Some
Quantum States of M-Branes under Hypothesis H

Urs Schreiber on joint work with Hisham Sati

NYU AD Science Division, Program of Mathematics

& Center for Quantum and Topological Systems

New York University, Abu Dhabi

talk at:

Centre for Research in String Theory

QMU London, Dec 9, 2021

slides and pointers at: `ncatlab.org/schreiber/show/Some+Quantum+States+of+M-Branes+under+Hypothesis+H`

Abstract.

Following a proposal by H. Sati, we have recently stated a hypothesis about the mathematical home of the quantum charges in M-theory. This "Hypothesis H" refines the traditional proposal for quantization of D-brane charge from K-theory to the non-abelian cohomology theory known as 4-*Cohomotopy*, whose classifying space is the 4-sphere.

Besides its motivation from homotopy-theoretic re-analysis of 11d supergravity and of the old brane scan, Hypothesis H is justified by its rigorous implication of a list of long-conjectured M-theoretic consistency conditions on C-field flux and M-brane charges – such as shifted C-field flux quantization, dual Page charge quantization and M2/M5-brane tadpole cancellation.

But if Hypothesis H is a correct assumption about the nature of M-theory, this suggests that quantum states of full M-theory should be reflected in the positive cohomology of the moduli space of Cohomotopy cocycles, much like quantum states of non-perturbative Chern-Simons theory are in the Dolbeault cohomology of moduli spaces of (flat) connections.

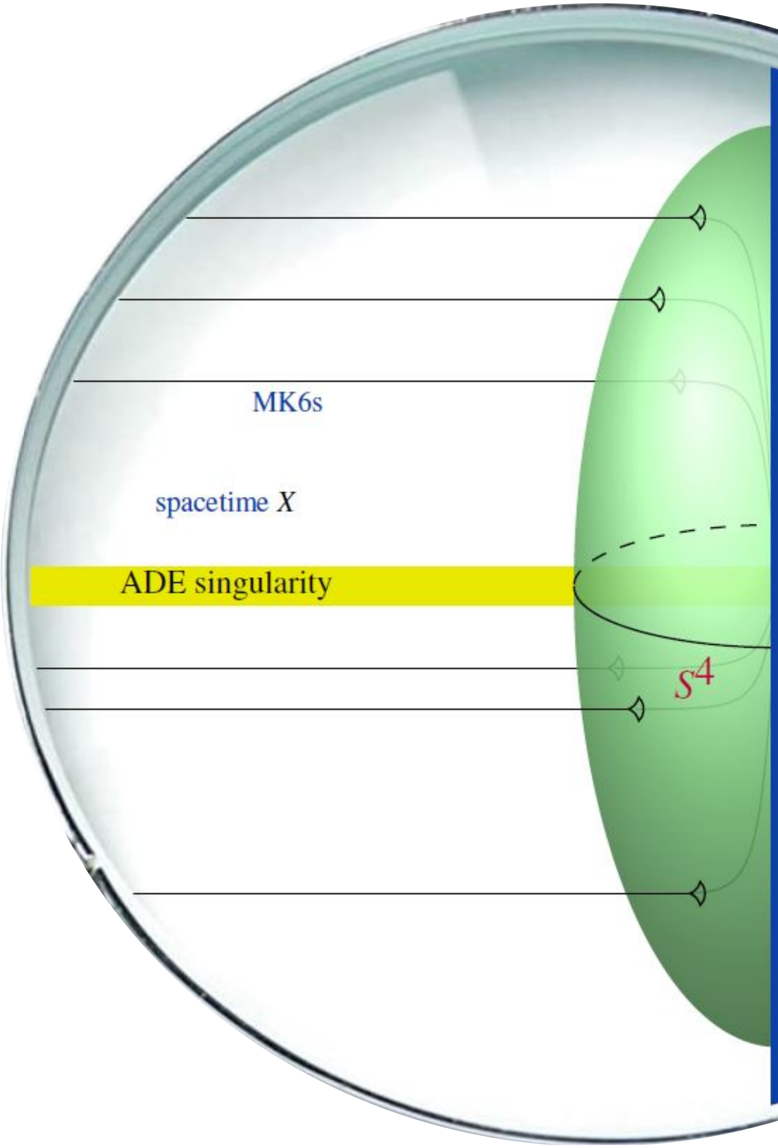
In this talk I discuss how, in the topological sector of D6 $\perp$ D8-brane intersections, such quantum states according to Hypothesis H are identified with *weight systems* on *horizontal chord diagrams*, and how these do reflect a range of phenomena expected from the traditional approaches to understanding these brane intersections, such as non-abelian DBI-theory, the BMN matrix model, Rozansky-Witten theory and Hanany-Witten theory.

Specifically, we have proven that the fundamental $\mathfrak{gl}_2(\mathbb{C})$-weight system satisfies the positivity condition that characterizes physical (i.e. non-ghost) quantum states. Under the above identification, this quantum state corresponds to an elementary squashed fuzzy funnel configuration & to the elementary M5-brane state in the BMN matrix model – both as expected for D6/D8-brane intersections.

Besides possible implications for the elusive formulation of M-theory, this result may provide a unifying explanation for the plethora of unexpected appearances that chord diagrams are recently making in fundamental high energy physics, notably in discussion of holographic entanglement entropy.

Based on `arXiv:1912.10425` & `arXiv:2105.02871`.

While perturbative string theory is ∼ well-defined (via worldsheet SCFT), defining non-perturbative brane physics (M-theory) has remained an open problem.



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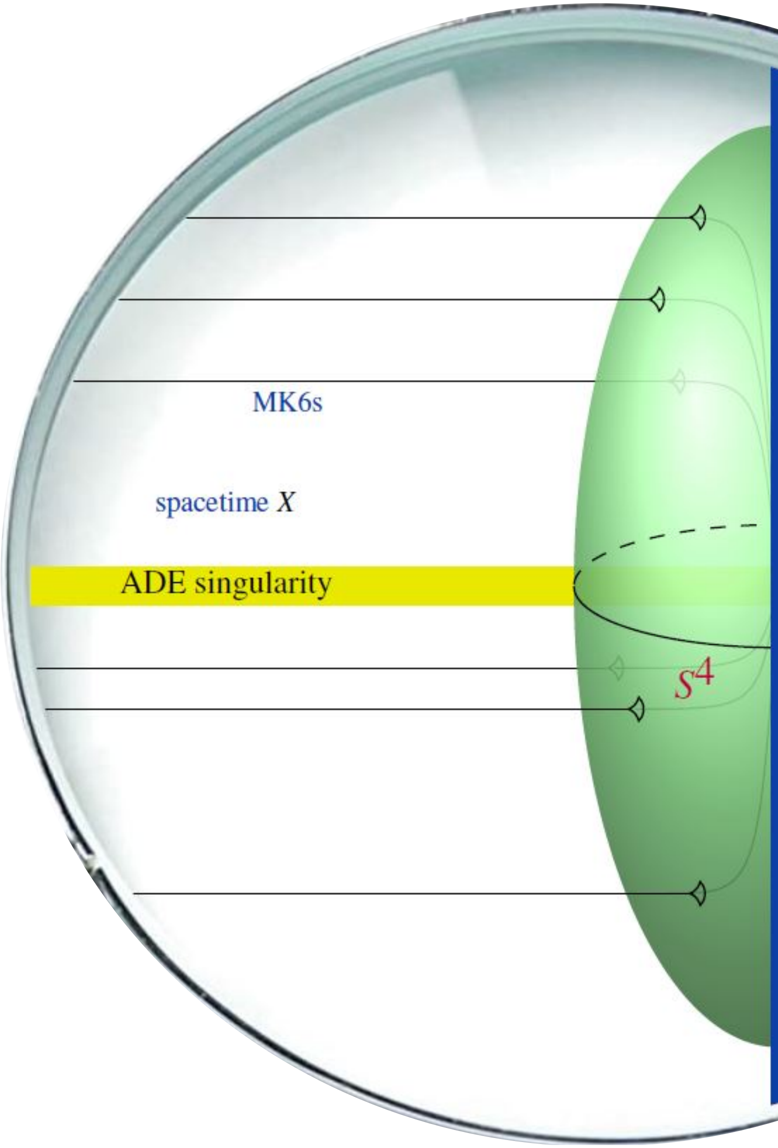

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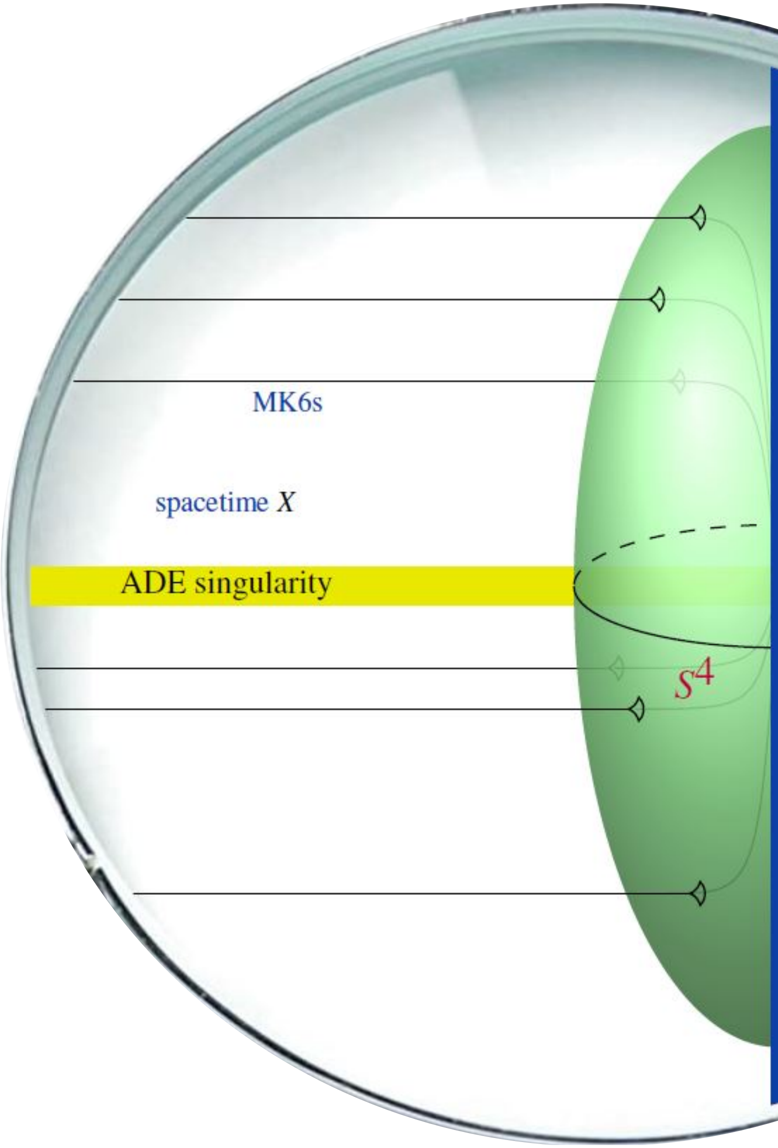

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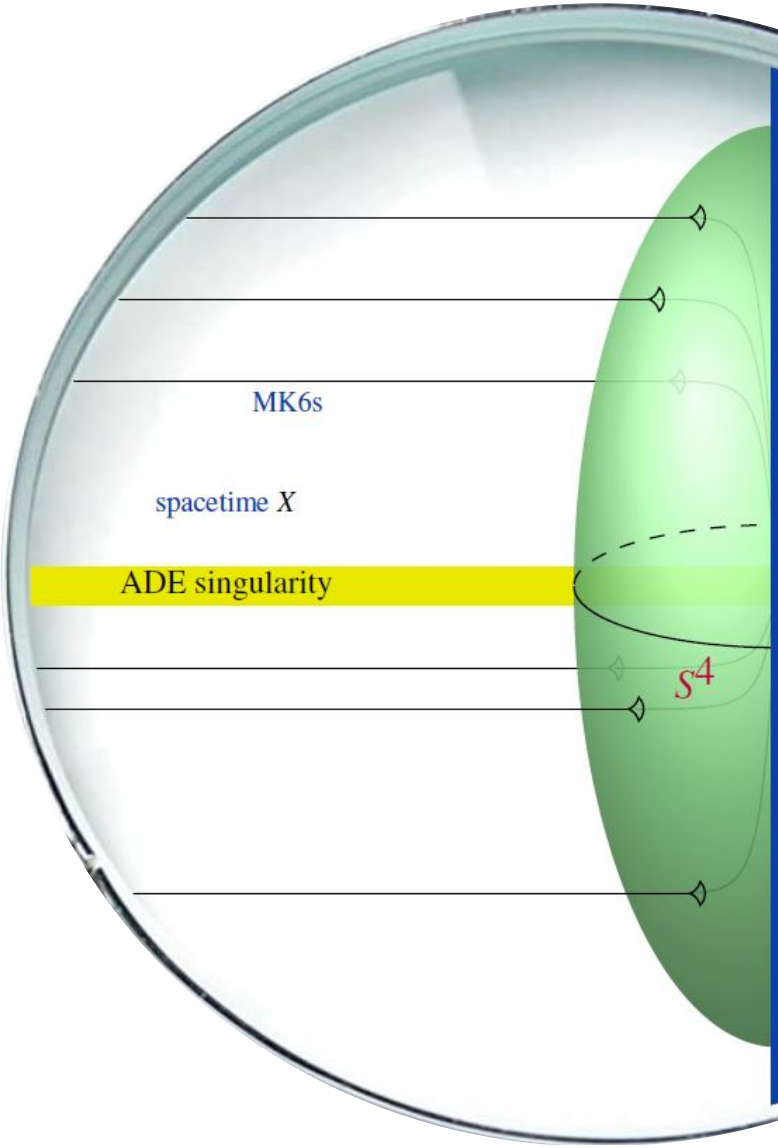

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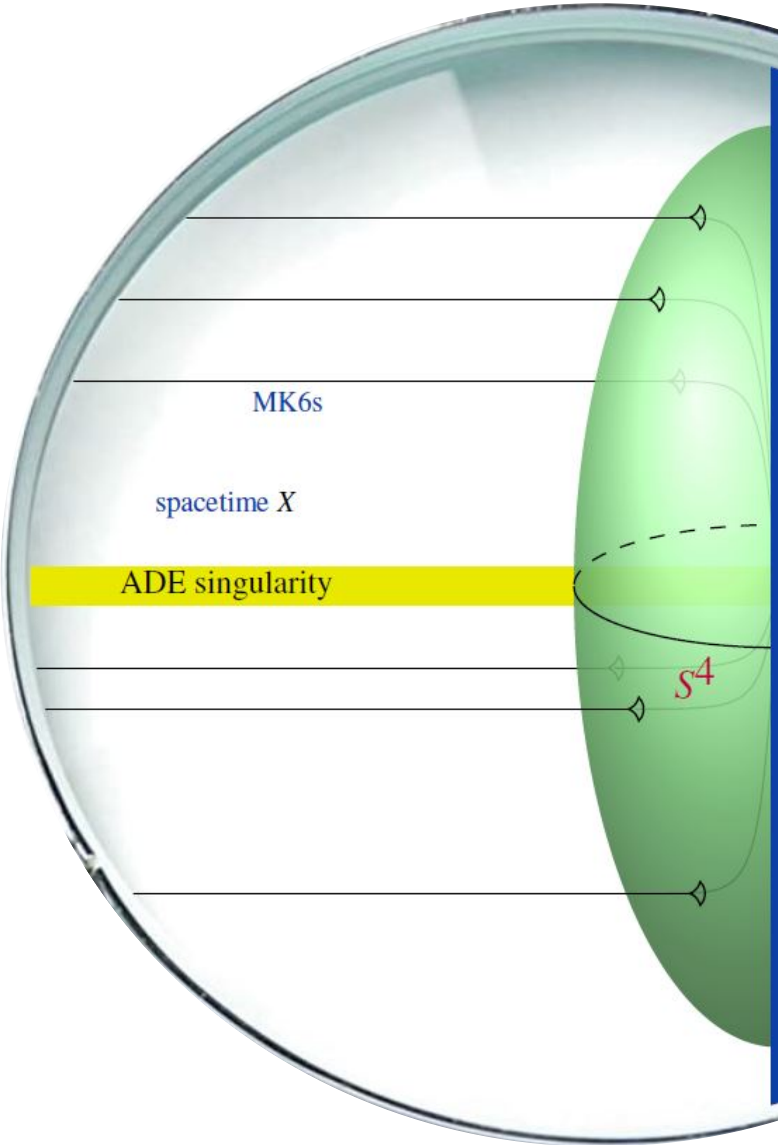

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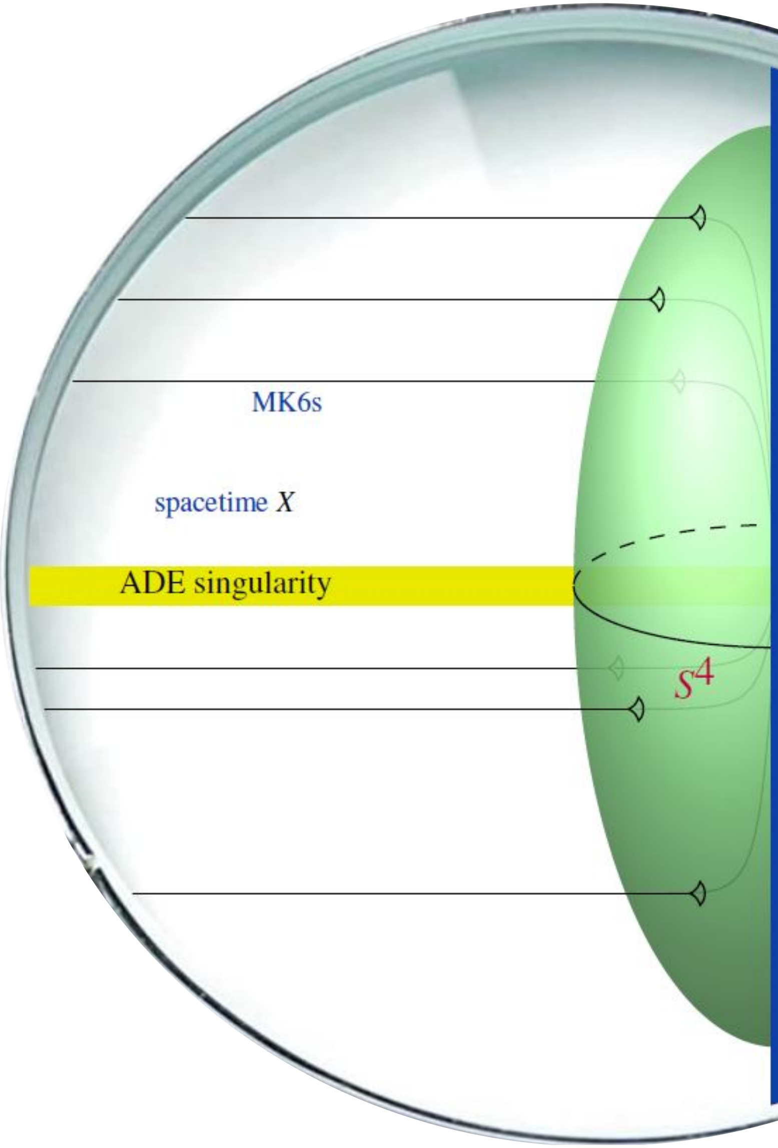

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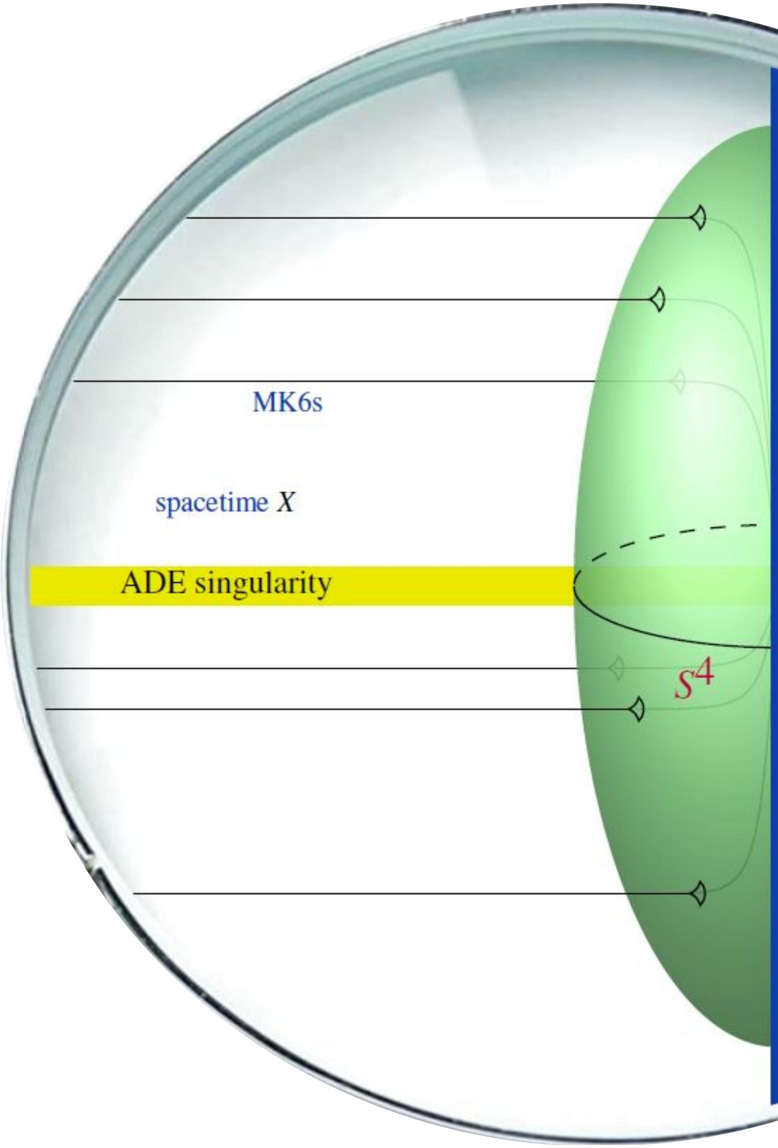

For more see `ncatlab.org/schreiber/show/Hypothesis+H`

0 – Cohesive Homotopy Theory

I – Quantum Charge of M-branes under Hypothesis H

II – Quantum Charge of D6 $\perp$ D8 under Hypothesis H

III – Quantum States of D6 $\perp$ D8 under Hypothesis H

String theory at its finest is, or should be, a new branch of geometry
... developed in the 21st century ... that fell by chance into the 20th century ...

To elucidate the proper generalization of geometry
[is] the central problem of string theory.

E. Witten (1988)

as quoted on p. 95, 102 in:

P C W Davis and J Brown (eds.)
Superstrings: A theory of everything?
Camb Univ Press 1988, 1991: Canto 1992

Non-perturbative quantum physics. [SS19-Quant][CSS21-Quant][Sc14-Quant]

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Lots of subtleties, but two general principles:

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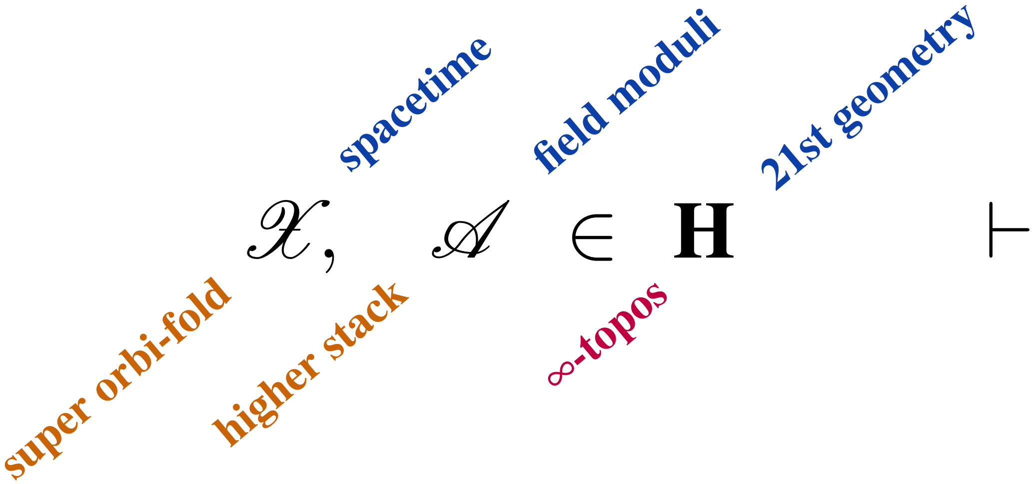

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Ex.: 3d CS theory with cpt gauge group: [Hit90] [APW91]

$\mathscr{X}$ = surface

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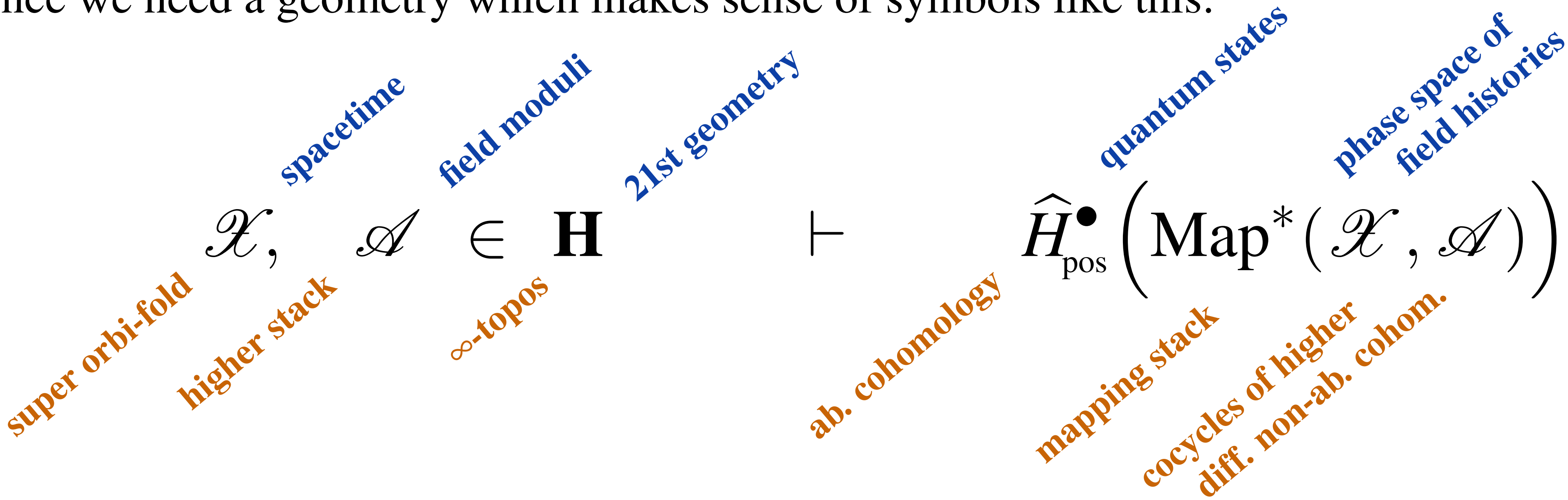

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Ex.: D6 ⊥ D8-branes via Hypothesis H
[SS19-Quant][CSS21-Quant]

$\mathscr{X}$ = transverse cptfd. space to branes
$\mathscr{A}$ = moduli stack of diff. 4-Cohomotopy
$\widehat{H}^{\bullet}_{\mathrm{pos}}$ = positive ordinary cohomology

The Higher Geometry of Physics.

[JSSW18-HigStrc][FSS19-RatM][SS20-OrbCoh]

The dictionary:

	physics	**mathematics**
	geometry	topos theory
+	gauge principle	homotopy theory
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The key point:

stringy geometry	↔	**higher homotopy**
p-brane charges		$\pi_{p+1}\big(\mathscr{A}\big) \in \mathrm{Grp}(\mathbf{H}_0)$
$p_1 \perp p_2$-intersections		higher k-invariants

[FSS13-Bouq, §3]
[HSS18-ADE, §2]
[FSS19-RatM, §7]

Higher gauged sigma-model targets.

[JSSW18-HigStrc][FSS19-RatM][SS20-OrbCoh]

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The idea:

Just as an emergent target space $\mathscr{X}$ seen via probes by worldvolumes Σ,

so an **∞-stack** $\mathscr{X}$ is a space bootstrapped by its gauged system of Σ-plots:

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moduli space of
Σ-model fields

$$\mathrm{Map}(\Sigma, \mathscr{X}) \;=\; \left\{ \begin{tikzcd}[column sep=huge] \text{probe } \Sigma \arrow[rr, bend left=60, "\text{trajectory}"{name=U}] \arrow[rr, bend right=60, "\text{trajectory}"'{name=D}] \arrow[from=U, to=D, Rightarrow, shift right=12, "\substack{\text{gauge transf.}\\ \text{homotopy}}"'{name=L}] \arrow[from=U, to=D, Rightarrow, shift left=12, "\substack{\text{gauge transf.}\\ \text{homotopy}}"{name=R}] \arrow[from=L, to=R, Rrightarrow, "\text{gauge-of-gauge}", "\text{transformations}"'] && \mathscr{X} \;\text{target} \end{tikzcd} \right\}$$

mapping ∞-stack

Cohesive homotopy theory of Smooth ∞-stacks. [SSS09][Sc13][ScSh14][SS20-OrbCoh]

Key example: Higher geometry locally modeled on $\mathrm{CartSp} = \left\{ \mathbb{R}^n \xrightarrow{\text{smooth}} \mathbb{R}^{n'} \right\}$:

$$\mathbf{H} \;=\; \mathrm{SmthGrpd}_\infty \;\coloneqq\; \mathrm{Sh}_\infty\big(\mathrm{CartSp}\big) \;\simeq\; \mathrm{Sh}_\infty\big(\mathrm{SmthMfd}\big)$$

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category of **topological spaces**

cartesian closed category of **compactly generated topological spaces**

cart. clsd. & locl. pres. category of **Δ-generated topological spaces**

quasi-topos of **diffeological spaces**

cohesive topos of **smooth sets**

cohesive ∞-topos of **smooth ∞-groupoids**

$$\begin{tikzcd}
\mathrm{TopSpc} \arrow[r, shift right=1.5, "k"'] \arrow[r, phantom, "\perp"] \arrow[rrrrrd, bend right=10] & \mathrm{kTopSpc} \arrow[l, hook', shift right=1.5] \arrow[r, shift right=1.5, "\mathrm{Cdfflg}"'] \arrow[r, phantom, "\perp"] \arrow[rrrrd, bend right=10] & \mathrm{DTopSpc} \arrow[l, hook', shift right=1.5] \arrow[r, hook, shift right=1.5] \arrow[r, phantom, "\perp"] \arrow[rrrd, bend right=10] & \mathrm{DfflSpc} \arrow[l, shift right=1.5, "\mathrm{Dtplg}"'] \arrow[r, hook, "i_{\sharp 0}"'] \arrow[rrd, bend right=10, "\mathrm{Pth}"'] & \mathrm{SmthGrpd}_0 \arrow[r, hook, "i_0"'] & \mathrm{SmthGrpd}_\infty \arrow[d, shift right=2, "\mathrm{Shp}"'] \arrow[d, phantom, "\int"] \\
 & & & & & \mathrm{Grpd}_\infty \arrow[u, shift right=2, "\mathrm{Dsc}"']
\end{tikzcd}$$

shape

underlying homotopy types are preserved

form path ∞-groupoids
(sing. simpl. compl.)

base ∞-topos of **bare ∞-groupoids**

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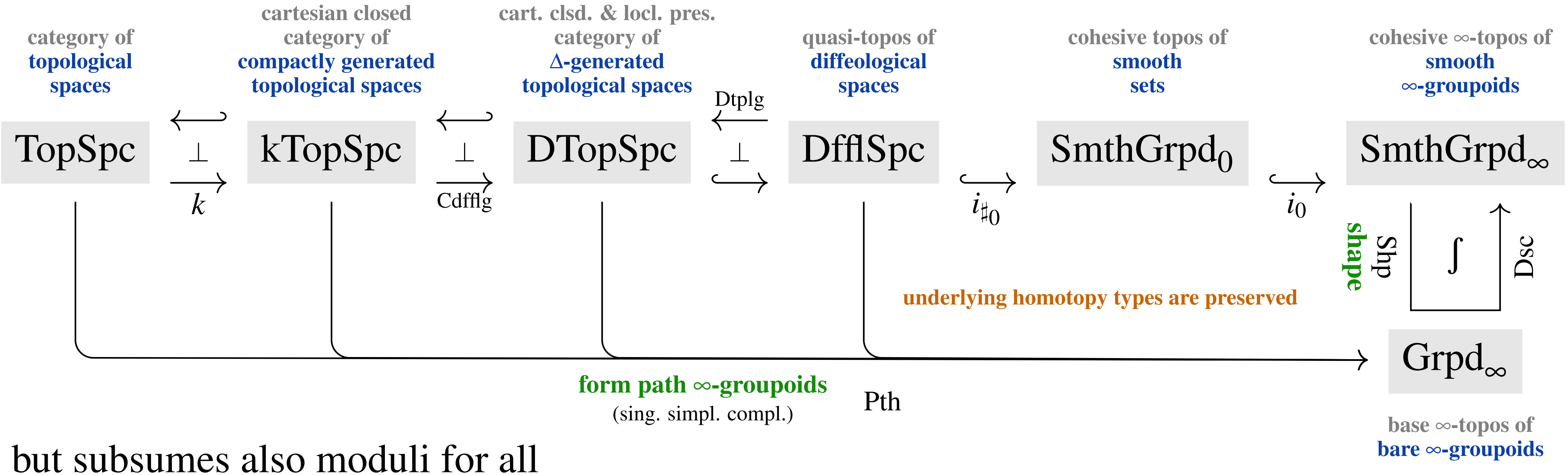

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Pth

base ∞-topos of **bare ∞-groupoids**

but subsumes also moduli for all
higher gauge fields $\leftrightarrow$ differential cohomology [FSS20-Char, §4.3]

in particular for *abelian* higher gauge fields:

$$\mathrm{Spectra}\big(\mathrm{SmthGrpd}_\infty\big) \;=\; \left\{ \begin{array}{c} \text{abelian generalized differential} \\ \text{cohomology theories} \end{array} \right\}$$

[Sc13] [BNV14],
review in [ADH21]

Higher symmetry. [NSS12-∞Bund][SS20-OrbCoh]

∞-Toposes $\mathbf{H}$ know all about *higher symmetries* – i.e. *n-group symmetries* for $n \in \mathbb{N} \sqcup \{\infty\}$):

higher groups

higher symmetry groups
group ∞-stacks

$$\begin{tikzcd}[column sep=huge]
\mathrm{Grp}(\mathbf{H}) \arrow[rr, shift right=2, "\mathbf{B}(-)" description, "\textbf{delooping}"'] \arrow[rr, phantom, "\sim"] && \mathbf{H}^{*/}_{\geq 1} \arrow[ll, shift right=2, "\Omega(-)" description, "\textbf{looping}"'] \\
\mathscr{G} \arrow[rr, mapsto] && \mathbf{B}\mathscr{G} \simeq * /\!\!/ \mathscr{G}
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pointed connected
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higher symmetries

higher $\mathcal{G}$-symmetries
$\mathcal{G}$ ∞-actions

$\mathcal{G}$-gauge field sectors
$\mathcal{G}$-principal ∞-bundles

modulating maps
slice over delooping

$$\begin{tikzcd}[column sep=large] \mathcal{G}\,\mathrm{Act}(\mathbf{H}) \arrow[r, "\sim", "(-)/\!\!/G"'] & \mathcal{G}\,\mathrm{PrnBdl}(\mathbf{H}) & \mathbf{H}_{/\mathbf{B}\mathcal{G}} \arrow[l, "\sim", "\mathrm{fib}"'] \\ G \circlearrowright P \arrow[r, mapsto] & \left(\begin{array}{c} P \\ \downarrow \\ P/\!\!/\mathcal{G} \end{array}\right) & \left(\begin{array}{c} P/\!\!/\mathcal{G} \\ \downarrow \\ \mathbf{B}\mathcal{G} \end{array}\right) \arrow[l, mapsto] \end{tikzcd}$$

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G \circlearrowright P \arrow[rr, mapsto] && \left(\begin{array}{c} P \\ \downarrow \\ P /\!\!/ \mathscr{G} \end{array}\right) && \left(\begin{array}{c} P /\!\!/ \mathscr{G} \\ \downarrow \\ \mathbf{B}\mathscr{G} \end{array}\right) \arrow[ll, mapsto]
\end{tikzcd}$$

Technical side remark. – The correspondence is enacted by homotopy cartesian squares of this form:

$\mathscr{G}$-principal
bundle

universal
$\mathscr{G}$-principal bundle

$$\begin{tikzcd}
\text{action } \mathscr{G} \circlearrowright P \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] && * \simeq \mathscr{G} /\!\!/ \mathscr{G} \arrow[d, "\mathrm{pt}_{\mathbf{B}\mathscr{G}}"] \\
P /\!\!/ \mathscr{G} \arrow[rr, "\vdash P", "\text{cocycle}"'] && \mathbf{B}\mathscr{G}
\end{tikzcd}$$

action quotient-
/ base-stack

universal
moduli stack

Higher moduli stacks.

[SS20-OrbCoh]

Maps *out* of an ∞-stack $\mathscr{X} \to \mathscr{A}$ encode $\mathscr{A}$-moduli on $\mathscr{X}$:

$$\underset{\textbf{mapping }\infty\textbf{-stack}}{\overset{\substack{\textbf{higher moduli stack}\\ \textbf{of }\mathscr{A}\textbf{-structures on }\mathscr{X}}}{\mathrm{Map}\big(\mathscr{X}, \mathscr{A}\big)}} \;=\; \left\{ \begin{tikzcd}[column sep=huge] \mathscr{X} \arrow[rr, bend left=70, "\textbf{modulating map}", ""{name=U, below}] \arrow[rr, bend right=70, "\textbf{modulating map}"', ""{name=D}] & & \mathscr{A} \arrow[from=U, to=D, Rightarrow, bend right=40, "\textbf{homotopy}"', ""{name=L, right}] \arrow[from=U, to=D, Rightarrow, bend left=40, "\textbf{homotopy}", ""{name=R, left}] \arrow[from=L, to=R, Rrightarrow, "\textbf{higher}", "\textbf{homotopies}"'] \end{tikzcd} \right\}$$

Ex.: $\mathrm{Map}(\mathscr{X}, \mathbf{B}\mathscr{G})$ is the moduli ∞-stack of $\mathscr{G}$-principal ∞-bundles on $\mathscr{X}$ (high. gauge field sect.)

Maps *out* of an ∞-stack $\mathscr{X} \to \mathscr{A}$ encode $\mathscr{A}$-moduli on $\mathscr{X}$:

higher moduli stack of $\mathscr{A}$-structures on $\mathscr{X}$

$$\mathrm{Map}(\mathscr{X}, \mathscr{A}) = \left\{ \begin{tikzcd}[column sep=huge] \mathscr{X} \arrow[rr, bend left=60, "\text{modulating map}"{name=U}] \arrow[rr, bend right=60, "\text{modulating map}"'{name=D}] & & \mathscr{A} \arrow[from=U, to=D, Rightarrow, shift right=4ex, bend right, "\text{homotopy}"'] \arrow[from=U, to=D, Rightarrow, shift left=4ex, bend left, "\text{homotopy}"] \arrow[from=U, to=D, phantom, "\overset{\text{higher}}{\underset{\text{homotopies}}{\Rrightarrow}}"] \end{tikzcd} \right\}$$

mapping ∞-stack

Ex.: $\mathrm{Map}(\mathscr{X}, \mathbf{B}\mathscr{G})$ is the moduli ∞-stack of $\mathscr{G}$-principal ∞-bundles on $\mathscr{X}$ (high. gauge field sect.)

<u>Fund. Thm.</u> [Lu09]: for $\mathscr{B} \in \mathbf{H}$ also $\mathbf{H}_{/\mathscr{B}}$ is an ∞-topos, whose objects are maps $\mathscr{X} \xrightarrow{c} \mathscr{B}$, with

$$\mathrm{Map}\big((\mathscr{X}, c), (\mathscr{A}, c')\big)_{\mathscr{B}} = \left\{ \begin{tikzcd} \mathscr{X} \arrow[rr, bend left=30, ""{name=U}] \arrow[rr, bend right=30, ""'{name=D}] \arrow[dr, "c"', ""{name=L}] & & \mathscr{A} \arrow[dl, "c'", ""'{name=R}] \arrow[from=U, to=D, Rightarrow, shift right=2ex, bend right] \arrow[from=U, to=D, Rightarrow, shift left=2ex, bend left] \arrow[from=U, to=D, phantom, "\Rrightarrow"] \\ & \mathscr{B} & \arrow[from=D, to=L, Rightarrow] \arrow[from=U, to=L, Rightarrow, dashed] \end{tikzcd} \right\}.$$

slice mapping ∞-stack

Ex.: $\mathrm{Map}(\mathscr{X}, \mathbf{B}\mathscr{G})_{\mathbf{B}\mathscr{Q}}$ is moduli ∞-stack of $\mathscr{G}$*-structures* on $\mathscr{Q}$-bundles (gravity/metrics, below)

Higher symmetry – Example: GS-mechanism. [SSS09][FSS12][FSS20-M5Str][SS20-M5GS]

$$\text{Under}\quad \overset{\textbf{Lie groups}}{\mathrm{Grp}(\mathrm{SmthMfd})} \hookrightarrow \overset{\textbf{diffeological groups}}{\mathrm{Grp}(\mathrm{DfflSpc})} \xhookrightarrow{\mathrm{Grp}(i_{0,\sharp_0})} \overset{\textbf{smooth } \infty\textbf{-groups}}{\mathrm{Grp}(\mathrm{SmthGrpd}_\infty)}$$

mapping stack into delooping...	**is moduli stack of**:

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Under $$\begin{tikzcd} \text{Lie groups} & \text{diffeological groups} & \text{smooth } \infty\text{-groups} \\ \mathrm{Grp}(\mathrm{SmthMfd}) \arrow[r, hook] & \mathrm{Grp}(\mathrm{DfflSpc}) \arrow[r, hook, "\mathrm{Grp}(i_{0,\sharp_0})"] & \mathrm{Grp}(\mathrm{SmthGrpd}_\infty) \end{tikzcd}$$

mapping stack into delooping...	**is moduli stack of:**
$\mathrm{Map}(X, \mathbf{B}\mathrm{U}(1))$	circle bundles

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Lie groups **diffeological groups** **smooth ∞-groups**

Under $\mathrm{Grp}(\mathrm{SmthMfd}) \hookrightarrow \mathrm{Grp}(\mathrm{DfflSpc}) \xhookrightarrow{\mathrm{Grp}(i_{0,\sharp_0})} \mathrm{Grp}(\mathrm{SmthGrpd}_\infty)$

mapping stack into delooping...	**is moduli stack of**:
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$\mathrm{Map}\big(\mathrm{X}, \mathbf{B}^{p+1}\mathrm{U}(1)\big)$	bundle p-gerbes

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$\mathrm{Map}\big(X, \mathbf{B}\mathrm{Spin}(n)\big)$	Spin-bundles
$\mathrm{Map}\big((X, \mathbf{c}_1), (\mathbf{B}\mathrm{SO}(n), \mathbf{w}_2)\big)_{\mathbf{B}\mathrm{U}(1)}$ $\simeq \mathrm{Map}\big(X, \mathbf{B}\mathrm{Spin}^{\mathbf{c}_1}(n)\big)$	Spin^c-bundles

Higher symmetry – Example: GS-mechanism. [SSS09][FSS12][FSS20-M5Str][SS20-M5GS]

Under

$$\begin{tikzcd} \text{Lie groups} & \text{diffeological groups} & \text{smooth } \infty\text{-groups} \\ \mathrm{Grp}(\mathrm{SmthMfd}) \arrow[r, hook] & \mathrm{Grp}(\mathrm{DfflSpc}) \arrow[r, hook, "\mathrm{Grp}(i_{0,\sharp_0})"] & \mathrm{Grp}(\mathrm{SmthGrpd}_\infty) \end{tikzcd}$$

mapping stack into delooping...	**is moduli stack of**:
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$\mathrm{Map}\big((\mathrm{X}, 0), (\mathbf{B}\mathrm{Spin}(n), \tfrac{1}{2}p_1)\big)_{\mathbf{B}^3\mathrm{U}(1)}$ $\simeq \mathrm{Map}\big(\mathrm{X}, \mathbf{B}\mathrm{String}(n)\big)$	String 2-bundles

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$\mathrm{Map}\big((\mathrm{X}, \mathbf{c}_2), (\mathbf{B}\mathrm{Spin}(n), \tfrac{1}{2}\mathbf{p}_1)\big)_{\mathbf{B}^3\mathrm{U}(1)}$ $\simeq\ \mathrm{Map}\big(\mathrm{X}, \mathbf{B}\mathrm{String}^{\mathbf{c}_2}(n)\big)$	twisted String 2-bundles (heterotic Green-Schwarz mech.)

Higher symmetry – Example: GS-mechanism. [SSS09][FSS12][FSS20-M5Str][SS20-M5GS]

$$\begin{tikzcd}
 & \text{Lie groups} & \text{diffeological groups} & \text{smooth } \infty\text{-groups} \\
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\end{tikzcd}$$

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Rem.: Different smooth ∞-groups $\mathscr{G}$ may have same shape $\int\mathscr{G}$ discrete ∞-group, e.g:

$$\begin{tikzcd}
\mathbf{B}\mathrm{PU}_\omega \arrow[r, "\eta^{\int}"] & B^3\mathbb{Z} \arrow[d, phantom, "\wr|"] & \mathbf{B}^2\mathrm{U}(1) \arrow[l, "\eta^{\int}"'] \\
 & K(\mathbb{Z},3) &
\end{tikzcd}$$

Higher symmetry – Example: Cohomotopy. Exs. 2.10, 2.16, 4.16 in [FSS20-Char]

Recall that *every connected* space is the classifying space of its loop ∞-group.

E.g., the 4-sphere encodes a rich ∞-higher symmetry

$$S^4 \;\simeq\; B(\Omega S^4) \;\;\in\; \mathrm{Grpd}_\infty\,, \qquad \Omega S^4 \;\in\; \mathrm{Grp}(\mathrm{Grpd}_\infty)\,.$$

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The corresponding moduli are classified by unstable/non-abelian **Cohomotopy**:

$$\underbrace{\pi^4(\mathrm{X})}_{\text{4-Cohomotopy}} := \pi_0 \mathrm{Map}\big(\mathrm{X}, S^4\big) \simeq \pi_0 \mathrm{Map}\big(\mathrm{X}, B(\Omega S^4)\big) \simeq \underbrace{H^1\big(\mathrm{X}; \Omega S^4\big)}_{\substack{\text{non-abelian cohomology} \\ \text{with coefficients in loop ∞-group}}}$$

Higher symmetry – Example: Cohomotopy.

Exs. 2.10, 2.16, 4.16 in [FSS20-Char]

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4-Cohomotopy

non-abelian cohomology

with coefficients in loop ∞-group

Incidentally, on 10-manifolds X^{10}, 4-Cohomotopy is stably equivalent to tmf^4 (cf below):

$$\begin{tikzcd}
S^4 \arrow[r] & \Omega^\infty \Sigma^\infty S^4 = \Omega^\infty \mathbb{S}^4 \arrow[r, "\beta", "\sim_{\leq 10}"'] & \Omega^\infty \mathrm{tmf}^4 \\
\pi^4\big(\mathbb{R}^{0,1} \times \mathrm{X}^{10}\big) \arrow[r, "\text{stabilization/ abelianization}"] & \mathbb{S}^4\big(\mathbb{R}^{0,1} \times \mathrm{X}^{10}\big) \arrow[r, "\sim"] & \mathrm{tmf}^4\big(\mathbb{R}^{0,1} \times \mathrm{X}^{10}\big)
\end{tikzcd}$$

Boardman homomorphism

unstable/ non-abelian 4-Cohomotopy

stable/ abelianized 4-Cohomotopy

elliptic 4-cohomology

Higher symmetry – Example: Cohomotopy. Exs. 2.10, 2.16, 4.16 in [FSS20-Char]

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4-Cohomotopy

non-abelian cohomology

with coefficients in loop ∞-group

- continue with more details on cohesive homotopy theory

- skip ahead to Quantum Charge of M-branes via Hypothesis H

Cohesive homotopy theory of Super ∞-stacks. [KhSc17], §3.1.2-3 in [SS20-OrbCoh]

Higher geometry locally modeled on

$$\mathrm{SupCartSp} \;=\; \Big\{ \overset{\text{super-Cartesian space}}{\mathbb{R}^{n|q}} \times \overset{\text{infinitesimal disk}}{\mathbb{D}} \xrightarrow{\text{smooth}} \mathbb{R}^{n'|q'} \times \mathbb{D}' \Big\}$$

$$\begin{tikzcd} \mathrm{Grpd}_\infty \arrow[r, shift right=2, "\mathrm{Dsc}"'] \arrow[r, phantom, "\int"] & \mathrm{SmthGrpd}_\infty \arrow[l, shift right=2, "\mathrm{Shp}"'] \arrow[r, shift right=2, "\mathrm{InfDsc}"'] \arrow[r, phantom, "\Im"] & \mathrm{SupSmthGrpd}_\infty \arrow[l, shift right=2, "\mathrm{InfShp}"'] \end{tikzcd} \;:=\; \underset{\text{super smooth }\infty\text{-groupoids}}{\mathrm{Sh}_\infty(\mathrm{SupCartSp})}$$

shape **formally étale**

lifts all fundamentals of differential geometry to higher geometry of super ∞-stacks, e.g.:

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Higher geometry locally modeled on

super-Cartesian space

infinitesimal disk

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$$\begin{tikzcd} \mathrm{Grpd}_\infty \arrow[r, shift right=2, "\mathrm{Dsc}"'] \arrow[r, phantom, "\int"] & \mathrm{SmthGrpd}_\infty \arrow[l, shift right=2, "\mathrm{Shp}"'] \arrow[r, shift right=2, "\mathrm{InfDsc}"'] \arrow[r, phantom, "\Im"] & \mathrm{SupSmthGrpd}_\infty \arrow[l, shift right=2, "\mathrm{InfShp}"'] \end{tikzcd} \;:=\; \mathrm{Sh}_\infty(\mathrm{SupCartSp})$$

shape **formally étale** **super smooth ∞-groupoids**

lifts all fundamentals of differential geometry to higher geometry of super ∞-stacks, e.g.:

A morphism of super ∞-stacks is a **local diffeomorphism** or *formally étale* if its $\Im$-unit is homotopy cartesian:

$$\begin{tikzcd} X \arrow[r, "\eta^{\Im}"] \arrow[d, "f"', "\text{ét}"] \arrow[dr, phantom, "\text{(pb)}"] & \Im X \arrow[d, "\Im f"] \\ Y \arrow[r, "\eta^{\Im}"] & \Im Y \end{tikzcd}$$

The **infinitesimal neighbourhood** around point x in a super ∞-stack X is the x-fiber of the $\Im$-unit:

$$\begin{tikzcd} \mathbb{D}_x X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & * \arrow[d, "\Im x"] \\ X \arrow[r, "\eta^{\Im}"] & \Im X \end{tikzcd}$$

$\mathscr{G}$-Structured super étale ∞-stacks. §4.2 in [SS20-OrbCoh]

For $V \in \mathrm{Grp}(\mathrm{SupSmthGrpd}_\infty)$ a super group stack such as super-Minkowski $V = \mathbb{R}^{d,1|\mathbf{N}}$:

Def.: A V**-fold** is an étale ∞-stack locally diffeomorphic to $V \overset{\text{ét}}{\twoheadleftarrow} \overset{\textbf{\textit{V}-atlas}}{U} \overset{\text{eff.epi}}{\twoheadrightarrow} \overset{\textbf{\textit{V}-fold}}{X}$.

(This cohesive **higher Cartan geometry** is formalized in Modal Homotopy Type Theory: [Sc15, §3][Cherubini (né Wellen) 17].)

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Thm.: A V-fold X naturally has a **frame ∞-bundle** $X \xrightarrow{\vdash \mathrm{Fr}(X)} \mathbf{B}\mathrm{Aut}(\mathbb{D}_{\mathrm{e}} V)$.

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Def.: For $\mathrm{str} : \mathscr{G} \longrightarrow \mathrm{Aut}(\mathbb{D}_{\mathrm{e}} V)$ we have
moduli of $\mathscr{G}$**-structures** on V-folds X:

$$\mathrm{Map}\big(X, \mathbf{B}\mathscr{G}\big)_{\mathbf{B}\mathrm{Aut}(\mathbb{D}_{\mathrm{e}} V)} = \left\{ \begin{tikzcd} X \arrow[rr, dashed, "\tau"] \arrow[dr, "\vdash \mathrm{Fr}(X)"'] & {} \arrow[d, Rightarrow, "\sim"', "\text{vielbein}"] & \mathbf{B}\mathscr{G} \arrow[dl, "\mathbf{B}\mathrm{str}"] \\ & \mathbf{B}\mathrm{Aut}(\mathbb{D}_{\mathrm{e}} V) & \end{tikzcd} \right\}$$

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$$\mathrm{Map}\big(X, \mathbf{B}\mathscr{G}\big)_{\mathbf{B}\mathrm{Aut}(\mathbb{D}_e V)} = \left\{ \begin{tikzcd} X \arrow[rr, dashed, "\tau"] \arrow[dr, "\vdash \mathrm{Fr}(X)"'] & \arrow[dl, phantom, "\overset{\sim}{\Swarrow}\ \text{vielbein}"] & \mathbf{B}\mathscr{G} \arrow[dl, "\mathbf{B}\mathrm{str}"] \\ & \mathbf{B}\mathrm{Aut}(\mathbb{D}_e V) & \end{tikzcd} \right\}$$

So for X a $\mathbb{R}^{d,1|\mathbf{N}}$-fold we have moduli of **super-vielbein fields**:

$$\mathrm{Map}\big(X, \mathbf{B}\mathrm{Spin}(d)\big)_{\mathbf{B}\mathrm{GL}(d,1|\mathbf{N})} = \left\{ \begin{tikzcd} X \arrow[rr, dashed] \arrow[dr, "\vdash \mathrm{Fr}(\mathscr{X})"'] & \arrow[dl, phantom, "\overset{\sim}{\Swarrow}\ \text{vielbein}"] & \mathbf{B}\mathrm{Spin}(d) \arrow[dl] \\ & \mathbf{B}\mathrm{GL}(d|\mathbf{N}) & \end{tikzcd} \right\}$$

(This cohesive **higher Cartan geometry** is formalized in Modal Homotopy Type Theory: [Sc15, §3][Cherubini (né Wellen) 17].)

Singular-Cohesive Homotopy Theory of orbi-∞-stacks. [Rezk14][SS20-OrbCoh]

Higher geometry locally modeled on orbi-singularities:

$$\mathrm{Snglrt} \;:=\; \Big\{ \overset{G}{\rotatebox[origin=c]{70}{$\prec$}} \;\Big\vert\; G \text{ fin. group} \Big\} \quad \text{with} \quad \mathrm{Map}\big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, \overset{G}{\rotatebox[origin=c]{70}{$\prec$}} \big) \;=$$

$$\begin{tikzcd} \Bigg\{ & \overset{K}{\rotatebox[origin=c]{70}{$\prec$}} \arrow[rr, bend left=40, "\text{group homomorph.}"{name=U}] \arrow[rr, bend right=40, "\text{group homomorph.}"'{name=D}] & & \overset{G}{\rotatebox[origin=c]{70}{$\prec$}} & \Bigg\} \arrow[from=U, to=D, Rightarrow, "\wr"', "\text{conjugation}"] \end{tikzcd}$$

$$\mathbf{H} \;=\; \underset{\text{orbi-singular super-}\infty\text{-stacks}}{\mathrm{GloSupSmthGrpd}_\infty} \;:=\; \mathrm{Sh}_\infty\big(\mathrm{SupCartSp} \times \mathrm{Snglrt} \big)$$

faithfully subsumes proper equivariant homotopy theory:

Singular-Cohesive Homotopy Theory of orbi-∞-stacks. [Rezk14][SS20-OrbCoh]

Higher geometry locally modeled on orbi-singularities:

$$\mathrm{Snglrt} \;:=\; \Big\{ \overset{G}{\prec} \;\Big\vert\; G \text{ fin. group} \Big\} \quad \text{with} \quad \mathrm{Map}\big(\overset{K}{\prec}, \overset{G}{\prec}\big) = \left\{ \begin{tikzcd}[column sep=huge] \overset{K}{\prec} \arrow[r, bend left=50, "\text{group homomorph.}"{name=U}] \arrow[r, bend right=50, "\text{group homomorph.}"'{name=D}] & \overset{G}{\prec} \arrow[from=U, to=D, Rightarrow, shorten=2mm, "\text{conjugation}", "\sim"'] \end{tikzcd} \right\}$$

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category of topological G-spaces

∞-topos of smooth G-∞-actions

slice of cohesive ∞-topos of smooth ∞-groupoids over $\mathbf{B}G$

slice of singular-cohesive ∞-topos of G-orbi-singular ∞-groupoids

$$\begin{tikzcd}[column sep=large]
G\,\mathrm{Act}(\mathrm{kTopSpc}) \arrow[r, "\mathrm{Cdfflg}"] \arrow[drrr, rounded corners] & G\,\mathrm{Act}(\mathrm{SmthGrpd}_\infty) \arrow[r, "(-)/\!\!/G", "\sim"'] \arrow[drr, rounded corners, "\text{underlying proper G-equivariant homotopy type}"'] & \mathrm{SmthGrpd}_{\infty/\mathbf{B}G} & \mathrm{GloSmthGrpd}_{\infty/\overset{G}{\prec}} \arrow[loop left, "\mathrm{Smth}", "\mathrm{Snglt}"', "\prec"{description}] \arrow[loop right, "G\mathrm{OrbSpc}", "G\mathrm{OrbSmth}"', "\cup_{\overset{G}{\prec}}"{description}] \arrow[d, shift right=3ex, "\mathrm{Shp}"'] \arrow[d, phantom, "\int"] & G\mathrm{SmthGrpd}_\infty \arrow[d, "\mathrm{Shp}"] \\
 & & & \mathrm{GloGrpd}_{\infty/\overset{G}{\prec}} \arrow[u, shift right=3ex, "\mathrm{Dsc}"'] \arrow[r, "G\mathrm{OrbSmth}"'] & G\mathrm{Grpd}_\infty
\end{tikzcd}$$

orbi-singular

shape

G-orbi-spatial

slice of base ∞-topos of global equivariant homotopy theory

sub-∞-topos of proper G-equivariant homotopy theory

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$$\begin{tikzcd}[column sep=large, row sep=large]
G\,\mathrm{Act}(\mathrm{kTopSpc}) \arrow[r, "\mathrm{Cdfflg}"] \arrow[drrr, rounded corners, "\text{underlying proper } G\text{-equivariant homotopy type}"'] & G\,\mathrm{Act}(\mathrm{SmthGrpd}_\infty) \arrow[r, "(-)/\!\!/G", "\sim"'] \arrow[drr, rounded corners] & \mathrm{SmthGrpd}_{\infty/\mathbf{B}G} \arrow[r, shift right, "\mathrm{Snglt}"', "\text{orbi-singular}"] \arrow[r, phantom, "\prec"] & \mathrm{GloSmthGrpd}_{\infty/\underset{\prec}{G}} \arrow[l, shift right, "\mathrm{Smth}"'] \arrow[d, shift right=2, "\mathrm{Shp}"', "\text{shape}"] \arrow[d, phantom, "\int"] \arrow[r, shift right, "G\mathrm{OrbSmth}"', "G\text{-orbi-spatial}"] \arrow[r, phantom, "\cup_{\underset{\prec}{G}}"] & G\mathrm{SmthGrpd}_\infty \arrow[l, shift right, "G\mathrm{OrbSpc}"'] \arrow[d, "\mathrm{Shp}"] \\
& & & \mathrm{GloGrpd}_{\infty/\underset{\prec}{G}} \arrow[u, shift right=2, "\mathrm{Dsc}"'] \arrow[r, "G\mathrm{OrbSmth}"'] & G\mathrm{Grpd}_\infty
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slice of base ∞-topos of **global equivariant homotopy theory**

sub-∞-topos of **proper G-equivariant homotopy theory**

Thm. (§4.1 in [SS20-OrbCoh])

Good orbifolds covered by $G \circlearrowright \mathrm{X}$ are equivalently $\prec(\mathrm{X}/\!\!/G) \;\in\; \mathbf{H} = \mathrm{GloSmthGrpd}_\infty$ and their proper-equivariant homotopy type is:

$$\int \prec(\mathrm{X}/\!\!/G) \;\simeq\; G\mathrm{OrbSpc}\Big(\textstyle\int\big(\mathrm{X}^{(-)}\big)\Big) \;\in\; G\mathrm{Grpd}_\infty \xhookrightarrow{\mathrm{Dsc}G\mathrm{OrbSpc}} \mathbf{H}_{/\underset{\prec}{G}}\,.$$

Equivariant classifying spaces. [SS22-EquBun]

Theorem.

If $G \circlearrowright \Gamma$ is a G-equivariant Hausdorff-topological group with $\int\Gamma$ truncated, then

$$\underset{\substack{\text{equivariant}\\ \text{classifying shape}}}{B_G\Gamma} \;:=\; \underset{G\text{-orbi-spatial}}{\cup_{\substack{G\\ \curlyvee}}}\; \underset{\text{shape}}{\int}\; \underset{\text{orbi-singular}}{\curlyvee}\; \underset{\text{delooping}}{\mathbf{B}}\; \Gamma \;\in\; \mathbf{H}\,.$$

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classifies G-equivariant Γ-principal bundles on G-orbifolds $\mathscr{X} \simeq \curlyvee(\mathrm{X} /\!\!/ G) \in \mathbf{H}_{/\overset{G}{\curlyvee}}$:

$$(G\,\mathrm{Equv}\,\Gamma\,\mathrm{PrnBdl}_{\mathrm{X}})_{/\sim_{\mathrm{iso}}} \;\simeq\; \tau_0\,\mathrm{Map}(\mathscr{X}, B_G\Gamma)_{\overset{G}{\curlyvee}}$$

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and its equivariant homotopy groups are given by non-abelian group cohomology:

$$\underset{\substack{\text{equivariant}\\ \text{homotopy groups}}}{\pi^H_\bullet} \big(\underset{\substack{\text{equivariant}\\ \text{classifying shape}}}{B_G\Gamma}\big) \;\simeq\; \underset{\substack{\text{non-abelian}\\ \text{group cohomology}}}{H^{1-\bullet}_{\mathrm{Grp}}} \big(\underset{\text{isotropy group}}{H}; \underset{\substack{\text{shape of}\\ \text{structure group}}}{\int\Gamma}\big)$$

B-fields on orbi-orientifolds. [SS22-EquBun]

Specifically, for

$$1 \to N \hookrightarrow G \twoheadrightarrow \mathbb{Z}_2 \to 1$$

and

$$\mathbb{Z}_2 \curvearrowright \mathrm{PU}^{\mathrm{gr}}_{\boldsymbol{\omega}} \;\in\; G\,\mathrm{Act}\big(\mathrm{Grp}(\mathrm{kTopSpc})\big)$$

the graded projective unitary group acted on by complex conjugation, the G-orbi-space

$$B_G\big(\mathrm{PU}^{\mathrm{gr}}_{\boldsymbol{\omega}}\big) \;\in\; \mathbf{H}$$

classifies type **IIA B-fields on G-orbi-orientifolds**

$$\left\{ \begin{array}{c} \text{type IIA } B_2\text{-fields on} \\ G\text{-orbi-orientifold } \mathscr{X} \end{array} \right\}_{\sim_{\mathrm{gauge}}} \;\simeq\; \tau_0 \mathrm{Map}\Big(\mathscr{X}\,, B_G\big(\mathrm{PU}^{\mathrm{gr}}_{\boldsymbol{\omega}}\big)\Big)_{\underset{\curvearrowright}{G}}\,.$$

with $\quad \pi^H_n\Big(B_G\big(\mathrm{PU}_{\omega}\big)\Big) \;\simeq\; H^{3-n}_{\mathrm{Grp}}\big(H;\mathbb{Z}\big)\,,$ $\quad$ reproducing [UrLü14, Thm. 15.17].

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Philosophical question:

But *why* coefficients like $B_G\mathrm{PU}_{\omega}$? *Are there god-given coefficients?* $\rightsquigarrow$

God-given coefficients – Tate spheres. Def. 5.19 in [SS20-OrbCoh]

For any line object $\mathbb{A}^1$ there are the *Tate spheres* (e.g. [VRO07, Rem. 2.22])

$$S^n_{\mathrm{Tate}} \;:=\; \mathrm{cof}\big(\mathbb{A}^n \backslash \{0\} \hookrightarrow \mathbb{A}^n\big) \quad \in \; \mathbf{H}$$

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Specifically for $\mathbb{A}^1 := \mathbb{R}^1 \in \mathrm{SmthGrpd}_\infty$ (incidentally $\int \simeq \mathrm{Loc}^{\mathbb{R}^1} \;:\; \mathbf{H} \to \mathbf{H}$)
we have the **smooth Tate spheres**

$$S^n_{\mathrm{Tate}} \;:=\; \mathrm{cof}\big(\mathbb{R}^n \backslash \{0\} \hookrightarrow \mathbb{R}^n\big) \quad \in \; \mathbf{H}\,.$$

Their shape is that of the ordinary n-spheres ([SS20-OrbCoh, Ex. 5.21]):

$$\int S^n_{\mathrm{Tate}} \;\simeq\; S^n \quad \in \; \mathrm{Grpd}_\infty \xhookrightarrow{\mathrm{Dsc}} \;\mathbf{H}$$

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More generally, for any

$$G \curvearrowright V \quad \in \; G\mathrm{Act}(\mathrm{VectorSpaces}_{\mathbb{R}}) \hookrightarrow G\mathrm{Act}(\mathrm{SmthMfd}) \hookrightarrow G\mathrm{Act}(\mathbf{H})$$

we have the **orbi-smooth V-Tate spheres** ([SS20-OrbCoh, Ex. 5.27])

$$\int \curlyvee \big(S^V_{\mathrm{Tate}} /\!\!/ G\big) \quad \in \; \mathbf{H}_{/\underset{\curlyvee}{G}}\,.$$

Example: The ADE-equivariant 4-sphere. §5.1 in [HSS18-ADE]; §3 in [SS19-TadCnc]

The G_{ADE}-equivariant Tate $\mathbf{4}$-sphere has equivariant homotopy type of the $\mathbf{4}$-representation sphere:

$$\textstyle\int \prec \big(S^{\mathbf{4}}_{\mathrm{Tate}} /\!\!/ G_{\mathrm{ADE}} \big) \underset{\substack{\text{stereogr.} \\ \text{project.}}}{\simeq} S\Big(\mathbb{R} \oplus \overset{\overset{G_{\mathrm{ADE}}}{\curvearrowright}}{\mathbb{H}} \Big) \;\in\; G_{\mathrm{ADE}}\mathrm{Grpd}_\infty \xhookrightarrow{\mathrm{Dsc}\; G_{\mathrm{ADE}}\mathrm{OrbSpc}} \mathbf{H}_{/\prec G_{\mathrm{ADE}}}$$

Example: The ADE-equivariant 4-sphere. §5.1 in [HSS18-ADE]; §3 in [SS19-TadCnc]

Consider the left multiplication action of $\mathrm{Sp}(1) = S(\mathbb{H})$ on the quaternions $\mathbb{H}$:

$$\mathrm{Sp}(1) \circlearrowright \mathbb{H} \;\simeq\; \mathrm{SU}(2)_L \circlearrowright \mathbb{C}^2 \;\simeq\; \mathrm{Spin}(3)_L \circlearrowright \mathbb{R}^4 .$$

The finite subgroups have a famous ADE-classification:

Label	$G_{\mathrm{ADE}} \underset{\mathrm{fin}}{\subset} \mathrm{SU}(2)$	**Order**	**Name**
$\mathbb{A}_n$	$\mathbb{Z}_{n+1}$	n	Cyclic
$\mathbb{D}_{n+4}$	$2\mathrm{D}_{n+2}$	$4(n+2)$	Binary dihedral
$\mathbb{E}_6$	2T	24	Binary tetrahedral
$\mathbb{E}_7$	2O	48	Binary octahedral
$\mathbb{E}_8$	2I	120	Binary icosahedral

Denote the restricted representation by $\mathbf{4} \;:=\; G_{\mathrm{ADE}} \circlearrowright \mathbb{R}^4 \quad \in\; \mathrm{RO}(G_{\mathrm{ADE}})$.

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Example: Super-Minkowski orbifolds.

Thm. 4.3 in [HSS18-ADE]

Theorem. Classification of subgroup actions of $\mathrm{Pin}^+(10,1) \curvearrowright \mathbb{R}^{10,1|\mathbf{32}}$ which fix $\geq 1/4$th of $\mathbf{32}$ such that all non-trivial subgroups have the same bosonic fixed locus:

Black brane species	**BPS**	**Fixed locus in $\mathbb{R}^{10,1\vert\mathbf{32}}$**	**Type of singularity in $\mathbb{R}^{10,1}$**	**Intersection law**							
				$\simeq$	$\mathbb{R}^{1,1}$	$\oplus$	$\mathbb{R}^4$	$\oplus$	$\mathbb{R}^4$	$\oplus$	$\mathbb{R}^1$
Elementary brane species			**Simple singularities**								
MO9	$1/2$	$\mathbb{R}^{9,1\vert\mathbf{16}}$	$\mathbb{Z}_2$	$=$	——						$(\mathbb{Z}_2)_{\mathrm{HW}}$
MO5	$1/2$	$\mathbb{R}^{5,1\vert 2\cdot\mathbf{8}}$	$\mathbb{Z}_2$	$\overset{\Delta}{\subset}$	——				$(\mathbb{Z}_2)_R$	$\times$	$(\mathbb{Z}_2)_{\mathrm{HW}}$
MO1	$1/2$	$\mathbb{R}^{1,1\vert 16\cdot\mathbf{1}}$	$\mathbb{Z}_2$	$\overset{\Delta}{\subset}$	——		$(\mathbb{Z}_2)_L$	$\times$	$(\mathbb{Z}_2)_R$	$\times$	$(\mathbb{Z}_2)_{\mathrm{HW}}$
MK6	$1/2$	$\mathbb{R}^{6,1\vert\mathbf{16}}$	$\mathbb{Z}_{n+1}, 2\mathbb{D}_{n+2},$ $2T, 2O, 2I$	$\subset$	——				$\mathrm{SU}(2)_R$		——
M2	$1/2 = 8/16$	$\mathbb{R}^{2,1\vert 8\cdot\mathbf{2}}$	$\mathbb{Z}_2$	$\overset{\Delta}{\subset}$	——		$\mathrm{SU}(2)_L$	$\times$	$\mathrm{SU}(2)_R$		——
M2	$6/16$	$\mathbb{R}^{2,1\vert 6\cdot\mathbf{2}}$	$\mathbb{Z}_{n+3}$	$\overset{\Delta}{\subset}$	——		$\mathrm{SU}(2)_L$	$\times$	$\mathrm{SU}(2)_R$		——
M2	$5/16$	$\mathbb{R}^{2,1\vert 5\cdot\mathbf{2}}$	$2\mathbb{D}_{n+2},$ $2T, 2O, 2I$	$\overset{\Delta}{\subset}$	——		$\mathrm{SU}(2)_L$	$\times$	$\mathrm{SU}(2)_R$		——
M2	$1/4 = 4/16$	$\mathbb{R}^{2,1\vert 4\cdot\mathbf{2}}$	$2\mathbb{D}_{n+2},$ $2O, 2I$	$\overset{(\mathrm{id},\tau)}{\subset}$	——		$\mathrm{SU}(2)_L$	$\times$	$\mathrm{SU}(2)_R$		——

$$\begin{tikzcd}
\mathbb{R}^{10,1} & \simeq_{\mathbb{R}} & \mathbb{R}^{1,1} & \oplus & \mathbb{R}^4 \arrow[loop above, "\mathrm{SU}(2)_L"] & \oplus & \mathbb{R}^4 \arrow[loop above, "\mathrm{SU}(2)_R"] & \oplus & \mathbb{R} \arrow[loop above, "(\mathbb{Z}_2)_{\mathrm{HW}}"]
\end{tikzcd}$$

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MO1	$1/2$	$\mathbb{R}^{1,1\vert 16\cdot\mathbf{1}}$	$\mathbb{Z}_2$	$\overset{\Delta}{\subset}$	——		$(\mathbb{Z}_2)_L$	$\times$	$(\mathbb{Z}_2)_R$	$\times$	$(\mathbb{Z}_2)_{\mathrm{HW}}$
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$$\begin{tikzcd}
\mathbb{R}^{10,1} \arrow[r, phantom, "\simeq_{\mathbb{R}}"] & \mathbb{R}^{1,1} \oplus \mathbb{R}^4 \arrow[loop above, "\mathrm{SU}(2)_L"] \oplus \mathbb{R}^4 \arrow[loop above, "\mathrm{SU}(2)_R"] \oplus \mathbb{R} \arrow[loop above, "(\mathbb{Z}_2)_{\mathrm{HW}}"]
\end{tikzcd}$$

yields super-Minkowski orbifolds, e.g.:

$$\mathscr{X}_{\mathrm{MK6}\perp\mathrm{K3}} = \mathbb{R}^{6,1|\mathbf{16}} \times \rotatebox[origin=c]{70}{$\prec$}(\mathbb{T}^{\mathbf{4}} /\!\!/ \mathbb{Z}_2^A) \;\in\; \mathbf{H}_{/\rotatebox[origin=c]{70}{$\prec$}\mathbb{Z}_2^A}$$

J-twisted proper orbifold Cohomotopy. Def. 5.28 in [SS20-OrbCoh]

Let $\mathscr{X}$ be an $\mathbb{R}^{d,1|\mathbf{N}}$-orbifold

with $\mathrm{Spin}(n)$-structure

$$\begin{tikzcd}
\mathscr{X} \arrow[rr, "\curlyvee\tau"] \arrow[dr, "\curlyvee\mathrm{Fr}(\mathscr{X})"'] & \arrow[d, phantom, "\overset{\sim}{\Leftarrow}\ \text{vielbein}"] & \curlyvee\mathbf{B}\mathrm{Spin}(n) \arrow[dl] \\
& \curlyvee\mathbf{B}\mathrm{GL}(d,1|\mathbf{N}) &
\end{tikzcd}$$

Def. ([SS20-OrbCoh, Ex. 5.29]) **J-twisted proper orbifold Cohomotopy** of $(\mathscr{X},\tau)$ is :

$$\overset{\substack{\text{tangentially J-twisted}\\ \text{proper orbifold Cohomotopy}}}{\pi^{\int\curlyvee\tau}(\mathscr{X})} := \pi_0 \mathrm{Map}\Big(\mathscr{X}, \overbrace{\textstyle\int\curlyvee(S^{\mathbf{n}}_{\mathrm{Tate}}/\!\!/\mathrm{Spin}(n))}^{\substack{\text{proper equivariant homotopy type}\\ \text{of canonical n-sphere}}}\Big)_{\underset{\text{proper equivariant tangential twist}}{\int\curlyvee\mathbf{B}\mathrm{Spin}(d)}}$$

$$= \pi_0 \left\{ \begin{tikzcd}
\mathscr{X} \arrow[rr, dashed, "\overset{\text{cocycle}}{c}"] \arrow[dr, "\underset{\text{twist}}{\eta^{\int}\circ\curlyvee\tau}"'] & \arrow[d, phantom, "\overset{\sim}{\Leftarrow}"] & \int\curlyvee(S^{n}_{\mathrm{Tate}}/\!\!/\mathrm{Spin}(n)) \arrow[dl, "\text{local coeffs.}"] \\
& \int\curlyvee\mathbf{B}\mathrm{Spin}(n) &
\end{tikzcd} \right\}$$

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& \curlyvee\mathbf{B}\mathrm{GL}(d,1|\mathbf{N}) &
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$$= \pi_0 \left\{ \begin{tikzcd}
\mathscr{X} \arrow[rr, dashed, "\overset{\text{cocycle}}{c}"] \arrow[dr, "\eta^{\int}\circ\curlyvee\tau"', "\text{twist}"] & \arrow[d, phantom, "\overset{\sim}{\Swarrow}"] & \int\curlyvee\big(S^{n}_{\mathrm{Tate}}/\!\!/\mathrm{Spin}(n)\big) \arrow[dl, "\text{local coeffs.}"] \\
& \int\curlyvee\mathbf{B}\mathrm{Spin}(n) &
\end{tikzcd} \right\}$$

In the case that $\mathscr{X} = \mathrm{X}$ is smooth (i.e. a manifold), this reduces to **J-twisted Cohomotopy**: [FSS19-HypH, Def. 2.1][FSS20-Char, Ex. 2.41], see also [Cru03, Lem. 5.2].

Orbifold Cohomotopy at and away from the singularities. Thm. 5.16 in [SS20-OrbCoh]

$$\begin{tikzcd}
\mathscr{X} \arrow[rr, dashed, "\text{cocycle}"] \arrow[dr, "\eta^{\int} \circ \prec\tau"'] \arrow[r, phantom, shift right=3ex, "\Downarrow\wr"] & & \int \prec \left(S^{\mathbf{n}|\mathbf{N}}_{\mathrm{Tate}} /\!\!/ \mathrm{Spin}(n) \right) \arrow[dl] \\
& \int \prec \mathbf{B}\mathrm{Spin}(n) &
\end{tikzcd}$$

Key Structure Theorem:

In limiting cases this reduces to:

(a) equiv Cohomotopy in RO-deg **n**

(b) tangent J-twisted Cohomotopy

tangentially J-twisted orbifold Cohomotopy

$$\begin{tikzcd}
& \boxed{\pi^{\int \prec \tau}\left(\mathscr{X}\right)} \arrow[dl, squiggly, "\mathscr{X} = \prec(\mathbb{T}^d /\!\!/ G)"] \arrow[dr, squiggly, "\mathscr{X} = X"'] & \\
\boxed{\pi^{\mathbf{n}}_G\left(\mathbb{T}^d\right)} & & \boxed{\pi^{\tau}\left(\mathrm{X}\right)}
\end{tikzcd}$$

(a) **on flat orbifolds**

(b) **on smooth orbifolds**

[HSS18-ADE]
[SS19-TadCnc]
[BSS19-FrcBrn]
[SS20-M5GS]

equivariant Cohomotopy in RO-degree n

tangentially J-twisted Cohomotopy

[FSS19-HypH]
[FSS19-M5WZ]
[FSS20-M5Str]
[SS20-M5Anom]
[FSS20-GSAnom]

$$\begin{tikzcd}
\prec(\mathbb{T}^d /\!\!/ G) \arrow[rr, dashed, "\text{cocycle}"] \arrow[dr, "\prec\tau"'] \arrow[r, phantom, shift right=3ex, "\Swarrow\wr"] & & \int \prec\left(S^{\mathbf{n}} /\!\!/ G\right) \arrow[dl] \\
& \underset{\prec}{G} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{X} \arrow[rr, dashed, "\text{cocycle}"] \arrow[dr, "\int\tau"'] \arrow[r, phantom, shift right=3ex, "\Swarrow\wr"] & & S^n /\!\!/ \mathrm{Spin}(n) \arrow[dl] \\
& B\mathrm{Spin}(n) &
\end{tikzcd}$$

0 – Cohesive Homotopy Theory

I – Quantum Charge of M-branes under Hypothesis H

II – Quantum Charge of D6 $\perp$ D8 under Hypothesis H

III – Quantum States of D6 $\perp$ D8 under Hypothesis H

Future historians may judge the late 20th century as a time when theorists were like children playing on the seashore, diverting themselves with the smoother pebbles or prettier shells of superstrings

while the great ocean of M-theory lay undiscovered before them.

M. Duff (1998)

closing sentence in:

M. Duff:
The Theory Formerly Known as Strings
Scient Amer 1998

Flux-quantization in non-abelian cohomology theory. [FSS20-Char]

Non-perturbative completion of a theory of charged objects
with flux densities satisfying Bianchi identities
involves choosing a non-abelian cohomology theory A
whose character image enforces these Bianchi identities:

Flux-quantization in non-abelian cohomology theory. [FSS20-Char]

Non-perturbative completion of a theory of charged objects
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charge “lattice”
non-abelian cohomology

non-abelian
de Rham cohomology

Whitehead L_∞-algebra

$$\begin{tikzcd} A(\mathrm{X}) \arrow[r, "\text{non-abelian character}", "\mathrm{ch}_A"'] & H_{\mathrm{dR}}\big(\mathrm{X};\, \mathfrak{l}A\big) \\ {[c]} \arrow[r, mapsto] & \Big[\big\{ F^{(a)}_{r_a} \in \Omega^{r_a}_{\mathrm{dR}}(X) \big\}_{1 \leq a \leq \dim[\pi_\bullet(A),\mathbb{R}]} \,\Big|\, d\,F^{(a)}_{r_a} = P_{r_a}\big(\{F^{(b)}_{r_b}\}_{b \leq a}\big) \Big] \end{tikzcd}$$

quantized flux/charge
class in A-cohomology

field strengths/flux densities
L_∞-algebra valued diff forms

higher Bianchi identities
L_∞-flatness condition

Flux-quantization in non-abelian cohomology theory.

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{[c]} \arrow[r, mapsto] & \left[\left\{ F^{(a)}_{r_a} \in \Omega^{r_a}_{\mathrm{dR}}(X) \right\}_{1 \leq a \leq \dim[\pi_\bullet(A),\mathbb{R}]} \,\middle|\, d\, F^{(a)}_{r_a} \;=\; P_{r_a}\big(\{F^{(b)}_{r_b}\}_{b \leq a}\big) \right]
\end{tikzcd}$$

quantized flux/charge
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higher Bianchi identities
L_∞-flatness condition

The choice of A is a **hypothesis** about the correct non-perturbative completion.

Flux-quantization in non-abelian cohomology theory. [FSS20-Char]

Non-perturbative completion of a theory of charged objects
with flux densities satisfying Bianchi identities
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charge "lattice"
non-abelian cohomology

non-abelian de Rham cohomology

Whitehead L_∞-algebra

$$\begin{tikzcd}[column sep=huge] A(\mathrm{X}) \arrow[r, "\text{non-abelian character}", "\mathrm{ch}_A"'] & H_{\mathrm{dR}}\big(\mathrm{X}; \mathfrak{l}A\big) \\ {[c]} \arrow[r, mapsto] & \left[\Big\{ F^{(a)}_{r_a} \in \Omega^{r_a}_{\mathrm{dR}}(X) \Big\}_{1 \leq a \leq \dim[\pi_\bullet(A),\mathbb{R}]} \,\middle|\, d\,F^{(a)}_{r_a} \;=\; P_{r_a}\Big(\big\{F^{(b)}_{r_b}\big\}_{b \leq a}\Big) \right] \end{tikzcd}$$

quantized flux/charge
class in A-cohomology

field strengths/flux densities
L_∞-algebra valued diff forms

higher Bianchi identities
L_∞-flatness condition

The choice of A is a **hypothesis** about the correct non-perturbative completion.

Given such a choice, the moduli ∞-stack of fields is a differential refinement of A:

$$\mathscr{A} \;=\; \widehat{A} \in \mathrm{SmthGrpd}_\infty$$

Hypothesis H. [Sa13, §2.5][FSS16-RatCoh][FSS19-HypH][SS21-MF]

Fact: The Bianchi identity of the **type IIA RR/B-fields**
is that enforced by the Whitehead L_∞-algebra of twisted KU-theory

$$H_{\mathrm{dR}}\big(\mathrm{X}, \mathfrak{l}(\mathrm{KU} /\!\!/ B\mathrm{U}(1))\big) \;\simeq\; \left\{ \begin{pmatrix} \{F_{2k}\}_k \\ H_3 \end{pmatrix} \in \Omega^\bullet_{\mathrm{dR}}(\mathrm{X}) \,\middle|\, \begin{array}{ll} d\,F_{2k} & = H_3 \wedge F_{2k-2} \\ d\,H_3 & = 0 \end{array} \right\}_{\sim_{\mathrm{conc}}}$$

The evident **hypothesis** here is the proposal by Minasian/Moore/Witten/Bouwknegt/Mathai:

The type IIA RR/B-field is flux-quantized in twisted K-theory.

It *must* be flux-quantized in something at least close, such as orbifold KR-theory.

Hypothesis H. [Sa13, §2.5][FSS16-RatCoh][FSS19-HypH][SS21-MF]

Fact: The Bianchi identity of the **type IIA RR/B-fields**
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The type IIA RR/B-field is flux-quantized in twisted K-theory.

It *must* be flux-quantized in something at least close, such as orbifold KR-theory.

Fact: The Bianchi identity of the **M-theory *C*-field** is
that enforced by the Whitehead L_∞-algebra of 4-Cohomotopy:

$$H_{\mathrm{dR}}\big(\mathrm{X},\, \mathfrak{l}S^4\big) \;\simeq\; \left\{ \begin{pmatrix} G_7 \\ G_4 \end{pmatrix} \in \Omega^{\bullet}_{\mathrm{dR}}(\mathrm{X}) \;\middle|\; \begin{aligned} d\,G_7 &= -\tfrac{1}{2} G_4 \wedge G_4 \\ d\,G_4 &= 0 \end{aligned} \right\}_{\sim_{\mathrm{conc}}}$$

The evident **hypothesis** here [Sa13, §2.5] we called *Hypothesis H*:

The M-theory C-field is flux-quantized in 4-Cohomotopy.

It *must* be charge-quantized in something at least close, such as J-twisted orbifold Cohomotopy.

Cohomotopy unifies M5/M2-brane charge. [HSS18-ADE, p. 17][FSS19-HypH, §2.3]

Cohomotopy is dual to Homotopy:
$$\underset{\substack{\textbf{4-co-homotopy group}\\\textbf{of spheres}}}{\pi^4\left(S^k\right)} \quad \simeq \quad \underset{\substack{\textbf{homotopy groups}\\\textbf{of 4-sphere}}}{\pi_k\left(S^4\right)}$$

Cohomotopy unifies M5/M2-brane charge. [HSS18-ADE, p. 17][FSS19-HypH, §2.3]

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Homotopy groups of the 4-sphere:

$k =$	1	2	3	4	5	6	7	8	9 $\cdots$
$\pi_k\left(S^4\right)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\overset{\text{all torsion}}{\cdots\cdots}$

Cohomotopy unifies M5/M2-brane charge. [HSS18-ADE, p. 17][FSS19-HypH, §2.3]

Cohomotopy is dual to Homotopy: $\underset{\text{4-co-homotopy group of spheres}}{\pi^4\left(S^k\right)} \simeq \underset{\text{homotopy groups of 4-sphere}}{\pi_k\left(S^4\right)}$

Homotopy groups of the 4-sphere:

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$\pi_k\left(S^4\right)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	all torsion $\cdots\cdots$

$\Longrightarrow$

4-Cohomotopy measures integer charges exactly around black BPS M2/M5-branes:

$$\pi^4\Big(\underbrace{\widehat{\mathrm{AdS}_7} \times S^4}_{\text{black M5-brane near horizon spacetime}}\Big) \simeq \pi^4\left(S^4\right) \simeq \pi_4\left(S^4\right) \simeq \mathbb{Z}$$

$$\pi^4\Big(\underbrace{\widehat{\mathrm{AdS}_4} \times S^7}_{\text{black M2-brane near horizon spacetime}}\Big) \simeq \pi^4\left(S^7\right) \simeq \pi_7\left(S^4\right) \simeq \mathbb{Z} \oplus \text{torsion}$$

Approximating Cohomotopy by K-theory.

[BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

J-twisted orbifold Cohomotopy
around an orbi-singularity

$\pi^{\mathbf{4}}_{G}\left(\mathbb{R}^{\mathbf{4}}_{\mathrm{cpt}}\right)$

un-stable/
non-linear!

equivariant generalized
cohomologies in RO-degree 4

equivariant generalized
cohomologies of the point

representation
rings

Approximating Cohomotopy by K-theory.

[BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

J-twisted orbifold Cohomotopy
around an orbi-singularity

un-stable/
non-linear!

stabilization/
linearization

equivariant
stable Cohomotopy

$$\begin{tikzcd}
\pi^{\mathbf{4}}_{G}\left(\mathbb{R}^{\mathbf{4}}_{\mathrm{cpt}}\right) \arrow[d, "\Sigma^{\infty}"] & \\
\mathbb{S}^{\mathbf{4}}_{G}\left(S^{\mathbf{4}}\right) \arrow[r, equal] & \mathbb{S}^{0}_{G}
\end{tikzcd}$$

equivariant generalized
cohomologies in RO-degree 4

equivariant generalized
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Approximating Cohomotopy by K-theory. [BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

J-twisted orbifold Cohomotopy around an orbi-singularity

un-stable/ non-linear!

$$\begin{tikzcd}
\pi^{\mathbf{4}}_G\left(\mathbb{R}^{\mathbf{4}}_{\mathrm{cpt}}\right) \arrow[d, "\Sigma^\infty", "\text{stabilization/ linearization}"'] & & & \\
\mathbb{S}^{\mathbf{4}}_G\left(S^{\mathbf{4}}\right) \arrow[r, equal] & \mathbb{S}^0_G \arrow[r, equal, "{[BP72][Se74]}", "{[De06][Gui06]}"'] & \mathrm{R}_{\mathbb{F}_1}(G) \arrow[r, equal, "{[Se71][tD79]}"] & A_G
\end{tikzcd}$$

equivariant stable Cohomotopy

Burnside ring

equivariant generalized cohomologies in RO**-degree 4**

equivariant generalized cohomologies of the point

representation rings

Approximating Cohomotopy by K-theory.

[BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

J-twisted orbifold Cohomotopy around an orbi-singularity

un-stable/ **non-linear!**

$$\begin{tikzcd}
\pi^{\mathbf{4}}_G\big(\mathbb{R}^{\mathbf{4}}_{\mathrm{cpt}}\big) \arrow[d, "\Sigma^\infty", "\text{stabilization/ linearization}"'] & & & \\
\mathbb{S}^{\mathbf{4}}_G\big(S^{\mathbf{4}}\big) \arrow[r, equal] \arrow[d] & \mathbb{S}^0_G \arrow[r, equal, "{[BP72][Se74]}", "{[De06][Gui06]}"'] \arrow[d, "\beta"] & \mathrm{R}_{\mathbb{F}_1}(G) \arrow[r, equal, "{[Se71][tD79]}"] \arrow[d, "\otimes_{\mathbb{F}_1}\mathbb{R}"] & A_G \\
\mathrm{KO}^{\mathbf{4}}_G\big(S^{\mathbf{4}}\big) \arrow[r, equal] & \mathrm{KO}^0_G \arrow[r, equal] & \mathrm{R}_{\mathbb{R}}(G) &
\end{tikzcd}$$

equivariant stable Cohomotopy

Burnside ring

Hurewicz-Boardman homomorphism
(initiality of $\mathbb{S}$)

equivariant orth. K-theory

equivariant generalized cohomologies in RO**-degree 4**

equivariant generalized cohomologies of the point

representation rings

Approximating Cohomotopy by K-theory. [BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

$$\begin{tikzcd}
\text{J-twisted orbifold Cohomotopy around an orbi-singularity} & \pi^{4}_{G}\big(\mathbb{R}^{4}_{\mathrm{cpt}}\big) \arrow[d, "\Sigma^\infty", "\text{stabilization/ linearization}"'] & \text{un-stable/ non-linear!} & & & \\
\text{equivariant stable Cohomotopy} & \mathbb{S}^{4}_{G}\big(S^{4}\big) \arrow[r, equal] \arrow[d, "\text{Hurewicz-Boardman homomorphism (initiality of } \mathbb{S})"] & \mathbb{S}^{0}_{G} \arrow[r, equal, "\text{[BP72][Se74]}", "\text{[De06][Gui06]}"'] \arrow[d, "\beta"] & \mathrm{R}_{\mathbb{F}_1}(G) \arrow[r, equal, "\text{[Se71][tD79]}"] \arrow[d, "\otimes_{\mathbb{F}_1}\mathbb{R}"] & A_G & \text{Burnside ring} \\
\text{equivariant orth. K-theory} & \mathrm{KO}^{4}_{G}\big(S^{4}\big) \arrow[r, equal] \arrow[d, "\text{further extension of scalars}"] & \mathrm{KO}^{0}_{G} \arrow[r, equal] \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"] & \mathrm{R}_{\mathbb{R}}(G) \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"] & & \\
\text{equivariant complex K-theory} & \mathrm{KU}^{4}_{G}\big(S^{4}\big) \arrow[r, equal] & \mathrm{KU}^{0}_{G} \arrow[r, equal] & \mathrm{R}_{\mathbb{C}}(G) & & \\
 & \text{equivariant generalized cohomologies in RO-degree 4} & \text{equivariant generalized cohomologies of the point} & \text{representation rings} & &
\end{tikzcd}$$

Approximating Cohomotopy by K-theory. [BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

J-twisted orbifold Cohomotopy around an orbi-singularity

un-stable/ non-linear!

$$\begin{tikzcd}
\pi^{\mathbf{4}}_G\big(\mathbb{R}^{\mathbf{4}}_{\mathrm{cpt}}\big) \arrow[d, "\Sigma^\infty", "\text{stabilization/ linearization}"'] & & & \\
\mathbb{S}^{\mathbf{4}}_G\big(S^{\mathbf{4}}\big) \arrow[r, equal] \arrow[d] & \mathbb{S}^0_G \arrow[r, equal, "{[BP72][Se74]}", "{[De06][Gui06]}"'] \arrow[d, "\beta"] & \mathrm{R}_{\mathbb{F}_1}(G) \arrow[r, equal, "{[Se71][tD79]}"] \arrow[d, "\otimes_{\mathbb{F}_1}\mathbb{R}"] & A_G \arrow[ddl, rounded corners, "\text{linearize virtual } G\text{-sets of fractional M-branes to virtual } G\text{-representations of fractional D-branes}"] \\
\mathrm{KO}^{\mathbf{4}}_G\big(S^{\mathbf{4}}\big) \arrow[r, equal] \arrow[d] & \mathrm{KO}^0_G \arrow[r, equal] \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"] & \mathrm{R}_{\mathbb{R}}(G) \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"] & \\
\mathrm{KU}^{\mathbf{4}}_G\big(S^{\mathbf{4}}\big) \arrow[r, equal] & \mathrm{KU}^0_G \arrow[r, equal] & \mathrm{R}_{\mathbb{C}}(G) &
\end{tikzcd}$$

equivariant stable Cohomotopy

Hurewicz-Boardman homomorphism (initiality of $\mathbb{S}$)

Burnside ring

equivariant orth. K-theory

further extension of scalars

equivariant complex K-theory

equivariant generalized cohomologies in RO-degree 4

equivariant generalized cohomologies of the point

representation rings

Approximating Cohomotopy by K-theory. [BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

$$\begin{tikzcd}
\text{J-twisted orbifold Cohomotopy around an orbi-singularity} & \pi^{\mathbf{4}}_G\left(\mathbb{R}^{\mathbf{4}}_{\mathrm{cpt}}\right) \arrow[d, "\Sigma^\infty", "\text{stabilization/ linearization}"'] & \text{un-stable/ non-linear!} & & & \\
\text{equivariant stable Cohomotopy} & \mathbb{S}^{\mathbf{4}}_G\left(S^{\mathbf{4}}\right) \arrow[r, equal] \arrow[d, "\text{Hurewicz-Boardman homomorphism (initiality of } \mathbb{S}\text{)}"] & \mathbb{S}^0_G \arrow[r, equal, "\text{[BP72][Se74]}", "\text{[De06][Gui06]}"'] \arrow[d, "\beta"] & \mathrm{R}_{\mathbb{F}_1}(G) \arrow[r, equal, "\text{[Se71][tD79]}"] \arrow[d, "\otimes_{\mathbb{F}_1}\mathbb{R}"] & A_G \arrow[ddl, "\text{linearize virtual } G\text{-sets of fractional M-branes to virtual } G\text{-representations of fractional D-branes}"] & \text{Burnside ring} \\
\text{equivariant orth. K-theory} & \mathrm{KO}^{\mathbf{4}}_G\left(S^{\mathbf{4}}\right) \arrow[r, equal] \arrow[d, "\text{further extension of scalars}"] & \mathrm{KO}^0_G \arrow[r, equal] \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"] & \mathrm{R}_{\mathbb{R}}(G) \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"] & & \\
\text{equivariant complex K-theory} & \mathrm{KU}^{\mathbf{4}}_G\left(S^{\mathbf{4}}\right) \arrow[r, equal] & \mathrm{KU}^0_G \arrow[r, equal] & \mathrm{R}_{\mathbb{C}}(G) & & \\
& \text{equivariant generalized cohomologies in RO-degree 4} & \text{equivariant generalized cohomologies of the point} & \text{representation rings} & &
\end{tikzcd}$$

Rem. [FSS20-Char, (353)].

The Boardman homomorphism exhibits exactly the identification $G_4 \mapsto F_4$ of [DMW00]:

$$\begin{tikzcd}
\pi^4 \arrow[r, "\Sigma^\infty", "\text{stabilization / linearization}"'] \arrow[d, bend right, "\mathrm{ch}"'] & \mathbb{S}^4 \arrow[r, "\beta", "\text{Boardman homomorphism}"'] & \mathrm{KU}^4 \arrow[r, equal, "\text{Bott per.}"'] & \mathrm{KU} \\
(G_4, G_7) \arrow[r, mapsto] & G_4 \arrow[r, mapsto] & F_4 & \\
& \text{C-field flux} & \text{RR-field flux} &
\end{tikzcd}$$

Approximating Cohomotopy by K-theory. [BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

J-twisted orbifold Cohomotopy around an orbi-singularity — **un-stable/ non-linear!**

$$\begin{tikzcd}
\pi^{\mathbf{4}}_G\left(\mathbb{R}^{\mathbf{4}}_{\mathrm{cpt}}\right) \arrow[d, "\Sigma^\infty", "\text{stabilization/ linearization}"'] & & & \\
\mathbb{S}^{\mathbf{4}}_G\left(S^{\mathbf{4}}\right) \arrow[r, equal] \arrow[d] & \mathbb{S}^0_G \arrow[r, equal, "{[BP72][Se74]}", "{[De06][Gui06]}"'] \arrow[d, "\beta"] & \mathrm{R}_{\mathbb{F}_1}(G) \arrow[r, equal, "{[Se71][tD79]}"] \arrow[d, "\otimes_{\mathbb{F}_1}\mathbb{R}"] & A_G \arrow[ddl, "\text{linearize virtual G-sets of fractional M-branes to virtual G-representations of fractional D-branes}"] \\
\mathrm{KO}^{\mathbf{4}}_G\left(S^{\mathbf{4}}\right) \arrow[r, equal] \arrow[d] & \mathrm{KO}^0_G \arrow[r, equal] \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"] & \mathrm{R}_{\mathbb{R}}(G) \arrow[d, "\otimes_{\mathbb{R}}\mathbb{C}"] & \\
\mathrm{KU}^{\mathbf{4}}_G\left(S^{\mathbf{4}}\right) \arrow[r, equal] & \mathrm{KU}^0_G \arrow[r, equal] & \mathrm{R}_{\mathbb{C}}(G) &
\end{tikzcd}$$

equivariant stable Cohomotopy

Hurewicz-Boardman homomorphism (initiality of $\mathbb{S}$)

Burnside ring

equivariant orth. K-theory

further extension of scalars

equivariant complex K-theory

equivariant generalized cohomologies in RO-degree 4 — **equivariant generalized cohomologies of the point** — **representation rings**

Rem. [FSS20-Char, (353)].

The Boardman homomorphism exhibits exactly the identification $G_4 \mapsto F_4$ of [DMW00]:

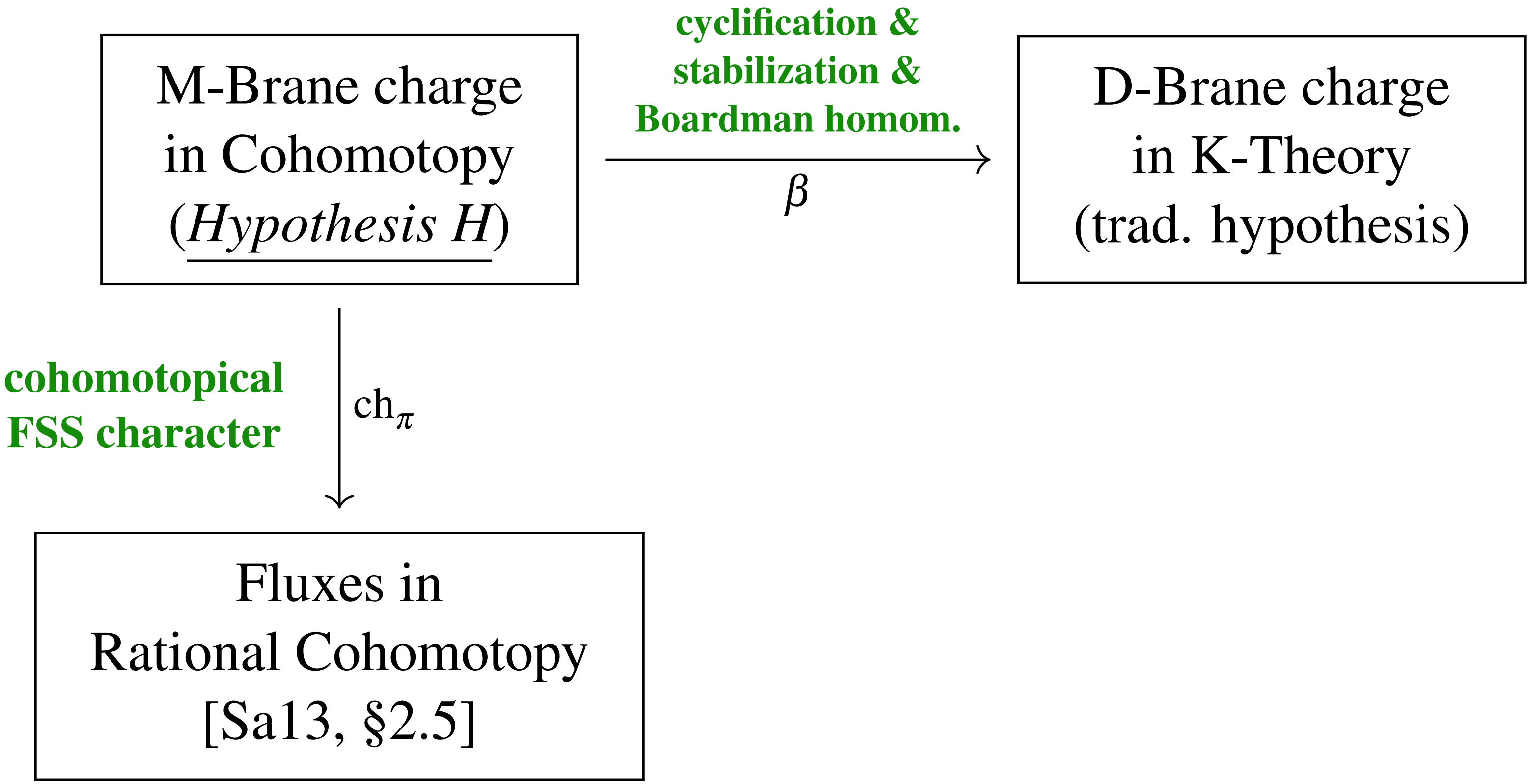
$$\begin{tikzcd}
\pi^4 \arrow[r, "\Sigma^\infty", "\text{stabilization / linearization}"'] \arrow[d, bend right, "\mathrm{ch}"'] & \mathbb{S}^4 \arrow[r, "\beta", "\text{Boardman homomorphism}"'] & \mathrm{KU}^4 \arrow[r, equal, "\text{Bott per.}"'] & \mathrm{KU} \\
(G_4, G_7) \arrow[r, mapsto] & G_4 \arrow[r, mapsto] & F_4 &
\end{tikzcd}$$

C-field flux — **RR-field flux**

However, β (and [DMW00]) misses the double dimensional reductions $G_4 \mapsto H_3$ and $G_7 \mapsto F_6$; these do appear from Cohomotopy via *cyclification* ([FSS16-RatCoh][FSS16-TDual][BSS19-RatSt]).

Approximating Cohomotopy by K-theory. [BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

These two approximations...



M-Brane charge
in Cohomotopy
(*Hypothesis H*)

cyclification &
stabilization &
Boardman homom.

β

D-Brane charge
in K-Theory
(trad. hypothesis)

cohomotopical
FSS character

ch_{π}

Fluxes in
Rational Cohomotopy
[Sa13, §2.5]

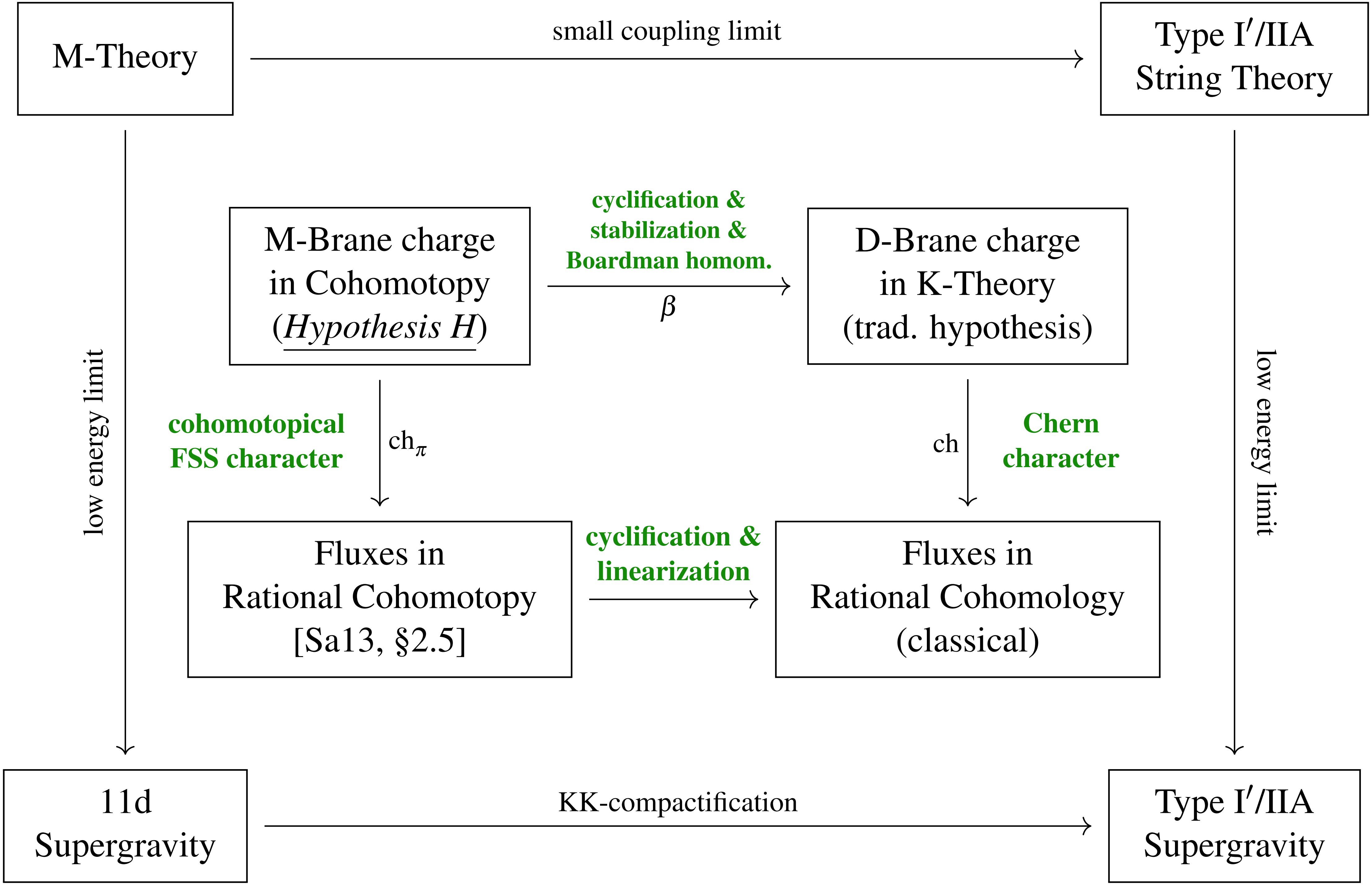
Approximating Cohomotopy by K-theory. [BSS19-FrcBrn][SS19-TadCnc][SS21-MF]

These two approximations are compatible with each other:

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{M-Brane charge}\\ \text{in Cohomotopy}\\ (\underline{\textit{Hypothesis H}})\end{array}} \arrow[r, "\begin{array}{c}\textbf{cyclification \&}\\ \textbf{stabilization \&}\\ \textbf{Boardman homom.}\end{array}", "\beta"'] \arrow[d, "\mathrm{ch}_\pi", "\begin{array}{c}\textbf{cohomotopical}\\ \textbf{FSS character}\end{array}"'] & \boxed{\begin{array}{c}\text{D-Brane charge}\\ \text{in K-Theory}\\ \text{(trad. hypothesis)}\end{array}} \arrow[d, "\mathrm{ch}"', "\begin{array}{c}\textbf{Chern}\\ \textbf{character}\end{array}"] \\
\boxed{\begin{array}{c}\text{Fluxes in}\\ \text{Rational Cohomotopy}\\ \text{[Sa13, §2.5]}\end{array}} \arrow[r, "\begin{array}{c}\textbf{cyclification \&}\\ \textbf{linearization}\end{array}"] & \boxed{\begin{array}{c}\text{Fluxes in}\\ \text{Rational Cohomology}\\ \text{(classical)}\end{array}}
\end{tikzcd}$$

Approximating Cohomotopy by K-theory.

[BSS19-FrcBrn][SS19-TadCnc][SS21-MF]



M-Theory

small coupling limit

Type I′/IIA
String Theory

low energy limit

M-Brane charge
in Cohomotopy
(*Hypothesis H*)

**cyclification &
stabilization &
Boardman homom.**

β

D-Brane charge
in K-Theory
(trad. hypothesis)

**cohomotopical
FSS character**

ch_π

ch

**Chern
character**

Fluxes in
Rational Cohomotopy
[Sa13, §2.5]

**cyclification &
linearization**

Fluxes in
Rational Cohomology
(classical)

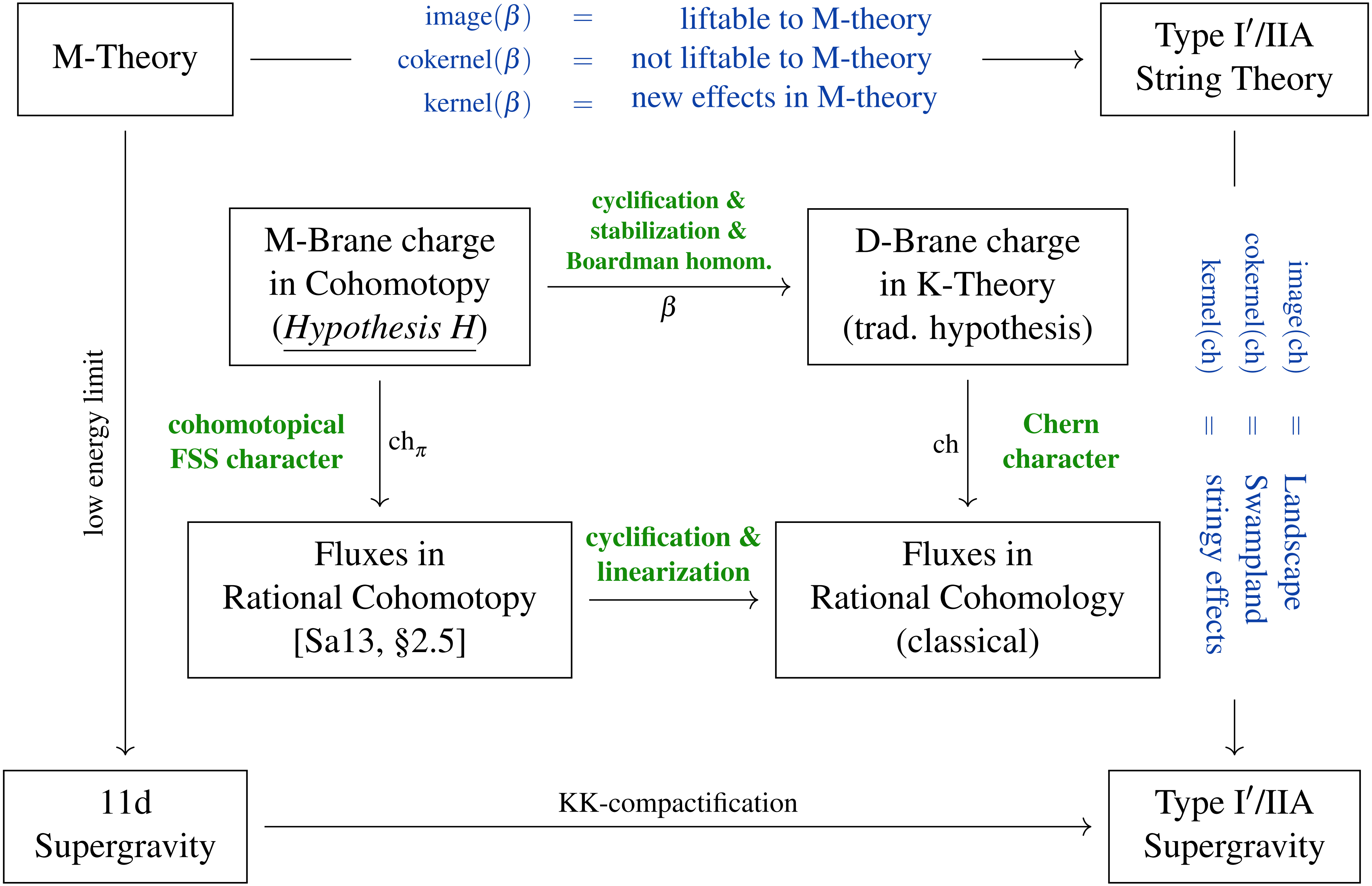
low energy limit

11d
Supergravity

KK-compactification

Type I′/IIA
Supergravity

Approximating Cohomotopy by K-theory. [BSS19-FrcBrn][SS19-TadCnc][SS21-MF]



M-Theory

image(β) = liftable to M-theory

cokernel(β) = not liftable to M-theory

kernel(β) = new effects in M-theory

Type I′/IIA String Theory

M-Brane charge in Cohomotopy (*Hypothesis H*)

cyclification & stabilization & Boardman homom.

β

D-Brane charge in K-Theory (trad. hypothesis)

low energy limit

cohomotopical FSS character

ch_π

ch

Chern character

image(ch) = Landscape

cokernel(ch) = Swampland

kernel(ch) = stringy effects

Fluxes in Rational Cohomotopy [Sa13, §2.5]

cyclification & linearization

Fluxes in Rational Cohomology (classical)

11d Supergravity

KK-compactification

Type I′/IIA Supergravity

Background: The black M5 in 11d SuGra.

Fact. [AFFH99, 5.2] [dMFF12, §8.3]:

All BPS black M5-brane solutions of 11D supergravity are $^1/_2$ BPS of this form:

$$\begin{tikzcd}
& \boxed{\begin{array}{c}\text{black M5}\\ (^1/_2\ \text{BPS})\end{array}} \arrow[dl, "\text{near horizon limit}"'] \arrow[dr, "\text{far-horizon limit}"] & \\
\mathrm{AdS}_7 \times S^{\mathbf{4}} /\!\!/ G_{\mathrm{ADE}} & & \mathbb{R}^{5,1} \times \mathbb{R}^1 \times \mathbb{R}^{\mathbf{4}} /\!\!/ G_{\mathrm{ADE}}
\end{tikzcd}$$

$$r \;\lesssim\; \ell_{\mathrm{Pl}} \sqrt[3]{N_{\mathrm{M5}}} \;\ll\; r$$

with $S^{\mathbf{4}} = S\big(\mathbb{R} \oplus \overset{G_{\mathrm{ADE}}\curvearrowright}{\mathbb{H}}\big)$
as above

as in the
above Thm.

**throat geometry intersecting
orbifold ADE-singularity**

**flat orbifold
with ADE-singularity**

Background: The black M5 in 11d SuGra.

Fact. [AFFH99, 5.2] [dMFF12, §8.3]:
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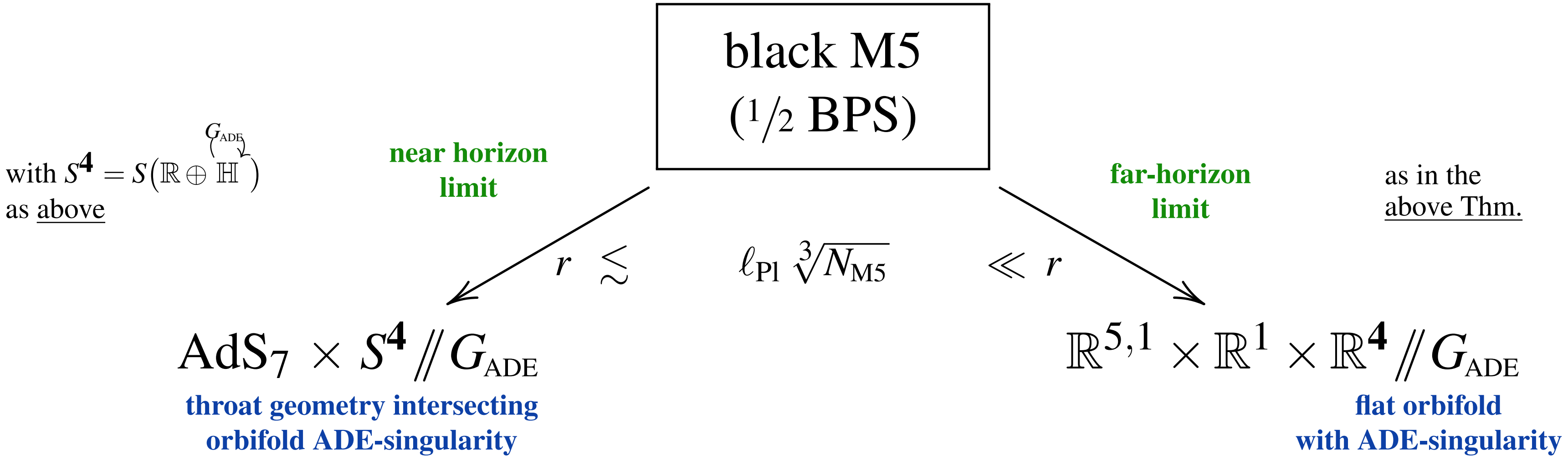


$$\boxed{\begin{array}{c}\text{black M5}\\ (^1/_2\ \text{BPS})\end{array}}$$

with $S^{\mathbf{4}} = S(\mathbb{R} \oplus \overset{G_{\mathrm{ADE}}\circlearrowright}{\mathbb{H}})$
as above

near horizon limit

far-horizon limit

as in the
above Thm.

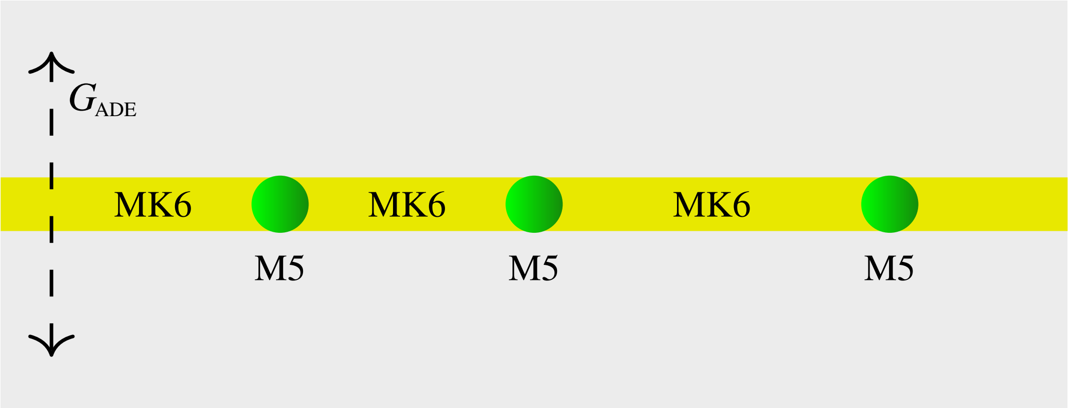
$$r \;\lesssim\; \ell_{\mathrm{Pl}} \sqrt[3]{N_{\mathrm{M5}}} \;\ll\; r$$

$$\mathrm{AdS}_7 \times S^{\mathbf{4}} /\!\!/ G_{\mathrm{ADE}} \qquad\qquad \mathbb{R}^{5,1} \times \mathbb{R}^1 \times \mathbb{R}^{\mathbf{4}} /\!\!/ G_{\mathrm{ADE}}$$

throat geometry intersecting orbifold ADE-singularity

flat orbifold with ADE-singularity

Consequence 1: Black BPS M5-branes are always domain walls inside an MK6-singularity:



E.g.: [ZHTV14, §3.1] [Fa17, §3.3.1]

Background: The black M5 in 11d SuGra.

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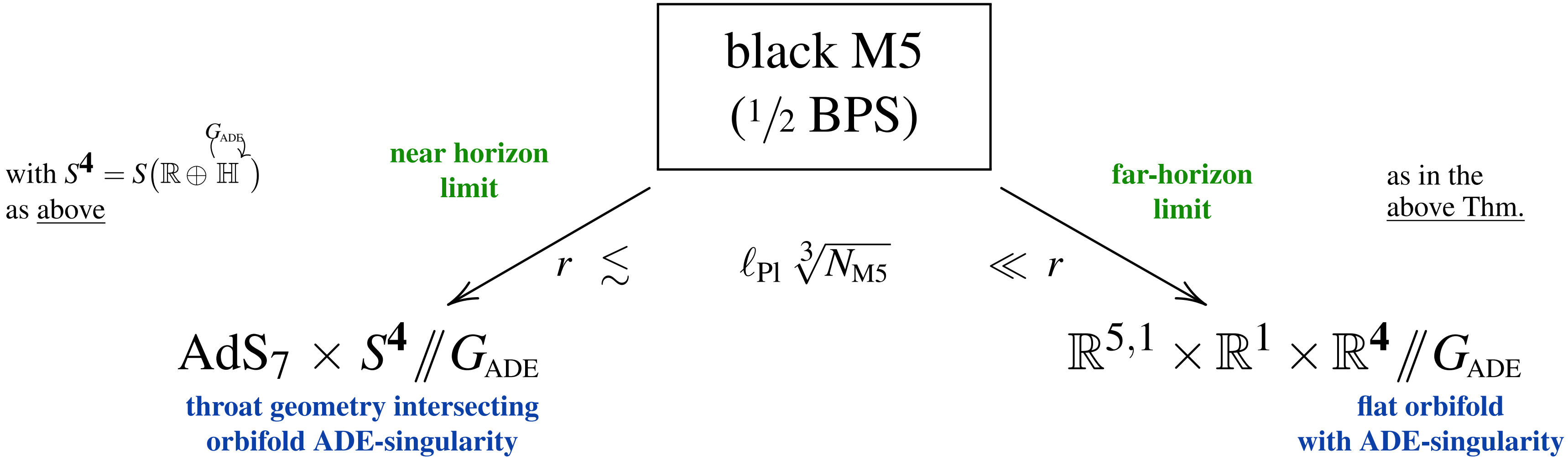


black M5
($^1\!/_2$ BPS)

with $S^{\mathbf{4}} = S(\mathbb{R} \oplus \overset{G_{\mathrm{ADE}}}{\mathbb{H}})$
as above

near horizon limit

far-horizon limit

as in the
above Thm.

$$r \lesssim \ell_{\mathrm{Pl}} \sqrt[3]{N_{\mathrm{M5}}} \ll r$$

$\mathrm{AdS}_7 \times S^{\mathbf{4}} /\!\!/ G_{\mathrm{ADE}}$

throat geometry intersecting orbifold ADE-singularity

$\mathbb{R}^{5,1} \times \mathbb{R}^1 \times \mathbb{R}^{\mathbf{4}} /\!\!/ G_{\mathrm{ADE}}$

flat orbifold with ADE-singularity

Consequence 2:

Individual M5-branes $N_{\mathrm{M5}} \sim \mathscr{O}(1)^3$ have Planck scale thickness $r \sim \ell_{\mathrm{Pl}}$
hence their **near geometry make no sense** as solutions of M-theory
due to infinite + unknown tower of higher curvature quantum corrections $\sim \big(\underset{\text{Planck area}}{\ell_{\mathrm{Pl}}^2} \cdot \underset{\text{Riem. curvature}}{R}\big)^k$.

Conversely:
The **M-meaningful far geometry** yields flat super-orbifold spacetimes
where all curvature is crammed into orbi-singularities
so that also all quantum effects must be hiding inside orbi-singularities –
plausibly detected as charges measured in a proper orbifold cohomology theory!

Background: The black M5 in 11d SuGra.

Fact. [AFFH99, 5.2] [dMFF12, §8.3]:
All BPS black M5-brane solutions of 11D supergravity are 1/2BPS of this form:

$$\begin{tikzcd}
 & \boxed{\begin{array}{c}\text{black M5}\\ (1/2\ \text{BPS})\end{array}} \arrow[dl, "\text{near horizon limit}"'] \arrow[dr, "\text{far-horizon limit}"] & \\
\mathrm{AdS}_7 \times S^{\mathbf{4}} /\!\!/ G_{\mathrm{ADE}} & & \mathbb{R}^{5,1} \times \mathbb{R}^1 \times \mathbb{R}^{\mathbf{4}} /\!\!/ G_{\mathrm{ADE}}
\end{tikzcd}$$

$$r \lesssim \ell_{\mathrm{Pl}} \sqrt[3]{N_{\mathrm{M5}}} \ll r$$

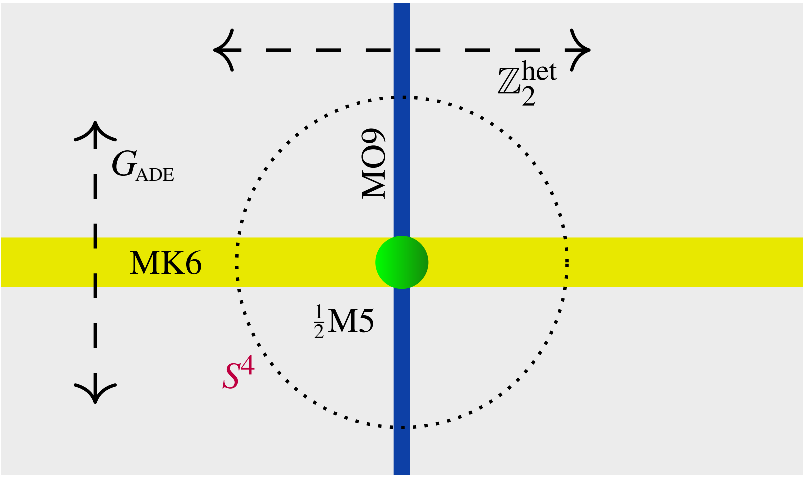
with $S^{\mathbf{4}} = S(\mathbb{R} \oplus \overset{G_{\mathrm{ADE}}\curvearrowright}{\mathbb{H}})$ as above

as in the above Thm.

throat geometry intersecting orbifold ADE-singularity

flat orbifold with ADE-singularity

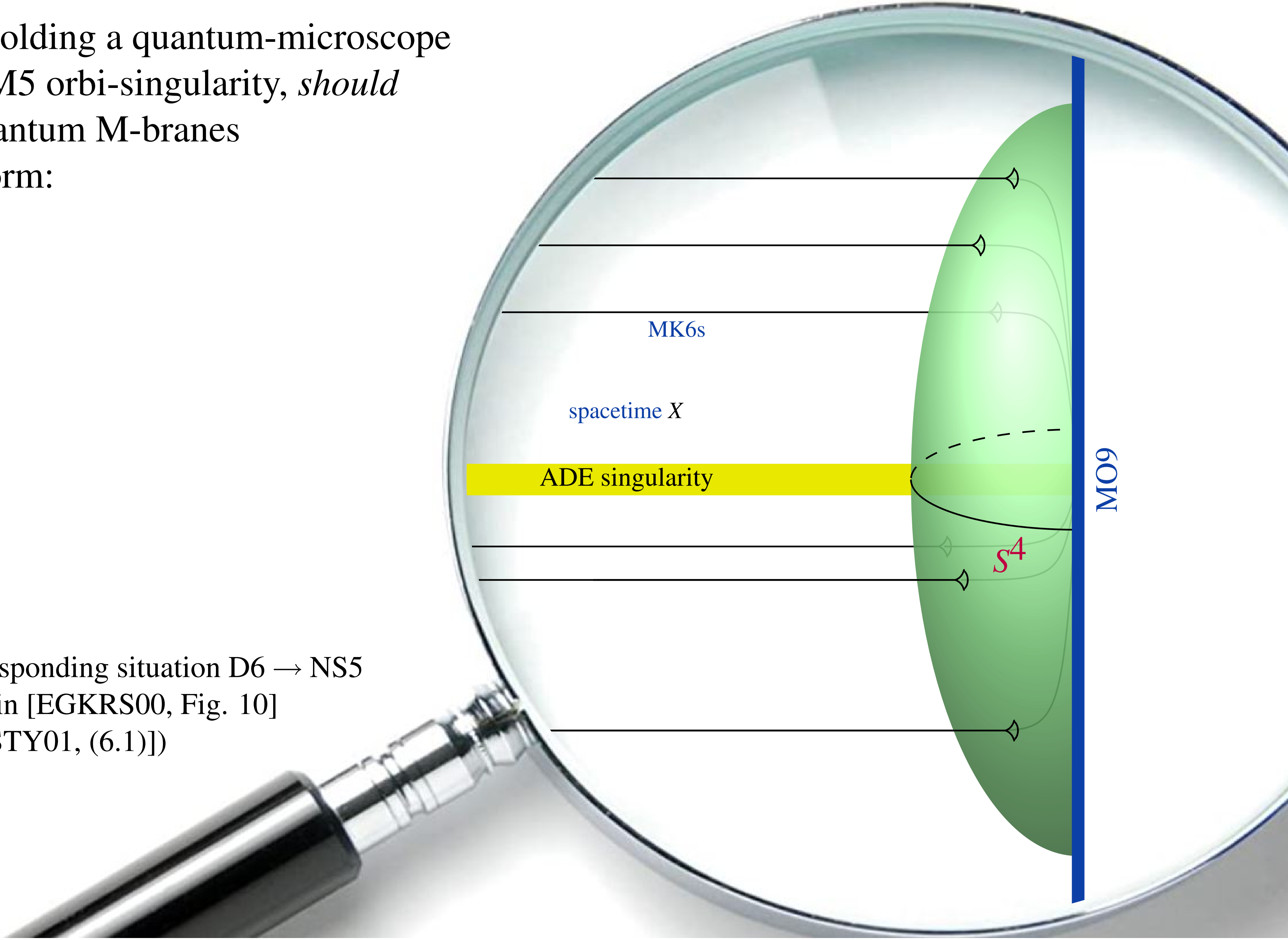
Consequence 3: An M5-shaped orbi-singularity must be $\underline{\mathrm{MK6} \perp \mathrm{MO9} =: \tfrac{1}{2}\mathrm{M5}}$:



E.g.: [GKSTY01, §6] [ZHTV14, §6] [GaTo14,§2.3]

Quantum M-branes?

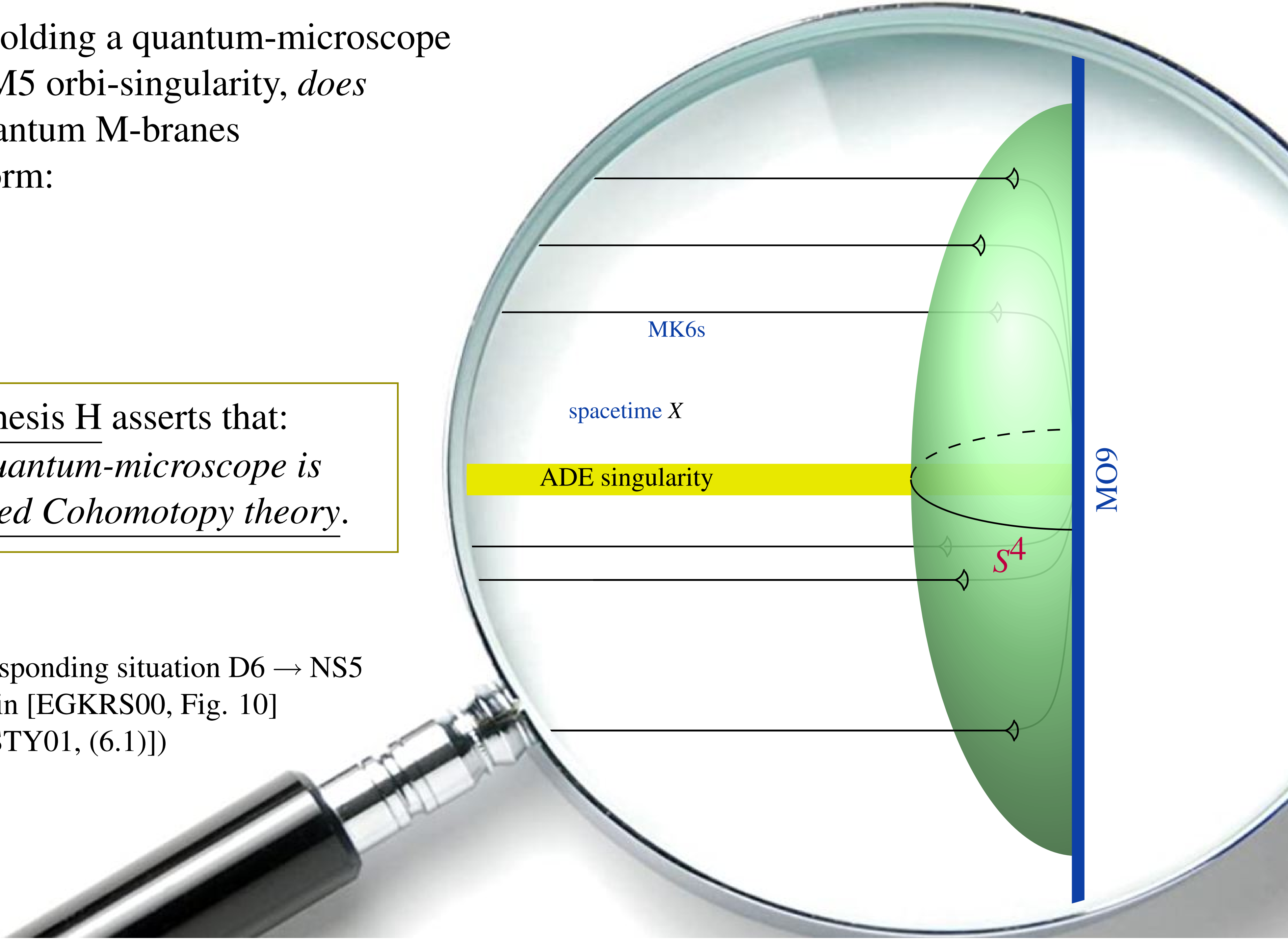
Hence, holding a quantum-microscope over a $\frac{1}{2}$M5 orbi-singularity, *should* show quantum M-branes of this form:



(the corresponding situation D6 → NS5 is shown in [EGKRS00, Fig. 10] and [GKSTY01, (6.1)])

Quantum M-branes!

Hence, holding a quantum-microscope over a $\frac{1}{2}$M5 orbi-singularity, *does* show quantum M-branes of this form:



Hypothesis H asserts that:
This quantum-microscope is J-twisted Cohomotopy theory.

(the corresponding situation D6 ⟶ NS5 is shown in [EGKRS00, Fig. 10] and [GKSTY01, (6.1)])

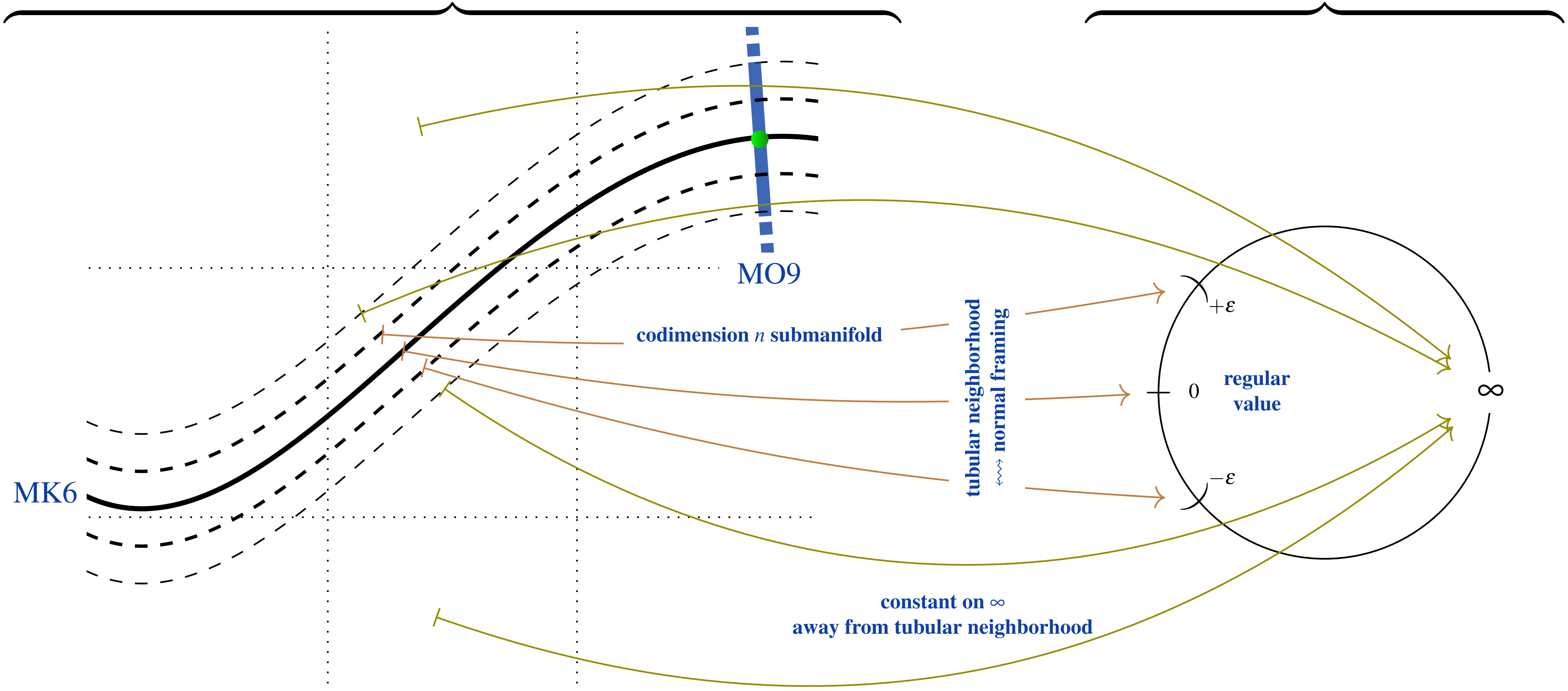
Cohomotopy charge map.

[SS19-TadCnc, §2.1] [SS21-MF, §2.2]

Namely, a small tubular neighbourhood of each MK6 carries *directed asymptotic transverse distance* from $\frac{1}{2}$M5 in MO9

⇒ *Cohomotopy charge*:

$$\underset{\text{spacetime manifold}}{X} \xrightarrow[\text{directed asymptotic transverse distance from MK6 loci in MO9-planes}]{\text{Cohomotopy charge}} \overset{\text{transversal space 1pt compactified}}{(\mathbb{R}^4)^{\mathrm{cpt}}} = \underset{\text{4-sphere Cohomotopy coefficient}}{S^4}$$



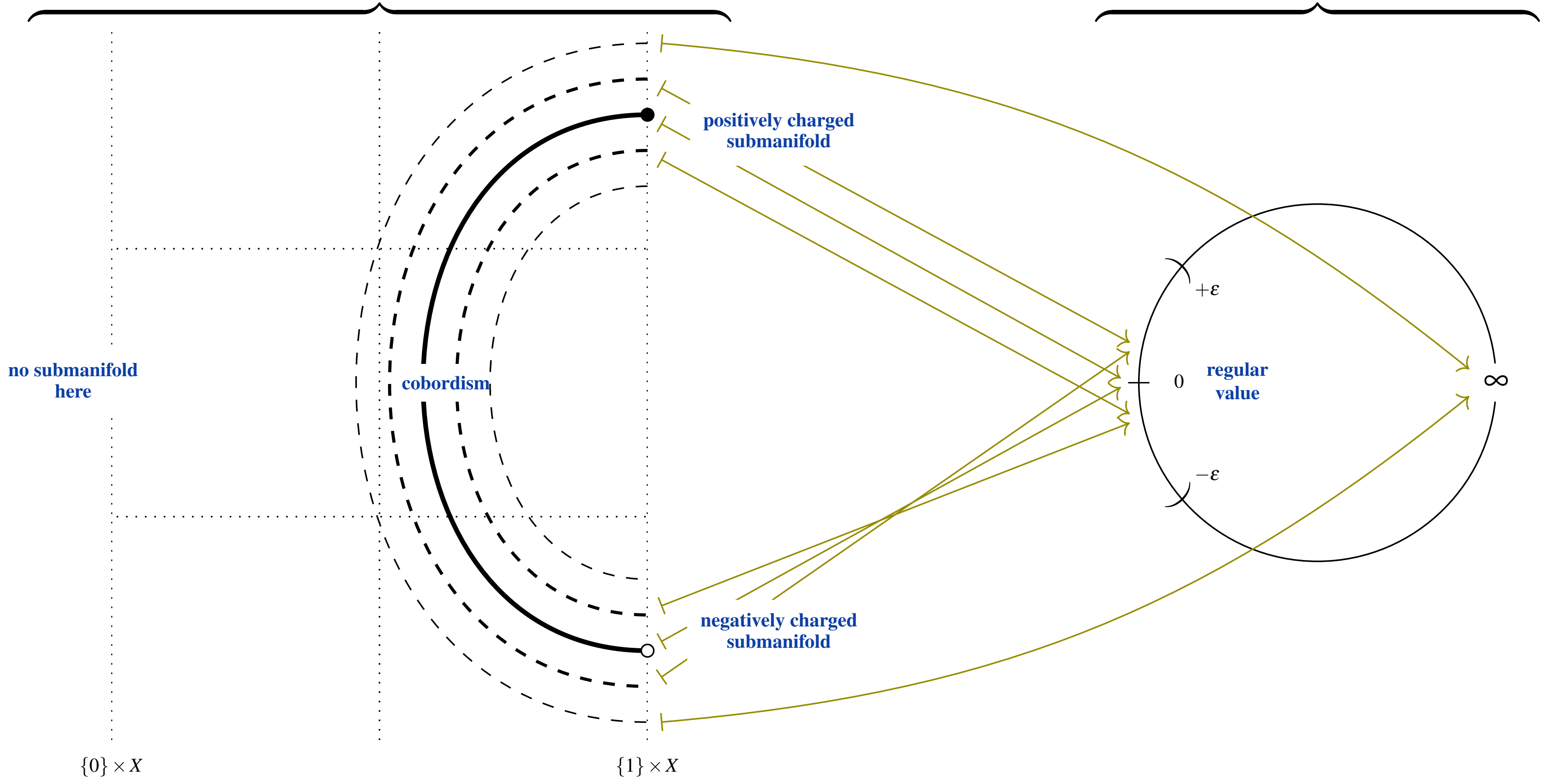
This construction and its reverse is *Pontrjagin's construction* ([Pon38], long before [Thom54]).

Anti-branes and Cobordism. [SS19-TadCnc, §2.1] [SS21-MF, §2.2]

Under the above Pontrjagin construction one finds that:

cobordisms exhibiting brane/anti-brane reactions ⟷ **coboundaries in Cohomotopy theory**

$$\underbrace{[0,1] \times X}_{\substack{\text{product space} \\ \text{of interval with manifold}}} \xrightarrow[\text{coboundary in Cohomotopy}]{0 \simeq (-1)+(+1)} \underbrace{(\mathbb{R}^n)^{\mathrm{cpt}} = S^n}_{\substack{n\text{-sphere} \\ \text{Cohomotopy coefficient}}}$$



Cohomotopy is framed Cobordism. [SS19-TadCnc, §2.1] [SS21-MF, §2.2]

Pontrjagin's theorem says that
4-Cohomotopy is the conserved charge of these M-brane/anti-brane reactions
in that the Cohomotopy charge map yields a *bijection* on cobordism classes:

Cohomotopy is framed Cobordism. [SS19-TadCnc, §2.1] [SS21-MF, §2.2]

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$$\overset{\text{unstable 4-Cohomotopy}}{\pi^4(\mathrm{X})} := \mathrm{Map}(\mathrm{X}, S^4)_{/\mathrm{htpy}} \; \underset{\text{Pontrjagin constr.}}{\overset{\text{Cohomotopy charge}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\sim}}}} \; \left\{ \begin{array}{l} \text{normally framed (nmly:} \\ \text{Cohomotopy-charged)} \\ \text{codim-4 submfds of X} \end{array} \right\}_{/\mathrm{cob}} =: \overset{\text{unstable framed 4-Cobordism}}{\mathrm{Cob}^4_{\mathrm{fr}}(\mathrm{X})}$$

quantized charges

quantum M-branes

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quantized charges

$$\begin{tikzcd}
\overset{\text{unstable 4-Cohomotopy}}{\pi^4(\mathrm{X})} := \mathrm{Map}(\mathrm{X}, S^4)_{/\mathrm{htpy}} \arrow[r, shift left=2, leftarrow, "\text{Cohomotopy charge}"] \arrow[r, phantom, "\sim"] \arrow[r, shift right=2, "\text{Pontrjagin constr.}"'] \arrow[d, "\text{1-step stabilization}"] & \left\{ \begin{array}{l} \text{normally framed (nmly:} \\ \text{Cohomotopy-charged)} \\ \text{codim-4 submfds of X} \end{array} \right\}_{/\mathrm{cob}} =: \overset{\text{unstable framed 4-Cobordism}}{\mathrm{Cob}^4_{\mathrm{fr}}(\mathrm{X})} \arrow[d, "\text{1-step stabilization}"', "\text{M5-brane moves away from MO9 into HW-bulk}"] \\
\pi^{4+1}(\mathrm{X} \times \mathbb{R}^1) \arrow[r, leftrightarrow, "\sim"] & \mathrm{Cob}^{4+1}_{\mathrm{fr}}(\mathrm{X} \times \mathbb{R}^1) \\
\text{remaining stabilization} &
\end{tikzcd}$$

quantum M-branes

Cohomotopy is framed Cobordism. [SS19-TadCnc, §2.1] [SS21-MF, §2.2]

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4-Cohomotopy is the conserved charge of these M-brane/anti-brane reactions
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quantized charges

$$\begin{tikzcd}
\text{unstable 4-Cohomotopy} & \text{unstable framed 4-Cobordism} \\
\pi^4(\mathrm{X}) := \mathrm{Map}(\mathrm{X}, S^4)_{/\mathrm{htpy}} \arrow[r, shift left=2, leftarrow, "\text{Cohomotopy charge}"] \arrow[r, phantom, "\sim"] \arrow[r, shift right=2, "\text{Pontrjagin constr.}"'] \arrow[d, "\text{1-step stabilization}"] & \left\{ \begin{array}{l} \text{normally framed (nmly:} \\ \text{Cohomotopy-charged)} \\ \text{codim-4 submfds of } \mathrm{X} \end{array} \right\}_{/\mathrm{cob}} =: \mathrm{Cob}^4_{\mathrm{fr}}(\mathrm{X}) \arrow[d, "\text{1-step stabilization}"', "\text{M5-brane moves away from MO9 into HW-bulk}"] \\
\pi^{4+1}(\mathrm{X} \times \mathbb{R}^1) \arrow[r, leftrightarrow, "\sim"] \arrow[d, dotted, "\text{remaining stabilization}"] & \mathrm{Cob}^{4+1}_{\mathrm{fr}}(\mathrm{X} \times \mathbb{R}^1) \arrow[d, dotted, "\text{remaining stabilization}"', "\text{often equivalence, e.g. for } \mathrm{X} \simeq S^7"] \\
\mathbb{S}^4(\mathrm{X}) \arrow[r, leftrightarrow, "\sim", "\text{Pontrjagin-Thom theorem}"'] & \mathrm{MFr}^4(\mathrm{X}) \\
\text{stable 4-Cohomotopy} & \text{stable framed 4-Cobordism}
\end{tikzcd}$$

quantum M-branes

Cohomotopy is framed Cobordism. [SS19-TadCnc, §2.1] [SS21-MF, §2.2]

Pontrjagin's theorem says that
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quantized charges

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\pi^{4+1}(\mathrm{X} \times \mathbb{R}^1) \arrow[r, leftrightarrow, "\sim"] \arrow[d, dotted, "\text{remaining stabilization}"] & \mathrm{Cob}^{4+1}_{\mathrm{fr}}(\mathrm{X} \times \mathbb{R}^1) \arrow[d, dotted, "\text{remaining stabilization}"', "\text{often equivalence, e.g. for } \mathrm{X} \simeq S^7"] \\
\underset{\text{stable 4-Cohomotopy}}{\mathbb{S}^4(\mathrm{X})} \arrow[r, leftrightarrow, "\sim", "\text{Pontrjagin-Thom theorem}"'] & \underset{\text{stable framed 4-Cobordism}}{\mathrm{MFr}^4(\mathrm{X})}
\end{tikzcd}$$

quantum M-branes

Rem.:

In particular this means that, in its stable = linearized approximation (cf above),
Hypothesis H says equivalently that M-brane charge is quantized in stable framed Cobordism.

This is reminiscent of discussion in [McNamara & Vafa 19], see [SS21-MF, §4] for more.

0 – Cohesive Homotopy Theory

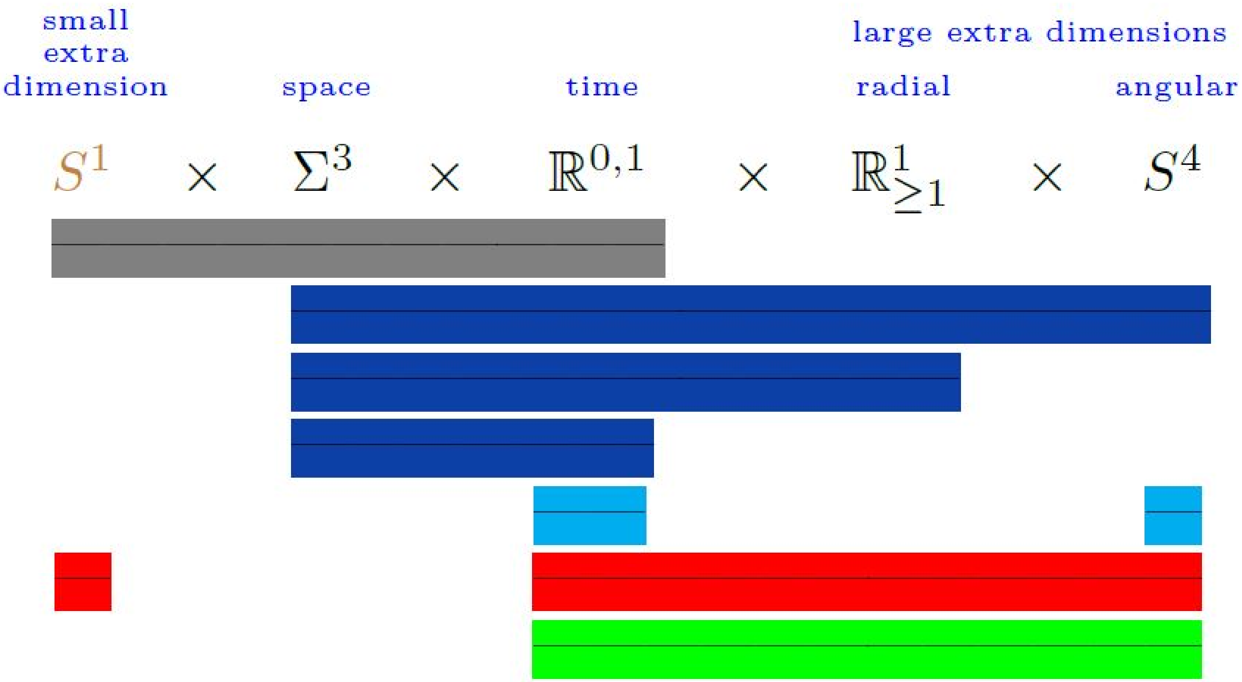
I – Quantum Charge of M-branes under Hypothesis H

II – Quantum Charge of D6 $\perp$ D8 under Hypothesis H

III – Quantum States of D6 $\perp$ D8 under Hypothesis H

Clay Millennium Problem:
To construct confined quantum chromodynamics (QCD).

D-brane configuration geometrically engineering quantum chromodynamics
(Witten-Sakai-Sugimoto model)



N_c color branes $\mathrm{D4}_{\mathrm{col}}$

N_f flavor branes $\mathrm{D8}_{\mathrm{fla}}$

meson fields $\mathrm{CS5}_{\mathrm{fla}}$ $\mathrm{WZ4}_{\mathrm{fla}}$

N_b baryon branes $\mathrm{D4}_{\mathrm{bar}}$

N_m monopole branes $\mathrm{D6}_{\mathrm{mon}}$

NS5

Cohomotopical D6 $\perp$ D8**-Charge.** §2 in [SS19-Quant]

The following slides survey how
D6 $\perp$ D8-brane moduli are seen in Cohomotopy theory.

The argument proceeds along these steps:

1. Reduction of cohom. M-branes to type IIA NS5/D-branes by cyclification.
2. Localized branes in flat spacetime via Cohomotopy vanishing-at-infinity.
3. Low codimension (defect-)branes from Cohomotopy in negative degrees.
4. The cohomotopical D$(9-d)$-brane moduli stack is identified with the configuration space of *labelled* points in transverse space, by Segal's theorem.
5. The intersection of these moduli for D6 $\perp$ D8-branes is equivalent to the configuration space of *ordered* points in D6-tranverse space.

(skip over all technicalities to punchline)

Double dimensional reduction of M-brane charge. [FSS16-TDual, §3][BSS19-RatSt, §2.2]

Prop.: $\mathscr{X} \in \mathbf{H}$ and $\mathscr{G} \in \mathrm{Grp}(\mathbf{H})$ $\quad\vdash\quad$ $$\begin{tikzcd} \mathbf{H} \arrow[r, phantom, "\perp"] & \mathbf{H}_{/\mathbf{B}\mathscr{G}} \arrow[l, bend right, "\mathrm{hofib}"'] \arrow[l, leftarrow, bend left, "\mathrm{Map}(\mathscr{G}, -)/\!\!/\mathscr{G}"'] \end{tikzcd}$$

$\mathscr{G}$-cyclification
∞-adjunction

Double dimensional reduction of M-brane charge. [FSS16-TDual, §3][BSS19-RatSt, §2.2]

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Ex.: $\mathscr{G} = S^1$ $\quad\vdash\quad$ $\mathscr{L}_{\mathrm{cyc}}(-) := \mathrm{Map}(S^1, -)/\!\!/S^1$ is the *cyclic loop space*.

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So, on an 11d spacetime circle bundle $S^1 \circlearrowright \mathscr{X} \longrightarrow \mathscr{X}/\!\!/S^1$ we have (see more exposition):

$$\pi^4(\mathscr{X}) = \left\{ \mathscr{X} \xrightarrow{\quad\text{cocycle}\quad} \textstyle\int S^4 \right\}_{/\mathrm{htpy}}$$

$$\simeq \left\{ \begin{tikzcd} \mathscr{X}/\!\!/S^1 \arrow[rr, "\substack{\text{cyclified}\\ \text{cocycle}}"] \arrow[dr, dashed, "\text{non-wrapped}"'] & & \mathscr{L}_{\mathrm{cyc}}\left(\int S^4\right) \\ & \int S^4 \arrow[ur, "\text{constant loops}"'] & \end{tikzcd} \right\}_{/\mathrm{htpy}}$$

$$\begin{tikzcd} \text{charges on 11d spacetime} \arrow[d, shift right=2, "\substack{\text{double dim}\\ \text{reduction}}"'] \\ \text{charges on 10d spacetime} \arrow[u, shift right=2, "\text{oxidation}"'] \end{tikzcd}$$

Double dimensional reduction of M-brane charge. [FSS16-TDual, §3][BSS19-RatSt, §2.2]

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$$\begin{tikzcd} \mathbf{H} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\mathrm{Map}(\mathscr{G},-)/\!\!/\mathscr{G}"'] & \mathbf{H}_{/\mathbf{B}\mathscr{G}} \arrow[l, bend right, "\text{hofib}"'] \end{tikzcd}$$

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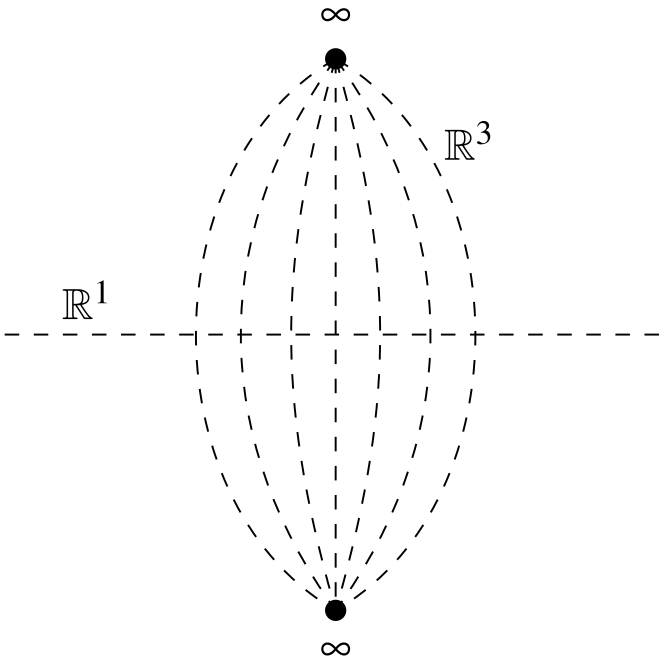
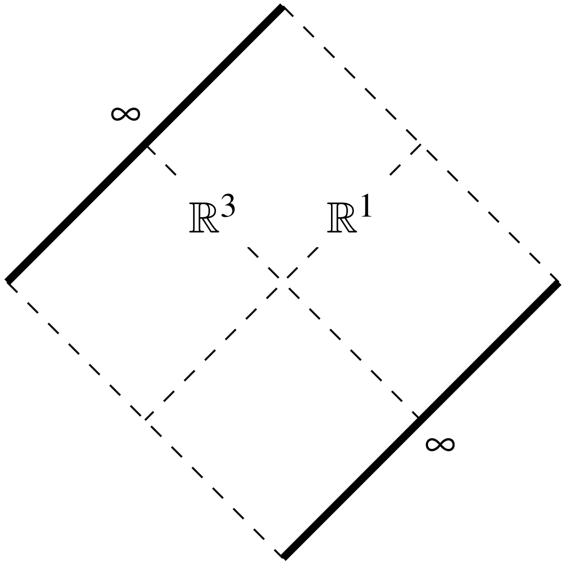
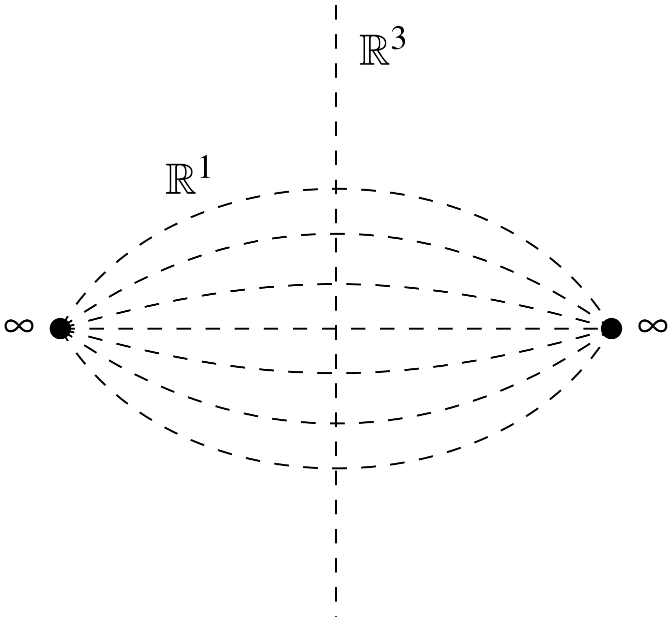
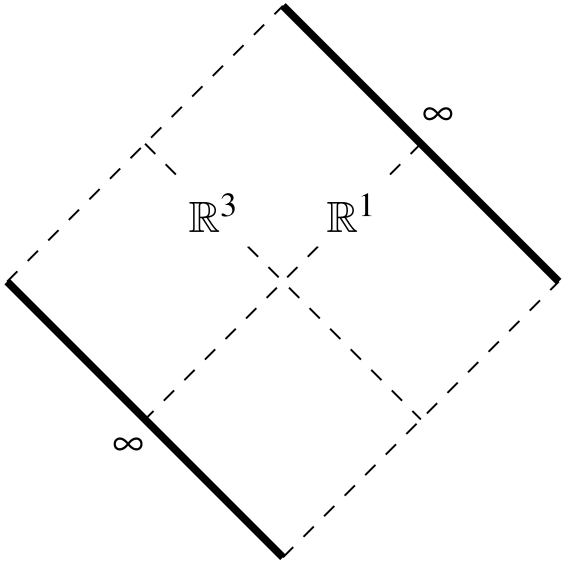
$$\xRightarrow{\text{HypothesisH}}$$

Dp/NS5**-brane charge** in 10d:
- in general is quantized in cyclified 4-Cohomotopy;
- for vanishing D(<5)-charge **is quantized again in 4-Cohomotopy**.

Charges vanishing at ∞ are seen by Cohomotopy of pointed spaces. [SS19-Quant, §2.1]

Smash product of pointed topological spaces	Visualization	
	with point at infinity	as Penrose diagram
cocycles vanish at infinity along these directions $\overbrace{\mathbb{R}^3_{\mathrm{cpt}}} \wedge \underbrace{\mathbb{R}^1_{+}}$ ...but not necessarily along these		
cocycles vanish at infinity along these direction $\underbrace{\mathbb{R}^3_{+}} \wedge \overbrace{\mathbb{R}^1_{\mathrm{cpt}}}$ ...but not necessarily along these		

Low codim branes from Cohomotopy charge in negative degree. [SS19-Quant, §2.3]

For transversal $d \leq 3$, all Cohomotopy charge vanishing at ∞ is trivial up to gauge:

$$\pi^4\big(\mathbb{R}_+^{11-d} \wedge \mathbb{R}_{\mathrm{cpt}}^{d\leq 3}\big) \;\simeq\; \pi^4\big(S^{d\leq 3}\big) \;\simeq\; \pi_{\leq 3}\big(S^4\big) \;\simeq\; \ast\,.$$

reflecting the absence of low codimension black p-branes in M-theory.

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But the full moduli space of Cohomotopy cocycles

$$\boldsymbol{\pi}^4\left(\mathbb{R}^d_{\mathrm{cpt}}\right) \quad := \quad \mathrm{Map}(\mathbb{R}^d_{\mathrm{cpt}}, \int S^4)$$

of which the manifest Cohomotopy charge is only the connected components witnesses a rich world of higher gauge solitons:

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Thm. ([Segal 1973, Thm. 3])
The Cohomotopy charge map identifies the above Cohomotopy moduli space with the configuration space of points in $\mathbb{R}^d$ with labels in $\mathbb{D}^{4-d}/\mathrm{bdr}$:

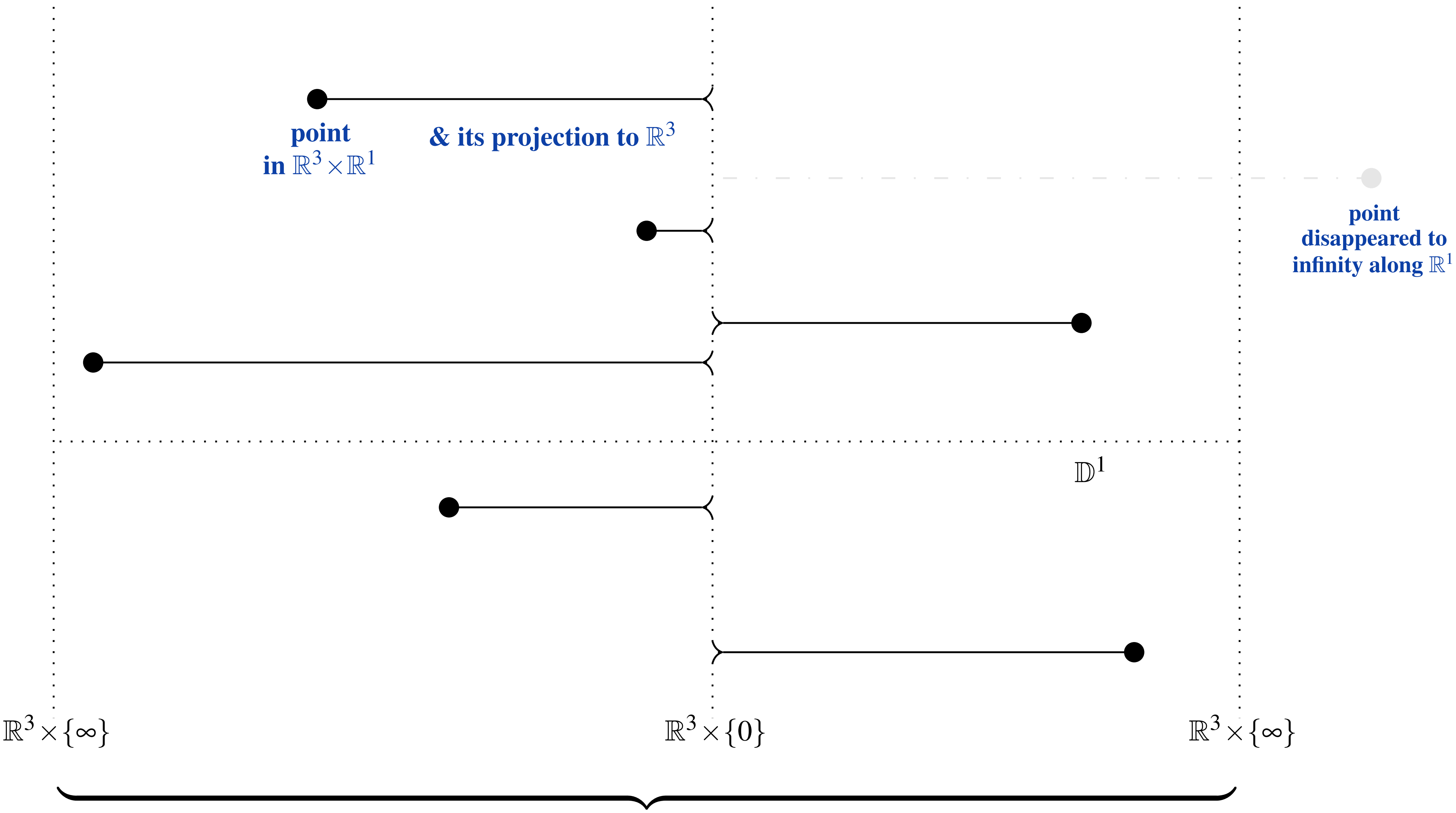
$$\mathrm{Conf}\big(\mathbb{R}^d, \mathbb{D}^{4-d}\big) \xrightarrow[\substack{\sim \\ \text{homotopy equivalence}}]{\text{Cohomotopy charge map}} \boldsymbol{\pi}^4\big(\mathbb{R}^d_{\mathrm{cpt}}\big)$$

configuration space of points
in $\mathbb{R}^d$ with labels in $\mathbb{D}^{4-d}$
disappearing at $\partial\mathbb{D}^{4-d}$

moduli space of
Cohomotopy cocycles

Low codim branes from Cohomotopy charge in negative degree.

[SS19-Quant, §2.3]



$$\mathrm{Conf}\big(\mathbb{R}^3, \mathbb{D}^{4-3}\big) \xrightarrow[\sim]{\text{Cohomotopy charge map}} \boldsymbol{\pi}^4\big(\mathbb{R}^3_{\mathrm{cpt}}\big)$$

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moduli space of
Cohomotopy cocycles

Configuration spaces are a form of differential Cohomotopy. [SS19-Quant, §2.3]

But the configuration space carries the geometric structure of a smooth 0-stack:

$$\mathrm{Conf}\big(\mathbb{R}^d, \mathbb{D}^{4-d}\big) \;\in\; \mathrm{DfflSpc} \overset{i_{0,\sharp_1}}{\hookrightarrow} \mathrm{SmthGrpd}_\infty\,,$$

while the plain Cohomotopy cocycle space is geometrically discrete:

$$\boldsymbol{\pi}^4\big(\mathbb{R}^d_{\mathrm{cpt}}\big) \;\in\; \mathrm{Grpd}_\infty \overset{\mathrm{Dsc}}{\hookrightarrow} \mathrm{SmthGrpd}_\infty\,.$$

Therefore, Segal's theorem says that the
configuration spaces constitute a *differential refinement* of Cohomotopy theory,
(on $\mathbb{R}^d_{\mathrm{cpt}}$s):

$$\overset{\text{a form of } \textbf{differential Cohomotopy}}{\boldsymbol{\pi}^4_{\mathrm{diff}}\big(\mathbb{R}^3_{\mathrm{cpt}}\big) := \underset{\substack{\textbf{configuration space of points}\\ \textbf{in } \mathbb{R}^3 \textbf{ with labels in } \mathbb{D}^{4-3}\\ \textbf{disappearing at } \partial\mathbb{D}^{4-3}}}{\mathrm{Conf}\big(\mathbb{R}^3, \mathbb{D}^{4-3}\big)}} \xrightarrow[\substack{\eta^{\int}\\ \textbf{shape unit}}]{\textbf{Cohomotopy charge map}} \underset{\substack{\textbf{moduli space of}\\ \textbf{Cohomotopy cocycles}}}{\boldsymbol{\pi}^4\big(\mathbb{R}^3_{\mathrm{cpt}}\big)}$$

Cohomotopical D6 $\perp$ D8**-Charge.** [SS19-Quant, Prop. 2.4, 2.11]

$$\boldsymbol{\pi}^4_{\mathrm{diff}}\big(\mathbb{R}^3_{\mathrm{cpt}} \wedge \mathbb{R}^1_{+} \cup \mathbb{R}^3_{+} \wedge \mathbb{R}^1_{\mathrm{cpt}}\big)$$

Therefore we now obtain
Cohomotopy charge of intersecting codim 3/1-branes

Cohomotopical D6 $\perp$ D8**-Charge.** [SS19-Quant, Prop. 2.4, 2.11]

$$\boldsymbol{\pi}^4_{\mathrm{diff}}\big(\mathbb{R}^3_{\mathrm{cpt}} \wedge \mathbb{R}^1_{+} \cup \mathbb{R}^3_{+} \wedge \mathbb{R}^1_{\mathrm{cpt}}\big) \quad = \boldsymbol{\pi}^4_{\mathrm{diff}}\big(\mathbb{R}^3_{\mathrm{cpt}} \wedge \mathbb{R}^1_{+}\big) \cap \boldsymbol{\pi}^4_{\mathrm{diff}}\big(\mathbb{R}^3_{+} \wedge \mathbb{R}^1_{\mathrm{cpt}}\big)$$

Therefore we now obtain
Cohomotopy charge of intersecting codim 3/1-branes as
the fiber product of their separate differential Cohomotopy charge.

$\rightsquigarrow$

Cohomotopical D6 $\perp$ D8**-Charge.** [SS19-Quant, Prop. 2.4, 2.11]

$$\boldsymbol{\pi}^4_{\mathrm{diff}}\big(\mathbb{R}^3_{\mathrm{cpt}} \wedge \mathbb{R}^1_{+} \cup \mathbb{R}^3_{+} \wedge \mathbb{R}^1_{\mathrm{cpt}}\big) \quad = \boldsymbol{\pi}^4_{\mathrm{diff}}\big(\mathbb{R}^3_{\mathrm{cpt}} \wedge \mathbb{R}^1_{+}\big) \cap \boldsymbol{\pi}^4_{\mathrm{diff}}\big(\mathbb{R}^3_{+} \wedge \mathbb{R}^1_{\mathrm{cpt}}\big)$$

$$\underset{\text{Segal's theorem}}{\simeq} \mathrm{Conf}\big(\mathbb{R}^3, \mathbb{D}^1\big) \cap \mathrm{Conf}\big(\mathbb{R}^1, \mathbb{D}^3\big)$$

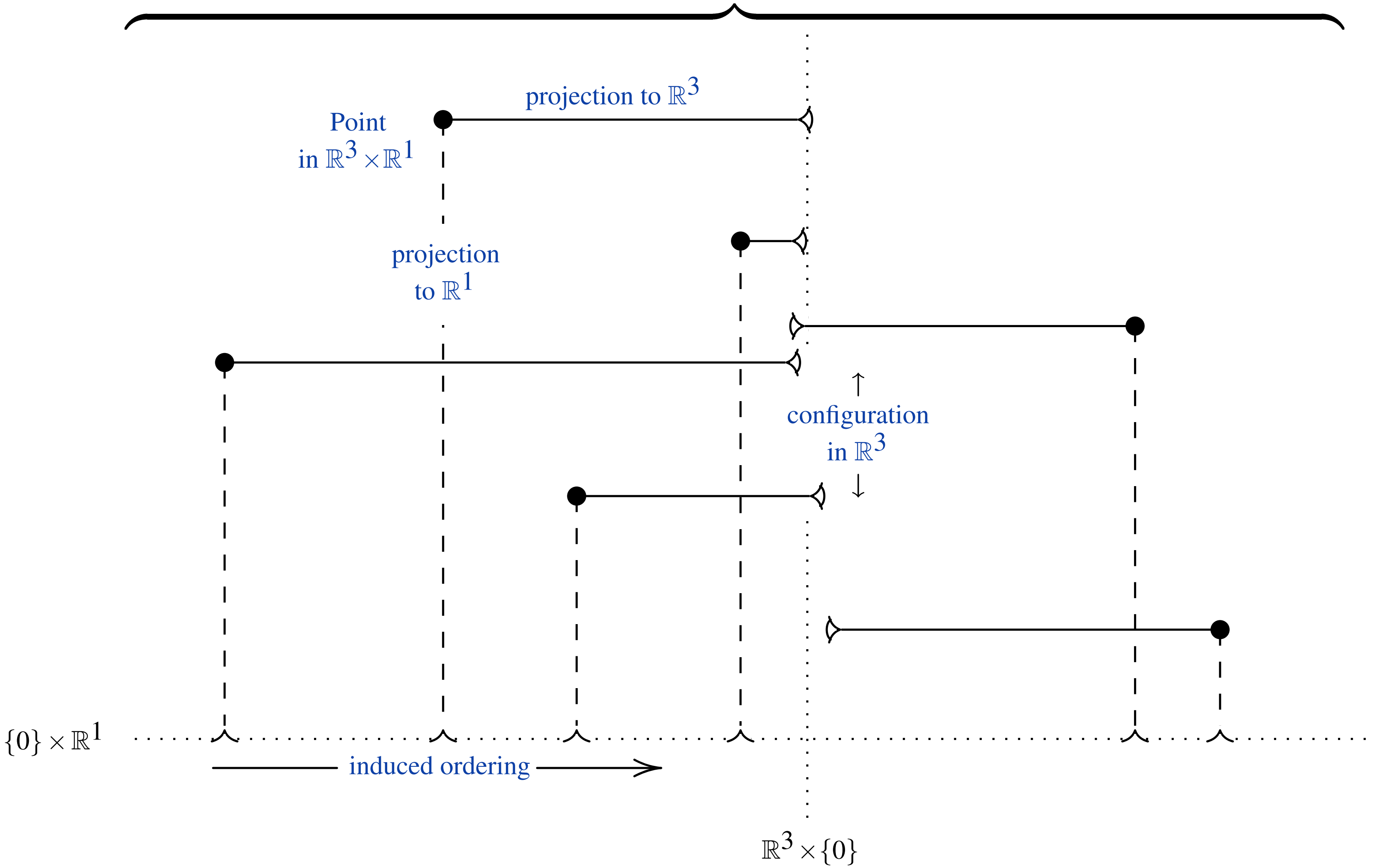
Cohomotopical D6 ⊥ D8**-Charge.**

[SS19-Quant, Prop. 2.4, 2.11]

$$\boldsymbol{\pi}^4_{\mathrm{diff}}\big(\mathbb{R}^3_{\mathrm{cpt}} \wedge \mathbb{R}^1_+ \cup \mathbb{R}^3_+ \wedge \mathbb{R}^1_{\mathrm{cpt}}\big) \quad = \boldsymbol{\pi}^4_{\mathrm{diff}}\big(\mathbb{R}^3_{\mathrm{cpt}} \wedge \mathbb{R}^1_+\big) \cap \boldsymbol{\pi}^4_{\mathrm{diff}}\big(\mathbb{R}^3_+ \wedge \mathbb{R}^1_{\mathrm{cpt}}\big)$$

$$\simeq \mathrm{Conf}\big(\mathbb{R}^3, \mathbb{D}^1\big) \cap \mathrm{Conf}\big(\mathbb{R}^1, \mathbb{D}^3\big) \quad \simeq \underset{n\in\mathbb{N}}{\sqcup} \mathrm{Conf}_{\{1,\cdots,n\}}\big(\mathbb{R}^3\big)$$

ordered configuration space



Cohomotopical D6 ⊥ D8**-Charge.** [SS19-Quant, Prop. 2.4, 2.11]

$$\boldsymbol{\pi}^4_{\mathrm{diff}}\big(\mathbb{R}^3_{\mathrm{cpt}} \wedge \mathbb{R}^1_{+} \cup \mathbb{R}^3_{+} \wedge \mathbb{R}^1_{\mathrm{cpt}}\big) \;=\; \boldsymbol{\pi}^4_{\mathrm{diff}}\big(\mathbb{R}^3_{\mathrm{cpt}} \wedge \mathbb{R}^1_{+}\big) \cap \boldsymbol{\pi}^4_{\mathrm{diff}}\big(\mathbb{R}^3_{+} \wedge \mathbb{R}^1_{\mathrm{cpt}}\big)$$

$$\simeq \mathrm{Conf}\big(\mathbb{R}^3, \mathbb{D}^1\big) \cap \mathrm{Conf}\big(\mathbb{R}^1, \mathbb{D}^3\big) \;\simeq\; \bigsqcup_{n\in\mathbb{N}} \underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{R}^3\big)$$

ordered configuration space

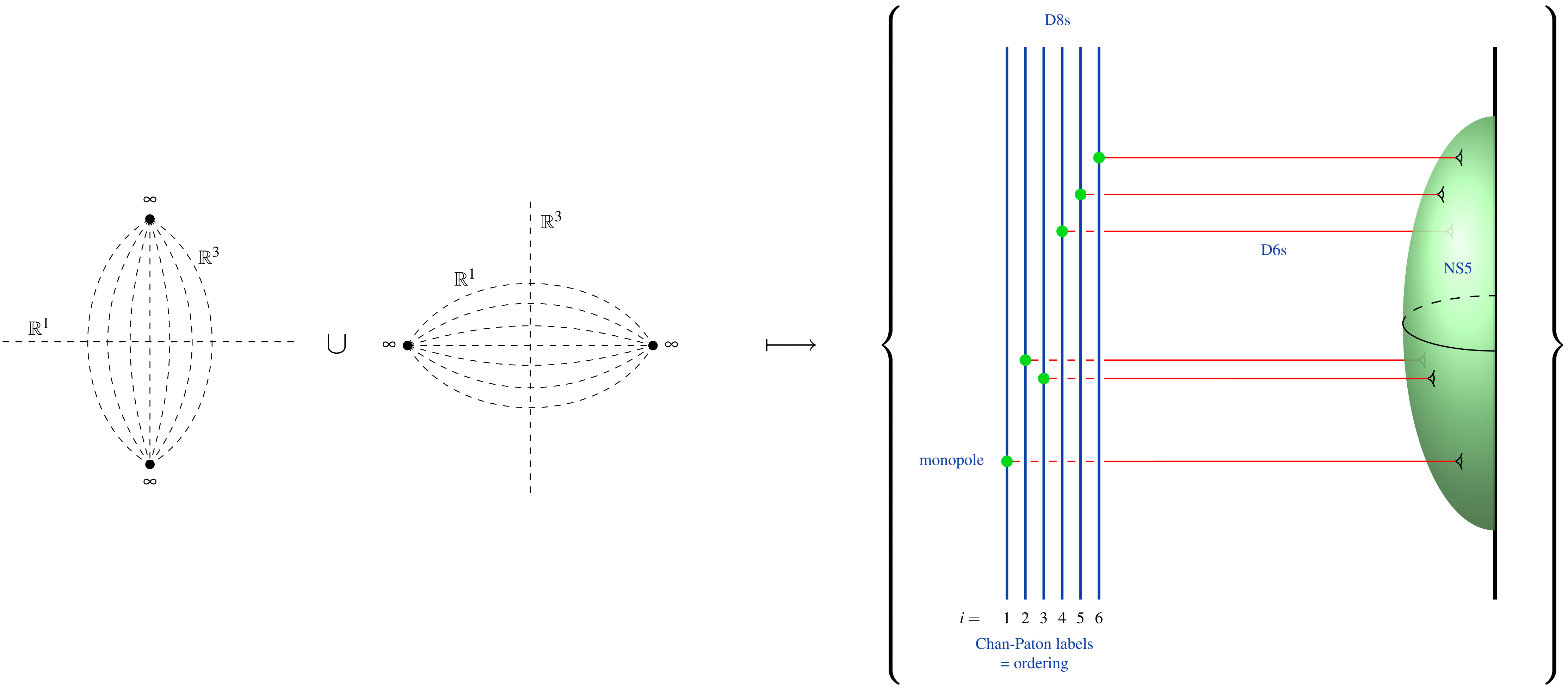
Transversal space to 3-codim branes hence to D6-branes

Transversal space to 1-codim branes hence to D8-branes

Differential Cohomotopy

Ordered configuration space

$$(\mathbb{R}^3)^{\mathrm{cpt}} \wedge (\mathbb{R}^1)_+ \;\cup\; (\mathbb{R}^3)_+ \wedge (\mathbb{R}^1)^{\mathrm{cpt}} \xmapsto{\quad \boldsymbol{\pi}^4_{\mathrm{diff}} \quad} \bigsqcup_{n\in\mathbb{N}} \underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{R}^3\big)$$



0 – Cohesive Homotopy Theory

I – Quantum Charge of M-branes under Hypothesis H

II – Quantum Charge of D6 $\perp$ D8 under Hypothesis H

III – Quantum States of D6 $\perp$ D8 under Hypothesis H

D6 $\perp$ D8-**states as pos. weight systems on hor. chord diagrams.** [SS19-Quant, §2.5,3.5]

Given the moduli stack of D6 $\perp$ D8-branes according to above discussion we turn to describing its quant. states & observables.

$$\boldsymbol{\pi}^4_{\mathrm{diff}}\left(\begin{array}{c} \mathbb{R}^3_{\mathrm{cpt}} \wedge \mathbb{R}^1_{+} \\ \cup \;\; \mathbb{R}^3_{+} \wedge \mathbb{R}^1_{\mathrm{cpt}} \end{array}\right)$$

by the above discussion:
moduli stack of
D6 $\perp$ D8**-branes**
by Hypothesis H

D6 $\perp$ D8-states as pos. weight systems on hor. chord diagrams. [SS19-Quant, §2.5,3.5]

The covariant phase space of any physical theory is the space of its field histories. Topologically these are loops of field configurations.

cov. phase space (topol. sect.)

$$\bigsqcup_{N \in \mathbb{N}} \Omega_N \, \boldsymbol{\pi}^4_{\mathrm{diff}} \left(\begin{array}{c} \mathbb{R}^3_{\mathrm{cpt}} \wedge \mathbb{R}^1_{+} \\ \cup \;\; \mathbb{R}^3_{+} \wedge \mathbb{R}^1_{\mathrm{cpt}} \end{array} \right)$$

D6 $\perp$ D8-**states as pos. weight systems on hor. chord diagrams.** [SS19-Quant, §2.5,3.5]

But by the above theorem this are equivalently loops of configurations of ordered points in the D6-transverse space.

cov. phase space (topol. sect.)

$$\begin{tikzcd}
\underset{N \in \mathbb{N}}{\sqcup} \Omega_N \boldsymbol{\pi}^4_{\mathrm{diff}} \left(\begin{array}{c} \mathbb{R}^3_{\mathrm{cpt}} \wedge \mathbb{R}^1_{+} \\ \cup \;\; \mathbb{R}^3_{+} \wedge \mathbb{R}^1_{\mathrm{cpt}} \end{array} \right) \arrow[d, "\wr", "\text{[SS19-Quant, §2.4], as above}"'] \\
\underset{N \in \mathbb{N}}{\sqcup} \Omega \left(\underset{\{1, \cdots N\}}{\mathrm{Conf}} (\mathbb{R}^3) \right)
\end{tikzcd}$$

conf. space of points

D6 $\perp$ D8-**states as pos. weight systems on hor. chord diagrams.** [SS19-Quant, §2.5,3.5]

Higher topol. observables on this phase space are its compactly supp. cohomology hence its homology (with complex coefficients).

cov. phase space (topol. sect.)

$$\begin{tikzcd}
 & \underset{N\in\mathbb{N}}{\sqcup}\,\Omega_N\,\boldsymbol{\pi}^4_{\mathrm{diff}}\left(\begin{array}{c}\mathbb{R}^3_{\mathrm{cpt}}\wedge\mathbb{R}^1_{+}\\ \cup\;\; \mathbb{R}^3_{+}\wedge\mathbb{R}^1_{\mathrm{cpt}}\end{array}\right) \arrow[d, "\wr", "\text{[SS19-Quant, §2.4], as above}"'] \\
 & \underset{N\in\mathbb{N}}{\sqcup}\,\Omega\Big(\underset{\{1,\cdots N\}}{\mathrm{Conf}}(\mathbb{R}^3)\Big) \arrow[dl, mapsto, "\text{homology}"'] \\
H_\bullet\Big(\underset{N\in\mathbb{N}}{\sqcup}\,\Omega\Big(\underset{\{1,\cdots N\}}{\mathrm{Conf}}(\mathbb{R}^3)\Big)\Big) &
\end{tikzcd}$$

conf. space of points

observables

D6 ⊥ D8-**states as pos. weight systems on hor. chord diagrams.** [SS19-Quant, §2.5,3.5]

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on this phase space are its
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cov. phase space (topol. sect.)

$$\begin{tikzcd}
 & & \bigsqcup_{N\in\mathbb{N}} \Omega_{N}\, \boldsymbol{\pi}^{4}_{\mathrm{diff}} \left(\begin{array}{c} \mathbb{R}^{3}_{\mathrm{cpt}} \wedge \mathbb{R}^{1}_{+} \\ \cup \;\; \mathbb{R}^{3}_{+} \wedge \mathbb{R}^{1}_{\mathrm{cpt}} \end{array} \right) \arrow[d, "\wr"', "\text{[SS19-Quant, §2.4], as above}"] & \\
 & & \bigsqcup_{N\in\mathbb{N}} \Omega\Big(\underset{\{1,\cdots N\}}{\mathrm{Conf}}\big(\mathbb{R}^{3}\big) \Big) \arrow[dl, mapsto, "\text{homology}"'] \arrow[dr, mapsto, "\text{cohomology}"] & \\
\text{observables} & H_{\bullet}\Bigg(\bigsqcup_{N\in\mathbb{N}} \Omega\Big(\underset{\{1,\cdots N\}}{\mathrm{Conf}}\big(\mathbb{R}^{3}\big) \Big) \Bigg) \arrow[rr, leftrightarrow, "(-)^{*}"] & & H^{\bullet}\Bigg(\bigsqcup_{N\in\mathbb{N}} \Omega\Big(\underset{\{1,\cdots N\}}{\mathrm{Conf}}\big(\mathbb{R}^{3}\big) \Big) \Bigg)
\end{tikzcd}$$

conf. space of points

D6 $\perp$ D8-**states as pos. weight systems on hor. chord diagrams.** [SS19-Quant, §2.5,3.5]

This homology algebra is that of hor. *chord diagrams* and its linear dual is that of *weight systems* (both explained in a moment)

$$\begin{tikzcd}
 & & \textbf{cov. phase space} \text{ (topol. sect.)} & & \\
 & & \underset{N \in \mathbb{N}}{\bigsqcup} \Omega_{N} \boldsymbol{\pi}^4_{\mathrm{diff}} \left(\begin{array}{c} \mathbb{R}^3_{\mathrm{cpt}} \wedge \mathbb{R}^1_{+} \\ \cup \;\; \mathbb{R}^3_{+} \wedge \mathbb{R}^1_{\mathrm{cpt}} \end{array} \right) \arrow[d, "\wr"', "\text{[SS19-Quant, §2.4], as above}"] & & \\
 & & \begin{array}{c} \underset{N \in \mathbb{N}}{\bigsqcup} \Omega \Big(\underset{\{1,\cdots N\}}{\mathrm{Conf}} (\mathbb{R}^3) \Big) \\ \textbf{conf. space of points} \end{array} \arrow[dl, mapsto, "\textbf{homology}"'] \arrow[dr, mapsto, "\textbf{cohomology}"] & & \\
\textbf{observables} & H_{\bullet} \left(\underset{N \in \mathbb{N}}{\bigsqcup} \Omega \Big(\underset{\{1,\cdots N\}}{\mathrm{Conf}} (\mathbb{R}^3) \Big) \right) \arrow[rr, leftrightarrow, "(-)^*"] \arrow[dd, equal] & & H^{\bullet} \left(\underset{N \in \mathbb{N}}{\bigsqcup} \Omega \Big(\underset{\{1,\cdots N\}}{\mathrm{Conf}} (\mathbb{R}^3) \Big) \right) \arrow[dd, equal] & \\
 & & \text{[FaHu 01] [Kohno 02] [SS19-Quant, Prop. 2.18]} \arrow[u, phantom, "\|"] \arrow[d, phantom, "\|"] & & \\
\textbf{hor. chord diagrams} & \underset{N \in \mathbb{N}}{\bigoplus} \mathscr{A}^{\mathrm{pb}}_{N} \arrow[rr, leftrightarrow, "(-)^*"'] & & \underset{N \in \mathbb{N}}{\bigoplus} \left(\mathscr{A}^{\mathrm{pb}}_{N} \right)^* & \textbf{weight systems}
\end{tikzcd}$$

D6 $\perp$ D8-states as pos. weight systems on hor. chord diagrams. [SS19-Quant, §2.5,3.5]

Since algebra of observables is canonically a star-algebra the quantum states on it are the pos. dual elmts. $\langle - \rangle$: i.e. such that $\langle \mathscr{O}^* \mathscr{O} \rangle \geq 0$.

cov. phase space (topol. sect.)

$$\begin{tikzcd}[column sep=small]
& \underset{N \in \mathbb{N}}{\sqcup} \Omega_N \boldsymbol{\pi}^4_{\mathrm{diff}} \left(\begin{array}{c} \mathbb{R}^3_{\mathrm{cpt}} \wedge \mathbb{R}^1_{+} \\ \cup \;\; \mathbb{R}^3_{+} \wedge \mathbb{R}^1_{\mathrm{cpt}} \end{array} \right) \arrow[d, "\wr"', "\text{[SS19-Quant, §2.4], as above}"] & \\
& \underset{N \in \mathbb{N}}{\sqcup} \Omega \Big(\underset{\{1,\cdots N\}}{\mathrm{Conf}} (\mathbb{R}^3) \Big) \arrow[dl, mapsto, "\text{homology}"'] \arrow[dr, mapsto, "\text{cohomology}"] & \\
H_\bullet \Big(\underset{N \in \mathbb{N}}{\sqcup} \Omega \Big(\underset{\{1,\cdots N\}}{\mathrm{Conf}} (\mathbb{R}^3) \Big) \Big) \arrow[rr, leftrightarrow, "(-)^*"] \arrow[d, equal] & {} \arrow[d, phantom, "\text{[FaHu 01] [Kohno 02] [SS19-Quant, Prop. 2.18]}"] & H^\bullet \Big(\underset{N \in \mathbb{N}}{\sqcup} \Omega \Big(\underset{\{1,\cdots N\}}{\mathrm{Conf}} (\mathbb{R}^3) \Big) \Big) \arrow[d, equal] \\
\underset{N \in \mathbb{N}}{\bigoplus} \mathscr{A}^{\mathrm{pb}}_N \arrow[rr, leftrightarrow, "(-)^*"'] & {} & \underset{N \in \mathbb{N}}{\bigoplus} \big(\mathscr{A}^{\mathrm{pb}}_N \big)^* \\
& & \underset{N \in \mathbb{N}}{\bigoplus} \big(\mathscr{A}^{\mathrm{pb}}_N \big)^*_{\mathrm{pos}} \arrow[u, hook, "\mathrm{conv}"', "\text{[SS19-Quant, §3.5]}"]
\end{tikzcd}$$

conf. space of points

observables

hor. chord diagrams

weight systems

quantum states

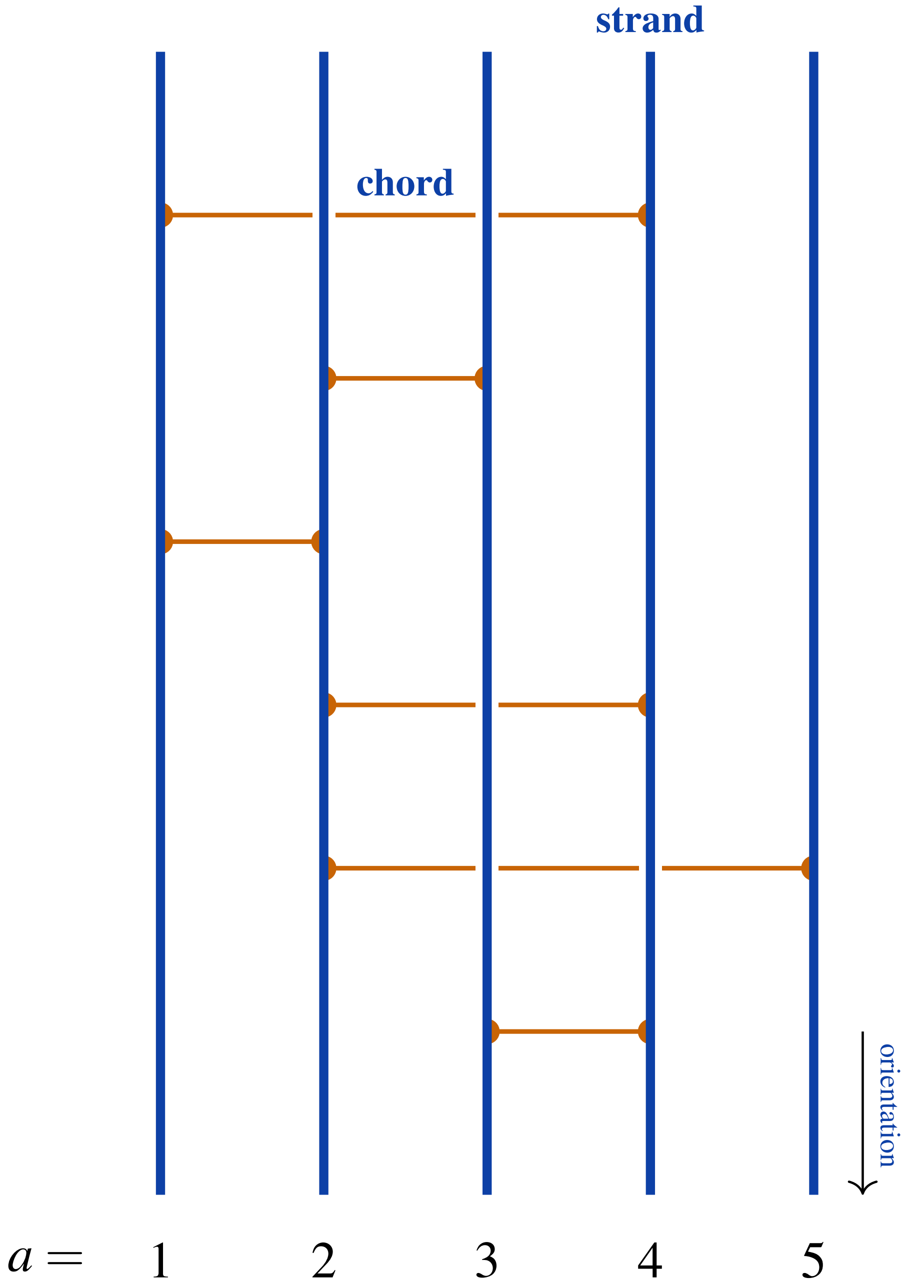
D6 ⊥ D8-states as pos. weight systems on hor. chord diagrams. [SS19-Quant, §2.5,3.5]

We proved that at least the fund. $\mathfrak{gl}_2(\mathbb{C})$ weight syst. is a non-ghost quantum state. Will explain below how these are sensible D6 ⊥ D8-states.

$$\begin{tikzcd}
 & & \textbf{cov. phase space}\ \text{(topol. sect.)} & & \\
 & & \underset{N\in\mathbb{N}}{\bigsqcup} \Omega_N \boldsymbol{\pi}^4_{\mathrm{diff}}\left(\begin{array}{c} \mathbb{R}^3_{\mathrm{cpt}} \wedge \mathbb{R}^1_{+} \\ \cup\ \ \mathbb{R}^3_{+} \wedge \mathbb{R}^1_{\mathrm{cpt}} \end{array}\right) \arrow[d, "\wr\quad\text{[SS19-Quant, §2.4], as above}"] & & \\
 & & \underset{N\in\mathbb{N}}{\bigsqcup} \Omega\Big(\underset{\{1,\cdots N\}}{\mathrm{Conf}}(\mathbb{R}^3)\Big) \arrow[ddl, mapsto, "\text{homology}"'] \arrow[ddr, mapsto, "\text{cohomology}"] & & \\
 & & \textbf{conf. space of points} & & \\
\textbf{observables} & H_\bullet\bigg(\underset{N\in\mathbb{N}}{\bigsqcup} \Omega\Big(\underset{\{1,\cdots N\}}{\mathrm{Conf}}(\mathbb{R}^3)\Big)\bigg) \arrow[rr, leftrightarrow, "(-)^*"] \arrow[dd, equal] & & H^\bullet\bigg(\underset{N\in\mathbb{N}}{\bigsqcup} \Omega\Big(\underset{\{1,\cdots N\}}{\mathrm{Conf}}(\mathbb{R}^3)\Big)\bigg) \arrow[dd, equal] & \\
 & & \begin{array}{c} \| \\ \text{[FaHu 01] [Kohno 02] [SS19-Quant, Prop. 2.18]} \\ \| \end{array} & & \\
\textbf{hor. chord diagrams} & \underset{N\in\mathbb{N}}{\bigoplus} \mathscr{A}^{\mathrm{pb}}_N \arrow[rr, leftrightarrow, "(-)^*"'] & & \underset{N\in\mathbb{N}}{\bigoplus} \big(\mathscr{A}^{\mathrm{pb}}_N\big)^* & \textbf{weight systems} \\
 & & & \underset{N\in\mathbb{N}}{\bigoplus} \big(\mathscr{A}^{\mathrm{pb}}_N\big)^*_{\mathrm{pos}} \arrow[u, hook, "\text{conv}"', "\text{[SS19-Quant, §3.5]}"] & \textbf{quantum states} \\
 & & \text{[CSS21-Quant, Thm. 1.2]} & w_{(\mathfrak{gl}_2(\mathbb{C}),\mathbf{2})} \arrow[u, phantom, "\rotatebox{90}{$\in$}"] & \begin{array}{c} \textbf{fund. } \mathfrak{gl}_2(\mathbb{C}) \textbf{ weight system /} \\ \textbf{elementary fuzzy funnel/} \\ 2\cdot \textbf{M5-brane state} \end{array}
\end{tikzcd}$$

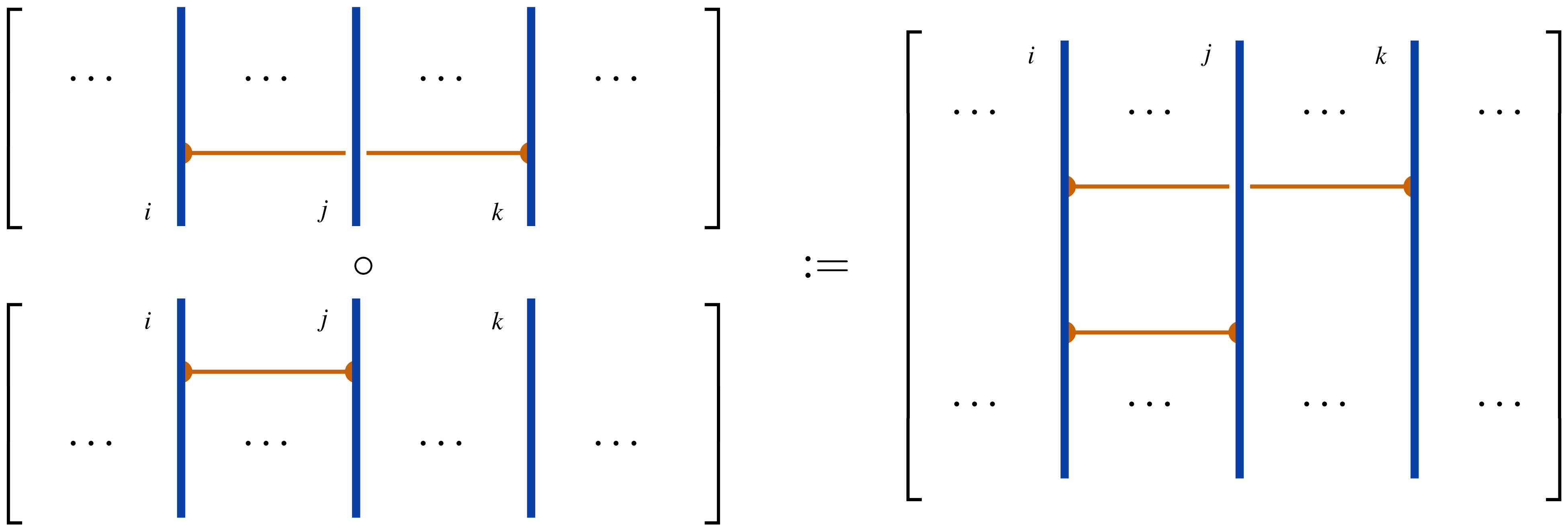
Horizontal chord diagrams.

A typical horizontal chord diagram, here with $N_{\mathrm{f}} = 5$ strands and degree $= 6$ chords:



Horizontal chord diagrams – monoid structure. §3.1 in [SS19-Quant]

The set of horizontal chord diagrams is a monoid under concatenation of strands:

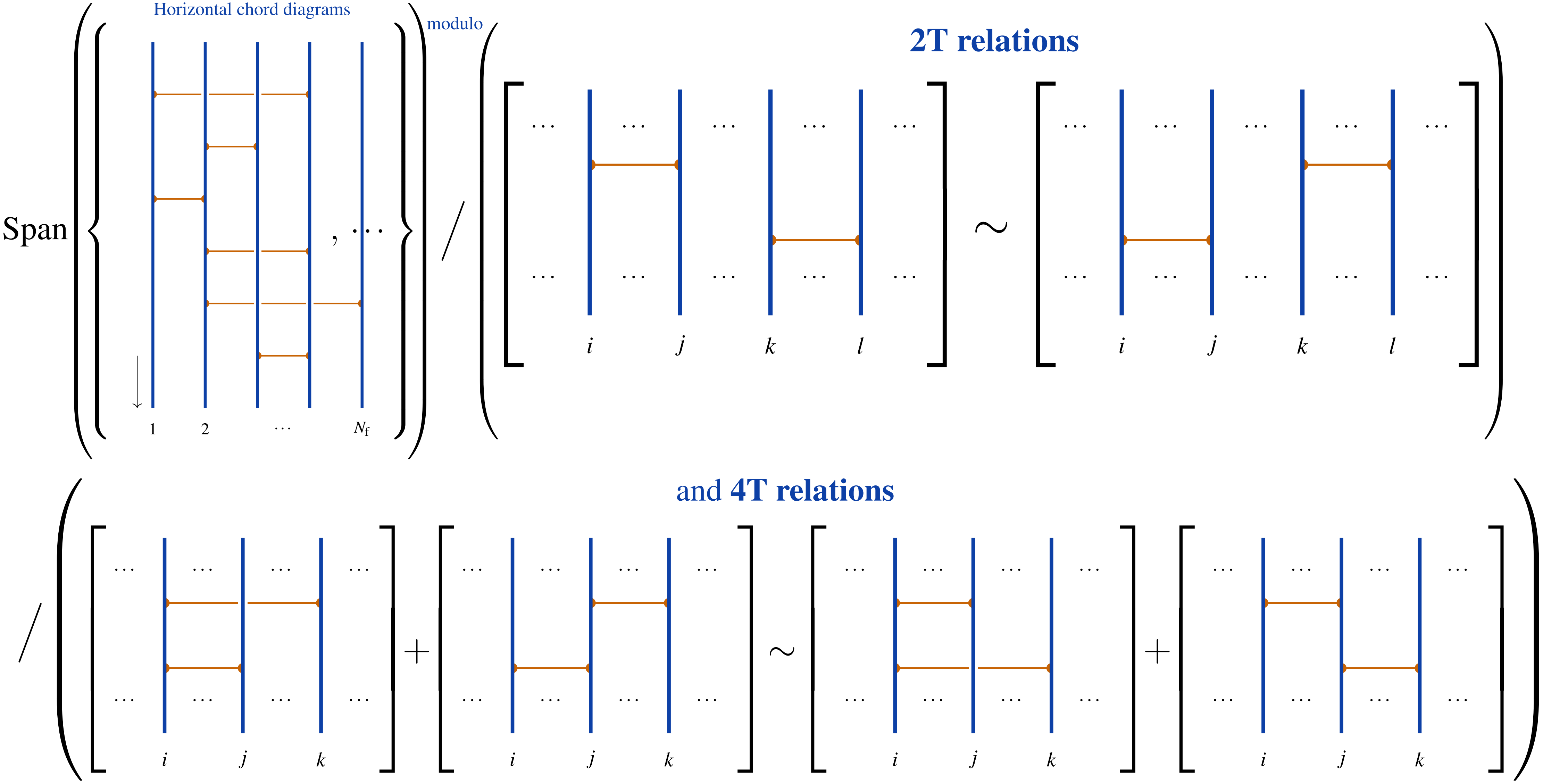


Horizontal chord diagrams – algebra structure.

§3.1 in [SS19-Quant]

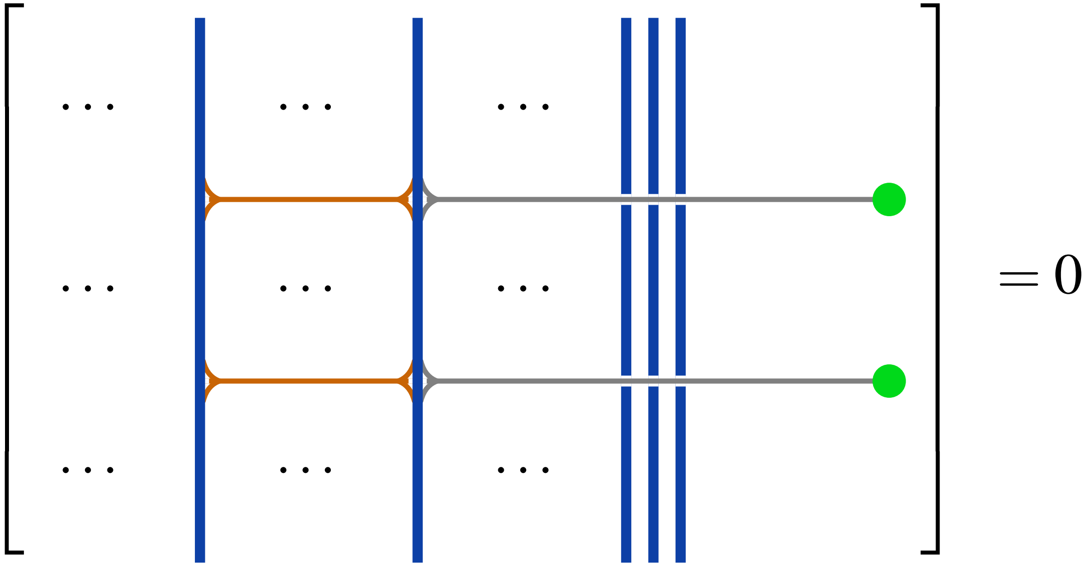
From this the algebra of hor. chord diagrams is obtained by dividing out relations:

$\mathscr{A}^{\mathrm{pb}}_{N_{\mathrm{f}}} :=$

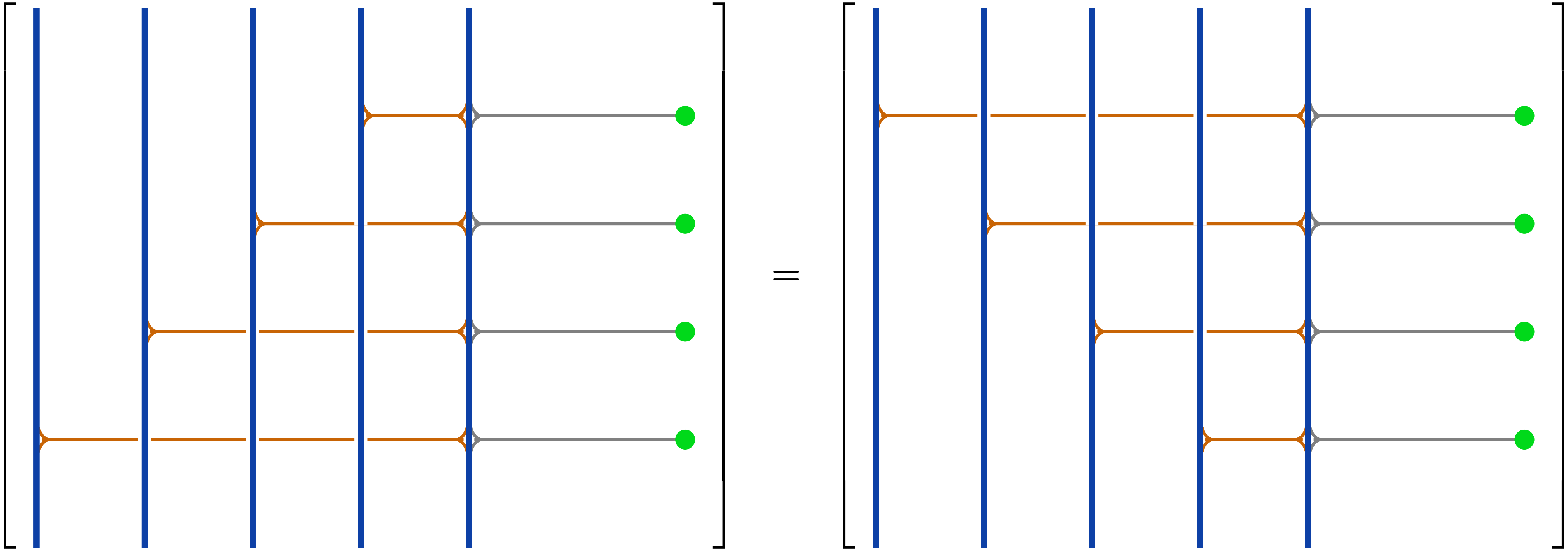


Horizontal chord diagrams – skew-algebra.

Denote skew-symmetric elements in the algebra of chord diagrams by attaching a green node as follows:

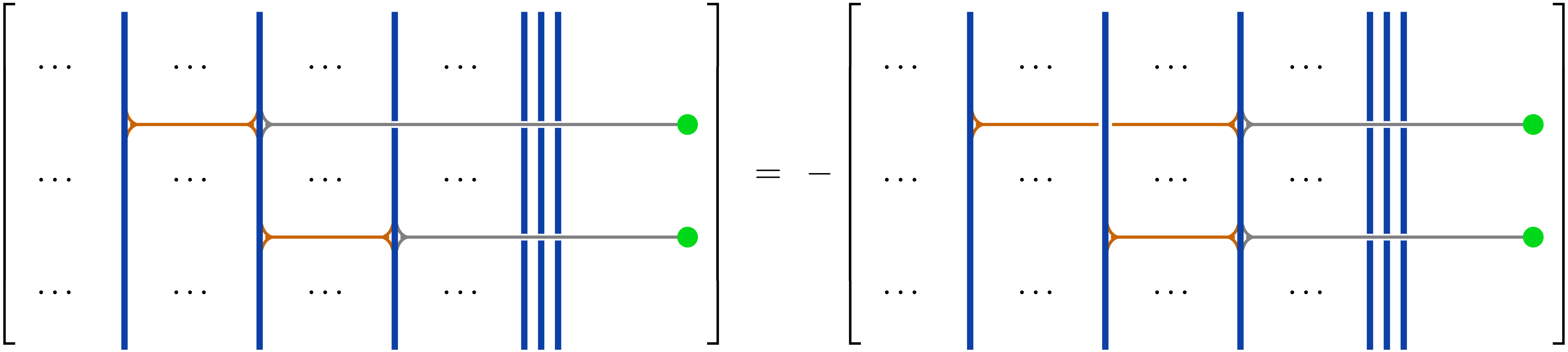


e.g.



Horizontal chord diagrams – skew-algebra.

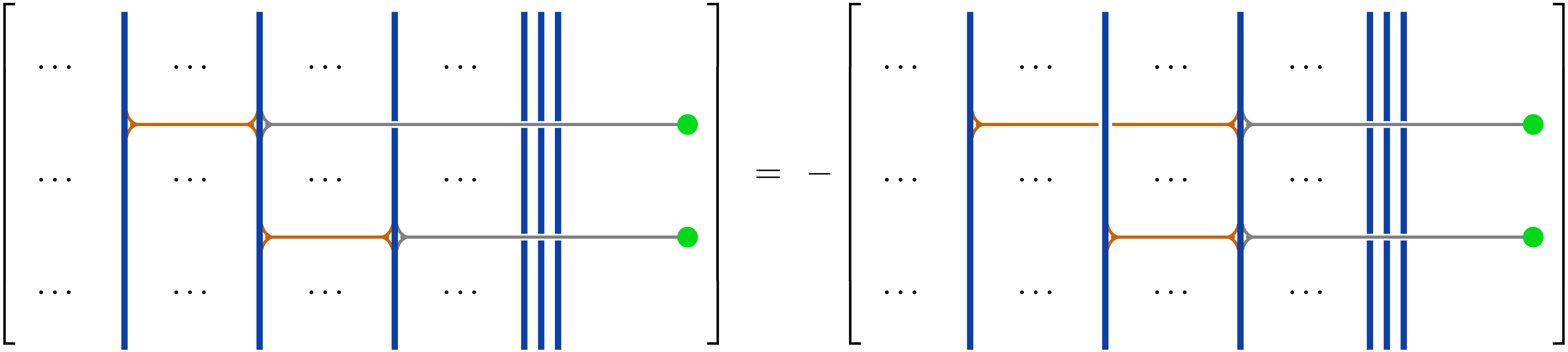
On skew chord diagrams, the 4T relation says the following:



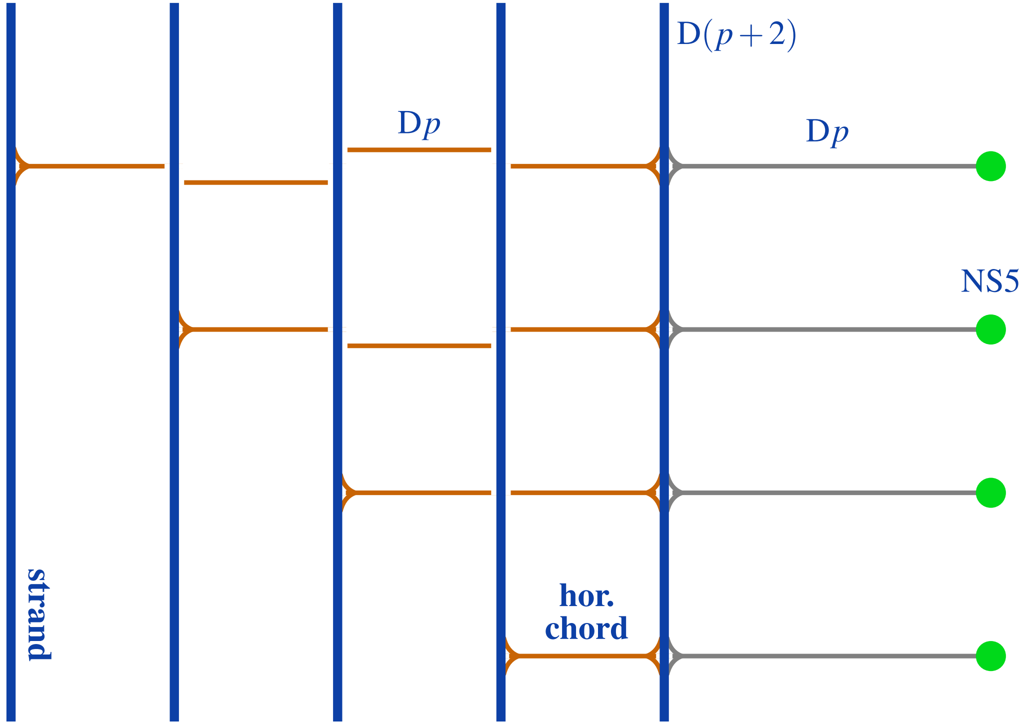
Horizontal chord diagrams – skew-algebra.

§4.10 in [SS19-Quant]

On skew chord diagrams, the 4T relation says the following:

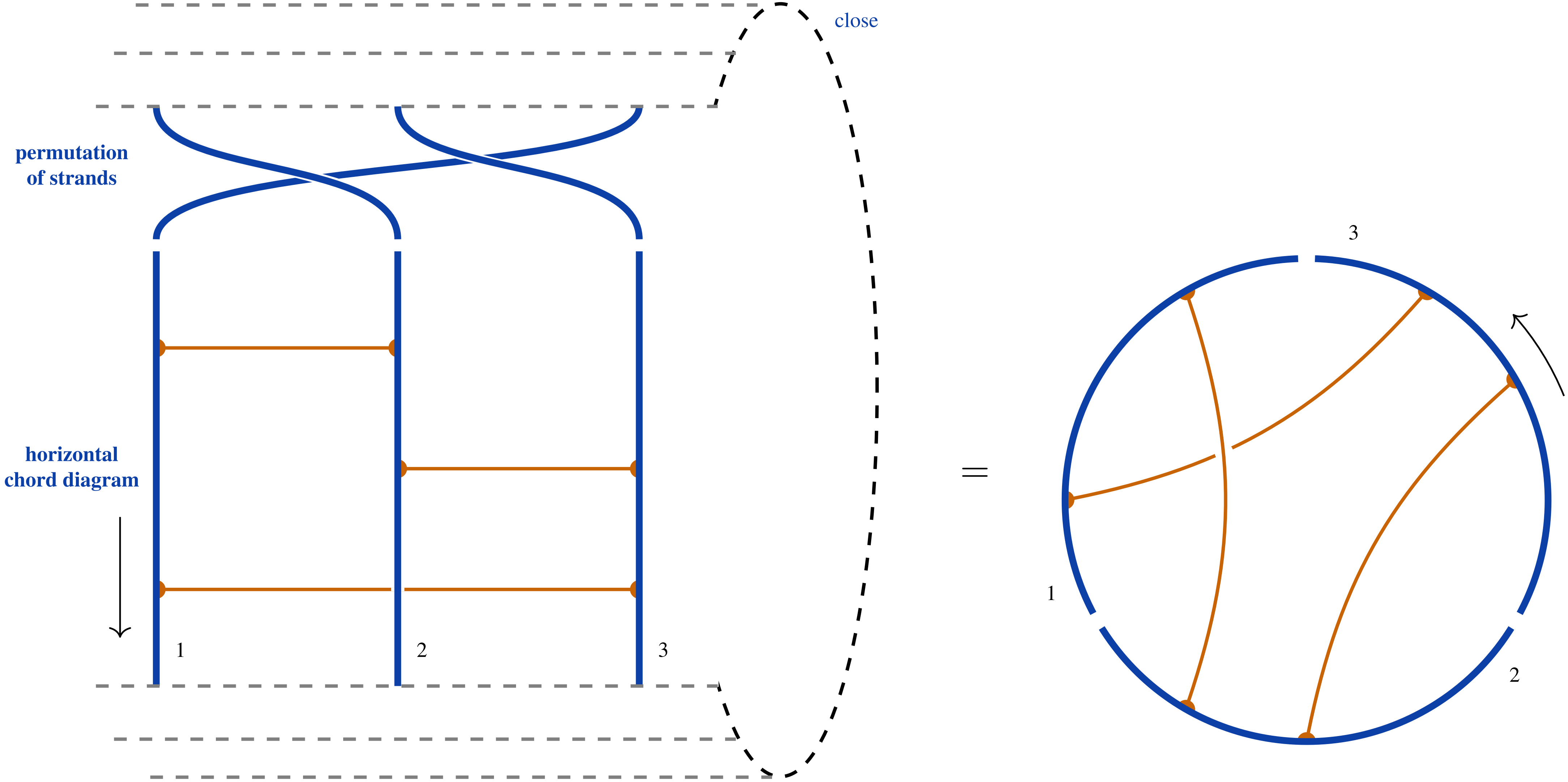


Hence skew-symmetric hor. chord diagrams look like $\mathrm{D}p \perp \mathrm{D}(p+2)$-branes according to the Hanany-Witten rules:



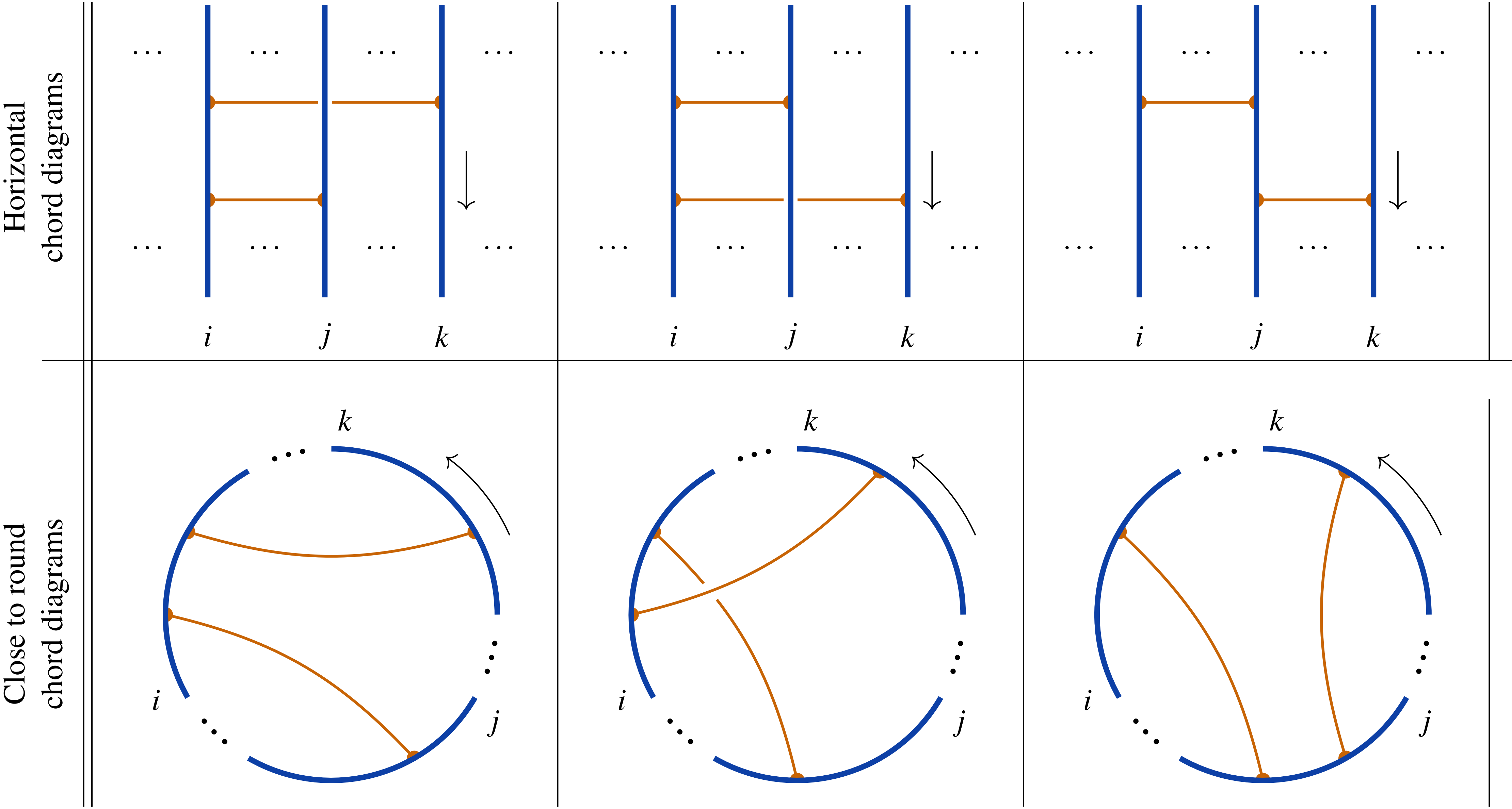
Round chord diagrams – from hor. chord diagrams.

Closing a horizontal chord diagram up to cyclic permutation of its strands yields a round chord diagram:



Round chord diagrams – from hor. chord diagrams.

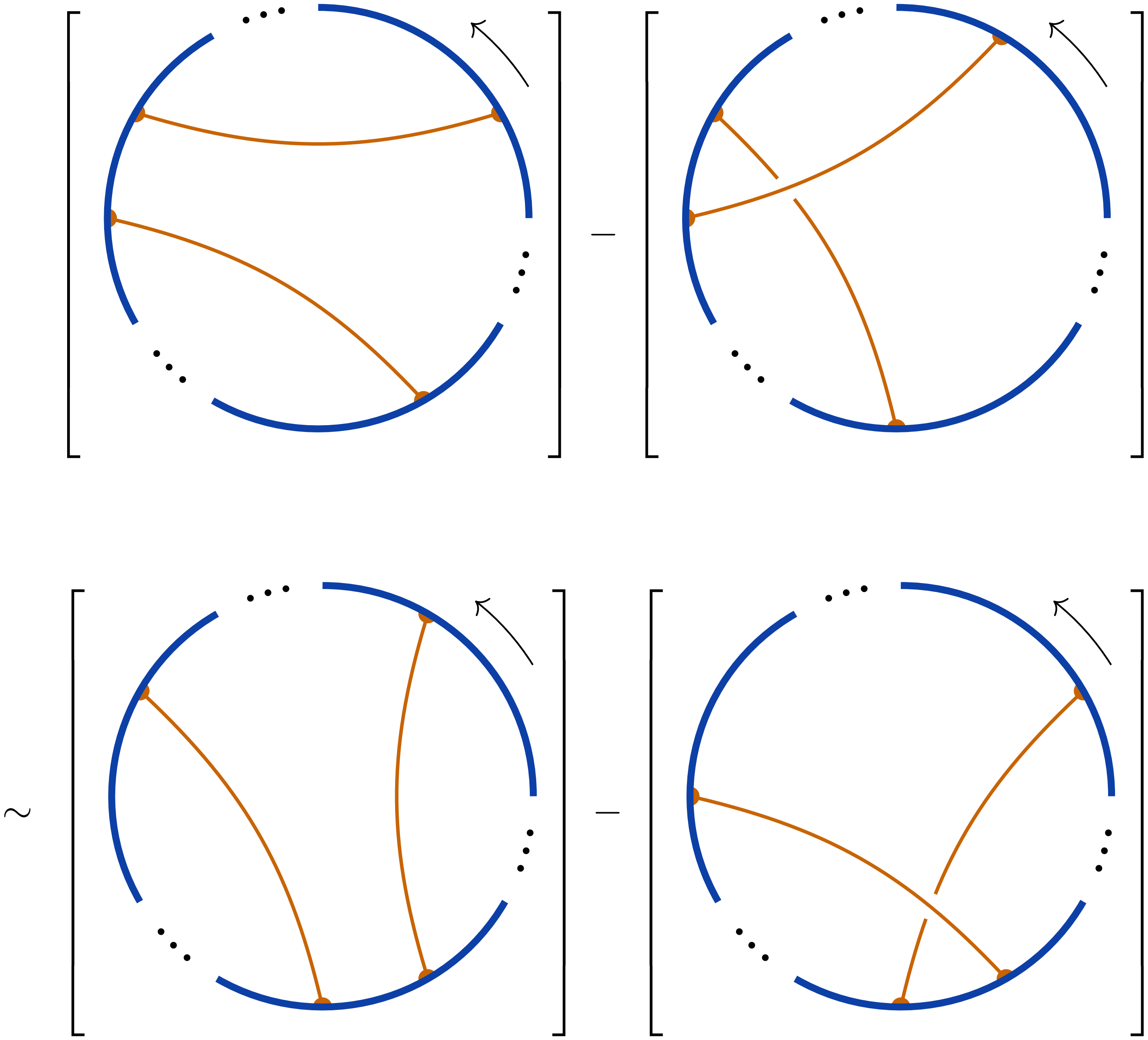
For example:



Round chord diagrams – from hor. chord diagrams.

§3.2 in [SS19-Quant]

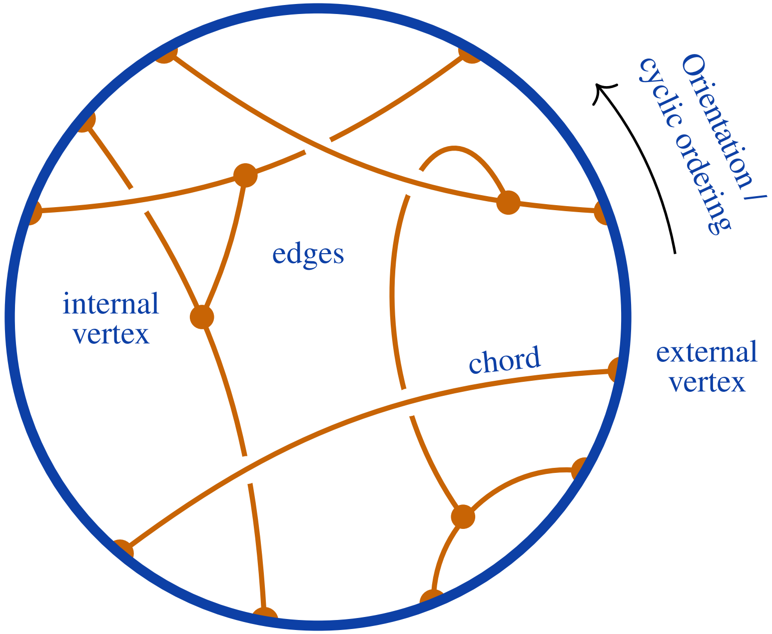
Thus, the 4T relation on hor. chord diagrams become the following relation on round diagr.:



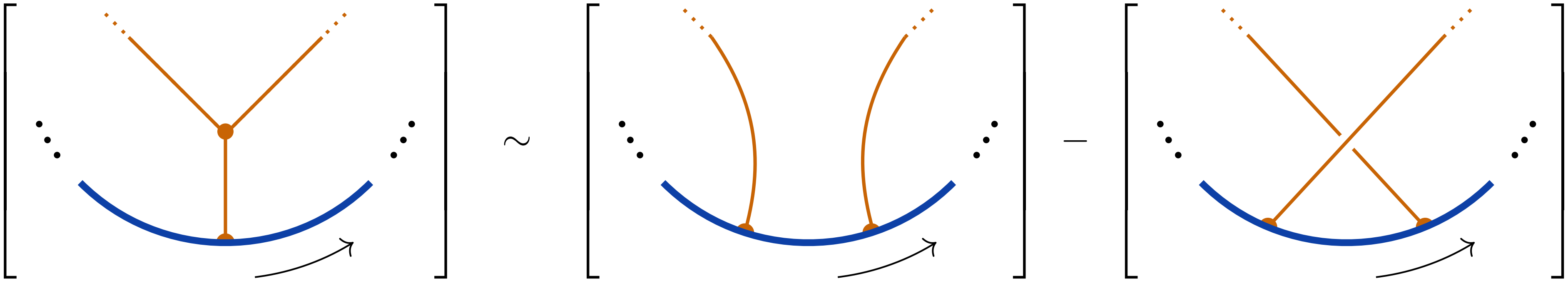
Round chord diagrams and Jacobi diagrams. §3.2 in [SS19-Quant]

These round 4T relations may be captured by *introducing a vertex*:

Prop. [BNa95]: The span of round chord diagrams modulo the above 4T relations
is equivalently the span of *Jacobi diagrams*



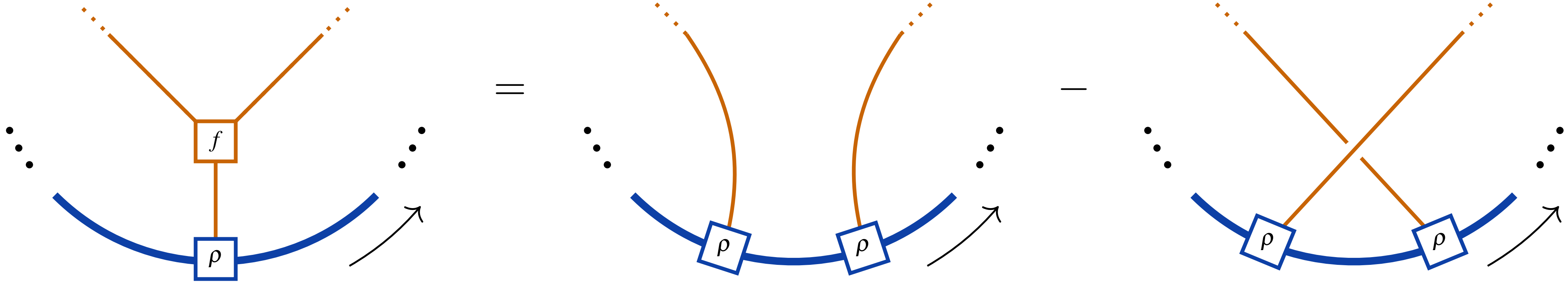
modulo the *STU-relations*:



Round chord diagrams and Lie algebras.

§3.3 in [SS19-Quant]

But this STU-relation is just the Jacobi identity / Lie action property in Penrose diagram notation for internal Lie theory:



$$\rho\big(f(x,y),z\big) \quad = \quad \rho\big(y,\rho(x,z)\big) \quad - \quad \rho\big(x,\rho(y,z)\big)$$

in a tensor category $\mathscr{C} \in \mathrm{TensorCat}$

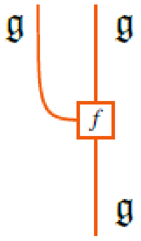
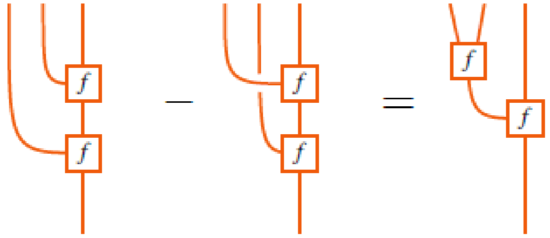
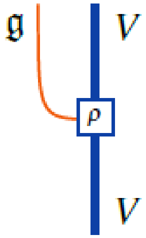
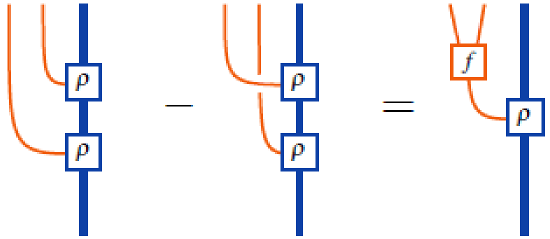
with a Lie action $\rho : \mathfrak{g} \otimes V \longrightarrow V$

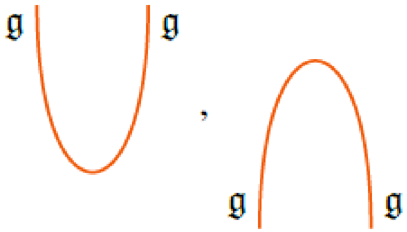
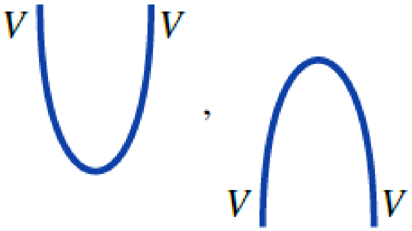
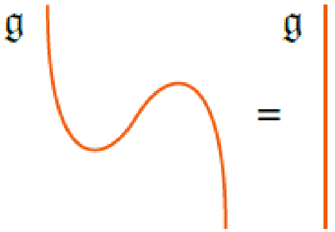
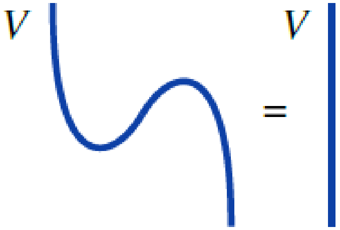
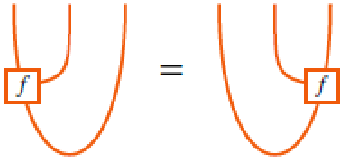
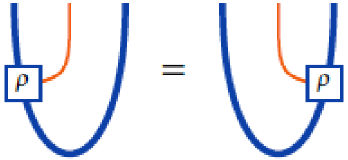
on a Lie module $V \in \mathscr{C}$

by a Lie algebra $\mathfrak{g} \in \mathscr{C}$

with Lie bracket $f : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}.$

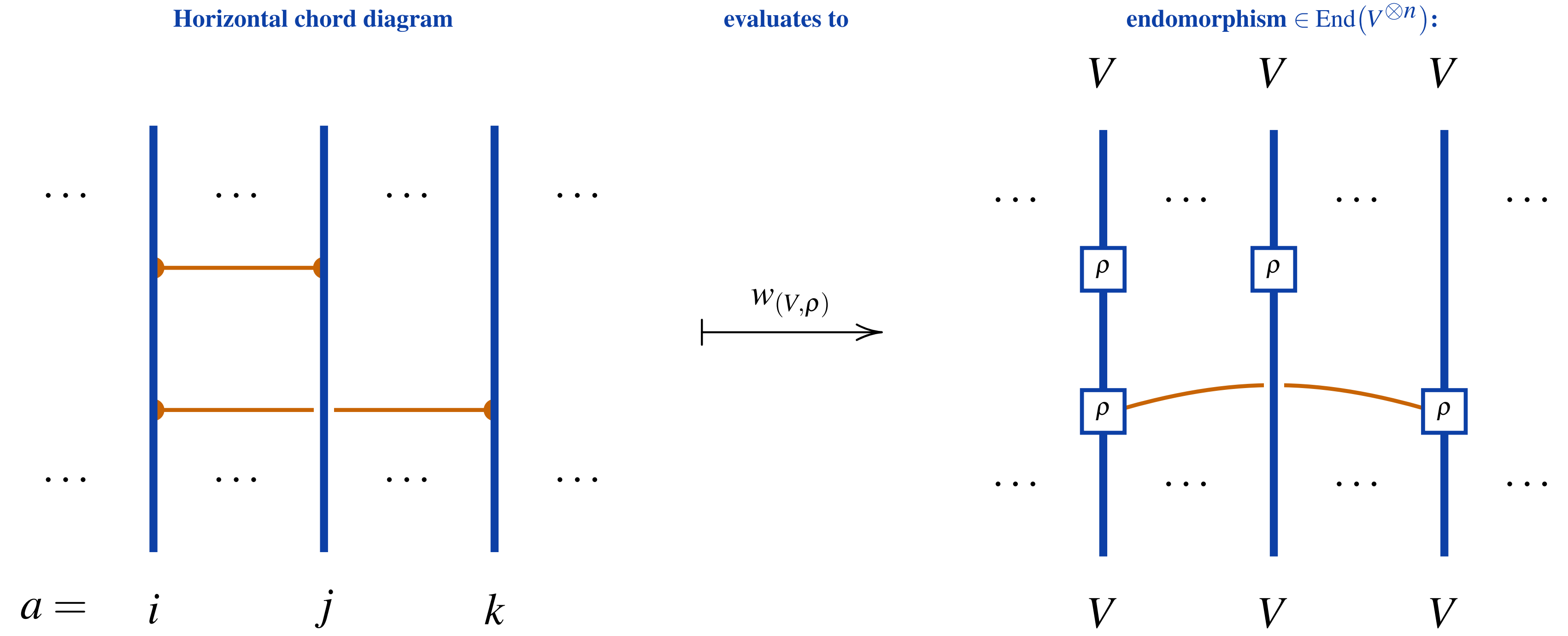
Round chord diagrams and Lie algebras. §3.3 in [SS19-Quant]

Data of metric Lie representation	Category notation	Penrose notation	Index notation
Lie bracket	$$\begin{tikzcd}\mathfrak{g}\otimes\mathfrak{g} \arrow[d, "f"] \\ \mathfrak{g}\end{tikzcd}$$		$f_{ab}{}^{c}$
Jacobi identity	$$\begin{tikzcd}\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \arrow[r, "\mathrm{id}\otimes f - f\otimes\mathrm{id}"] \arrow[d, "\sigma_{213} \circ (\mathrm{id}\otimes f)"'] & \mathfrak{g}\otimes\mathfrak{g} \arrow[d, "f"] \\ \mathfrak{g}\otimes\mathfrak{g} \arrow[r, "f"'] & \mathfrak{g}\end{tikzcd}$$		$f_{ae}{}^{d} f_{bc}{}^{e} - f_{be}{}^{d} f_{ac}{}^{e}$ $= f_{ec}{}^{d} f_{ab}{}^{e}$
Lie action	$$\begin{tikzcd}\mathfrak{g}\otimes V \arrow[d, "\rho"] \\ V\end{tikzcd}$$		$\rho_{a}{}^{i}{}_{j}$
Lie action property	$$\begin{tikzcd}\mathfrak{g}\otimes\mathfrak{g}\otimes V \arrow[r, "\mathrm{id}\otimes\rho - f\otimes\mathrm{id}"] \arrow[d, "\sigma_{213} \circ (\mathrm{id}\otimes\rho)"'] & \mathfrak{g}\otimes V \arrow[d, "\rho"] \\ \mathfrak{g}\otimes V \arrow[r, "\rho"'] & V\end{tikzcd}$$		$\rho_{a}{}^{j}{}_{l}\,\rho_{b}{}^{l}{}_{i} - \rho_{b}{}^{j}{}_{l}\,\rho_{a}{}^{l}{}_{i}$ $= f_{ab}{}^{c}\,\rho_{c}{}^{j}{}_{i}$

Metric	$\begin{tikzcd} \mathfrak{g}\otimes\mathfrak{g} \arrow[d, "g"] \\ \mathbf{1} \end{tikzcd}$, $\begin{tikzcd} \mathbf{1} \arrow[d, "g^{-1}"] \\ \mathfrak{g}\otimes\mathfrak{g} \end{tikzcd}$		g_{ab} , g^{ab}
	$\begin{tikzcd} V\otimes V \arrow[d, "k"] \\ \mathbf{1} \end{tikzcd}$, $\begin{tikzcd} \mathbf{1} \arrow[d, "k^{-1}"] \\ V\otimes V \end{tikzcd}$		k_{ij} , k^{ij}
Metric property	$\begin{tikzcd} \mathfrak{g} \arrow[r, "\simeq"] \arrow[ddrr, "\mathrm{id}"'] & \mathfrak{g}\otimes\mathbf{1} \arrow[r, "\mathrm{id}\otimes g^{-1}"] & \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \arrow[d, "g\otimes\mathrm{id}"] \\ & & \mathbf{1}\otimes\mathfrak{g} \arrow[d, "\simeq"] \\ & & \mathfrak{g} \end{tikzcd}$		$g_{ac}\,g^{cb} = \delta_a^b$
	$\begin{tikzcd} V \arrow[r, "\simeq"] \arrow[ddrr, "\mathrm{id}"'] & V\otimes\mathbf{1} \arrow[r, "\mathrm{id}\otimes k^{-1}"] & V\otimes V\otimes V \arrow[d, "k\otimes\mathrm{id}"] \\ & & \mathbf{1}\otimes V \arrow[d, "\simeq"] \\ & & V \end{tikzcd}$		$k_{il}\,k^{lj} = \delta_i^j$
Metricity of Lie bracket	$\begin{tikzcd} \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g} \arrow[r, "\mathrm{id}\otimes f"] \arrow[d, "f\otimes\mathrm{id}"'] & \mathfrak{g}\otimes\mathfrak{g} \arrow[d, "g"] \\ \mathfrak{g}\otimes\mathfrak{g} \arrow[r, "g"'] & \mathfrak{g} \end{tikzcd}$		$f_{ab}{}^{d}\,g_{dc} = f_{bc}{}^{d}\,g_{ad}$
Metricity of Lie action	$\begin{tikzcd} V\otimes\mathfrak{g}\otimes V \arrow[r, "\mathrm{id}\otimes\rho"] \arrow[d, "\rho\otimes\mathrm{id}"'] & V\otimes V \arrow[d, "k"] \\ V\otimes V \arrow[r, "k"'] & V \end{tikzcd}$		$\rho_a{}^{l}{}_{i}\,k_{lj} = \rho_a{}^{l}{}_{j}\,k_{li}$

Lie algebra weight systems. §3.3 in [SS19-Quant]

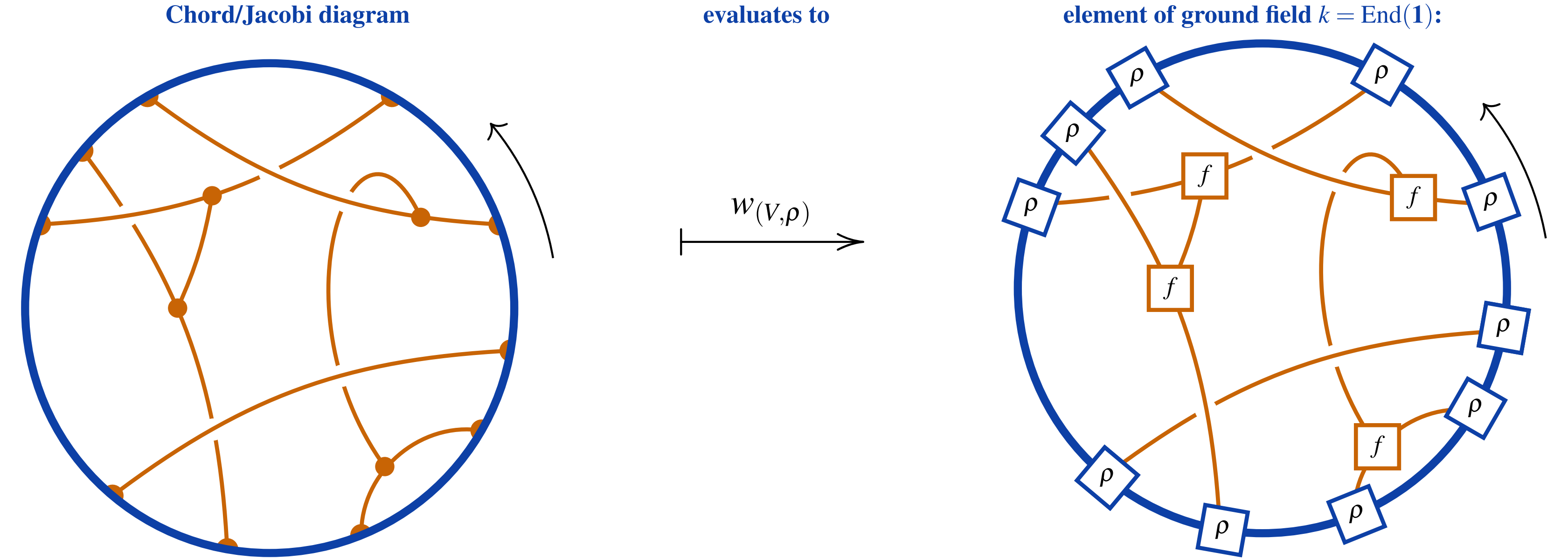
This means that every metric Lie algebra module yields a *weight system* on chord diagrams, namely a linear dual respecting the 4T relations:



Horizontal chord diagram

evaluates to

endomorphism $\in \mathrm{End}\left(V^{\otimes n}\right)$**:**



Chord/Jacobi diagram

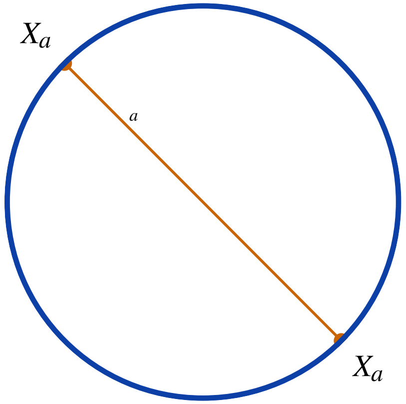
evaluates to

element of ground field $k = \mathrm{End}(\mathbf{1})$**:**

Lie algebra weight systems – Fuzzy 2-sphere states. §4.2 in [SS19-Quant]

$\int_{S^2_N} (R^2)$ 

$= \frac{4\pi}{\sqrt{N^2-1}} \mathrm{Tr}\big(X_a \cdot X^a\big)$

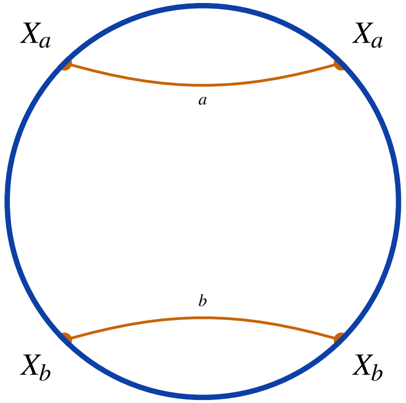


Ex.: The weight system given by the Lie module $\mathbf{N}$ of the Lie algebra $\mathfrak{g} = \mathfrak{su}(2)$ equipped with its Killing form metric yields the radius observables of the N-bit fuzzy 2-sphere/fuzzy funnel.

[Papageorgakis, Ramgoolam, Spence, McNamara 04-05]

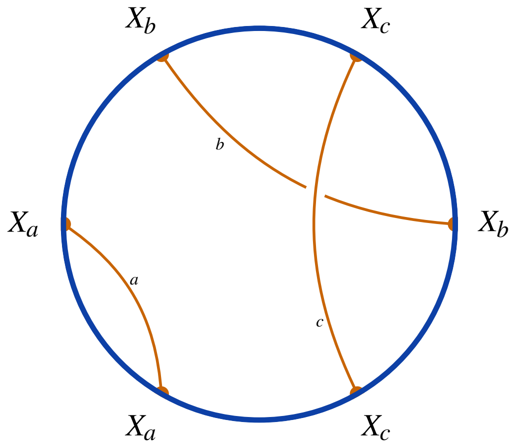
$\int_{S^2_N} (R^2)^2$ 

$= \frac{4\pi}{\sqrt{N^2-1}} \mathrm{Tr}\big(X_a \cdot X^a \cdot X_b \cdot X^b\big)$



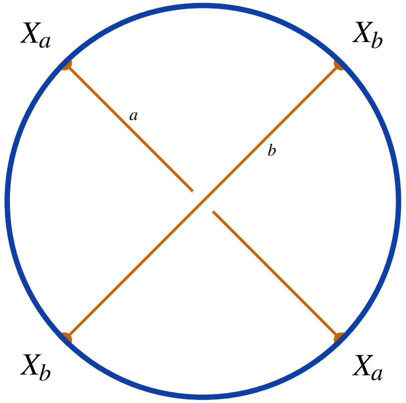
$\int_{S^2_N} (R^2)^3$ 

$= \frac{4\pi}{\sqrt{N^2-1}} \mathrm{Tr}\big(X_a \cdot X^a \cdot X_b \cdot X_c \cdot X^b \cdot X^c\big)$



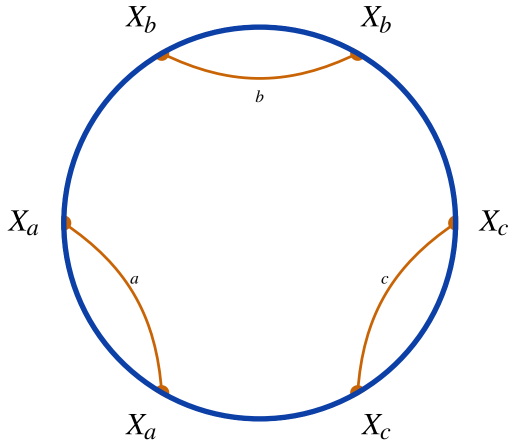
$\int_{S^2_N} (R^2)^2$ 

$= \frac{4\pi}{\sqrt{N^2-1}} \mathrm{Tr}\big(X_a \cdot X_b \cdot X^a \cdot X^b\big)$



$\int_{S^2_N} (R^2)^3$ 

$= \frac{4\pi}{\sqrt{N^2-1}} \mathrm{Tr}\big(X_a \cdot X^a \cdot X_b \cdot X^b \cdot X_c \cdot X^c\big)$



Lie algebra weight systems – BMN 5-brane states.

Hence if we fix the Lie algebra to $\mathfrak{g} = \mathrm{su}(2)$ then

Lie algebra weight systems – BMN 5-brane states.

§4.9 in [SS19-Quant]

Hence if we fix the Lie algebra to $\mathfrak{g} = \mathrm{su}(2)$ then

Lie algebra weight systems are labelled by iso classes of $\mathfrak{su}(2)$-modules
hence by i-indexed sums of $N_i^{(\mathrm{M2})} \in \mathbb{N}_+$ many copies of the irrep $\mathbf{N}_i^{(\mathrm{M5})} \in \mathfrak{su}(2)\mathrm{Mod}$.

Lie algebra weight systems – BMN 5-brane states. §4.9 in [SS19-Quant]

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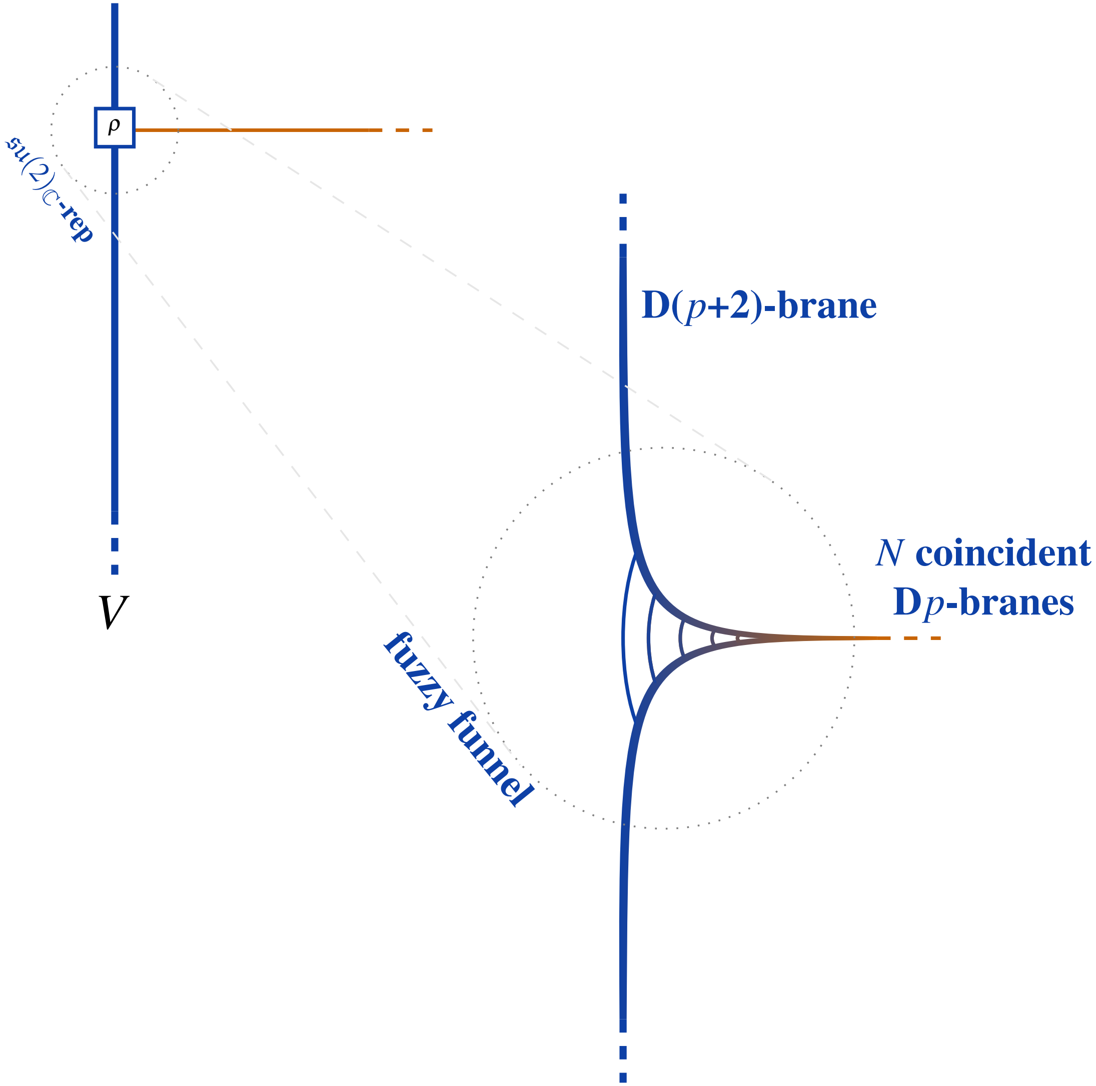
Just this kind of data is thought to describe

1. fuzzy funnel $\mathrm{D}p \perp \mathrm{D}(p+2)$-intersections



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Just this kind of data is thought to describe

1. fuzzy funnel $\mathrm{D}p \perp \mathrm{D}(p+2)$-intersections
2. M2/M5-brane bound states in the BMN matrix model:

$$V := \bigoplus_i \Big(\overbrace{N_i^{(\mathrm{M2})} \cdot \mathbf{N}_i^{(\mathrm{M5})}}^{\substack{\text{M2/M5-brane charge up to } i\text{th stack} \\ (i\text{th irrep with multiplicity})}} \Big) \in \mathfrak{su}(2)_{\mathbb{C}}\mathrm{Mod}_{/\sim}$$

	Stacks of macroscopic		
	M2-branes	*M5-branes*	
If for all i:	$N_i^{(\mathrm{M5})} \to \infty$	$N_i^{(\mathrm{M2})} \to \infty$	(the relevant large N limit)
with fixed	$N_i^{(\mathrm{M2})}$	$N_i^{(\mathrm{M5})}$	(the number of coincident branes up to the ith stack)
and fixed	$N_i^{(\mathrm{M2})}/N_{\mathrm{tot}}$	$N_i^{(\mathrm{M5})}/N_{\mathrm{tot}}$	(the charge/LC-momentum carried up to the ith stack)

[MSJVR02, Fig. 2][AIST17 (1.2)-(1.4)]

Lie algebra weight systems – from $\mathfrak{su}(2)$ to $\mathfrak{gl}(2)$. (67) in [SS19-Quant]

In fact, the fuzzy funnel $\mathrm{D}p \perp \mathrm{D}(p+2)$-states involve, in addition to the transverse $\vec{x} \in \mathfrak{su}(2)$, a field $A_y \in \mathbb{C}$, commuting with $\vec{x}$ [GW08, §3.1.1], whence the appropriate Lie algebra is $\mathfrak{gl}(2)$:

Lie algebra weight systems – from $\mathfrak{su}(2)$ to $\mathfrak{gl}(2)$.

(67) in [SS19-Quant]

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$$\begin{tikzcd}
\mathfrak{sl}(2,\mathbb{C}) \simeq \mathfrak{su}(2)_{\mathbb{C}} \arrow[r, hook, "\iota=(\mathrm{id},0)"] & \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathbb{C} \simeq \mathfrak{gl}(2,\mathbb{C})
\end{tikzcd}$$

$$\begin{tikzcd}
\underbrace{\overbrace{\mathfrak{su}(2)_{\mathbb{C}}}^{\substack{\text{scalar fields}\\ \text{on } \mathrm{D}p\\ \langle X_1,X_2,X_3\rangle}}\mathrm{MetMod}}_{\substack{\text{values of scalar fields}\\ \text{at } \mathrm{D}p\perp\mathrm{D}(p+2)}} \arrow[r, shift right, "A_y:=1"'] & \underbrace{\Big(\mathfrak{su}(2)_{\mathbb{C}} \oplus \overbrace{\mathbb{C}}^{\substack{\text{gauge field}\\ \text{on } \mathrm{D}p\\ \langle A_y\rangle}}\Big)\mathrm{MetMod}}_{\substack{\text{values of scalars \& gauge field}\\ \text{at } \mathrm{D}p\perp D(p+2)}} \arrow[l, shift right, "0 \mapsfrom A_y"', "\iota^*"]
\end{tikzcd}$$

Lie algebra weight systems – from $\mathfrak{su}(2)$ to $\mathfrak{gl}(2)$. (67) in [SS19-Quant]

In fact, the fuzzy funnel $\mathrm{D}p \perp \mathrm{D}(p+2)$-states involve, in addition to the transverse $\vec{x} \in \mathfrak{su}(2)$, a field $A_y \in \mathbb{C}$, commuting with $\vec{x}$ [GW08, §3.1.1], whence the appropriate Lie algebra is $\mathfrak{gl}(2)$:

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\end{tikzcd}$$

$$\begin{tikzcd}
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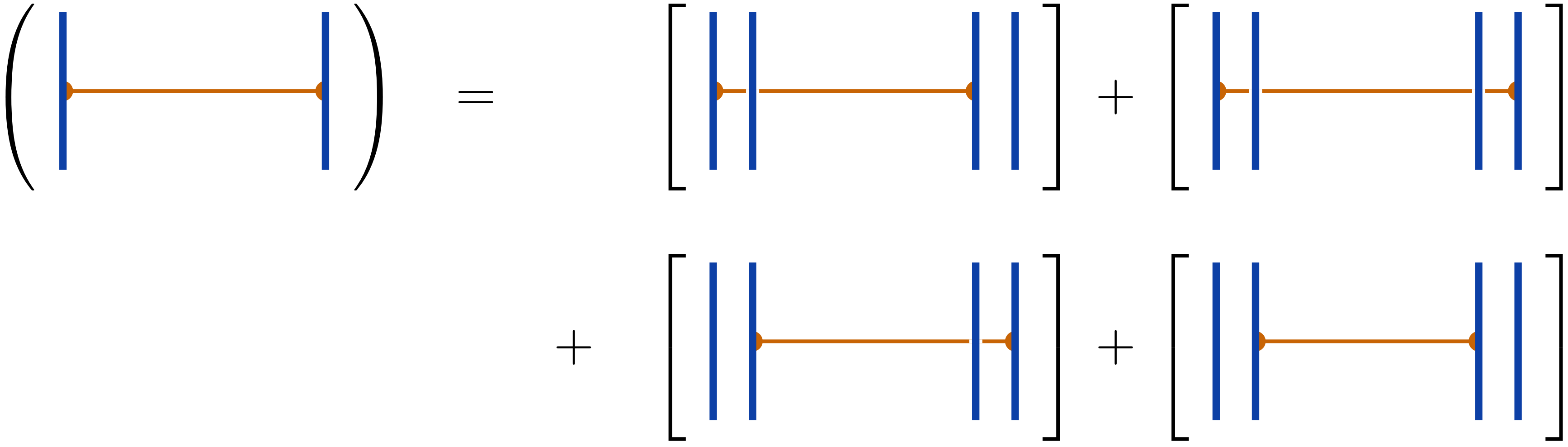
\end{tikzcd}$$

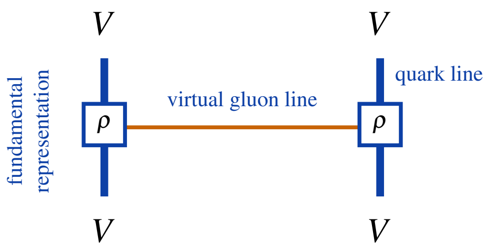
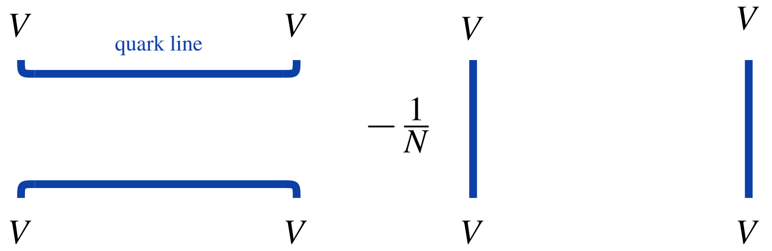
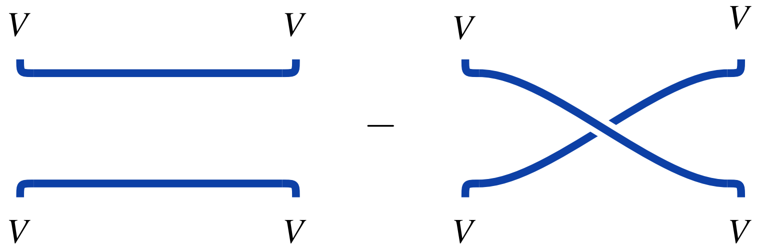
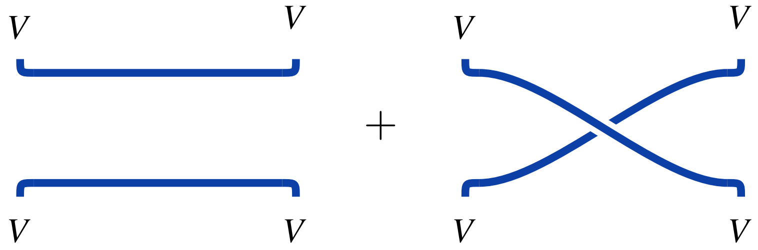
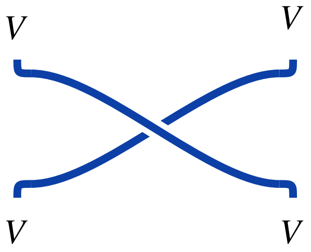
A theorem of [BNa96] shows that *all weight systems are spanned* by the fundamental $\mathfrak{gl}(n)$-weight system $\mathbf{n}$ via permutations and resolving of stacks of coincident strands,

e.g.: $\Delta^{(2,2)}$



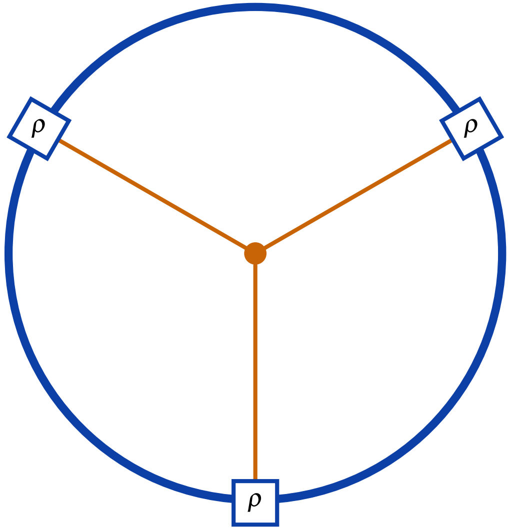
Lie algebra weight systems – 't Hooft double lines.

§4.6 in [SS19-Quant]

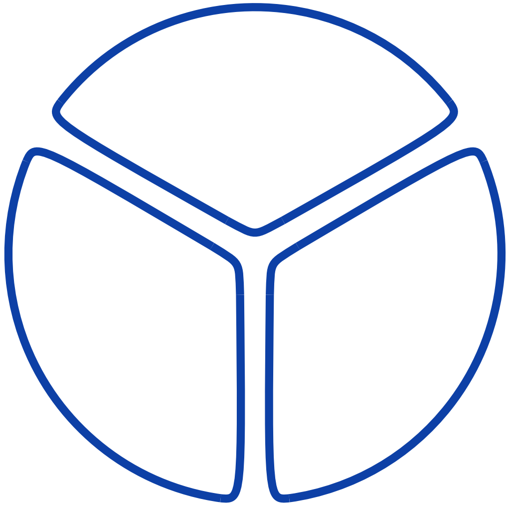
Metric Lie algebra $\mathfrak{g}$	**Metric contraction of fundamental action tensors**  $=$	In many Lie weight systems, chords evaluate to double strands:
$\mathfrak{su}(N)$		Killing metric $g(x,y) = \mathrm{tr}\big(\mathrm{ad}_x \circ \mathrm{ad}_y\big)$
$\mathfrak{so}(N)$		
$\mathfrak{sp}(N)$		
$\mathfrak{gl}(N)$		fundamental metric $g(x,y) = \mathrm{tr}(x \circ y)$ (squashed 2-sphere)

Lie algebra weight systems – ’t Hooft double lines. §4.6 in [SS19-Quant]

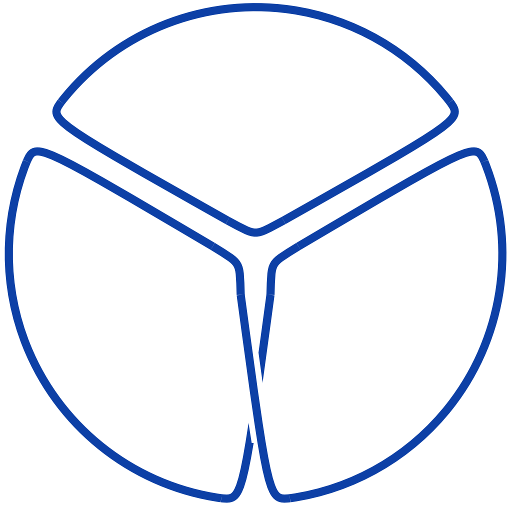
Ex.: For $\mathfrak{g} = \mathfrak{so}(n)$:



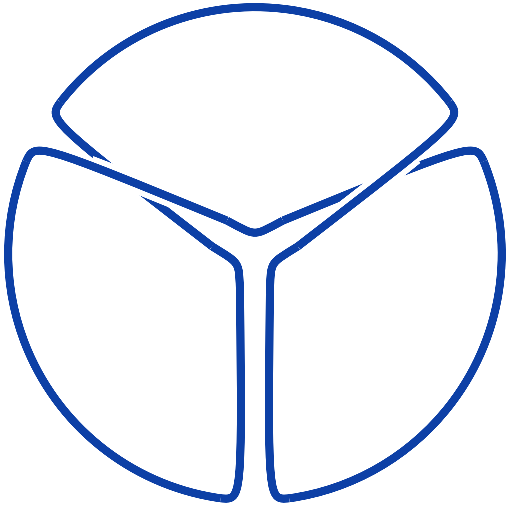
$=$



$- \quad 3\cdot$



$+ \quad 3\cdot$



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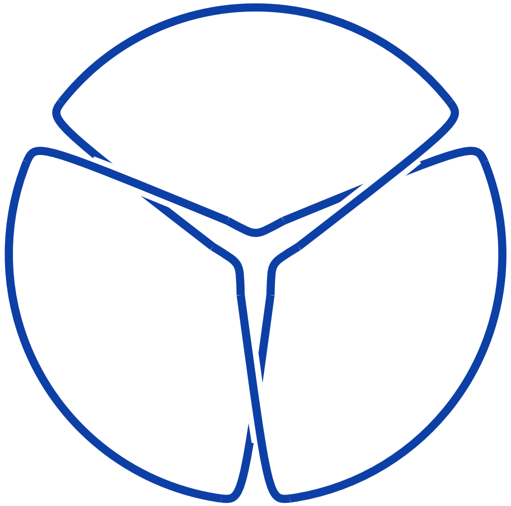
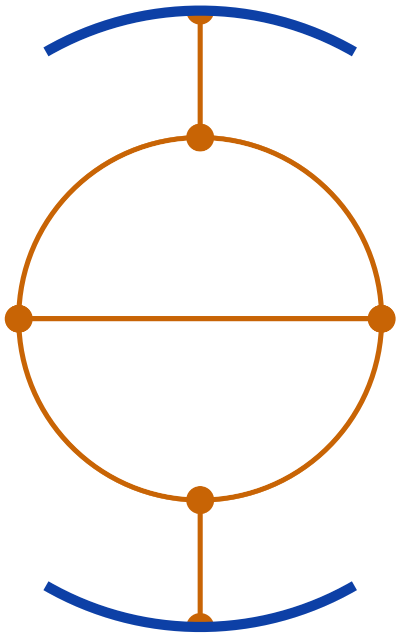


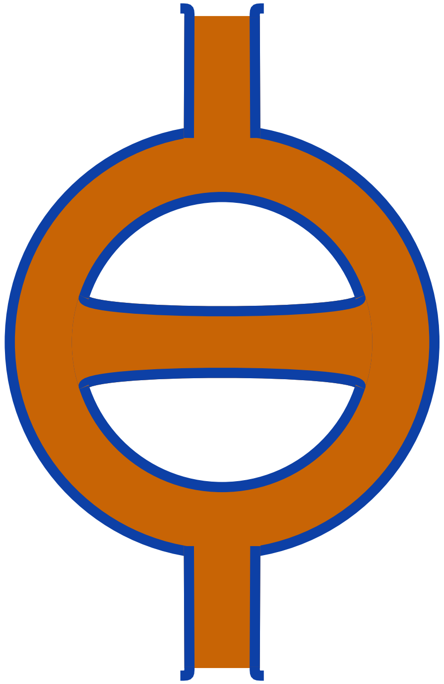
Lie algebra weight systems – ’t Hooft double lines.

§4.6 in [SS19-Quant]

Ex.: For $\mathfrak{g} = \mathfrak{so}(n)$:



$\longmapsto$



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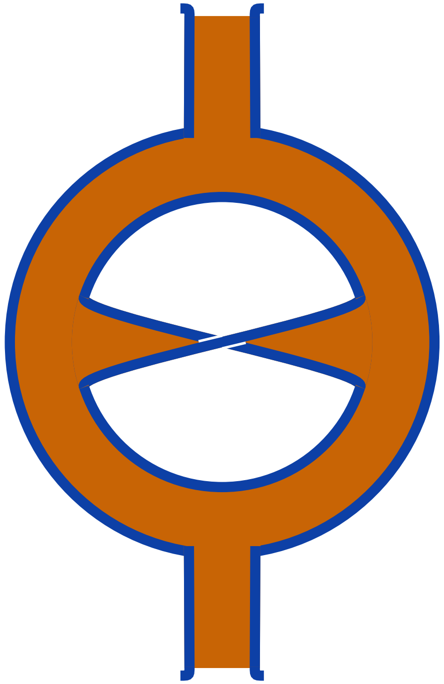
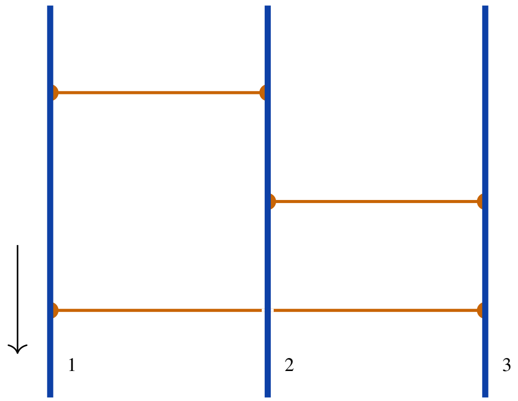


$+ \cdots$

Lie algebra weight systems – ’t Hooft double lines.

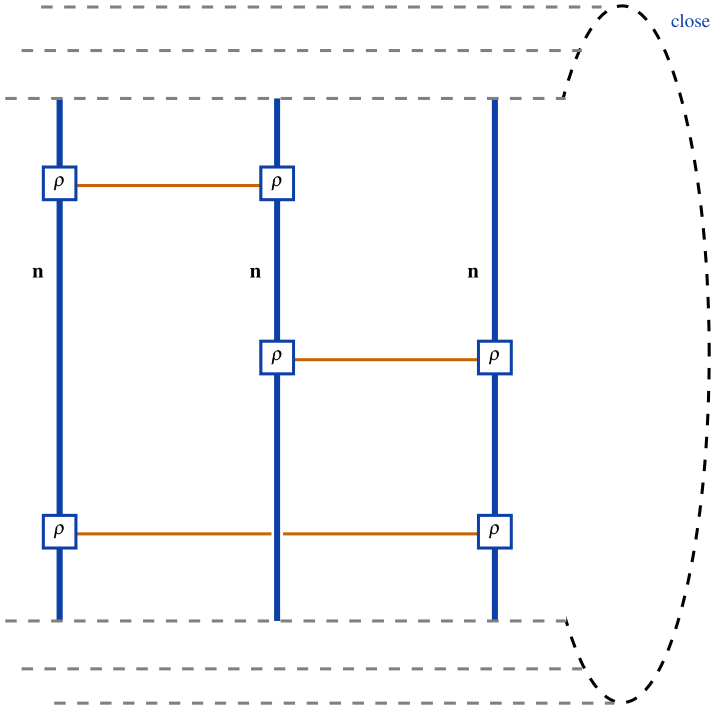
§4.6 in [SS19-Quant]

Ex.: For $\mathfrak{g} = \mathfrak{gl}(n)$ ([BNa96]):



$$\xmapsto{n^3 \cdot w_{(\mathfrak{gl}(n),\mathbf{n})}(-)}$$

fundamental $\mathfrak{gl}(n)$-weight system



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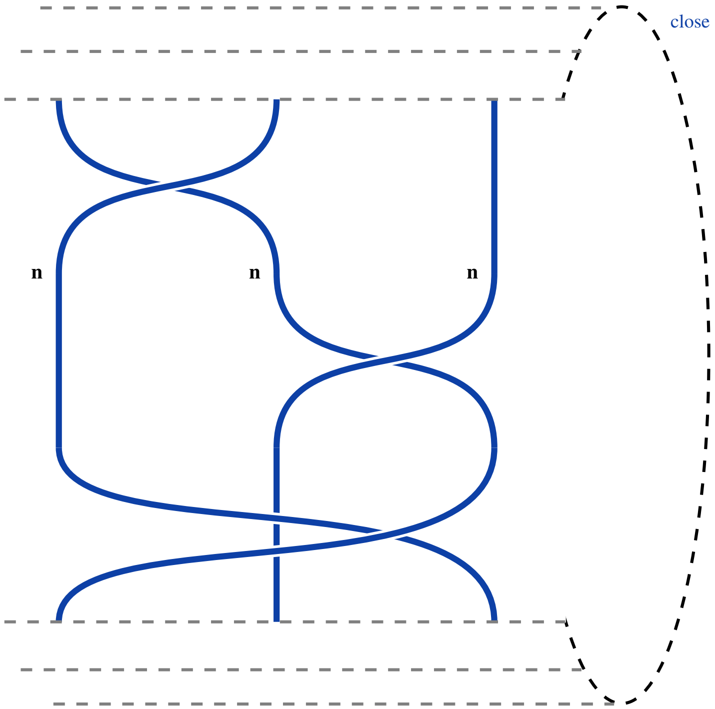

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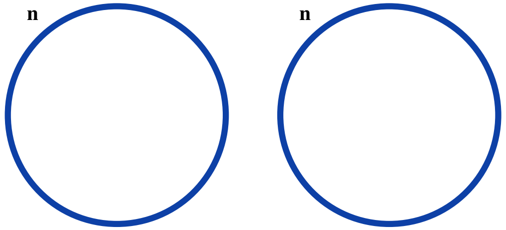
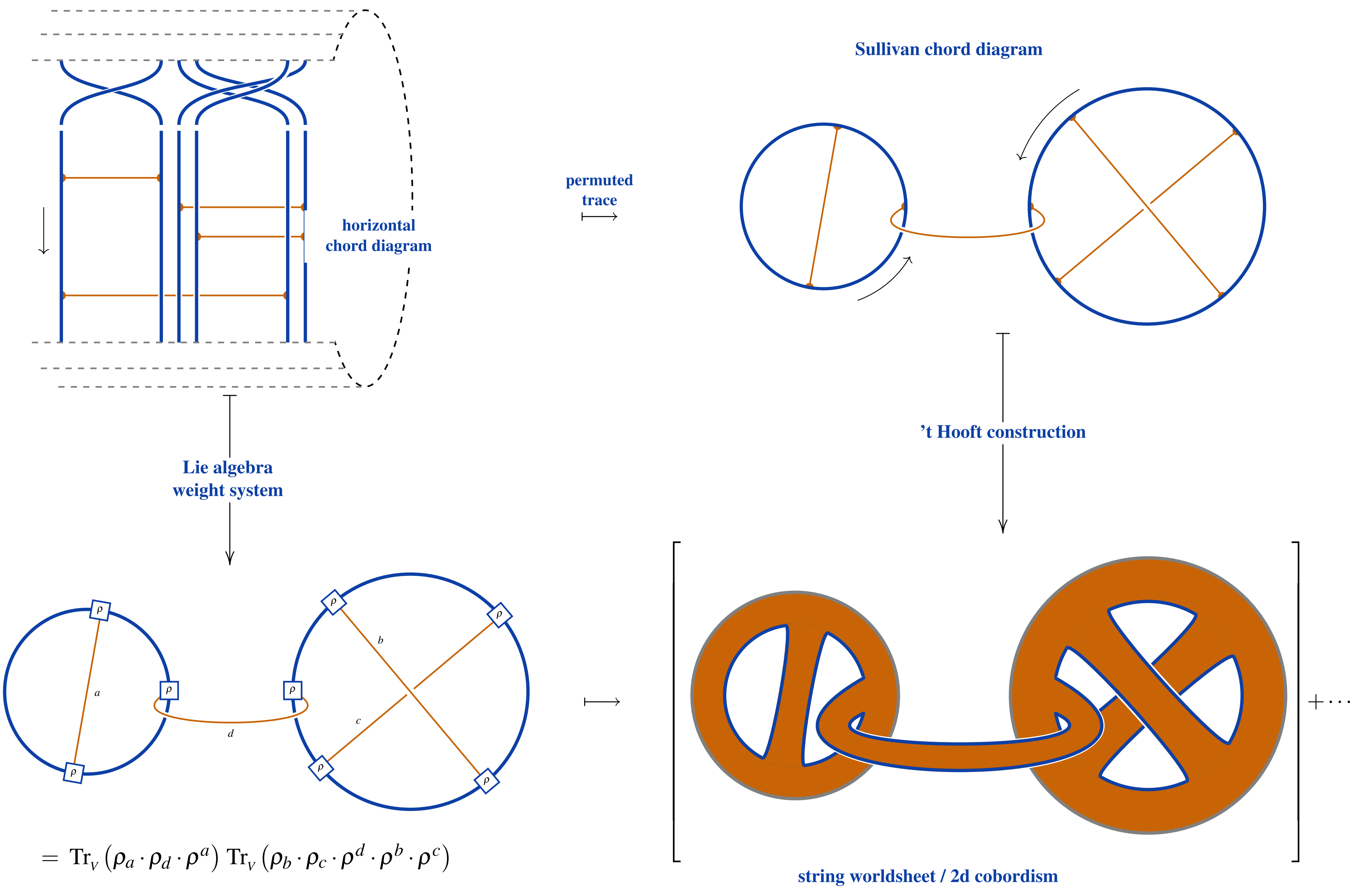


$= n^2$

Lie algebra weight systems – Summary.



horizontal chord diagram

permuted trace

$\longmapsto$

Sullivan chord diagram

't Hooft construction

Lie algebra weight system

$$= \mathrm{Tr}_V\left(\rho_a \cdot \rho_d \cdot \rho^a\right)\,\mathrm{Tr}_V\left(\rho_b \cdot \rho_c \cdot \rho^d \cdot \rho^b \cdot \rho^c\right)$$

multi-trace observable

$\longmapsto$

$+\cdots$

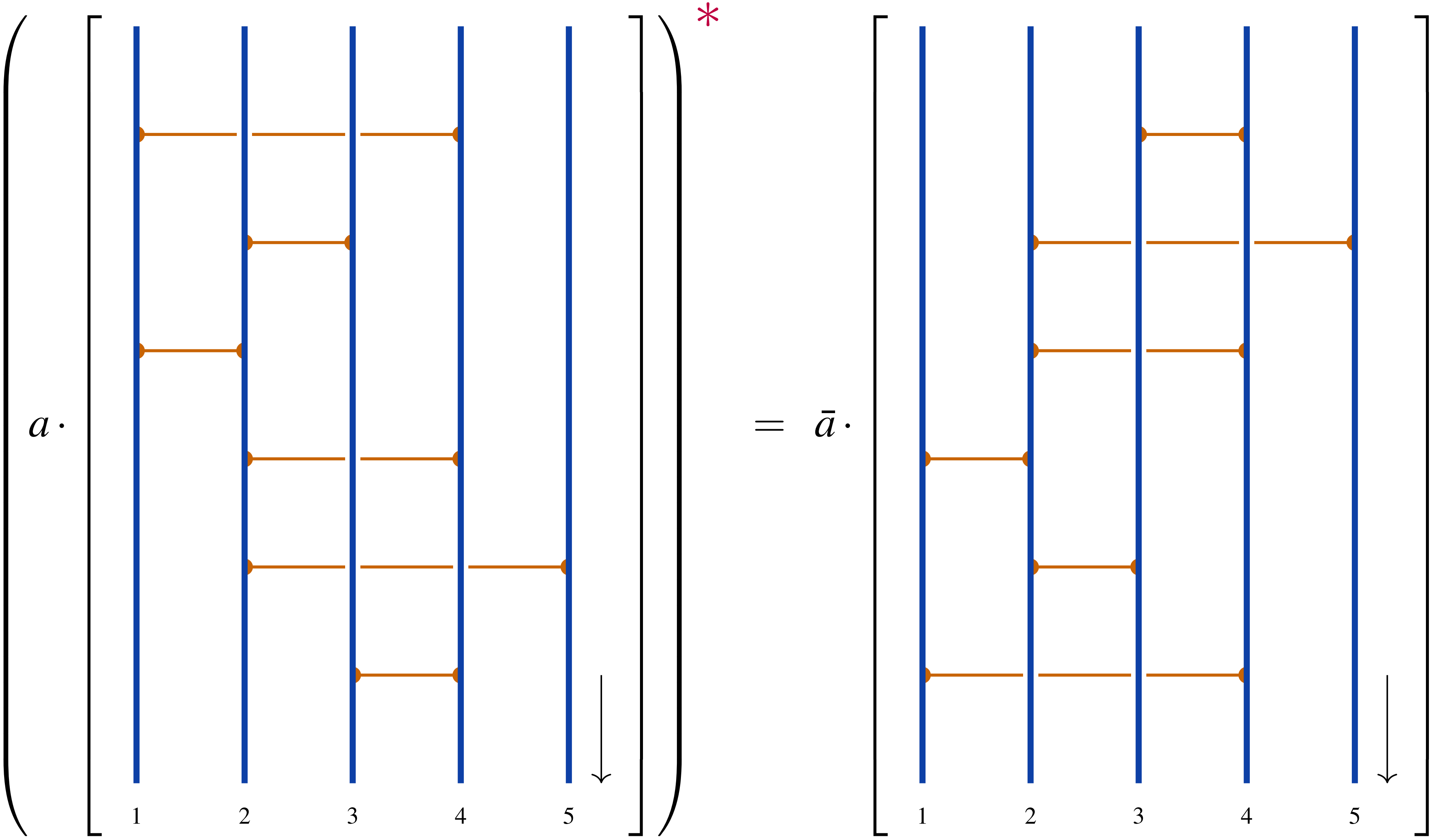
string worldsheet / 2d cobordism

Horizontal chord diagrams as observables.

[CSS21-Quant], §3.5 in [SS19-Quant]

The algebra of horizontal chord diagrams is canonically a star-algebra under reversal of strands (↔ reversal of loops in configuration space):

e.g.:



Weight systems as quantum states. [CSS21-Quant], §3.5 in [SS19-Quant]

Def. (e.g. [Mey95, §I.1.1][Lan17, Def. 2.4]):

Given a star-algebra $(\mathscr{A}, (-)^*)$, a *quantum state* is a complex-linear function

$$\rho \,:\, \mathscr{A} \longrightarrow \mathbb{C}$$

which satisfies:

(1) (positivity): $\rho\big(A^* A\big) \geq 0 \in \mathbb{R} \subset \mathbb{C}$ for all $A \in \mathscr{A}$;

(2) (normalization): $\rho(\mathbf{1}) = 1$ for $\mathbf{1} \in \mathscr{A}$ the algebra unit.

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Thm. [CSS21-Quant, §Thm. 1.2]:

The fundamental $\mathfrak{gl}(n)$-weight systems, for all $n \in \mathbb{N}_+$,
are quantum states on the star-algebra of horizontal chord diagrams,
hence so are all their convex combinations.

Weight systems as quantum states.

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Rem. 1:

There should be many more quantum states on hor. chord diagrams
but this is the first class rigorously identified so far.
Moreover, this class is suggestively singled out by Bar-Natan's theorem.

Weight systems as quantum states. [CSS21-Quant], §3.5 in [SS19-Quant]

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Rem. 2:

Under the above identifications, the quantum state which is the fundamental $\mathfrak{gl}(2) \simeq \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathbb{C}$-weight system corresponds to the elementary M2/M5-brane state in the BMN matrix model.

Aside – Chord diagrams controlling holographic entanglement entropy.

$$S_A = (\#\text{ dimers between } A \text{ and } A^{\mathrm{C}}) \times \frac{\log 2}{2}\,.$$

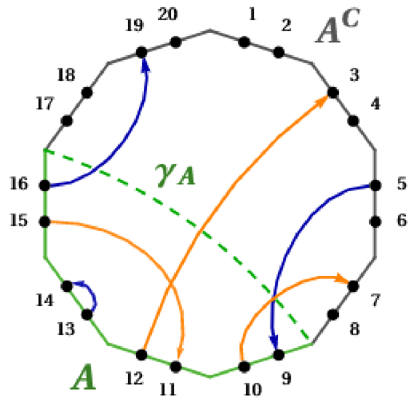
Curiously,

round chord diagrams also capture the

RT formula for holographic entanglement entropy

by reducing tensor networks like the HaPPY code to

Majorana dimer codes with chords geodesics in AdS_2



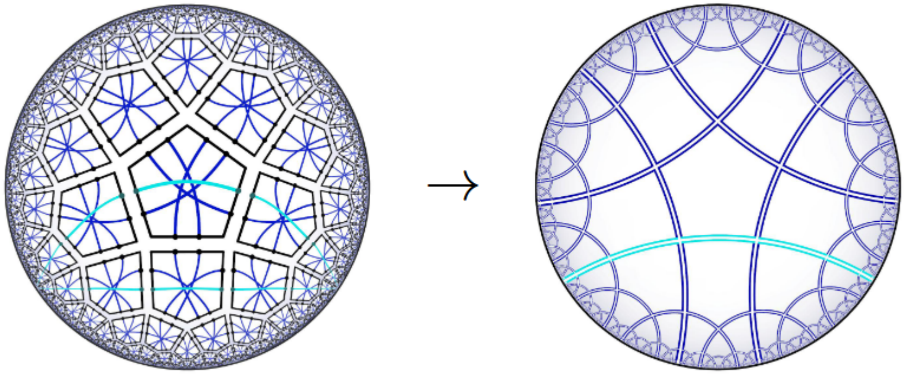


Figure 9: The $\{5,4\}$ HaPPY code in terms of Majorana dimers for a local $\bar{0}$ input on all tiles, shown for the uncontracted states on each pentagon (left) and the full contraction (right). The full contraction contains only paired dimers, an example pair and its constituent dimer parts in the contracted system are highlighted.

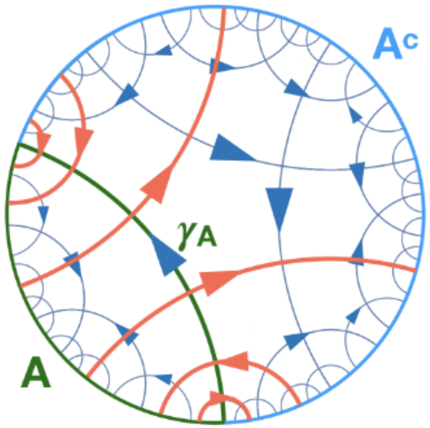


FIG. 1. Universal picture of holographic toy models: bit-threads distributed evenly on the hyperbolic lattice. In the continuous case it is bit-threads distributed homogeneously and isotropically in AdS space. The bit-threads connecting boundary subregion A and its complement A^c are highlighted in orange. Their number is proportional to the length of covering geodesic γ_A, which yields the Ryu-Takayanagi formula (Eq. (1)).

([Jahn, Gluza, Pastawski and Eisert 19][Yan 20])

Outlook – Further predictions.

This concludes my survey of one prediction of Hypothesis H on flat spacetimes.

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Among the predictions in the other limit, of smooth but curved spacetimes X, stands out the shifted 4-flux quantization [FSS19-HypH, Prop. 3.13][FSS20-Char, §5.3]:

$$\pi^{\tau}(\mathrm{X}) \xrightarrow{\ \mathrm{ch}\ } \left\{ \begin{array}{l} G_7, \\ G_4 \end{array} \in \Omega^{\bullet}(\mathrm{X}) \,\middle|\, \begin{array}{l} d\,G_7 = -\tfrac{1}{2} G_4 \wedge G_4 + \cdots \\ d\,G_4 = 0,\ \left[G_4 + \tfrac{1}{4} p_1(\omega)\right] \in H^4(\mathrm{X};\mathbb{Z}) \end{array} \right\}\Big/\!\sim_{\mathrm{conc}}$$

(where τ is $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$-structure on X and ω is a compatible connection/field of gravity).

That this shifted flux quantization should hold in M-theory is a famous proposal [Wi96a, 96b] & general. cohomology to capture this one condition has been purpose-built: [HS05][DFM07].

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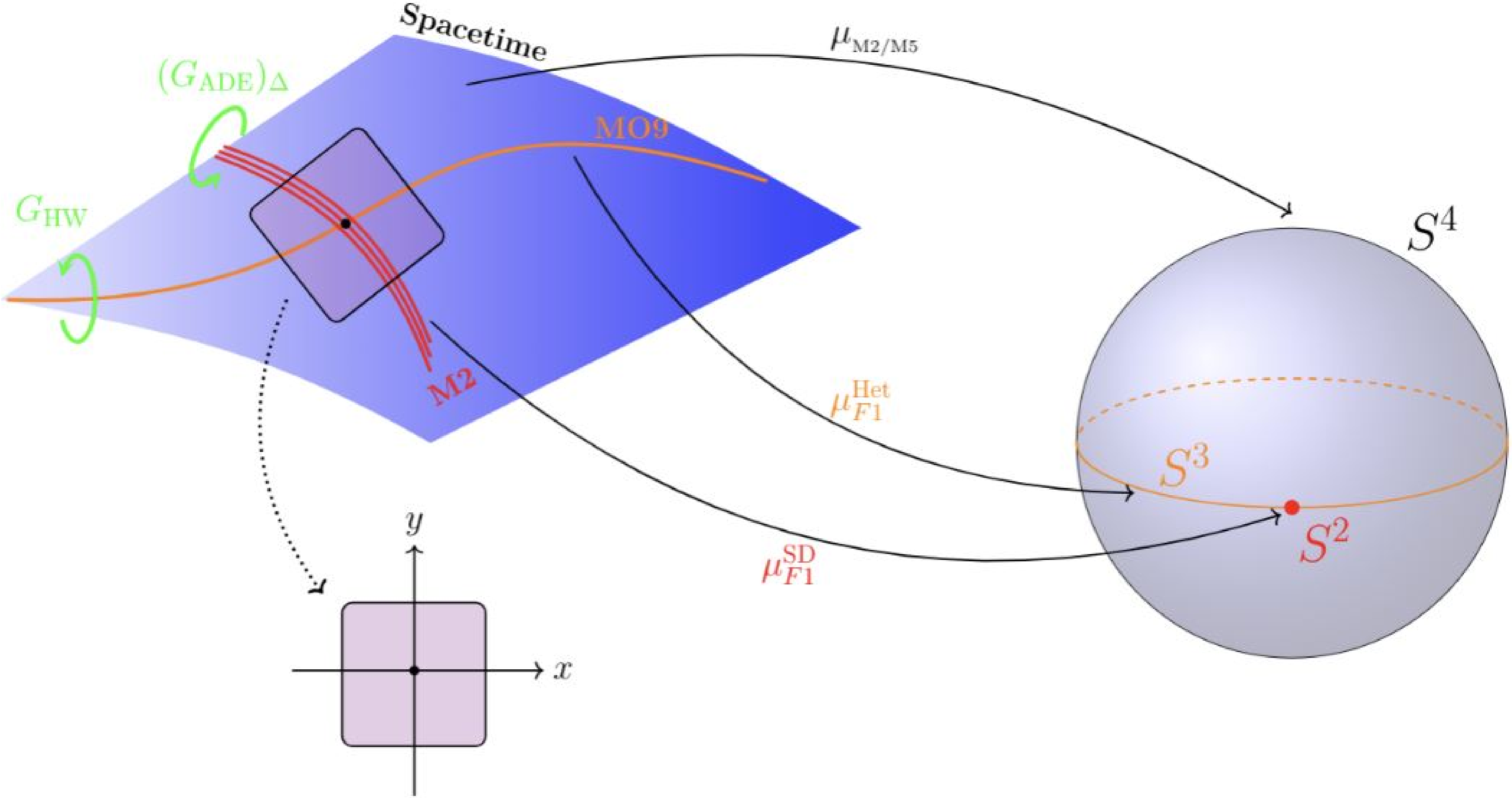
That this shifted flux quantization should hold in M-theory is a famous proposal [Wi96a, 96b] & general. cohomology to capture this one condition has been purpose-built: [HS05][DFM07].

But various further consistency conditions on M-flux are expected, e.g.

Page charge quantization of G_7. Hypothesis H implies this, too: [FSS19-M5WZ, Thm. 4.8].

these slides and further pointers are available at:

`ncatlab.org/schreiber/show/Some+Quantum+States+of+M-Branes+under+Hypothesis+H`



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