Structured Homotopy Theory
from
String Theory

Urs Schreiber

(Czech Academy of Science, Prague & NYU, Abu Dhabi)

a couple of talks at

Geometry, Topology & Physics

NYU AD, April 2018

Based on joint work with:

V. Braunack-Mayer, D. Fiorenza, J. Huerta, H. Sati,

Abstract.

Homotopy theory is extremely rich, with various structured variants such as equivariant, graded, parameterized and stable homotopy theory. Powerful tools from differential-graded algebra, particularly in the rational approximation, serve for concrete computations.

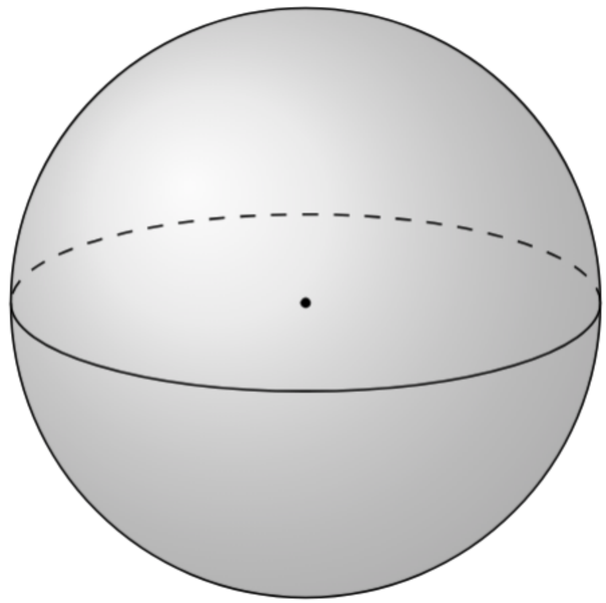
Also string/M-theory is extremely rich, revealing a system of higher dimensional objects (branes) with subtle inter-relations.

In these talks I survey recent insights into a close relation between the two, which provides structured homotopy theory with curious new examples and sheds light on the elusive foundations of string/M-theory.

The 4-sphere will play a surprisingly central role, as initially realized by Hisham Sati.

The 4-sphere

$$S^4$$



We will discuss its incarnation
in various flavours of *homotopy theory*:

1)

Super homotopy theory

and

fundamental M2/M5-branes

2)

Parameterized stable homotopy theory

and

gauge enhancement

3)

Equivariant homotopy theory

and

black M-branes

1)
Super homotopy theory
and
fundamental M2/M5-branes

based on FSS 13, FSS 15, FSS 16, HS 17

Classical homotopy theory

$$\mathrm{Ho}(\mathrm{Spaces}) \; := \; \mathrm{Spaces}\left[\{\text{Isos on all homotopy groups}\}^{-1}\right]$$

see ncatlab.org/nlab/show/Introduction+to+Homotopy+Theory

For example homotopy groups of spheres:

$$\pi_k(S^n) := \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Spaces})}\left(S^k, S^n\right)$$

These exhibit an endlessly rich pattern.
But most of them are torsion groups.

For example for S^4 two of them non-torsion:
generated by the *quaternionic Hopf fibration*:

Quaternionic Hopf fibration

$$
\begin{array}{ccccc}
S^7 & \simeq & S(\mathbb{H} \oplus \mathbb{H}) & (x, y) & |x|^2 + |y|^2 = 2 \\
\Big\downarrow{\scriptstyle \mathrm{Hopf}_{\mathbb{H}}} & & \Big\downarrow & \Big\downarrow & \\
S^4 & \simeq & S(\mathbb{R} \oplus \mathbb{H}) & \begin{pmatrix} t := 1 - |x|^2, \\ z := x \cdot y \end{pmatrix} & t^2 + |z|^2 = 1
\end{array}
$$

$$
\pi_\bullet\left(S^4\right) \;\simeq\; \mathbb{Z}\underbrace{\langle \mathrm{id}_{S^4} \rangle}_{\mathrm{deg}=4} \;\oplus\; \mathbb{Z}\underbrace{\langle \mathrm{Hopf}_{\mathbb{H}} \rangle}_{\mathrm{deg}=7} \;\oplus\; \text{torsion}
$$

Rational homotopy theory

disregards all torsion information:

$$\mathrm{Ho}(\mathrm{Spaces}_{\mathbb{Q}}) \;:=\; \mathrm{Spaces}\left[\{\text{Isos on rationalized homotopy groups}\}^{-1}\right]$$

$$\pi_k(S^n) \otimes \mathbb{Q} = \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Spaces}_{\mathbb{Q}})}\left(S^k, S^n\right)$$

$$\pi_\bullet\left(S^4\right) \otimes \mathbb{Q} \;=\; \mathbb{Q}\underbrace{\langle \mathrm{id}_{S^4}\rangle}_{\deg=4} \;\oplus\; \mathbb{Q}\underbrace{\langle \mathrm{Hopf}_{\mathbb{H}}\rangle}_{\deg=7}$$

dg-Algebraic model for rational homotopy theory.

$$\text{dgcAlg} \;:=\; \{\text{differential graded-commutative algebras over } \mathbb{R}\}$$

Homotopy theory of dg-algebras:

$$\begin{gathered}\text{Ho}\left(\text{dgcAlg}^{\text{op}}\right) \\ := \\ \text{dgcAlg}^{\text{op}}\left[\{\text{Isos on all cohomology groups}\}^{-1}\right]\end{gathered}$$

Quillen-Sullivan equivalence:

$$\begin{array}{ccc} \text{Ho}(\text{Spaces}) & \underset{\mathcal{S}}{\overset{\mathcal{O}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} & \text{Ho}(\text{dgcAlg}^{\text{op}}) \\ \uparrow & & \uparrow \\ \text{Ho}\left(\text{Spaces}_{\mathbb{Q},\text{nil},\text{fin}}\right) & \underset{\mathcal{S}}{\overset{\mathcal{O}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} & \text{Ho}\left(\text{dgcAlg}^{\text{op}}_{\text{cn},\text{fin}}\right) \end{array}$$

dg-Algebra model for the 4-sphere

Classical fact of rational homotopy theory:

$$\mathcal{O}(S^4) \;\simeq\; \mathbb{R}[\underbrace{\omega_4}_{\text{deg}=4}, \underbrace{\omega_7}_{\text{deg}=7}]/\begin{pmatrix} d\omega_4 = 0 \\ d\omega_7 = -\tfrac{1}{2}\omega_4 \wedge \omega_4 \end{pmatrix}$$

This equation

$$\begin{aligned} dG_4 &= 0 \\ dG_7 + \tfrac{1}{2}G_4 \wedge G_4 \;&=\; 0 \end{aligned}$$

is also the equations of motion in 11-dimensional supergravity for the M-brane flux.

Conjecture (Sati 13, Sect. 1.5):
The cohomology theory classifying M-branes is some flavour of degree-4 cohomotopy.

Super homotopy theory.

Via Quillen-Sullivan we may immediatly generalize homotopy theory to *superspaces*:

$$\mathrm{sdgcAlgebras} \;:=\; \left\{ \begin{array}{c} \text{differential graded algebras} \\ \text{with} \\ \underbrace{\mathbb{Z}}_{\substack{\text{cohomological} \\ \text{grading}}} \times \underbrace{\mathbb{Z}/2}_{\substack{\text{super} \\ \text{grading}}} \text{-grading} \end{array} \right\}$$

In supergravity, cofibrant sdgc-algebras are known as "FDA"s.

We consider the corresponding homotopy theory:

$$\begin{array}{c} \mathrm{Ho}(\mathrm{SuperSpaces}_{\mathbb{Q},\mathrm{nil},\mathrm{fin}}) \\ := \\ \mathrm{sdgcAlg}^{\mathrm{op}}_{\mathrm{cn},\mathrm{fin}}\left[\{\text{Isos on all cohomology groups}\}^{-1}\right] \end{array}$$

see ncatlab.org/nlab/show/geometry+of+physics+–+superalgebra

Example: Super-Minkowski spacetimes.

The nilpotency condition on the fundamental group allows precisely the mild non-abelianness that goes with super-Minkowski spacetimes:

$$\mathcal{O}\Big(\underbrace{\mathbb{T}^{p,1|\mathbf{N}}}_{\substack{\text{toroidal}\\ \text{super-Minkowski}\\ \text{spacetime}}} \Big) \;:=\; \mathbb{R}\Big[\underbrace{\underbrace{(e^a)_{a=0}^p}_{\text{deg}=(1,\text{even})} \;,\; \underbrace{(\psi^\alpha)_{\alpha=1}^N}_{(1,\text{odd})}}_{\text{super vielbein}} \Big] \Big/ \underbrace{\begin{pmatrix} de^a = \overline{\psi}\Gamma^a\psi \\ d\psi^\alpha = 0 \end{pmatrix}}_{\substack{\text{torsion}\\ \text{constraint}}}$$

see ncatlab.org/nlab/show/geometry+of+physics+–+supersymmetry

Extension tower of super-Minkowski spacetime.

These appear in a sequence of iterated maximal R-symmetry invariant central extensions:

$$
\begin{array}{ccccc}
 & & \mathbb{R}^{10,1|\mathbf{32}} & & \\
 & & & \searrow & \\
\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \leftleftarrows & \mathbb{R}^{9,1|\mathbf{16}} & \rightrightarrows & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
 & & & \searrow & \\
 & & \mathbb{R}^{5,1|\mathbf{8}} & \rightrightarrows & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
 & \swarrow & & & \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \leftleftarrows & \mathbb{R}^{3,1|\mathbf{4}} & & \\
 & \swarrow & & & \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \leftleftarrows & \mathbb{R}^{2,1|\mathbf{2}} & & \\
 & \swarrow & & & \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \leftleftarrows & \mathbb{R}^{0|\mathbf{1}} & & .
\end{array}
$$

The M2/M5-Brane cocycle.

On the top-dimensional 11d super Minkowski spacetime, there is a unique element in Spin(10, 1)-invariant rational cohomotopy in degree 4.

$$\mathbb{T}^{10,1|\mathbf{32}} \xrightarrow{\mu_{M2/M5}} S^4$$

$\in \mathrm{Ho}(\mathrm{SuperSpaces}_{\mathbb{R}})^{\mathrm{Spin}(10,1)}$

In string/M theory this statement
characterizes fundamental M2-brane and M5-branes
via the WZW terms of their Green-Schwarz-type sigma-models.

2)

Parameterized stable homotopy theory

and

gauge enhacement

based on BSS 18

Central extensions are homotopy fibers.

That 11d super-spacetime is an extension
of 10d super-spacetime
means that it is a *homotopy fiber* of a 2-cocyle:

$$
\begin{array}{ccc}
\mathrm{Ext}_{\overline{\psi}\Gamma_{10}\psi}\left(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}\right) & = & \mathbb{T}^{10,1|\mathbf{32}} \\
 & & \downarrow{\scriptstyle \text{homotopy fiber}} \\
 & & \mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} \\
 & & \searrow{\scriptstyle \text{2-cocyle}}\ \ \overline{\psi}\Gamma_{10}\psi \\
 & & \qquad BS^1
\end{array}
$$

$\in \mathrm{SuperSpaces}_{\mathbb{Q}}$

The Ext/Cyc-Adjunction.

Proposition. The extension/homotopy fiber functor has a right adjoint

$$\mathrm{Ho}\,(\mathrm{Spaces}) \underset{\mathrm{Cyc}}{\overset{\mathrm{Ext}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Ho}\left(\mathrm{Spaces}_{/BS^1}\right)$$

given by forming *cyclic loop spaces*:

$$\mathrm{Cyc}(X) \;:=\; \mathrm{Maps}(S^1, X) \,/\!\!/\, S^1$$

i.e. the homotopy quotient of the free loop space by the rigid rotation of loops.

Example: cyclification of the 4-Sphere

The cyclification of the 4-sphere is

$$\mathcal{O}\left(\mathrm{Cyc}\left(S^4\right)\right) \;=\; \mathbb{Q}[h_3, h_7,\; \omega_2, \omega_4, \omega_6]/\left(\begin{array}{l} dh_7 = -\tfrac{1}{2}\omega_4 \wedge \omega_4 \\ \qquad + \omega_2 \wedge \omega_6 \\ dh_3 = 0 \\ d\omega_{2p} = h_3 \wedge \omega_{2p-2} \end{array}\right)$$

Curiously, the terms in blue exhibit a truncation of rationalized twisted K-theory.

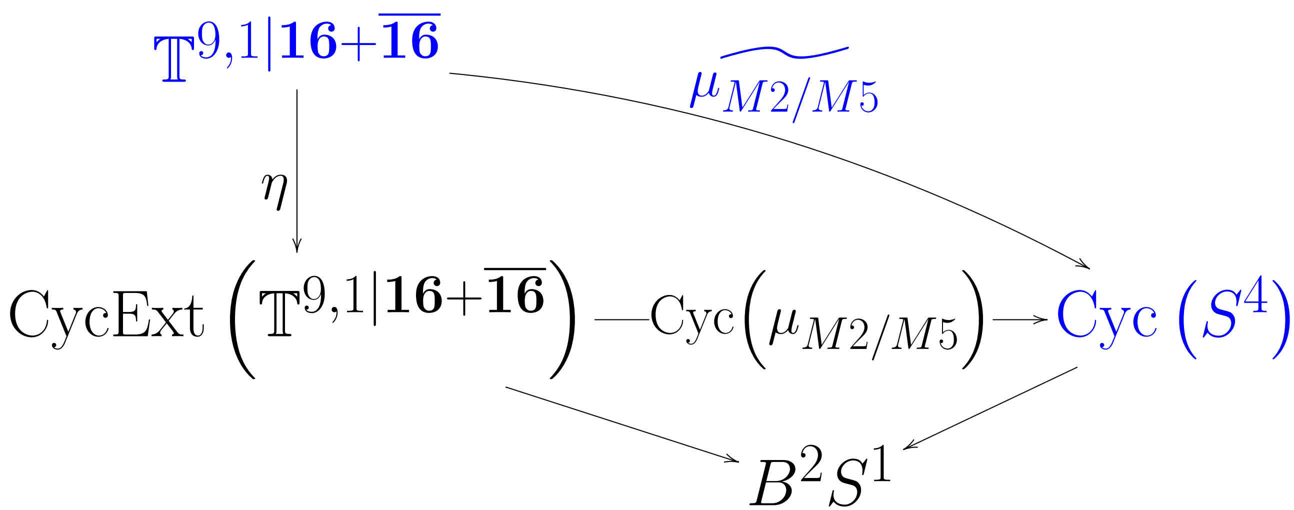
Below we will find also the rest of rationalized K-theory from the 4-sphere...

The Ext/Cyc-adjunct of $\mu_{M2/M5}$

Hence the M2/M5-cocycle

$$\mathbb{T}^{10,1|\mathbf{32}} \xrightarrow{\mu_{M2/M5}} S^4$$

induces its Ext/Cyc-adjunct



see ncatlab.org/schreiber/show/Super+Lie+n-algebra+of+Super+p-branes

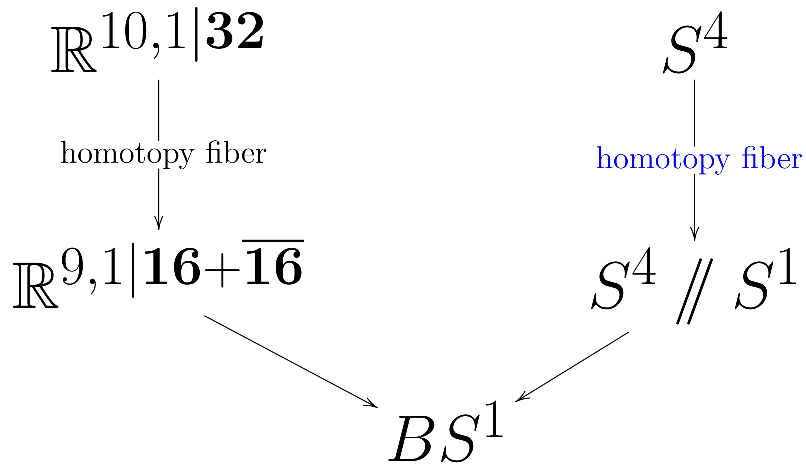
This is double dimensional reduction of M2/M5-cocycle

to the F1/NS5/D0/D2/D4 branes of type IIA string theory:

Sullivan algebra of cyclified 4-sphere	$dh_7 = -\frac{1}{2}\omega_4 \wedge \omega_4 + \omega_2 \wedge \omega_6$ $dh_3 = 0$ $d\omega_{2p} = h_3 \wedge \omega_{2p-2}$	$2p \in \{0, 2, 4\}$
Bianchi identities of NS1/NS5-flux and D($p \leq 4$) RR-fluxes:	$dH_7 = -\frac{1}{2}F_4 \wedge F_4 + F_2 \wedge F_6$ $dH_3 = 0$ $dF_{2p} = H_3 \wedge F_{2p-2}$	

Extensions and actions

To make the double dimensional reduction more symmetric, we ask also S^4 to be an S^1-extension



The base $S^4 /\!/ S^1$ of such an extension is necessarily the homotopy quotient by some S^1-action on S^4.

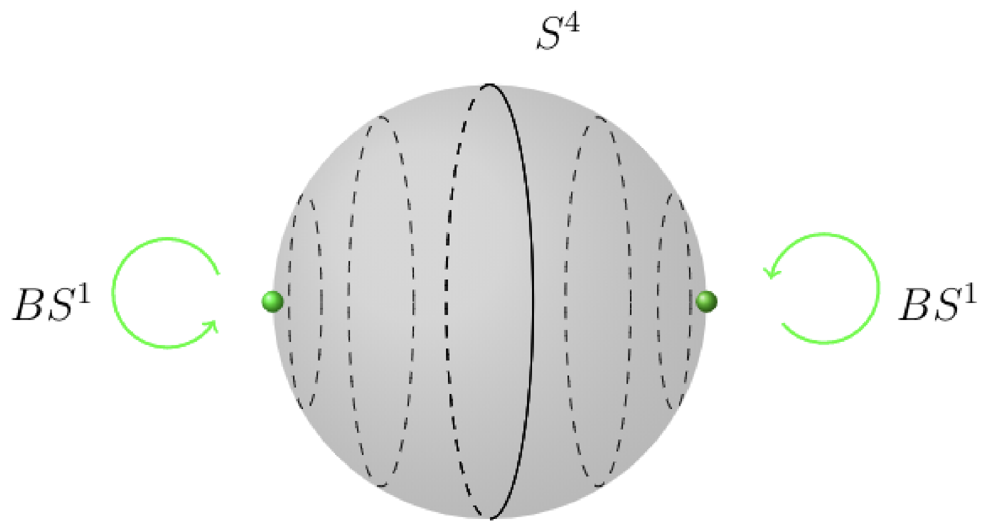
Example: Suspended Hopf action on S^4

The identifications

$$S^1 \simeq U(1) \subset \mathrm{SU}(2) \simeq S(\mathbb{H})$$

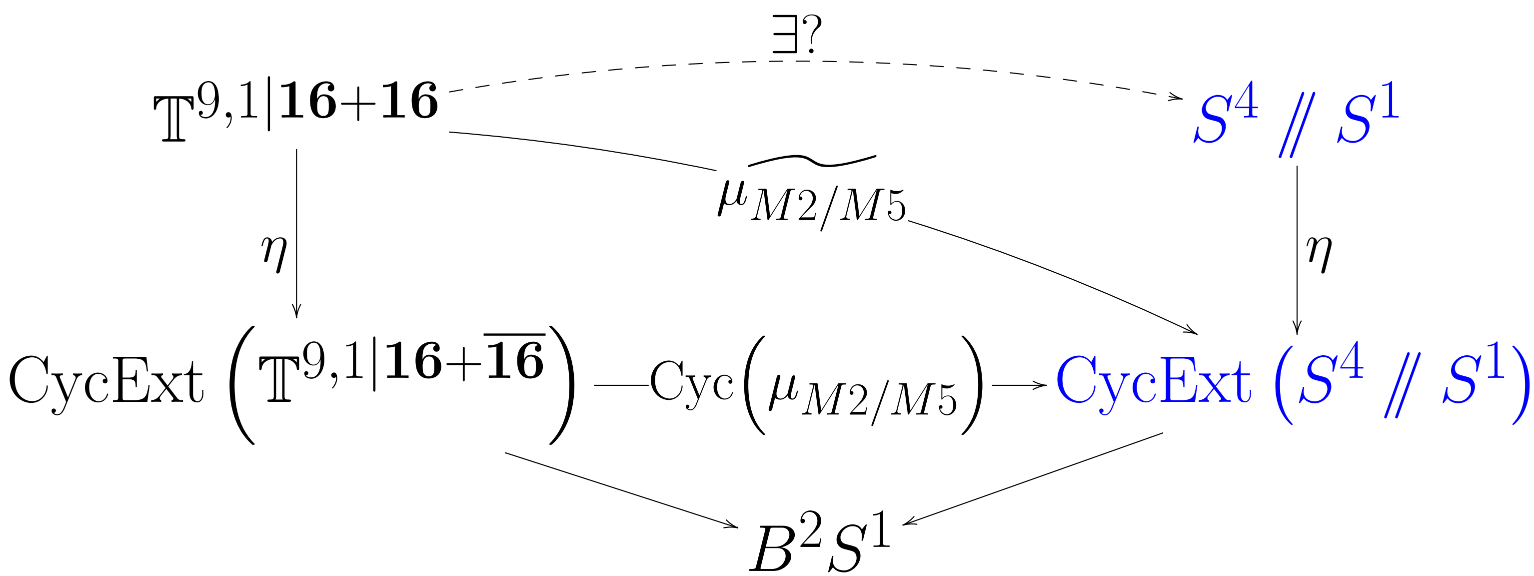
$$S^4 \simeq S(\mathbb{R} \oplus \mathbb{H})$$

induce an S^1-action on S^4.



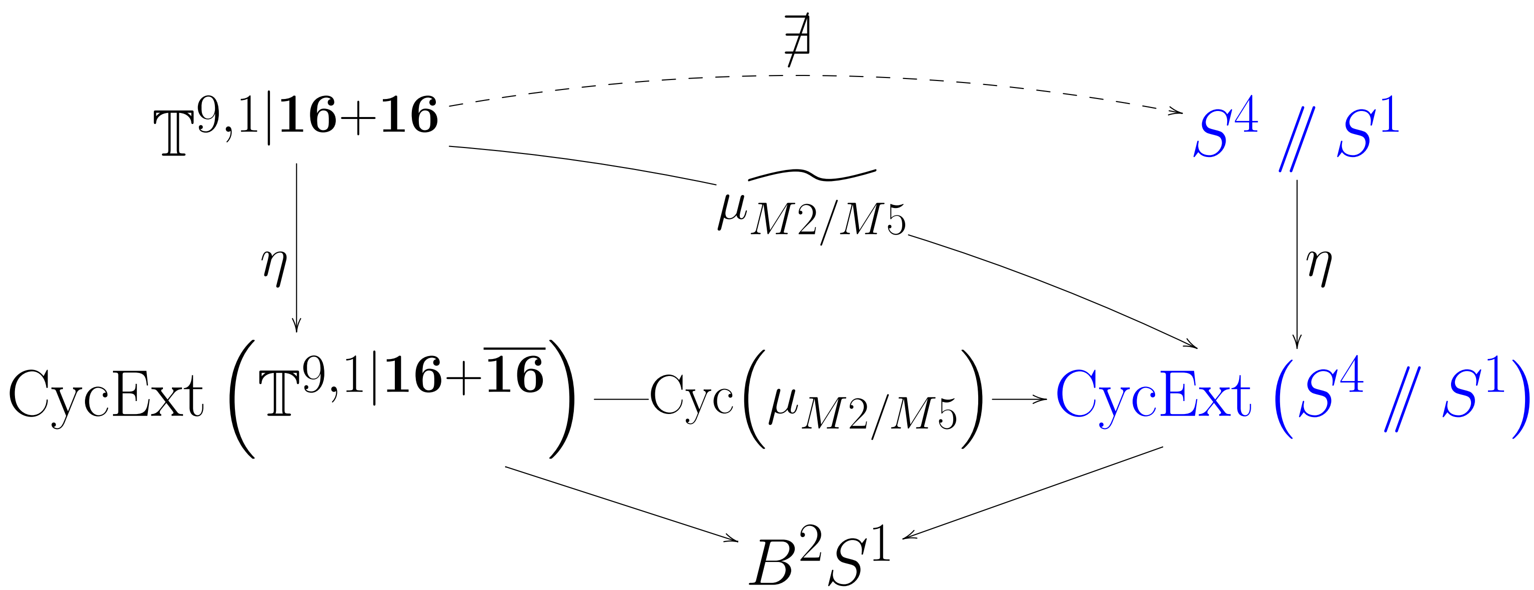
Attempted lift in Super homotopy theory

For any choice of S^1-action on S^4
we may ask for a lift of
the double dimensional reduction of the M2/M5-cocyle:



Attempted lift in Super homotopy theory

For any choice of S^1-action on S^4
we may ask for a lift of
the double dimensional reduction of the M2/M5-cocyle:



Prop. *Such a lift does not exist, for any choice of action.*
But it may exist *to first linear order.*

Stable homotopy theory

$$\mathrm{Ho}(\mathrm{Spectra}) \;:=\; \mathrm{Spectra}\left[\{\text{ Isos on all stable homotopy groups }\}^{-1}\right]$$

Spectra stabilize the operation of forming loop spaces:

$$\begin{array}{ccc}
\mathrm{Ho}\left(\mathrm{Spaces}^{*/}\right) & \underset{\Omega}{\overset{\Sigma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} & \mathrm{Ho}\left(\mathrm{Spaces}^{*/}\right) \\
{\scriptstyle \Sigma^\infty}\downarrow \dashv \uparrow{\scriptstyle \Omega^\infty} & & {\scriptstyle \Sigma^\infty}\downarrow \dashv \uparrow{\scriptstyle \Omega^\infty} \\
\mathrm{Ho}(\mathrm{Spectra}) & \underset{\Omega}{\overset{\Sigma}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} & \mathrm{Ho}(\mathrm{Spectra})
\end{array}$$

Loop spaces are the groups of homotopy theory.

Double loop space are the first-order commutative groups.

⋮

Hence: Spectra are the abelian groups of homotopy theory,

Hence: Σ^∞ is linearization in homotopy theory.

see ncatlab.org/nlab/show/Introduction+to+Stable+Homotopy+Theory

Parameterized stable homotopy theory

For $X \in \mathrm{Ho}\,(\mathrm{Spaces})$

$$\mathrm{Ho}\,(\mathrm{Spectra}_X) \;:=\; \mathrm{Spectra}_X \left[\left\{ \begin{array}{c} \text{Isos on all stable homotopy groups} \\ \text{for all homotopy fibers over } X \end{array} \right\}^{-1} \right]$$

Parameterized spectra stabilize forming homotopy-fiber wise loop spaces:

$$\begin{array}{ccc}
\mathrm{Ho}\left(\left(\mathrm{Spaces}_{/X} \right)^{X/} \right) & \underset{\Omega_X}{\overset{\Sigma_X}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} & \mathrm{Ho}\left((\mathrm{Spaces}_X)^{X/} \right) \\
{\scriptstyle \Sigma^\infty_X} \downarrow \dashv \uparrow {\scriptstyle \Omega^\infty_X} & & {\scriptstyle \Sigma^\infty_X} \downarrow \dashv \uparrow {\scriptstyle \Omega^\infty_X} \\
\mathrm{Ho}\left(\mathrm{Spectra}_{/X} \right) & \underset{\Omega_X}{\overset{\Sigma_X}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} & \mathrm{Ho}\,(\mathrm{Spectra}_X)
\end{array}$$

Hence: Parameterized spectra are the bundles of abelian groups in homotopy theory.

Rational parameterized stable homotopy theory

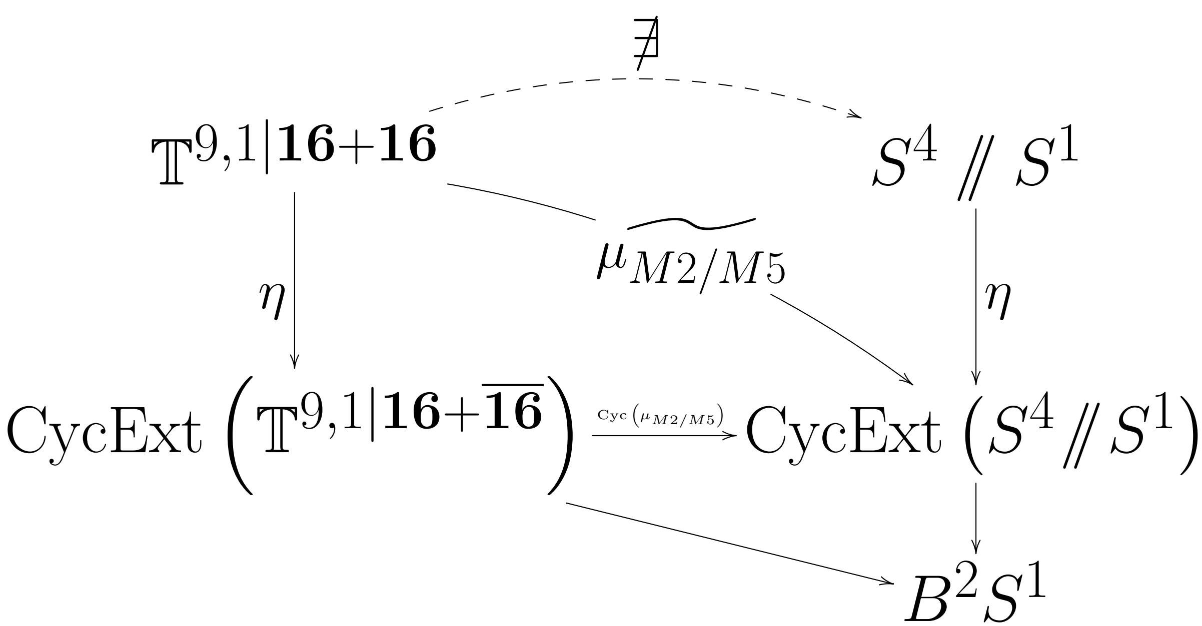
Theorem (Braunack-Mayer 18) **:**

The Quillen-Sullivan dg-Model for rational homotopy theory generalizes to parameterized stable homotopy theory by modeling parameterized spectra by dg-modules:

$$
\begin{array}{ccc}
\mathrm{Ho}\left(\mathrm{Spectra}_{\mathcal{S}(A)}\right) & \underoverset{\underset{\mathcal{S}}{\longleftarrow}}{\overset{\mathcal{O}}{\longrightarrow}}{\bot} & \mathrm{Ho}\left(\mathrm{dgMod}_A\right)^{\mathrm{op}} \\
\Big\uparrow & & \Big\uparrow \\
\mathrm{Ho}\left(\left(\mathrm{Spectra}_{\mathcal{S}(A)}\right)_{\mathbb{Q},\mathrm{fin}}\right) & \underoverset{\underset{\mathcal{S}}{\longleftarrow}}{\overset{\mathcal{O}}{\longrightarrow}}{\simeq} & \mathrm{Ho}\left(\mathrm{dgMod}_{A,\mathrm{bd}}\right)^{\mathrm{op}}
\end{array}
$$

Linearized lift of Ext/Cyc-adjunct of M2/M5-Cocycle.

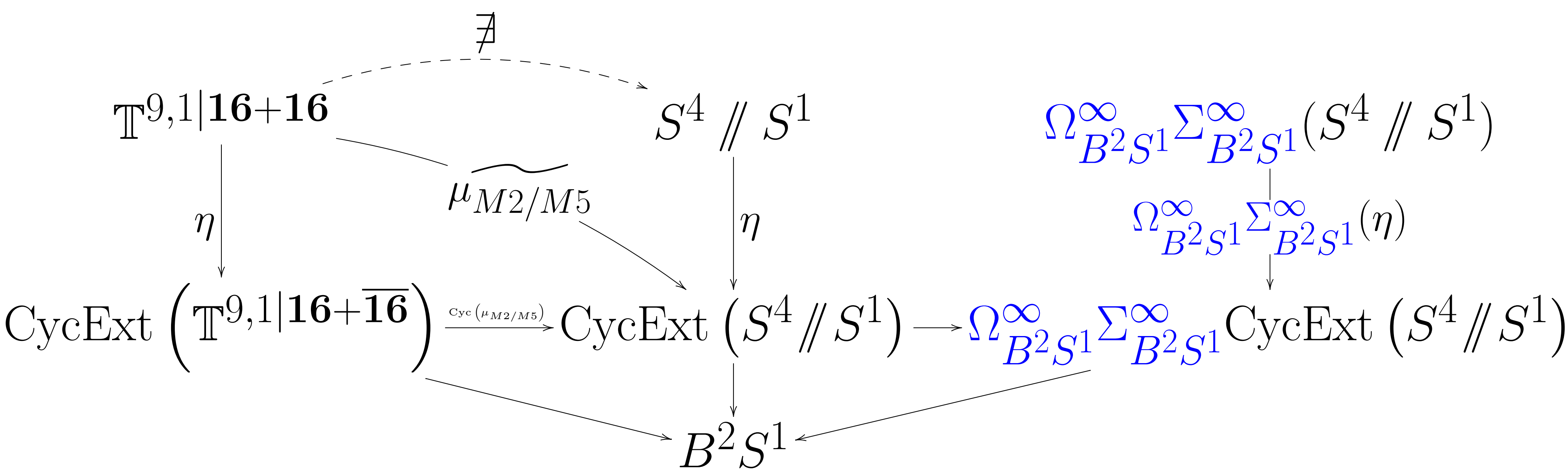
Remember that we were asking for a full lift,
which, however, does not exist:



$\in \mathrm{Ho}\left(\mathrm{SuperSpaces}_{\mathbb{R}}\right)$

Linearized lift of Ext/Cyc-adjunct of M2/M5-Cocycle.

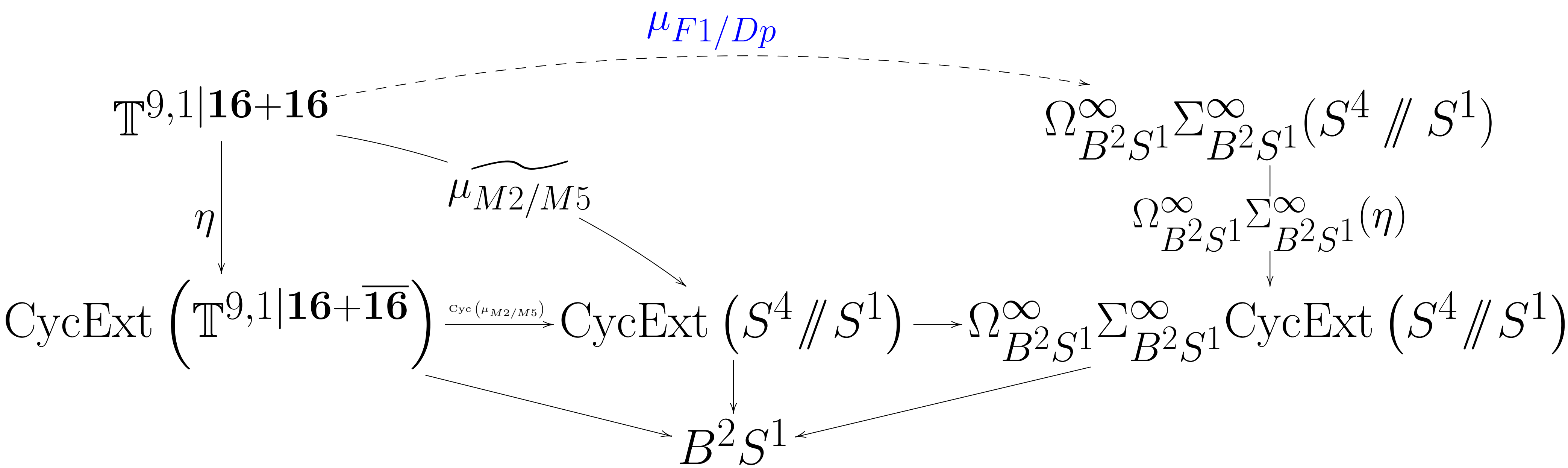
But now we may linearize the coefficients:



$\in \mathrm{Ho}\left(\mathrm{SuperSpaces}_{\mathbb{R}}\right)$

Linearized lift of Ext/Cyc-adjunct of M2/M5-Cocycle.

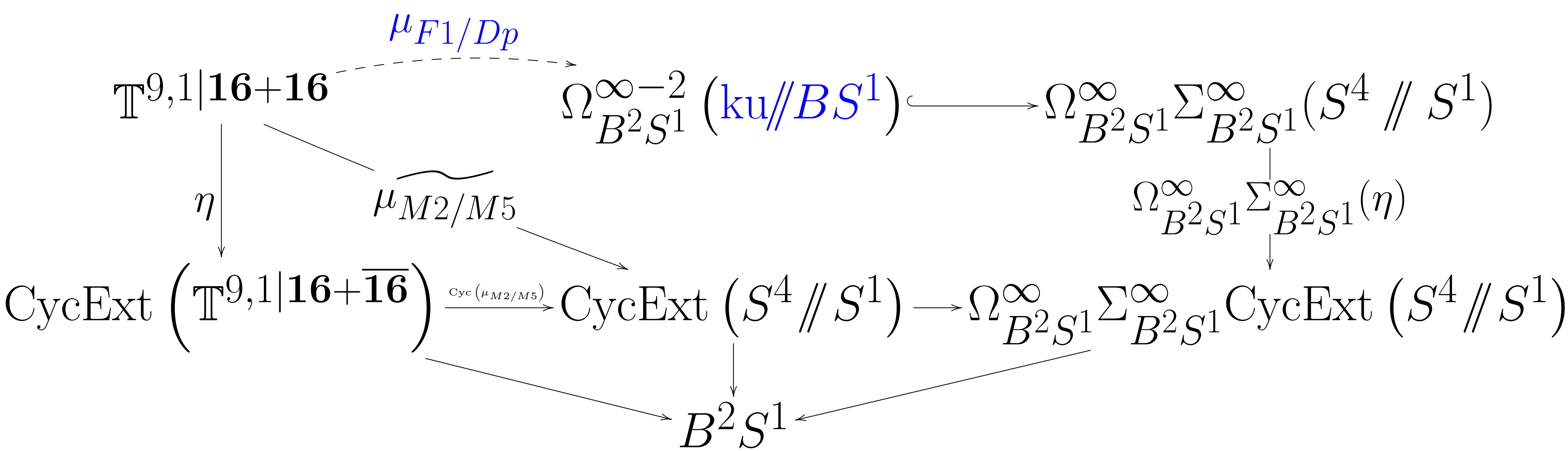
Theorem: With the suspended Hopf S^1-action on S^4 the linearized lifting problem has a unique solution:



$\in \mathrm{Ho}\left(\mathrm{SuperSpaces}_{\mathbb{R}}\right)$

Linearized lift of Ext/Cyc-adjunct of M2/M5-Cocycle.

Theorem: This solution factors through a summand which is the (rationalized) twisted K-theory spectrum:



$\in \mathrm{Ho}\left(\mathrm{SuperSpaces}_{\mathbb{R}}\right)$

This is gauge enhancement of M2/M5-cocycle

to the full F1/Dp branes of type IIA string theory:

dg-module for fiberwise stabilized suspended Hopf quotient of S^4 $\Sigma^\infty_{B^2 S^1}\left(S^4 /\!\!/ S^1\right)$	$dh_3 = 0$ $d\omega_{2p} = h_3 \wedge \omega_{2p-2}$	$2p \in \mathbb{Z}$
Bianchi identities of NS1-flux and Dp RR-fluxes:	$dH_3 = 0$ $dF_{2p} = H_3 \wedge F_{2p-2}$	

Beyond rational approximation?

We would like to eventually
solve the open problem
of lifting this discussion
beyond the rational approximation.

As a first step
towards solving this problem
we now lift the $S^1 \subset \mathrm{SU}(2)$-action beyond rational
while keeping the spaces acted on rationally.

This hybrid approach
is
equivariant rational homotopy theory.

3)
Equivariant homotopy theory
and
black M-branes

based on HSS18

Equivariant homotopy theory

For G a compact Lie group, the evident definition is:

$\mathrm{Ho}(G\mathrm{Spaces}) :=$

$G\mathrm{CWComplexes}\left[\{G\text{-equivariant homotopy equivalences}\}^{-1}\right]$

The equivariant Whitehead theorem
relates this to fixed-point loci:

$$\mathrm{Ho}(G\mathrm{Spaces}) \simeq$$
$$G\mathrm{Spaces}\left[\left\{\begin{array}{c}\text{Isos on all homotopy groups}\\ \text{after restriction to } H\text{-fixed points}\\ \text{for all closed subgroups } H \subset G\end{array}\right\}^{-1}\right]$$

Systems of fixed point loci

But the H-fixed points are equivalently
the G-equivariant maps out of the orbit space G/H:

$$X^H \;=\; \mathrm{Maps}(G/H, X)^G$$

Hence if we form the category of all possible G-orbit spaces

$$\mathrm{Orb}_G \;:=\; \{G/H\}_{\underset{\text{closed}}{H \subset G}} \subset G\mathrm{Spaces}$$

Then the *system of fixed point loci* is extracted as

$$G\mathrm{Spaces} \xrightarrow{\;Y\;} \mathrm{PSh}(\mathrm{Orb}_G, \mathrm{Spaces})$$

$$\mathrm{X} \;\mapsto\; \left(\begin{array}{ccc} G/H_1 & & X^{H_1} \\ \big\downarrow{\scriptstyle f} & \mapsto & \big\uparrow{\scriptstyle \mathrm{Maps}(f, X)^G} \\ G/H_2 & & X^{H_2} \end{array} \right)$$

Equivariant homotopy theory is about fixed point loci.

Elmendorf's theorem: Equivariant homotopy theory is plain homotopy theory of the systems of fixed point loci:

$$\mathrm{Ho}\left(G\mathrm{Spaces}\right) \underset{\simeq}{\overset{Y}{\longrightarrow}} \mathrm{Ho}\left(\mathrm{PSh}\left(\mathrm{Orb}_G, \mathrm{Spaces}\right)\right)$$

This induces in particular
equivariance for non-classical homotopy theories:
Equivariant rational homotopy theory.

$$\mathrm{Ho}\left(G\mathrm{Spaces}_{\mathbb{Q},\mathrm{nil},\mathrm{fin}}\right) \simeq \mathrm{Ho}\left(\mathrm{PSh}\left(\mathrm{Orb}_G\,, \mathrm{dgcAlg}^{\mathrm{op}}_{\mathrm{cn},\mathrm{fin}}\right)\right)$$

Equivariant rational super homotopy theory.

$$\mathrm{Ho}\left(G\mathrm{Super}\mathrm{Spaces}_{\mathbb{Q},\mathrm{nil},\mathrm{fin}}\right) \simeq \mathrm{Ho}\left(\mathrm{PSh}\left(\mathrm{Orb}_G\,, \mathrm{sdgcAlg}^{\mathrm{op}}_{\mathrm{cn},\mathrm{fin}}\right)\right)$$

Example: $\mathrm{SU}(2)$**-Action on** $S^4_{\mathbb{Q}}$

Prop. The Hopf S^1-action on S^4 is trivial on the rational homotopy type of S^4, but in equivariant rational homotopy theory it is visible via its fixed point locus.

In fact this holds for the full $\mathrm{SU}(2)$-action:

$$
(S^4)^{(-)} \quad : \quad
\begin{array}{c}
\mathrm{SU}(2)/1 \\
\big\downarrow{\scriptstyle g} \\
\mathrm{SU}(2)/1 \\
\big\downarrow \\
\mathrm{SU}(2)/\mathrm{SU}(2)
\end{array}
\quad \longmapsto \quad
\begin{array}{c}
S^4 \\
\big\uparrow{\scriptstyle \mathrm{id}} \\
S^4 \\
\big\uparrow \\
S^0
\end{array}
$$

$\in \mathrm{Spaces}_{\mathbb{Q}}$

G_{HW}-action on S^4

There is also a $\mathbb{Z}_2$-action on S^4 which does act non-trivially in rational homotopy theory, this is a reflection

$$
\begin{array}{ccc}
S^4 & \xrightarrow{\text{reverse}} & S^4 \\
(x_1, \cdots, x_4, x_5) & \longmapsto & (x_1, \cdots, x_4, -x_5)
\end{array}
$$

The corresponding system of fixed points:

$$
(S^4)^{(-)} \quad : \quad
\begin{array}{c}
\mathbb{Z}_2/1 \\
\downarrow \sigma \\
\mathbb{Z}_2/1 \\
\downarrow \\
\mathbb{Z}_2/\mathbb{Z}_2
\end{array}
\quad \longmapsto \quad
\begin{array}{c}
S^4 \\
\uparrow \text{reverse} \\
S^4 \\
\uparrow \\
S^3
\end{array}
$$

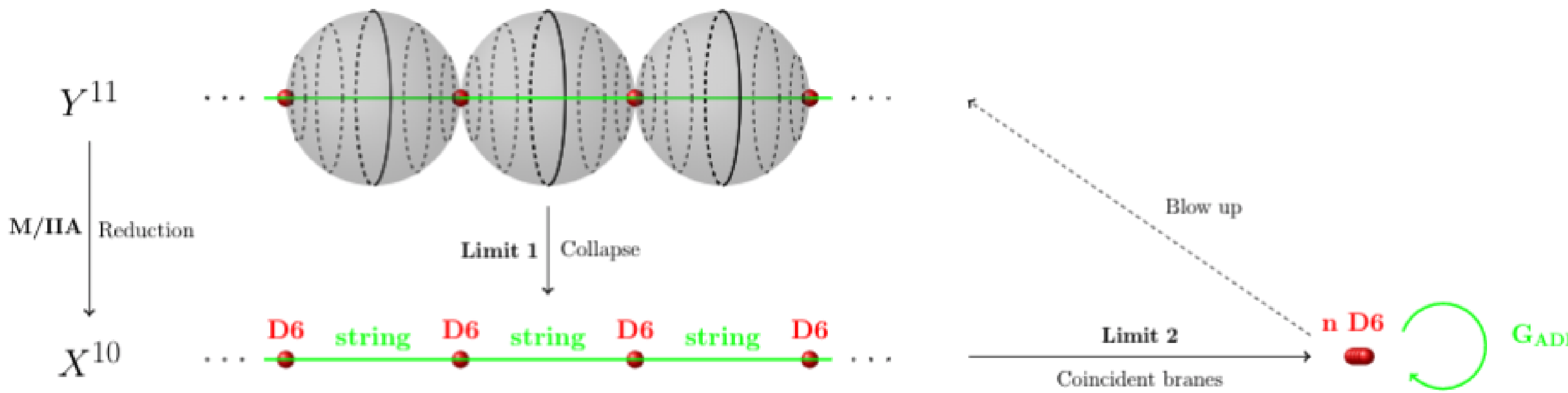
We will denote this action by G_{HW}.

ADE-classification of the finite subgroups of $\mathrm{SU}(2)$:

Dynkin label	Finite subgroup $G_{\mathrm{ADE}} \subset \mathrm{SU}(2)$	Name of group
$\mathbb{A}_n$	$\mathbb{Z}_{n+1}$	Cyclic
$\mathbb{D}_{n+4}$	$2\mathrm{D}_{n+2}$	Binary dihedral
$\mathbb{E}_6$	$2\mathrm{T}$	Binary tetrahedral
$\mathbb{E}_7$	$2\mathrm{O}$	Binary octahedral
$\mathbb{E}_8$	$2\mathrm{I}$	Binary icosahedral

Resolution of ADE-singularities

Fact (du Val). The blow-up of an orbifold singularity
fixed by a finite subgroup of SU(2)
is a system of spheres touching according to a Dynkin diagram:



Sen had suggested an interpretation in terms of M-brane physics.
But the mathematical formulation of M-branes
had remained an open problem.

Equivariant enhancement?

Hence we have lifted the 4-sphere to an object in $G_{\mathrm{ADE}} \times G_{\mathrm{HW}}$-equivariant rational (super) homotopy theory.

Therefore we are now entitled to the following

Question: Does the M2/M5-cocyle have a corresponding equivariant enhancement?

$$\begin{array}{ccc} \mathbb{R}^{10,1|\mathbf{32}} \xrightarrow{\mu_{M2/M5}} S^4 & \in \mathrm{Ho}\left(G_{\mathrm{ADE,HW}}\mathrm{SuperSpaces}_{\mathbb{R}}\right) \\ & \big\downarrow{\scriptstyle \text{forget equivariance}} \\ \mathbb{R}^{10,1|\mathbf{32}} \xrightarrow{\mu_{M2/M5}} S^4 & \in \mathrm{Ho}\left(\mathrm{SuperSpaces}_{\mathbb{R}}\right) \end{array}$$

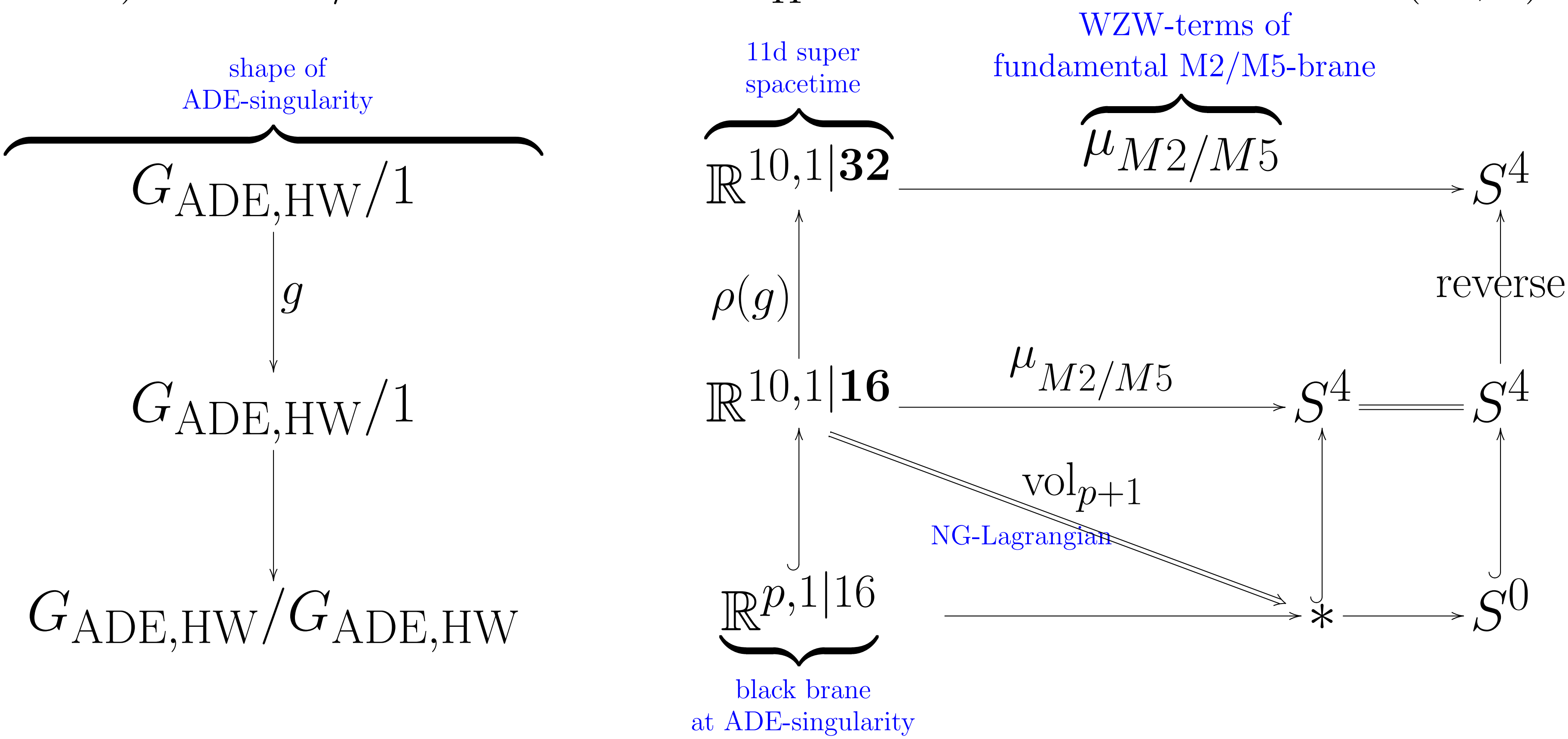
First, this requires identifying $G_{\mathrm{ADE}} \times G_{\mathrm{HW}}$-actions on $\mathbb{R}^{10,1|\mathbf{32}}\ldots$

Theorem: ADE-Singularities in 11d super spacetime.

Group action on $\mathbb{R}^{10,1\vert\mathbf{32}}$	**Possible singular locus**					**BPS**
type H	$\mathrm{NS1_H}$	M2	$1/2\mathrm{M5_H}$	MKK6	$\mathrm{M9_H}$	
type I	E1		$1/2\mathrm{M5_I}$		$\mathrm{M9_I}$	
G_{ADE}		$\mathbb{R}^{2,1\vert 8\cdot\mathbf{2}}$		$\mathbb{R}^{6,1\vert\mathbf{8}+\mathbf{8}}$		$1/2$
$G_{\mathrm{HW}} = \mathbb{Z}_2$					$\mathbb{R}^{9,1\vert\mathbf{16}}$	$1/2$
$G_{\mathrm{ADE,HW}} = \mathbb{Z}_2$	$\mathbb{R}^{1,1\vert 16\cdot\mathbf{1}}$		$\mathbb{R}^{5,1\vert\mathbf{8}+\mathbf{8}}$			$1/2$
$G_{\mathrm{ADE}} \times G_{\mathrm{HW}}$	$\mathbb{R}^{1,1\vert 8\cdot\mathbf{1}}$		$\mathbb{R}^{5,1\vert\mathbf{8}}$			$1/4$

Theorem: There are two $G_{\mathrm{ADE,HW}}$-equivariant enhancements of the M2/M5-brane cocycle, exhibiting…

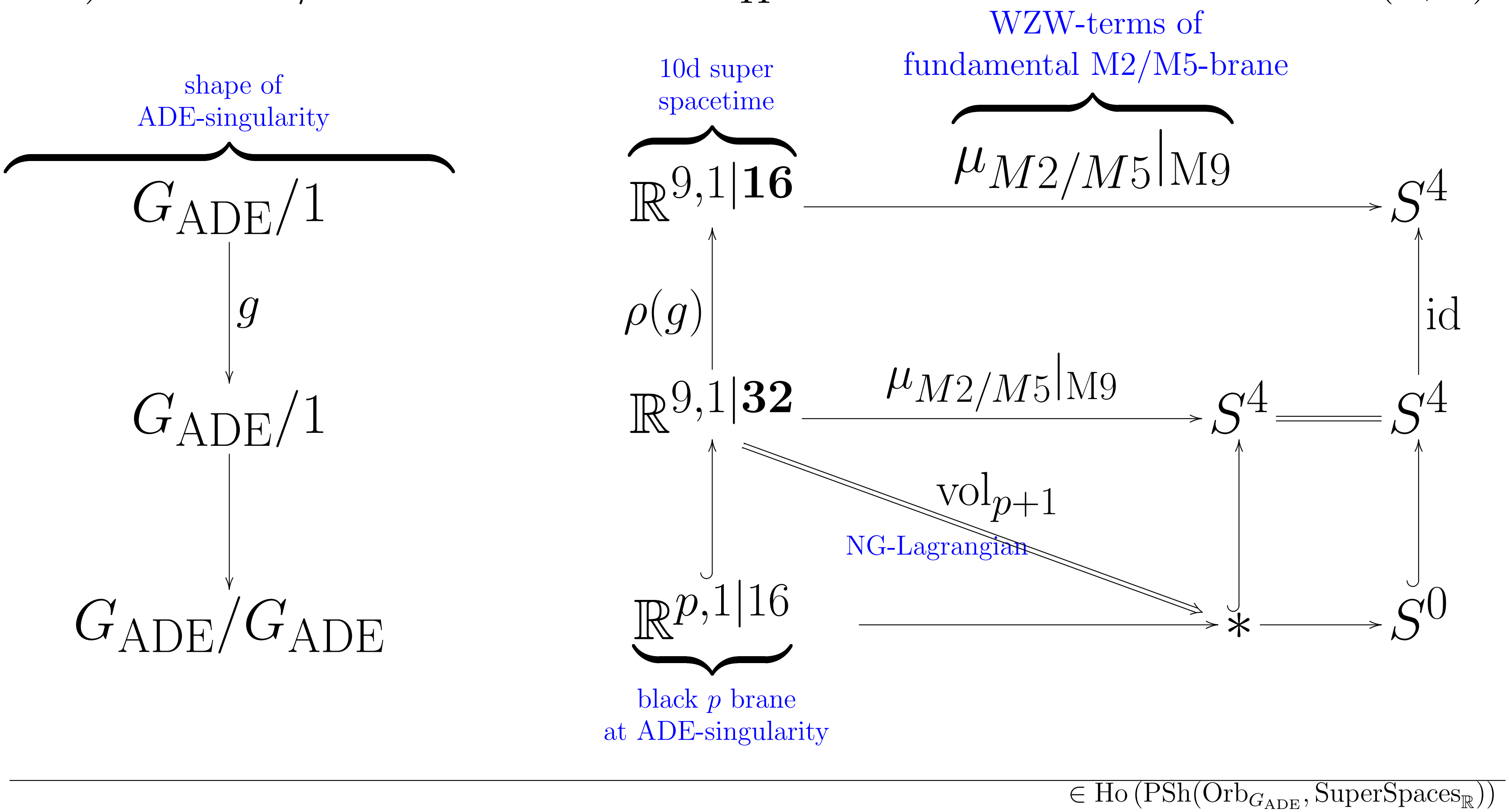
1) …the ½-BPS black ½**M5** of $D = 5+1$ and $N = (2,0)$
2) …the ½-BPS black $\mathbf{NS1}_{\mathrm{H}}$ of $D = 1+1$ and $N = (16,0)$



$\in \mathrm{Ho}\left(\mathrm{PSh}(\mathrm{Orb}_{G_{\mathrm{ADE,HW}}}, \mathrm{SuperSpaces}_{\mathbb{R}})\right)$

Theorem: There are two G_{ADE}-equivariant enhancements of the M2/M5-brane cocycle restricted to $\mathrm{M9}_H$, exhibiting…

1) …the $^1/_4$-BPS black $^1/_2\mathbf{M5}$ of $D = 5+1$ and $N = (1,0)$
2) …the $^1/_4$-BPS black $\mathbf{NS1}_{\mathrm{H}}$ of $D = 1+1$ and $N = (8,0)$



Conclusion:

A fair bit of
the expected structure of M-theory
emerges out of the superpoint $\mathbb{R}^{0|1}$
in rational equivariant super homotopy theory.

Evident Conjecture:

The full theory emerges
once passing beyond the rational approximation
in full super-geometric homotopy theory.
(arXiv:1310.7930).

see also ncatlab.org/schreiber/show/StringMath2017

Epilogue

In full super-geometric homotopy theory
the superpoint $\mathbb{R}^{0|1}$ itself
emerges from $\varnothing$

$$
\begin{array}{ccccccccc}
\mathrm{id} & \dashv & \mathrm{id} & & & & & & \\
\vee & & \vee & & & & & & \\
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \boxed{\color{blue}{\mathbb{R}^{0|1}}} & & & & \\
 & & \vee & & \vee & & & & \\
 & & \Re & \dashv & \boxed{\mathbb{D}} & \dashv & \mathrm{Et} & & \\
 & & & & \vee & & \vee & & \\
 & & & & \boxed{\mathbb{R}} & \dashv & \flat & \dashv & \sharp \\
 & & & & & & \vee & & \vee \\
 & & & & & & \varnothing & \dashv & \ast
\end{array}
$$

(Schreiber 16, FOMUS proceedings)