M. Cavaglià and P. Moniz. Canonical and quantum FRW cosmological solutions in M-theory. `gr-qc/0010280`, 2000.

[47] M. Cavaglià and P. Moniz. FRW cosmological solutions in M-theory. `gr-qc/0011098`, 2000.

[48] A. Cheng, P. D'Eath, and P. Moniz. Quantization of the Bianchi type-IX model in supergravity with a cosmological constant. DAMTP R 93/35, 1993.

[49] A. Cheng and P. Moniz. Canonical quantization of Bianchi class A models in $N = 2$ supergravity. *Mod. Phys. Lett. A*, 11(3):227–245, 1996.

[50] S. Chern. *Complex Manifolds without Potential Theory.* Springer-Verlag, 1995.

[51] J. Chisholm and R. Farwell. Gauge transformations of spinors with Clifford algebraic structure. *J. Phys. A*, 32:2805, 1999.

[52] Y. Choquet-Bruhat and C. DeWitt-Morette. *Analysis, manifolds and physics, Part II.* North-Holland Physics, 1989.

[53] M. Claudson and M. B. Halpern. Supersymmetric ground state wave functions. *Nucl. Phys. B*, 250:689–715, 1984.

[54] S. Coleman, J. Hartle, T. Piran, and S. Weinberg. *Quantum Cosmology and Baby universes.* World Scientific, 1991.

[55] A: Connes. *Noncommutative Geometry.* Academic Press, 1994.

[56] F. Cooper and B. Freedman. Aspects of supersymmetric quantum mechanics. *Ann. Phys.*, 146:262–288, 1983.

[57] S. Cordes, G. Moore, and S. Ramgoolam. Lectures on 2D Yang-Mills theory, equivariant cohomology and topological field theories. `hep-th/9411210`, 1994.

[58] I. Cotăescu and M. Visinescu. Schrödinger quantum modes on the Taub-NUT background. `hep-th/9911914`, 1999.

[59] I. Cotăescu and M. Visinescu. The Dirac field in the Taub-NUT background. `hep-th/0008181`, 2000.

[60] I. Cotăescu and M. Visinescu. Dynamical algebra and Dirac quantum modes in Taub-NUT background. `hep-th/0102083`, 2001.

[61] I. Cotăescu and M. Visinescu. Hierarchy of Dirac, Pauli and Klein-Gordon conserved operators in Taub-NUT background. `hep-th/0107205`, 2001.

[62] I. Cotăescu and M. Visinescu. Non-existence of f-symbolfs in generalized Taub-NUT spacetimes. `hep-th/0105201`, 2001.

[63] I. Cotăescu and M. Visinescu. Runge-Lenz operator for Dirac field in Taub-NUT background. `hep-th/0101163`, 2001.

[64] I. Cotăescu and M. Visinescu. Supersymmetry and discrete transformations of the Dirac operators in Taub-NUT geometry. `hep-th/0202034`, 2002.

[65] E. Cremmer and B. Julia. . *Phys. Lett. B*, 80:48, 1978.

[66] E. Cremmer and B. Julia. . *Phys. Lett. B*, 156:141, 1979.

[67] E. Cremmer, B. Julia, and J. Scherk. . *Phys. Lett. B*, 76:409, 1978.

[68] A. Csordás and R. Graham. Exact quantum states for $N = 1$ supergravity. `gr-qc/9507008`, 1995.

[69] A. Csordás and R. Graham. Nontrivial fermion states in supersymmetric minisuperspace. `gr-qc/9503054`, 1995.

[70] A. Csordás and R. Graham. Supersymmetric minisuperspace with non-vanishing fermion number. `gr-qc/9502004`, 1995.

[71] T. Damour. String cosmology and chaos. `hep-th/0204017`, 2002.

[72] T. Damour and M. Henneaux. Chaos in superstring cosmology. *Phys. Rev. Lett.*, 85(5):920–923, 2000.

[73] T. Damour and M. Henneaux. Oscillatory behaviour in homogeneous string cosmology models. *Phys. Lett. B*, 488:108–116, 2000.

[74] T. Damour, M. Henneaux, A. Rendall, and M. Weaver. Kasner-like behaviour for subcritical Einstein-matter systems. `gr-qc/0202069`, 2002.

[75] A. Das, J. Maharana, and S. Roy. The Neveu-Schwarz-Ramond string in background fields: Nilpotency of the BSRT charge. *Nucl. Phys. B*, 331:573–591, 1990.

[76] A. Dasgupta, H. Nicolai, and J. Plefka. An introduction to the quantum supermembrane. `hep-th/0201182`, 2002.

[77] S. De Leo, Z. Oziewicz, W. Rodrigues, Jr., and J. Vaz, Jr. The Dirac-Hestenes lagrangian. `hep-ph/9906243`, 1999.

[78] B. de Wit, H.-J. Matschull, and H. Nicolai. Physical states in $D = 3, N = 2$ supergravity. *Phys. Lett. B*, 318, 1993.

[79] B. De Witt. *Supermanifolds.* Cambridge University Press, 1984.

[80] P. D'Eath. Canonical quantization of supergravity. *Phys. Rev. D*, 29(10):2199, 1983.

[81] P. D'Eath. Quantization of Bianchi type IX model in supergravity. *Phys. Rev. D*, 48:713, 1993.

[82] P. D'Eath. Physical states in $N = 1$ supergravity. *Phys. Lett. B*, 321:368, 1994.

[83] P. D'Eath. *Supersymmetric Quantum Cosmology.* Cambridge University Press, 1996.

[84] P. D'Eath, S. Hawking, and O. Obregón. Supersymmetric Bianchi models and the square root of the Wheeler-DeWitt equation. *Phys. Lett. B*, 300:44, 1993.

[85] P. D'Eath and D. Hughes. Mini-superspace with local supersymmetry. *Nucl. Phys. B*, 378, 1992.

[86] S. Deser and B. Zumino. A complete action for the spinning string. *Phys. Lett. B*, 65(4), 1976.

[87] A. Dolgallo and K. Ilinski. Generalized supersymmetric quantum mechanics on Riemannian surfaces with meromorphic superpotentials. *J. Math. Phys.*, 35(5):2074, 1994.

[88] A. Dolgallo and K. Ilinski. Holomorphic supersymmetric quantum mechanics, generalized supersymmetry, and parasupersymmetry. *Ann. Phys.*, 236:219, 1994.

[89] E. Donets, M. Tentyukov, and M. Tsulaia. Towards $N = 2$ SUSY homogeneous quantum cosmology; Einstein-Yang-Mills systems. *Phys. Rev. D*, 59, 1999.

[90] M. Duff and B. Nilsson. Kaluza-Klein supergravity. *Phys. Rep.*, 130(1,2):1–142, 1986.

[91] D. Edelen. *Applied exterior calculus.* John Wiley and Sons, 1985.

[92] T. Eguchi, P. Gilkey, and A. Hanson. Gravitation, gauge theories and differential geometry. *Phys. Rep.*, 66(6):213–393, 1980.

[93] G. Esposito. *Dirac operators and spectral geometry.* Cambridge University Press, 1998.

[94] F. De Jonghe, A. Macfarlane, K. Peeters, and J. van Holten. New supersymmetry of the monopole. `hep-th/9507046`, 1995.

[95] L. Faddeev and A. Slavnov. *Gauge Fields, Introduction to Quantum Theory.* Benjamin/Cummings, 1980.

[96] J. Feng, K. Matchev, and F. Wilczek. Particle and astroparticle searches for supersymmetry. `hep-ph/0111295`, 2001.

[97] R. Feynman and A. Hibbs. *Quantum mechanics and path integrals.* McGraw-Hill, 1965.

[98] T. Frankel. *The Geometry of Physics.* Cambridge University Press, 1997.

[99] P. Freund. Kaluza-Klein cosmologies. *Nucl. Phys. B*, 209:146–156, 1982.

[100] P. Freund. *Introduction to Supersymmetry.* Cambridge University Press, 1986.

[101] J. Fröhlich, O. Grandjean, and A. Recknagel. Supersymmetric quantum theory and (non-commutative) differential geometry. `hep-th/9612205`, 1996.

[102] J. Fröhlich, O. Grandjean, and A. Recknagel. Supersymmetric quantum theory, non-commutative geometry, and gravitation. `hep-th/9706132`, 1997.

[103] G. Fülöp. BRST–co-BRST quantization of reparametrization-invariant theories. *Int. J. Theor. Phys.*, 13:3629–3647, 1997.

[104] G. Fülöp and R. Marnelius. Gauge fixing and coBRST. *Nucl. Phys. B*, 456:443–472, 1995.

[105] B. Gaveau, T. Jaconson, M. Kac, and L. Schulman. Relativistic extension of the analogy between quantum mechanics and brownian motion. *Phys. Rev. Lett.*, 54(5):419–422, 1984.

[106] H. Gersch. Feynman's relativistic chessboard as an Ising model. *Int. J. Theor. Phys.*, 20:491–5001, 1981.

[107] G. Gibbons and G. Papadopoulos. HKT and OKT geometries on soliton black hole moduli spaces. `hep-th/9706207`, 1997.

[108] G. Gibbons, R. Rietdijk, and J. van Holten. SUSY in the sky. `hep-th/9303112`, 1993.

[109] J. Gilbert and M. Murray. *Clifford algebras and Dirac operators in harmonic analysis.* Cambridge University Press, 1991.

[110] R. Graham. Supersymmetric Bianchi type IX cosmology. *Phys. Rev. Lett.*, 67(11):1381, 1991.

[111] R. Graham. Supersymmetric general Bianchi type IX cosmology with a cosmological term. *Phys. Lett. B*, 277:393, 1992.

[112] R. Graham. Anisotropic diagonal Bianchi type-IX minisuperspace with $N = 4$ supersymmetry. *Phys. Rev. D*, 48(4):1602, 1993.

[113] R. Graham and J. Bene. Supersymmetric Bianchi type IX cosmology with a scalar field. *Phys. Lett. B*, 302:183, 1993.

[114] R. Graham and H. Luckock. Nicolai maps for quantum cosmology. *Phys. Rev. D*, 49:2786–2791, 1994.

[115] M. Green, J. Schwarz, and E. Witten. *Superstring Theory*. Cambridge University Press, 1987.

[116] S. Gukov. Solitons, superpotentials and calibration. `hep-th/9911011`, 1996.

[117] S. Gull, A. Lasenby, and C. Doran. Imaginary numbers are not real – the geometric algebra of spacetime. *Found. Phys.*, 1993.

[118] J. Halliwell. Derivation of the Wheeler-DeWitt equation from a path integral for minisuperspace models. *Phys. Rev. D*, 38(8):2468–2481, 1988.

[119] B. Hatfield. *Quantum Field Theory of Point Particles and Strings*. Addison-Wesley, 1992.

[120] S. Hawking, R. Laflamme, and G. Lyons. Origin of time asymmetry. *Phys. Rev. D*, 47(12):5342–5356, 1993.

[121] M. Henneaux. Poisson brackets of the constraints in the Hamiltonian formulation of tetrad gravity. *Phys Rev. D*, 27:986–989, 1983.

[122] M. Henneaux. Lectures on string theory, with emphasis on hamiltonian and BRST methods. In L. Brink and C. Teitelboim, editors, *Principles of String Theory*, pages 93–292. Plenum Press, 1988.

[123] M. Henneaux and C. Teitelboim. Relativistic quantum mechanics of supersymmetric particles. *Ann. Phys.*, 143:127, 1982.

[124] M. Henneaux and C. Teitelboim. *Quantization of Gauge Systems*. Princeton University Press, 1992.

[125] D. Hestenes. *Space-time algebra*. Gordon and Breach science publishers, 1966.

[126] D. Hestenes. Local observables in the Dirac theory. *J. Math. Phys.*, 14, 1973.

[127] D. Hestenes. Observables, operators, and complex numbers in the Dirac theory. *J. Math. Phys.*, 16:556–572, 1975.

[128] D. Hestenes. *Spacetime Calculus*. (in preparation, draft available at `modelingts.la.asu.edu/HTML/STC.html`), 2002.

[129] D. Hestenes and R. Gurtler. Local observables in the quantum theory. *Am. J. Phys.*, 39:1028–1038, 1971.

[130] D. Hestenes and G. Sobczyk. *Clifford Algebra to Geometric Calculus*. D. Reidel Publishing Company, 1984.

[131] E. Hoker and D. Phong. Lectures on supersymmetric Yang-Mills theory and integrable systems. `hep-th/9912271`, 1999.

[132] J. W. van Holten. Aspects of brst quantization. `hep-th/0201124`, 2002.

[133] C. Hull. The geometry of supersymmetric quantum mechanics. `hep-th/9910028`, 1999.

[134] G. Imponente and G. Montani. On the quantum origin of the Mixmaster chaos covariance. `gr-qc/0111051`, 2001.

[135] G. Imponente and G. Montani. Covariant Mixmaster dynamics. `gr-qc/0201103`, 2002.

[136] G. Imponente and G. Montani. The Mixmaster model as a cosmological framework and aspects of its quantum dynamics. `gr-qc/0201102`, 2002.

[137] C. J. Isham. Canonical quantum gravity and the problem of time. `gr-qc/9210011`, 1992.

[138] J. van Holten. Supersymmetry and the geometry of Taub-NUT. `hep-th/9409139`, 1994.

[139] J. van Holten and A. Waldron. An index theorem for non-standard Dirac operators. `hep-th/9901163`, 1999.

[140] T. Jacobson. New variables for canonical supergravity. *Phys. Rev. D*, 29:2911, 1984.

[141] T. Jacobson. . *Class. Quant. Grav.*, 5:923, 1988.

[142] T. Jacobson and L. Schulman. Quantum stochastics: The passage from a relativistic to a non-relativistic path integral. *J. Phys. Math.*, 17:375–383, 1983.

[143] T. Jacobson and L. Smolin. Nonperturbative quantum geometries. *Nucl. Phys. B*, 299:295, 1988.

[144] G. Junker. Quantum and classical stochastic dynamics: Exactly solvable models by supersymmetric methods. `quant-ph/9810070`, 1998.

[145] W. Kalau and J. W. van Holten. BRST cohomology and BRST gauge fixing. *Nucl. Phys. B*, 361:233–252, 1991.

[146] H. Kalka and G. Soff. *Supersymmetrie*. Teubner Studienbücher, 1997.

[147] L. Kauffman. Discrete physics and the Dirac equation. *Phys. Lett. A (also* `hep-th/9603202`*)*, 218:139–146, 1996.

[148] C. Kiefer. On the meaning of path integrals in quantum cosmology. *Ann. Phys.*, 207:53–70, 1991.

[149] C. Kiefer. Decoherence in quantum electrodynamics and quantum gravity. *Phys. Rev. D*, 46(4):1658–1670, 1992.

[150] C. Kiefer. Topology, decoherence, and semiclassical gravity. *Phys. Rev. D*, 47(12):5414–5421, 1993.

[151] C. Kiefer. Conceptual issues in quantum cosmology. `gr-qc/9906100`, 1999.

[152] C. Kiefer and D. Polarski. Quantum-to-classical transition for fluctuations in the early universe. `gr-qc/9802003`, 1998.

[153] C. Kiefer and H. Zeh. Arrow of time in a recollapsing quantum universe. *Phys. Rev. D (*`gr-qc/9402036`*)*, 51(8), 1994.

[154] E. Kiritsis. Introduction to superstring theory. `hep-th/9709062`, 1998.

[155] E. Kolb and R. Slansky. Dimensional reduction in the early universe: Where have all the massive particles gone? *Phys. Lett. B*, 135(5,6):378–382, 1984.

[156] J. Kress. Generalised conformal Killing-Yano tensors: Applications to electrodynamics. *PhD thesis*, 1997.

[157] M. Kreuzer. Einführung in die Superstring-Theorie. `http://www.itp.tuwien.ac.at/~kreuzer/inc/est.ps.gz`, 2001.

[158] M. Kreuzer. Einführung in die Superstring-Theorie II. `http://www.itp.tuwien.ac.at/~kreuzer/inc/esst.ps.gz`, 2001.

[159] K. Kuchar and C. Torre. World sheet diffeomorphism and the canonical string. *J. Math. Phys.*, 30(8):1769 – 1793, 1997.

[160] J. Kupsch and O. Smolyanov. Functional representation for fock superalgebras. `hep-th/9708069`, 1997.

[161] R. Laflamme and A. Matacz. Decoherence functional and inhomogeneities in the early universe. `gr-qc/9303036`, 1993.

[162] A. Lahanas and D. Nanopoulos. Combining supersymmetric dark matter with recent accelerator data. `hep-ph/0112134`, 2001.

[163] L. Landau and E. Lifshitz. *The classical theory of fields.* Pergamon, New York, 1975.

[164] A. Lasenby, C. Doran, and S. Gull. A multivector derivative approach to Lagrangian field theory. *Found. Phys.*, 23(10):1295, 1993.

[165] H. Lawson and M. Michelsohn. *Spin Geometry.* Princeton University Press, 1989.

[166] J. Lidsey and J. Maharana. Dualities and hidden supersymmetry in string quantum cosmology. `gr-qc/9801090`, 1998.

[167] U. Lindström. Supersymmetry, a biased review. `hep-th/0204016`, 2002.

[168] Y. Ling and L. Smolin. Supersymmetric spin networks and quantum supergravity. `hep-th/9904016`, 1999.

[169] D. Long and G. Shore. The schrödinger wave functional and vacuum state in curved spacetime. `hep-th/9605004`, 1996.

[170] D. Long and G. Shore. The schrödinger wave functional and vacuum state in curved spacetime II – boundaries and foliations. `gr-qc/9607032`, 1996.

[171] P. Lounesto. Clifford algebras and Hestenes spinors. *Found. Phys.*, 23(9):1203, 1993.

[172] P. Lounesto. Clifford algebras and spinor operators. In W. Baylis, editor, *Clifford (geometric) algebras*, pages 5–35. Birkhäuser Boston, 1996.

[173] H. Luckock. Supersymmetric quantum cosmology. In A. Lun, L. Brewin, and E. Chow, editors, *Proceedings of the Fourth Monash General Relativity Workshop*, pages 99–107, 1994.

[174] H. Luckock. Quantum cosmology, supersymmetry, and the problem of time. In D. J. Wiltshire, editor, *Proceedings of the First Australasian Conference on General Releativity and Gravitation*, pages 175–190, 1996.

[175] A. Macfarlane and A. Mountain. Construction of supercharges for the one-dimensional supersymmetric non-linear sigma model. `JHEP06(1999)005`, 1999.

[176] A. Macfarlane and P. Popat. The quantum mechanics of the $N = 2$ extended supersymmetric nonlinear σ-model. *J. Math. Phys. A*, 17:2955–2961, 1984.

[177] A. Macfarlane and P. Popat. The quantum mechanics of the supersymmetric nonlinear σ-model. *J. Math. Phys. A*, 17:2945, 1984.

[178] W. Machin. Dynamics of the spinning particle in one dimension. `hep-th/0109052`, 2001.

[179] A. Macías, O. Obregón, and M. Ryan. Quantum cosmology: the supersymmetric square root. *Class. Quant. Gravity*, 4:1477, 1987.

[180] A. Macías, O. Obregón, and J. Socorro. . *Int. J. Mod. Phys. A*, 8:4291, 1993.

[181] R. Marnelius. An important bigrading in general BRST quantization. *Nucl. Phys. B*, 370:165–182, 1991.

[182] R. Marnelius. Simple BRST quantization of general gauge models. *Nucl. Phys. B*, 395:647–660, 1993.

[183] R. Marnelius. Gauge fixing and abelianization in simple BRST quantization. *Nucl. Phys. B*, 412:817–833, 1994.

[184] R. Marnelius. Proper BRST quantization of relativistic particles. *Nucl. Phys. B*, 418:353–378, 1994.

[185] R. Marnelius and M. Ögren. New inner products for physical states in BRST quantization. *Nucl. Phys. B*, 351:474–490, 1991.

[186] R. Marnelius and N. Sandström. Basics of BRST quantization on inner product spaces. `hep-th/9810125`, 1999.

[187] R. Marnelius and N. Sandström. Quantum BRST properties of reparametrization invariant theories. `hep-th/0006175`, 200.

[188] D. Marolf. Group averaging and refined algebraic quantization: Where are we now? `gr-qc/0011112`, 2000.

[189] D. Marolf and C. Rovelli. Relativistic quantum measurement. `gr-qc/0203056`, 2002.

[190] H.-J. Matschull. About loop states in supergravity. `gr-qc/9403034`, 1994.

[191] H.-J. Matschull and H. Nicolai. Canonical quantum supergravity in three dimensions. *Nuc. Phys. B*, 411:609–646, 1994.

[192] D. McKeon and G. Ord. Time reversal in stochastic processes and the Dirac equation. *Phys. Rev. Lett.*, 69:3–4, 1992.

[193] J. Michelson and A. Strominger. The geometry of (super) conformal quantum mechanics. `hep-th/9907191`, 2000.

[194] D. Miralles. On Dirac equations in curved space-times. `ftp://www.clifford.org/clf-alg/preprints/1996/mira9601.latex`, 1996.

[195] C. Misner, K. Thorne, and J. Wheeler. *Gravitation.* Freedman and Company, 1973.

[196] P. Moniz. FRW model with vector fields in $N = 1$ supergravity. *Helv. Phys. Acta*, 69:293–296, 1996.

[197] P. Moniz. Supersymmetric quantum cosmology. `gr-qc/9604025`, 1996.

[198] P. Moniz. Wave function of a supersymmetric FRW model with vector fields. *Int. J. Mod. Phys. D*, 6(4):465–478, 1997.

[199] P. Moniz. Why *two* is more attractive than one... or Bianchi class-A models and Reissner-Nordström black holes in quantum $N = 2$ supergravity. *Nucl. Phys. B (Proc. Suppl.)*, 57:307–311, 1997.

[200] P. Moniz. Origin of structure in supersymmetric quantum cosmology. *Phys. Rev. D*, 57(12):R7071–R7074, 1998.

[201] P. V. Moniz. Conserved currents in supersymmetric quantum cosmology? `gr-qc/9606047`, 1997.

[202] M. Montesino and C. Rovelli. Statistical mechanics of generally covariant quantum theories: A Boltzmann-like approach. *Class. Quantum Grav*, 18:555–569 (also `gr--qc/0002024`), 2001.

[203] A. Mostafazadeh. A genuine probability interpretation for Klein-Gordon fields and its implication for quantum cosmology. `gr-qc/0205049`, 2002.

[204] J. Nelsen and C. Teitelboim. Hamiltonian formulation of the theory of interacting gravitational and electron fields. *Ann. Phys. (N.Y.)*, 116:86–104, 1978.

[205] H. Nicolai and R. Helling. Supermembranes and M(atrix) theory. `hep-th/9809103`, 1998.

[206] O. Obregón, J. Pullin, and M. Ryan. Bianchi cosmologies: New variables and a hidden supersymmetry. *Phys. Rev. D*, 48(12):5642–5647, 1993.

[207] O. Obregón, J. Socorro, and J. Benítez. Supersymmetric quantum cosmology proposals and the Bianchi type-II model. *Phys. Rev. D*, 47(10):4471–4475, 1993.

[208] Y. Obukhov and F. Hehl. Spacetime metrics from linear electrodynamics. `gr-qc/9904067`, 1999.

[209] G. Ord. Classical analog of quantum phase. *Int. J. Theor. Phys.*, 31:1177–1195, 1992.

[210] G. Ord. A reformulation of the Feynman chessboard model. *J. Stat. Phys.*, 66:647–659, 1992.

[211] G. Ord. Classical particles and the Dirac equation with an electromagnetic field. *Chaos, Solitons and Fractals*, 8:727–741, 1997.

[212] G. Ord and J. Gualtieri. The Feynman propagator from a single path. `quant-ph/0109092`, 2002.

[213] G. Ord and D. McKeon. On the Dirac equation in $3+1$ dimensions. *Ann. Phys.*, 222:244–253, 1992.

[214] D. Page. Inconsistency of canonically quantized $N = 1$ supergravity. `hep-th/9306032`, 1993.

[215] G. Papadopoulos. KT and HKT geometries in strings and in black hole muduli spaces. `hep-th/0201111`, 2002.

[216] J. Park. Cohomological field theories with kähler structure. `hep-th/9910209`, 1999.

[217] M. Pavšič. How the Geometric Calculus resolves the ordering ambiguity of quantum theory in curvedspace. `gr-qc/0111092`, 2001.

[218] K. Peeters. Geometrical aspects of supersymmetry: Spinning particles, string dualities and the eleven-dimensional supermembrane. *PhD Thesis*, 1998.

[219] G. Pete. Morse theory. `http://www.math.u-szeged.hu/gpete/morse.ps`, 1999.

[220] W. Pezzaglia and A. Differ. A Clifford dyadic superfield from bilateral interactions of geometric multispin Dirac theory. `gr-qc/9311015`, 1993.

[221] M. Pilati. . *Nucl. Phys. B*, 132:138, 1978.

[222] M. Plyushchay. On the nature of fermion-monopole supersymmetry. `hep-th/0005122`, 2000.

[223] J. Polchinski. *String Theory*. Cambridge University Press, 1998.

[224] M. Riesz. Sur certain notions fondamental en theorie quantique relativiste. In L. Garding and L. Hormander, editors, *Marcel Riesz, Collected Papers*, pages 545–570. Springer, 1988.

[225] R. Rietdijk and J. van Holten. Killing tensors and a new geometric duality. `hep-th/9511166`, 1995.

[226] W. Rodrigues, J. Vaz, and M. Pavsic. The Clifford bundle and the dynamics of the superparticle. *Banach Center Publications*, 37:295–315, 1991.

[227] G. Roepsdorff and Ch. Vehns. Generalized Dirac operators and superconnections. *math-ph/9911006*, 1999.

[228] A. Rogers. Gauge fixing and BFV quantization. `hep-th/9902133`, 1999.

[229] A. Rogers. Supersymmetry and Brownian motion on supermanifolds. `quant-ph/0201006`, 2002.

[230] H. Rosu and J. Socorro. One-parameter familiy of closed, radiation-filled Friedmann-Robertson-Walker "quantum" universes. *Phys. Lett. A*, 223:28–30, 1997.

[231] C. Rovelli. The projector on physical states in loop quantum gravity. `gr-qc/9806121`, 2001.

[232] C. Rovelli. Spacetime states and covariant quantum theory. `gr-qc/0111016`, 2001.

[233] O. Rudolph. Super Hilbert spaces. `math-ph/9910047`, 2000.

[234] M. Ryan. *Hamiltonian Cosmology*. Springer-Verlag, 1972.

[235] M. Ryan and L. Shepley. *Homogeneous Relativistic Cosmologies*. Princeton University Press, 1975.

[236] D. Sahdev. Towards a realistic Kaluza-Klein cosmology. *Phys. Lett. B*, 137(3,4):155–159, 1984.

[237] K. Sakamoto and K. Shiraishi. Conformal quantum mechanics in two black hole moduli space. `hep-th/0203152`, 2002.

[238] T. Sano. The Ashtekar formulation and WKB wave function of $N = 1, 2$ supergravity. `hep-th/9211103`, 1992.

[239] T. Sano and J. Shiraishi. The non-perturbative canonical quantization of the $N = 1$ supergravity. *Nucl. Phys. B*, 410:423, 1993.

[240] U. Schreiber. Überblick zur stochastischen Interpretation der Quantenmechanik . `http://www-stud.uni-essen.de/~sb0264/stochastic.ps.gz`, 2000.

[241] M. Seriu. Evolution of the discrepancy between a universe and its model. `gr-qc/0111060`, 2001.

[242] Q. Shafi and C. Wetterich. Cosmology from higher-dimensional gravity. *Phys. Lett.*, 129(6):387–391, 1983.

[243] T. Smith. HyperDiamond Feynamn checkerboard in 4-dimensional spacetime. `quant-ph/9503012`, 1995.

[244] L. Smolin. Covariant quantization of membrane dynamics. `hep-th/9710191`, 1997.

[245] J. Socorro, O. Obregón, and A. Macías. Supersymmetric microsuperspace quantization for the Taub model. *Phys. Rev. D*, 45(6):2026–2032, 1992.

[246] M. Spiegelglas. Q_{BRST} cohomology. *Nucl. Phys. B*, 283:205–216, 1986.

[247] P. Srivastava. *Supersymmetry, Superfields and Supergravity.* Adam Hilger, 1986.

[248] A. Strohmaier. On noncommutative and semi-Riemannian geometry. `math-ph/0110001`, 2002.

[249] A. Strominger. Baby universes. In S. Coleman, J. Hartle, T. Piran, and S. Weinberg, editors, *Quantum Cosmology and Baby universes*, pages 269–346. World Scientific, 1991.

[250] R. Szabo. BUSSTEPP lectures on string theory. `hep-th/0207142`, 2002.

[251] R. Tabensky and C. Teitelboim. The square root of general relativity. *Phys. Lett. B*, 69:453, 1977.

[252] Y. Tanii. Introduction to supergravities in diverse dimensions. `hep-th/9802138`, 1998.

[253] M. Tanimoto. The role of killing-yano tensors in supersymmetric mechanics on a curved manifold. `gr-qc/9501006`, 1995.

[254] W. Taylor. Lectures on D-branes, gauge theory and M(atrices). `hep-th/9801182`, 1998.

[255] W. Taylor. M(atrix) theory: Matrix quantum mechanics as a fundamental theory. `hep-th/0101126`, 2001.

[256] C. Teitelboim. Supergravity and the square roots of constraints. *Phys. Rev. Lett.*, 38:1106, 1977.

[257] C. Teitelboim. Quantum mechanics of the gravitational field. *Phys. Rev. D*, 25(12):3159, 1982.

[258] G. Trayling and W. Baylis. A geometric basis for the standard-model gauge group. `hep-th/0103137`, 2001.

[259] K. Ülker. $N = 1$ SYM action and BRST cohomology. `hep-th/0108062`, 2001.

[260] D. Vaman and M. Visinescu. Spinning particles in Taub-NUT space. `hep-th/9707175`, 1997.

[261] D. Vaman and M. Visinescu. Supersymmetries and constants of motion in Taub-NUT spinning space. `hep-th/9805116`, 1998.

[262] J. Varilly. An introduction to noncommutative geometry. `physics/9709045`, 1997.

[263] J. Vaz and W. Rodrigues. Clifford algebras, supercalculus and spinning particle models. In J. Dowling, editor, *Electron theory and Quantum Electrodynamics - 100 years later*, volume 358 of *NATO ASI series, series B: Physics*, pages 201–222. Plenum Press, 1997.

[264] J. Vaz, Jr. and W. Rodigues, Jr. Maxwell and Dirac theories as already unified theory. `hep-th/9511181`, 1995.

[265] M. Visinescu. Generalized Killing equations for spinning spaces and the role of Killing-Yano tensors. `hep-th/9610097`, 1996.

[266] M. Visinescu. Generalized Taub-NUT metrics and Killing-Yano tensors. `hep-th/9911126`, 2000.

[267] W. Rodrigues, Jr and J. Vaz, Jr. Spinors as differential forms, and applications to electromagnetism and quantum mechanics. In P. Pronin and G. Sardanahvily, editors, *Gravity, Particles and Space-Time*, pages 307–344. World Scientific, 1996.

[268] W. Rodrigues, Jr., Q. De Souza, J. Vaz, Jr., and P. Lounesto. Dirac-Hestenes spinor fields in Riemann-Cartan spacetime. `hep-th/9607073`, 1996.

[269] R. Wald. *General Relativity*. University of Chicago Press, 1984.

[270] D. Wands. String-inspired cosmology. `hep-th/0203107`, 2002.

[271] J. Wess and J. Bagger. *Supersymmetry and Supergravity*. Princeton University Press, 1992.

[272] P. West. Supergravity, brane dynamics and string duality. `hep-th/9811101`, 1998.

[273] D. L. Wiltshire. An introduction to quantum cosmology. `gr-qc/0101003`, 2001.

[274] E. Witten. Constraints on supersymmetry breaking. *Nucl. Phys. B*, 202:253–316, 1982.

[275] E. Witten. Supersymmetry and morse theory. *J. Diff. Geom.*, 17:661–692, 1982.

[276] E. Witten. On flux quantization in M-theory and effective action. `hep-th/9609122`, 1996.

[277] R. Woodard. Enforcing the Wheeler-DeWitt constraint the easy way. *Class. Quant. Grav.*, 10:483–496, 1992.

[278] J. Wosiek. Spectra of supersymmetric Yang-Mills quantum mechanics. `hep-th/0203116`, 2002.

[279] H. Zeh. *The physical basis of the direction of time*. Springer, 2001.

[280] B. Zumino. Supersymmetry and Kähler manifolds. *Phys. Lett. B*, 87:203–206, 1979.