1 Background

1.1 Tensor conventions

Our tensor conventions are standard, but since superspace computations crucially depend on the corresponding prefactors, here to briefly make them explicit:

- The Einstein summation convention applies throughout: Given a product of terms indexed by some $i \in I$, with the index of one factor in superscript and the other in subscript, then a sum over I is implied: $x_i\, y^i := \sum_{i \in I} x_i\, y^i$.
- Our Minkowski metric is "mostly plus"
$$\big(\eta_{ab}\big)_{a,b=0}^{d} \;=\; \big(\eta^{ab}\big)_{a,b=0}^{d} \;:=\; \big(\mathrm{diag}(-1,+1,+1,\cdots,+1)\big)_{a,b=0}^{d}\,. \tag{1}$$
- Shifting position of frame indices always refers to contraction with the Minkowski metric (1):
$$V^a \;:=\; V_b\, \eta^{ab}\,, \quad V_a \;=\; V^b \eta_{ab}\,.$$
- Skew-symmetrization of indices is denoted by square brackets ($(-1)^{|\sigma|}$ is sign of the permutation σ):
$$V_{[a_1 \cdots a_p]} \;:=\; \tfrac{1}{p!} \sum_{\sigma \in \mathrm{Sym}(n)} (-1)^{|\sigma|} V_{a_{\sigma(1)} \cdots a_{\sigma(p)}}\,.$$
- We normalize the Levi-Civita symbol to
$$\epsilon_{012\cdots} \;:=\; +1 \quad \text{hence} \quad \epsilon^{012\cdots} \;:=\; -1\,. \tag{2}$$
- We normalize the Kronecker symbol to
$$\delta^{a_1 \cdots a_p}_{b_1 \cdots b_p} \;:=\; \delta^{[a_1}_{[b_1} \cdots \delta^{a_p]}_{b_p]} \;=\; \delta^{a_1}_{[b_1} \cdots \delta^{a_p}_{b_p]} \;=\; \delta^{[a_1}_{b_1} \cdots \delta^{a_p]}_{b_p} \tag{3}$$
so that
$$V_{a_1 \cdots a_p} \delta^{a_1 \cdots a_p}_{b_1 \cdots b_p} \;=\; V_{[b_1 \cdots b_p]} \quad \text{and} \quad \epsilon^{c_1 \cdots c_p a_1 \cdots a_q}\, \epsilon_{c_1 \cdots c_p b_1 \cdots b_q} \;=\; -\,p! \cdot q!\, \delta^{a_1 \cdots a_q}_{b_1 \cdots b_q}\,. \tag{4}$$

1.2 Super-algebra

In *homological* super-algebra, where a homological degree $n \in \mathbb{Z}$ (such as of flux densities) interacts with a super-degree $\sigma \in \mathbb{Z}_2$ there are – beware – two different *sign rules* in use (cf. [DM99, p. 62]), whose relation is a little subtle. The traditional sign rule in supergravity (e.g. [CDF91, (II.2.106-9)]) that we follow here comes from $\mathbb{Z} \times \mathbb{Z}_2$-*bi-grading*. (The alternative sign rule which collapses this bi-degree to a single "parity" degree in $\mathbb{Z}_2$ is popular with authors who say the word "Q-manifold").

Sign rule. For homological super-algebra we consider bigrading in the direct product ring $\mathbb{Z} \times \mathbb{Z}_2$ — where the first factor $\mathbb{Z}$ is the homological degree and the second $\mathbb{Z}_2 \simeq \{\mathrm{evn}, \mathrm{odd}\}$ the super-degree – with sign rule
$$\mathrm{deg}_1 = (n_1, \sigma_1),\ \mathrm{deg}_2 = (n_2, \sigma_2) \,\in\, \mathbb{Z} \times \mathbb{Z}_2 \qquad \vdash \qquad \mathrm{sgn}\big(\mathrm{deg}_1,\, \mathrm{deg}_2\big) \;:=\; (-1)^{n_1 \cdot n_2 + \sigma_1 \cdot \sigma_2}\,. \tag{5}$$

For $(v_i)_{i \in I}$ a set of generators with bi-degrees $(\mathrm{deg}_i)_{i \in I}$ we write:

(i) $\mathbb{R}\big\langle (v_i)_{i \in I} \big\rangle$ for the graded super-vector space spanned by these elements,

(ii) $\mathbb{R}\big[(v_i)_{i \in I}\big]$ for the graded-commutative polymonial algebra generated by these elements, hence the tensor algebra on $|I|$ generators modulo the relation
$$v_1 \cdot v_2 \;=\; (-1)^{\mathrm{sgn}(\mathrm{deg}_1, \mathrm{deg}_2)}\, v_2 \cdot v_1\,, \tag{6}$$
hence the (graded, super) *symmetric algebra* on the above super-vector space:
$$\mathbb{R}\big[(v_i)_{i \in I}\big] \;\;:=\;\; \mathrm{Sym}\big(\mathbb{R}\big\langle (v_i)_{i \in I} \big\rangle\big).$$

(iii) $\mathbb{R}_{\mathrm{d}}\big[(v_i)_{i\in I}\big]$ for the (free) differential graded-commutative algebra (dgca) generated by these elements and their *differentials*

$$(\mathrm{d}v_i)_{i\in I}$$

treated as primitive elements with $\deg(\mathrm{d}e_i) = \deg(e_i) + (1, \mathrm{evn})$ and modulo the corresponding relation (6), with differential defined by

$$e_i \longmapsto \mathrm{d}e_i \qquad , \qquad \mathrm{d}e_i \longmapsto 0$$

and extended as a (graded) derivation, hence the dgca

$$\mathbb{R}_{\mathrm{d}}\big[(v_i)_{i\in I}\big] \;\;:=\;\; \Big(\mathrm{Sym}\big(\mathbb{R}\big\langle (v_i)_{i\in I},\, (\mathrm{d}v_i)_{i\in I}\big\rangle\big),\, \mathrm{d}\Big). \tag{7}$$

1.3 Spinors in 11d

We briefly record the following standard facts about the Majoran spinor representation **32** of $\mathrm{Spin}(1,10)$ (proofs and references may be found in [MiSc06, §2.5][GSS24a, §2.2.1]).

(We may and do take this to be the only spinor representation that we construct from "from scratch"; all other spin representations we extract via simple algebra from this one. For instance the **16** and $\overline{\mathbf{16}}$ of $\mathrm{Spin}(1,9)$ are convenienetly identified with the images $P(\mathbf{32})$ and $\overline{P}(\mathbf{32})$ of **32** under the projector $P := \frac{1}{2}(1+\Gamma_{10})$ and its adjoint, respectively cf. (55) below.)

There exists an irreducible $\mathbb{R}$-linear representation **32** of $\mathrm{Pin}^+(1,10)$ with Clifford generators to be denoted

$$\Gamma_a \;:\; \mathbf{32} \to \mathbf{32} \tag{8}$$

and equipped with a $\mathrm{Spin}(1,10)$-equivariant skew-symmetric and non-degenerate bilinear form

$$\big(\overline{(-)}(-)\big) \;:\; \mathbf{32}\otimes\mathbf{32} \longrightarrow \mathbb{R} \tag{9}$$

satisfying all of the following properties.

In stating these we use the following notation:

- We denote, as usual, the skew-symmetrized product of k Clifford generators by
$$\Gamma_{a_1\cdots a_k} \;:=\; \tfrac{1}{k!}\sum_{\sigma\in\mathrm{Sym}(k)} \mathrm{sgn}(\sigma)\,\Gamma_{a_{\sigma(1)}}\cdot\Gamma_{a_{\sigma(2)}}\cdots\Gamma_{a_{\sigma(n)}} \;: \tag{10}$$
- The spinor pairing (9) serves as the *spinor metric* whose components – being the odd partner of the Minkowski metric (1) – we denote by $\big(\eta_{\alpha\beta}\big)_{\alpha,\beta=1}^{32}$:
$$\psi^\alpha\,\eta_{\alpha\beta}\,\phi^\beta \;\;:=\;\; \big(\overline{\psi}\,\phi\big)\,. \tag{11}$$
These are skew symmetric in their indices
$$\eta_{\alpha\beta} \;=\; -\eta_{\beta\alpha} \tag{12}$$
which together with the inverse matrix $(\eta^{\alpha\beta})$ is used to lower and raise spinor indices by contraction "from the right" (the position of the terms is irrelevant, since the components $\eta_{\alpha\beta}$ are commuting numbers, but the order of the indices matters due to the skew-symmetry):
$$\psi_\alpha \;:=\; \psi^{\alpha'}\eta_{\alpha'\alpha}\,,\qquad \psi^\alpha \;=\; \psi_{\alpha'}\eta^{\alpha'\alpha}\,,\qquad \psi_\alpha\phi^\alpha \;=\; -\psi^\alpha\phi_\alpha\,. \tag{13}$$

Now, conventions may be chosen such that all of the following holds true:

- The Clifford generators (8) square to the mostly plus Minkowski metric (1)
$$\Gamma_a\Gamma_b + \Gamma_b\Gamma_a \;\;=\;\; +2\,\eta_{ab}\,\mathrm{id}_{\mathbf{32}}\,. \tag{14}$$
- The Clifford product is given on the basis elements (10) as
$$\Gamma^{a_j\cdots a_1}\,\Gamma_{b_1\cdots b_k} \;\;=\;\; \sum_{l=0}^{\min(j,k)} \pm l!\binom{j}{l}\binom{k}{l}\,\delta^{[a_1\cdots a_l}_{[b_1\cdots b_l}\,\Gamma^{a_j\cdots a_{l+1}]}{}_{b_{l+1}\cdots b_k]}\,. \tag{15}$$
- The Clifford volume form equals the Levi-Civita symbol (2):
$$\Gamma_{a_1\cdots a_{11}} \;=\; \epsilon_{a_1\cdots a_{11}}\mathrm{id}_{\mathbf{32}}\,. \tag{16}$$
- The trace of all positive index Clifford basis elements vanishes:
$$\mathrm{Tr}(\Gamma_{a_1\cdots a_p}) \;=\; \left\{\begin{array}{ccl} 32 & | & p=0 \\ 0 & | & p>0\,. \end{array}\right. . \tag{17}$$

- The Hodge duality relation on Clifford elements is:

$$\Gamma^{a_1 \cdots a_p} \;=\; \tfrac{(-1)^{(p+1)(p-2)/2}}{(11-p)!}\, \epsilon^{a_1 \cdots a_p\, b_1 \cdots a_{11-p}}\, \Gamma_{b_1 \cdots b_{11-p}} \,. \tag{18}$$

  For instance:

$$\begin{aligned} \Gamma^{a_1 \cdots a_{11}} &= \epsilon^{a_1 \cdots a_{11}} \mathrm{Id}_{\mathbf{32}}\,, & \Gamma^{a_1 \cdots a_6} &= +\tfrac{1}{5!}\, \epsilon^{a_1 \cdots a_6\, b_1 \cdots b_5}\, \Gamma_{b_1 \cdots b_5}\,, \\ \Gamma^{a_1 \cdots a_{10}} &= \epsilon^{a_1 \cdots a_{10} b}\, \Gamma_b\,, & \Gamma^{a_1 \cdots a_5} &= -\tfrac{1}{6!}\, \epsilon^{a_1 \cdots a_5\, b_1 \cdots b_6}\, \Gamma_{b_1 \cdots b_6}\,. \end{aligned} \tag{19}$$

- The Clifford generators are skew self-adjoint with respect to the pairing (9)

$$\overline{\Gamma_a} \;=\; -\Gamma_a \qquad \text{in that} \qquad \underset{\phi, \psi \in \mathbf{32}}{\forall} \; \big(\overline{(\Gamma_a \phi)}\, \psi\big) \;=\; -\big(\overline{\phi}\, (\Gamma_a \psi)\big)\,, \tag{20}$$

  so that generally

$$\overline{\Gamma_{a_1 \cdots a_p}} \;=\; (-1)^{p + p(p-1)/2}\, \Gamma_{a_1 \cdots a_p}\,. \tag{21}$$

- The $\mathbb{R}$-vector space of $\mathbb{R}$-linear endomorphisms of $\mathbf{32}$ has a linear basis given by the $\leq$ 5-index Clifford elements

$$\mathrm{End}_{\mathbb{R}}(\mathbf{32}) \;=\; \big\langle 1,\, \Gamma_{a_1},\, \Gamma_{a_1 a_2},\, \Gamma_{a_1 a_2 a_3},\, \Gamma_{a_1 \cdots a_4},\, \Gamma_{a_1 \cdots a_5} \big\rangle_{a_i = 0,1,\cdots}\,, \tag{22}$$

- The $\mathbb{R}$-vector space of *symmetric* bilinear forms on $\mathbf{32}$ has a linear basis given by the expectation values with respect to (9) of the 1-, 2-, and 5-index Clifford basis elements:

$$\mathrm{Hom}_{\mathbb{R}}\Big((\mathbf{32} \otimes \mathbf{32})_{\mathrm{sym}},\, \mathbb{R} \Big) \;\simeq\; \Big\langle \big((\overline{-})\Gamma_a(-)\big)\,,\; \big((\overline{-})\Gamma_{a_1 a_2}(-)\big)\,,\; \big((\overline{-})\Gamma_{a_1 \cdots a_5}(-)\big) \Big\rangle_{a_i = 0,1,\cdots\,,} \tag{23}$$

  which means in components that these Clifford generators are symmetric in their lowered indices (13):

$$\Gamma^a_{\alpha\beta} \;=\; \Gamma^a_{\beta\alpha}\,, \quad \Gamma^{a_1 a_2}_{\alpha\beta} \;=\; \Gamma^{a_1 a_2}_{\beta\alpha}\,, \quad \Gamma^{a_1 \cdots a_5}_{\alpha\beta} \;=\; \Gamma^{a_1 \cdots a_5}_{\beta\alpha}\,, \tag{24}$$

  while a basis for the *skew-symmetric* bilinear forms is given by

$$\mathrm{Hom}_{\mathbb{R}}\Big((\mathbf{32} \otimes \mathbf{32})_{\mathrm{skew}},\, \mathbb{R} \Big) \;\simeq\; \Big\langle \big((\overline{-})(-)\big)\,,\; \big((\overline{-})\Gamma_{a_1 a_2 a_3}(-)\big)\,,\; \big((\overline{-})\Gamma_{a_1 \cdots a_4}(-)\big) \Big\rangle_{a_i = 0,1,\cdots\,,} \tag{25}$$

  which means in components that these Clifford generators are skew-symmetric in their lowered indices (13):

$$\eta_{\alpha\beta} \;=\; -\eta_{\beta\alpha}\,, \quad \Gamma^{a_1 a_2 a_3}_{\alpha\beta} \;=\; -\Gamma^{a_1 a_2 a_3}_{\beta\alpha}\,, \quad \Gamma^{a_1 \cdots a_5}_{\alpha\beta} \;=\; -\Gamma^{a_1 \cdots a_5}_{\beta\alpha} \tag{26}$$

- Any linear endomorphism $\phi \in \mathrm{End}_{\mathbb{R}}(\mathbf{32})$ is uniquely a linear combination of Clifford elements as:

$$\phi \;=\; \tfrac{1}{32} \sum_{p=0}^{5} \tfrac{(-1)^{p(p-1)/2}}{p!} \mathrm{Tr}\big(\phi \circ \Gamma_{a_1 \cdots a_p}\big) \Gamma^{a_1 \cdots a_p}\,. \tag{27}$$

- which implies in particular the Fierz expansion

$$\big(\overline{\phi}_1\, \psi\big)\big(\overline{\psi}\, \phi_2\big) \;=\; \tfrac{1}{32}\Big(\big(\overline{\psi}\, \Gamma^a\, \psi\big)\big(\overline{\phi}_1\, \Gamma_a\, \phi_2\big) - \tfrac{1}{2}\big(\overline{\psi}\, \Gamma^{a_1 a_2}\, \psi\big)\big(\overline{\phi}_1\, \Gamma_{a_1 a_2}\, \phi_2\big) + \tfrac{1}{5!}\big(\overline{\psi}\, \Gamma^{a_1 \cdots a_5}\, \psi\big)\big(\overline{\phi}_1\, \Gamma_{a_1 \cdots a_5}\, \phi_2\big) \Big). \tag{28}$$

Proposition 1.1 (**The general Fierz identities** [DF82, (3.1-3) & Table 2][CDF91, (II.8.69) & Table II.8.XI]**).**
(i) *The* Spin(1, 10)*-irrep decomposition of the first few symmetric tensor powers of* **32** *is:*

$$\begin{aligned} \big(\mathbf{32} \otimes \mathbf{32}\big)_{\mathrm{sym}} &\cong \mathbf{11} \oplus \mathbf{55} \oplus \mathbf{462} \\ \big(\mathbf{32} \otimes \mathbf{32} \otimes \mathbf{32}\big)_{\mathrm{sym}} &\cong \mathbf{32} \oplus \mathbf{320} \oplus \mathbf{1408} \oplus \mathbf{4424} \\ \big(\mathbf{32} \otimes \mathbf{32} \otimes \mathbf{32} \otimes \mathbf{32}\big)_{\mathrm{sym}} &\cong \mathbf{1} \oplus \mathbf{165} \oplus \mathbf{330} \oplus \mathbf{462} \oplus \mathbf{65} \oplus \mathbf{429} \oplus \mathbf{1144} \oplus \mathbf{17160} \oplus \mathbf{32604}\,. \end{aligned} \tag{29}$$

(ii) *In more detail, the irreps appearing on the right are tensor-spinors spanned by basis elements*

$$\begin{aligned} &\big\langle \Xi^{\alpha}_{a_1 \cdots a_p} \;=\; \Xi^{\alpha}_{[a_1 \cdots a_p]} \big\rangle_{a_i \in \{0, \cdots, 10\}, \alpha \in \{1, \cdots 32\}} \;\in\; \mathrm{Rep}_{\mathbb{R}}\big(\mathrm{Spin}(1,10)\big) \\ &\text{with} \quad \Gamma^{a_1} \Xi_{a_1 a_2 \cdots a_p} \;=\; 0 \end{aligned} \tag{30}$$

(jointly to be denoted $\Xi^{(N)}$ for the case of the irrep $\mathbf{N}$) *such that:*

$$\begin{aligned} \psi\big(\overline{\psi}\, \Gamma_a\, \psi\big) &= \tfrac{1}{11}\, \Gamma_a\, \Xi^{(32)} + \Xi^{(320)}_a\,, \\ \psi\big(\overline{\psi}\, \Gamma_{a_1 a_2}\, \psi\big) &= \tfrac{1}{11}\, \Gamma_{a_1 a_2}\, \Xi^{(32)} - \tfrac{2}{9}\, \Gamma_{[a_1}\, \Xi^{(320)}_{a_2]} + \Xi^{(1408)}_{a_1 a_2}\,, \\ \psi\big(\overline{\psi}\, \Gamma_{a_1 \cdots a_5}\, \psi\big) &= -\tfrac{1}{77} \Gamma_{a_1 \cdots a_5} \Xi^{(32)} + \tfrac{5}{9} \Gamma_{[a_1 \cdots a_4} \Xi^{(320)}_{a_5]} + 2\, \Gamma_{[a_1 a_2 a_3}\, \Xi^{(1408)}_{a_4 a_5]} + \Xi^{(4224)}_{a_1 \cdots a_5}\,. \end{aligned} \tag{31}$$

Background formulas for 11d Supergravity. Our notation and conventions for super-geometry and for on-shell 11d supergravity on super-space follow [GSS24a, §2.2 & §3], to which we refer for further details and exhaustive referencing. We denote the local data of a super-Cartan connection on (a surjective submersion $\widetilde{X}$ of) (super-

)spacetime X, representing a super-gravitational field configuration, as[1]

$$
\begin{array}{lllll}
\textbf{Graviton} & \big(E^a\big)_{a=0}^{D-1} & \in & \Omega^1_{\mathrm{dR}}\big(\widetilde{X};\, \mathbb{R}^{1,D-1}\big) \\
\textbf{Gravitino} & \big(\Psi^\alpha\big)_{\alpha=1}^{N} & \in & \Omega^1_{\mathrm{dR}}\big(\widetilde{X};\, \mathbf{N}_{\mathrm{odd}}\big) \\
\textbf{Spin-connection} & \big(\Omega^{ab} = -\Omega^{ba}\big)_{a,b=0}^{D-1} & \in & \Omega^1_{\mathrm{dR}}\big(\widetilde{X};\, \mathfrak{so}(1,D-1)\big)
\end{array}
\tag{32}
$$

and the corresponding Cartan structural equations (cf. [GSS24a, Def. 2.78]) for the supergravity field strengths as

$$
\begin{array}{llllll}
\textbf{Super-Torsion} & \big(T^a & := & \mathrm{d}\,E^a & - \,\Omega^a{}_b\,E^b - \big(\,\overline{\Psi}\,\Gamma^a\,\Psi\big)\big)_{a=0}^{D-1} \\
\textbf{Gravitino field strength} & \big(\rho & := & \mathrm{d}\,\Psi & - \,\tfrac{1}{4}\Omega^{ab}\,\Gamma_{ab}\psi\,\big)_{\alpha=1}^{N} \\
\textbf{Curvature} & \big(R^{ab} & := & \mathrm{d}\,\Omega^{ab} & - \,\Omega^a{}_c\,\Omega^{cb}\,\big)_{a,b=0}^{D-1}\,.
\end{array}
\tag{33}
$$

Finally, we denote the corresponding components in the given local super-coframe (E,Ψ) by [GSS24a, (127-8)]:

$$
\begin{array}{lll}
T^a & \equiv & 0 \\
\rho & =: & \tfrac{1}{2}\rho_{ab}\,E^a\,E^b \,+\, H_a\Psi\,E^a \\
R^{a_1a_2} & =: & \tfrac{1}{2}R^{a_1a_2}{}_{b_1b_2}\,E^{a_1}\,E^{a_2} \,+\, \big(\overline{J}^{a_1a_2}{}_b\Psi\big)E^b \,+\, \big(\overline{\Psi}\,K^{a_1a_2}\,\Psi\big)\,,
\end{array}
\tag{34}
$$

where all components not explicitly appearing vanish identically by the superspace torsion constraints [GSS24a, (121), (137)]. In addition, in the main text we consider the situation that also $\rho_{ab} = 0$ whence also $J^{a_1a_2}{}_b = 0$.

1.4 Super-L_∞ algebra

We recall (from [FSS15, §2][FSS18a, §2][FSS19, (21)][HSS19, §3.2][Sc21, p 33, 48]) the notion of higher (meaning: categorified symmetry) super-Lie algebras (of finite type) and their identification with the "FDA"s from the supergravity literature ([vN83][CDF91, §III.6], cf. [AD24]).

Our ground field is the real numbers $\mathbb{R}$ and all super-vector spaces are assumed to be finite-dimensional.

Given a finite dimensional super-Lie algebra $\mathfrak{g} \simeq \mathfrak{g}_{\mathrm{evn}} \oplus \mathfrak{g}_{\mathrm{odd}}$, the linear dual of the super-Lie bracket map

$$
[\text{-},\text{-}] \;:\; \mathfrak{g} \vee \mathfrak{g} \longrightarrow \mathfrak{g}
$$

may be understood to map the first to the second exterior power of the underlying dual super-vector space, and as such it extends uniquely to a $\mathbb{Z}\times\mathbb{Z}_2$-graded derivation d of degree=(1, evn) on the exterior super-algebra (where the minus sign is just a convention)

$$
\begin{tikzcd}
\wedge^1 \mathfrak{g}^* \arrow[r, "{-[\text{-},\text{-}]^*}"] \arrow[d, hook] & \wedge^2 \mathfrak{g}^* \arrow[d, hook] \\
\wedge^\bullet \mathfrak{g}^* \arrow[r, "\mathrm{d}"] & \wedge^\bullet \mathfrak{g}^*
\end{tikzcd}
$$

With this, the condition $d \circ d = 0$ is equivalently the super-Jacobi identity on [-, -], and the resulting differential graded super-commutative algebra is know as the *Chevalley-Eilenberg algebra* of $\mathfrak{g}$:

$$
\mathrm{CE}\big(\mathfrak{g},\, [\text{-},\text{-}]\big) \;:=\; \big(\wedge^\bullet\,\mathfrak{g}^*,\, \mathrm{d}\big)\,.
$$

This construction is a *fully faithful formal duality*

$$
\begin{tikzcd}
\mathrm{sLieAlg} \arrow[rr, hook', "\mathrm{CE}"] & & \mathrm{sDGCAlg}^{\mathrm{op}} \\
\big(\underbrace{V}_{\text{super-vector space}},\, [\text{-},\text{-}]\big) \arrow[rr, mapsto] & & \big(\wedge^\bullet\, V^*,\, \mathrm{d} = -[\text{-},\text{-}]^*\big)\,,
\end{tikzcd}
\tag{35}
$$

in that

- **(i)** for every super-vector space V a choice of such differential d on $\wedge^\bullet V^*$ uniquely comes from a super-Lie bracket [-, -] on V this way, and
- **(ii)** super-Lie homomorphisms $\phi : \mathfrak{g} \to \mathfrak{g}'$ are in bijection with super-dg-algebra homomorphisms $\phi^* : \mathrm{CE}(\mathfrak{g}') \to \mathrm{CE}(\mathfrak{g})$.

---

[1] Our use of different letters for the even and odd components of a super co-frame follows e.g. [CDF91]. Other authors write "E^α" for what we denote "Ψ^α", e.g. [BaSo23]. While it is of course part of the magic of supergravity that E^a and E^α/Ψ^α are unified into a single super-coframe field E, we find that for reading and interpreting formulas it is helpful to use different symbols for its even and odd components.

More concretely, given $(T_i)_{i=1}^n$ a linear basis for $\mathfrak{g}$ with corresponding structure constants $\big(f_{ij}^k \in \mathbb{R}\big)_{i,j,k=1}^n$, then the Chevalley-Eilenberg algebra is equivalently the graded-commutative polynomial algebra

$$\mathrm{CE}\big(\mathfrak{g}, [\text{-},\text{-}]\big) \;\simeq\; \big(\mathbb{R}\big[t^1, \cdots, t^1\big], \mathrm{d}\big)$$

on generators of degree $(1, \sigma_i)$ with corresponding structure constants for its differential:

	super Lie algebra	**super dgc-algebra**
Generators	$\big(\underbrace{T_i}_{\deg = (0,\sigma_i)}\big)_{i=1}^n$	$\big(\underbrace{t^i}_{\deg = (1,\sigma_i)}\big)_{i=1}^n$
Relations	$[T_i, T_j] \;=\; f_{ij}^k\, T_k$	$\mathrm{d}\, t^k \;=\; -\tfrac{1}{2} f_{ij}^k\, t^i t^j$

(36)

This dual perspective via the CE-algebra is most convenient for passing from super-Lie to *strong homotopy super-Lie algebras*, also known as *super Lie ∞-algebra* (subsuming Lie 2-algebras, Lie 3-algebras etc., hence infinitesimal "categorified symmetry" algebras), and also known as *super-L_∞ algebras*, for short: These are obtained simply by dropping the assumption that the CE-generators are in degree 1:

Namely for a *$\mathbb{Z}$-graded* super-vector space $V_\bullet$ (degree-wise finite-dimensional by our running assumption, hence "of finite type"), a sequence of higher arity super-skew-commutative brackets is dually a map from the degreewise dual $V^\vee$ (with $V_n^\vee := (V_n)^*$) to its graded Grassmann algebra:

$$\mathrm{d} \;:\; \wedge^1 V^\vee \longrightarrow \wedge^\bullet V^\vee$$

and the higher super-Jacobi identity is dually simply the statement that this map, extended uniquely as a super-graded derivation to all of $\wedge^\bullet V^\vee$, is a differential

$$\mathrm{d} \;:\; \wedge^\bullet V^\vee \longrightarrow \wedge^\bullet V^\vee$$

in that it squares to zero: $\mathrm{d} \circ \mathrm{d} \;=\; 0$. (This is the evident super-algebraic enhancement of the characterization of finite-type L_∞-algebras in [SSS09, §6.1].)

This way, super-L_∞ algebras are equivalently nothing but super dgc-algebras whose underlying super-graded algebra is of the form $\wedge^\bullet V^\vee$, with super L_∞-homomorphisms identified as homomorphisms of these super dgc-algebras going in the *opposite* direction ("pullback"):

$$\begin{tikzcd}
\mathrm{sLieAlg}_\infty \arrow[rr, "\mathrm{CE}"] & & \mathrm{sDGCAlg}^{\mathrm{op}} \\
\big(\underbrace{V}_{\substack{\text{graded super-}\\ \text{vector space}}}, [\text{-}], [\text{-},\text{-}], [\text{-},\text{-},\text{-}], \cdots\big) \arrow[rr, mapsto] & & \big(\wedge^\bullet\, V^\vee,\, \mathrm{d} = -[\text{-}]^* - [\text{-},\text{-}]^* - [\text{-},\text{-},\text{-}]^* - \cdots\big)
\end{tikzcd}$$

More concretely, by a choice of linear basis $(T_i)_{i \in I}$ for its underlying graded super vector space V, the CE-algebra of a super-L_∞-algebra may be written as:

$$\mathrm{CE}(\mathfrak{g}) \;\;\simeq\;\; \mathbb{R}_{\mathrm{d}}\big[(t^i)_{i \in I}\big] \big/ \big(\mathrm{d}\, t^i \;=\; P^i(\vec{t}\,)\big)_{i \in I} \tag{37}$$

where

- $\deg(t^i) \;=\; \deg(T_i) + (1, \mathrm{evn})$
- $\mathbb{R}_{\mathrm{d}}\big[(t^i)_{i \in I}\big]$ is the free differential $(\mathbb{Z} \times \mathbb{Z}_2)$-graded symmetric algebra on these generators and their differentials (7), whose product is subject only to the sign rule (5).
- $P^i(\vec{t}\,)$ are graded-symmetric polynomials in the generators,
- d is extended from generators to polynomials as a super-graded derivation of degree $(1, \mathrm{evn})$,
- the consistency condition is (only) that $\mathrm{d} \circ \mathrm{d} = 0$.

Accordingly, a homomorphism of super L_∞-algebras $f : \mathfrak{g} \to \mathfrak{h}$ with dual linear basis $(e^i)_{i \in I}$ and $(t^j)_{j \in J}$ is dually given by an algebra homomorphism $f^* : \mathrm{CE}(\mathfrak{h}) \to \mathrm{CE}(\mathfrak{g})$ pulling back the generators t^j to polynomials $f^*(t^j) \in \wedge^\bullet(\mathfrak{g}^\vee)$ in the generators e^i such that the differential is respected:

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "f"] & \mathfrak{h} \\
\mathrm{CE}(\mathfrak{g}) & \mathrm{CE}(\mathfrak{h}) \arrow[l, "f^*"'] \\
f^*(t^j) & t^j \arrow[l, mapsto]
\end{tikzcd}
\;,\qquad \text{such that} \qquad \underset{j \in J}{\forall}\; \mathrm{d} f^*(t^j) \;=\; f^*(\mathrm{d} t^j)\,. \tag{38}$$

As such, we may recognize the CE-algebras (37) as the "free differential algebras" of the supergravity literature

[vN83][CDF91, §III.6]:

Remark 1.2 (**CE-algebras are differential quotients of free differential graded-commutative algebras.**). The quotient notation in (37), following [FSS23, §4], is justified by thinking of

- $\mathbb{R}_{\mathrm{d}}\big[(e^i)_{i\in I}\big]$ as the (actual) *free differential* super-graded-commutative algebra, hence with each $\mathrm{d}e^i$ being a new generator subject to no relation (except super-graded commutativity),
- $\Big(\mathrm{d}\,e^i \;=\; P^i\big((e^j)_{j\in I}\big)\Big)_{i\in I}$ as a differential ideal,
- the quotient hence enforcing these equations on the previously free differential.

The base example in supergravity is the following Ex. 1.3:

Example 1.3 (**Supersymmetry algebras).** For $d \in \mathbb{N}$ and $\mathbf{N} \in \mathrm{Rep}_{\mathbb{R}}(\mathrm{Spin}(1,d))$ a real spin-representation equipped with a $\mathrm{Spin}(1,d)$-equivariant linear map

$$\big(\overline{(\text{-})}\Gamma(\text{-})\big) \;:\; \mathbf{N} \underset{\mathrm{sym}}{\otimes} \mathbf{N} \longrightarrow \mathbb{R}^{1,d}\,, \tag{39}$$

the corresponding super-translation super-Lie algebra $\mathbb{R}^{1,d\,|\,\mathbf{N}}$ is given by

$$\mathrm{CE}\big(\mathbb{R}^{1,d\,|\,\mathbf{N}}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\Big[\big(\underbrace{\psi^\alpha}_{\deg=(1,\mathrm{odd})}\big)_{\alpha=1}^{N},\;\; \big(\underbrace{e^a}_{\deg=(1,\mathrm{evn})}\big)_{a=0}^{d}\Big]\Big/\left(\begin{array}{ll}\mathrm{d}\,\psi^\alpha & = 0\\ \mathrm{d}\,e^a & = \big(\overline{\psi}\,\Gamma^a\,\psi\big)\end{array}\right). \tag{40}$$

We further discuss examples of this kind in §1.6 below.

The assumed $\mathrm{Spin}(1,d)$-equivariance implies that the ordinary Lorentz Lie algebra $\mathfrak{so}_{1,d}$ acts automrophically on $\mathbb{R}^{1,d\,|\,\mathbf{N}}$. The corresponding semidirect product super-Lie algebra is the *super-Poincaré Lie algebra*, the actual "supersymmetry algebra" in these dimensions:

$$\mathrm{CE}\Big(\mathbb{R}^{1,d\,|\,\mathbf{N}} \rtimes \mathfrak{so}_{1,d}\Big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\Big[\big(\underbrace{\psi^\alpha}_{\deg=(1,\mathrm{odd})}\big)_{\alpha=1}^{N},\;\big(\underbrace{e^a}_{\deg=(1,\mathrm{evn})}\big)_{a=0}^{d},\;\big(\underbrace{\omega^{ab}=-\omega^{ba}}_{\deg=(1,\mathrm{evn})}\big)_{a,b=0}^{d}\Big]\Big/\left(\begin{array}{ll}\mathrm{d}\,\psi^\alpha & = 0\\ \mathrm{d}\,e^a & = \big(\overline{\psi}\,\Gamma^a\,\psi\big) \;+\; \omega^a{}_b\, e^b\\ \mathrm{d}\,\omega^{ab} & = \omega^a{}_c\omega^{cb}\end{array}\right)$$

Important examples among higher Lie-algebras are the following:

Example 1.4 (**Real Whitehead L_∞-algebras.** cf. [FSS23, Prop. 5.11]**).** Given a topological space X —- which is (1.) connected, (2.) nilpotent, e.g., in that its fundamental group is abelian, and (3.) whose $\mathbb{R}$-cohomology $H^\bullet(X;\mathbb{R})$ is degreewise finite-dimensional — there is an L_∞-algebra, $\mathfrak{l}X$, characterized by the following two properties:

(i) The underlying graded vector space is the $\mathbb{R}$-rationalization of the homotopy groups $\pi_\bullet(X)$ of X:

$$\mathrm{CE}\big(\mathfrak{l}X\big) \;\;\simeq\;\; \Big(\wedge^\bullet\, \underbrace{\big(\pi_\bullet(X)\otimes_{\mathbb{Z}}\mathbb{R}\big)^\vee}_{\deg\,(\bullet,\mathrm{evn})},\; \mathrm{d}\Big)\,.$$

(ii) The cochain cohomology of its CE-algebra reproduces the ordinary cohomology of X:

$$H^\bullet\big(\mathrm{CE}(\mathfrak{l}X),\,\mathrm{d}\big) \;\;\simeq\;\; H^\bullet(X;\mathbb{R})\,.$$

In rational homotopy theory the dg-algebra $\mathrm{CE}\big(\mathfrak{l}X\big)$ is known (reviewed in [FSS23, §5]) as the *minimal Sullivan model* of the topological space X, retaining exactly the information of its rational homotopy type.

Example 1.5 (**Line Lie n-algebra.**). For $n \in \mathbb{N}$ and X an integral Eilenberg-MacLane space

$$X \underset{\mathrm{hmtp}}{\simeq} B^n U(1) \underset{\mathrm{hmtp}}{\simeq} K(\mathbb{Z}, n+1)$$

(classifying ordinary integral cohomology in degree $n+1$ and equivalently classifying complex line bundles (for $n=1$), line bundle gerbes (for $n=2$) and generally principal circle n-bundles) its real Whitehead L_∞-algebra (Ex. 1.4)

$$b^n\mathbb{R} \;:=\; \mathfrak{l}\big(B^n\mathrm{U}(1)\big)$$

is given by

$$\mathrm{CE}\big(\mathfrak{l}B^n\mathrm{U}(1)\big) \;\;\simeq\;\; \mathbb{R}_{\mathrm{d}}\big[\underbrace{\omega_{n+1}}_{\deg=(n+1,\mathrm{evn})}\big]\Big/\big(\mathrm{d}\,\omega_{n+1} \;=\; 0\big)\,.$$

This means that super-L_∞ homomorphisms (38) into these higher Lie algebras are equivalently $(n+1)$-cocycles:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\alpha_{n+1}"] & \mathfrak{l}\big(B^n \mathrm{U}(1)\big) \\ \alpha_{n+1} & \omega_{n+1} \arrow[l, mapsto] \end{tikzcd} \qquad \Leftrightarrow \qquad \left\{ \begin{array}{l} \alpha_{n+1} \,\in\, \mathrm{CE}(\mathfrak{g}) \\ \deg(\alpha_{n+1}) \,=\, (n+1,\, \mathrm{evn}), \\ \mathrm{d}\, \alpha_{n+1} \,=\, 0 \end{array} \right.$$

Example 1.6 (**Real Whitehead L_∞-algebra of the 4-sphere**, cf. [FSS23, Ex. 5.3])**.** The Whitehead L_∞-algebra (Ex. 1.4) of the 4-sphere, $\mathfrak{l}S^4$, is given by

$$\mathrm{CE}\big(\mathfrak{l}X\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\begin{bmatrix}\omega_4 \\ \omega_7\end{bmatrix} \Big/ \left(\begin{array}{lcl} \mathrm{d}\,\omega_4 & = & 0 \\ \mathrm{d}\,\omega_7 & = & \tfrac{1}{2}\,\omega_4\,\omega_4 \end{array} \right).$$

The Lie 7-algebra of Ex. 1.6 (a 6-fold "categorified symmetry" algebra) is noteworthy because it provides the correct coefficients for the duality-symmetric C-field super-flux densities in 11d supergravity (for more on this see [GSS24a]):

Example 1.7 (**4-Sphere valued super-flux of 11d SuGra** [FSS15, p 5][FSS17, Cor. 2.3][GSS24a, Ex. 2.30] following [Sa13, §2.5])**.** On the 11d super-Minkowski algebra $\mathbb{R}^{1,10\,|\,\mathbf{32}}$ (Ex. 1.3) the super-invariants

$$\left. \begin{array}{lcl} G_4^0 & := & \tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1 a_2}\,\psi\big)e^{a_1}e^{a_2} \\ G_7^0 & := & \tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1 \cdots a_5}\,\psi\big)e^{a_1}\cdots e^{a_5} \end{array} \right\} \;\in\; \mathrm{CE}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big)\,, \qquad \begin{array}{lcl} \mathrm{d}\,G_4^0 & = & 0 \\ \mathrm{d}\,G_7^0 & = & \tfrac{1}{2}G_4^0\,G_4^0\,, \end{array}$$

are identified with a homomorphism (38) of super-L_∞-algebras from super-Minkowski space to $\mathfrak{l}S^4$ (Ex. 1.6):

$$\begin{tikzcd} \mathbb{R}^{1,10\,|\,\mathbf{32}} \arrow[r, "{(G_4^0,\, G_7^0)}"] & \mathfrak{l}S^4 \\ G_4^0 & \omega_4\,, \arrow[l, mapsto] \\ G_7^0 & \omega_7\,. \arrow[l, mapsto] \end{tikzcd} \tag{41}$$

Example 1.8 (**Real Whitehead L_∞-algebra of twisted K-theory spectrum** cf. [FSS17, §4][FSS23, Ex. 5.7, 6.6])**.** The real Whitehead L_∞-algebras (Ex. 1.4) of the classifying spectra KU_0 and KU_1 for complex topological K-theory canonically homotopy-quotiented by $\mathrm{PU}(\mathcal{H}) \simeq B\mathrm{U}(1)$ have a generator in degree 3 together with generators in every even (every odd) degree, with differential of the form known from 3-twisted de Rham cohomology

$$\begin{array}{l} \mathrm{CE}\Big(\mathfrak{l}\big(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1)\big)\Big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\Big[\overbrace{h_3}^{\deg\,=\,(3,\mathrm{evn})},\, \big(\underbrace{f_{2k}}_{\deg\,=\,(2k,\mathrm{evn})}\big)_{k\in\mathbb{Z}} \Big] \Big/ \left(\begin{array}{lcl} \mathrm{d}\,h_3 & = & 0 \\ \mathrm{d}\,f_{2k+2} & = & h_3\,f_{2k} \end{array} \right) \\ \\ \mathrm{CE}\Big(\mathfrak{l}\big(\mathrm{KU}_1 /\!\!/ B\mathrm{U}(1)\big)\Big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\Big[\overbrace{h_3}^{\deg\,=\,(3,\mathrm{evn})},\, \big(\underbrace{f_{2k+1}}_{\deg\,=\,(2k+1,\mathrm{evn})}\big)_{k\in\mathbb{Z}} \Big] \Big/ \left(\begin{array}{lcl} \mathrm{d}\,h_3 & = & 0 \\ \mathrm{d}\,f_{2k+3} & = & h_3\,f_{2k+1} \end{array} \right). \end{array} \tag{42}$$

Since the general h_3 here is closed, these L_∞-algebras are canonically fibered over the line Lie 2-algebra (Ex. 1.5):

$$\begin{tikzcd} \mathfrak{l}\Big(\mathrm{KU}_i /\!\!/ B\mathrm{U}(1)\Big) \arrow[d, two heads] & h_3 \\ \mathfrak{l}\,B^2\mathrm{U}(1) & \omega_3 \arrow[u, mapsto] \end{tikzcd}$$

In rational homotopy theory this is the model for the fibration classifying 3-twisted complex-topological K-theory (cf. [FSS23, Ex. 3.4, Prop. 6.11, Prop. 10.1]).

1.5 The Ext/Cyc adjunction

With (extended) super-spacetimes understood — via their translational super-symmetry (Ex. 1.3) — as (higher) super-Lie algebras, fundamental constructions of super-Lie theory have (rational/infinitesimal) geometric significance. Notably the process of *central extension* (Def. 1.9) of super-L_∞ algebras by 2-cocycles corresponds in the super-geometric interpretation to the emergence of extra dimensions by 0-brane condensation ([CdAIP00, §2][FSS15, Rem. 3.11][HS18], see Ex. 1.20, 1.21 below).

One may hence ask for the (higher super) Lie-theoretic incarnation of the geometrically expected process of double [2] dimensional Kaluza-Klein reduction — and conversely: oxidation — along such extensions. Remarkably, this is given by the process of *cyclification* (passage to loop spaces homotopy-quotiented by loop rotation, as known from cyclic cohomology and from the geometric motivation for the Witten genus): On the rational-homotopy level of super-L_∞-algebras this is due to [FSS18a, §3][FSS18b, §2.6], recalled as Def. 1.11 and Prop. 1.13 below (for exposition see [Sc16, §4], for more in the context of U-duality see [SV23][SV23], for the topological globalization see [BMSS19, §2.2][SS24a] and for its application to double-field theory see [Al20][Al21]).

Definition 1.9 (**Central extension of super-L_∞ algebra by 2-cocycle).** Given $\mathfrak{g} \in \mathrm{sLieAlg}_\infty$ and a 2-cocycle

$$\omega_2 \,\in\, \mathrm{CE}(\mathfrak{g})\,, \;\; \deg(\omega_2) \,=\, (1,\mathrm{evn})\,, \;\; \mathrm{d}\,\omega_2 \,=\, 0 \qquad \Leftrightarrow \qquad \mathfrak{g} \xrightarrow{\;\;\omega_2\;\;} b\mathbb{R}$$

then the corresponding *central extension* $\widehat{\mathfrak{g}} \in \mathrm{sLieAlg}_{\mathbb{R}}$ is that super-Lie algebra whose CE-algebra is that of $\mathfrak{g}$ with one more generator e' adjoined whose differential is ω_2:

$$\mathrm{CE}\big(\widehat{\mathfrak{g}}\big) \;=\; \mathrm{CE}(\mathfrak{g})\big[\underbrace{\,e'\,}_{\mathrm{deg}=(1,\mathrm{evn})}\big]\big/\big(\mathrm{d}\,e' \;=\; \omega_2\big) \qquad \Leftrightarrow \qquad \begin{tikzcd} \widehat{\mathfrak{g}} \arrow[d, two heads, "p \,:=\, \mathrm{hofib}(\omega_2)"] & \\ \mathfrak{g} \arrow[r, "\omega_2"] & b\mathbb{R}\,. \end{tikzcd}$$

Remark 1.10 (**Basic and fiber forms on a centrally extended super-L_∞ algebra).** Given a central extension as in Def. 1.9, every element in its CE-algebra decomposes uniquely as the sum

$$\alpha \;=\; \alpha_{\mathrm{bas}} \,+\, e'\, p_*(\alpha) \tag{43}$$

of a *basic form* (not involving the generator e, hence in the image of p^*)

$$\alpha \,\in\, p^*\big(\mathrm{CE}(\mathfrak{g})\big)$$

and the product of the generator e' with the image of α under *fiber integration* p_*, which is a super-graded derivation of degree $(-1,\mathrm{evn})$:

$$p_* \;:\; \mathrm{CE}(\widehat{\mathfrak{g}}) \longrightarrow \mathrm{CE}(\mathfrak{g})\,.$$

The differential of a general element is given in this decomposition in terms of (the image under p^* of) the differential $\mathrm{d}_{\mathfrak{g}}$ by:

$$\begin{aligned} \mathrm{d}_{\widehat{\mathfrak{g}}}\big(\alpha_{\mathrm{bas}} \,+\, e'\,p_*\alpha\big) \;&=\; \mathrm{d}_{\widehat{\mathfrak{g}}}\,\alpha_{\mathrm{bas}} \,+\, \big(\mathrm{d}_{\widehat{\mathfrak{g}}}\,e'\big)\,p_*\alpha \,-\, e'\,\mathrm{d}_{\widehat{\mathfrak{g}}}\,p_*\alpha \\ &=\; \big(\mathrm{d}_{\mathfrak{g}}\alpha_{\mathrm{bas}} \,+\, \omega_2\,p_*\alpha\big) \,-\, e'\,\mathrm{d}_{\mathfrak{g}}p_*\alpha\,. \end{aligned} \tag{44}$$

Definition 1.11 (**Cyclification of super L_∞-algebras**, cf. [FSS17, Prop. 3.2][FSS18a, Def. 3.3]**).** Given $\mathfrak{h} \,\in\, \mathrm{sLieAlg}_\infty^{\mathrm{fin}}$ with

$$\mathrm{CE}(\mathfrak{h}) \;\simeq\; \mathbb{R}\big[(e^i)_{i\in I}\big]\big/\big(\mathrm{d}\,e^i = P^i(\vec{e}\,)\big)_{i\in I}\,,$$

its *cyclification* $\mathrm{cyc}(\mathfrak{h}) \,\in\, \mathrm{sLieAlf}_\infty$ is given by

$$\mathrm{CE}\big(\mathrm{cyc}(\mathfrak{h})\big) \;\;:=\;\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{l} \big(e^i\big)_{i\in I},\, \overbrace{\omega_2,}^{\mathrm{deg}\,=\,(2,\mathrm{evn})} \\ \big(\underbrace{\,\mathrm{s}e^i\,}_{\substack{\mathrm{deg}\;= \\ \mathrm{deg}(e^i)\,-\,(1,\mathrm{evn})}}\big)_{i\in I} \end{array}\right]\Big/\left(\begin{array}{ll} \mathrm{d}\,\omega_2 & =\; 0 \\ \mathrm{d}\,e^i & =\; \mathrm{d}_{\mathfrak{h}}\,e^i + \omega_2\,\mathrm{s}e^i \\ \mathrm{d}\,\mathrm{s}e^i & =\; -\mathrm{s}\big(\mathrm{d}_{\mathfrak{h}}\,e^i\big) \end{array}\right)\,, \tag{45}$$

where in the last line on the right the shift is understood as uniquely extended to a super-graded derivation of degree $(-1,\mathrm{evn})$:

$$\begin{array}{ccc} \mathrm{s} : \mathrm{CE}\big(\mathrm{cyc}(\mathfrak{h})\big) & \longrightarrow & \mathrm{CE}\big(\mathrm{cyc}(\mathfrak{h})\big) \\ \omega_2 & \longmapsto & 0\,, \\ e^i & \longmapsto & \mathrm{s}e^i\,, \\ \mathrm{s}e^i & \longmapsto & 0\,. \end{array}$$

To check that this is well-defined:

---

[2] The term "double dimensional reduction" originates with [DHIS87], referring to the fact that for Kaluza-Klein reduction of target spaces for p-branes both the target spacetime as well as the worldvolume of *wrapping* branes reduces in dimension – or, essentially equivalently, that also the corresponding flux densities decrease in degree upon integration over the fiber spaces. This is of cours ethe very mechanism that underlies the emergence of fields with enhanced/exceptional symmetry in lower dimensions.

Lemma 1.12 (**Differential and shift in cyclification).** *In Def. 1.11 the differential* d *and shift* s *square to zero and anti-commute with each other:*

$$\mathrm{d}\,\mathrm{d} \;=\; 0\,, \quad \mathrm{s}\,\mathrm{s} \;=\; 0\,, \quad \mathrm{s}\,\mathrm{d} + \mathrm{d}\,\mathrm{s} \;=\; 0\,. \tag{46}$$

Proof. First, that s squares to zero is immediate from the definition. Moreover, since we are dealing with (graded) derivations and their (graded) commutator, it is sufficient to check all these statements on generators.

The anticommutativity is thus seen as:

$$\begin{aligned}
\mathrm{s}\,\mathrm{d}\,\omega_2 + \mathrm{d}\,\mathrm{s}\,\omega_2 \;&=\; 0 + 0 \;=\; 0\,,\\
\mathrm{s}\,\mathrm{d}\,e^i \,+\, \mathrm{d}\,\mathrm{s}\,e^i \;&=\; \mathrm{s}\big(\mathrm{d}_{\mathfrak{h}} e^i + \omega_2\, \mathrm{s} e^i\big) - \mathrm{s}\,\mathrm{d}_{\mathfrak{h}} e^i \;=\; 0\,,\\
\mathrm{s}\,\mathrm{d}\,\mathrm{s} e^i + \mathrm{d}\,\mathrm{s}\,\mathrm{s} e^i \;&=\; -\mathrm{s}\,\mathrm{s}\,\mathrm{d}_{\mathfrak{h}} e^i \;=\; 0\,.
\end{aligned}$$

For nilpotency of d we first trivially have $\mathrm{d}\,\mathrm{d}\,\omega_2 \;=\; 0$, then

$$\begin{aligned}
\mathrm{d}\,\mathrm{d}\,e^i \;&=\; \mathrm{d}\big(\mathrm{d}_{\mathfrak{h}} e^i + \omega_2\, \mathrm{s} e^i\big)\\
&=\; \underbrace{\mathrm{d}_{\mathfrak{h}}\,\mathrm{d}_{\mathfrak{h}}}_{=0}\, e^i \;+\; \omega_2\, s\,\mathrm{d}_{\mathfrak{h}} e^i \;+\; \omega_2\, \mathrm{d}(\mathrm{s}\, e^i) + \omega_2\, \omega_2 \underbrace{\mathrm{s}\,\mathrm{s}}_{=0}\; e^i\\
&=\; \omega_2\big(\mathrm{s}\,\mathrm{d}_{\mathfrak{h}} - \mathrm{s}\,\mathrm{d}_{\mathfrak{h}}\big) e^i \;=\; 0\,.
\end{aligned}$$

From this, finally:

$$\mathrm{d}\,\mathrm{d}\,\mathrm{s}\,e^i \;\;=\; \mathrm{s}\,\mathrm{d}\,\mathrm{d}\; e^i \;=\; 0\,.$$

□

Proposition 1.13 (**The Ext/Cyc-adjunction** [FSS18a, Thm. 3.8])**.** *Given* $\mathfrak{g}, \mathfrak{h} \in \mathrm{sLieAlg}_\infty$ *with a 2-cocycle* [3] $c_1 \,\in\, \mathrm{CE}(\mathfrak{g})$*, there is a bijection between:*
(i) *maps into* $\mathfrak{h}$ *out of the central extension* $\widehat{\mathfrak{g}}$ *classified by the 2-cocycle* (Def. 1.9)*,*
(ii) *maps out of* $\mathfrak{g}$ *into the cyclification of* $\mathfrak{h}$ (Def. 1.11) *that preserve the 2-cocycle:*

$$\begin{tikzcd}
\Big\{\, \widehat{\mathfrak{g}} \xrightarrow{\qquad f \qquad} \mathfrak{h} \,\Big\} \arrow[r, shift left, "\overset{\textbf{reduction}}{\mathrm{rdc}_{c_1}}"] \arrow[r, phantom, "\sim"] & \Bigg\{\, \mathfrak{g} \arrow[l, shift left, "\underset{\textbf{oxidation}}{\mathrm{oxd}_{c_1}}"] \arrow[rr, "\widetilde{f}"] \arrow[dr, "c_1"'] & & \mathrm{cyc}(\mathfrak{h}) \,\Bigg\} \arrow[dl, "\omega_2"] \\
 & & b\mathbb{R} &
\end{tikzcd} \tag{47}$$

given by

$$\begin{tikzcd}
 & & & \mathfrak{g} \arrow[r, "\widetilde{f}"] & \mathrm{cyc}(\mathfrak{h}) \\
\widehat{\mathfrak{g}} \arrow[r, "f"] & \mathfrak{h} & & \alpha^i_{\mathrm{bas}} & e^i \arrow[l, mapsto] \\
\alpha^i_{\mathrm{bas}} + e'\, p_* \alpha^i & e^i \arrow[l, mapsto] & \leftrightsquigarrow & -p_*\alpha^i & \mathrm{s}e^i \arrow[l, mapsto] \\
 & & & c_1 & \omega_2\,. \arrow[l, mapsto]
\end{tikzcd} \tag{48}$$

Proof. The assignment (48) is manifestly a bijection of maps of underlying graded super-algebras. Hence it suffices to show that if one of these is moreover a homomorphism of dg-algebras (in that it preserves the differential) then so is its image. To that end, first note that when the map on the left of (48) is a dg-homomorphism then this implies that

$$\begin{aligned}
f^*(\mathrm{d}_{\mathfrak{h}} e^i) \;&=\; \mathrm{d}_{\widehat{\mathfrak{g}}} f^*(e^i) && \text{by homomorphy}\\
&=\; \mathrm{d}_{\widehat{\mathfrak{g}}}(\alpha^i_{\mathrm{bas}} + e'\, p_*\alpha^i) && \text{by (48)}\\
&=\; (\mathrm{d}_{\mathfrak{g}} \alpha^i_{\mathrm{bas}} + \omega_2\, p_*\alpha^i) - e'\, \mathrm{d}_{\mathfrak{g}}\, p_*\alpha^i && \text{by (44)}\,,
\end{aligned} \tag{49}$$

while the map on the right being an algebra homomorphism already implies (seen e.g. by expanding in generators):

$$\begin{aligned}
\widetilde{f}^*\big(\mathrm{d}_{\mathfrak{h}} e^i\big) \;&=\; \big(f^*(\mathrm{d}_{\mathfrak{h}} e^i)\big)_{\mathrm{bas}}\\
\widetilde{f}^*(\mathrm{s}\,\mathrm{d}_{\mathfrak{h}} e^i) \;&=\; -p_*\big(f^*(\mathrm{d}_{\mathfrak{h}} e^i)\big)\,.
\end{aligned} \tag{50}$$

If the map $\widetilde{f}$ on the right is moreover a dg-homomorphism then this implies that the map f on the left is so, as

---

[3] We usually give all algebra generators a subscript indicative of their degree. But here we write "c_1" since this is the standard symbol for the 1st Chern class of a line bundle, namely here for the Lie-theoretic line bundle $\widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$.

follows:

$$
\begin{aligned}
f^*(\mathrm{d}_{\mathfrak{h}} e^i) &= \big(f^*(\mathrm{d}_{\mathfrak{h}} e^i)\big)_{\mathrm{bas}} + e'\, p_* f^*(\mathrm{d}_{\mathfrak{h}} e^i) && \text{by (43)} \\
&= \widetilde{f}^*(\mathrm{d}_{\mathfrak{h}} e^i) - e'\, \widetilde{f}^*\big(\mathrm{s}\, \mathrm{d}_{\mathfrak{h}} e^i\big) && \text{by (50)} \\
&= \widetilde{f}^*(\mathrm{d}_{\mathrm{cyk}(\mathfrak{h})} e^i - \omega_2\, \mathrm{s}e^i) - e'\, \widetilde{f}^*\big(\mathrm{s}\, \mathrm{d}_{\mathrm{cyc}(\mathfrak{h})} e^i\big) && \text{by (45)} \\
&= \widetilde{f}^*(\mathrm{d}_{\mathrm{cyk}(\mathfrak{h})} e^i - \omega_2\, \mathrm{s}e^i) + e'\, \widetilde{f}^*\big(\mathrm{d}_{\mathrm{cyc}(\mathfrak{h})}\, \mathrm{s}e^i\big) && \text{by (46)} \\
&= \mathrm{d}_{\mathfrak{g}} \widetilde{f}^*(e^i) - \widetilde{f}^*(\omega_2\, \mathrm{s}e^i) + e'\, \mathrm{d}_{\mathfrak{g}} \widetilde{f}^*\big(\mathrm{s}e^i\big) && \text{by homomorphy} \\
&= \mathrm{d}_{\mathfrak{g}} \alpha^i_{\mathrm{bas}} + \omega_2\, p_* \alpha^i - e'\, \mathrm{d}_{\mathfrak{g}} p_* \alpha^i && \text{by (48)} \\
&= \mathrm{d}_{\widehat{\mathfrak{g}}}\big(\alpha^i_{\mathrm{bas}} + e'\, p_* \alpha_i\big) && \text{by (44)} \\
&= \mathrm{d}_{\widehat{\mathfrak{g}}}\, f^*(e^i) && \text{by (48).}
\end{aligned}
\tag{51}
$$

Conversely, when f on the left of (48) is a dg-homomorphism, then it implies respect for the differential on the right, because:

$$
\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\widetilde{f}"] & & \mathrm{cyc}(\mathfrak{h}) \\
\alpha^i_{\mathrm{bas}} \arrow[d, mapsto, "\mathrm{d}"] & & e^i \arrow[ll, mapsto] \arrow[dd, mapsto, "\mathrm{d}"] \\
\mathrm{d}_{\mathfrak{g}} \alpha^i_{\mathrm{bas}} \arrow[dr, equal, "(49)"'] & & \\
& \big(f^*(\mathrm{d}_{\mathfrak{h}} e^i)\big)_{\mathrm{bas}} - \omega_2\, p_* \alpha^i & \mathrm{d}_{\mathfrak{h}} e^i + \omega_2\, \mathrm{s}e^i \arrow[l, mapsto, "(50)"'] \\
-p_* \alpha^i \arrow[d, mapsto, "\mathrm{d}"] & & \mathrm{s}e^i \arrow[ll, mapsto] \arrow[dd, mapsto, "\mathrm{d}"] \\
-\mathrm{d}_{\mathfrak{g}}\, p_* \alpha^i \arrow[dr, equal, "(49)"'] & & \\
& p_* f^*(\mathrm{d}_{\mathfrak{h}} e^i) & -\mathrm{s}\big(\mathrm{d}_{\mathfrak{h}} e^i\big) \arrow[l, mapsto, "(50)"'] \\
\omega_2 \arrow[d, mapsto, "\mathrm{d}"] & & \omega_2 \arrow[ll, mapsto] \arrow[d, mapsto, "\mathrm{d}"] \\
0 & & 0\,. \arrow[ll, mapsto]
\end{tikzcd}
$$

□

Example 1.14 (**Cyclification of the 4-Sphere** [FSS17, Ex. 3.3])**.** The cyclification (Def. 1.11) of the real Whitehead L_∞-algebra of the 4-sphere (Ex. 1.4) is:

$$
\mathrm{CE}\big(\mathrm{cyc}(\mathfrak{l}S^4)\big) \;\simeq\; \mathbb{R}_{\mathrm{d}} \left[\begin{array}{c} \omega_2 \\ \omega_4 \\ \mathrm{s}\omega_4 \\ \omega_7 \\ \mathrm{s}\omega_7 \end{array} \right] \Big/ \left(\begin{array}{rl} \mathrm{d}\,\omega_2 &= 0 \\ \mathrm{d}\,\omega_4 &= \omega_2\, \mathrm{s}\omega_4 \\ \mathrm{d}\,\mathrm{s}\omega_4 &= 0 \\ \mathrm{d}\,\omega_7 &= \tfrac{1}{2}\omega_4\, \omega_4 + \omega_2\, \mathrm{s}\omega_4 \\ \mathrm{d}\,\mathrm{s}\omega_7 &= -\omega_4\, \mathrm{s}\omega_4 \end{array} \right)
$$

Example 1.15 (**Cyclification of twisted K-spectra**)**.** The cyclifications (Def. 1.11) of the real Whitehead L_∞-algebra of the twisted K-theory spectra (Ex. 1.8) are identified by isomorphisms (38) (several, we indicate two):

$$\begin{tikzcd}[row sep=large]
\mathrm{CE}\Big(\mathrm{cyc}\,\mathfrak{l}\big(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1)\big)\Big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} \omega_2 \\ h_3 \\ \mathrm{s}h_3 \\ (f_{2k})_{k\in\mathbb{Z}} \\ (\mathrm{s}f_{2k})_{k\in\mathbb{Z}} \end{array}\right] \Big/ \left(\begin{array}{rcl} \mathrm{d}\,\omega_2 & = & 0 \\ \mathrm{d}\,h_3 & = & \omega_2\,\mathrm{s}h_3 \\ \mathrm{d}\,\mathrm{s}h_3 & = & 0 \\ \mathrm{d}\,f_{2k+2} & = & h_3\,f_{2k} + \omega_2\,\mathrm{s}f_{2k+2} \\ \mathrm{d}\,\mathrm{s}f_{2k+2} & = & -(\mathrm{s}h_3)f_{2k} + h_3\,\mathrm{s}f_{2k} \end{array}\right)
\\
\mathrm{CE}\Big(\mathrm{cyc}\,\mathfrak{l}\big(\mathrm{KU}_1 /\!\!/ B\mathrm{U}(1)\big)\Big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} \omega_2 \\ h_3 \\ \mathrm{s}h_3 \\ (f_{2k+1})_{k\in\mathbb{Z}} \\ (\mathrm{s}f_{2k+1})_{k\in\mathbb{Z}} \end{array}\right] \Big/ \left(\begin{array}{rcl} \mathrm{d}\,\omega_2 & = & 0 \\ \mathrm{d}\,h_3 & = & \omega_2\,\mathrm{s}h_3 \\ \mathrm{d}\,\mathrm{s}h_3 & = & 0 \\ \mathrm{d}\,f_{2k+1} & = & h_3\,f_{2k-1} + \omega_2\,\mathrm{s}f_{2k+1} \\ \mathrm{d}\,\mathrm{s}f_{2k+3} & = & -(\mathrm{s}h_3)f_{2k+1} + h_3\,\mathrm{s}f_{2k+1} \end{array}\right)
\arrow[u, shift right=8ex, "{\begin{array}{ccccc} \omega_2 & \mathrm{s}h_3 & h_3 & f_{2k} & \mathrm{s}f_{2k+2} \\ \mapsup & \mapsup & \mapsup & \mapsup & \mapsup \\ -\mathrm{s}h_3 & -\omega_2 & h_3 & \mathrm{s}f_{2k+1} & f_{2k+1} \end{array}}", "\wr"']
\arrow[u, shift left=8ex, "{\begin{array}{ccccc} \omega_2 & \mathrm{s}h_3 & h_3 & f_{2k} & \mathrm{s}f_{2k+2} \\ \mapsup & \mapsup & \mapsup & \mapsup & \mapsup \\ \mathrm{s}h_3 & \omega_2 & h_3 & \mathrm{s}f_{2k+1} & -f_{2k+1} \end{array}}"', "\wr"]
\end{tikzcd} . \tag{52}$$

Remark 1.16 (**Towards T-duality.**). Since the isomorphism (52) swaps the Chern class ω_2 with the dimensional reduction $\mathrm{s}h_3$ of the 3-form, it does not preserve the fibering (47) of the cyclification over $b\mathbb{R} \simeq \mathfrak{l}\big(B\mathrm{U}(1)\big)$. This effect of "swapping the spacetime extension" (from that classified by ω_2 to that classified by $\mathrm{s}h_3$) reflects the phenomenon of superspace T-duality in §1.7.

Example 1.17 (**Double cyclification**).

$$\mathrm{CE}\big(\mathrm{cyc}^2(\mathfrak{h})\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} \dot\omega_2, \\ \omega_2, \\ \dot{\mathrm{s}}\omega_2, \\ (e_i)_{i\in I}, \\ (\dot{\mathrm{s}}e^i)_{i\in I} \\ (\mathrm{s}e^i)_{i\in I}, \\ (\dot{\mathrm{s}}\mathrm{s}e^i)_{i\in I} \end{array}\right] \Big/ \left(\begin{array}{rcc} \mathrm{d}\,\dot\omega_2 & = & 0 \\ \mathrm{d}\,\omega_2 & = & \dot\omega_2\,\dot{\mathrm{s}}\omega_2 \\ \mathrm{d}\,\dot{\mathrm{s}}\omega_2 & = & 0 \\ \mathrm{d}\,e^i & = & \mathrm{d}_{\mathfrak{h}}e^i + \omega_2\,\mathrm{s}e^i + \dot\omega_2\,\dot{\mathrm{s}}e^i \\ \mathrm{d}\,\dot{\mathrm{s}}e^i & = & -\dot{\mathrm{s}}\mathrm{d}_{\mathfrak{h}}e^i - (\dot{\mathrm{s}}\omega_2)(\mathrm{s}e^i) - \omega_2\,\dot{\mathrm{s}}\mathrm{s}e^i \\ \mathrm{d}\,\mathrm{s}e^i & = & -\mathrm{s}\mathrm{d}_{\mathfrak{h}}e^i + \dot\omega_2\,\dot{\mathrm{s}}\mathrm{s}e^i \\ \mathrm{d}\,\dot{\mathrm{s}}\mathrm{s}e^i & = & \dot{\mathrm{s}}\mathrm{s}\,\mathrm{d}_{\mathfrak{h}}\,e^i \end{array}\right)$$

Discarding $\dot{\mathrm{s}}\omega_2$ in these formulas gives the 2-toroidification:

$$\mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} \dot\omega_2, \\ \omega_2, \\ (e_i)_{i\in I}, \\ (\dot{\mathrm{s}}e^i)_{i\in I} \\ (\mathrm{s}e^i)_{i\in I}, \\ (\dot{\mathrm{s}}\mathrm{s}e^i)_{i\in I} \end{array}\right] \Big/ \left(\begin{array}{rcc} \mathrm{d}\,\dot\omega_2 & = & 0 \\ \mathrm{d}\,\omega_2 & = & 0 \\ \mathrm{d}\,e^i & = & \mathrm{d}_{\mathfrak{h}}e^i + \omega_2\,\mathrm{s}e^i + \dot\omega_2\,\dot{\mathrm{s}}e^i \\ \mathrm{d}\,\dot{\mathrm{s}}e^i & = & -\dot{\mathrm{s}}\mathrm{d}_{\mathfrak{h}}e^i - \omega_2\,\dot{\mathrm{s}}\mathrm{s}e^i \\ \mathrm{d}\,\mathrm{s}e^i & = & -\mathrm{s}\mathrm{d}_{\mathfrak{h}}e^i + \dot\omega_2\,\dot{\mathrm{s}}\mathrm{s}e^i \\ \mathrm{d}\,\dot{\mathrm{s}}\mathrm{s}e^i & = & \dot{\mathrm{s}}\mathrm{s}\,\mathrm{d}_{\mathfrak{h}}\,e^i \end{array}\right)$$

This has an automorphism given by swapping the two fiber directions.

Example 1.18 (Double cyclification of the 4-sphere).

$$
\mathrm{CE}\big(\mathrm{cyc}^2(\mathfrak{l}S^4)\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}
\left[
\begin{array}{c}
\dot{\omega}_2 \\
\omega_2 \\
\dot{\mathrm{s}}\omega_2 \\
\omega_4 \\
\dot{\mathrm{s}}\omega_4 \\
\mathrm{s}\omega_4 \\
\dot{\mathrm{s}}\mathrm{s}\omega_4 \\
\omega_7 \\
\dot{\mathrm{s}}\omega_7 \\
\mathrm{s}\omega_7 \\
\dot{\mathrm{s}}\mathrm{s}\omega_7
\end{array}
\right]
\Big/
\left(
\begin{array}{rcl}
\mathrm{d}\,\dot{\omega}_2 &=& 0 \\
\mathrm{d}\,\omega_2 &=& \dot{\omega}_2\,\dot{\mathrm{s}}\omega_2 \\
\mathrm{d}\,\dot{\mathrm{s}}\omega_2 &=& 0 \\
\mathrm{d}\,\omega_4 &=& \omega_2\,\mathrm{s}\omega_4 + \dot{\omega}_2\,\dot{\mathrm{s}}\omega_4 \\
\mathrm{d}\,\dot{\mathrm{s}}\omega_4 &=& -(\dot{\mathrm{s}}\omega_2)(s\omega_4) - \omega_2\,\dot{\mathrm{s}}\mathrm{s}\omega_4 \\
\mathrm{d}\,\mathrm{s}\omega_4 &=& \dot{\omega}_2\,\dot{\mathrm{s}}\mathrm{s}\omega_4 \\
\mathrm{d}\,\dot{\mathrm{s}}\mathrm{s}\omega_4 &=& 0 \\
\mathrm{d}\,\omega_7 &=& \tfrac{1}{2}\omega_4\omega_4 + \omega_2\,\mathrm{s}\omega_7 + \dot{\omega}_2\,\dot{\mathrm{s}}\omega_7 \\
\mathrm{d}\,\dot{\mathrm{s}}\omega_7 &=& -\omega_4\,\dot{\mathrm{s}}\omega_4 - (\dot{\mathrm{s}}\omega_2)(\mathrm{s}\omega_7) - \omega_2\,\dot{\mathrm{s}}\mathrm{s}\omega_7 \\
\mathrm{d}\,\mathrm{s}\omega_7 &=& -\omega_4\,\mathrm{s}\omega_4 + \dot{\omega}_2\,\dot{\mathrm{s}}\mathrm{s}\omega_7 \\
\mathrm{d}\,\dot{\mathrm{s}}\mathrm{s}\omega_7 &=& (\dot{\mathrm{s}}\omega_4)(\mathrm{s}\omega_4) + \omega_4\,\dot{\mathrm{s}}\mathrm{s}\omega_4
\end{array}
\right)
$$

Discarding $\dot{\mathrm{s}}\omega_2$ in these formulas gives the 2-toroidification of the 4-sphere:

$$
\mathbb{R}_{\mathrm{d}}
\left[
\begin{array}{c}
\dot{\omega}_2 \\
\omega_2 \\
\omega_4 \\
\dot{\mathrm{s}}\omega_4 \\
\mathrm{s}\omega_4 \\
\dot{\mathrm{s}}\mathrm{s}\omega_4 \\
\omega_7 \\
\dot{\mathrm{s}}\omega_7 \\
\mathrm{s}\omega_7 \\
\dot{\mathrm{s}}\mathrm{s}\omega_7
\end{array}
\right]
\Big/
\left(
\begin{array}{rcl}
\mathrm{d}\,\dot{\omega}_2 &=& 0 \\
\mathrm{d}\,\omega_2 &=& 0 \\
\mathrm{d}\,\omega_4 &=& \omega_2\,\mathrm{s}\omega_4 + \dot{\omega}_2\,\dot{\mathrm{s}}\omega_4 \\
\mathrm{d}\,\dot{\mathrm{s}}\omega_4 &=& -\omega_2\,\dot{\mathrm{s}}\mathrm{s}\omega_4 \\
\mathrm{d}\,\mathrm{s}\omega_4 &=& \dot{\omega}_2\,\dot{\mathrm{s}}\mathrm{s}\omega_4 \\
\mathrm{d}\,\dot{\mathrm{s}}\mathrm{s}\omega_4 &=& 0 \\
\mathrm{d}\,\omega_7 &=& \tfrac{1}{2}\omega_4\omega_4 + \omega_2\,\mathrm{s}\omega_7 + \dot{\omega}_2\,\dot{\mathrm{s}}\omega_7 \\
\mathrm{d}\,\dot{\mathrm{s}}\omega_7 &=& -\omega_4\,\dot{\mathrm{s}}\omega_4 - \omega_2\,\dot{\mathrm{s}}\mathrm{s}\omega_7 \\
\mathrm{d}\,\mathrm{s}\omega_7 &=& -\omega_4\,\mathrm{s}\omega_4 + \dot{\omega}_2\,\dot{\mathrm{s}}\mathrm{s}\omega_7 \\
\mathrm{d}\,\dot{\mathrm{s}}\mathrm{s}\omega_7 &=& (\dot{\mathrm{s}}\omega_4)(\mathrm{s}\omega_4) + \omega_4\,\dot{\mathrm{s}}\mathrm{s}\omega_4
\end{array}
\right)
$$

1.6 Super-space-times

We discuss the translational supersymmetry algebras (Ex. 1.3) for $D = 10$ and of "type II", in terms of the algebraic data provided by the $D = 11$ supersymmetry algebra. This is immediate for type IIA, but for type IIB it requires a little bit of finesse. With that in hand, though, the superspace T-duality in §1.7 flows very naturally.

Dimensional reduction of 11d super Minkowski spacetime. Consider the projection operators

$$\left.\begin{array}{rcl} P & := & \tfrac{1}{2}\big(1 + \Gamma_{10}\big) \\ \overline{P} & := & \tfrac{1}{2}\big(1 - \Gamma_{10}\big) \end{array}\right\} \in \mathrm{End}_{\mathbb{R}}(\mathbf{32}) \tag{53}$$

satisfying the following immediate but consequential relations:

$$\begin{array}{rcl} P\,P & = & P, \\ \overline{P}\,\overline{P} & = & \overline{P}, \\ P\,\overline{P} & = & 0, \\ \overline{P}\,P & = & 0, \end{array} \qquad \begin{array}{rclcl} \Gamma_{\leq 9}\,P & = & \overline{P}\,\Gamma_{\leq 9}, & & \\ \Gamma_{\leq 9}\,\overline{P} & = & P\,\Gamma_{\leq 9}, & & \\ \Gamma_{10}\,P & = & P\,\Gamma_{10} & = & +P, \\ \Gamma_{10}\,\overline{P} & = & \overline{P}\,\Gamma_{10} & = & -\overline{P}, \end{array} \qquad P + \overline{P} = \mathrm{id}, \qquad \begin{array}{rcl} \overline{P\psi} & = & \overline{\psi}\,\overline{P}, \\ \overline{\overline{P}\psi} & = & \overline{\psi}\,P\,. \end{array} \tag{54}$$

Canonically identifying actions of spin subgroups on **32**

$$\mathrm{Spin}(1,8) \hookrightarrow \mathrm{Spin}(1,9) \hookrightarrow \mathrm{Spin}(1,10)$$

by restriction of the Clifford algebra to products of its first $1 + d$ generators $\Gamma_0, \Gamma_1, \cdots \Gamma_d$, the projectors (53) carve out two $\mathrm{Spin}(1,9)$-representations from the $\mathrm{Spin}(1,10)$-rep **32**:

$$\begin{array}{rcrcl} \mathbf{16} & := & P(\mathbf{32}) & \in & \mathrm{Rep}_{\mathbb{R}}\big(\mathrm{Spin}(1,9)\big) \\ \overline{\mathbf{16}} & := & \overline{P}(\mathbf{32}) & \in & \mathrm{Rep}_{\mathbb{R}}\big(\mathrm{Spin}(1,9)\big)\,, \end{array} \tag{55}$$

and hence we have a branching of representations of this form:

$$\begin{array}{ccc} \mathrm{Spin}(1,10) & \hookleftarrow & \mathrm{Spin}(1,9) \\ \mathbf{32} & \longrightarrow & \mathbf{16} \oplus \overline{\mathbf{16}} \\ \psi & \longmapsto & \underbrace{P\psi}_{\psi_1} + \underbrace{\overline{P}\psi}_{\psi_2}\,. \end{array} \tag{56}$$

Under this branching and decomposition ($\psi = \psi_1 + \psi_2 = P(\psi_1) + \overline{P}(\psi_2)$), the spinor pairing $\big(\overline{(-)}(-)\big)$ on **32** translates into pairings on $\mathbf{16} = P(\mathbf{32})$ and on $\overline{\mathbf{16}} = \overline{P}(\mathbf{32})$, by evaluating on pairs of spinors belonging in each of the projected subspaces respectively. In terms of the two projected representations the vector-valued spinor pairing $\big(\overline{(-)}\Gamma(-)\big)$ decomposes as follows:

Lemma 1.19 (Decomposed vectorial spinor pairing).

$$\begin{array}{rcll} \big(\overline{\psi}\,\Gamma_a\,\phi\big) & = & \big(\overline{\psi}_1\,\Gamma_a\,\phi_1\big) + \big(\overline{\psi}_2\,\Gamma_a\,\phi_2\big)\,, & \text{for } a \neq 10, \\ \big(\overline{\psi}\,\Gamma_{10}\,\phi\big) & = & \big(\overline{\psi}_2\,\phi_1\big) - \big(\overline{\psi}_1\,\phi_2\big). & \end{array} \tag{57}$$

Proof. The first line in (57) follows since the mixed terms vanish due to the decomposition (56) and using the relations (54):

$$\begin{array}{rcl} \big(\overline{\psi_1}\,\Gamma_a\,\phi_2\big) & = & \big(\overline{P\psi_1}\,\Gamma_a\,\overline{P}\phi_2\big) \\ & = & \big(\overline{\psi_1}\,\overline{P}\,\Gamma_a\,\overline{P}\phi_2\big) \\ & = & \big(\overline{\psi_1}\,\Gamma_a\,P\overline{P}\phi_2\big) \\ & = & 0\,, \end{array} \qquad \begin{array}{rcl} \big(\overline{\psi_2}\,\Gamma_a\,\phi_1\big) & = & \big(\overline{\overline{P}\psi_2}\,\Gamma_a\,P\phi_1\big) \\ & = & \big(\overline{\psi_1}\,P\,\Gamma_a\,P\phi_2\big) \\ & = & \big(\overline{\psi_1}\,\Gamma_a\,\overline{P}P\phi_2\big) \\ & = & 0\,, \end{array} \qquad \text{for } a \neq 10.$$

Similarly but complementarily, for the second line these relations give:

$$\begin{array}{rcl} \big(\overline{\psi}\,\Gamma_{10}\,\phi\big) & = & \big(\overline{\psi}_1\,\Gamma_{10}\,\phi_2\big) + \big(\overline{\psi}_2\,\Gamma_{10}\,\phi_1\big) + \big(\overline{\psi}_1\,\Gamma_{10}\,\phi_1\big) + \big(\overline{\psi}_2\,\Gamma_{10}\,\phi_2\big) \\ & = & \big(\overline{P\psi_1}\,\Gamma_{10}\,\overline{P}\phi_2\big) + \big(\overline{\overline{P}\psi_2}\,\Gamma_{10}\,P\phi_1\big) + \big(\overline{P\psi_1}\,\Gamma_{10}\,P\phi_1\big) + \big(\overline{\overline{P}\psi_2}\,\Gamma_{10}\,\overline{P}\phi_2\big) \\ & = & \big(\overline{\psi_1}\,\overline{P}\Gamma_{10}\,\overline{P}\phi_2\big) + \big(\overline{\psi_2}\,P\Gamma_{10}\,P\phi_1\big) + \big(\overline{\psi_1}\,\overline{P}\Gamma_{10}\,P\phi_1\big) + \big(\overline{\psi_2}\,P\Gamma_{10}\,\overline{P}\phi_2\big) \\ & = & -\big(\overline{\psi_1}\,\overline{P}\,\overline{P}\phi_2\big) + \big(\overline{\psi_2}\,PP\phi_1\big) + \big(\overline{\psi_1}\,\overline{P}P\phi_1\big) + \big(\overline{\psi_2}\,P\overline{P}\phi_2\big) \\ & = & -\big(\overline{\psi_1}\,\phi_2\big) + \big(\overline{\psi_2}\,\phi_1\big). \end{array}$$

□

This gives:

Example 1.20 **(The M/IIA super-space extension).** We have a $\mathrm{Spin}(1,9)$-equivariant isomorphism

$$\mathrm{CE}\big(\mathbb{R}^{1,10\,|\,\mathbf{32}}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}} \left[\begin{array}{c} \big(\psi_1^\alpha\big)_{\alpha=1}^{16}, \\ \big(\psi_2^\alpha\big)_{\alpha=1}^{16}, \\ \big(e^a\big)_{a=0}^{9}, \\ e^{10} \end{array} \right] \Big/ \left(\begin{array}{rcll} \mathrm{d}\,\psi_1 &=& 0 & \\ \mathrm{d}\,\psi_2 &=& 0 & \\ \mathrm{d}\,e^a &=& \big(\overline{\psi}_1\,\Gamma_a\,\psi_1\big) + \big(\overline{\psi}_2\,\Gamma_a\,\psi_2\big) & \text{for } a \neq 10 \\ \mathrm{d}\,e^{10} &=& \underbrace{\big(\overline{\psi}_2\,\psi_1\big) - \big(\overline{\psi}_1\,\psi_2\big)}_{\big(\overline{\psi}\,\Gamma_{10}\,\psi\big)} & \end{array} \right),$$

which exhibits $\mathbb{R}^{1,10\,|\,\mathbf{32}}$ as a central extension (Def. 1.9) of $\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}$ whose classifying cocycle we denote [4] c_1^M:

$$\begin{tikzcd} \underset{\substack{\text{D = 11} \\ \text{super-Minkowski} \\ \text{spacetime}}}{\mathbb{R}^{1,10|\mathbf{32}}} \arrow[r, two heads, "\text{hofib}", "\text{extension of}"'] & \underset{\substack{\text{D = 10 type IIA} \\ \text{super-Minkowski} \\ \text{spacetime}}}{\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}} \arrow[r, "c_1^M \,:=\, (\overline{\psi}\,\Gamma_{10}\,\psi)", "\text{classified by first Chern class}"'] & b\mathbb{R}\,. \end{tikzcd} \tag{58}$$

Reducing one dimension further, the two $\mathrm{Spin}(1,9)$-reps (55) in turn become isomorphic when restricted to $\mathrm{Spin}(1,8)$-representations, the isomorphism given by acting with Γ_9:

$$\begin{tikzcd} \mathbf{16} \arrow[r, phantom, "\equiv"] & P(\mathbf{32}) \arrow[loop above, "\mathrm{Spin}(1,8)"] \arrow[r, "\Gamma_9", "\sim"'] \arrow[d, "\Gamma_{ab}"'] & \overline{P}(\mathbf{32}) \arrow[loop above, "\mathrm{Spin}(1,8)"] \arrow[d, "\Gamma_{ab}"'] \arrow[r, phantom, "\equiv"] & \overline{\mathbf{16}} \\ & P(\mathbf{32}) \arrow[r, "\Gamma_9", "\sim"'] & \overline{P}(\mathbf{32}) & \end{tikzcd} \qquad a, b \leq 8. \tag{59}$$

This gives:

Example 1.21 **(The IIA/9D super-space extension).** We have a $\mathrm{Spin}(1,8)$-equivariant isomorphism

$$\mathrm{CE}\big(\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}} \left[\begin{array}{c} (\psi_1^\alpha)_{\alpha=1}^{16}, \\ (\psi_2^\alpha)_{\alpha=1}^{16}, \\ (e^a)_{a=0}^{8}, \\ e^{9} \end{array} \right] \Big/ \left(\begin{array}{rcll} \mathrm{d}\,\psi_1 &=& 0 & \\ \mathrm{d}\,\psi_2 &=& 0 & \\ \mathrm{d}\,e^a &=& \big(\overline{\psi}_1\,\Gamma^a\,\psi_1\big) + \big(\overline{\psi}_2\,\Gamma^a\,\psi_2\big) & \text{for } a \neq 9 \\ \mathrm{d}\,e^{9} &=& \underbrace{\big(\overline{\psi}_1\,\Gamma^9\,\psi_1\big) + \big(\overline{\psi}_2\,\Gamma^9\,\psi_2\big)}_{\big(\overline{\psi}\,\Gamma^{9}\,\psi\big)} & \end{array} \right)$$

which exhibits $\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}$ as a central extension (Def. 1.9) of $\mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}}$ by a 2-cocycle to be denoted c_1^A (cf. ftn. 4):

$$\begin{tikzcd} \underset{\substack{\text{D = 10 type IIA} \\ \text{super-Minkowski} \\ \text{spacetime}}}{\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}} \arrow[r, two heads, "\text{hofib}", "\text{extension of}"'] & \underset{\substack{\text{D = 9 type II} \\ \text{super-Minkowski} \\ \text{spacetime}}}{\mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}}} \arrow[r, "c_1^A \,:=\, \big(\overline{\psi}\,\Gamma_9\,\psi\big)", "\text{classified by first Chern class}"'] & b\mathbb{R}\,. \end{tikzcd} \tag{60}$$

The type IIB super-spacetime. We need a presentation of the type IIB super-spacetime analogous to the IIA case above, namely expressed in terms of the 11d spinors in $\mathbf{32}$ and their spinor pairing $\big(\overline{(-)}(-)\big) : \mathbf{32} \otimes \mathbf{32} \to \mathbb{R}$. However, since along $\mathfrak{so}_{1,9} \hookrightarrow \mathfrak{so}_{1,10}$ this representation branches as $\mathbf{32} \mapsto \mathbf{16} \oplus \overline{\mathbf{16}}$ (56) we have to re-express the given $\overline{\mathbf{16}}$ as a $\mathbf{16}$.

Observe that this may be achieved by "compensating with a group automorphism": The same diagram (59) which shows how $P(\mathbf{32})$ and $\overline{P}(\mathbf{32})$ are isomorphic as $\mathrm{Spin}(1,8)$-representations also shows which transformation on the Lie algebra generators occurs when comparing them as $\mathrm{Spin}(1,9)$-representations, namely the Lorentz-

---

[4] Read c_1^M as: The first Chern class classifying 11d super-spacetime as a super-Lie line bundle over 10d type IIA super-spacetime.

generators with an index=9 pick up a minus sign:

$$\begin{tikzcd}
\mathbf{16} & \equiv & P(\mathbf{32}) \arrow[loop above, "\mathrm{Spin}(1{,}8)"] \arrow[rr, "\Gamma_9", "\sim"'] \arrow[d, shift right, "\Gamma_{ab}"'] \arrow[d, shift left, "\Gamma_{a9}"] & & \overline{P}(\mathbf{32}) \arrow[loop above, "\mathrm{Spin}(1{,}8)"] \arrow[d, shift right, "\Gamma_{ab}"'] \arrow[d, shift left, "-\Gamma_{a9}"] & \equiv & \overline{\mathbf{16}} \\
& & P(\mathbf{32}) \arrow[rr, "\Gamma_9", "\sim"'] & & \overline{P}(\mathbf{32})\,.
\end{tikzcd}$$

$a, b \leq 8$ (61)

This means that the **16** of $\mathrm{Spin}(1, 9)$ is the pullback of $\overline{\mathbf{16}}$ along the group homomorphism which on Lie algebras is given by this sign change (59):

$$\begin{tikzcd}
\mathfrak{so}_{1,9} \arrow[r, "\sim"] \arrow[rr, bend left, "\mathbf{16}"] & \mathfrak{so}_{1,9} \arrow[r, "\overline{\mathbf{16}}"] & \mathfrak{gl}_{16} \\
J_{ab<9} \arrow[r, mapsto] & J_{ab} \arrow[r, mapsto] & \frac{1}{2}\Gamma_{ab}|_{\overline{P}(\mathbf{32})} \\
J_{a9} \arrow[r, mapsto] & -J_{a9} \arrow[r, mapsto] & -\frac{1}{2}\Gamma_{a9}|_{\overline{P}(\mathbf{32})} \\
J_{ab} \arrow[rr, mapsto] & & \frac{1}{2}\Gamma^B_{ab}|_{\overline{P}(\mathbf{32})}
\end{tikzcd}$$

$(a, b < 9)$ (62)

Here in the last line we have summarized this situation by introducing the following notation – recalling that our undecorated "Γ_a" are always those of the 11d Clifford algebra (8):

$$\begin{array}{lcll}
\Gamma^B_a & := & \Gamma_a & \text{for } a < 9 \\
\Gamma^B_9 & := & \Gamma_9\Gamma_{10} & \\
\Gamma^B_{ab} & := & \Gamma^B_a\Gamma^B_b & \text{for } a < b \leq 9 \\
\Gamma^B_{ba} & := & -\Gamma^B_a\Gamma^B_b & \text{for } a < b \leq 9\ ,
\end{array} \tag{63}$$

which works since $\Gamma_{10}|_{\overline{P}(\mathbf{32})} = -\mathrm{id}$, by (54). But by the same relation also $\Gamma_{10}|_{P(\mathbf{32})} = +\mathrm{id}$, so that the Γ^B operators reduce to the original Clifford generators Γ (8) when restricted on $\mathbf{16} \equiv P(\mathbf{32})$ and hence encoding also a copy of the original representation

$$\begin{tikzcd}
\mathfrak{so}_{1,9} \arrow[r, "\mathbf{16}"] & \mathfrak{gl}_{16} \\
J_{ab} \arrow[r, mapsto] & \frac{1}{2}\Gamma^B_{ab}|_{P(\mathbf{32})} \equiv \frac{1}{2}\Gamma_{ab}|_{P(\mathbf{32})}
\end{tikzcd}$$

In total we produced the type IIB spinor representation $\mathbf{16} \oplus \mathbf{16}$ of $\mathrm{Spin}(1, 9)$ as a pullback of the type IIA spinor representation $\mathbf{16} \oplus \overline{\mathbf{16}}$, in terms of the 11d Clifford algebra expression (63) as:

$$\begin{tikzcd}
\mathfrak{so}_{1,9} \arrow[r, "\sim"] \arrow[rr, bend left, "\mathbf{16} \oplus \mathbf{16}"] & \mathfrak{so}_{1,9} \arrow[r, "\mathbf{16} \oplus \overline{\mathbf{16}}"] & \mathfrak{gl}_{32} \\
J_{ab} \arrow[rr, mapsto] & & \frac{1}{2}\Gamma^B_{ab}
\end{tikzcd}$$

$(a < b \leq 9)$ (64)

It is this transformation which turns out to make manifest superspace T-duality below.

Remark 1.22 (**Subtleties with type IIB Clifford elements.**).
(i) Beware that the operators Γ^B in (63) do not generate a Clifford algebra (and not a $\mathrm{Pin}(1, 9)$-group), but they do generate the correct $\mathrm{Spin}(1, 9)$-group and -representation.
(ii) As we will see in a moment, this defect is in a sense compensated by another defect, namely that Γ^B_9 is not skew-self-adjoint as the other Clifford generators (20):

$$\overline{\Gamma^B_{ab}} \;=\; \begin{cases} -\Gamma^B_{ab} & \text{for } a, b < 9 \\ +\Gamma^B_{ab} & \text{if } b = 9\,, \end{cases} \tag{65}$$

where the second line comes about as

$$\overline{\Gamma^B_{a9}} \;\equiv\; \overline{\Gamma_a\Gamma_9\Gamma_{10}} \;=\; \overline{\Gamma_{10}}\,\overline{\Gamma_9}\,\overline{\Gamma_a} \;=\; (-1)^3\Gamma_{10}\Gamma_9\Gamma_a \;=\; (-1)^{3+1}\Gamma_a\Gamma_9\Gamma_{10} \;=\; \Gamma^B_{a9} \quad \text{for } a < 9\,.$$

With this notation we may set:

Definition 1.23 (**Type IIB super-Minkowski super-Lie algebra.**). The 10d type IIB super-Minkowski Lie algebra is given by

$$\mathrm{CE}\big(\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}}\big) \;=\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{l}(\psi_1^\alpha)_{\alpha=1}^{16},\\ (\psi_2^\alpha)_{\alpha=1}^{16},\\ (e^a)_{a=0}^{9}\end{array}\right] \Big/ \left(\begin{array}{lcl}\mathrm{d}\,\psi_1 &=& 0\\ \mathrm{d}\,\psi_2 &=& 0\\ \mathrm{d}\,e^a &=& \big(\overline{\psi}\,\Gamma_B^a\,\psi\big)\end{array}\right) \tag{66}$$

where the pairing is that of spinors in the **32** of 11d (!) under the identification (56) and where, to compensate this, Γ_B^a is from (63).

Since it is evident that the differential in (66) is at least Spin$(1,8)$-equivariant, we have following analog of Ex. 1.21:

Example 1.24 (**The IIB/9D super-space extension).** We have a Spin$(1,8)$-equivariant isomorphism

$$\mathrm{CE}\big(\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c}(\psi_1^\alpha)_{\alpha=1}^{16},\\ (\psi_2^\alpha)_{\alpha=1}^{16},\\ (e^a)_{a=0}^{8},\\ e^9\end{array}\right] \Big/ \left(\begin{array}{lcll}\mathrm{d}\,\psi_1 &=& 0 & \\ \mathrm{d}\,\psi_2 &=& 0 & \\ \mathrm{d}\,e^a &=& \big(\overline{\psi}_1\,\Gamma^a\,\psi_1\big) + \big(\overline{\psi}_2\,\Gamma^a\,\psi_2\big) & \text{for } a<9\\ \mathrm{d}\,e^9 &=& \underbrace{\big(\overline{\psi}_1\,\Gamma^9\,\psi_1\big) - \big(\overline{\psi}_2\,\Gamma^9\,\psi_2\big)}_{\big(\overline{\psi}\,\Gamma_B^9\,\psi\big) = \big(\overline{\psi}\,\Gamma^9\Gamma_{10}\,\psi\big)} & \end{array}\right)$$

which exhibits $\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}}$ as a central extension (Def. 1.9) of $\mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}}$ classified by a 2-cocycle to be denoted c_1^B (cf. ftn. 4):

$$\begin{tikzcd} \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}} \arrow[r, two heads, "\mathrm{hofib}", "\text{extension of}"'] & \mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}} \arrow[r, "c_1^B := \big(\overline{\psi}\,\Gamma_B^9\,\psi\big) = \big(\overline{\psi}\,\Gamma^9\Gamma_{10}\,\psi\big)", "\text{classified by first Chern class}"'] & b\mathbb{R}\,. \\ \substack{\mathbf{D = 10 \text{ type IIB}}\\ \textbf{super-Minkowski}\\ \textbf{spacetime}} & \substack{\mathbf{D = 9 \text{ type II}}\\ \textbf{super-Minkowski}\\ \textbf{spacetime}} & \end{tikzcd} \tag{67}$$

What is less evident is that (66) is also Spin$(1,9)$-equivariant under the action (64), since the $\big\{\Gamma_B^a\big\}_{a=0}^{9} \subset \mathrm{Cl}(1,10)$ by themselves do not generate a Clifford sub-algebra, by Rem. 1.22. However, the failure of Γ_B^9 to be skew-self-adjoint (65) compensates this defect, as follows:

Lemma 1.25 (**Lorentz-equivariance of the type IIB spacetime).** *The differential in* (66) *is indeed equivariant* (39) *under the* Spin$(1,9)$*-action* (64)*.*

Checking this is straightforward, but we spell out the proof because this statement was omitted in [FSS18a]:

Proof. For a Lie action of $\mathfrak{so}_{1,10}$ on ψ by

$$J^{ab}\psi \;=\; -\tfrac{1}{2}\Gamma_B^{ab}\psi$$

we need to check that the term $\big(\overline{\psi}\,\Gamma_B\,\psi\big)$ transforms in the vector representation, namely that

$$J^{ab}\big(\overline{\psi}\,\Gamma_B^c\,\psi\big) \;\equiv\; \big(\overline{J^{ab}\psi}\,\Gamma_B^c\,\psi\big) + \big(\overline{\psi}\,\Gamma_B^c\,J^{ab}\psi\big) \;=\; \eta^{bc}\big(\overline{\psi}\,\Gamma_B^a\,\psi\big) - \eta^{ac}\big(\overline{\psi}\,\Gamma_B^b\,\psi\big)\,.$$

In the case where $a,b,c<9$ this is, by (63), just the ordinary case which, just to recall, works out as usual:

$$J^{ab}\big(\overline{\psi}\,\Gamma_B^c\,\psi\big) \;=\; \big(\overline{\psi}\,[\tfrac{1}{2}\Gamma^{ab},\Gamma^c]\,\psi\big) \;=\; \big(\overline{\psi}(\eta^{bc}\Gamma^a - \delta^{ac}\Gamma^b)\psi\big)\,, \qquad \text{for } a,b,c<9\,,$$

where in the first step we use – via the first line in (65) – that $\overline{\tfrac{1}{2}\Gamma_{ab}} \;=\; -\tfrac{1}{2}\Gamma_{ab}$, which in the second step gives rise to the commutator [-, -] (in the 11d Clifford algebra).

Next, the case where $a,b<9$ but $c=9$ does involve the modified $\Gamma_B^9 = \Gamma^9\Gamma^{10}$, but gives the correct answer trivially since the same kind of commutator appears and evidently vanishes:

$$J^{ab}\big(\overline{\psi}\,\Gamma_B^9\,\psi\big) \;=\; \big(\overline{\psi}\,[\tfrac{1}{2}\Gamma^{ab},\Gamma^9\Gamma^{10}]\,\psi\big) \;=\; 0\,, \qquad \text{for } a,b<9.$$

The interesting effect is in the next case, where one of the first two indices take the value 9. Here the modified adjointness relation in the second line of (65) makes instead an anti-commutator {-, -} appear, which however becomes a commutator after pulling out the factor of Γ^{10} that comes with Γ_B^9, and that again gives the correct result:

$$J^{a9}\big(\overline{\psi}\,\Gamma_B^c\,\psi\big) \;=\; -\big(\overline{\psi}\,\{\tfrac{1}{2}\Gamma^{a9}\Gamma^{10},\Gamma^c\}\,\psi\big) \;=\; \big(\overline{\psi}\,[\tfrac{1}{2}\Gamma^{a9},\Gamma^c]\Gamma^{10}\,\psi\big) \;=\; -\eta^{ac}\big(\overline{\psi}\,\Gamma_B^9\,\psi\big) \qquad \text{for } a,c<9\,.$$

Finally, a similar computation passing through an anti-commutator also confirms the last remaining case:

$$J^{a9}\big(\overline{\psi}\,\Gamma_B^9\,\psi\big) \;=\; -\big(\overline{\psi}\{\tfrac{1}{2}\Gamma^{a9}\Gamma^{10},\Gamma^9\Gamma^{10}\}\psi\big) \;=\; \big(\overline{\psi}\,\Gamma_B^a\,\psi\big) \qquad \text{for } a<9. \qquad \square$$

Observing that the Clifford elements

$$\begin{array}{rcl} \sigma_1 & := & \Gamma_9 \\ \sigma_2 & := & \Gamma_{10} \end{array} \quad \text{(whose product we denote } \sigma_3 := \sigma_1 \sigma_2 \; = \; \Gamma_9 \Gamma_{10}\text{)} \tag{68}$$

anti-commute with *all* the $(\Gamma^B_a)_{a=0}^9$ (63) we also have:

Proposition 1.26 (**R-Symmetry of type IIB** [FSS18a, Rem. 2.11])**.** *The elements* (68) *generate a* $\mathrm{Pin}(2)$*-action on* $\mathbf{16} \oplus \mathbf{16}$*, which commutes with the* $\mathrm{Spin}(1,9)$*-action* (64)*, making a direct product action of* $\mathrm{Spin}(1,9) \times \mathrm{Pin}(2)$*.*

(This effectively "12-dimensional" spin-action is seen to be related to the "F-theory"-perspective on type IIB, see blow...)

Putting these pieces together, we have more generally that:

Proposition 1.27 (**Lorentz-invariants on type IIB super-spacetime**)**.** *Setting*

$$\Gamma^B_{a_1 \cdots a_p} \; := \; \left\{ \begin{array}{ll} (-1)^{\mathrm{sgn}(\sigma)} \, \Gamma^B_{\sigma(a_1)} \cdots \Gamma^B_{\sigma(a_p)} & \Big\vert \;\; \exists \; \sigma \in \mathrm{Sym}(n) \text{ s.t. } \; a_{\sigma(1)} < \cdots < a_{\sigma(n)} \\ 0 & \Big\vert \;\; \text{otherwise} \end{array} \right. \tag{69}$$

the following expressions are invariants for the $\mathrm{Spin}(1,9)$*-action* (64) *type IIB super-spacetime* (66)*:*

$$\left. \begin{array}{l} \big(\overline{\psi} \, \Gamma^B_{a_1 \cdots a_p} \, \psi\big) \quad e^{a_1} \cdots e^{a_p} \\ \big(\overline{\psi} \, \Gamma^B_{a_1 \cdots a_p} \, \sigma_i \, \psi\big) e^{a_1} \cdots e^{a_p} \end{array} \right\} \; \in \; \mathrm{CE}\big(\mathbb{R}^{1,9 \,\vert\, \mathbf{16} \oplus \mathbf{16}}\big) \tag{70}$$

for σ_i *from* (68)*.*

(Of course, many of these expressions vanish, notably by (25).)

Proof. The statement for the second line follows immediately from that for the first line by Prop. 68. We proceed to prove the statement for the second line.

For $p = 0$ the statement holds trivially, since given expression vanishes, by (25).

From $p = 1$ on we shall prove the stronger statement that $\big(\overline{\phi} \, \Gamma_{a_1 \cdots a_9} \, \psi\big) e^{a_1} \cdots e^{a_p}$ is invariant for ϕ possibly any other element transforming as $J^{ab} \phi \; = \; -\frac{1}{2} \Gamma^{ab}_B \phi$:

The case $p = 1$ follows verbatim as in Lem. 1.25, with the first factor of "ψ" there generalized to "ϕ".

For the remaining cases we may, by (69), assume without restriction of generality that $a_1 < \cdots < a_{p+1}$, hence in particular that $a_n < 9$ whenever $n \leq p$, and we need to show that the following is invariant:

$$\begin{array}{rcll} \big(\overline{\phi} \, \Gamma^B_{a_1 \cdots a_9} \, \psi\big) e^{a_1} \cdots e^{a_p} & = & \big(\overline{\phi} \, \Gamma^B_{a_1 \cdots a_p} \Gamma^B_{a_{p+1}} \, \psi\big) e^{a_1} \cdots e^{a_p} e^{a_{p+1}} & \text{by assmptn} \\ & = & \pm \big(\overline{\Gamma^B_{a_1 \cdots a_p} \phi} \, \Gamma^B_{a_{p+1}} \, \psi\big) e^{a_1} \cdots e^{a_p} \; e^{a_{p+1}} & \text{by (21)} \end{array}$$

Now observe that the expression

$$\phi' \; := \; \Gamma^B_{a_1 \cdots a_p} \phi \, e^{a_1} \cdots e^{a_p}$$

transforms as a spinor under any J^{ab}: For $a, b < 9$ this is the standard situation, and then for J^{a9} it follows since (69) makes any Γ_{10}-factor "stay on the right". But with this we are reduced to seeing that $\big(\overline{\phi'} \, \Gamma_{a_{p+1}} \, \psi\big) e^{a_{p+1}}$ is invariant, which is the case $p = 1$ already proven. ☐

1.7 Superspace T-duality

We give a streamlined review of the core part of the formulation (and in fact a derivation) from [FSS18a] of T-duality between super-invariant super-flux densities on super-Minkowski spacetime. [5]

The key observations driving this are that:

(1.) The type IIA super-flux densities on super-Minkowki spacetime satisfying their Bianchi identities are equivalently (Prop. 1.31) super-L_∞ cocycles with coefficients in the real Whitehead L_∞-algebra of the twisted K-theory spectrum KU (Ex. 1.8).

---

[5] Note that the focus on Minkowski super-spacetime is only superficially a specialization: The torsion constraints that govern supergravity theories say that – in the manner of Cartan geometry – the actual super-flux densities on on-shell super-spacetimes are tangentspace-wise constrained to have fermionic components given by these Minkowskian super-invariants – and in fact in 11d SuGra that condition is equivalent to the supergravity equations of motion [GSS24a, Thm. 3.1]. In this the super-invariants on super-Minkowski spacetimes are the archetypes that govern full on-shell supergravity theories.

(2.) Double dimensional reduction of super-flux densities on super-Minkowski spacetime is equivalently given by *cyclifying* (as in *cyclic cohomology*) their coefficient L_∞-algebra (Prop. 1.13).

(3.) The cyclifications of twisted KU_0 is equivalent to that of twisted KU_1 by swapping the Chern class with the wrapping mode of the 3-form (Ex. 1.15).

(4.) The type IIA flux densities are equivalent to the IIA flux densities after reduction via cyclification to 9d, whereby the type IIA spacetime is swapped for the type IIB spacetime (Thm. **??**).

M/IIA duality. Recall from Ex. 1.7that the C-field super-flux densities on 11D super-Minkowski spacetime are encoded by a super-L_∞ homomorphism of the form:

$$\begin{array}{rcc}
\mathbb{R}^{1,10\,|\,\mathbf{32}} & \xrightarrow{\quad (G_4,\,G_7) \quad} & \mathfrak{l}S^4 \\
\tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1 a_2}\,\psi\big)e^{a_1}e^{a_2} \;=:\; G_4 & \longmapsfrom & \omega_4 \\
\tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\,\psi\big)e^{a_1}\cdots e^{a_5} \;=\; G_7 & \longmapsfrom & \omega_7
\end{array}
\tag{71}$$

Example 1.28 (**IIA-Reduction of C-field super-flux** [FSS17, pp 11], cf. [SS24c, Ex. 2.13])**.** Under reduction via the Ext/Cyc-adjunction (Prop. 1.13) with respect to the M/IIA extension (Ex. 1.20)

$$\begin{tikzcd}
\mathbb{R}^{1,10\,|\,\mathbf{32}} \arrow[r, "{(G_4,\,G_7)}"] & \mathfrak{l}S^4 & \leftrightarrow & \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[rr, "{\mathrm{rdc}_{c_1^M}(G_4,G_7)}"] \arrow[dr, "{(\overline{\psi}\,\Gamma_{10}\,\psi)}"'] & & \mathrm{cyc}(\mathfrak{l}S^4) \arrow[dl, "\omega_2"] \\
& & & & b\mathbb{R} &
\end{tikzcd}$$

the flux densities from 11D (71) become:

$$\begin{array}{rcc}
\mathbb{R}^{1,10\,|\,\mathbf{32}} & \xrightarrow{\quad (G_4,\,G_7) \quad} & \mathfrak{l}S^4 \\
\tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1 a_2}\,\psi\big)e^{a_1}e^{a_2} & \longmapsfrom & \omega_4 \\
\tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\,\psi\big)e^{a_1}\cdots e^{a_5} & \longmapsfrom & \omega_7
\end{array}
\quad\rightsquigarrow\quad
\begin{array}{rcc}
\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} & \xrightarrow[\qquad]{\substack{\mathrm{rdc}_{c_1^M}(G_4,G_7)\,= \\ (F_2,F_4,F_6,H_3^A,H_7^A)}} & \mathrm{cyc}\big(\mathfrak{l}S^4\big) \\
\big(\overline{\psi}\,\Gamma_{10}\psi\big) \;=:\; \mathrm{F}_2 & \longmapsfrom & \omega_2 \\
\tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1 a_2}\,\psi\big)e^{a_1}\,e^{a_2} \;=:\; \mathrm{F}_4 & \longmapsfrom & \omega_4 \\
\big(\overline{\psi}\,\Gamma_a\Gamma_{10}\,\psi\big)e^{a} \;=:\; H_3^A & \longmapsfrom & \mathrm{s}\omega_4 \\
\tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\,\psi\big)e^{a_1}\cdots e^{a_5} \;=:\; H_7^A & \longmapsfrom & \omega_7 \\
-\tfrac{1}{4!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_4}\Gamma_{10}\,\psi\big)e^{a_1}\cdots e^{a_4} \;=:\; -F_6 & \longmapsfrom & \mathrm{s}\omega_7
\end{array}
\tag{72}$$

satisfying:

$$\left.\begin{array}{rcl}
\mathrm{d}\,G_4 & = & 0 \\
\mathrm{d}\,G_7 & = & \tfrac{1}{2}\,G_4\,G_4
\end{array}\right\}
\quad\rightsquigarrow\quad
\left\{\begin{array}{rcl}
\mathrm{d}\,F_2 & = & 0 \\
\mathrm{d}\,F_4 & = & H_3^A\,F_2 \\
\mathrm{d}\,F_6 & = & H_3^A\,F_4 \\
\mathrm{d}\,H_3^A & = & 0 \\
\mathrm{d}\,H_7^A & = & \tfrac{1}{2}\,F_4\,F_4 \;-\; F_2\,F_6\,.
\end{array}\right.
\tag{73}$$

However, on the type IIA super-spacetime there appear further/higher super-invariants satisfying analogous differential equations — this observation is essentially due to [CdAIP00, §6.1], except that we also consider F_{12} [6]

Definition 1.29 (**Higher type IIA super-flux densities.**)**.** Consider the following super-invariants, beyond

---

[6] A 12-form term like F_{12} in (74) – nominally the WZW term for "D10-branes" – is rarely considered in the literature (an exception is [CS09, p 30]) since it is evidently invisible on ordinary (bosonic) spacetimes. But on super-space it is non-vanishing and must be considered [BMSS19, Rem. 4.3] to complete the flux densities $F_{2\bullet}$ to an $\mathfrak{l}\mathrm{KU}_0$-valued cocycle. On the other hand, the yet higher degree fluxes of this form $F_{2k+2} = \frac{1}{(2k)!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_{2k}}\,\psi\big)e^{a_1}\cdots e^{a_{2k}}$ do vanish on $\mathbb{R}^{1,9\,|\,\mathbf{N}}$ and hence need not be further considered.

those appearing via reduction from 11d in Ex. 1.28 [7]

$$\left.\begin{array}{rcl} F_8 &:=& +\tfrac{1}{6!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_6}\,\psi\big)e^{a_1}\cdots e^{a_6} \\ F_{10} &:=& +\tfrac{1}{8!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_8}\Gamma_{10}\,\psi\big)e^{a_1}\cdots e^{a_8} \\ F_{12} &:=& +\tfrac{1}{10!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_{10}}\,\psi\big)e^{a_1}\cdots e^{a_{10}} \end{array}\right\} \in \mathrm{CE}\big(\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}\big). \tag{74}$$

Lemma 1.30 (Bianchi identities for higher IIA super-fluxes). *The higher super-flux densities* (74) *satisfy*

$$\begin{array}{rcl} \mathrm{d}\,F_8 &=& H_3^A\,F_6 \\ \mathrm{d}\,F_{10} &=& H_3^A\,F_8 \\ \mathrm{d}\,F_{12} &=& H_3^A\,F_{10} \\ 0 &=& H_3^A\,F_{12} \end{array} \tag{75}$$

with H_3^A from (72).

Proof. The proof for the first equation in (75) is also given in [CdAIP00, §B], which we follow. First to note that the closure of H_3^A from (73)

$$\begin{array}{rcl} 0 &=& \mathrm{d}\,H_3^0 \\ &=& \mathrm{d}\Big(\big(\overline{\psi}\,\Gamma_a\Gamma_{10}\,\psi\big)e^a\Big) \qquad\qquad a < 10 \\ &=& \big(\overline{\psi}\,\Gamma_a\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma^a\,\psi\big) \end{array}$$

means in components that

$$\begin{array}{rcl} 0 &=& \big(\Gamma_a\Gamma_{10}\big)_{(\alpha\beta}\Gamma^a_{\gamma\delta)} \\ &=& \tfrac{2\cdot 3!}{4!}\Big(\big(\Gamma_a\Gamma_{10}\big)_{(\alpha\beta}\Gamma^a_{\delta)\gamma} + \big(\Gamma_a\Gamma_{10}\big)_{\gamma(\alpha}\Gamma^a_{\beta\delta)}\Big)\,, \end{array} \qquad a < 10 \tag{76}$$

where in the second line we used (23) that $(\Gamma_a\Gamma_{10})_{\alpha\beta}$ and $(\Gamma_a)_{\gamma\delta}$ both already are symmetric in their spinor indices.

With this in hand we compute as follows:

$$\begin{array}{rcll} \mathrm{d}\,F_8 &=& \mathrm{d}\,\tfrac{1}{6!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_6}\,\psi\big)e^{a_1}\cdots e^{a_6} & \text{by (74)} \\ &=& -\tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5\,b}\,\psi\big)\big(\overline{\psi}\,\Gamma^b\,\psi\big)e^{a_1}\cdots e^{a_5} & \text{by (40)} \\ &=& -\tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\Gamma_b\,\psi\big)\big(\overline{\psi}\,\Gamma^b\,\psi\big)e^{a_1}\cdots e^{a_5} & \text{by (15) \& (25)} \\ &=& +\tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\Gamma_{10}\,\Gamma_b\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma^b\,\psi\big)e^{a_1}\cdots e^{a_5} & \text{by (14)} \\ &=& +\tfrac{1}{5!}\big(\Gamma_{a_1\cdots a_5}\Gamma_{10}\big)_{(\alpha|\kappa|}\big(\Gamma_b\Gamma_{10}\big)^{\kappa}{}_{\beta}\big(\Gamma^b\big)_{\gamma\delta)}\;\psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\,e^{a_1}\cdots e^{a_5} & \text{matrix multip.} \\ &=& -\tfrac{1}{5!}\big(\Gamma_{a_1\cdots a_5}\Gamma_{10}\big)_{(\alpha|\kappa|}\big(\Gamma_b\Gamma_{10}\big)_{\gamma\beta}\big(\Gamma^b\big)^{\kappa}{}_{\delta)}\;\psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\,e^{a_1}\cdots e^{a_5} & \text{by (76)} \\ &=& +\tfrac{1}{5!}\big(\Gamma_{a_1\cdots a_5}\Gamma_b\Gamma_{10}\big)_{(\alpha\delta}\big(\Gamma^b\Gamma_{10}\big)_{\gamma\beta)}\;\psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\,e^{a_1}\cdots e^{a_5} & \text{matrix multip.} \\ &=& +\tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\Gamma_b\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma^b\Gamma_{10}\,\psi\big)\,e^{a_1}\cdots e^{a_5} & \\ &=& +\tfrac{1}{4!}\big(\overline{\psi}\,\Gamma_{[a_1\cdots a_4}\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma_{a_5]}\Gamma_{10}\,\psi\big)\,e^{a_1}\cdots e^{a_5} & \text{by (15) \& (25)} \\ &=& +\,F_6\,H_3^A & \text{by (72)} \end{array}$$

---

[7] In the string theory lore the higher flux densities related to the higher super-invariants (74) and corresponding to (1.) the D6-brane, (2.) the D8-brane and (3.) the "D10-brane", are meant to have 11d ancestors given, more or less informally, by (1.) the 11d Kaluza-Klein monopole, (2.) a Scherk-Schwarz compactification to massive type IIA theory, respectively, while the M-theory lift of (3.) the "D10-brane" seems not to have been discussed (cf. ftn. 6).

More in the spirit of the rigorous derivations here, we have shown in [BMSS19] that the relevant higher generators appear when the 4-sphere coefficient (Ex. 1.7) for the fluxes in 11d are subjected to fiberwise "stabilization" over the 3-sphere (in the sense of stable homotopy theory).

The remaining two cases (and in fact all cases) work analogously:

$$
\begin{aligned}
\mathrm{d}\,F_{10} &= \mathrm{d}\,\tfrac{1}{8!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_8}\Gamma_{10}\,\psi\big)e^{a_1}\cdots e^{a_8} && \text{by (74)}\\
&= -\tfrac{1}{7!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_7\,b}\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma^b\,\psi\big)e^{a_1}\cdots e^{a_7} && \text{by (40)}\\
&= -\tfrac{1}{7!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_7}\Gamma_b\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma^b\,\psi\big)e^{a_1}\cdots e^{a_7} && \text{by (15) \& (25)}\\
&= -\tfrac{1}{7!}\big(\Gamma_{a_1\cdots a_7}\big)_{(\alpha|\kappa|}\big(\Gamma_b\Gamma_{10}\big)^{\kappa}{}_{\beta}\big(\Gamma^b\big)_{\gamma\delta)}\;\psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\,e^{a_1}\cdots e^{a_7} && \text{matrix multip.}\\
&= +\tfrac{1}{7!}\big(\Gamma_{a_1\cdots a_7}\big)_{(\alpha|\kappa|}\big(\Gamma_b\Gamma_{10}\big)_{\gamma\beta}\big(\Gamma^b\big)^{\kappa}{}_{\delta)}\;\psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\,e^{a_1}\cdots e^{a_7} && \text{by (76)}\\
&= +\tfrac{1}{7!}\big(\Gamma_{a_1\cdots a_7}\Gamma_b\big)_{(\alpha\delta}\big(\Gamma^b\Gamma_{10}\big)_{\gamma\beta)}\;\psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\,e^{a_1}\cdots e^{a_7} && \text{matrix multip.}\\
&= +\tfrac{1}{7!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_7}\Gamma_b\,\psi\big)\big(\overline{\psi}\,\Gamma^b\Gamma_{10}\,\psi\big)\,e^{a_1}\cdots e^{a_7} && \\
&= +\tfrac{1}{6!}\big(\overline{\psi}\,\Gamma_{[a_1\cdots a_6}\,\psi\big)\big(\overline{\psi}\,\Gamma_{a_7]}\Gamma_{10}\,\psi\big)\,e^{a_1}\cdots e^{a_7} && \text{by (15) \& (25)}\\
&= +F_8\,H_3^A && \text{by (72)}
\end{aligned}
$$

and:

$$
\begin{aligned}
\mathrm{d}\,F_{12} &= \mathrm{d}\,\tfrac{1}{10!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_{10}}\,\psi\big)e^{a_1}\cdots e^{a_{10}} && \text{by (74)}\\
&= -\tfrac{1}{9!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_9\,b}\,\psi\big)\big(\overline{\psi}\,\Gamma_b\,\psi\big)e^{a_1}\cdots e^{a_9} && \text{by (40)}\\
&= -\tfrac{1}{9!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_9}\Gamma_b\,\psi\big)\big(\overline{\psi}\,\Gamma^b\,\psi\big)e^{a_1}\cdots e^{a_9} && \text{by (15) \& (25)}\\
&= +\tfrac{1}{9!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_9}\Gamma_{10}\,\Gamma_b\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma^b\,\psi\big)e^{a_1}\cdots e^{a_9} && \text{by (14)}\\
&= +\tfrac{1}{9!}\big(\Gamma_{a_1\cdots a_9}\Gamma_{10}\big)_{(\alpha|\kappa|}\big(\Gamma_b\Gamma_{10}\big)^{\kappa}{}_{\beta}\big(\Gamma^b\big)_{\gamma\delta)}\;\psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\,e^{a_1}\cdots e^{a_9} && \text{matrix multip.}\\
&= -\tfrac{1}{9!}\big(\Gamma_{a_1\cdots a_9}\Gamma_{10}\big)_{(\alpha|\kappa|}\big(\Gamma_b\Gamma_{10}\big)_{\gamma\beta}\big(\Gamma^b\big)^{\kappa}{}_{\delta)}\;\psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\,e^{a_1}\cdots e^{a_9} && \text{by (76)}\\
&= +\tfrac{1}{9!}\big(\Gamma_{a_1\cdots a_9}\Gamma_b\Gamma_{10}\big)_{(\alpha\delta}\big(\Gamma^b\Gamma_{10}\big)_{\gamma\beta)}\;\psi^\alpha\psi^\beta\psi^\gamma\psi^\delta\,e^{a_1}\cdots e^{a_9} && \text{matrix multip.}\\
&= +\tfrac{1}{9!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_9}\Gamma_b\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma^b\Gamma_{10}\,\psi\big)\,e^{a_1}\cdots e^{a_9} && \\
&= +\tfrac{1}{8!}\big(\overline{\psi}\,\Gamma_{[a_1\cdots a_8}\Gamma_{10}\,\psi\big)\big(\overline{\psi}\,\Gamma_{a_9]}\Gamma_{10}\,\psi\big)\,e^{a_1}\cdots e^{a_9} && \text{by (15) \& (25)}\\
&= +F_{10}\,H_3^A && \text{by (72)}\,.
\end{aligned}
$$

□

It is worth summarizing this state of affairs in super-L_∞ algebraic language:

Proposition 1.31 **(The type IIA super-cocycles** [FSS18a, Prop. 4.8]**).** *On the type IIA super-Minkowski spacetime* $\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}$ *we have the following super-invariants*

$$
\left.
\begin{aligned}
H_3^A &:= \big(\overline{\psi}\,\Gamma_a\Gamma_{10}\,\psi\big)e^a\\
F_{-2k} &:= \;\; 0\,,\quad k\in\mathbb{N}\\
F_2 &:= \big(\overline{\psi}\,\Gamma_{10}\,\psi\big)\\
F_4 &:= \tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1a_2}\,\psi\big)e^{a_1}e^{a_2}\\
F_6 &:= \tfrac{1}{4!}\big(\overline{\psi}\,\Gamma_{10}\Gamma_{a_1\cdots a_4}\,\psi\big)e^{a_1}\cdots e^{a_4}\\
F_8 &:= \tfrac{1}{6!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_6}\,\psi\big)e^{a_1}\cdots e^{a_6}\\
F_{10} &:= \tfrac{1}{8!}\big(\overline{\psi}\,\Gamma_{10}\Gamma_{a_1\cdots a_8}\,\psi\big)e^{a_1}\cdots e^{a_8}\\
F_{12} &:= \tfrac{1}{10!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_{10}}\,\psi\big)e^{a_1}\cdots e^{a_{10}}\\
F_{14+2k} &= \;\; 0\,,\quad k\in\mathbb{N}
\end{aligned}
\right\}
\;\in\; \mathrm{CE}\big(\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}}\big)
\quad \text{s.t.} \quad
\begin{cases}
\mathrm{d}\,H_3^A \;=\; 0\\
\mathrm{d}\,F_{2\bullet+2} \;=\; H_3^A\,F_{2\bullet}\,,
\end{cases}
\tag{77}
$$

hence equivalently constituting a super-L_∞ *homomorphism* (38) *to the real Whitehead* L_∞*-algebra of twisted* KU_0 (42)*:*

$$
\begin{tikzcd}
\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[rr, "{(H_3^A,(F_{2k})_{k\in\mathbb{Z}})}"] \arrow[dr, "H_3^A"'] & & \mathfrak{l}\big(\mathrm{KU}_0/\!\!/B\mathrm{U}(1)\big) \arrow[dl, "h_3"] \\
& \mathfrak{l}\big(B^2\mathrm{U}(1)\big)\,. &
\end{tikzcd}
\tag{78}
$$

Example 1.32 (**Reduction of IIA super-cocycles to 9d).** The reduction of of the type IIA cocycles (Prop. 1.31) via the Ext/Cyc-adjunction (Prop. 1.13) along the IIA/9D extension (Ex. 1.21)

$$\begin{tikzcd}
\mathbb{R}^{1,9\,\vert\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[r, "{(H_3^A,(F_{2k})_{k\in\mathbb{Z}})}"] & \mathfrak{l}\big(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1)\big) \arrow[r, phantom, "\leftrightsquigarrow"] & \mathbb{R}^{1,8\,\vert\,\mathbf{16}\oplus\mathbf{16}} \arrow[rr, "{\mathrm{rdc}_{c_1^A}(H_3^A,(F_{2k})_{k\in\mathbb{Z}})}"] \arrow[dr, swap, "{c_1^A = (\overline{\psi}\,\Gamma^9\,\psi)}"] & & \mathrm{cyc}\,\mathfrak{l}\big(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1)\big) \arrow[dl, "\omega_2"] \\
& & & b\mathbb{R} &
\end{tikzcd}$$

gives the following system of super-invariants in 9D, where on the right we show their equivalent incarnation as having coefficients either in the cyclification of twisted KU_0 or of twisted KU_1, via (52):

$$\begin{array}{rcccc}
\mathbb{R}^{1,8\,\vert\,\mathbf{16}\oplus\mathbf{16}} & \xrightarrow[(78)]{\mathrm{rdc}_{c_1^A}(H_3^A,(F_{2k})_{k\in\mathbb{Z}})} & \mathrm{cyc}\,\mathfrak{l}\big(\mathrm{KU}_{0} /\!\!/ B\mathrm{U}(1)\big) & \xrightarrow[(52)]{\sim} & \mathrm{cyc}\,\mathfrak{l}\big(\mathrm{KU}_{1} /\!\!/ B\mathrm{U}(1)\big) \\
c_1^A \;=\; \big(\overline{\psi}\,\Gamma_9\,\psi\big) & \longmapsfrom & \omega_2 & \longmapsfrom & -\mathrm{s}h_3 \\
\big(\overline{\psi}\,\Gamma_a\Gamma_{10}\,\psi\big)e^a & \longmapsfrom & h_3 & \longmapsfrom & h_3 \\
c_1^B \;=\; -\big(\overline{\psi}\,\Gamma_9\Gamma_{10}\,\psi\big) & \longmapsfrom & \mathrm{s}h_3 & \longmapsfrom & -\omega_2 \\
0 & \longmapsfrom & f_{\leq 0} & \longmapsfrom & \mathrm{s}f_{\leq 1} \\
0 & \longmapsfrom & \mathrm{s}f_{\leq 0} & \longmapsfrom & f_{\leq 1} \\
\big(\overline{\psi}\,\Gamma_{10}\,\psi\big) & \longmapsfrom & f_2 & \longmapsfrom & \mathrm{s}f_3 \\
0 & \longmapsfrom & \mathrm{s}f_2 & \longmapsfrom & f_1 \\
\tfrac{1}{2}\big(\overline{\psi}\,\Gamma_{a_1a_2}\,\psi\big)e^{a_1}e^{a_2} & \longmapsfrom & f_4 & \longmapsfrom & \mathrm{s}f_5 \\
\big(\overline{\psi}\,\Gamma_a\Gamma_9\,\psi\big)e^a & \longmapsfrom & \mathrm{s}f_4 & \longmapsfrom & f_3 \\
\tfrac{1}{4!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_4}\Gamma_{10}\,\psi\big)e^{a_1}\cdots e^{a_4} & \longmapsfrom & f_6 & \longmapsfrom & \mathrm{s}f_7 \\
\tfrac{1}{3!}\big(\overline{\psi}\Gamma_{a_1a_2a_3}\Gamma_9\Gamma_{10}\,\psi\big)e^{a_1}e^{a_2}e^{a_3} & \longmapsfrom & \mathrm{s}f_6 & \longmapsfrom & f_5 \\
\tfrac{1}{6!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_6}\,\psi\big)e^{a_1}\cdots e^{a_6} & \longmapsfrom & f_8 & \longmapsfrom & \mathrm{s}f_9 \\
\tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\Gamma_9\,\psi\big)e^{a_1}\cdots e^{a_5} & \longmapsfrom & \mathrm{s}f_8 & \longmapsfrom & f_7 \\
\tfrac{1}{8!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_8}\Gamma_{10}\,\psi\big)e^{a_1}\cdots e^{a_8} & \longmapsfrom & f_{10} & \longmapsfrom & \mathrm{s}f_{11} \\
\tfrac{1}{7!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_7}\Gamma_9\Gamma_{10}\,\psi\big)e^{a_1}\cdots e^{a_7} & \longmapsfrom & \mathrm{s}f_{10} & \longmapsfrom & f_9 \\
\tfrac{1}{10!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_{10}}\,\psi\big)e^{a_1}\cdots e^{a_{10}} & \longmapsfrom & f_{12} & \longmapsfrom & \mathrm{s}f_{13} \\
\tfrac{1}{9!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_9}\Gamma_9\,\psi\big)e^{a_1}\cdots e^{a_9} & \longmapsfrom & \mathrm{s}f_{12} & \longmapsfrom & f_{11} \\
0 & \longmapsfrom & f_{\geq 14} & \longmapsfrom & \mathrm{s}f_{\geq 15} \\
0 & \longmapsfrom & \mathrm{s}f_{\geq 14} & \longmapsfrom & f_{\geq 13}
\end{array}
\tag{79}$$

T-dualization. Using the super-L_∞ machinery, we now obtain with mechanical ease the T-dual version of Prop. 1.31, cf. [Sak00, §2]:

Proposition 1.33 (**The type IIB super-cocycles** [FSS18a, Prop. 4.10])**.** *On the type IIB super-Minkowski spacetime* $\mathbb{R}^{1,9\,\vert\,\mathbf{16}\oplus\mathbf{16}}$ (Def. 1.23) *we have the following super-invariants*

$$\left.\begin{array}{rcl}
H_3^B &=& \big(\overline{\psi}\,\Gamma^B_a\,\Gamma_{10}\,\psi\big)e^a \\
F_{\leq 1} &=& 0 \\
F_3 &=& \big(\overline{\psi}\,\Gamma^B_a\Gamma_9\,\psi\big)e^a \\
F_5 &=& \tfrac{1}{3!}\big(\overline{\psi}\,\Gamma^B_{a_1a_2a_3}\Gamma_9\Gamma_{10}\,\psi\big)e^{a_1}e^{a_2}e^{a_3} \\
F_7 &=& \tfrac{1}{5!}\big(\overline{\psi}\,\Gamma^B_{a_1\cdots a_5}\Gamma_9\,\psi\big)e^{a_1}\cdots e^{a_5} \\
F_9 &=& \tfrac{1}{7!}\big(\overline{\psi}\,\Gamma^B_{a_1\cdots a_7}\Gamma_9\Gamma_{10}\,\psi\big)e^{a_1}\cdots e^{a_7} \\
F_{11} &=& \tfrac{1}{7!}\big(\overline{\psi}\,\Gamma^B_{a_1\cdots a_9}\Gamma_9\,\psi\big)e^{a_1}\cdots e^{a_9}
\end{array}\right\}
\in \mathrm{CE}\big(\mathbb{R}^{1,9\,\vert\,\mathbf{16}\oplus\mathbf{16}}\big) \quad \textit{s.t.} \quad
\left\{\begin{array}{rcl}
\mathrm{d}\,H_3^B &=& 0 \\
\mathrm{d}\,F_{2\bullet+1} &=& H_3^B\,F_{2\bullet-1}\,,
\end{array}\right.
\tag{80}$$

where the $\Gamma^B_{a_1\cdots a_p}$ *are from* (69)*, hence equivalently constituting a super-*L_∞ *homomorphism* (38) *to the real White-*

head L_∞-algebra of twisted KU_1 (42)*:*

$$\begin{tikzcd}
\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}} \arrow[rr, "{\left(H_3^B, (F_{2k+1})_{k\in\mathbb{Z}}\right)}"] \arrow[dr, "H_3^B"'] & & \mathfrak{l}\big(\mathrm{KU}_1 /\!\!/ B\mathrm{U}(1)\big) \arrow[dl, "h_3"] \\
& \mathfrak{l}\big(B^2\mathrm{U}(1)\big)\,. &
\end{tikzcd} \tag{81}$$

Proof. We claim that this is the *T-dual* statement of Prop. 1.31 in that the super-invariants in (80) are the result of:

(1.) reducing (47) the IIA super-invariants (78) along the type IIA fibration (60) to 9d,

(2.) equivalently re-regarding their coefficients in the cyclification of twisted KU_1 instead of twisted KU_0, via (52), noticing that this swaps the Chern class from that classifying the type IIA extension to that classifying the IIB extension (67),

(3.) re-oxidizing (47) the result, but now along the IIB fibration (67):

$$\begin{tikzcd}
& \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}} \arrow[rr, "{\mathrm{oxd}_{c_1^B} T\left(\mathrm{rdc}_{c_1^A}\left(H_3^A, (F_{2k})_{k\in\mathbb{Z}}\right)\right)}", "{=\left(H_3^B, (F_{2k+1})_{k\in\mathbb{Z}}\right)}"'] \arrow[dd, ->>] & & \mathfrak{l}\big(\mathrm{KU}_1 /\!\!/ B\mathrm{U}(1)\big) \\
\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[rr, "{\left(H_3^A, (F_{2k})_{k\in\mathbb{Z}}\right)}"] \arrow[dr, ->>] & & \mathfrak{l}\big(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1)\big) & \\
& \mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}} \arrow[rr, "{T\left(\mathrm{rdc}_{c_1^A}\left(H_3^A, (F_{2k})_{k\in\mathbb{Z}}\right)\right)}"] \arrow[dr, "{\mathrm{rdc}_{c_1^A}\left(H_3^A, (F_{2k})_{k\in\mathbb{Z}}\right)}"'] & & \mathrm{cyc}\,\mathfrak{l}\big(\mathrm{KU}_1 /\!\!/ B\mathrm{U}(1)\big) \\
& & \mathrm{cyc}\,\mathfrak{l}\big(\mathrm{KU}_0 /\!\!/ B\mathrm{U}(1)\big) \arrow[ur, "T", "\sim"'] &
\end{tikzcd}$$

By the Ext/Cyc-adjunction (Prop. 1.13) the result of this process is guaranteed to be a super-L_∞ homomorphism of the form shown in the top right of the above diagram, which implies the claimed Bianchi identities (81).

Hence all that remains to be shown is that the super-invariants produced by this process are indeed those shown in (80). This is a straightforward matter of plugging the 9d super-invariants (79) into the oxidation formula (48):

$$
\begin{aligned}
F_1 &= \overbrace{0}^{f_1} - e^9 \overbrace{0}^{\mathrm{s}f_1} \\
&= 0 \\
F_3 &= \overbrace{\sum_{a<9} \big(\overline{\psi}\,\Gamma_a \Gamma_9\,\psi\big) e^a}^{f_3} - e^9 \overbrace{\underbrace{\big(\overline{\psi}\,\Gamma_{10}\,\psi\big)}_{-\big(\overline{\psi}\;\underbrace{\Gamma_9\Gamma_{10}}_{\Gamma_9^B}\,\Gamma_9\,\psi\big)}}^{\mathrm{s}f_3} \\
&= \big(\overline{\psi}\,\Gamma^B_a \Gamma_9\,\psi\big) e^a \\
F_5 &= \overbrace{\sum_{a<9} \tfrac{1}{3!}\big(\overline{\psi}\,\Gamma_{a_1 a_2 a_2}\Gamma_9\Gamma_{10}\,\psi\big) e^{a_1} e^{a_2} e^{a_3}}^{f_5} - e^9 \overbrace{\sum_{a<9} \tfrac{1}{2}\, \underbrace{\big(\overline{\psi}\,\Gamma_{a_1 a_2}\,\psi\big)}_{-\big(\overline{\psi}\,\Gamma_{a_1 a_2}\,\underbrace{\Gamma_9\Gamma_{10}}_{\Gamma_9^B}\,\Gamma_9\Gamma_{10}\,\psi\big)} e^{a_1} e^{a_2}}^{\mathrm{s}f_5} \\
&= \tfrac{1}{3!}\big(\overline{\psi}\Gamma^B_{a_1 a_2 a_3}\Gamma_9\Gamma_{10}\psi\big) e^{a_1} e^{a_2} e^{a_3} \\
F_7 &= \overbrace{\sum_{a<9} \tfrac{1}{5!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_5}\Gamma_9\,\psi\big) e^{a_1}\cdots e^{a_5}}^{f_7} - e^9 \overbrace{\underbrace{\tfrac{1}{4!} \sum_{a<9} \big(\overline{\psi}\,\Gamma_{a_1\cdots a_4}\Gamma_{10}\,\psi\big)}_{-\big(\overline{\psi}\,\Gamma_{a_1\cdots a_4}\,\underbrace{\Gamma_9\Gamma_{10}}_{\Gamma_9^B}\,\Gamma_9\,\psi\big)} e^{a_1}\cdots e^{a_4}}^{\mathrm{s}f_7} \\
&= \tfrac{1}{5!}\big(\overline{\psi}\,\Gamma^B_{a_1\cdots a_5}\Gamma_9\,\psi\big) e^{a_1}\cdots e^{a_5} \\
F_9 &= \overbrace{\sum_{a_i<9} \tfrac{1}{7!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_7}\Gamma_9\Gamma_{10}\,\psi\big) e^{a_1}\cdots e^{a_7}}^{f_9} - e^9 \overbrace{\sum_{a_i<9} \tfrac{1}{6!}\, \underbrace{\big(\overline{\psi}\,\Gamma_{a_1\cdots a_6}\,\psi\big)}_{-\big(\overline{\psi}\,\Gamma_{a_1\cdots a_6}\,\underbrace{\Gamma_9\Gamma_{10}}_{\Gamma_9^B}\,\Gamma_9\Gamma_{10}\,\psi\big)} e^{a_1\cdots a_6}}^{\mathrm{s}f_9} \\
&= \tfrac{1}{7!}\big(\overline{\psi}\,\Gamma^B_{a_1\cdots a_7}\Gamma_9\Gamma_{10}\,\psi\big) e^{a_1}\cdots e^{a_7} \\
F_{11} &= \overbrace{\sum_{a_i<9} \tfrac{1}{9!}\big(\overline{\psi}\,\Gamma_{a_1\cdots a_9}\Gamma_9\,\psi\big) e^{a_1}\cdots e^{a_9}}^{f_{11}} - e^9 \overbrace{\sum_{a_i<9} \tfrac{1}{8!}\, \underbrace{\big(\Gamma_{a_1\cdots a_8}\Gamma_{10}\big)}_{-\big(\overline{\psi}\,\Gamma_{a_1\cdots a_8}\,\underbrace{\Gamma_9\Gamma_{10}}_{\Gamma_9^B}\,\Gamma_9\,\psi\big)} e^{a_1}\cdots e^{a_8}}^{\mathrm{s}f_{11}} \\
&= \tfrac{1}{9!}\big(\overline{\psi}\,\Gamma^B_{a_1\cdots a_9}\Gamma_9\,\psi\big) e^{a_1}\cdots e^{a_9}
\end{aligned}
$$

☐

Remark 1.34 (Lorentz invariance of IIB super-fluxes). While the proof of Prop. 1.33 does not make manifest that the resulting super-translation invariants (80) are also Spin(1, 9)-invariant, this is immediate by Prop. 1.27.

Remark 1.35 (T-Dual NS-Flux [FSS18a, Rem. 5.4]**).** The action of the T-duality operation from Prop. 1.33 on the NS-flux densities $H_3^{A/B}$ is particularly interesting: Note that both these fluxes come out as the sum of (1.) the basic H_3-flux in 9d (pulled back to 10d along the corresponding fibration) with (2.) the fiber form e^9 times the Chern class classifying the *other* extension:

$$
\begin{tikzcd}
\begin{array}{rcl} H_3^A &=& \overbrace{H_3}^{h_3} + e^9_A \overbrace{\big(\overline{\psi}\,\Gamma_9\Gamma_{10}\,\psi\big)}^{-\mathrm{s}h_3} \\ &=& H_3 + e^9 \quad c_1^B \end{array} \arrow[rr, mapsto, "\text{superspace T-duality}"] \arrow[dr, mapsto, "\mathrm{rdc}_{c_1^A}", "\text{reduction along IIA extension}"'] & & \begin{array}{rcl} H_3^B &=& H_3 + e^9\big(\overline{\psi}\,\Gamma_9\,\psi\big) \\ &=& H_3 + e^9_B \quad c_1^A \end{array} \\
& \begin{array}{rcl} H_3 &=& \sum\limits_{a<9} e^a \big(\overline{\psi}\,\Gamma_a\Gamma_{10}\,\psi\big) \\ c_1^A &=& \big(\overline{\psi}\,\Gamma_9\,\psi\big) \\ c_1^B &=& \big(\overline{\psi}\,\Gamma_9^B\,\psi\big) \end{array} \arrow[ur, mapsto, "\mathrm{oxd}_{c_1^B}", "\text{oxidation along IIB extension}"'] &
\end{tikzcd}
\tag{82}
$$

In other words this says that the fiber integration of H_3^A from the type IIA spacetime down to 9d is the Chern class classifying the type IIB extension, and vice versa:

$$
p^A_* \, H^A_3 \; = \; c_1^B \qquad \text{and} \qquad p^B_* \, H^B_3 \; = \; c_1^A \,.
$$

The analogous phenomenon in ordinary T-duality (i.e., not on super-flux densities over super-spacetime as considered here) was originally proposed in [BEM04, (1.8)] and gave rise to the mathematical notion of "topological T-duality".

While the formalism of topological T-duality has worked wonders, its actual relation to string/M-theory rests on educated guesses (though much progress was recently made when [Wa24] related it to the Buscher rules). Here it is interesting that we find (82) this relation being hard-coded in the DNA of supergravity.

Doubled super-spacetime. This suggests to turn attention to the "doubled" or "correspondence space" of the IIA/B superspacetimes:

Definition 1.36 (**$\mathbb{R}^1$-Doubled super-space** [FSS18a, Def. 6.1])**.** Write $\mathbb{R}^{1,8=(1+1)\,|\,32}$ for the super-Lie algebra given by extending the 9d super type II spacetime by *both* the Chern class classifying the IIA extension as well as that classifying the IIB extension:

$$\mathrm{CE}\big(\mathbb{R}^{1,8+(1+1)\,|\,32}\big) \;\simeq\; \mathbb{R}_{\mathrm{d}}\left[\begin{array}{c} (\psi^\alpha)_{\alpha=0}^{32} \\ (e^a)_{a=0}^{8} \\ e^9_A, e^9_B \end{array}\right] \Big/ \left(\begin{array}{ccc} \mathrm{d}\,\psi & = & 0 \\ \mathrm{d}\,e^a & = & \big(\overline{\psi}\,\Gamma^a\,\psi\big) \\ \mathrm{d}\,e^9_{A/B} & = & \big(\overline{\psi}\,\Gamma^9_{A/B}\,\psi\big) \end{array}\right) \tag{83}$$

This may be understood as the fiber product of the IIA- with the IIB extension, making a Cartesian square of super-Lie algebras, as follows:

$$\begin{tikzcd}
 & \mathbb{R}^{1,8+(1+1)\,|\,32} \arrow[dl, two heads, "p_B"'] \arrow[dr, two heads, "p_A"] \arrow[dd, phantom, "\text{(pb)}"] & \\
\mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} \arrow[dr, two heads] & & \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\mathbf{16}} \arrow[dl, two heads] \\
 & \mathbb{R}^{1,8\,|\,\mathbf{16}\oplus\mathbf{16}} &
\end{tikzcd}$$

We say that

$$P_2 \;:=\; e^9_A\, e^9_B \;\in\; \mathrm{CE}\big(\mathbb{R}^{1,8+(1+1)\,|\,32}\big) \tag{84}$$

is the *twisted Poincaré super 2-form* (or just *Poincaré form*, for short) on the doubled super spacetime.

With this we may concisely re-state Rem. 1.35 as follows:

Proposition 1.37 (**Poincaré form trivializes difference of T-dual NS super-fluxes** [FSS18a, Prop. 6.2])**.** *The pullbacks of the NS super-flux densities to the doubled super-spacetime* (83) *differ by the differential of the twisted Poincaré 2-form* (84)

$$\mathrm{d}P_2 \;=\; p_B^\ast H_3^B - p_A^\ast H_3^A\,. \tag{85}$$

Proof. Via Prop. 1.33 and Rem. 1.35 this follows straightforwardly:

$$\begin{aligned}
p_B^\ast H_3^B - p_A^\ast H_3^A \;&=\; \Big(\sum_{a<9} e^a\big(\overline{\psi}\,\Gamma_a\,\psi\big) + e^9_B\,\big(\overline{\psi}\,\Gamma_9\,\psi\big)\Big) \;-\; \Big(\sum_{a<9} e^a\big(\overline{\psi}\,\Gamma_a\,\psi\big) + e^9_A\,\big(\overline{\psi}\,\Gamma_9\Gamma_{10}\,\psi\big)\Big) \\
&=\; e^9_B\,\big(\overline{\psi}\,\Gamma_9\,\psi\big) \;-\; e^9_A\,\big(\overline{\psi}\,\Gamma_9\Gamma_{10}\,\psi\big) \\
&=\; \mathrm{d}\big(e^9_A\,e^9_B\big) \;\equiv\; \mathrm{d}\,P_2\,.
\end{aligned}$$

□

Remark 1.38 (**Poincaré 2-form in T-duality literature** [FSS18a, Rem. 6.3])**.** That the analogue of the relation (85) should hold for ordinary T-duality (i.e. disregarding super-flux densities on super-spacetimes as considered here) was originally proposed by [BEM04, (1.12)]. As previously in Rem. 1.35, here it is interesting to find these phenomena hard-coded in the DNA of supergravity.

In fact, understanding the super-Lie algebraic content of Prop. 1.37 through the lens of (super-)rational homotopy theory (essentially via Ex. 1.4) then it reproduces the image under rationalization of topological T-duality in the form proposed in [BRS06, Def. 2.8].

A transparent understanding of how the (twisted) Poincaré 2-form and its Bianchi identity (85) controls the *Buscher rules* of T-duality was more recently obtained in [Wa24, Lem. 3.3.1(c)].

F-Theory super-spacetime. Given that the derivation began in (71) on 11D super-spacetime, going through its reduction to 10D IIA super-spacetime, to arrive at the "doubled" version of 10d super-spacetime (83), it is natural to ask for the doubling to take place already in 11D, hence for extending 9D super-spacetime by *all three* extra dimensions: (1.) the IIA fiber, (2.) the IIB fiber and (3.) the M fiber.

In terms of super-Lie algebraic local model spaces this request is immediate to satisfy:

Definition 1.39 (**F-theory super-spacetime** [FSS18a, Def. 8.1, Prop. 8.3])**.** Write $\mathbb{R}^{1,9+(1+1)\,|\,32}$ for the super-Lie algebra given by

$$\mathrm{CE}\big(\mathbb{R}^{1,9+(1+1)\,|\,32}\big) \;\simeq\; \mathbb{R}\left[\begin{array}{c} (\psi^\alpha)_{\alpha=1}^{32} \\ (e^a_A)_{a=1}^{9} \\ e^{10}, e^9_A, e^9_B \end{array}\right] \Big/ \left(\begin{array}{lll} \mathrm{d}\,\psi^\alpha & = 0 & \\ \mathrm{d}\,e^a & = \big(\overline{\psi}\,\Gamma^a\,\psi\big) & \\ \mathrm{d}\,e^9_A & = \big(\overline{\psi}\,\Gamma^9\,\psi\big) & = \big(\overline{\psi}\,\sigma_1\,\psi\big) \\ \mathrm{d}\,e^{10} & = \big(\overline{\psi}\,\Gamma^{10}\,\psi\big) & = \big(\overline{\psi}\,\sigma_2\,\psi\big) \\ \mathrm{d}\,e^9_B & = \big(\overline{\psi}\,\Gamma^9\Gamma^{10}\,\psi\big) & = \big(\overline{\psi}\,\sigma_3\,\psi\big) \end{array}\right) \tag{86}$$

(using the notation (68) on the right) which is equivalently the homotopy-fiber product of the 11D super-spacetime with the doubled 10D super-spacetime:

$$\begin{tikzcd}
 & \mathbb{R}^{1,9+(1+1)\,|\,32} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\text{(pb)}"] & \\
\mathbb{R}^{1,10\,|\,\mathbf{32}} \arrow[dr, two heads, "p_M"'] & & \mathbb{R}^{1,8+(1+1)\,|\,32} \arrow[dl, two heads, "p_A"] \\
 & \mathbb{R}^{1,9\,|\,\mathbf{16}\oplus\overline{\mathbf{16}}} &
\end{tikzcd}$$

By inspection one sees that (cf. also [Sak00]):

Proposition 1.40 (**Superspace S-duality on F-theory super-spacetime** [FSS18a, Prop. 8.6])**.** *The group* Pin(2) *of Prop. 1.26 acts by super-Lie automorphisms on the F-theory super-spacetime* (86) *under which (the pullback of) flux densities* H_3^B *and* F_3 (from Prop. 1.33: the F1- and the D1-string couplings) *span the 2-dimensional vector representation*

$$\begin{array}{ccc}
\mathbb{R}^{1,9+(1+1)\,|\,32} & \xrightarrow{\;\exp(\frac{t}{2}\sigma_3)\;} & \mathbb{R}^{1,9+(1+1)\,|\,32} \\
e^{\frac{t}{2}\Gamma_9\Gamma_{10}}\psi & \longleftarrow\!\shortmid & \psi \\
\cos(t)\,e^9 + \sin(t)\,e^{10} & \longleftarrow\!\shortmid & e^9 \\
\cos(t)\,e^{10} - \sin(t)\,e^9 & \longleftarrow\!\shortmid & e^{10} \\
\cos(t)\,F_3 + \sin(t)\,H_3^A & \longleftarrow\!\shortmid & F_3 \\
\cos(t)\,H_3^A - \sin(t)\,F_3 & \longleftarrow\!\shortmid & H_3^A
\end{array}$$

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