UNIVERSITEIT UTRECHT
Master in Mathematics

Master thesis, defended on August 31, 2010

∞-Stacks and their Function Algebras

with applications to ∞-Lie theory

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Abstract

For T any abelian Lawvere theory, we establish a Quillen adjunction between model category structures on cosimplicial T-algebras and on simplicial presheaves over duals of T-algebras, whose left adjoint forms algebras of functions with values in the canonical T-line object. We find mild general conditions under which this descends to the local model structure that models ∞-stacks over duals of T-algebras.

For T the theory of commutative algebras this reproduces the situation in Toën's *Champs Affines*. We consider the case where T is the theory of C^∞-rings: the case of synthetic differential geometry. In particular, we work towards a definition of smooth ∞-vector bundles with flat connection. To that end we analyse the tangent category of the category of C^∞-rings and Kock's simplicial model for synthetic combinatorial differential forms which may be understood as an ∞-categorification of Grothendieck's de Rham space functor.

Introduction

I will describe three ideas that are strongly related to this thesis, and after that explain where one is to place this thesis amongst them. The first two have been around for about half a century and the last for much less. The three ideas are 1: topos theory 2: synthetic differential geometry 3: higher topos theory. Topos theory is the mother subject of all three and this thesis is part of an attempt to apply the ideas of higher topos theory to synthetic differential geometry.

1. In the context of algebraic geometry, Grothendieck (and the community around him) suggested studying sheaves over certain small sites of duals of rings. This gave a way of embedding varieties, which in this context are identified with the duals of their coördinate rings, into a category with convenient closure properties. The concept of an elementary topos and the theory of the inner logic of elementary topoi were subsequently developed by several people. One important figure in this more abstract development is Bill Lawvere.

2. Inside certain topoi defined over sites not of duals of rings, but duals of C^∞-rings, authors like Anders Kock argue using the internal logic of the topos, as developed in the theory of elementary topoi. There is mathematical value in the freedom that is provided by working in the topos, as intuitive constructions that for example physicists want to use can be given rigorous meaning inside these topoi. One calls this field Synthetic Differential Geometry.

3. Certain generalizations of the notion of sheaf are useful They are called stacks. Where a sheaf is a contravariant functor, in some situations one encounters not a functor, but only a functor 'up to isomorphism', i.e. such that the image of a composite is, up to isomorphism, equal the composition of their images. Here we talk of isomorphism of arrows, i.e. certain kinds of morphisms between morphisms. One can subsequently consider morphisms between those morphisms and so on. This leads to the (imprecise) concept of an ∞-stack. Grothendieck himself was interested in ∞-stacks and wrote a famous letter to Daniel Quillen concerning them, which is entitled 'À la poursuite des champs'.

Joyal answered the call for a definition by suggesting that a model category on simplicial presheaves he constructed could model what Grothendieck was looking for. This idea has been applied fruitfully, for example by Morel and Voevodsky in their study of the A^1-homotopy theory of schemes, as well as by Toën and Vezzosi in their investigation of derived algebraic geometry.

Jacob Lurie, building on unpublished work of Joyal published, in 2009, an impressive piece of work 'Higher Topos Theory' [17] where ∞-stacks become objects of certain ∞-categories called ∞-topoi. Moreover, he proves that Joyal's model categories of simplicial presheaves provide 'models' for his ∞-topoi. Lurie develops derived geometry grounded in this theory in his works entitled 'Derived Algebraic

Geometry'.

The place of this thesis amongst the ideas 1, 2 and 3 is between 2 and 3. The program is to understand smooth derived geometry. We take our example from Toën's work 'Champs Affines' [23] in which it is established that there is a Quillen adjunction between cosimplicial R-algebras and simplicial presheaves (with a local model structure) over a small site of duals of R-algebras for any commutative ring R. We interpret the left adjoint as sending a simplicial presheaf X to the cosimplicial R-algebra $\mathcal{O}(R)$ of functions from X to some line object O. There is space for generalization here, which we occupy in order to apply this idea to the context of C^{∞}-rings. Thus we show, for any Lawvere theory T under the Lawvere theory of abelian groups, that there is a line object in the model category of simplicial presheaves over any suitable small site of duals of C^{∞}-rings and that homming into this line object gives the left adjoint of a Quillen adjunction.

Another result in this text is the construction of a cosimplicial C^{∞}-ring that is closely related to Kock's infinitesimal simplices and the proof that its normalization under the Dold-Puppe equivalence is the De Rham complex. There is still work to do in this part, but intuitively the De Rham complex is the function algebra on the ∞-stack of infinitesimal paths on a manifold, surely a pleasing mental image. A more rounded off piece of work is the description of the tangent category of C^{∞}-rings (intuitively the category of all modules over all C^{∞}-rings) in terms of the tangent category of commutative rings, with the important corollary that the category of simplicial modules over a simplicial C^{∞}-ring R is equivalent to the category of simplicial modules over its underlying simplicial ring $U(R)$. It is expected that these results will lead to a good notion of ∞-vector bundle with flat connection on a manifold.

Other work in the area of smooth derived geometry is that of David Spivak. In his PhD-thesis and the article 'Derived Smooth Manifolds' ([22]) based on that he discusses 'spaces' that are locally equivalent to duals of simplicial C^{∞}-rings. He then proves how this gives a good intersection theory of generalized smooth manifolds. Our approach can be seen as complementary to Spivak's. Thus, morally speaking, Spivak considers a simplicial site, whereas we consider simplicial presheaves. Ultimately one would to seek to combine these in derived geometry.

Outline

This work is divided into two parts. In the first part we derive the main theorem, in the second part we present some applications.

We begin by proving that there is an enriched adjunction between cosimplicial T-algebras and simplicial presheaves on certain small categories $\mathcal{C}$ of opposites of T-algebras for Lawvere theories T lying under the Lawvere theory of abelian groups. The left adjoint sends a simplicial presheaf to the cosimplicial T-algebra of functions from it to the line object. This enriched adjunction is shown to be a Quillen adjunction, when we regard the simplicial presheaves with their projective model category structure and the cosimplicial algebras with a structure where weak equivalences induce isomorphisms in cohomology. Moreover, under mild cohomological conditions, this Quillen adjunction is proven to descend to the (hyper-)local structure on simplicial presheaves for subcanonical Grothendieck topologies on $\mathcal{C}$.

The second part explores the application of this theory to the context of C^∞-rings, and therefore to synthetic differential geometry. We characterize the tangent category for C^∞-rings in terms of the one for rings (the tangent category of the category of rings is the category of all modules over rings). Next we see how the De Rham complex on a manifold emerges naturally as the normalization of the cosimplicial C^∞-ring of functions on the "infinitesimal path ∞-groupoid" on the manifold. This is a rerooting of work done by Kock in the synthetic differential context on infinitesimal simplices. The context provided by me is performed not in some topos based on duals of C^∞-rings, but on the category of duals of C^∞-rings itself. Moreover, the relationship between said ∞-groupoid and the De Rham complex is shown to be described by Dold-Puppe's normalization functor. We end this second and last part of the thesis by proving that the adjunction from the main theorem descends to a model category localized at a suitable set of Čech covers. This means that the image of the 'infinitesimal path ∞-groupoid" under the derived right adjoint of our Quillen adjunction is the smooth ∞-stack of infinitesimal paths.

Contents

Part I

General Theory

Chapter 1

Cosimplicial T-algebras

1.1 Lawvere theories

In this section I review the notion of a Lawvere theory. An algebra for a Lawvere theory is intuitively a set with some set of operations that satisfy axioms that use only the operations on the set and equality in their formulations. For example, the category of abelian groups is the category of structures $(A, +, -, 0)$ such that appropriate axioms hold. This in contrast to structures $(A, +)$ such that appropriate axioms hold. However, the categories are equivalent, so we ignore the distinction in this text.

Let us get to it.

1. **Definition.** A Lawvere theory is a category T isomorphic to a category having all natural numbers $n \in \mathbb{Z}_{\geq 0}$ as objects, such that n is an n-fold categorical power of 1. An algebra for a Lawvere theory T is a product preserving functor from T to Set. The category of all T-algebras as objects and morphisms all natural transformations between them is written T-alg.

If T is a Lawvere theory then the category of T-algebras is equivalent to a category of equationally defined universal algebras. Below I translate the basics of congruences to the setting of Lawvere theories.

2. **Definition.** Let T be a Lawvere theory, A a T-algebra. A congruence $\sim$ on A is an equivalence relation on $A(1)$ such that the following holds for any $f : n \to 1$ in arr T. If $a, b \in A(n)$ such that for each $i = 1, ..., n$ we have $A(\pi_i)(a) \sim A(\pi_i)(b)$ then also $A(f)(a) \sim A(f)(b)$.

3. **Proposition.** Let T be a Lawvere theory and A a T-algebra. Then for any relation R on $A(1)$ there is a smallest congruence on A containing R.

Proof: The intersection of a set of congruences is a congruence. Take our desired congruence to be the intersection of all congruences of A containing R. $\square$

4. **Proposition.** Let T be a Lawvere theory, A a T-algebra, C a congruence on A. Then the relations $A/C(f) \subseteq (A/C)^n \times A/C$ induced by $A(f)$ for each $f : n \to 1$ in arr T are functions and define a T-algebra.

Proof: $\square$

This proposition is a good opportunity to introduce some notation:

5. **Definition.** If $R \subseteq A(1) \times A(1)$ then $\langle R \rangle$ denotes the smallest congruence containing R. Write A/C for the algebra proven to exist in the above proposition.

6. **Proposition.** Let $\phi, \psi : A \to B$ be two homomorphisms of T-algebras, $C_{\phi,\psi}$ the smallest congruence on B containing the relation that relates $\phi(a)$ and $\psi(a)$ for every $a \in A(1)$. Then the canonical map $B \to B/C_{\phi,\psi}$ is a coequalizer of ϕ and ψ.

Proof: Straightforward and similar to the case of groups. $\square$

7. **Remark.** Let T and S be Lawvere theories, and let $f : T \to S$ be a product preserving functor between them. Then write $f^* : S\text{-alg} \to T\text{-alg}$ for the functor defined by right composition with f. This functor preserves filtered colimits, since in each category these are computed pointwise.

If T is the category with objects $n \in \mathbb{Z}_{\geq 0}$ with n the n-th power of 1 and such that T is generated by the projections, then there is always a product preserving functor $T \to S$ for any Lawvere theory and we denote the corresponding forgetful functor $U_S : S\text{-alg} \to \text{Set}$.

8. **Proposition.** For any Lawvere theory T the forgetful functor $U_T : T\text{-alg} \to \text{Set}$ has a left adjoint which I denote by F_T.

Proof: If S is a set, let $F_T(S)$ be the algebra with underlying set all formal expressions $f(s_1, ..., s_n)$ for $s_1, ..., s_n \in S$ and f an arrow of T with the obvious action of arrows. $\square$

The following theorem is due to Lawvere in [16] (in this form). We need the special case where T is the theory of abelian groups in the proof for the model category structure on the category of algebras for a Lawvere theory S such that there is a product preserving functor from T to S.

9. **Proposition.** Let T and S be Lawvere theories, and let $f : T \to S$ be a product preserving functor between them. Then the functor $f^* : S\text{-alg} \to T\text{-alg}$ defined by right composition with f has a left adjoint.

Proof: Let $F_S \dashv U_S$ and $F_T \dashv U_T$ be the adjunctions from proposition 8. Let A be a T-algebra. Then $A \cong F_T U_T(A)/\Gamma$ for some T-congruence Γ. Now $U_T F_T U_T(A) \subseteq$

$U_SF_SU_T(A)$, so $\Gamma \subseteq U_SF_SU_T(A)$ and $F_SU_T(A)/\langle\Gamma\rangle$ is an S-algebra, where $\langle\Gamma\rangle$ is the smallest S-congruence containing Γ. This $F_SU_T(A)/\Gamma$ is the free S-algebra on A. $\square$

10. **Remark.** There is a theory of Sets. It consists of all natural numbers starting from 0 as objects, with only projections as arrows. The adjunction $U_S \vdash F_S$ is a special case of the above proposition, where we take T to be this theory of sets.

Fix a Lawvere theory T.

11. **Proposition.** The category of T-algebras is complete and cocomplete.

Proof: Completeness follows by taking pointwise limits. For cocompleteness it remains only to show that T-alg has coproducts since coequalizers are given by proposition 6.

The $\mathcal{S}$-indexed coproduct of a set $\{F_T(S)/\Gamma_S; S \in \mathcal{S}\}$ of T-algebras is

$$F_T\left(\coprod \mathcal{S}\right)/\langle \cup_{S\in\mathcal{S}}\iota_S\Gamma_S\rangle,$$

where $\iota_S\Gamma_S$ is the image of Γ_S in $F_T(\coprod\mathcal{S})$. $\square$

1.2 Model category structure

12. **Definition.** Let T be a Lawvere theory. We say that T is abelian if there is a product preserving functor $\mathcal{A} \to T$ from the Lawvere theory for abelian groups to T.

Fix an abelian Lawvere theory T. Write $T\text{-alg}^{\Delta}$ for the category of cosimplicial T-algebras. Say a morphism of $T\text{-alg}^{\Delta}$ is a fibration if it is a componentwise surjection and say it is a weak equivalence if it induces an isomorphism on the cohomology of the associated cochain complexes of abelian groups. Say a morphism is a cofibration if it has the left lifting property with respect to the class of all acyclic fibrations. In this subsection I prove that this gives a simplicial model structure on the category $T\text{-alg}^{\Delta}$. Note that the two out of three axiom for weak equivalences is trivial and our category is complete and cocomplete.

The strategy for proving that there is a simplicial model category structure on the simplicial category of cosimplicial T-algebras is to prove first that there is such a structure on the category of cosimplicial abelian groups and use the “transfer theorem” which states that under some conditions on a right adjoint we may conclude that a simplicial model category exists in the domain of this right adjoint, given a model category structure on the codomain.

I thought/learned of the model category structure on cosimplicial abelian groups in this section from reading the note [13] and theorem 2.1.2. in [23], the ideas presented here are slight variations of their's, which are heavily inspired by [3]. Moreover, since the proof seems to also work in the case of simplicial abelian groups and the model structure presented here might therefore be considered standard even without the above references, I have relegated it to the appendix.

We consider both Ab_s^Δ and T-alg_s^Δ as simplicially enriched categories as defined in [10], by virtue of Ab and T-alg being both complete and cocomplete. Our first subject is the category of cosimplicial abelian groups.

13. **Theorem.** Call a morphism of cosimplicial abelian groups a fibration if it degreewise a surjection and a weak equivalence if it is a quasi-isomorphism. This gives Ab^Δ the structure of a simplicial model category.

Proof: See the appendix. □

14. **Theorem.** Say a morphism in T-alg^Δ is a fibration if its image under the forgetful functor to cosimplicial abelian groups is and say it is a weak equivalence if its image under the forgetful functor is. Then this gives a simplicial model structure on the category T-alg_s^Δ.

Proof: Since by construction Ab^Δ is cofibrantly generated we apply the 'Transfer principle' and Quillen's path argument on page 811 of [2]. We need to check that there is a functorial path object and that the forgetful functor preserves filtered colimits to conclude that the described structure is a model category structure. The forgetful functor to abelian groups preserves filtered colimits by remark 7. For a functorial path object, take $A \mapsto A^{\Delta[1]}$; applying the forgetful functor to

$$A \cong A^{\Delta[0]} \to A^{\Delta[1]} \to A^{\Delta[0]+\Delta[0]} \cong A \times A$$

yields a path object in Ab^Δ, since by simplicial enrichment A^- carries acyclic cofibrations between cofibrant objects to weak equivalences, by the Factorization Lemma (Corollary 7.7.2 [11]), by the fact that the simplicial unit interval $\Delta[1]$ is a cylinder object for the point $\Delta[0]$ and using that every object A is fibrant.

It is left to verify axiom 'SM7' for enriched model categories (the "pushout-product axiom"). By [17] A.3.1.6 we need only show that for $i : C \to C'$ a cofibration of simplicial sets and $j : X \to Y$ a fibration in T-alg^Δ the map $f : X^{C'} \to X^C \times_{Y^C} Y^{C'}$ is a fibration which is trivial if either i or j is. Apply the forgetful functor T-$\mathrm{alg}^\Delta \to \mathrm{Ab}^\Delta$ to f. Note that exponentiation is preserved under the forgetful functor and so are pullbacks. Since fibrations and weak equivalences are preserved and reflected by the forgetful functor, the map $X^{C'} \to X^C \times_{Y^C} Y^{C'}$ is as desired. □

Chapter 2

Function T-algebras on stacks

In algebraic geometry one considers small subsites $\mathcal{C}$ of Ring^{op} and sees sheaves on such sites as general spaces. There is an adjuction $\text{Spec} \vdash \mathcal{O}$ where $\text{Spec} : R \mapsto \text{Ring}(R, -) : \text{Ring} \to \text{Sh}(\mathcal{S})$ and $\mathcal{O} : X \mapsto \text{Sh}(X, \text{Spec}(\mathbb{Z}[x]))$, assuming $\mathbb{Z}[x]$ is in $\mathcal{S}$ and its spectrum $\text{Spec}(\mathbb{Z}[x])$ is a sheaf.

In [23] Toën shows that one can do something analogous replacing sheaves by stacks and rings by cosimplicial algebras, with appropriate model category structures. In this chapter we show that the argument goes through for T-algebras of a particular kind, under mild cohomological conditions. The line object $\text{Spec}(\mathbb{Z}[x])$ is generalized to $T\text{-alg}(F_T(1), -)$, the non-simplicial spectrum of the free T-algebra on one generator. The main theorem is that there is a Quillen adjunction between cosimplicial T-algebras and simplicial presheaves on some small subsite of $T\text{-alg}^{\text{op}}$ when the latter is endowed with the projective model category structure.

Throughout this chapter $\mathcal{C}$ will stand for a small full subcategory of $T\text{-alg}^{\text{op}}$.

2.1 Simplicial presheaves

We first review some model category structures on categories of contravariant functors from some small category $\mathcal{C}$ to the category of simplicial sets. The two most prominent model category structures on $\left(\text{Set}^{\Delta^{\text{op}}}\right)_s^{\mathcal{C}^{\text{op}}}$ which are called global are the injective and the projective model category structure. We will be concerned with the projective structure, which I now describe. Call a morphism $f : X \to Y$ of simplicial presheaves a global fibration (weak equivalence) if for each object A of $\mathcal{C}$ we have that f_A is a fibration (weak equivalence) of simplicial sets in the Quillen model category structure. Say a map f of simplicial presheaves is a global cofibration if it has the left lifting property with respect to all global fibrations that are also global weak equivalences.

15. **Proposition.** The above definitions make $\left(\text{Set}^{\Delta^{\text{op}}}\right)_s^{\mathcal{C}^{\text{op}}}$ into a simplicially enriched model category.

Proof: The Quillen model structure on the simplicially enriched category of simplicial sets is cofibrantly generated, so we may apply Theorem 11.7.3 in [11]. □

The model category structure we are interested in on the simplicial category of simplicial presheaves is not the projective category structure, rather it is a left Bousfield localization of this category. For its definition we use coskeleta. Let $\mathcal{C}$ be a small category.

16. **Definition.** Let $n \in \mathbb{Z}_{\geq 0}$. Write Δ_n for the full subcategory of Δ on the objects of the form $[k]$ for $k \leq n$. The n-truncation functor is the functor $\left(\mathrm{Set}^{\mathcal{C}}\right)^{\Delta^{\mathrm{op}}} \to \left(\mathrm{Set}^{\mathcal{C}}\right)^{\Delta_n^{\mathrm{op}}}$ given by restriction along the inclusion $\Delta_n \to \Delta$.

The truncation functor clearly preserves colimits so it has a right adjoint. Pick one.

17. **Definition.** Let $n \in \mathbb{Z}_{\geq n}$. The functor $\mathrm{cosk}_n : \left(\mathrm{Set}^{\mathcal{C}}\right)^{\Delta_n^{\mathrm{op}}} \to \left(\mathrm{Set}^{\mathcal{C}}\right)^{\Delta^{\mathrm{op}}}$ that is right adjoint to the n-truncation functor is called the n-th coskeleton functor.

18. **Definition.** Let $(\mathcal{C}, \Gamma)$ be a site, X an object of $\mathcal{C}$ and $f : U \to \underline{hX}$ a map of simplicial presheaves. Then f is said to be a hypercover if
1) In each degree U is a coproduct of representables.
2) For every $n \in \mathbb{Z}_{\geq 0}$ we have that the map $f([n+1]) \to \mathrm{cosk}_n(f)([n+1])$ induces an epimorphism on the associated sheaves.

19. **Definition.** A hypercover is said to be split if its domain X satisfies the following condition. There exist subpresheaves $N([k]) \subseteq X([k])$ for $[k] \in \Delta$ such that the following holds:

$$\coprod_{\sigma} N(\mathrm{dom}(\sigma)) \to X([n]),$$

where σ ranges over all surjections onto $[n]$ is an isomorphism.

20. **Remark.** The domain of a split hypercover is cofibrant in the projective structure on simplicial presheaves. We consider them because since their domains and codomains are cofibrant a morphism on homotopy function complexes induced by a hypercover f into a fibrant object is naturally weakly equivalent to the morphism induced by f on the simplicially enriched hom-objects.

21. **Definition.** A hypercover of height n is a hypercover f for which the canonical map $f \to \mathrm{cosk}_{n+1} f$ is an isomorphism. When there exists some n for which f is of height n, f is called bounded. It is called a Čech cover if it has height 0.

22. **Theorem.** The left-Bousfield localization of the projective model category structure on simplicial presheaves on some small site at the class of all hypercovers exists. The localization at all Čech covers also exists. Both are simplicial model categories.

Proof: [8], theorem 6.2 for the first part of the theorem. Since there is only a set of Čech covers, and the projective model category structure is simplicial, combinatorial and left-proper, the localization at Čech covers exists by theorem A.3.7.3. of [17]. The fact that these are simplicial model category follows from theorem 4.1.1. in [11]. □

2.2 The $\mathrm{Set}^{\Delta^{\mathrm{op}}}$-enriched adjunction

In this section I define the Set-enriched functors $\mathcal{O}$ and Spec, which are the objects of study for this chapter, and show that they are adjoint to each other. I thank Urs Schreiber for pointing out this proof of the adjunction. I was unfamiliar with the end calculus and the previous proof spanned over five pages and relied on less obvious facts.

23. **Definition.** We define the functor $\mathrm{Spec} : \left(T\text{-alg}^{\Delta}\right)^{\mathrm{op}} \to \left(\mathrm{Set}^{\Delta^{\mathrm{op}}}\right)^{\mathcal{C}}$ by

$$\begin{array}{lll} R & \mapsto & T\text{-alg}_s^{\Delta}(R, \underline{\ \ }) \\ f & \mapsto & T\text{-alg}_s^{\Delta}(f, \underline{\ \ }), \end{array}$$

where $\underline{\ \ }$ denotes the functor sending a T-algebra $A \in \mathrm{ob}\,\mathcal{C}$ to $\underline{A}$, the constant cosimplicial T-algebra with value A.

24. **Definition.** Define the functor $\mathcal{O} : \left(\mathrm{Set}^{\Delta^{\mathrm{op}}}\right)^{\mathcal{C}} \to \left(T\text{-alg}^{\Delta}\right)^{\mathrm{op}}$ as

$$\begin{array}{lll} M & \mapsto & (n \mapsto \mathrm{Set}^{\mathcal{C}}(M(n), U_T)) \\ f & \mapsto & (n \mapsto \mathrm{Set}^{\mathcal{C}}(f_n, U_T)) \end{array}$$

25. **Remark.** Note that we can endow U_T with a T-algebra structure by sending an arrow $f : m \to n$ of T to the natural transformation $A \mapsto A(f)$. So the specification above does indeed yield a functor.

26. **Remark.** An easy Yoneda style argument shows that $n \mapsto \mathrm{Hom}_{\mathrm{Set}^{\mathcal{C}}}(M(n), U_T)$ is isomorphic to $\left(\mathrm{Set}^{\Delta^{\mathrm{op}}}\right)^{\mathcal{C}}_s(M, \underline{U_T})$. Moreover, $\underline{U_T}$ is isomorphic to the simplicial presheaf $\mathrm{Spec}(\underline{F_T(1)})$. So we are "homming into the line object".

Likewise we could describe $\mathrm{Spec}(A)$ as $T\text{-alg}_s^{\Delta}(A, -)$.

However, it is a nuisance to keep unraveling the Yoneda lemma in the following computation, which is why I prefer the definition given above.

27. **Theorem.** There exists a simplicially enriched adjunction $\mathrm{Spec} \vdash \mathcal{O}$.

Proof: The proof is split into two parts. In the first I prove that there is an adjunction, and in the second part I use the first result to prove that there is a simplicial set enriched adjunction. To show there is an adjunction, note that there is a sequence of natural isomorphisms (writing $k = 1^k$ for an object of T):

$$
\begin{aligned}
T\text{-alg}^{\Delta}(A,\mathcal{O}(X)) &\cong \int_{[n]\in\Delta}\int_{k\in T} \mathrm{Set}(A([n])(k), \mathrm{Set}^{\mathcal{C}}(X([n]), U_T^k))\\
&\cong \int_{[n]\in\Delta}\int_{k\in T}\int_{B\in\mathcal{C}} \mathrm{Set}(A([n])(k), \mathrm{Set}(X([n])(B), B(k)))\\
&\cong \int_{[n]\in\Delta}\int_{B\in\mathcal{C}} \mathrm{Set}(X([n])(B), \int_{k\in T} \mathrm{Set}(A([n])(k), B(k)))\\
&\cong \int_{[n]\in\Delta}\int_{B\in\mathcal{C}} \mathrm{Set}(X([n])(B), T\text{-alg}(A([n]), B))\\
&\cong \left(\mathrm{Set}^{\mathcal{C}}\right)^{\Delta^{\mathrm{op}}}(X, \mathrm{Spec}(A))
\end{aligned}
$$

For the simplicial enrichment of the adjunction we have the following sequence of natural isomorphisms, where the notation A^S for S a simplicial set and A an object is cotensoring as in [10], and likewise $A\cdot S$ is tensoring:

$$
\begin{aligned}
T\text{-alg}^{\Delta}\left(A\cdot\Delta([n]), \mathcal{O}(X)\right) &\cong \mathrm{Set}^{\mathcal{C}\times\Delta^{\mathrm{op}}}\left(X, \mathrm{Spec}(A\cdot\Delta[n])\right)\\
&\cong \mathrm{Set}^{\mathcal{C}\times\Delta^{\mathrm{op}}}\left(X, T\text{-alg}_s^{\Delta}(\Delta[n]\cdot A, \underline{\ \ })\right)\\
&\cong \int_{B\in\mathcal{C}} \mathrm{Set}^{\Delta^{\mathrm{op}}}\left(X(B), T\text{-alg}_s^{\Delta}(\Delta[n]\cdot A, \underline{B})\right)
\end{aligned}
$$

Now note that $\mathrm{Set}^{\Delta^{\mathrm{op}}}\left(X(B), T\text{-alg}_s^{\Delta}(\Delta[n]\cdot A, \underline{B})\right)$ is naturally isomorphic to

$$
\left(\mathrm{Set}_s^{\Delta^{\mathrm{op}}}\left(X(B), T\text{-alg}_s^{\Delta}(\Delta[n]\cdot A, \underline{B})\right)\right)_0
$$

and from that we conclude that the end above is naturally isomorphic to

$$
\mathrm{Set}^{\mathcal{C}\times\Delta^{\mathrm{op}}}\left(X\cdot\Delta[n], T\text{-alg}_s^{\Delta}(A, \underline{\ \ })\right) \cong \mathrm{Set}^{\mathcal{C}\times\Delta^{\mathrm{op}}}(X, \mathrm{Spec}(A))([n]). \quad \square
$$

2.3 The Quillen adjunction

This section is devoted to showing how the adjunction from last section is actually a Quillen adjunction in two different ways. The basis is the following theorem, stating that if we take the projective structure on simplicial presheaves, we obtain a Quillen adjunction.

28. **Theorem.** The functor $\mathrm{Spec} : \left(T\text{-alg}_s^{\Delta}\right)^{\mathrm{op}} \to (\mathrm{Set}^{\mathcal{C}\times\Delta^{\mathrm{op}}})_{\mathrm{proj}}$ is a right Quillen functor.

Proof: Let $f : X \to Y$ be an (acyclic) cofibration in $T\text{-alg}^{\Delta}$, in other words an (acyclic) fibration in $\left(T\text{-alg}^{\Delta}\right)^{\mathrm{op}}$. We need to show that $(A,[n]) \mapsto T\text{-alg}^{\Delta}\left(f, A^{\Delta[n]}\right)$ is a (acyclic) fibration in the projective model structure; that is to say, that for each $A \in \mathrm{ob}\,\mathcal{C}$, the morphism of simplicial sets $[n] \mapsto T\text{-alg}^{\Delta}\left(f, A^{\Delta[n]}\right)$ is an (acyclic) Kan fibration.

Note that all objects, thus in particular $\underline{A} : [n] \mapsto A$ for $A \in \mathrm{ob}\,\mathcal{C} \subseteq T$-alg, in $T\text{-alg}^{\Delta}$ are fibrant. Therefore, by [17] A.3.1.6 (2'),

$$T\text{-alg}_s^{\Delta}(Y, \underline{A}) \to T\text{-alg}_s^{\Delta}(X, \underline{A}) \times_{T\text{-alg}_s^{\Delta}(X,*)} T\text{-alg}_s^{\Delta}(Y, *)$$

($*$ denotes the terminal object) is an (acyclic) fibration. But this morphism is isomorphic to $T\text{-alg}_s^{\Delta}(f, \underline{A}) : [n] \mapsto T\text{-alg}^{\Delta}(f, A^{y([n])})$. $\square$

29. **Lemma.** Let K be a class of split hypercovers and suppose $H^p(\mathcal{O}(f))$ is an isomorphism for every hypercover $f \in K$. If the left-Bousfield localization of $(\mathrm{Set}^{\mathcal{C}\times\Delta^{\mathrm{op}}})_{\mathrm{proj}}$ at the class K exists then Spec is a right Quillen functor with respect to this localization.

Proof: By [11] 13.1.2 and the fact that in $T\text{-alg}^{\Delta}$ all objects are fibrant we know $\left(T\text{-alg}^{\Delta}\right)^{\mathrm{op}}$ is left proper. Therefore, by [17] A.3.7.2 it suffices to check that fibrant objects are sent to fibrant objects. So let B be cofibrant in $T\text{-alg}^{\Delta}$. We want to check that $\mathrm{Spec}(B)$ is fibrant, i.e. that $\mathrm{Spec}(B)$ is fibrant in the global structure and that $\mathrm{Spec}(B)$ is local. The first follows from the fact that Spec is right Quillen to the global model structure and hence sends fibrations to fibrations in the global structure.

For the second condition, let $f \in K$. Since both the domain and codomain of f are cofibrant ([7] corollary 9.4[1]) it is enough to show that

$$\left(\mathrm{Set}^{\Delta^{\mathrm{op}}}\right)_s^{\mathcal{C}}(f, \mathrm{Spec}(B)) \cong T\text{-alg}^{\Delta}(B, \mathcal{O}(f))$$

is a weak equivalence of simplicial sets. But we know that $\mathcal{O}(f)$ is a weak equivalence in $T\text{-alg}^{\Delta}$. Since B is cofibrant $T\text{-alg}_s^{\Delta}(B, -)$ sends acyclic fibrations to acyclic fibrations by 'SM7'. Since all objects are fibrant in this model category, and since the factorization lemma (Corollary 7.7.2 [11]), $T\text{-alg}_s^{\Delta}(B, -)$ sends weak equivalences between fibrant objects to weak equivalences, $T\text{-alg}_s^{\Delta}(B, \mathcal{O}(f))$ is a weak equivalence of simplicial sets. $\square$

30. **Theorem.** Assume $F_T(*) \in \mathcal{C}$ and let J be a subcanonical Grothendieck topology on $\mathcal{C}^{\mathrm{op}}$ and suppose K is a class of split hypercovers for J such that for all $f \in K$ we have $H^1(\mathcal{O}(\mathrm{dom} f)) = 0$. Then Spec is right Quillen with respect to the left-Bousfield localization at K of $(\mathrm{Set}^{\mathcal{C}\times\Delta^{\mathrm{op}}})_{\mathrm{proj}}$ (if this structure exists).

Proof: The next three paragraphs of this proof show that $\mathcal{O}(f)$ induces an isomorphism in cohomology. By the lemma above this is sufficient.

First a note on $\mathrm{Sh}(\mathcal{C}^{\mathrm{op}}, J)/hY$. By Lemma C.2.2.17 in [14] $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}/hY \simeq \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}/Y}$. Also, if J_Y is the Grothendieck topology on $\mathcal{C}^{\mathrm{op}}/Y$ that calls a family a covering if its domains are a covering family, then J_Y is subcanonical and there is an

[1]The condition in the premise is satisfied by split hypercovers as defined in [8]

equivalence $\epsilon : \mathrm{Sh}(\mathcal{C}^{\mathrm{op}}/Y, J_Y) \to \mathrm{Sh}(\mathcal{C}^{\mathrm{op}}, J)/hY$ that sends $h_{\mathcal{C}^{\mathrm{op}}/Y}(g : A \to Y)$ to $h_{\mathcal{C}^{\mathrm{op}}}(g) : hA \to hY$.

Now I reproduce Verdier's ([24], appendix to exposé V, section 2) definition of homology in our case for $f \in \left(\mathrm{Set}^{\Delta^{\mathrm{op}}}\right)^{\mathcal{C}} / \underline{hY}$. Take the free abelian group F on f, this is a simplicial object of $\mathrm{Ab}\left(\mathrm{Set}^{\mathcal{C}}/hY\right)$. Since $\mathrm{Ab}\left(\mathrm{Set}^{\mathcal{C}}/hY\right)$ is abelian, we may take the homology of this object and define its abelian group objects to be the homology of f. By 3) of théorème 3.2. in [24], the associated sheaves of the homology of f are $0 \in \mathrm{Ab}(\mathrm{Sh}(\mathcal{C}^{\mathrm{op}}, J)/hY)$ in degree strictly greater than 0; in dimension 0 the associated sheaf is the constant sheaf on the integers.

By the Freyd-Mitchell embedding theorem [9], we may embed the category of abelian groups in $\mathrm{Sh}(\mathcal{C}^{\mathrm{op}}, J)/hY$ through some exact i into the category of modules over some commutative ring R. Write $\tilde{F}$ for the chain iNF in R-Mod. We may then apply the universal coëfficient theorem for cohomology (Cartan-Eilenberg's Homological Algebra theorem VI.3.3a) and get a short exact sequence (we shall need it for $n > 1$):

$$0 \longrightarrow \mathrm{Ext}^1(H_{n-1}(\tilde{F}), iC) \longrightarrow H^n(R\text{-Mod}(\tilde{F}, iC)) \longrightarrow \mathrm{Ab}(H_n(\tilde{F}), iC) \longrightarrow 0.$$

Now by Verdier's result we have that the sheaf of abelian groups $H_n(N(F)) \cong 0$ for $n \geq 1$. Since i is exact therefore $0 \cong iH_n(N(F)) \cong H_n(iN(F))$ for $n \geq 1$, so we see $H^n(R\text{-Mod}(\tilde{F}, iC)) \cong 0$ and since i is full and faithful for any abelian group object C in $\mathrm{Sh}(\mathcal{C}^{\mathrm{op}}, J)/hY$. In particular we may take, as coefficients C, the obvious group on $U_T \times hY$. Note that, since i is full and faithful we have

$$\begin{array}{ll}
R\text{-Mod}(iNF, iC) & \cong \\
\mathrm{Ch}^{+}\left(\mathrm{Ab}\left(\mathrm{Sh}(\mathcal{C}^{\mathrm{op}}, J)/hY\right)\right)\left(NF, \underline{U_T \times hY}\right) & \cong \\
\mathrm{Ab}\left(\mathrm{Sh}(\mathcal{C}^{\mathrm{op}}, J)/hY\right)^{\Delta}\left(F, \underline{U_T \times hY}\right) & \cong \\
\mathrm{Sh}(\mathcal{C}^{\mathrm{op}}, J)/hY^{\Delta}(f, \underline{U_T \times hY}) & \cong \\
\mathrm{Sh}(\mathcal{C}^{\mathrm{op}}, J)^{\Delta}(X, \underline{U_T}). &
\end{array}$$

All that is left is to show is that $H^0(\mathcal{O}(f))$ is an isomorphism. But this is an easy consequence of the fact that U_T is a sheaf (being represented by $F_T(*) \in \mathcal{C}$). $\square$

Part II

Applications to C^{∞}-rings.

Chapter 3

C^∞-rings

3.1 C^∞-rings

In this section I introduce the category of C^∞-rings, mostly following [18]. A C^∞-ring is an algebra for the Lawvere theory of Cartesian spaces and smooth maps, and therefore part I applies. There is an injective, full and faithful functor from the opposite category of C^∞-manifolds to the category of C^∞-rings that preserves intersections of transversal submanifolds. Our main motivation for studying this extension is that this category contains what we call infinitesimal objects that behave as we want them to.

Write $\mathbb{E}$ for the category with as objects all finite cartesian powers of the real line and as arrows all infinitely differentiable functions between them.

31. **Definition.** A C^∞-ring is a product preserving functor from $\mathbb{E}$ to Set and C^∞-ring is full subcategory of $\text{Set}^{\mathbb{E}}$ consisting of C^∞-rings. Morphisms of C^∞-rings are called homomorphisms (of C^∞-rings).

32. **Example.** Let $\mathcal{M}$ be a C^∞ manifold. Define a C^∞-ring $C^\infty(\mathcal{M})$ by sending $\mathbb{R}^n$ to $C^\infty(\mathcal{M}, \mathbb{R}^n)$. For a smooth map $f : \mathbb{R}^n \to \mathbb{R}$ let $C^\infty(M)(f)$ be the function sending a smooth map $\phi : \mathcal{M} \to \mathbb{R}^n$ to $f \circ \phi : \mathcal{M} \to \mathbb{R}$.

33. *The underlying* $\mathbb{R}$*-algebra of a* C^∞*-ring.* Let A be a C^∞-ring. Then the ordered triple $\mathcal{U}(A) := (A(\mathbb{R}), A(\cdot), A(+))$ is a ring. If $r \in \mathbb{R}$ then there is a smooth function $\underline{r} : \mathbb{R}^0 \to \mathbb{R}$ sending the only element $* \in \mathbb{R}^0$ to r. Using this, we construct a ring homomorphism $\mathbb{R} \to \mathcal{U}(A)$, sending some $r \in \mathbb{R}$ to the element $A(\underline{r})(*)$, where now $*$ is the unique element of $A(\mathbb{R}^0)$. This gives us, for every C^∞-ring an underlying $\mathbb{R}$-algebra.

We will have to do computations with C^∞-rings and a big chunk of these computations will in fact occur in the underlying $\mathbb{R}$-algebras. In these cases we will suppress the A in front of ring-theoretic operations, treating a C^∞-ring as an $\mathbb{R}$-algebra supplied with additional operations. So $A(+)(a_1, a_2)$ will become $a_1 + a_2$

and so on.

A very useful property of the category of C^∞-rings is that the quotient of the underlying $\mathbb{R}$-algebra of a C^∞-ring by one of its ideals is again canonically a C^∞-ring. This follows from the following theorem.[1]

Let $p \in \mathbb{R}^n$ for some natural number n. A starlike neighborhood U of p in $\mathbb{R}^n$ is an open neighborhood of p in $\mathbb{R}^n$ such that any straight line segment in $\mathbb{R}^n$ connecting p to some point in U is contained in U.

34. **Theorem.** Any infinitely differentiable function f in a starlike neighborhood U of a point $p \in \mathbb{R}^k$ satisfies, for any natural number n the formula

$$f(x) = \sum_{\tau \in T} \left(\prod_{i=1}^{k} \left(\frac{1}{\tau_i!}(x_i - p_i)^{\tau_i} \right) \frac{\partial^{\tau_1}}{\partial x_1^{\tau_1}} \cdots \frac{\partial^{\tau_k}}{\partial x_k^{\tau_k}} f\Big|_p \right) + \sum_{\sigma \in S} \left(\prod (x_i - p_i)^{\sigma_i} \right) g_\sigma(x),$$

for some $g_\sigma \in C^\infty(U, \mathbb{R})$; I wrote T for $\{(\tau_1, ..., \tau_k) \in \mathbb{Z}^k_{\geq 0}; \sum_{i=1}^k \tau_i \leq n\}$ and $S := \{\sigma \in \mathbb{Z}^k_{\geq 0}; \sum_{i=1}^k \sigma_i = n + 1\}$.

Proof: Corollary 2.4 of [19]. □

In the sequel it is mostly through theorem 34 that we put the infinite differentiability of functions at work. An important special case is when n is set equal to zero, which we apply in the following proposition.

35. **Proposition.** Let A be a C^∞-ring. If I is an ideal of the underlying ring $\mathcal{U}(A)$ then the quotient A/I exists and $\mathcal{U}(A/I) \cong \mathcal{U}(A)/I$.

Proof: We use the case $n = 0$ of theorem 34 to show that the ideal I is also a C^∞-ring congruence. I spell this out now.

Define $A/I : \mathbb{R}^n \mapsto A(\mathbb{R}^n)/\sim_{I,n}$, where $\sim_{I,n} \subseteq A(\mathbb{R}^n)$ is the congruence induced by I. That is to say, for $x, y \in A(\mathbb{R}^n)$ we set $x \sim_{I,n} y$ iff for every $i = 1, ..., n$ we have $\pi_i(x) - \pi_i(y) = 0$ in A/I. For $f : \mathbb{R}^n \to \mathbb{R}$ a smooth function define a relation $(A/I(f))_0 \subseteq \mathbb{R}^n \times \mathbb{R}$ by $[y]_{\sim_{I,n}} \simeq (A/I(f))_0 \, [x]_{\sim_{I,n}}$ iff $f(x) \sim_{I,1} y$. To obtain a function we must now check that if $x \sim_{I,n} y$ also $f(x) \sim_{I,1} f(y)$.

To do this we apply theorem 34. Let us consider the case $n = 0$ for f, and let $p, q \in \mathbb{R}^n$ be any point in $\mathbb{R}^n$. Since f is smooth everywhere we obtain from theorem 34 that

$$f(q) - f(p) = \sum_{i=1}^{n} (q_i - p_i) g_i(q, p)$$

[1] Lawvere theories satisfying this theorem are called Fermat theories ([6]).

for smooth $g_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. This yields a corresponding equation in the category $\mathbb{E}$ and applying A to this equation we obtain, for $x, y \in A(\mathbb{R}^n)$ that

$$A(f)(x) - A(f)(y) = \sum_{i=1}^{n} (\pi_i(x) - \pi_i(y)) A(g_i)(x, y).$$

If now $x \sim_{I,n} y$ then by definition of $\sim_{I,n}$ the righthand side is equivalent to zero, hence so is the left hand side.

The statement about the underlying $\mathbb{R}$-algebras follows by construction. □

36. **Definition.** We call a C^∞-ring finitely generated if it is of the form $C^\infty(\mathbb{R}^n)/I$ for some ideal I. Such a C^∞-ring is said to be finitely presented if I is a finitely generated ideal.

3.1.1 Weil algebras and infinitesimals

The ring of dual numbers $k[X]/(X^2)$ plays an important role in algebraic geometry. It has only one prime ideal and its spectrum therefore has only one point. It is said that algebraic geometers view its spectrum $\mathrm{Spec}(k[X]/(X^2))$ as a point with a small arrow sticking out. There is a way to make this precise, namely by expressing things in categorial logic.

Indeed, finitely presented k-algebras (equiped with the structure of a site, see [21]) are the building blocks of a topos. The embedding of the category of finitely presented k-algebras into the Zariski topos gives a way to talk about the opposites of k-algebras using the full power of intuitionistic higher order logic[2]. But this is a bit of an overkill for our purposes. It will suffice to stay in the category of schemes over k.

The functor $\mathrm{Spec} : k\text{-alg}^{\mathrm{op}} \to \mathrm{Sch}(k)$ has a left adjoint, and therefore Spec preserves limits. This implies that the image of a colimit in k-alg is a limit in $\mathrm{Sch}(k)$. Now the following diagram is easily seen to be a coequalizer:

$$\begin{tikzcd} k[X] \arrow[r, shift left, "X \mapsto X^2"] \arrow[r, shift right, "X \mapsto 0"'] & k[X] \arrow[r] & k[X]/(X^2). \end{tikzcd}$$

Therefore the image under Spec

$$\begin{tikzcd} \mathrm{Spec}(k[X]/(X^2)) \arrow[r] & \mathbb{A}^1_k \arrow[r, shift left, "x \mapsto x^2"] \arrow[r, shift right, "x \mapsto 0"'] & \mathbb{A}^1_k \end{tikzcd}$$

is an equalizer. In categorial logic we express this by saying that the monomorphism $\mathrm{Spec}(k[X]/(X^2)) \to \mathbb{A}^1_k$ is isomorphic to the subobject $\{x \in \mathbb{A}^1_k | x^2 = 0\}$ of

[2]This is a rather vague statement, I refer to the sequent calculus that can be found in [14]. This in contrast to, for example, Martin-Löf's conception of intuitionistic logic.

$\mathbb{A}^1_k$. If we now call elements infinitesimally small when their square is zero we see that indeed $\mathrm{Spec}(k[X]/(X^2))$ is (up to isomorphism) the object of infinitesimals in $\mathbb{A}^1_k$.

37. **Proposition.** Let k be a natural number and $I \subseteq \mathbb{R}[x_1, ..., x_n]$ be an ideal containing all monomials of degree k. Then there is a unique C^∞-ring $\overline{A}$ with underlying $\mathbb{R}$-algebra equal to $A := \mathbb{R}[x_1, ..., x_n]/I$. There is an isomorphism of C^∞-rings $\overline{A} \to C^\infty(\mathbb{R}^n)/\langle I\rangle$ induced by the homomorphism of rings $\mathbb{R}[x_1, ..., x_n] \hookrightarrow C^\infty(\mathbb{R})$ sending $x_i \mapsto \pi_i$.

Proof: The pivot of this proof is once again theorem 34. We first prove uniqueness of $\overline{A}$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Is its action on an n-tuple of polynomials $p = ([p_1], ..., [p_n])$ fixed?

Write $c(p_i)$ for the constant coëfficient of p_i. Applying theorem 34 to f around $(c(p_1), ..., c(p_n))$ up to degree k, we obtain $\overline{A}(f)(p) =$

$$\sum_{\tau\in T}\left(\prod_{i=1}^{k}\left(\frac{1}{\tau_i!}\left([p_i]-[c(p_i)]\right)^{\tau_i}\right)\cdot\overline{A}\left(\frac{\partial^{\tau_1}}{\partial x_1}\cdots\frac{\partial^{\tau_k}}{\partial x_k^{\tau_k}}f\right)(c(p_1),...,c(p_n))\right)$$
$$+\sum_{\sigma\in S}\left(\prod_{i=1}^{k}\left([p_i]-[c(p_i)]\right)^{\sigma_i}\right)\cdot\overline{A}(g_\sigma)([p_i]-[c(p_i)]).$$

The second term of this formula vanishes by assumption on I, and we obtain an expression for the action of f.

To prove the existence of $\overline{A}$ we need only show that the induced homomorphism of rings $A \to C^\infty(\mathbb{R}^n)/\langle I\rangle$ is an isomorphism. Apply theorem 34 again to see that any function $f \in C^\infty(\mathbb{R}^n)$ is equivalent modulo $\langle I\rangle$ to the function

$$x \mapsto \sum_{\tau\in T}\left(\prod_{i=1}^{k}\left(\frac{1}{\tau_i!}x_i^{\tau_i}\right)\frac{\partial^{\tau_1}}{\partial x_1^{\tau_1}}\cdots\frac{\partial^{\tau_k}}{\partial x_k^{\tau_k}}f\Big|_p\right),$$

which is the image of the obvious polynomial. Use this polynomial to define a two sided inverse to the induced map. □

38. **Definition.** A Weil algebra is a C^∞-ring who's underlying $\mathbb{R}$-algebra is isomorphic to some $\mathbb{R}[x_1, ..., x_n]/I$ where for some k all monomials of degree k are in I.

For some characterisations of Weil algebras, see [18] Theorem I.3.17. Weil algebras are used extensively in the section on the infinitesimal path ∞-groupoid.

3.2 The tangent category

39. **Definition.** Let $\mathcal{C}$ be a category with finite pullbacks. For an object $C \in \mathrm{ob}\,\mathcal{C}$ write $\mathrm{Ab}(\mathcal{C}/C)$ for the category of abelian group objects in the category over C.

We define $T_{\mathcal{C}}$; an object is an object in $\mathrm{Ab}(\mathcal{C}/C)$ for some C. If $X \in \mathrm{Ab}(\mathcal{A})$ and $Y \in \mathrm{Ab}(\mathcal{B})$ a morphism $X \to Y$ is a pair (f, ϕ) with $f : A \to B$ an arrow in $\mathcal{C}$ and $\phi : X \to f^*Y$ an arrow in $\mathrm{Ab}(\mathcal{C}/A)$.

40. **Definition.** We define a category Mod of modules over commutative rings with multiplicative unit. As objects, take all modules over such rings. If M is an R-module and N is an S-module then a morphism $M \to N$ is a pair (f, ϕ) of a ring homomorphism $f : R \to S$ and an R-linear map $\phi : M \to N_f$, where N_f is the R-module with structure map $R \to S \to \mathrm{End}_{\mathrm{Ab}}(N)$.

41. **Proposition.** $\mathrm{Mod} \simeq T_{\mathrm{Ring}}$.

Proof: We construct a functor $F : \mathrm{Mod} \to T_{\mathrm{Ring}}$ and show that it is full, faithful and surjective on objects.

As for the definition of F, let us first define F_0 as follows. If M is an R-module then $R \times M$ is an abelian group that can be given the structure of a ring by $(r_0, m_0)(r_1, m_1) = (r_0r_1, r_0m_1 + r_1m_0)$. The projection $\pi_R : R \times M \to R$ can be equiped with an abelian group structure in Ring/R by pulling back the group structure in Set of M along $R \to 1$; one needs only check that this structure is in fact in $\mathrm{Ab}(\mathrm{Ring}/R)$, i.e. that the pullback of the arrows in Set become arrows in Ring. This defines F on objects. If M is an R-module and N an S-module and $(f, \phi) : M \to N$ is an arrow in Mod then take $F(f, \phi)$ to be (f, ϕ'), where $\phi' : F_0(M) \to f^*(F_0(N)) : (r, m) \mapsto (r, \phi(m))$.

It is clear that F is full and faithful. Indeed, for M a module over R and N a module over S the inverse of $F_{M,N}$ is given by sending $(f, \phi') \mapsto (f, \pi \circ \phi')$, where $\pi : R \times_S (S \times N) \to S \times N \to N$ is the projection from the vertex of the pullback $f^*F_0(N)$ followed by the projection $S \times N \to N$.

To show that it is essentially surjective on objects, let $a : G \to R$ be an abelian group object in Ring/R (by abuse of notation). Then the kernel of a is an R-module and $\phi : G \to F_0(\ker(a)) : g \mapsto (a(g), g - \eta(a(g)))$ is an isomorphism of rings over R, where I wrote η for the 'unit element' of a. It is easily seen that it respects the group structure. $\square$

42. **Proposition.** Let R be a C^∞-ring and write $U(R)$ for its underlying ring. Then the forgetful functor $\mathcal{U} : \mathrm{Ab}(C^\infty\text{-ring}/R) \to \mathrm{Ab}(\mathrm{Ring}/U(R))$ is one half of an equivalence of categories.

Proof: The hardest part of this proof is to show that the functor $\mathcal{U}$ is injective on objects. In the process of doing so we obtain a closed formula for the action of an abelian group in the overcategory of some R on some smooth function $f : \mathbb{R}^k \to \mathbb{R}$ in terms of R and the ring structure on the abelian group. The argument is analogous to the proof of proposition 37.

Let R be a C^∞-ring and let $a : M \to R$ be an abelian group in C^∞-ring$/R$. Up to isomorphism M sends $\mathbb{R}^n$ to $M(\mathbb{R})^n$. To see what it does on arrows, note that the underlying abelian group over the ring $U(R)$ is in the essential image of the functor Mod $\to T_{\text{Ring}}$ and the ring that is the domain of $\mathcal{U}(a)$ is therefore isomorphic to $R \times M_0$ (for some module M_0) with the ring structure as defined in the proof of proposition 41. Along this isomorphism we could write (r, m) for a general element of $M(\mathbb{R})$, but I think it is better to save our formulas from too many commas and parentheses and write $r \oplus m$ for the same element.

Now let $f : \mathbb{R}^k \to \mathbb{R}$ be a smooth function. We use theorem 34 for $n = 2$ and p some point in $\mathbb{R}^k$. We then obtain the formula for $f \circ + : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$:

$$f(p + w) = f(p) + \sum_{l=1}^{k} w_l \cdot (\partial f/\partial x_l)(p) + \sum_{(i,j)\in\{1,\ldots,k\}^2} w_i \cdot w_j \cdot h_{ij}(p, w).$$

for smooth functions h_{ij}. Thus, if $r = (r_i \oplus 0; 1 \leq i \leq k)$ and $m = (0 \oplus m_i; 1 \leq i \leq k)$ then $M(f \circ +)(r, m) =$

$$M(f)(r) + \sum_{l=1}^{k} m_l \cdot M(\partial f/\partial x_l)(r) + \sum_{(i,j)\in k\times k} m_i \cdot m_j \cdot M(h_{ij})(y, w).$$

The last term in this formula is zero, by definition of the ring structure. Since M is a C^∞-ring over R we may replace the occurences of M in other terms on the right hand side by R to obtain:

$$M(f)(r_i \oplus m_i; 1 \leq i \leq k) = R(f)(r) \oplus \sum_{l=1}^{k} m_l \cdot R(\partial f/\partial x_l)(r). \tag{3.1}$$

This gives the closed formula I promised and injectivity on objects as a corollary.

For essential surjectivity, just check that the above closed formula for M yields a C^∞-ring for each module N over $U(R)$; the abelian group structure is trivially present and the underlying module is isomorphic to N.

The functor $\mathcal{U}$ is obviously faithful and therefore injective on arrows. To show that $\mathcal{U}$ if full, consider, for $\phi : M \to N$ be a $U(R)$-linear map the assignment $\mathbb{R}^n \mapsto \phi^n$. Once it is checked that it is natural using the above closed formula one immediately sees that it is sent by $\mathcal{U}$ to ϕ. □

43. **Proposition.** $T_{C^\infty\text{-ring}} \simeq C^\infty\text{-ring} \times_{\text{Ring}} T_{\text{Ring}}$.

Proof: On objects an inverse is provided by proposition 42. As for arrows, let $(\alpha, (f, \phi)) : (R, \mathfrak{A}) \to (S, \mathfrak{B})$ be an arrow in the pullback. We seek an arrow $(f', \phi') : \mathcal{U}_R^{-1}(\mathfrak{A}) \to \mathcal{U}_S^{-1}(\mathfrak{B})$. Take $f' := \alpha$ and define $\phi'_{\mathbb{R}}$ to be ϕ. We must check a naturality square for $g : \mathbb{R}^n \to \mathbb{R}$ an arbitrary smooth function. This is easy using the closed formula (3.1) in the proof of proposition 42:

$$\phi\left(R(g)(r) \oplus \sum_{l=1}^{k} m_l \cdot R(\partial f/\partial x_l)(r)\right) = R(g)(r) \oplus \sum_{l=1}^{k} \phi(m_l) \cdot R(\partial f/\partial x_l)(r),$$

as desired. □

44. **Remark.** I originally formulated this theorem as an isomorphism for something called a nasal C^∞-ring, a functor preserving products on the nose. Later I figured out that this is not a useful notion, see [12], and that the theorem was false[3].

45. **Corollary.** For any simplicial C^∞-ring R, the category of abelian groups over R is isomorphic to the category of abelian groups over the underlying simplicial ring.

Let me now give some additional motivation for the notion of module given here. Call an abelian group object over some C^∞-ring a module. Then derivations, after Quillen [20] are sections of the projection onto the C^∞-ring, i.e. the underlying object in the overcategory. One easily computes that for the case of rings this yields derivations. In the case of C^∞-rings we recover Dubuc and Kock's notion of module in [6]. In that text a module is just a module over the underlying ring. A derivation is a linear map $d : R \to M$ for some module M over R such that for every smooth function $f : \mathbb{R}^n \to \mathbb{R}$ we have that

$$d(R(f)(r_1, ..., r_n)) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \cdot d(r_i).$$

This coincides with Quillen's notion of derivation applied to C^∞-rings.

Additionally we have that the initial object in the category of $C^\infty(M)$-derivations for some manifold M is isomorphic to the module of De Rham 1-forms, a strong indication that we should be looking more at C^∞-rings and their Quillen derivations.

3.3 The infinitesimal path ∞-groupoid of a Manifold

The opposite $\mathbb{L}$[4] of C^∞-rings is what one is really interested in from the point of view of studying smooth geometry. There are several sites defined on different subcategories of the opposite category of finitely generated C^∞-rings, and in the next section we will study one. In the present section the motivation is to show how

[3] I believe the theorem above is true.

[4] We deviate from the usage in [18] here, in that book $\mathbb{L}^{\mathrm{op}}$ is the category of finitely generated C^∞-rings.

one can reconstruct the De Rahm complex by definitions in the internal language of $\mathbb{L}$. We write ℓ in front of something in C^∞-ring to indicate that we consider it as part of $\mathbb{L}$. If there is something in $\mathbb{L}$ that we wish to view as something in C^∞-ring we write C^∞ in front of it.

The following definitions are inspired on Kock's construction, in [15]. His construction occurs within a smooth topos, and ours does not. However, the construction given here in this section can be viewed as occuring in one of the topoi of [18] by the Yoneda embedding (which preserves limits). Reasoning internally in a smooth topos, as always, Kock proves an internal version ([15] theorem 4.7.1) of theorem 51. Something that is not made fully explicit in Kock's version, and is natural to do in our context is to compute the normalized cocomplex of $C^\infty\Pi_{\mathrm{inf}}(M)$. In this section 'normalized cocomplex' of a cosimplicial abelian group refers to the dual of the Dold-Puppe correspondence ([5]). The Dold-Puppe correspondence ([1], section 3, or see [10], III.2) is an equivalence between simplicial abelian groups and chain complexes of abelian groups. So when I tell the reader to take the normalized cochain of a cosimplicial C^∞-ring I am abusing notation, the reader should take the normalized cochain complex of the underlying cosimplicial abelian group.

Write $(R, +, \cdot, -, 0, 1)$ for the image of the ring $(\mathbb{R}, +, \cdot, -, 0, 1)$ under $\ell C^\infty(-, \mathbb{R}) : \mathrm{Diff} \to \mathbb{L}$. Since $\mathbb{L}$ has all finite limits the following definitions make sense for all $n, m \in \mathbb{Z}_{\geq 0}$. The object

$$D(n) := \left\{ x \in R^n ;\ \bigwedge_{k=1}^{n} \bigwedge_{l=1}^{n} x_k \cdot x_l = 0 \right\}$$

I call the object of first order infinitesimals in R^n.

$$\widetilde{D}(m, n) := \left\{ x \in D(n)^m ;\ \bigwedge_{k=1}^{m} \bigwedge_{l=1}^{m} x_k - x_l \in D(n) \right\}.$$

If we think of $D(n)$ as the type of infinitesimally small vectors in R^n, then we may think of $\widetilde{D}(m, n)$ as the type of m-tuples of vectors that are all infinitesimally close to the origin and in addition are pairwise infinitesimally close to each other. Also,

$$R^n_{<m>} := \left\{ x \in (R^n)^{m+1} ;\ \bigwedge_{k=1}^{m+1} \bigwedge_{l=1}^{m+1} x_k - x_l \in D(n) \right\}$$

can then be thought of as the type of $m+1$ tuples of pairwise infinitesimally close vectors in R^n.

46. **Proposition.** For $n, m \in \mathbb{Z}_{\geq 0}$ we have $R^n \times \widetilde{D}(m, n) \cong R^n_{<m>}$.

Proof: Working out the definitions one easily computes that

$$R^n \times \widetilde{D}(m,n) \cong \ell\left(C^\infty((\mathbb{R}^n)^{m+1},\mathbb{R})/\sum_{k=2}^{m+1}\sum_{l=2}^{m+1} I\circ(\pi_k-\pi_l)+\sum_{k=2}^{m+1} I\circ\pi_k\right)$$

and that

$$R^n_{<m>} \cong \ell\left(C^\infty((\mathbb{R}^n)^{m+1},\mathbb{R})/\sum_{k=1}^{m+1}\sum_{l=1}^{m+1} I\circ(\pi_k-\pi_l)\right),$$

where $I := \langle \pi_i\cdot\pi_j;\ i,j=1,...,n\rangle \subseteq C^\infty(\mathbb{R}^n,\mathbb{R})$. The image under $\ell C^\infty(-,\mathbb{R})$ of $\phi : (\mathbb{R}^n)^{m+1} \to (\mathbb{R}^n)^{m+1} : (x_1,...,x_{m+1}) \mapsto (x_1, x_2-x_1,...,x_{m+1}-x_1)$ then induces an isomorphism from $R^n_{<m>}$ to $R^n\times\widetilde{D}(m,n)$. One just checks that the dual map is onto and has the desired kernel. □

47. **Remark.** The assignment $[m]\mapsto R^n_{<m>}$ has a natural structure of a simplicial locus. We define it by defining a cosimplicial structure on its dual. Write $\overline{n} := \{1,...,n\}$ and F for the free C^∞-ring functor. Then $F(\Delta([0],-)\times\overline{n})$ is a cosimplicial C^∞-ring and $\mathcal{I} : [m]\mapsto \sum_{k=1}^{m+1}\sum_{l=1}^{m+1} I\circ(\pi_k-\pi_l)$ is a cosimplicial ideal of $F(\Delta([0],-)\times\overline{n})$. Write Inf_n for $F(\Delta([0],-)\times\overline{n})/\mathcal{I}$. Then $\ell\mathrm{Inf}_n$ is a simplicial locus which in degree $[m]$ is isomorphic to $R^n_{<m>}$.

48. **Definition.** For any natural number $n\geq 0$ define $\Pi_{\mathrm{inf}}(R^n) := \ell\mathrm{Inf}_n$. Choosing[5] an open cover for each manifold M such that all covering opens and each intersection of pairs of such opens is diffeomorphic to $\mathbb{R}^n$ we express M as a colimit of copies of $U_i\cong\mathbb{R}^n$ in the category of manifolds and set

$$\Pi_{\mathrm{inf}}(M) \cong \mathrm{colim}_i\Pi_{\mathrm{inf}}(U_i).$$

49. **Remark.** Note that this is, up to isomorphism, independent of the choice of covering of the manifold. Indeed, in each degree $[m]$ we have $\mathrm{colim}_i\Pi_{\mathrm{inf}}(U_i) \cong \mathrm{colim}_i(U_i\times\tilde{D}(m,n))$ and since $\tilde{D}(m,n)$ is a Weil algebra it is exponentiable, by theorem II.1.13 in [18] we get

$$\Pi_{\mathrm{inf}}(M) \cong (\mathrm{colim}_iU_i)\times\tilde{D}(m,n) \cong M\times\tilde{D}(m,n).$$

50. **Remark.** The following theorem gives motivation for definition 48. Given some $f\in C^\infty((\mathbb{R}^n)^{m+1},\mathbb{R})$ we consider the m-form

$$\sum_{\alpha:\{2,...,m+1\}\hookrightarrow\{1,...,n\}} \frac{\partial^\alpha f}{\partial\xi^\alpha}\Big|_{(x,0,...,0)}\cdot\bigwedge_{i=1}^m d\pi_{\alpha(i+1)},$$

where the α range over all injections. It induces an isomorphism in degree m from the normalized chain complex of $C^\infty\Pi_{\mathrm{inf}}(\mathbb{R}^n)$ to the De Rham complex of $\mathbb{R}^n$. If we cover some M by an atlas and apply this isomorphism locally we obtain an isomorphism as in the following theorem.

[5] using a Grothendieck universe argument

51. **Theorem.** Let M be a manifold. The normalized cochain complex of the underlying cosimplicial abelian group of $C^\infty(\Pi_{\text{inf}}(-))$ is isomorphic to the De Rham complex of M.

Proof: Our strategy will be to first prove this for $M = \mathbb{R}^n$ and then glue the isomorphism at the end.

Suppose $M = \mathbb{R}^n$. For the purposes of this proof, we use the isomorphism

$$\Omega^m(\mathbb{R}^n) \cong C^\infty\mathrm{Man}\left(\mathbb{R}^n, \bigwedge_{i=1}^m (\mathbb{R}^n)^*\right),$$

where $\bigwedge_{i=1}^m (\mathbb{R}^n)^*$ denotes the space of alternating tensors.

To compute the normalized cochain complex of $C^\infty\Pi_{\text{inf}}(\mathbb{R}^n)$ we must first analyse the cofaces, some of whose images we will divide out by. The cosimplicial structure was obtained from the one on $C^\infty R^n_{<->}$ by transport of structure. One checks that for $i \geq 1$ the (i-th) coface map of the corresponding cosimplicial C^∞-ring sends π_k to π_k if $k < i$ and π_k to π_{k+1} if $k \geq i$. The 0-th coface map sends π_1 to $\pi_1 + \pi_2$ and π_k to π_{k+1} for all $k > 1$. We compute the normalized cochain complex of this cosimplicial $\mathbb{R}$-module by dividing out $C^\infty(R^n \times \widetilde{D}(m,n))$ by the sub-vectorspace generated by the images of all but the 0-th coface map. To facilitate this procedure we note that there is an equality of ideals

$$\textstyle\sum_{l=2}^{m+1}\sum_{k=2}^{m+1} I \circ (\pi_k - \pi_l) + \sum_{k=2}^{m+1} I \circ \pi_k =$$

$$\langle \pi_{i,j}\pi_{i',j'} + \pi_{i,j'}\pi_{i',j};\ i,i' = 2,...,m+1 \ \&\ j,j' = 1,...,n\rangle$$

as the reader can check for herself.

Now consider, for $[m] \in \Delta_0$ the function $\phi_{[m]} : C^\infty((\mathbb{R}^n)^{m+1}, \mathbb{R}) \to \Omega^m(\mathbb{R}^n)$ that sends f to $x \mapsto$

$$\sum_{\alpha:\{2,...,m+1\}\hookrightarrow\{1,...,n\}} \frac{\partial^\alpha f}{\partial \xi^\alpha}\Big|_{(x,0,...,0)} \cdot \bigwedge_{i=1}^m \pi_{\alpha(i+1)},$$

where the $\alpha : \{2,...,m+1\} \hookrightarrow \{1,...,n\}$ range over injections. Firstly, this is a well defined $\mathbb{R}$-linear function. Second, the ideal $\langle \pi_{i,j}\pi_{i',j'} + \pi_{i,j'}\pi_{i',j};\ i,i' = 2,...,m+1 \ \&\ j,j' = 1,...,n\rangle$ is in the kernel of ϕ (we omit the subscript $[m]$). This one proves by using the definition of wedge product and noting that

$$x \mapsto \frac{\partial^\alpha (f \cdot (\pi_{i,j}\pi_{i',j'} + \pi_{i,j'}\pi_{i',j}))}{\partial \xi^\alpha}\Big|_{(x,0,...,0)}$$

is fixed under transposition of two of the non-x variables. Also, the images of all but the 0-th coface map are contained in $\ker(\phi)$. Indeed, if some function f does not depend on the ($1 <$) i-th (vector valued) variable, then

$$\frac{\partial^\alpha f}{\partial \xi^\alpha}\Big|_{(x,0,...,0)}$$

will be zero. So this induces a linear map $\overline{\phi}$ from $N(C^\infty(\Pi_{\mathrm{inf}}(\mathbb{R}^n)))_m$ to $\Omega^m(\mathbb{R}^n)$. To go back, take a form ω and send it to the coset of $\psi(\omega) : (x, y_1, ..., y_m) \mapsto \omega(x)(y_1, ..., y_m)$.

We must check that $\psi(\phi(f))$ is in the same coset as f. Taking the Taylor expansion of $(y_1, ..., y_m) \mapsto f(x, y_1, ..., y_m)$ up to order $m+1$ around 0 and using Hadamard's lemma for the rest term we note that since $g \cdot \pi_{i,j}\pi_{i,j'}$ is in $\ker(\phi)$ for smooth g and since all terms of lower degree depend on less than m variables of $y_1, ..., y_m$, the function f is equivalent modulo $\ker\phi$, to

$$(x, y_1, ..., y_m) \mapsto \sum_{\alpha:\{1,...,m\}\hookrightarrow\{1,...,n\}} \frac{\partial^\alpha f}{\partial \xi^\alpha}\Big|_{(x,0,...,0)} \cdot \prod_{i=1}^m y_{i,\alpha(i)},$$

modulo $\ker(\phi)$. The symmetrization of this map equals $\psi(\phi(f))$. Since for any smooth g we have $g \cdot (\pi_{i,j}\pi_{i',j'} + \pi_{i,j'}\pi_{i',j}) \in \ker(\phi)$ we obtain $\psi(\phi(f)) \sim f$.

Now we verify that $\phi(\psi(\omega)) = \omega$ for any alternating m-form ω. If ω is such a form then

$$\frac{\partial^\alpha \psi(\omega)}{\partial \xi^\alpha}\Big|_{(x,0,...,0)} = \omega(x)(e_{1,\alpha(1)}, ..., e_{m,\alpha(m)}),$$

where $e_{l,k}$ is a standard basisvector of $(\mathbb{R}^n)^m$; this can be checked expressing $(y_1, ..., y_m) \mapsto \omega(x)(y_1, ..., y_m)$ as a linear combination of maps of the form $\prod_{i=1}^m \pi_{i,p(i)}$ with $p : \{1, ..., m\} \hookrightarrow \{1, ..., n\}$. Thus ω is equal to $x \mapsto$

$$(y_1, ..., y_m) \mapsto \sum_\alpha \frac{\partial^\alpha \psi(\omega)}{\partial \xi^\alpha}\Big|_{(x,0,...,0)} \prod_{i=1}^m y_{i,\alpha(i)}.$$

Its symmetrization is $\phi(\psi(\omega))$, but since each $\omega(x)$ is an alternating multilinear map $\phi(\psi(\omega)) = \omega$.

I now show that $\phi \circ d^0 \circ \psi = d$, where d^0 is the 0-th face map and d is the exterior derivative. Let $\omega \in \Omega^m(\mathbb{R}^n)$. Then for some $\{f_\alpha \in C^\infty(\mathbb{R}^n, \mathbb{R});\ \alpha : \{1, ..., m\} \hookrightarrow \{1, ..., n\}\}$ we have

$$\omega : x \mapsto \sum_\alpha \left(f_\alpha(x) \cdot \bigwedge_{i=1}^m \pi_{i,\alpha(i)} \right).$$

Thus, $d^0(\psi(\omega)) = \sum_\alpha \left(f_\alpha \circ (\pi_1 + \pi_2) \right) \cdot \bigwedge_{i=1}^m \pi_{i+2,\alpha(i)} \big)$

$$\sim_{\ker(\phi_{m+1})} \sum_\alpha \left((f_\alpha \circ (\pi_1 + \pi_2)) \cdot \prod_{i=1}^m \pi_{i+2,\alpha(i)} \right) =: \chi.$$

We have $\phi_{m+1}(\chi) = \phi_{m+1}(d^0(\psi_m(\omega)))$. One computes, for $\beta : \{2, ..., m+2\} \hookrightarrow \{1, ..., n\}$, that

$$\frac{\partial^\beta \chi}{\partial \xi^\beta}\Big|_{(x,0,...,0)} = \frac{\partial f_{\beta'}}{\partial \xi_{\beta(2)}}\Big|_x,$$

where $\beta' : \{1, ..., m\} \hookrightarrow \{1, ..., n\} : j \mapsto \beta(j+2)$. Consequently,

$$\phi_{m+1}(\chi) : x \mapsto \sum_{\beta:\{2,...,m+2\}\hookrightarrow\{1,...,n\}} \frac{\partial f_{\beta'}}{\partial \xi_{\beta(2)}}\Big|_x \cdot \bigwedge_{i=1}^{m+1} \pi_{i,\beta(i+1)} =$$

$$\sum_{\alpha:\{1,...,m\}\hookrightarrow\{1,...,n\}} D_x f \wedge \bigwedge_{i=2}^{m+1} \pi_{i,\alpha(i-1)} = d(\omega)(x).$$

This concludes the proof for $M = \mathbb{R}^n$.

For general M all we need to do is show that the linear function $\phi_{[m]}$ respects the charts. This is a straightforward computation using the standard charts on the m-fold exterior power of the cotangent bundle and using the expression of $\phi_{[m]}$ given in remark 50. □

3.4 Thickened Cartesian spaces and the Spec $\vdash \mathcal{O}$ adjunction

In this section we define the site of thickened cartesian spaces and prove that the Spec $\vdash \mathcal{O}$ adjunction is Quillen with respect to the Čech local structure on simplicial presheaves. As a corollary, using theorem **??** the adjunction is Quillen to the hyperlocal model structure. The site we consider is locally connected, and this is crucial for envisioned applications.

We consider the full subcategory Thk $\subseteq \mathbb{L}$ on all loci isomorphic to $\mathcal{R}^n \times lW$ for some Weil algebra W. Take the Grothendieck topology generated by the following notion of coverage:

52. **Definition.** For some set of arrows γ in Thk with codomain $\mathcal{R}^n \times W$ we set that γ is a covering iff all arrows $\gamma_i \in \gamma$ are of the form $g_i \times \mathrm{id}_W : \mathcal{R}^n \times W \to \mathcal{R}^n \times W$ such that the $h^{-1}(g_i)$ form an open cover $\mathbb{R}^n$, any finite intersection of their images is diffeomorphic to some $\mathbb{R}^k$ and all the images have compact closure. Call the resulting site (Thk, Γ).

53. **Theorem.** Endow $\left(\mathrm{Set}^{\mathrm{Thk}^{\mathrm{op}}}\right)_s^{\Delta^{\mathrm{op}}}$ with the model category structure which is the left Bousfield localization of the projective model category structure at the set of morphisms K such that for each $f \in K$ the following holds. The

morphism $f : \mathcal{U} \to \mathcal{R}^m \times \ell W$ is a Čech cover such that there is a coverage $\langle f_i : \ell C^\infty(U_i) \times \ell W \to \mathcal{R}^m \times \ell W; i \in \alpha \rangle$ for which

$$f([0]) : \coprod_{i \in \alpha} h\ell C^\infty(U_i) \times h\ell W \to \mathcal{R}^m \times \ell W$$

that is induced by $h(f_i)$ on the components of the domain. Then Spec is a right Quillen functor with respect to this model category structure.

Proof: By lemma 29 it suffices to prove that the cohomology of $\mathcal{O}(f)$ induces an isomorphism in cohomology for every $f \in K$ since all Čech covers are split. In degree 0 follows from the fact that the presheaf O is representable (by $\mathcal{R}$) and therefore a sheaf. In higher degrees it suffices to prove that $\mathcal{O}(\mathrm{dom} f) \cong 0$, since the cohomology of the codomain is clearly 0.

Showing the higher cohomology groups are zero is small variation (due to the presence of ℓW) on a standard argument. First we rewrite $A \cong \mathcal{O}(\mathrm{dom}(f))$ as having

$$A_n = \prod_{I \in \alpha^{n+1}} C^\infty\text{-ring}\left(C^\infty(\mathbb{R}), C^\infty(\cap_{i \in n+1} U_{I(i)}) \otimes_\infty W\right)$$

and define the cochain map by transport of structure. Suppose ξ is in the kernel of the cochain map $\partial^n : A_{n-1} \to A_n$ for $n > 1$. Let $\langle \rho_i; i \in \alpha \rangle$ be a partition of unity subordinate to the covering $\{U_i; i \in \alpha\}$ of $\mathbb{R}^m$. Then the element $\zeta \in A_{n-1}$ defined by

$$\left\langle g \mapsto \sum_{i \in \alpha} \xi_{I_i}(g) \cdot \iota_{C^\infty(\mathbb{R}^m)}(\rho_i); I \in \alpha^n \right\rangle,$$

where I_i means $0 \mapsto i$ and $j + 1 \mapsto I(j)$ and $\iota_{C^\infty(\mathbb{R}^m)}$ denotes the inclusion into the coproduct. One then uses $\partial^n(\xi) = 0$ to conclude that $\partial^{n-1}(\zeta)$ is, up to a sign, equal to ξ. $\Box$

We are now in a position to define the ∞-stack of infinitesimal paths on a manifold M. Namely as $\mathrm{Spec} \circ P(\Pi_{\mathrm{inf}}(M))$, where P is a chosen fibrant replacement functor.

Appendix

This appendix is devoted to proving theorem 13. We split the proof in two. In the first part we prove the model category structure exists and in the second that it is simplicially enriched.

54. **Theorem.** Call a morphism of cosimplicial abelian groups a fibration if it is degreewise a surjection. Say it is a weak equivalence if it induces an isomorphism in cohomology. Then this gives the structure of a model category on the category of cosimplicial abelian groups.

It is easy to verify the two-out-of-three axiom and the category of cosimplicial abelian groups is complete and cocomplete. It is a simple exercise to show that the classes of fibrations, cofibrations and weak equivalences are stable under taking retracts. The focus is on the other two axioms. All that follows is part of a proof of these axioms.
We define, for any $n \in \mathbb{Z}_{\geq 0}$, the cochain complex $\mathbb{Z}[n]$ with $\mathbb{Z}$ in degree n and 0 everywhere else. Then $S^n := \Gamma\mathbb{Z}[n]$, where Γ is the functor from cochain complexes to cosimplicial objects given by the Dold-Puppe correspondence. Also set $\mathbb{Z}[n-1, n]$ to have $\mathbb{Z}$ in degree $n-1$ and n, the identity between them, and 0 everywhere else except $\mathbb{Z}[-1,0] := 0$. $D^n := \Gamma\mathbb{Z}[n-1,n]$.

55. **Lemma.** $0 \to S^n$ and $S^n \to D^n$ are cofibrations for all $n \in \mathbb{Z}_{\geq 0}$.

Proof: We prove that $S^n \to D^n$ is a cofibration. Since the Dold-Puppe functor N from cosimplicial objects to cochain complexes sends degreewise surjections to degreewise surjections ([10], lemma III.2.11.(1)) we need to check that if $\phi : A \to B$ is a degreewise surjection between cochain complexes of abelian groups that induces an isomorphism in cohomology then in any square as below that commutes, there is a diagonal filler as drawn

$$\begin{tikzcd} \mathbb{Z}[n] \arrow[r, "f"] \arrow[d] & A \arrow[d, "\iota"] \\ \mathbb{Z}(n-1,n) \arrow[r, "g"'] \arrow[ur, dashed] & B \end{tikzcd} .$$

Analysing this one finds we seek a following lift (for $n > 0$)

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr] & & \mathbb{Z} \arrow[dl, dashed] \arrow[dr, "f_n"] \arrow[ddl, "g_{n-1}"'] \arrow[ddr, "g_n"] & \\ & A_{n-1} \arrow[rr, "\partial_A^n"'] \arrow[d, "\iota_{n-1}"'] & & A_n \arrow[d, "\iota_n"] \\ & B_{n-1} \arrow[rr, "\partial_B^n"'] & & B_n \end{tikzcd} .$$

This is equivalent to finding an element α of A_{n-1} for which $\partial_A^n(\alpha) = f_n(1)$ and $\iota_{n-1}(\alpha) = g_{n-1}(1)$. Now we know that $f_n(1) \in \ker \partial_A^{n+1}$ and since ι is a quasi-isomorphism we have $\left(\iota_n \restriction \ker \partial_A^{n+1}\right)^{-1}(\mathrm{im}\ \partial_B^n) = \mathrm{im}\ \partial_A^n$. Since $\iota_n(f_n(1)) = \partial_B^n(g_{n-1}(1))$ this implies that there is some $z \in A_{n-1}$ such that $\partial_A^n(z) = f_n(1)$, let z be such. Then $\iota_{n-1}(z) - g_{n-1}(1) \in \ker\ \partial_B^n$. Therefore, there is some $b \in B_{n-2}$ and some $a \in \ker\ \partial_A^n$ such that $\iota_{n-1}(a) - \partial_B^{n-1}(b) = \iota_{n-1}(z) - g_{n-1}(1)$. Since ι_{n-2} is onto there is some $b' \in A_{n-2}$ such that $\iota_{n-1}(a) - \iota_{n-1}(\partial_A^{n-1}(b')) = \iota_{n-1}(z) - g_{n-1}(1)$. Write $-a' := \partial_A^{n-1}(b') - a$. Then $\iota_{n-1}(z - a') = \iota_{n-1}(z) - \iota_{n-1}(a') = \iota_{n-1}(z) - \iota_{n-1}(a) + \iota_{n-1}\partial_A^{n-1}(b') = g_{n-1}(1)$ and $\partial_A^n(z - a') = \partial_A^n(z) = f_n(1)$.

It is left to prove the $n = 0$ case. In that case we must show that if $\mathbb{Z}(0) \to A \to B$ is zero then $\mathbb{Z}(0) \to A$ is already zero. Suppose $f_0(1) \neq 0$. Since $\iota_0(f_0(1)) = 0$ we have $f_0(1) \sim_{\mathrm{im}\ \partial_0} 0$, but $\mathrm{im}\ \partial_0 = 0$, so the result follows.

Now $0 \to S^n$. We must find a filler in

$$\begin{tikzcd}
& A_{n-1} \arrow[rr, "\partial_A^n"] \arrow[dd, "\iota_{n-1}"] & & A_n \arrow[rr, "\partial_A^{n+1}"] \arrow[dd, "\iota_n"] & & A_{n+1} \arrow[dd, "\iota_{n+1}"] \\
0 \arrow[ur] \arrow[dr] \arrow[rr] & & \mathbb{Z} \arrow[ur, dashed] \arrow[dr, "g_n"'] \arrow[rr] & & 0 \arrow[ur] \arrow[dr] & \\
& B_{n-1} \arrow[rr, "\partial_B^n"'] & & B_n \arrow[rr, "\partial_B^{n+1}"'] & & B_{n+1}
\end{tikzcd} .$$

This is equivalent to finding an element $a_0 \in \ker\ \partial_A^{n+1}$ such that $\iota_n(a_0) = g_n(1)$. By assumption on ι we may take some $a' \in A_{n-1}$ and $a \in \ker\ \partial_A^{n+1}$ such that $\partial_B^n(\iota_{n-1}(a')) - g_n(1) = \iota_n(a)$. Take $a_0 = \partial_A(a') - a$. $\square$

56. **Lemma.** $0 \to D^n$ has, for each $n \in \mathbb{Z}_{\geq 0}$, the left lifting property with respect to all fibrations.

Proof: We seek fillers in

$$\begin{tikzcd}
& A_{n-2} \arrow[rr, "\partial_A^{n-1}"] \arrow[dd, "\iota_{n-2}"] & & A_{n-1} \arrow[rr, "\partial_A^n"] \arrow[dd, "\iota_{n-1}"] & & A_n \arrow[rr, "\partial_A^{n+1}"] \arrow[dd, "\iota_n"] & & A_{n+1} \arrow[dd, "\iota_{n+1}"] \\
0 \arrow[ur] \arrow[dr] \arrow[rr] & & \mathbb{Z} \arrow[ur, dashed] \arrow[dr, "g_{n-1}"'] \arrow[rr] & & \mathbb{Z} \arrow[ur, dashed] \arrow[dr, "g_n"'] \arrow[rr] & & 0 \arrow[ur] \arrow[dr] & \\
& B_{n-2} \arrow[rr, "\partial_B^{n-1}"'] & & B_{n-1} \arrow[rr, "\partial_B^n"'] & & B_n \arrow[rr, "\partial_B^{n+1}"'] & & B_{n+1}
\end{tikzcd} .$$

Upon inspection of this diagram one notices that it is equivalent to prove there is an element $a \in A_{n-1}$ with $\iota_{n-1}(a) = g_{n-1}(1)$. There is such an element because ι_{n-1} is onto by assumption. $\square$

57. **Lemma.** A map $f : A \to B$ is an acyclic fibration iff it is a fibration and has the right lifting property with respect to all maps of the form $0 \to S^n$ and

$S^n \to D^n$.

Proof: One direction ($\Rightarrow$) follows by lemma 55. For the converse, let $n \in \mathbb{Z}_{\geq 0}$. Then there is always a filler in

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ \mathbb{Z}[n] \arrow[r, "1 \mapsto b"'] \arrow[ur, dashed] & B \end{tikzcd},$$

for $b \in \ker \partial_B^{n+1}$. Hence $\mathrm{im}\,(f \restriction \ker \partial_A^{n+1}) = \ker \partial_B^{n+1}$, so a forteriori $H^n(f)$ is onto.

For injectivity suppose $a \in \ker \partial_A^{n+1}$ is mapped to $b \in \mathrm{im}\, \partial_B^n$. Let b' be such that $\partial_B^n(b') = b$ and let $\beta : \mathbb{Z}(n-1, n) \to B$ send $1 \in \mathbb{Z}(n-1,n)_{n-1}$ to b'. Then the diagram

$$\begin{tikzcd} \mathbb{Z}[n] \arrow[r, "1 \mapsto a"] \arrow[d] & A \arrow[d, "f"] \\ \mathbb{Z}[n-1, n] \arrow[r, "\beta"'] \arrow[ur, dashed] & B \end{tikzcd}$$

must have a filler, implying that $a \in \mathrm{im}\, \partial_A^n$. $\square$

58. **Lemma.** Any map $f : A \to B$ in Ab^Δ can be factored as $f = qj$, where j has the left lifting property with respect to all fibrations and is a weak equivalence, and q is a fibration.

Proof:

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "f"'] & A \coprod \left(\coprod_{n \in \mathbb{Z}_{\geq 0}} \coprod_{b \in B_n} D^n \right) \arrow[d] \\ & B \end{tikzcd};$$

$\coprod_{n \in \mathbb{Z}_{\geq 0}} \coprod_{b \in B_n} D^n$ is a sum of objects having the left lifting property with respect to all fibrations, and hence has that lifting property itself. Its cohomology in degree i is $\coprod_{n \in \mathbb{Z}_{\geq 0}} \coprod_{b \in B_n} H^i(D^n) = 0$ by proposition V.9.3 in [4]. It is then clear that j is a quasi-isomorphism. It also enjoys the lifting property, being a pushout of a morphism having the left lifting property with respect to all fibrations. $\square$

59. **Lemma.** Any map $f : A \to B$ in Ab^Δ may be factored $f = pi$, where p is a trivial fibration and i a cofibration.

Proof: It suffices, by lemma 58 to prove this for f a fibration. The result in that case follows from a small object argument using lemmata 55 56 57. $\square$

60. **Lemma.** Suppose that the map $i : A \to B$ in Ab^{Δ} is a cofibration and a weak equivalence. Then i has the left lifting property with respect to all fibrations.

Proof: This is a consequence of lemma 58 . Find a factorisation

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[dr, "i"'] & \overline{B} \arrow[d, "q"] \\ & B \end{tikzcd}$$

as in lemma 58, so that j has the left lifting property with respect to all fibrations and is a weak equivalence, and q is a fibration. Then q is a trivial fibration, so the dotted arrow exists making the diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & \overline{B} \arrow[d, "q"] \\ B \arrow[r, "\mathrm{id}_B"'] \arrow[ur, dashed] & B \end{tikzcd}$$

commute. The map i is therefore a retract of j, so i has the desired lifting property.
$\square$

61. **Theorem.** The model category on cosimplicial abelian groups is simplicially enriched.

Proof: It remains to show the 'SM7' axiom. We must show that for any cofibration of simplicial sets $i : C \to C'$ and any fibration $j : X \to Y$ of cosimplicial abelian groups the induces map

$$k : X^{C'} \to X^{C} \times_{Y^C} Y^{C'}$$

is a fibration of cosimplicial abelian groups, which is acyclic if either i or j is acyclic. One easily checks that k is a fibration under these circumstances.

For acyclicity we need a lemma. Consider an arbitrary cosimplicial abelian group A and a weak equivalence i of simplicial sets. I claim that the map A^i is a weak equivalence. For any $L \in \mathrm{Set}^{\Delta^{\mathrm{op}}}$ the cosimplicial abelian group A^L is the diagonal of the bi-cosimplicial abelian group

$$([m], [n]) \mapsto \prod_{x \in K([m])} A([n]).$$

The cohomology of this diagonal is naturally isomorphic to its homotopy groups ([25] theorem 8.3.8) which are naturally isomorphic to the cohomology of the total complex (theorem 8.5.1 in [25]). Since taking the total complex of a double complex preserves quasi-isomorphisms and the map induced by i induces a quasi-isomorphism on the cochain-complexes associated to the bi-cosimplicial object displayed above by the Dold-Puppe correspondence, we may conclude A^i is a quasi-isomorphism for any A. An analogous argument serves to show that $j^{C'}$ is a quasi-isomorphism.

We now wrap up by noting we have an exact sequence of cosimplicial abelian groups

$$0 \longrightarrow X^C \times_{Y^C} Y^{C'} \longrightarrow X^C \oplus Y^{C'} \xrightarrow{f} Y^C \longrightarrow 0$$

where $f : (x, y) \mapsto j^C(x) - Y^i(y)$. This then gives rise to a long exact sequence in cohomology

$$\cdots \to H^p(X^C \times_{Y^C} Y^{C'}) \to H^p(X^C) \oplus H^p(Y^{C'}) \to \\ H^p(Y^C) \to H^{p+1}(X^C \times_{Y^C} Y^{C'}) \to \cdots$$

(using that cohomology preserves coproducts [4]). Using the above paragraph, if i is a weak equivalence the map $H^p(X^C \times_{Y^C} Y^{C'}) \to H^p(X^C) \oplus H^p(Y^{C'})$ is an isomorphism onto $H^p(X^C)$ and therefore $X^C \times_{Y^C} Y^{C'} \to X^C$ is a weak equivalence. And if moreover $X^i : X^{C'} \to X^C$ is a weak equivalence, then by the two out of three axiom for weak equivalences, k would also be a weak equivalence. A similar argument works to prove that if j is a weak equivalence of cosimiplicial abelian groups then so is k. $\square$

Acknowledgements

Urs Schreiber's help has been exceptional. He taught me much over these last months, on many different levels. Meeting *at least* once a week in the last 3 months before the thesis, his enormous enthusiasm was every time contagious. I am grateful that he had the courage to introduce me to ∞-stacks, a topic that can only be studied by someone like me when there is an expert constantly steering me and filling gaps in my understanding. With Urs this was possible, because of the excessive time he put into talking to me.

I wish to also thank Jaap van Oosten for helpful advice. Without interaction with the groups at the Higher category-reading group and the Topos Theory seminar, all this would have been much less enjoyable. Thanks go out to all those at the topology group that made being in Utrecht pleasant.

My mother Amparo Garcia Celma has supported and encouraged me unconditionally in my studies. She has always provided everything I needed and together with George Vis listened to all problems I had.

Luckily she was not alone in supporting me. There is of course the Stel family: Adriana, Donna, Harry, Mathijs and Teddy. There are my dear friends in Utrecht: Adriana, Frans, Margarita, Margreet, Rick, Sara and Sylvestre all of whom were always happy to help me with whatever means they had to their disposal. I thank Francesca Pallina Diana, for being my companion in everything, Ansten Mørch Klev for inspiring me throughout the years, Maurits Bartstra and Ivanka Dimitrievic for very good times, Ties Olivier Schelfhout for his unending support and Mario Veen for help with existential questions.

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