Structured Homotopy Theory
from
String Theory

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Based on joint work with:

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Abstract.

Homotopy theory is extremely rich, with various structured variants such as equivariant, graded, parameterized and stable homotopy theory. Powerful tools from differential-graded algebra, particularly in the rational approximation, serve for concrete computations.

Also string/M-theory is extremely rich, revealing a system of higher dimensional objects (branes) with subtle inter-relations.

In these talks I survey recent insights into a close relation between the two, which provides structured homotopy theory with curious new examples and sheds light on the elusive foundations of string/M-theory.

The 4-sphere will play a surprisingly central role, as initially realized by Hisham Sati.
The 4-sphere

\[ S^4 \]

We will discuss its incarnation in various flavours of *homotopy theory*: 
1) Super homotopy theory and fundamental M2/M5-branes

2) Parameterized stable homotopy theory and gauge enhancement

3) Equivariant homotopy theory and black M-branes
1) Super homotopy theory and fundamental M2/M5-branes

based on [FSS 13, FSS 15, FSS 16, HS 17]
Classical homotopy theory

$$\text{Ho}(\text{Spaces}) := \text{Spaces} \left[ \{ \text{Isos on all homotopy groups} \}^{-1} \right]$$

For example homotopy groups of spheres:

$$\pi_k(S^n) := \text{Hom}_{\text{Ho}(\text{Spaces})} \left( S^k, S^n \right)$$

These exhibit an endlessly rich pattern.
But most of them are torsion groups.

For example for $S^4$ two of them non-torsion:
generated by the *quaternionic Hopf fibration*: 
Quaternionic Hopf fibration

\[
\begin{align*}
S^7 & \simeq S(H \oplus H) \\
\text{Hopf}_H & \downarrow \\
S^4 & \simeq S(R \oplus H) \\
(x, y) & \downarrow \\
& |x|^2 + |y|^2 = 2 \\
\end{align*}
\]

\[
\begin{align*}
& (t := 1 - |x|^2, ) \\
& z := x \cdot y \\
& t^2 + |z|^2 = 1 \\
\end{align*}
\]

\[
\pi \left( S^4 \right) \simeq \mathbb{Z} \langle \text{id}_{S^4} \rangle \oplus \mathbb{Z} \langle \text{Hopf}_H \rangle \oplus \text{torsion}
\]

\[
\begin{align*}
\text{deg}=4 & \oplus \text{deg}=7 & \text{torsion}
\end{align*}
\]
Rational homotopy theory
disregards all torsion information:

\[ \text{Ho}(\text{Spaces}_\mathbb{Q}) := \text{Spaces} \left[ \{ \text{Isos on rationalized homotopy groups} \}^{-1} \right] \]

\[ \pi_k(S^n) \otimes \mathbb{Q} = \text{Hom}_{\text{Ho}(\text{Spaces}_\mathbb{Q})} \left( S^k, S^n \right) \]

\[ \pi_* \left( S^4 \right) \otimes \mathbb{Q} = \mathbb{Q}\langle \text{id}_{S^4} \rangle \oplus \mathbb{Q}\langle \text{Hopf}_{\text{II}} \rangle \]

\[ \text{deg}=4 \quad \text{deg}=7 \]
**dg-Algebraic model for rational homotopy theory.**

\[ \text{dgcAlg} := \{ \text{differential graded-commutative algebras over } \mathbb{R} \} \]

Homotopy theory of dg-algebras:

\[ \text{Ho} (\text{dgcAlg}^{\text{op}}) := \text{dgcAlg}^{\text{op}} \left[ \{ \text{Isos on all cohomology groups} \}^{-1} \right] \]

Quillen-Sullivan equivalence:

\[
\begin{array}{ccc}
\text{Ho} (\text{Spaces}) & \xrightarrow{\mathcal{O}} & \text{Ho} (\text{dgcAlg}^{\text{op}}) \\
\downarrow & \Phi & \downarrow \\
\text{Ho} \left( \text{Spaces}_{\mathbb{Q}, \text{nil}, \text{fin}} \right) & \xrightarrow{\mathcal{O}} & \text{Ho} \left( \text{dgcAlg}_{\text{cn}, \text{fin}}^{\text{op}} \right)
\end{array}
\]

\[ \cong \]

\[
\begin{array}{ccc}
\text{Ho} (\text{dgcAlg}^{\text{op}}) & \xrightarrow{S} & \text{Ho} (\text{dgcAlg}^{\text{op}}) \\
\downarrow & \Phi & \downarrow \\
\text{Ho} \left( \text{dgcAlg}_{\text{cn}, \text{fin}}^{\text{op}} \right) & \xrightarrow{\mathcal{O}} & \text{Ho} \left( \text{dgcAlg}^{\text{op}}_{\text{cn}, \text{fin}} \right)
\end{array}
\]
**dg-Algebra model for the 4-sphere**

Classical fact of rational homotopy theory:

\[ \mathcal{O}(S^4) \cong \mathbb{R}[\omega_4, \omega_7]/\begin{pmatrix} d\omega_4 &= 0 \\
\omega_7 &= -\frac{1}{2}\omega_4 \wedge \omega_4 \end{pmatrix} \]

This equation

\[ dG_4 = 0 \]
\[ dG_7 + \frac{1}{2}G_4 \wedge G_4 = 0 \]

is also the equations of motion in 11-dimensional supergravity for the M-brane flux.

**Conjecture** ([Sati 13, Sect. 1.5]):

The cohomology theory classifying M-branes is some flavour of degree-4 cohomotopy.
Super homotopy theory.

Via Quillen-Sullivan we may immediately generalize homotopy theory to *superspaces*:

\[
\text{sdgcAlgebras} := \left\{ \begin{array}{c}
\text{differential graded algebras with} \\
\mathbb{Z} \times \mathbb{Z}/2 \text{-grading}
\end{array} \right. \\
\text{cohomological grading} \\
\text{super grading}
\right\}
\]

In supergravity, cofibrant sdgc-algebras are known as “FDA”s. We consider the corresponding homotopy theory:

\[
\text{Ho}(\text{SuperSpaces}_{\mathbb{Q}, \text{nil,fin}}) := \text{sdgcAlg}_{\text{cn,fin}}^{\text{op}} \left[ \{ \text{Isos on all cohomology groups} \}^{-1} \right]
\]

see [ncatlab.org/nlab/show/geometry+of+physics+++superalgebra](ncatlab.org/nlab/show/geometry+of+physics+++superalgebra)
Example: Super-Minkowski spacetimes.

The nilpotency condition on the fundamental group allows precisely the mild non-abelianness that goes with super-Minkowski spacetimes:

\[ \mathcal{O}(\mathbb{T}^{p,1}\vert N) := \mathbb{R}\left[ (e^a)^p_{a=0}, (\psi^\alpha)^N_{\alpha=1}\right]/\left(\frac{de^a}{d\psi^\alpha} = 0, \frac{d\psi^\alpha}{\overline{\psi}}\Gamma^{\alpha} \psi\right) \]

\[ \text{toroidal super-Minkowski spacetime} \]

\[ \text{super vielbein} \]

\[ \text{torsion constraint} \]

See [ncatlab.org/nlab/show/geometry+of+physics++supersymmetry](ncatlab.org/nlab/show/geometry+of+physics++supersymmetry)
Extension tower of super-Minkowski spacetime.

These appear in a sequence of iterated maximal R-symmetry invariant central extensions:

\[
\begin{align*}
\mathbb{R}^{10,1|32} & \\
\mathbb{R}^{9,1|16+16} & \rightarrow \mathbb{R}^{9,1|16} \rightarrow \mathbb{R}^{9,1|16+16} \\
\mathbb{R}^{5,1|8} & \rightarrow \mathbb{R}^{5,1|8+8} \\
\mathbb{R}^{3,1|4+4} & \rightarrow \mathbb{R}^{3,1|4} \\
\mathbb{R}^{2,1|2+2} & \rightarrow \mathbb{R}^{2,1|2} \\
\mathbb{R}^{0|1+1} & \rightarrow \mathbb{R}^{0|1} .
\end{align*}
\]
The **M2/M5-Brane cocycle.**

On the top-dimensional 11d super Minkowski spacetime, there is a unique element in Spin(10, 1)-invariant rational cohomotopy in degree 4.

\[
\mathbb{T}^{10,1\mid 32} \xrightarrow{\mu_{M2/M5}} S^4
\]

In string/M theory this statement characterizes fundamental M2-brane and M5-branes via the WZW terms of their Green-Schwarz-type sigma-models.

\[\in \text{Ho}(\text{SuperSpaces}_R)^{\text{Spin}(10,1)}\]
2) Parameterized stable homotopy theory and gauge enhancement

based on [BSS 18]
Central extensions are homotopy fibers.

That 11d super-spacetime is an extension
of 10d super-spacetime
means that it is a homotopy fiber of a 2-cocyle:

$$\text{Ext}_{\bar{\psi} \Gamma_10 \psi} \left( T^{9,1|16+16} \right) \rightarrow T^{10,1|32}$$
The Ext/Cyc-Adjunction.

**Proposition.** The extension/homotopy fiber functor has a right adjoint

\[ \text{Ho} \left( \text{Spaces} / BS^1 \right) \rightleftarrows \text{Ext} \]

\[
\begin{array}{c}
\text{Ho} \left( \text{Spaces} \right) \\
\text{Cyc}
\end{array}
\]

given by forming *cyclic loop spaces*:

\[ \text{Cyc}(X) := \text{Maps}(S^1, X) \parallel S^1 \]

i.e. the homotopy quotient of the free loop space by the rigid rotation of loops.
Example: cyclification of the 4-Sphere

The cyclification of the 4-sphere is

\[ \mathcal{O} \left( \text{Cyc} \left( S^4 \right) \right) = \mathbb{Q}[h_3, h_7, \omega_2, \omega_4, \omega_6] / \begin{pmatrix} dh_7 = -\frac{1}{2} \omega_4 \wedge \omega_4 \\ + \omega_2 \wedge \omega_6 \\ dh_3 = 0 \\ d\omega_{2p} = h_3 \wedge \omega_{2p-2} \end{pmatrix} \]

Curiously, the terms in blue exhibit a truncation of rationalized twisted K-theory.

Below we will find also the rest of rationalized K-theory from the 4-sphere...
The Ext/Cyc-adjunct of $\mu_{M2/M5}$

Hence the M2/M5-cocycle

$$\mathbb{T}^{10,1|32} \xrightarrow{\mu_{M2/M5}} S^4$$

induces its Ext/Cyc-adjunct

$$\mathbb{T}^{9,1|16+16} \xrightarrow{\eta} \text{CycExt} \left( \mathbb{T}^{9,1|16+16} \right) \xrightarrow{\mu_{M2/M5}} \text{Cyc} \left( \mu_{M2/M5} \right) \xrightarrow{\text{Cyc}} \text{Cyc} \left( S^4 \right) \xrightarrow{B^2 S^1}$$

see ncatlab.org/schreiber/show/Super+Lie+n-algebra+of+Super+p-branes
This is **double dimensional reduction** of M2/M5-cocycle to the F1/NS5/D0/D2/D4 branes of type IIA string theory:

| Sullivan algebra of cyclified 4-sphere | $dh_7 = -\frac{1}{2}\omega_4 \wedge \omega_4$
|                                           | $+ \omega_2 \wedge \omega_6$
|                                           | $dh_3 = 0$
|                                           | $d\omega_{2p} = h_3 \wedge \omega_{2p-2}$ |
| Bianchi identities of NS1/NS5-flux and D($p \leq 4$) RR-fluxes: | $dH_7 = -\frac{1}{2}F_4 \wedge F_4$
|                                           | $+ F_2 \wedge F_6$
|                                           | $dH_3 = 0$
|                                           | $dF_{2p} = H_3 \wedge F_{2p-2}$ |

$2p \in \{0, 2, 4\}$

FSS 16b
Extensions and actions

To make the double dimensional reduction more symmetric, we ask also $S^4$ to be an $S^1$-extension

$$\mathbb{R}^{10,1}|32 \quad \text{homotopy fiber} \quad \mathbb{R}^{9,1}|16+\overline{16} \quad \mathbb{R}^{9,1}|16+\overline{16} \quad BS^1$$

The base $S^4 \sslash S^1$ of such an extension is necessarily the homotopy quotient by some $S^1$-action on $S^4$. 
Example: Suspended Hopf action on $S^4$

The identifications

$$S^1 \simeq U(1) \subset SU(2) \simeq S(\mathbb{H})$$

$$S^4 \simeq S(\mathbb{R} \oplus \mathbb{H})$$

induce an $S^1$-action on $S^4$. 
Attempted lift in Super homotopy theory

For any choice of $S^1$-action on $S^4$ we may ask for a lift of the double dimensional reduction of the M2/M5-cocyle:

\[
\begin{align*}
\mathbb{T}^{9,1|16+16} & \xrightarrow{\eta} S^4 \parallel S^1 \\
\text{CycExt} \left( \mathbb{T}^{9,1|16+16} \right) & \xrightarrow{\mu_{M2/M5}} \text{Cyc} \left( \mu_{M2/M5} \right) \xrightarrow{\text{CycExt} \left( S^4 \parallel S^1 \right)} B^2 S^1
\end{align*}
\]
Attempted lift in Super homotopy theory

For any choice of $S^1$-action on $S^4$ we may ask for a lift of the double dimensional reduction of the M2/M5-cocyle:

\[
\begin{align*}
\mathbb{T}^{9,1}|16 + 16 & \twoheadrightarrow \mathbb{C}yc\left(\mathbb{T}^{9,1}|16 + 16\right) \twoheadrightarrow \mathbb{C}yc\left(\mu_{M2/M5}\right) \twoheadrightarrow \mathbb{C}yc\Ext\left(S^4 \sslash S^1\right) \\
\eta & \quad \mu_{M2/M5} \quad \eta
\end{align*}
\]

\[
B^2 S^1 \quad \mathcal{F}
\]

**Prop.** Such a lift does not exist, for any choice of action. But it may exist to first linear order.
Stable homotopy theory

\[ \text{Ho}(\text{Spectra}) := \text{Spectra} \left[ \left\{ \text{Isos on all stable homotopy groups} \right\}^{-1} \right] \]

Spectra stabilize the operation of forming loop spaces:

\[
\begin{align*}
\Sigma & \quad \Delta \quad \Omega \\
\Sigma^\infty & \quad \Delta \quad \Omega^\infty
\end{align*}
\]

\[
\begin{align*}
\text{Ho}(\text{Spaces}^*/\!\!/) & \xleftarrow{\Sigma} \xrightarrow{\Omega} \text{Ho}(\text{Spaces}^*/\!\!/) \\
\text{Ho}(\text{Spectra}) & \xleftarrow{\Sigma^\infty \equiv \Omega} \xrightarrow{\Omega^\infty} \text{Ho}(\text{Spectra})
\end{align*}
\]

Loop spaces are the groups of homotopy theory.
Double loop space are the first-order commutative groups.

Hence: Spectra are the abelian groups of homotopy theory,
Hence: \(\Sigma^\infty\) is linearization in homotopy theory.

see [ncatlab.org/nlab/show/Introduction+to+Stable+Homotopy+Theory](ncatlab.org/nlab/show/Introduction+to+Stable+Homotopy+Theory)
Parameterized stable homotopy theory

For $X \in \text{Ho}(\text{Spaces})$

$$\text{Ho}(\text{Spectra}_X) := \text{Spectra}_X \left[ \left\{ \text{Isos on all stable homotopy groups for all homotopy fibers over } X \right\}^{-1} \right]$$

Parameterized spectra stabilize forming homotopy-fiber wise loop spaces:

$$\text{Ho} \left( \left( \text{Spaces}/X \right)^{X/} \right) \xrightarrow{\Sigma_X} \text{Ho} \left( \left( \text{Spaces}_X \right)^{X/} \right)$$

Hence: Parameterized spectra are the bundles of abelian groups in homotopy theory.
Rational parameterized stable homotopy theory

**Theorem** *(Braunack-Mayer 18)*:

The Quillen-Sullivan dg-Model for rational homotopy theory generalizes to parameterized stable homotopy theory by modeling parameterized spectra by dg-modules:

\[
\text{Ho} \left( \text{Spectra}_{S(A)} \right) \xrightarrow{\mathcal{O}} \text{Ho} \left( \text{dgMod}_{A} \right)^{\text{op}}
\]

\[
\text{Ho} \left( \left( \text{Spectra}_{S(A)} \right)_{Q, \text{fin}} \right) \xrightarrow{\mathcal{O}} \text{Ho} \left( \text{dgMod}_{A, \text{bd}} \right)^{\text{op}}
\]
Linearized lift of Ext/Cyc-adjunct of M2/M5-Cocycle.

Remember that we were asking for a full lift, which, however, does not exist:

\[
\begin{align*}
\mathbb{T}^{9,1}|_{16+16} & \xrightarrow{\eta} S^4 \parallel S^1 \\
\text{CycExt} \left( \mathbb{T}^{9,1}|_{16+16} \right) & \xrightarrow{\mu_{M2/M5}} \text{CycExt} \left( S^4 \parallel S^1 \right) \\
& \xrightarrow{\eta} B^2 S^1
\end{align*}
\]

$\in \text{Ho}(\text{SuperSpaces}_R)$
Linearized lift of Ext/Cyc-adjunct of M2/M5-Cocycle.

But now we may linearize the coefficients:

\[
\begin{align*}
\mathbb{T}^{9,1|16+16} & \xrightarrow{\eta} \mathbb{T}^{9,1|16+16} & \overset{\mu_{M2/M5}}{\longrightarrow} & S^4 \parallel S^1 \\
\text{CycExt} \left( \mathbb{T}^{9,1|16+16} \right) & \overset{\text{CycExt}(\mu_{M2/M5})}{\longrightarrow} \text{CycExt} \left( S^4 \parallel S^1 \right) & \overset{\eta}{\longrightarrow} & \Omega^\infty_{B^2S^1 \Sigma^\infty_{B^2S^1}}(S^4 \parallel S^1) \\
 & \downarrow & \downarrow & \downarrow \\
 & \in \Ho(SuperSpaces_{\mathbb{R}}) & \in \Ho(SuperSpaces_{\mathbb{R}}) & \in \Ho(SuperSpaces_{\mathbb{R}})
\end{align*}
\]
Linearized lift of Ext/Cyc-adjunct of M2/M5-Cocycle.

**Theorem:** With the suspended Hopf $S^1$-action on $S^4$ the linearized lifting problem has a unique solution:
Linearized lift of Ext/Cyc-adjunct of M2/M5-Cocycle.

**Theorem:** This solution factors through a summand which is the (rationalized) twisted K-theory spectrum:

\[
\begin{array}{c}
\mathbb{T}^{9,1}\mid 16+16 \\
\downarrow \eta \\
\text{CycExt} \left( \mathbb{T}^{9,1}\mid 16+16 \right)_{\text{cyc (M2/M5)}} \\
\downarrow \\
B^2 S^1
\end{array}
\xrightarrow{\mu \text{F1/Dp}}
\begin{array}{c}
\Omega_{B^2 S^1}^\infty (\text{ku} / B S^1) \\
\downarrow \\
\Omega_{B^2 S^1}^\infty \Sigma_{B^2 S^1}^\infty (S^4 / S^1)
\end{array}
\xrightarrow{\mu \text{M2/M5}}
\begin{array}{c}
\text{CycExt} (S^4 / S^1) \\
\downarrow \\
\Omega_{B^2 S^1}^\infty \Sigma_{B^2 S^1}^\infty \text{CycExt} (S^4 / S^1)
\end{array}
\in \text{Ho (SuperSpaces}_\mathbb{R}^\text{)}
This is **gauge enhancement** of M2/M5-cocycle to the full F1/Dp branes of type IIA string theory:

| dg-module for fiberwise stabilized suspended Hopf quotient of $S^4$ $\Sigma_B^{\infty} B^2 S^1 (S^4 \sslash S^1)$ | $dh_3 = 0$  
$\omega_{2p} = h_3 \wedge \omega_{2p-2}$ |
|---|---|
| Bianchi identities of NS1-flux and Dp RR-fluxes: | $dH_3 = 0$  
$F_{2p} = H_3 \wedge F_{2p-2}$ |

$2p \in \mathbb{Z}$
Beyond rational approximation?

We would like to eventually solve the open problem of lifting this discussion beyond the rational approximation.

As a first step towards solving this problem we now lift the $S^1 \subset SU(2)$-action beyond rational while keeping the spaces acted on rationally.

This hybrid approach is 

equivariant rational homotopy theory.
3) Equivariant homotopy theory and black M-branes

based on HSS18
Equivariant homotopy theory

For $G$ a compact Lie group, the evident definition is:

$$\text{Ho}(G\text{Spaces}) := G\text{CWComplexes} \left[ \{G\text{-equivariant homotopy equivalences}\}^{-1} \right]$$

The equivariant Whitehead theorem relates this to fixed-point loci:

$$\text{Ho}(G\text{Spaces}) \cong G\text{Spaces} \left[ \left\{ \begin{array}{l} \text{Isos on all homotopy groups} \\ \text{after restriction to } H\text{-fixed points} \\ \text{for all closed subgroups } H \subset G \end{array} \right\}^{-1} \right]$$
Systems of fixed point loci

But the $H$-fixed points are equivalently the $G$-equivariant maps out of the orbit space $G/H$:

$$X^H = \text{Maps}(G/H, X)^G$$

Hence if we form the category of all possible $G$-orbit spaces

$$\text{Orb}_G := \{G/H\}_{H \subset G \subset \text{closed} \text{GSpaces}}$$

Then the system of fixed point loci is extracted as

$$\text{GSpaces} \xrightarrow{Y} \text{PSh(Orb}_G, \text{Spaces})$$

$$X \mapsto \begin{pmatrix} G/H_1 & X^{H_1} \\ \downarrow f & \downarrow \text{Maps}(f, X)^G \\ G/H_2 & X^{H_2} \end{pmatrix}$$
Equivariant homotopy theory is about fixed point loci.

**Elmendorf’s theorem:** Equivariant homotopy theory is plain homotopy theory of the systems of fixed point loci:

\[
\text{Ho}(G\text{Spaces}) \xrightarrow{Y} \text{Ho}(\text{PSh}(\text{Orb}_G, \text{Spaces}))
\]

This induces in particular equivariance for non-classical homotopy theories:

**Equivariant rational homotopy theory.**

\[
\text{Ho}(G\text{Spaces}_{\mathbb{Q}, \text{nil}, \text{fin}}) \simeq \text{Ho}\left(\text{PSh}\left(\text{Orb}_G, \text{dgcAlg}^{\text{op}}_{\text{cn}, \text{fin}}\right)\right)
\]

**Equivariant rational super homotopy theory.**

\[
\text{Ho}(G\text{SuperSpaces}_{\mathbb{Q}, \text{nil}, \text{fin}}) \simeq \text{Ho}\left(\text{PSh}\left(\text{Orb}_G, \text{sdgcAlg}^{\text{op}}_{\text{cn}, \text{fin}}\right)\right)
\]
Example: SU(2)-Action on $S^4_Q$

Prop. The Hopf $S^1$-action on $S^4$ is trivial on the rational homotopy type of $S^4$, but in equivariant rational homotopy theory it is visible via its fixed point locus.

In fact this holds for the full SU(2)-action:

\[
\begin{array}{ccc}
SU(2)/1 & \xrightarrow{g} & S^4 \\
(S^4)(-) : & SU(2)/1 & \mapsto S^4 \\
& SU(2)/SU(2) & \xrightarrow{id} S^0 \\
\end{array}
\]
\textbf{G}_{\text{HW}}\text{-action on } S^4

There is also a \( \mathbb{Z}_2 \)-action on \( S^4 \) which does act non-trivially in rational homotopy theory, this is a reflection

\[
S^4 \xrightarrow{\text{reverse}} S^4
\]

\[
(x_1, \cdots, x_4, x_5) \xrightarrow{} (x_1, \cdots, x_4, -x_5)
\]

The corresponding system of fixed points:

\[
\begin{array}{ccc}
\mathbb{Z}_2 / 1 & \xrightarrow{\sigma} & S^4 \\
\downarrow \sigma & & \downarrow \text{reverse} \\
(S^4)(-) : \mathbb{Z}_2 / 1 & \hookrightarrow & S^4 \\
\downarrow & & \downarrow \\
\mathbb{Z}_2 / \mathbb{Z}_2 & & S^3
\end{array}
\]

We will denote this action by \( G_{\text{HW}} \).
ADE-classification of the finite subgroups of $SU(2)$:

<table>
<thead>
<tr>
<th>Dynkin label</th>
<th>Finite subgroup $G_{ADE} \subset SU(2)$</th>
<th>Name of group</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$</td>
<td>$\mathbb{Z}_{n+1}$</td>
<td>Cyclic</td>
</tr>
<tr>
<td>$D_{n+4}$</td>
<td>$2D_{n+2}$</td>
<td>Binary dihedral</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$2T$</td>
<td>Binary tetrahedral</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$2O$</td>
<td>Binary octahedral</td>
</tr>
<tr>
<td>$E_8$</td>
<td>$2I$</td>
<td>Binary icosahedral</td>
</tr>
</tbody>
</table>
Resolution of ADE-singularities

**Fact (du Val).** The blow-up of an orbifold singularity fixed by a finite subgroup of SU(2) is a system of spheres touching according to a Dynkin diagram:

Sen had suggested an interpretation in terms of M-brane physics. But the mathematical formulation of M-branes had remained an open problem.
**Equivariant enhancement?**

Hence we have lifted the 4-sphere to an object in $G_{\text{ADE}} \times G_{\text{HW}}$-equivariant rational (super) homotopy theory.

Therefore we are now entitled to the following

**Question:** Does the M2/M5-cocyle have a corresponding equivariant enhancement?

\[
\mathbb{R}^{10,1|32} \xrightarrow{\mu_{M2/M5}} S^4 \in \text{Ho} \left( G_{\text{ADE,HW}} \text{SuperSpaces}_\mathbb{R} \right)
\]

First, this requires identifying $G_{\text{ADE}} \times G_{\text{HW}}$-actions on $\mathbb{R}^{10,1|32} \ldots$
Theorem: ADE-Singularities in 11d super spacetime.

| Group action on $\mathbb{R}^{10,1|32}$ | Possible singular locus | BPS |
|----------------------------------------|-------------------------|-----|
| type H                                 | $\text{NS}_1\text{H}$ | $\text{M}2$ | $\frac{1}{2}\text{M}5\text{H}$ | $\text{MKK}6$ | $\frac{1}{2}\text{M}5\text{I}$ | $\frac{\text{M}9\text{H}}{\text{M}9\text{I}}$ |
| type I                                 | $\text{E}1$            |             |
| $G_{\text{ADE}}$                      | $\mathbb{R}^{2,1|8\cdot2}$ | $\mathbb{R}^{6,1|8+8}$ | $\frac{1}{2}$ |
| $G_{\text{HW}} = \mathbb{Z}_2$       |                         |             | $\mathbb{R}^{9,1|16}$ | $\frac{1}{2}$ |
| $G_{\text{ADE,HW}} = \mathbb{Z}_2$   | $\mathbb{R}^{1,1|16\cdot1}$ | $\mathbb{R}^{5,1|8+8}$ | $\frac{1}{2}$ |
| $G_{\text{ADE}} \times G_{\text{HW}}$| $\mathbb{R}^{1,1|8\cdot1}$ | $\mathbb{R}^{5,1|8}$ | $\frac{1}{4}$ |
**Theorem:** There are two $G_{\text{ADE,HW}}$-equivariant enhancements of the M2/M5-brane cocycle, exhibiting . . .

1) . . . the $1/2$-BPS black $1/2\text{M5}$ of $D = 5 + 1$ and $N = (2, 0)$
2) . . . the $1/2$-BPS black $\text{NS1}_H$ of $D = 1 + 1$ and $N = (16, 0)$

\[ \begin{array}{c}
\xymatrix{
G_{\text{ADE,HW}/1} \ar[d]^g & \mathbb{R}^{10,1|32} \ar[l] & S^4 \\
G_{\text{ADE,HW}/1} & \mathbb{R}^{10,1|16} \ar[u]^\rho(g) & S^4 \\
G_{\text{ADE,HW}/G_{\text{ADE,HW}}} & \mathbb{R}^{p,1|16} \ar[u]_\rho & S^0 \\
\end{array} \]

- shape of ADE-singularity
- WZW-terms of fundamental M2/M5-brane
- 11d super spacetime
- reverse
- black brane at ADE-singularity
- NG-Lagrangian

$\in \mathcal{H}_0(\text{PSh}(\text{Orb}_{G_{\text{ADE,HW}}}, \text{SuperSpaces}_\mathbb{R}))$
**Theorem:** There are two $G_{\text{ADE}}$-equivariant enhancements of the M2/M5-brane cocycle restricted to $M9_H$, exhibiting...

1) ... the $\frac{1}{4}$-BPS black $\frac{1}{2}\text{M5}$ of $D = 5 + 1$ and $N = (1, 0)$
2) ... the $\frac{1}{4}$-BPS black $\text{NS1}_H$ of $D = 1 + 1$ and $N = (8, 0)$

\[
\begin{array}{cccc}
\mathbb{R}^{9,1|16} & \xrightarrow{\rho(g)} & \mathbb{R}^{9,1|32} & \xrightarrow{\mu_{M2/M5\mid M9}} \mathbb{R}^{p,1|16} \\
\xrightarrow{\mu_{M2/M5\mid M9}} & & \xrightarrow{\text{vol}_{p+1}} & \xrightarrow{\text{id}} \\
S^4 & & & S^4 \\
\xrightarrow{\text{NG-Lagrangian}} & & & S^4 \\
\xrightarrow{\text{black p brane at ADE-singularity}} & & & S^0 \\
\end{array}
\]

$\in \text{Ho (PSh}(\text{Orb}_{G_{\text{ADE}}, \text{SuperSpaces}_R}))$
Conclusion:

A fair bit of the expected structure of M-theory emerges out of the superpoint $\mathbb{R}^{0|1}$ in rational equivariant super homotopy theory.

Evident Conjecture:

The full theory emerges once passing beyond the rational approximation in full super-geometric homotopy theory. (arXiv:1310.7930).

see also ncatlab.org/schreiber/show/StringMath2017
Epilogue

In full super-geometric homotopy theory
the superpoint $\mathbb{R}^{0|1}$ itself
emerges from $\emptyset$

\[
\begin{array}{c}
\text{id} \rightarrow \text{id} \\
\vee \quad \vee \\
\Rightarrow \rightarrow \rightsquigarrow \rightarrow \mathbb{R}^{0|1} \\
\vee \quad \vee \\
\mathcal{R} \rightarrow \mathbb{D} \rightarrow \mathbb{E} \\
\vee \quad \vee \\
\mathbb{R} \rightarrow \mathbb{b} \rightarrow \mathbb{f} \\
\vee \quad \vee \\
\emptyset \rightarrow \ast
\end{array}
\]

(Schreiber 16, FOMUS proceedings)