TED K-Theory of Cohomotopy Moduli Spaces and Anyonic Topological Order

Urs Schreiber on joint work with Hisham Sati

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New York University, Abu Dhabi

Center for Quantum & Topological Systems

talk at:

Higher Structures and Field Theory @ ESI Vienna, 25 Aug 2022

slides and pointers at: `ncatlab.org/schreiber/show/TED-K+of+Cohomotopy+and+Anyons`

Introduction

(1) – TED K-Theory
via Cohesive ∞-Topos Theory

(2) – Interacting enhancement
via Hypothesis H

(3) – Anyon braiding
via Cohesive Homotopy Type Theory

Summary

It is largely folklore that:

Topological K-theory

fully Twisted & Equivariant & Differential (TED)

classifies

free topological phases
in condensed matter theory

stable D-branes
in string theory

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anyon statistics for topological quantum gates		$N \sim 1$ YM theory for hadrodynamics

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(1) Systematic construction of TED K-theory using cohesive ∞-topos theory

(for finite equivariance as befits the "very good" orbifolds appearing in CMT and ST)

[arX:2008.01101][arX:2009.11909][arX:2011.06533][arX:2203.11838][SS22-TEC]

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$$
\begin{tikzcd}
 & \mathrm{Fred} /\!\!/ \mathrm{PU}(\mathcal{H}) \arrow[d] \\
\underset{\text{orbi-singularity}}{X \times * /\!\!/ G} \arrow[r, "\tau"', "\text{equivariant twist}"'] \arrow[ur, dashed, "\text{TE K-cocycle}"] & \underset{\substack{\text{universal stacky} \\ \text{Fredholm bundle}}}{\mathbf{B}\mathrm{PU}(\mathcal{H})}
\end{tikzcd}
\quad \overset{\substack{\text{mapping stack} \\ \text{adjunction}}}{\longleftrightarrow} \quad
\begin{tikzcd}
 & \mathrm{Fred}^G /\!\!/ G^* \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & \mathrm{Maps}(\mathbf{B}G, \mathrm{Fred} /\!\!/ \mathrm{PU}(\mathcal{H}))^{\mathrm{stbl}} \arrow[d] \\
X \arrow[r, "\text{inner local system}", "\text{inside singularity}"'] \arrow[ur, dashed, "\substack{\text{adjunct} \\ \text{TE K-cocycle}}"] \arrow[rr, bend right, "\text{adjunct equivariant twist}"'] & \mathbf{B}G^* \arrow[r, "\text{non-trivial}", "\text{equivariant } \pi_1"'] & \mathrm{Maps}(\mathbf{B}G, \mathbf{B}\mathrm{PU}(\mathcal{H}))^{\mathrm{stbl}}
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Evaluate TED K-cohomology not on Brillouin torus/spacetime-orbifold itself,
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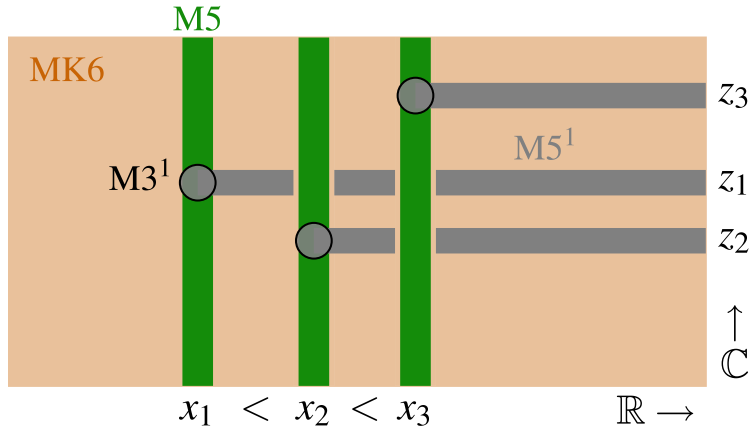
(see below)

$$\underbrace{\coprod_n \underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\mathbb{C})}_{\substack{\text{Configuration space of}\\ \text{ordered points in the plane}}} \;\simeq\; \overbrace{\bigcup_n \mathrm{Conf}_n\big(\mathbb{C};\, \mathbb{R}_{\mathrm{cpt}}\big)}^{\substack{\text{3-Cohomotopy cocycle space}\\ \text{for codim=1 branes}\\ \mathrm{Map}^*(\mathbb{R}_+ \wedge \mathbb{C}_{\mathrm{cpt}},\, S^3)\,\simeq}} \underset{\substack{\bigcup_n \mathrm{Conf}_n\big(\ast;\, (\mathbb{R}\times\mathbb{C})_{\mathrm{cpt}}\big)\\ \text{Fiber product of respective configuration spaces}\\ \text{(of un-ordered points escaping to transverse infinity)}\\ \text{reflecting the brane intersections}}}{\times} \overbrace{\bigcup_n \mathrm{Conf}_n\big(\mathbb{R};\, \mathbb{C}_{\mathrm{cpt}}\big)}^{\substack{\text{3-Cohomotopy cocycle space}\\ \text{for codim-2 branes}\\ \mathrm{Map}^*(\mathbb{R}_{\mathrm{cpt}} \wedge \mathbb{C}_+,\, S^3)\,\simeq}}$$

e.g.: $\underset{\{1,\cdots,3\}}{\mathrm{Conf}}(\mathbb{C}) \;\simeq\; \left\{ \right.$  $\left. \right\}$

The moduli space of flat M3-branes according to *Hypothesis H* is the configuration space of ordered points in their transverse plane.

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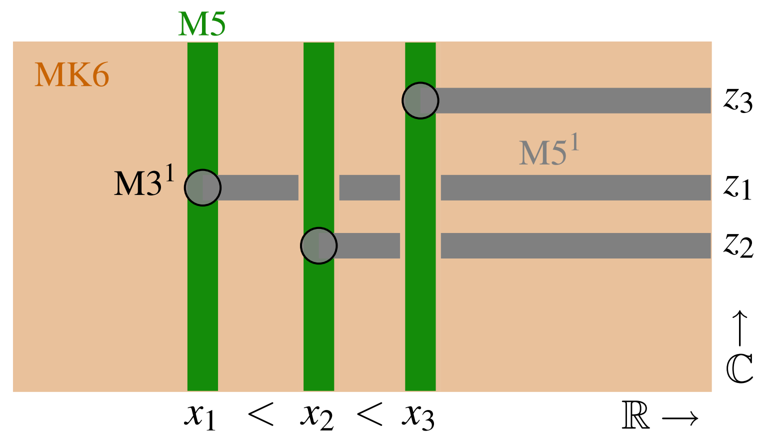

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Claim: The TED K-cohomology of n-point configurations in Brillouin torus classifies valence bundle of n-electron interacting states [`arX:2206.13563`]

(3) Concrete implementation of topological quantum gates via `TED-K` in cohesive homotopy type theory:

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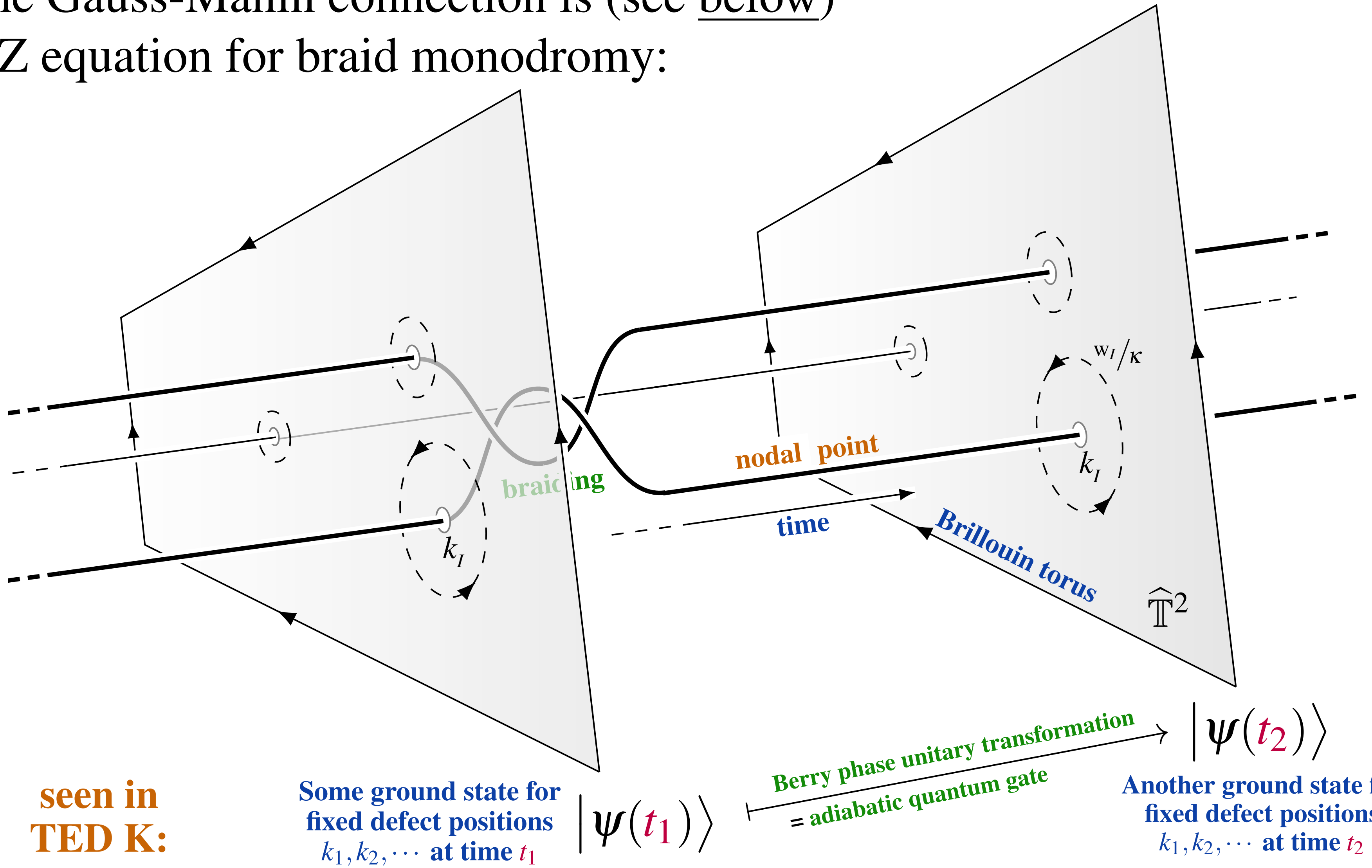

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Remarkably, for such constructions in cohesive ∞-topos theory there is developed a programming language: "cohesive HoTT" [EPTCS **158** (2014)] [`arX:1402.7041`]

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Programming platform:

Cohesive Homotopy Type Theory with dependent linear types

$\xrightarrow[\text{(1)}]{\text{implements}}$

Library/Module:

TED-K-cohomology of defect configurations in crystallographic orbifolds

$\xrightarrow[\text{(2)}]{\text{emulates}}$

Hardware platform:

Anyonic quantum states in topological phases of quantum materials

$\downarrow$ runs (3)

Topological quantum braid gates and circuits

Cohesive Homotopy Type Theory $\xrightarrow{\text{topological quantum programming}}$ Topological quantum braid gates and circuits

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(2) – Interacting enhancement
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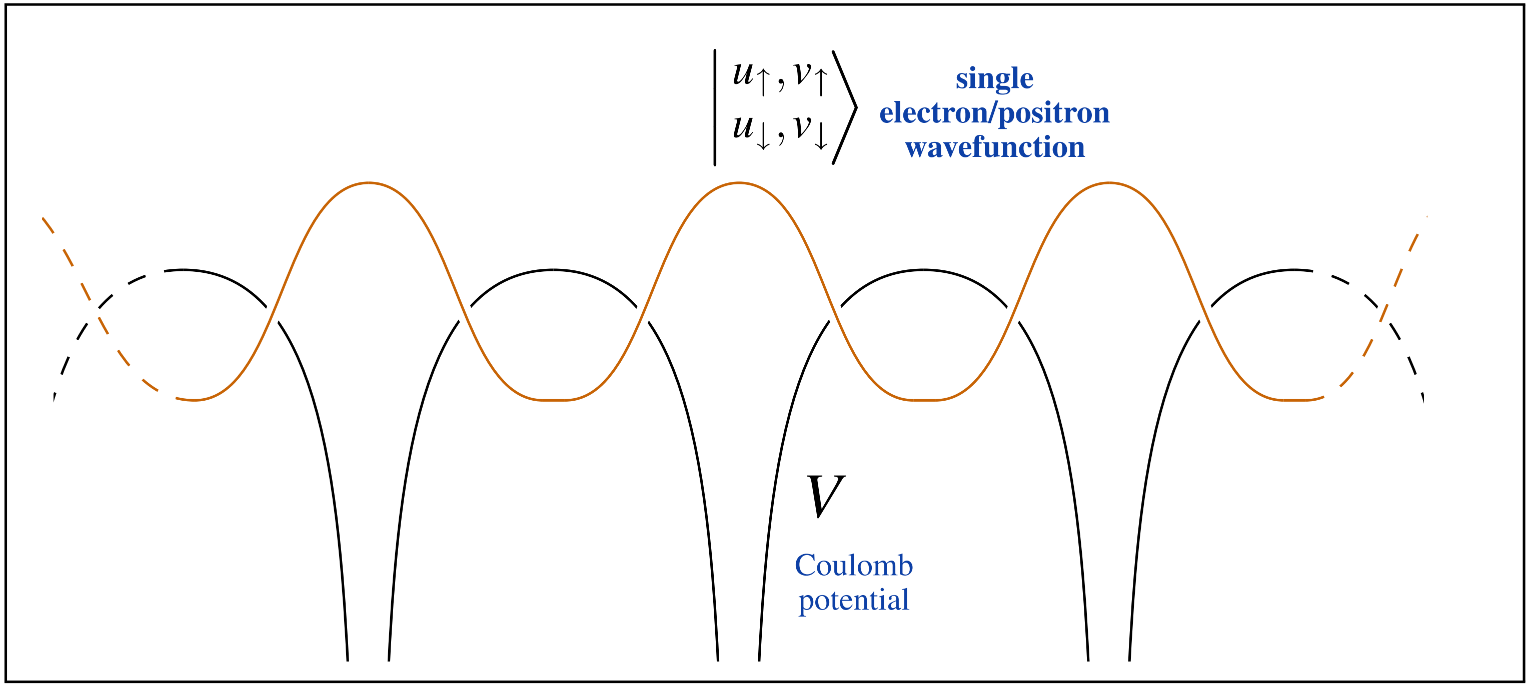
Summary

This part is a
quick motivation and exposition of
TED K-theory
following these articles:

Proper Orbifold Cohomology	[arX:2008.01101]
The twisted non-abelian character map	[arX:2009.11909]
Equivariant Principal ∞-bundles	[arX:2112.13654]
Anyonic Defect Branes in TED-K-Theory	[arX:2203.11838]
The twisted equivariant character map	[SS22-TEC]

Vacua of electron/positron field in Coulomb background.

Fact ([KS77][CHO82]). The vacua of the free Dirac quantum field in a classical Coulomb background...



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$$
\overbrace{\ker(F)}^{\text{finite-dimensional kernel}} \hookrightarrow \mathcal{H} \xrightarrow[\text{bounded linear}]{\overset{\text{Fredholm operator}}{F}} \mathcal{H} \twoheadrightarrow \overbrace{\operatorname{coker}(F)}^{\text{finite-dimensional cokernel}}
$$

$$
\underbrace{\psi \in \mathcal{H} \,\big|\, \forall_\phi \ \langle \phi | F | \psi \rangle = 0}_{\ker(F)} \qquad \underbrace{\psi \in \mathcal{H} \,\big|\, \forall_\phi \ \langle \psi | F | \phi \rangle = 0}_{\operatorname{coker}(F)}
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on the single-electron/positron Hilbert space:

$$
\begin{tikzcd}
\overset{\text{electron states in dressed vacuum}}{\ker(F)} \arrow[r, hook] & \overset{\text{single electron Hilbert space}}{\mathcal{H}} \arrow[ddrr, "F"] & & \mathcal{H} & \\
& \oplus & \text{dressed vacuum Fredholm operator} & \oplus & \\
& \underset{\text{single positron Hilbert space}}{\mathcal{H}} \arrow[uurr, "F^*"] & & \mathcal{H} \arrow[r, two heads] & \overset{\text{positron states in dressed vacuum}}{\operatorname{coker}(F)}
\end{tikzcd}
$$

$$
\begin{aligned}
\overset{\text{total charge in dressed vacuum}}{\operatorname{ind}(F)} &= \overset{\text{number of electrons in dressed vacuum state}}{\dim\big(\ker(F)\big)} - \overset{\text{number of positrons in dressed vacuum state}}{\dim\big(\operatorname{coker}(F)\big)} \\
&= \dim\big(\operatorname{coker}(F^*)\big) - \dim\big(\ker(F^*)\big)
\end{aligned}
$$

Quantum symmetries.

On these dressed vacua of electron/positron states
the following *CPT-twisted projective group*

$$
\overset{\text{even projective unitary group}}{\frac{\mathrm{U}(\mathcal{H}) \times \mathrm{U}(\mathcal{H})}{\mathrm{U}(1)}} \rtimes \Big(\underbrace{\overset{\text{grading involution}}{\mathbb{Z}_2}}_{\{\mathrm{e}, P\}} \times \underbrace{\overset{\text{complex conjugation}}{\mathbb{Z}_2}}_{\{\mathrm{e}, T\}} \Big)
$$

group of quantum symmetries

$$
C := PT\,, \quad P \cdot \big[U_+, U_-\big] := \big[U_-, U_+\big] \cdot P\,, \qquad T \cdot \big[U_+, U_-\big] := \big[\overline{U}_+, \overline{U}_-\big] \cdot T
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naturally acts by conjugation:

$$\begin{array}{rccc} [U_+, U_-] & : & F & \longmapsto & U_+^{-1} \circ F \circ U_- \\ C \cdot [U_+, U_-] & : & F & \longmapsto & U_-^{-1} \circ F^t \circ U_+ \\ P \cdot [U_+, U_-] & : & F & \longmapsto & U_-^{-1} \circ F^* \circ U_+ \\ T \cdot [U_+, U_-] & : & F & \longmapsto & U_+^{-1} \circ \overline{F} \circ U_- \end{array}$$

Twisted equivariant KR-theory – As a single diagram of smooth groupoids.

Homotopy classes of quantum-symmetry equivariant families
of such self-adjoint odd Fredholm operators
constitute *twisted equivariant* KR-*cohomology*:

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\mathrm{KR}^{\tau}_{G}(\mathrm{X}) \;:=\; \left\{ \begin{tikzcd} & \overset{\text{space of self-adjoint odd Fredholm operators}}{\mathrm{Fred}^0_{\mathbb{C}}} /\!\!/ \overset{\text{group of quantum symmetries}}{\left(\frac{\mathrm{U}(\mathcal{H})\times \mathrm{U}(\mathcal{H})}{\mathrm{U}(1)} \rtimes \{\mathrm{e},P\}\times\{\mathrm{e},T\}\right)} \arrow[d] \\ \overset{\text{orbi-orientifold}}{\mathrm{X}/\!\!/G} \arrow[ur, dashed, "\text{equivariant family of Fredholm operators}", "\text{cocycle in TE-K-theory}"'] \arrow[r, "\text{twist } \tau"'] & {} \end{tikzcd} \right\} \Big/ \sim_{\mathrm{htpy}}
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$$

Free topological phases of matter.

$\Rightarrow$ Idea: *Single*-particle valence bundle of electrons in crystalline insulator
classified by topological K-theory of Brillouin torus
equivariant wrt quantum symmetries [Kitaev 09] [FM12]

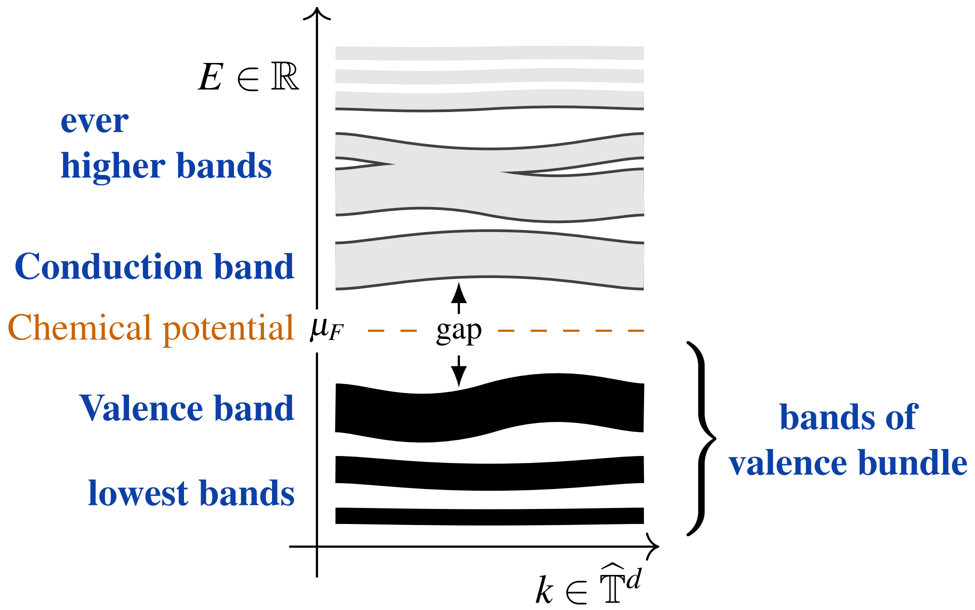
Single particle valence bundle

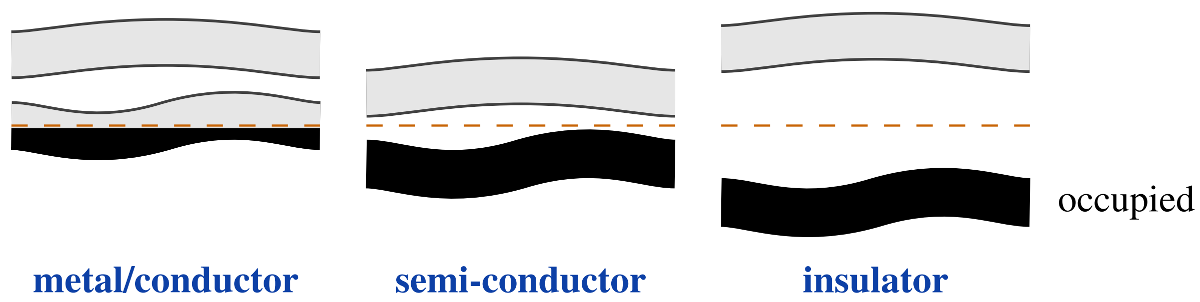
$$\mathcal{V} = \left\{ k \in \widehat{\mathbb{T}}^d, |\psi\rangle \in \mathcal{H} \oplus \mathcal{H} \;\middle|\; \big|\langle\psi|H_k|\psi\rangle\big| \leq \mu_F \right\} \subset \mathcal{B}$$

Bundle of all relativistic Bloch states

$$\begin{array}{c} \mathcal{V} \\ \downarrow \\ \widehat{\mathbb{T}}^d \end{array}$$

Brillouin torus of momenta in crystal





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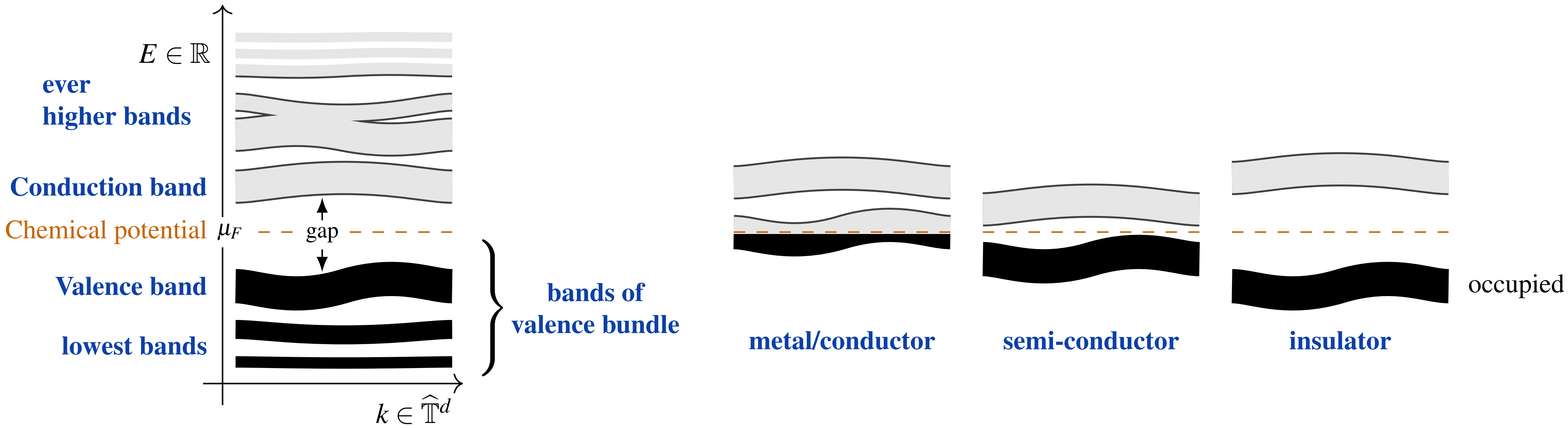
$\boxed{\to \underline{n\text{-particle story}}}$



CPT Quantum symmetries.

$$
\begin{tikzcd}
\mathbf{B}\big(\{\mathrm{e}, T\}\big) \arrow[rr, "{\substack{\textbf{pure quantum T-symmetry} \\ T \longmapsto \widehat{T}}}"] \arrow[dr, hook] & & \mathbf{B}\left(\dfrac{\mathrm{U}(\mathcal{H}) \times \mathrm{U}(\mathcal{H})}{\mathrm{U}(1)} \rtimes \{\mathrm{e}, T\} \right) \arrow[r] \arrow[dl] & \mathbf{B}\big(\mathbf{B}\mathrm{U}(1) \rtimes \{\mathrm{e}, T\}\big) \\
& \mathbf{B}\big(\{\mathrm{e}, P\} \times \{\mathrm{e}, T\}\big) & &
\end{tikzcd}
$$

Let's use the previous machinery to compute the possible quantum T-symmetries...

CPT Quantum symmetries.

$$
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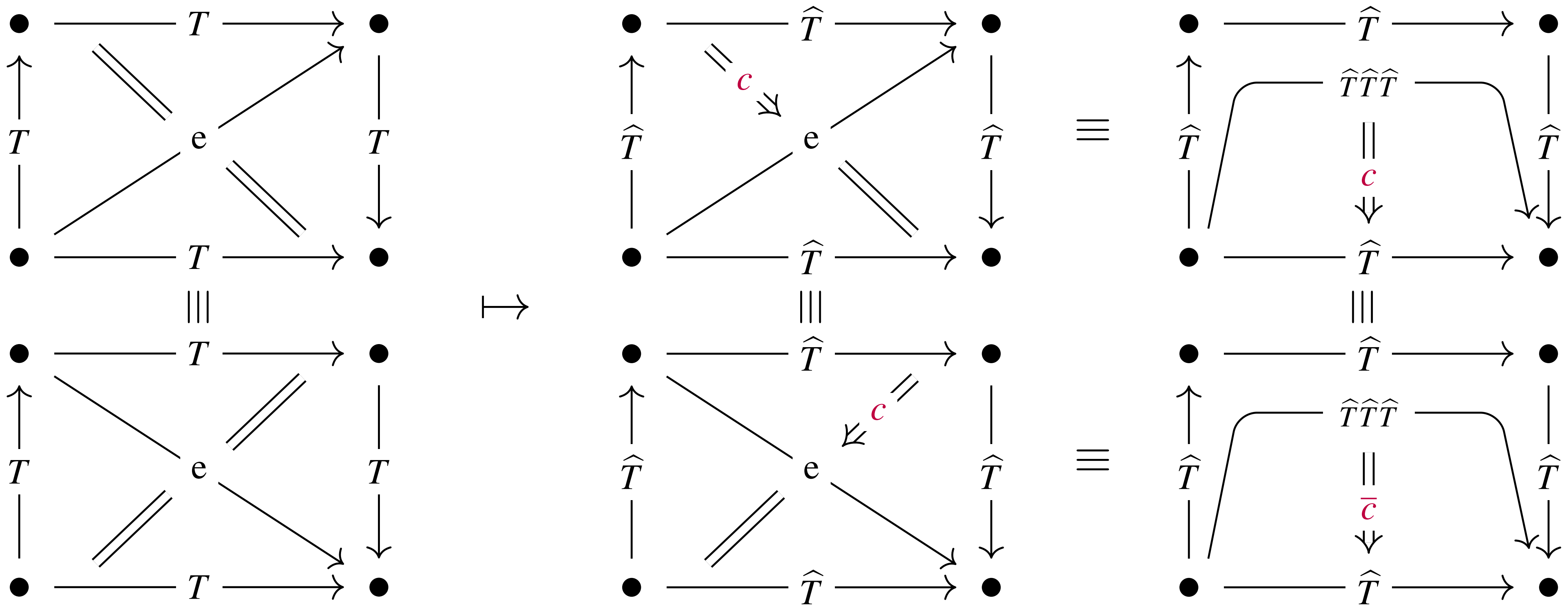
& \mathbf{B}\big(\{\mathrm{e}, P\} \times \{\mathrm{e}, T\}\big) & &
\end{tikzcd}
$$



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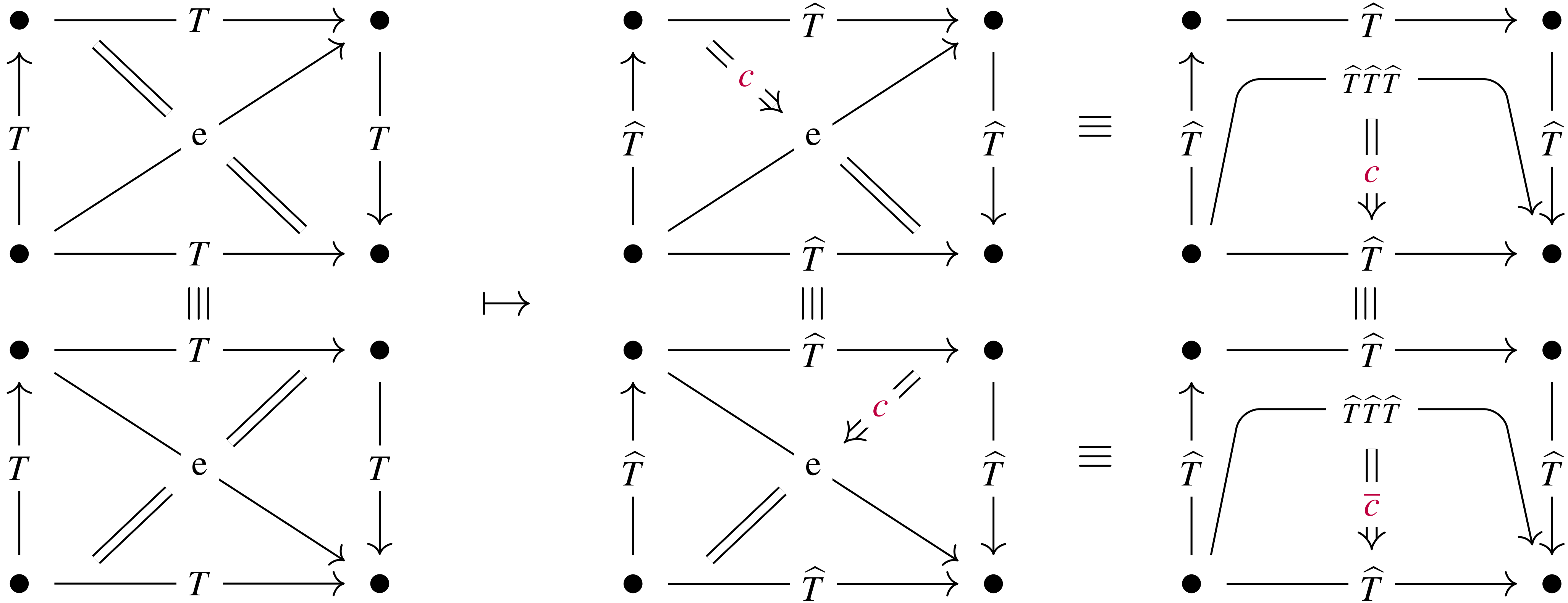
\end{tikzcd}
$$



So $\overline{c} = c$ and hence there are two choices for quantum T-symmetry, up to homotopy:

$$
\widehat{T}^2 = \pm 1 \quad \text{and similarly} \quad \widehat{C}^2 = \pm 1 \,.
$$

Example – Orientifold KR-theory

Let I be *I*nversion action on 2-torus $\widehat{\mathbb{T}}^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$ and trivial action on observables

$$
\begin{array}{ccc}
\mathbb{T}^2 & \xrightarrow{\;\;I\;\;} & \mathbb{T}^2 \\
k & \longmapsto & -k\,,
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathrm{Fred}^0_{\mathbb{C}} & \xrightarrow{\;\;I\;\;} & \mathrm{Fred}^0_{\mathbb{C}} \\
F & \longmapsto & F\,.
\end{array}
$$

If T acts as I on $\mathbb{T}^2$, then $\mathrm{KR}^{\widehat{T}^2 = +1}$ is *Atiyah's Real K-theory* aka *orienti-fold* K-theory:

$$
\mathrm{KR}\big(\widehat{\mathbb{T}}^{0,2}\big) \;\simeq\; \left\{
\begin{tikzcd}
 & & \mathrm{Fred}^0_{\mathbb{C}} /\!\!/ \big(\mathrm{U}(\mathcal{H}) \rtimes \{\mathrm{e}, T\}\big) \arrow[d] \\
\mathbb{T}^2 /\!\!/ \{\mathrm{e}, I\} \arrow[urr, dashed] \arrow[rr, "\widehat{T}^2 = +1"] \arrow[dr, "I \mapsto T"', "\text{inversion of space}"] & {} \arrow[d, phantom, "\text{combined with}"] & \mathbf{B}\big(\mathrm{U}(\mathcal{H}) \rtimes \{\mathrm{e}, T\}\big) \arrow[dl, "\text{complex conj. of observables}"'] \\
 & \mathbf{B}\{\mathrm{e}, T\} &
\end{tikzcd}
\right\}_{\big/ \sim_{\mathrm{htpy}}}
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 & \mathbf{B}\{\mathrm{e},T\} &
\end{tikzcd}
\right\}_{\big/\sim_{\mathrm{htpy}}}
$$

But what happens on I-fixed loci i.e. on "orientifolds" ? $\longrightarrow$

CPT Quantum symmetries – 10 global choices. (following [FM12, Prop. 6.4])

Equivariance group	$G =$	$\{\mathrm{e}\}$	$\{\mathrm{e}, P\}$	$\{\mathrm{e}, T\}$		$\{\mathrm{e}, C\}$		$\{\mathrm{e}, T\} \times \{\mathrm{e}, C\}$			
Realization as quantum symmetry τ :	$\widehat{T}^2 =$			$+1$	-1			$+1$	-1	-1	$+1$
	$\widehat{C}^2 =$					$+1$	-1	$+1$	$+1$	-1	-1
Maximal induced Clifford action anticommuting with all G-invariant odd Fredholm operators	$E_{-3} =$								$\mathrm{i}\widehat{T}\widehat{C}\beta$		
	$E_{-2} =$					$\mathrm{i}\widehat{C}\beta$			$\mathrm{i}\widehat{C}\beta$		
	$E_{-1} =$		$\widehat{P}\beta$			$\widehat{C}\beta$		$\widehat{C}\beta$	$\widehat{C}\beta$		
	$E_{+0} =$	β	β	β	$\begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}$	β	β	β	β	β	β
	$E_{+1} =$				$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$		$\widehat{C}\beta$			$\widehat{C}\beta$	$\widehat{C}\beta$
	$E_{+2} =$				$\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$		$\mathrm{i}\widehat{C}\beta$			$\mathrm{i}\widehat{C}\beta$	
	$E_{+3} =$				$\begin{pmatrix} 0 & -\widehat{T} \\ \widehat{T} & 0 \end{pmatrix}$					$\mathrm{i}\widehat{T}\widehat{C}\beta$	
	$E_{+4} =$				$\begin{pmatrix} 0 & \mathrm{i}\widehat{T} \\ \mathrm{i}\widehat{T} & 0 \end{pmatrix}$						
τ-twisted G-equivariant KR-theory of fixed loci	$\mathrm{KR}^\tau =$	KU^0	KU^1	KO^0	KO^4	KO^2	KO^6	KO^1	KO^3	KO^5	KO^7

$$
\left\{
\begin{array}{lll|cc ll}
\text{bounded opers.} & \widehat{F} : \mathcal{H}^2 \xrightarrow[\mathbb{K}-\text{linear}]{\text{bounded}} \mathcal{H}^2 & & & & \text{bounded oper.} & E_0, \cdots, E_p : \mathcal{H}^2 \xrightarrow[\mathbb{K}-\text{linear}]{\text{bounded}} \mathcal{H}^2 \\
\text{self-adjoint} & \widehat{F}^* = \widehat{F} := F + F^* & & \begin{array}{c}\text{graded comm.} \\ E_i \circ \widehat{F} = -\widehat{F} \circ E_i\end{array} & \text{with} & \text{(anti-)self-adjoint} & (E_i)^* = \mathrm{sgn}_i \cdot E_i \\
\text{Fredholm} & \dim\big(\ker\big(\widehat{F}\big)\big) < \infty & & & & \text{Clifford gen.} & E_i \circ E_j + E_j \circ E_i = 2\mathrm{sgn}_i \cdot \delta_{ij}
\end{array}
\right\}
$$

$=: \mathrm{Fred}^{-p}_{\mathbb{C}}$

$$
\text{[Karoubi 70]:} \quad \left\{ X \xrightarrow[\text{cnts}]{} \mathrm{Fred}^p_{\mathbb{K}} \right\} \Big/_{\sim \text{htpy}} \quad = \quad \left\{ \begin{array}{ll} \mathrm{KU}^p(X) = \mathrm{KU}^{p+2}(X) & \mid \quad \mathbb{K} = \mathbb{C} \\ \mathrm{KO}^p(X) = \mathrm{KO}^{p+8}(X) & \mid \quad \mathbb{K} = \mathbb{R} \end{array} \right.
$$

	$E_{-3} =$								$\mathrm{i}\widehat{T}\widehat{C}\beta$		
	$E_{-2} =$					$\mathrm{i}\widehat{C}\beta$			$\mathrm{i}\widehat{C}\beta$		
	$E_{-1} =$		$\widehat{P}\beta$			$\widehat{C}\beta$		$\widehat{C}\beta$	$\widehat{C}\beta$		
Maximal induced Clifford action anticommuting with all G-invariant odd Fredholm operators	$E_{+0} =$	β	β	β	$\begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}$	β	β	β	β	β	β
	$E_{+1} =$				$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$		$\widehat{C}\beta$			$\widehat{C}\beta$	$\widehat{C}\beta$
	$E_{+2} =$				$\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$		$\mathrm{i}\widehat{C}\beta$			$\mathrm{i}\widehat{C}\beta$	
	$E_{+3} =$				$\begin{pmatrix} 0 & -\widehat{T} \\ \widehat{T} & 0 \end{pmatrix}$					$\mathrm{i}\widehat{T}\widehat{C}\beta$	
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τ-twisted G-equivariant KR-theory of fixed loci	$\mathrm{KR}^\tau =$	KU^0	KU^1	KO^0	KO^4	KO^2	KO^6	KO^1	KO^3	KO^5	KO^7

Example – TI-equivariant KR-theory is KO^0-theory.

The combination $T \cdot I$ acts trivially on the domain space and by complex conjugation on observables.

Hence $(T \cdot I)$-equivariant $(\widehat{T}^2 = +1)$-twisted KR-theory is KO^0-theory:

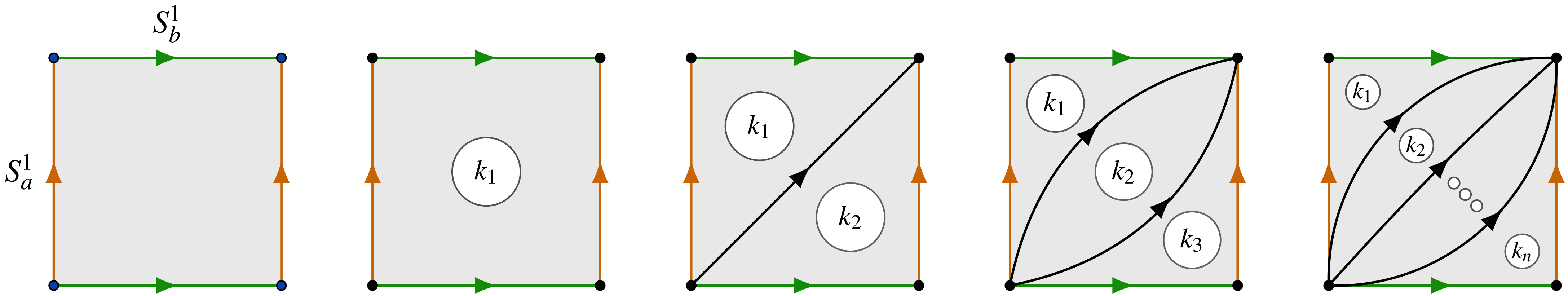
$$
\mathrm{KO}^0(X) \;\simeq\; \left\{
\begin{tikzcd}
 & & \mathrm{Fred}^0_{\mathbb{C}} /\!\!/ \big(\mathrm{U}(\mathcal{H}) \rtimes \{\mathrm{e}, T\}\big) \arrow[d] \\
X \times * /\!\!/ \{\mathrm{e}, TI\} \arrow[urr, dashed] \arrow[rr, "\widehat{T}^2 = +1"] \arrow[dr, "\substack{\text{no action}\\ \text{on space}}", "TI \mapsto T"'] & & \mathbf{B}\big(\mathrm{U}(\mathcal{H}) \rtimes \{\mathrm{e}, T\}\big) \arrow[dl, "\substack{\text{complex conj.}\\ \text{of observables}}"'] \\
 & \mathbf{B}\{\mathrm{e}, T\} \arrow[u, phantom, "\substack{\text{combined}\\ \text{with}}"] &
\end{tikzcd}
\right\} \Big/ \sim_{\mathrm{htpy}}
$$

$n =$	0	1	2	3	4	5	6	7	8	9	$\cdots$
$\mathrm{KO}^0(S^n_*) =$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\cdots$

Example – TI-equivariant KR-theory of punctured torus.

So the TI-equivariant $(\widehat{T}^2 = +1)$-twisted KR-theory of the N-punctured torus is

$$
\begin{aligned}
&\mathrm{KR}^{(\widehat{T}^2 = +1)}\left(\widehat{\mathbb{T}}^2 \setminus \{k_1, \cdots, k_N\}\right) \\
&\simeq \mathrm{KO}^0\left(\widehat{\mathbb{T}}^2 \setminus \{k_1, \cdots, k_N\}\right) \\
&\simeq \mathrm{KO}^0\left(\bigvee_{1+N} S^1_*\right) \qquad (N \geq 1) \\
&\simeq \bigoplus_{1+N} \mathbb{Z}_2
\end{aligned}
$$



$\widehat{\mathbb{T}}^2 \underset{\text{stbl}}{\simeq} S^1_a \vee S^1_b \vee S^2_{\text{bulk}}$

$\widehat{\mathbb{T}}^2 \setminus \{k_1\} \underset{\text{htpy}}{\simeq} S^1_a \vee S^1_b$

$\widehat{\mathbb{T}}^2 \setminus \{k_1, k_2\} \underset{\text{htpy}}{\simeq} S^1_a \vee S^1_b \vee S^1$

$\widehat{\mathbb{T}}^2 \setminus \{k_1, k_2, k_3\} \underset{\text{htpy}}{\simeq} S^1_a \vee S^1_b \vee S^1 \vee S^1$

$\widehat{\mathbb{T}}^2 \setminus \{k_1, \cdots, k_n\} \underset{\text{htpy}}{\simeq} \bigvee_{1+n} S^1$

The B-field twist.

Besides these CPT-quantum symmetries,
K-theory generically admits the famous *twisting by a B-field*:

The homotopy fiber sequence of 2-stacks discussed before

$$\mathbf{B}\mathrm{U}(\mathcal{H}) \longrightarrow \mathbf{B}\big(\mathrm{U}(\mathcal{H})/\mathrm{U}(1)\big) \xrightarrow[\text{}]{\overset{\textbf{universal Dixmier-Douady class}}{\mathrm{DD}}} \mathbf{B}^2\mathrm{U}(1)$$

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induces a surjection of equivalence classes of equivariant higher bundles

equivariant projective bundles **equivariant bundle gerbes**

$$\pi_0\ \mathrm{Maps}\Big(\widehat{\mathrm{X}/\!\!/G}, \mathbf{B}\big(\mathrm{U}(\mathcal{H})/\mathrm{U}(1)\big)\Big) \xrightarrow{\ \mathrm{DD}_*\ } \!\!\!\!\!\rightarrow \pi_0\,\mathrm{Maps}\Big(\widehat{\mathrm{X}/\!\!/G}, \mathbf{B}^2\mathrm{U}(1)\Big)$$

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$$\underset{\text{equivariant projective bundles}}{\pi_0 \ \mathrm{Maps}\Big(\widehat{\mathrm{X}/\!\!/G}, \mathbf{B}\big(\mathrm{U}(\mathcal{H})/\mathrm{U}(1)\big)\Big)} \xrightarrow{\mathrm{DD}_*} \underset{\text{equivariant bundle gerbes}}{\pi_0 \,\mathrm{Maps}\Big(\widehat{\mathrm{X}/\!\!/G}, \mathbf{B}^2\mathrm{U}(1)\Big)}$$

which has a natural section:

$$\underset{\text{equivariant bundle gerbes}}{\overset{\text{“stable twists”}}{\pi_0\mathrm{Maps}\big(\widehat{\mathrm{X}/\!\!/G}, \mathbf{B}^2\mathrm{U}(1)\big)}} \hookrightarrow \underset{\text{full quantum-symmetry twists}}{\pi_0 \,\mathrm{Maps}\bigg(\widehat{\mathrm{X}/\!\!/G}, \mathbf{B}\Big(\tfrac{\mathrm{U}(\mathcal{H})\times\mathrm{U}(\mathcal{H})}{\mathrm{U}(1)} \rtimes \big(\{\mathrm{e}, C\} \times \{\mathrm{e}, P\}\big)\Big)\bigg)}.$$

The B-field twist – Inner local systems.

On fixed loci (orbi-singularities)

$$\mathrm{X}/\!\!/G \;\simeq\; \mathrm{X} \times \ast /\!\!/ G \;=\; \mathrm{X} \times \mathbf{B}G$$

the B-field twist induces *secondary* twists by "inner local systems":

stable twists over fixed locus

$$\mathrm{Maps}\big(\mathrm{X} \times \ast /\!\!/ G,\, \mathbf{B}^2\mathrm{U}(1)\big) \simeq \mathrm{Maps}\big(\mathrm{X} \times \mathbf{B}G,\, \mathbf{B}^2\mathrm{U}(1)\big)$$

$$\simeq \mathrm{Maps}\big(\mathrm{X},\, \mathrm{Maps}(\mathbf{B}G,\, \mathbf{B}^2\mathrm{U}(1))\big)$$

The B-field twist – Inner local systems.

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Here we are assuming $G \underset{\mathrm{fin}}{\subset} \mathrm{SU}(2)$ so that $H^2_{\mathrm{Grp}}\big(G, \mathrm{U}(1)\big) = 0$.

And $G^\ast := \mathrm{Hom}\big(G, \mathrm{U}(1)\big)$ denotes the Pontrjagin-dual group.

The B-field twist – Inner local systems.

On fixed loci (orbi-singularities)

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$$\simeq \mathrm{Maps}\big(\mathrm{X},\, \mathrm{Maps}(\mathbf{B}G,\, \mathbf{B}^2\mathrm{U}(1))\big)$$

$$\simeq \mathrm{Maps}\big(\mathrm{X},\, \mathbf{B}G^* \times \mathbf{B}^2\mathrm{U}(1)\big)$$

$$\simeq \underbrace{\mathrm{Maps}\big(\mathrm{X},\, \mathbf{B}G^*\big)}_{\textbf{inner local systems}} \times \underbrace{\mathrm{Maps}\big(\mathrm{X},\, \mathbf{B}^2\mathrm{U}(1)\big)}_{\textbf{bundle gerbes}}$$

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Hence the

inner local system-twisted KU-*cohomology*
of a G-orbi-singularity of shape X

arises as follows:

$$\mathrm{KU}_G^{n+[\omega_1]}(X) \;=\; \left\{ \begin{tikzcd} & \mathrm{Fred}^n_{\mathbb{C}} /\!\!/ \mathrm{PU}(\mathcal{H}) \arrow[d] \\ X \times * /\!\!/ G \arrow[ur, dashed, "\text{cocycle}"] \arrow[r, "\tau"', "\text{inner local system twist}"] & \mathbf{B}\mathrm{PU}(\mathcal{H}) \end{tikzcd} \right\} \Big/ \sim_{\mathrm{htpy}}$$

The B-field twist – Inner local systems – The diagrammatics.

Hence the

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X \arrow[ur, dashed, "\text{cocycle}"] \arrow[r, "\widetilde{\tau}", "\text{adjunct twist}"'] & \mathrm{Maps}\big(\mathbf{B}G,\, \mathbf{B}\mathrm{PU}(\mathcal{H})\big)
\end{tikzcd}
\right\} \Big/ \sim_{\mathrm{htpy}}
$$

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$$
\mathrm{KU}_G^{n+[\omega_1]}(X) \;=\; \left\{
\begin{tikzcd}
 & & \mathrm{Maps}\big(\mathbf{B}G, \mathrm{Fred}^n_{\mathbb{C}} /\!\!/ \mathrm{PU}(\mathcal{H})\big) \arrow[d] \\
X \arrow[rr, dashed, "\text{cocycle}"] \arrow[r, "\omega_1"', "\text{inner local system}"] & \mathbf{B}G^{*} \arrow[r, "\text{automorphisms of univ. stable equiv. twist}"'] & \mathrm{Maps}\big(\mathbf{B}G, \mathbf{B}\mathrm{PU}(\mathcal{H})\big)
\end{tikzcd}
\right\}_{\big/\sim_{\mathrm{htpy}}}
$$

The B-field twist – Inner local systems – The diagrammatics.

Hence the

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$$
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$$

For the proof we consider the following diagram [SS22-Bun, Ex. 4.1.56][SS22, §3]:

$$
\begin{tikzcd}
\mathbf{B}G \arrow[rr, bend left, "\mathrm{stbl}_0"] \arrow[rr, bend right, "\mathrm{stbl}_0"'] \arrow[rr, phantom, "\Downarrow \rho \in G^*"] & & \mathbf{B}\mathrm{PU}(\mathcal{H})
\end{tikzcd}
$$

$$
\begin{tikzcd}
\bullet \arrow[d, "g"] & & \underset{\substack{[\rho_i] \in \\ \mathrm{Irr}(G)}}{\bigoplus} \rho_i \otimes \ell^2(\mathbb{C}) \arrow[r, "v \mapsto 1_\rho \otimes v"] \arrow[d, "\underset{[\rho_i]}{\oplus}(\rho_i(g)\otimes \mathrm{id})"'] & \underset{\substack{[\rho_i] \in \\ \mathrm{Irr}(G)}}{\bigoplus} \rho_i \otimes \ell^2(\mathbb{C}) \arrow[d, "\underset{[\rho_i]}{\oplus}(\rho_i(g)\otimes \mathrm{id})"] \\
\bullet & \longmapsto & \underset{\substack{[\rho_i] \in \\ \mathrm{Irr}(G)}}{\bigoplus} \rho_i \otimes \ell^2(\mathbb{C}) \arrow[r, "v \mapsto 1_\rho \otimes v"'] \arrow[ur, Rightarrow, "\rho(g)(1_\rho)"] & \underset{\substack{[\rho_i] \in \\ \mathrm{Irr}(G)}}{\bigoplus} \rho_i \otimes \ell^2(\mathbb{C})
\end{tikzcd}
$$

The B-field twist – Inner local systems – The proof.

For the proof we consider the following diagram [SS22-Bun, Ex. 4.1.56][SS22, §3]:

$$
\begin{tikzcd}[column sep=huge]
\mathbf{B}G \arrow[r, bend left, "\mathrm{stbl}_0"{name=U}] \arrow[r, bend right, "\mathrm{stbl}_0"'{name=D}] & \mathbf{B}\mathrm{PU}(\mathcal{H})
\arrow[from=U, to=D, phantom, "\Downarrow \rho \in G^*"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\overset{\text{stable } G\text{-representation}}{\bigoplus_i \rho_i \otimes \ell^2(\mathbb{C})} \arrow[rr, "v \mapsto 1_\rho \otimes v"] \arrow[dr, "F", "\text{Fredholm operator}"'] \arrow[dd, "\oplus_i \rho_i \otimes \ell^2(\mathbb{C})"] & & \bigoplus_i \rho_i \otimes \ell^2(\mathbb{C}) \arrow[dr, "{[\rho] \cdot F}"] \arrow[d, phantom, "\text{action of group character on equivariant Fredholm operator}"] & \\
& \bigoplus_i \rho_i \otimes \ell^2(\mathbb{C}) \arrow[rr, "v \mapsto 1_\rho \otimes v", "\text{tensoring with unit of group character}"'] \arrow[dd, "\oplus_s \rho_i(g) \otimes \mathrm{id}"] \arrow[dl, phantom, "\text{equivariance of Fredholm operator}"] & & \bigoplus_i \rho_i \otimes \ell^2(\mathbb{C}) \arrow[dd, "\oplus_s \rho_i(g) \otimes \mathrm{id}"] \arrow[ddl, phantom, "\text{projective intertwining action of group character}"] \\
\bigoplus_i \rho_i \otimes \ell^2(\mathbb{C}) \arrow[dr, "F"'] & & & \\
& \bigoplus_i \rho_i \otimes \ell^2(\mathbb{C}) \arrow[rr, "v \mapsto 1_\rho \otimes v"'] & & \bigoplus_i \rho_i \otimes \ell^2(\mathbb{C})
\end{tikzcd}
$$

The B-field twist – Inner local systems – Chern character.

One aspect of these twistings becomes transparent under the *Chern character*:

complex K-theory

periodic de Rham cohomology

$$\mathrm{KU}^0(\mathrm{X}) \xrightarrow[\text{Chern character}]{} \mathrm{KU}^0(\mathrm{X};\mathbb{C}) \;\simeq\; \bigoplus_{d\in\mathbb{N}} H^{2d}\Big(\Omega^\bullet_{\mathrm{dR}}\big(\mathrm{X};\mathbb{C}\big),\mathrm{d}\Big)$$

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For twist by B-field $\widehat{B}_2$ there is a closed differential 3-form H_3 such that:

$$\underset{\text{plain B-field -twisted K-theory}}{\mathrm{KU}^{n+\widehat{B}_2}(\mathrm{X})} \xrightarrow[\textbf{Chern character}]{\textbf{twisted}} \mathrm{KU}^{\widehat{B}_2}(\mathrm{X};\mathbb{C}) \;\simeq\; \overset{\text{3-twisted periodic de Rham cohomology}}{\bigoplus_{d\in\mathbb{Z}} H^{n+2d}\Big(\Omega^\bullet_{\mathrm{dR}}\big(\mathrm{X};\mathbb{C}\big),\mathrm{d}+H_3\wedge\Big)}$$

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$$\underbrace{\mathrm{KU}^0(\mathrm{X})}_{\text{complex K-theory}} \xrightarrow[\text{Chern character}]{} \mathrm{KU}^0(\mathrm{X};\mathbb{C}) \;\simeq\; \overbrace{\bigoplus_{d\in\mathbb{N}} H^{2d}\Big(\Omega^\bullet_{\mathrm{dR}}(\mathrm{X};\mathbb{C}),\mathrm{d}\Big)}^{\text{periodic de Rham cohomology}}$$

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For twist by inner C_κ-local system, there is closed 1-form ω_1 with holon. in $C_\kappa \subset \mathrm{U}(1)$ such that:

$$\underbrace{\overbrace{\mathrm{KU}^{n+[\omega_1]}_{C_\kappa}(\mathrm{X})}^{\text{inner local system -twisted K-theory}}}_{\text{of A-type singularity}} \xrightarrow[\text{Chern character}]{\text{twisted equivariant}} \overbrace{\bigoplus_{\substack{d\in\mathbb{Z}\\ 1\leq r\leq\kappa}} H^{n+2d}\Big(\Omega^\bullet_{\mathrm{dR}}(\mathrm{X};\mathbb{C}),\mathrm{d}+r\cdot\omega_1\wedge\Big)}^{\text{1-twisted periodic de Rham cohomology}}$$

The B-field twist – Inner local systems – Chern character.

One aspect of these twistings becomes transparent under the Chern character:

This is the hidden 1-twisting in TED-K – that we will next relate to anyons. $\longrightarrow$

$$\underset{\text{of A-type singularity}}{\overset{\substack{\text{inner local system}\\ \text{-twisted K-theory}}}{\mathrm{KU}^{n+[\omega_1]}_{C_\kappa}(\mathrm{X})}} \xrightarrow[\text{Chern character}]{\text{twisted equivariant}} \overset{\text{1-twisted periodic de Rham cohomology}}{\bigoplus_{\substack{d \in \mathbb{Z}\\ 1 \leq r \leq \kappa}} H^{n+2d}\Big(\Omega^\bullet_{\mathrm{dR}}\big(\mathrm{X};\mathbb{C}\big), \mathrm{d} + r \cdot \omega_1 \wedge\Big)}$$

References

[Karoubi 70] M. Karoubi:
Espaces Classifiants en K-Théorie,
Trans. of the Amer. Math. Soc. **147** (1970), 75-115,
[doi:10.2307/1995218]

[KS77] M. Klaus and G. Scharf:
The regular external field problem in quantum electrodynamics,
Helv. Phys. Acta **50** (1977)
[doi:10.5169/seals-114890]

[CHO82] A. L. Carey, C. A. Hurst, and D. M. O'Brien:
Automorphisms of the canonical anticommutation relations and index theory
J. Funct. Anal. **48** (1982) 360-393
[doi:10.1016/0022-1236(82)90092-1]

[FM12] D. S. Freed and G. W. Moore:
Twisted equivariant matter,
Ann. Henri Poincaré **14** (2013), 1927–2023,
[doi:10.1007/s00023-013-0236-x] [arXiv:1208.5055]

Introduction

(1) – TED K-Theory
via Cohesive ∞-Topos Theory

(2) – Interacting enhancement
via Hypothesis H

(3) – Anyon braiding
via Cohesive Homotopy Type Theory

Summary

This part is a lightning indication
of the basic idea in these articles:

Framed M-branes and topological invariants	[arX:1310.1060]
ADE-Equivariant Cohomotopy and M-branes	[arX:1805.05987]
The rational higher structure of M-theory	[arX:1903.02834]
Cohomotopy implies M-theory anom. canc.	[arX:1904.10207]
Cohomotopy implies M5-brane WZ term	[arX:1906.07417]
Cohomotopy implies tadpole cancellation	[arX:1909.12277]
Cohomotopy implies intersecting brane obs.	[arX:1912.10425]
Cohomotopy implies M5-brane anom. canc.	[arX:2002.07737]
Cohomotopy implies String structure on M5	[arX:2002.11093]
Cohomotopy implies GS-mechanism	[arX:2008.08544]
Cohomotopy implies GS-mechanism on M5	[arX:2011.06533]
M/F-Theory as Mf-theory	[arX:2103.01877]

Interacting n-electron wavefunctions are functions on the space of n points in Bri-torus

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Pauli exclusion $\Rightarrow$ these span vector bundle away from the locus of coinciding points:

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Slater determinants of Bloch states

Slater-Bloch valence bundle of interacting n-electron states

$$\mathscr{V}_n \subset \coprod_{(k^1,\cdots,k^n)} \mathrm{Span}\Big\{ \Psi_{i_1,\cdots,i_n}\Big(\big(k^1,s^1\big),\cdots,\big(k^n,s^n\big)\Big)\Big\}_{\substack{(i_1,\cdots,i_n)\\(s^1,\cdots,s^n)}}$$

$$\big\downarrow$$

configuration space of n "probe" points

$$\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\Big(\underbrace{\widehat{\mathbb{T}}^d \setminus \{k_1,\cdots,k_N\}}_{\text{in complement of } N \text{ "nodal" points inside the Brillouin torus}}\Big) = \Big\{ (k^1,\cdots,k^n) \in \big(\widehat{\mathbb{T}}^d\big)^n \,\Big|\, \underbrace{\underset{i\neq j}{\forall}\, k^i \neq k^j}_{\text{Pauli exclusion}} \;\text{ and }\; \underbrace{\underset{i,I}{\forall}\, k^i \neq k_I}_{\text{nodal singularities}} \Big\}.$$

Interacting n-electron wavefunctions are functions on the space of n points in Bri-torus
Pauli exclusion $\Rightarrow$ these span vector bundle away from the locus of coinciding points:

$$\begin{array}{ccc}
\underset{\text{interacting } n\text{-electron states}}{\text{Slater-Bloch valence bundle of}} & \mathscr{V}_n \subset & \displaystyle\coprod_{(k^1,\cdots,k^n)} \mathrm{Span}\overbrace{\Big\{ \Psi_{i_1,\cdots,i_n}\Big(\big(k^1,s^1\big),\cdots,\big(k^n,s^n\big)\Big)\Big\}}^{\text{Slater determinants of Bloch states}}_{\substack{(i_1,\cdots,i_n)\\(s^1,\cdots,s^n)}} \\
& \big\downarrow & \\
\underset{n \text{ “probe” points}}{\text{configuration space of}} & \underset{\{1,\cdots,n\}}{\mathrm{Conf}}\underbrace{\Big(\widehat{\mathbb{T}}^d \setminus \{k_1,\cdots,k_N\}\Big)}_{\substack{\text{in complement of } N \text{ “nodal”}\\ \text{points inside the Brillouin torus}}} & = \left\{ (k^1,\cdots,k^n) \in \big(\widehat{\mathbb{T}}^d\big)^n \,\middle|\, \underbrace{\underset{i\neq j}{\forall}\, k^i \neq k^j}_{\substack{\text{Pauli}\\\text{exclusion}}} \;\; \text{and} \;\; \underbrace{\underset{i,I}{\forall}\, k^i \neq k_I}_{\substack{\text{nodal}\\\text{singularities}}} \right\}.
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This locus is known as the **configuration space of n points**.

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Deep theorems (Hopf, Pontrjagin, Segal $\to$ next slides) relate configurations of points to *Cohomotopy* theory – a *non-abelian* generalized cohomology theory:

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$$

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$$
\begin{array}{cccc}
\text{Cohomotopy} & \pi^n(X) & = & \mathrm{Maps}\Big(X, \overbrace{S^n}^{\text{sphere}}\Big)\Big/_{\text{htpy}} \\
& & & \Big\downarrow \Big(S^n \to K(\mathbb{Z},n)\Big)_* \\
\text{ordinary cohomology} & H^n(X;\mathbb{Z}) & = & \mathrm{Maps}\Big(X, \underbrace{K(\mathbb{Z},n)}_{\text{E.-M.-space}}\Big)\Big/_{\text{htpy}}
\end{array}
$$

Pontrjagin's theorem. Normal. framed submanifolds carry charge in Cohomotopy:

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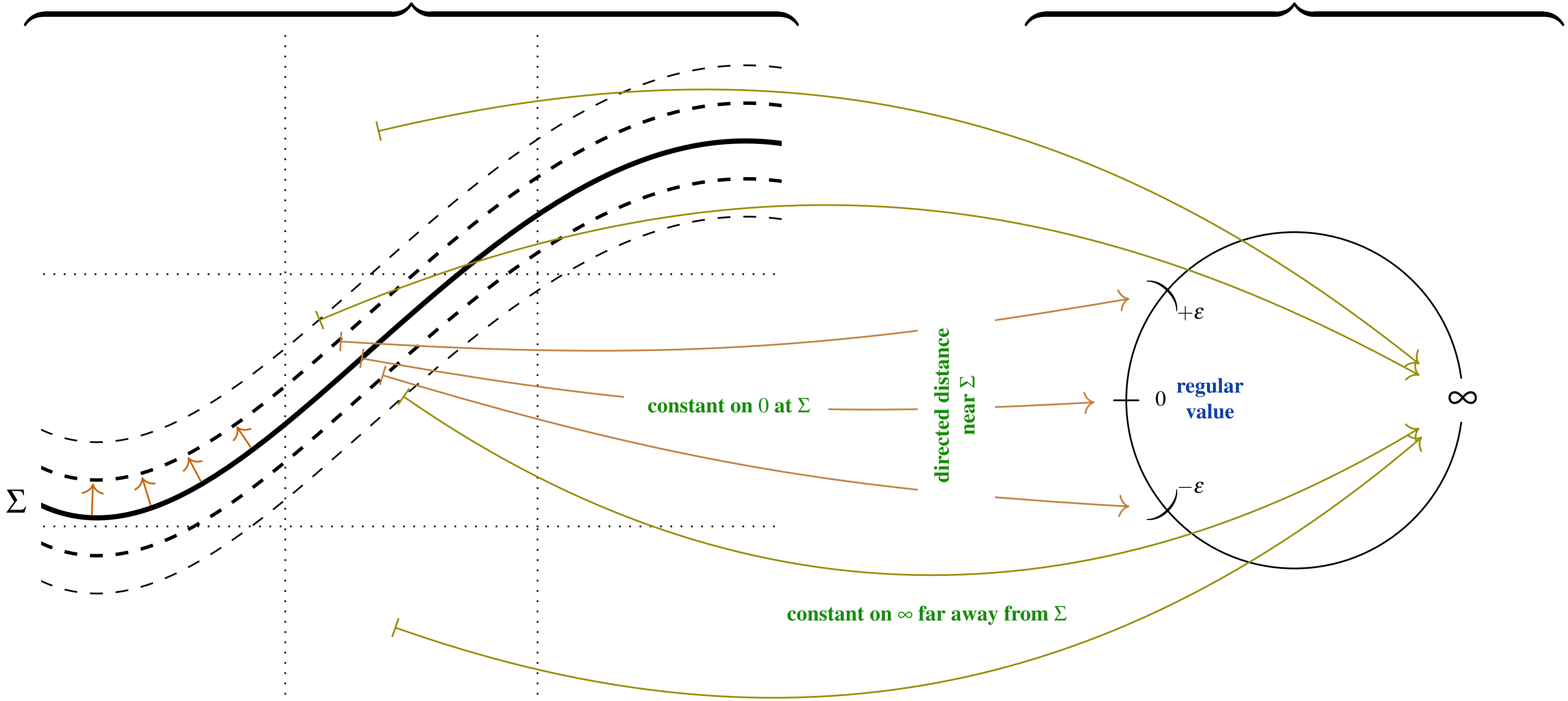
$$\underset{\substack{\text{closed submanifold,}\\ \text{normally framed}}}{\Sigma^{d-n}} \hookrightarrow \underset{\text{manifold}}{M^d} \xrightarrow[\substack{\text{directed asymptotic distance from }\Sigma\\ \simeq\ \text{cocycle representing Cohomotopy charge of }\Sigma}]{c} \underset{\substack{\text{Cohomotopy classifying space}\\ (n\text{-sphere})}}{(\mathbb{R}^n)^{\mathrm{cpt}} = S^n}$$



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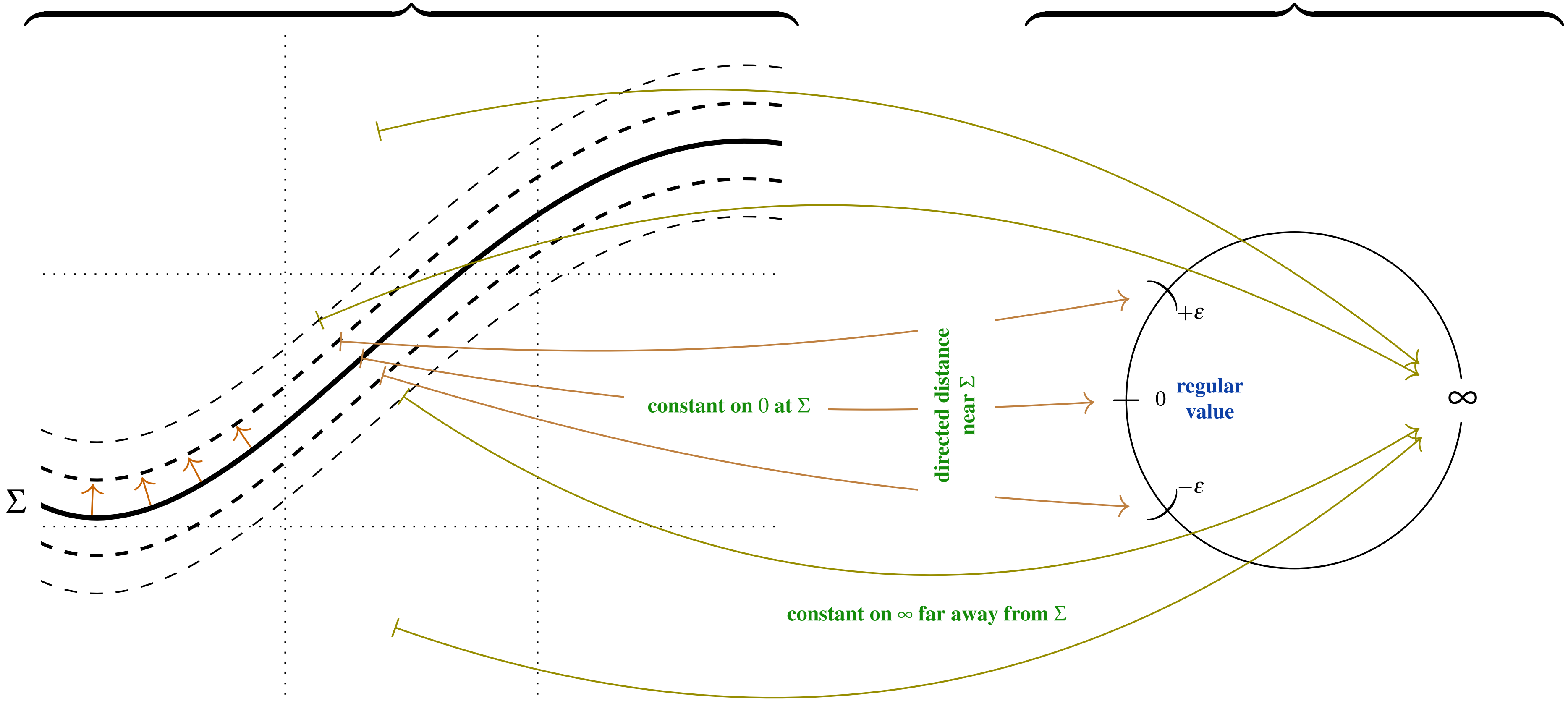
closed submanifold, normally framed — manifold — directed asymptotic distance from Σ $\simeq$ cocycle representing Cohomotopy charge of Σ — Cohomotopy classifying space (n-sphere)



which identifies them with their Cobordism class:

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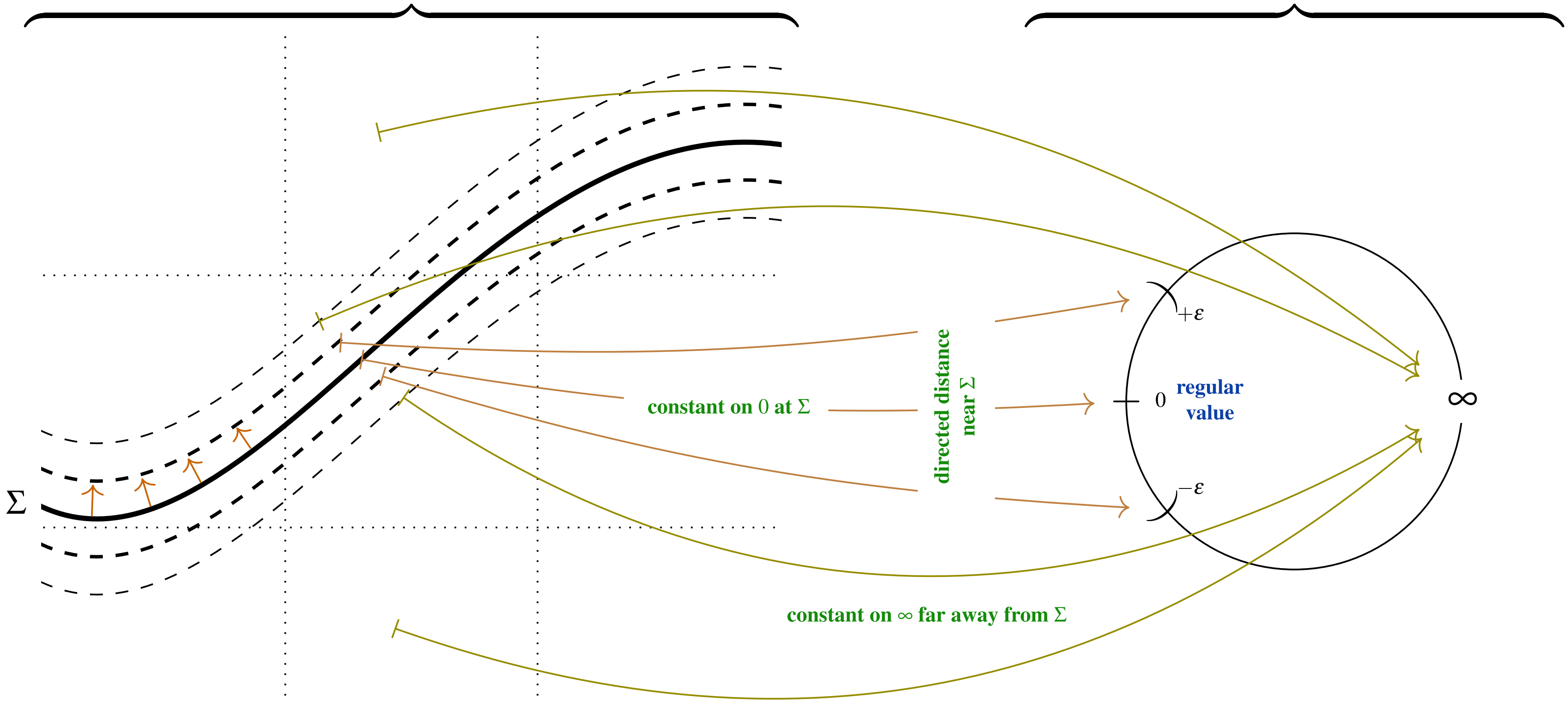

which identifies them with their Cobordism class:

$$\begin{tikzcd}
\underset{\text{framed unstable } n\text{-Cobordism of } M^d}{\mathrm{Cob}^n_{\mathrm{Fr}}\big(M^d\big)} := \mathrm{NFrmSubMflds}_{d-n}\big(M^d\big)_{/\mathrm{brdsm}} \arrow[rr, shift left, "\text{directed asymptotic distance}"] \arrow[rr, phantom, "\simeq"] \arrow[rrr, bend left, "\text{assign Cohomotopy charge}"] & & \mathrm{Maps}\big(M^d, S^n\big)_{/\mathrm{hmtpy}} =: \underset{\text{unstable } n\text{-Cohomotopy of } M^d}{\pi^n\big(M^d\big)} \arrow[ll, shift left, "\text{pre-image of regular value 0}"] \arrow[lll, bend left, "\text{reconstruct submanifold from its charge}"]
\end{tikzcd}$$

Special case: Hopf degree theorem.

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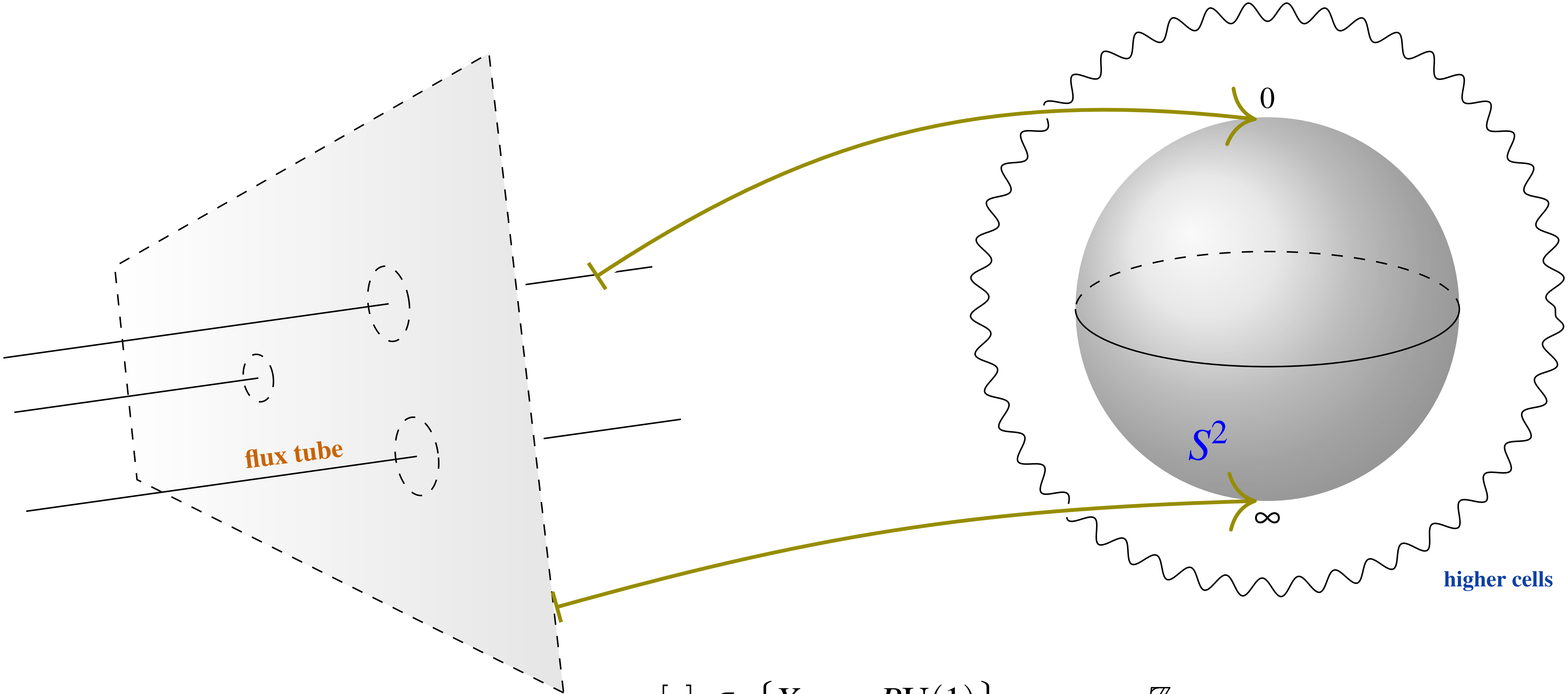
Special case: Hopf degree theorem.
On n-manifolds, n-Cohomotopy agrees with integral n-cohomology
and the Pontrjagin construction is the assignment of **instanton charge**:

$$\underbrace{\underset{\text{spacetime seen by fields vanishing at transversal infinity}}{X := \mathbb{R}^{1,1} \times \left(\mathbb{R}^2\right)^{\mathrm{cpt}} \simeq \mathbb{R}^{1,1} \times S^2}} \xrightarrow[\substack{\text{magnetic flux} \\ \text{through transversal plane}}]{c} \underbrace{\underset{\text{classifying space for ordinary cohomology}}{B\mathrm{U}(1) = K(\mathbb{Z},2)}}$$



$$\underset{\text{total flux = homotopy class}}{[c] \in \left\{X \longrightarrow B\mathrm{U}(1)\right\}_{/\mathrm{hmpty}}} \simeq \underset{\substack{\text{charge} \\ \text{lattice}}}{\mathbb{Z}}$$

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$$\underbrace{\overbrace{\mathrm{Conf}\Big(\mathbb{R}^d,\, \mathbb{R}^{p-d}_{\mathrm{cpt}}\Big)}^{\textbf{configurations of points in } \mathbb{R}^p}}_{\substack{\textbf{which are distinct in } \mathbb{R}^p \\ \textbf{\& may vanish to } \infty \textbf{ along } \mathbb{R}^{p-d}}} \xrightarrow[\substack{\sim \\ \textbf{homotopy equivalence}}]{\textbf{cohomotopy charge map}} \overbrace{\mathrm{Maps}\Big(\mathbb{R}^d_{\mathrm{cpt}},\, S^p\Big)}^{\textbf{Cohomotopy moduli space}}$$

Segal's theorem generalizes Pontrjagin from Cohomotopy classes to **moduli**:

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Here an element of $\mathrm{Conf}\big(\mathbb{R}^d, \mathbb{R}^1_{\mathrm{cpt}}\big)$:

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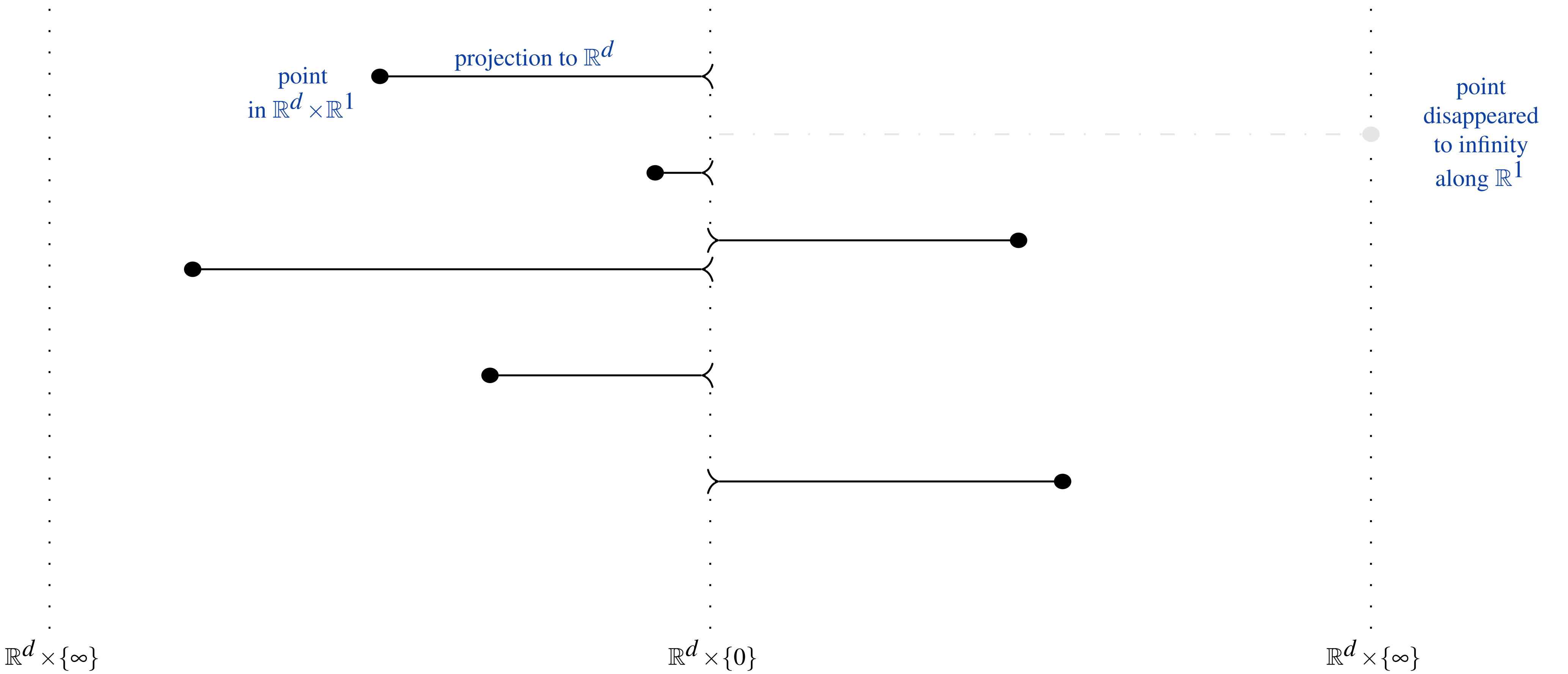

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$$
\begin{tikzcd}
\pi^4(X) \arrow[r, "\underset{\text{gen. character map}}{\mathrm{ch}_\pi^n}"] \arrow[d, "\left(S^4 \to K(\mathbb{Z},4)\right)_*"] & \left\{ \begin{pmatrix} G_7 \\ G_4 \end{pmatrix} \in \Omega^\bullet_{\mathrm{dR}}(X) \;\middle|\; \overset{\text{Bianchi identity for supergravity C-field}}{\begin{aligned} \mathrm{d}\,G_7 &= -\tfrac{1}{2} G_4 \wedge G_4 \\ \mathrm{d}\,G_4 &= 0 \end{aligned}} \right\}_{/\mathrm{cncrd}} \arrow[d, "\begin{smallmatrix} G_7 \mapsto 0 \\ G_4 \mapsto G_4 \end{smallmatrix}"] \\
H^4(X;\mathbb{Z}) \arrow[r] & \left\{ G_4 \in \Omega^\bullet_{\mathrm{dR}}(X) \;\middle|\; \mathrm{d}\,G_4 = 0 \right\}_{/\mathrm{cncrd}}
\end{tikzcd}
$$

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$$\begin{tikzcd}
\pi^4(X) \arrow[r, "\mathrm{ch}^n_\pi", "\text{gen. character map}"'{above=12pt}] \arrow[d, "\left(S^4 \to K(\mathbb{Z},4)\right)_*"] & \left\{ \begin{pmatrix} G_7 \\ G_4 \end{pmatrix} \in \Omega^\bullet_{\mathrm{dR}}(X) \,\middle|\, \overset{\text{Bianchi identity for supergravity C-field}}{\begin{aligned} \mathrm{d}\,G_7 &= -\tfrac{1}{2} G_4 \wedge G_4 \\ \mathrm{d}\,G_4 &= 0 \end{aligned}} \right\}_{/\mathrm{cncrd}} \arrow[d, "{\substack{G_7 \mapsto 0 \\ G_4 \mapsto G_4}}"] \\
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In fact, tangentially twisted 4-Cohomotopy, coupling this to the spacetime metric, implies a list of subtle topological conditions expected to hold in M-theory.

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$$
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H^4(X; \mathbb{Z}) \arrow[rr] && \left\{ G_4 \in \Omega^\bullet_{\mathrm{dR}}(X) \;\middle\vert\; \mathrm{d}\, G_4 = 0 \right\}_{\big/ \mathrm{cncrd}}
\end{tikzcd}
$$

Bianchi identity for supergravity C-field

In fact, tangentially twisted 4-Cohomotopy, coupling this to the spacetime metric, implies a list of subtle topological conditions expected to hold in M-theory.

Moreover, generalized cohomology *of* intersecting Cohomotopy moduli spaces reflects intersecting brane observables expected from non-abelian DBI action.

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\Big\downarrow{\scriptstyle \left(S^4 \to K(\mathbb{Z},4)\right)_*} & & \Big\downarrow{\scriptstyle \begin{array}{l} G_7 \mapsto 0 \\ G_4 \mapsto G_4 \end{array}} \\
H^4(X;\mathbb{Z}) & \longrightarrow & \left\{ G_4 \in \Omega^\bullet_{\mathrm{dR}}(X) \;\middle|\; \mathrm{d}\,G_4 = 0 \right\}_{\Big/ \mathrm{cncrd}}
\end{array}
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$\rightsquigarrow$ **Hypothesis H**:

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\pi^4(X) \arrow[r, "\overset{\textbf{gen. character map}}{\mathrm{ch}^n_\pi}"] \arrow[d, "\left(S^4 \to K(\mathbb{Z},4)\right)_*"] & \left\{ \begin{pmatrix} G_7 \\ G_4 \end{pmatrix} \in \Omega^\bullet_{\mathrm{dR}}(X) \,\middle\vert\, \overset{\textbf{Bianchi identity for supergravity C-field}}{\begin{aligned} \mathrm{d}\, G_7 &= -\tfrac{1}{2} G_4 \wedge G_4 \\ \mathrm{d}\, G_4 &= 0 \end{aligned}} \right\}_{/\mathrm{cncrd}} \arrow[d, "\substack{G_7 \mapsto 0 \\ G_4 \mapsto G_4}"] \\
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In fact, tangentially twisted 4-Cohomotopy, coupling this to the spacetime metric, implies a list of subtle topological conditions expected to hold in M-theory.

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Introduction

(1) – TED K-Theory
via Cohesive ∞-Topos Theory

(2) – Interacting enhancement
via Hypothesis H

(3) – Anyon braiding
via Cohesive Homotopy Type Theory

Summary

This part is a brief indication
of a few aspects discussed in:

Anyonic Defect Branes in TED-K [arX:2203.11838]
Anyonic Topological Order in TED-K [arX:2206.13563]
Topological Quantum Programming in TED-K [PlanQC **2022** 33]

K-Theory classifies non-perturbative vacua.

Solid state physics	K-theory	String theory

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Single electron state	Line bundle	Single D-brane
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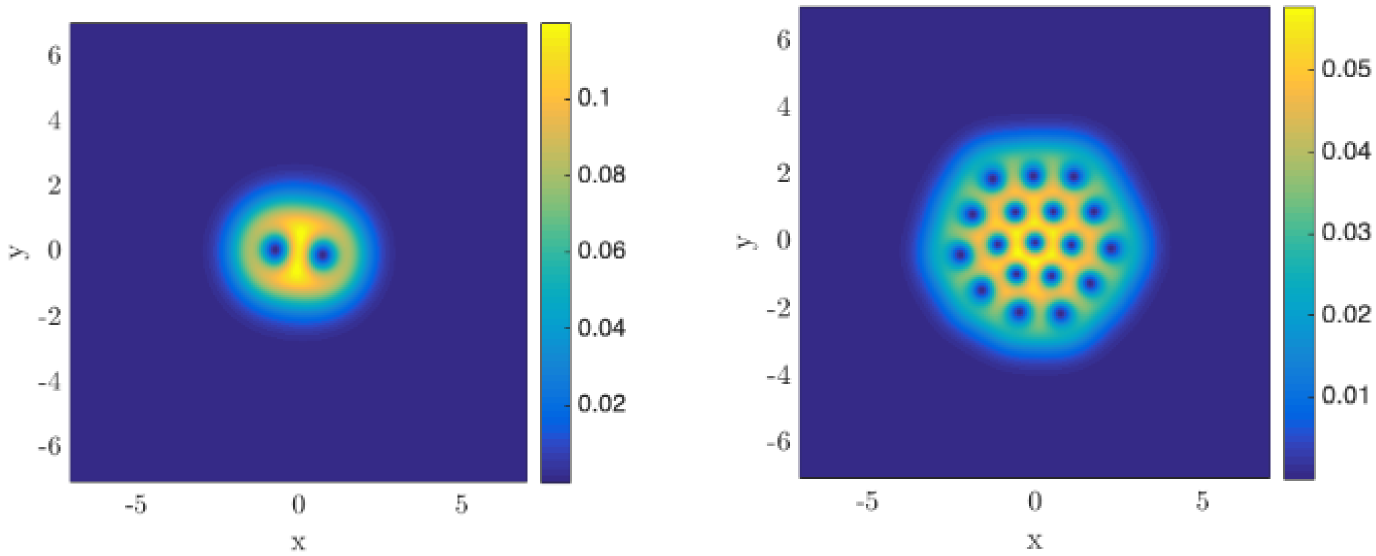
Anyons	Punctures	Defect branes

Anyons in condensed matter & string theory.

In solid state physics

anyons are presumed pointlike defects in gapped topological phases of effectively 2-dimensional materials whose adiabatic dynamics is that of Wilson lines in $\mathfrak{su}(2)$-CS theory.



(numerical simulation from `arXiv:1901.10739`)

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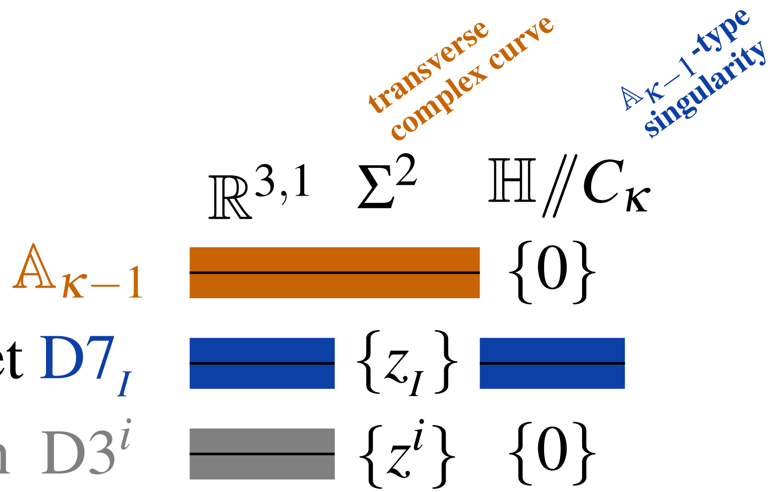

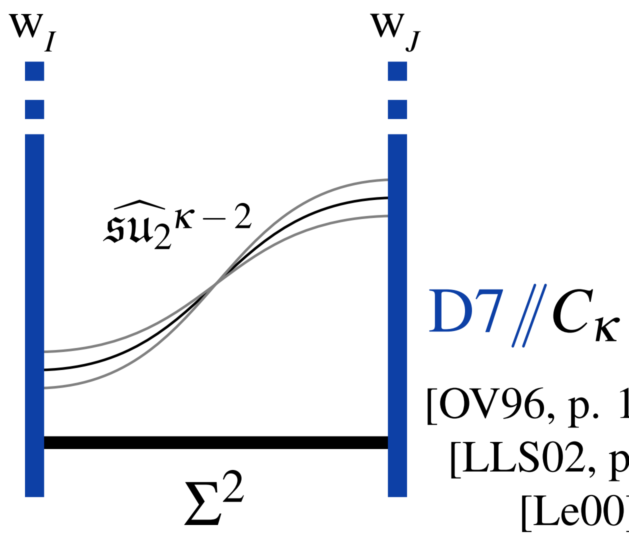

[OV96, p. 10-12]
[LLS02, p. 4s]
[Le00]

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Yes! $\longrightarrow$

TED-Cohomological incarnation of Conformal blocks.

Consider

$$\kappa \quad := \quad k+2 \qquad \text{“level”}$$

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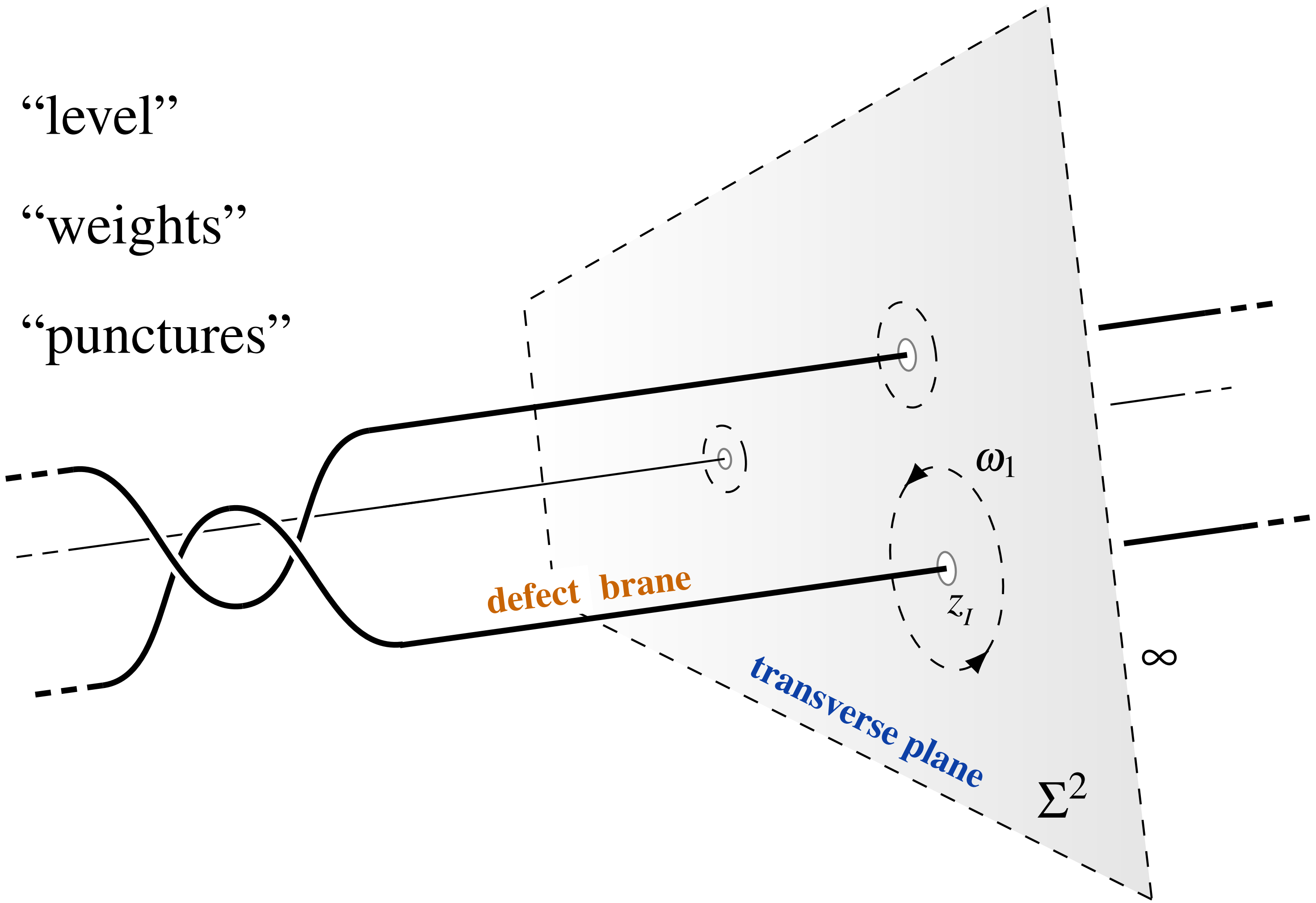

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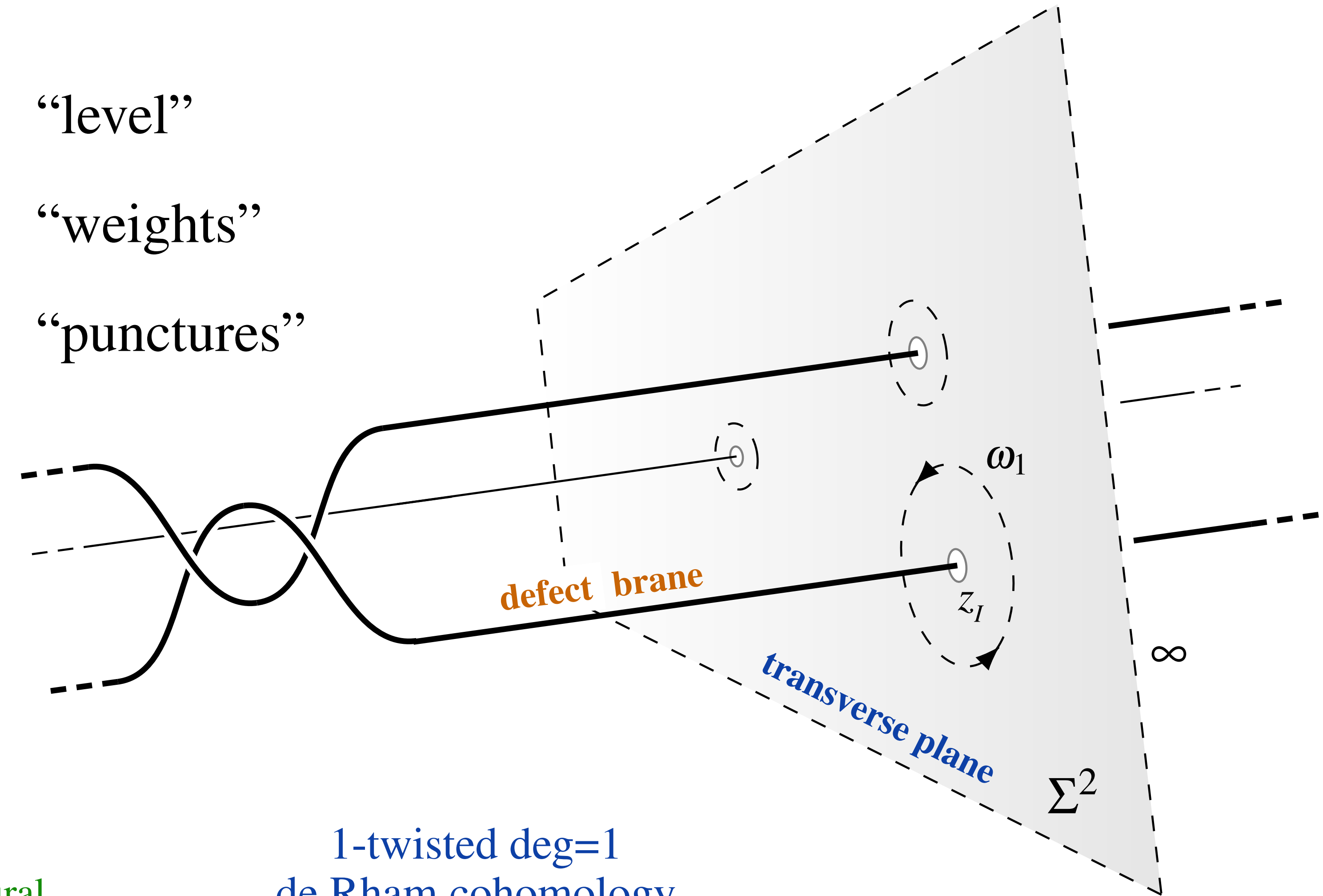

$\mathfrak{su}(2)$-affine deg=1 conformal blocks $\xhookrightarrow{\text{natural inclusion}}$ 1-twisted deg=1 de Rham cohomology

$$\mathrm{CnfBlck}^1_{\widehat{\mathfrak{sl}_2}^k}(\vec{\mathrm{w}}, \vec{z}) \hookrightarrow H^1\Big(\Omega^\bullet_{\mathrm{dR}}\big(\mathbb{C}\setminus\{\vec{z}\}\big), \mathrm{d} + \omega_1 \wedge\Big)$$

[FSV94, Cor. 3.4.2]

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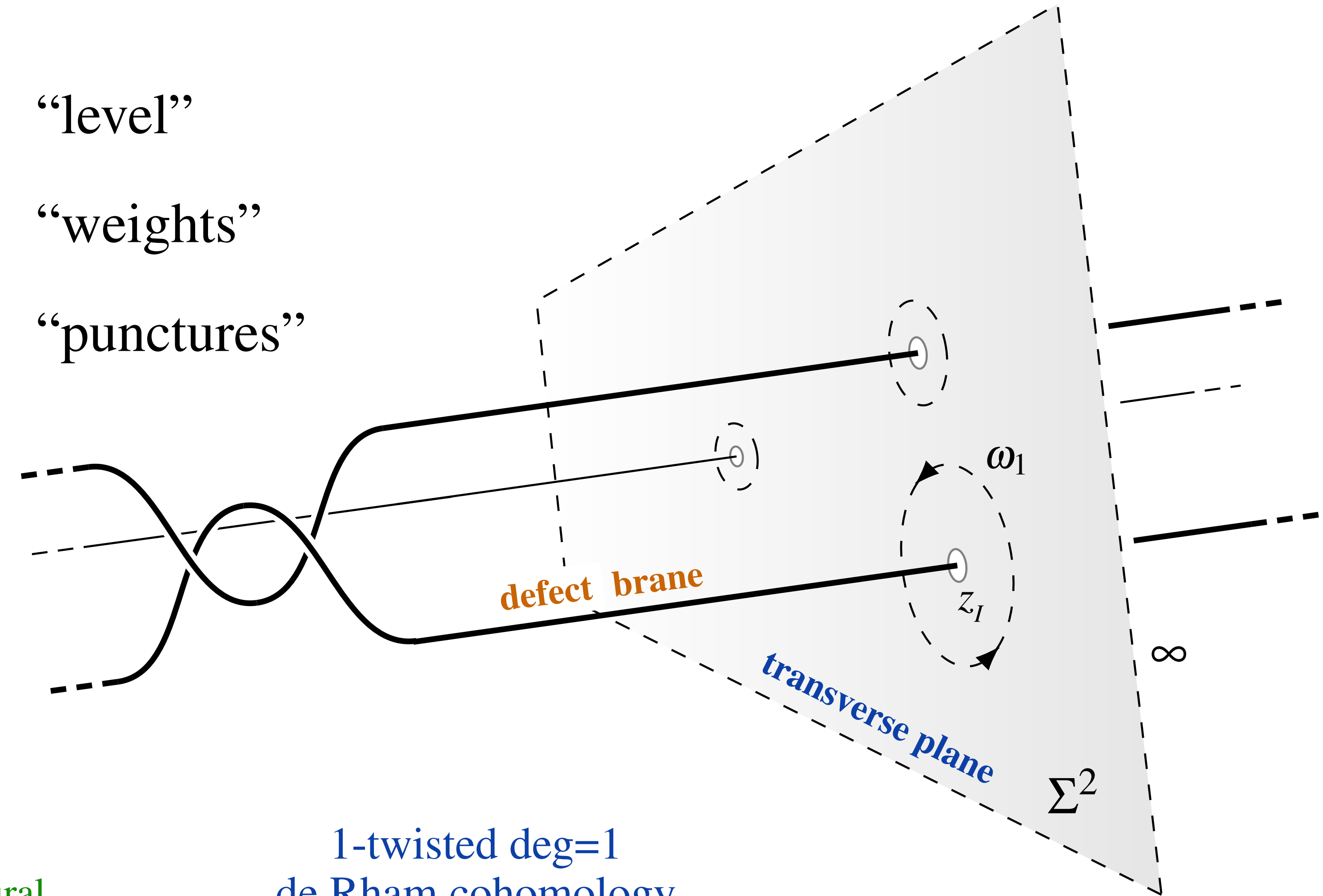

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$$

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\xhookrightarrow{\text{natural inclusion}} \mathrm{KU}^{1+\omega_1}\Big(\big(\mathbb{C} \setminus \{\vec{z}\}\big) \times * /\!\!/ C_\kappa; \mathbb{C}\Big) \qquad \text{[SS22, Prop. 2.16]}
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inner local system-twisted deg=1 K-theory of $\mathbb{A}_{\kappa-1}$-singularity

(as explained above)

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Generally, consider *configuration spaces of points* (e.g. [SS19, §2.2])

$$\underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathrm{X}\big) \;:=\; \Big\{ z^1, \cdots, z^n \in \mathrm{X} \,\Big\vert\, \underset{i<j}{\forall}\; z^i \neq z^j \Big\}.$$

with $\quad \omega_1 := \sum_{1 \leq i \leq n} \sum_I -\frac{\mathrm{w}_I}{\kappa} \frac{\mathrm{d}z}{z - z_I} + \sum_{1 \leq i < j \leq n} \frac{2}{\kappa} \frac{\mathrm{d}z}{z^i - z^j} \quad$ on $\quad \underset{\{1,\cdots,n\}}{\mathrm{Conf}}\big(\mathbb{C} \setminus \{\vec{z}\}\big)$

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The previous statement is subsumed since $\underset{\{1\}}{\mathrm{Conf}}(\mathrm{X}) = \mathrm{X}$.

Conclusion.

The commonly expected $\widehat{\mathfrak{su}_2}^k$-charges of anyons and defect branes *are* reflected in the TED-K-theory of *configuration spaces of points* in 2-dimensional transverse spaces *inside* $\mathbb{A}_{k+1}$*-orbi-singularities*.

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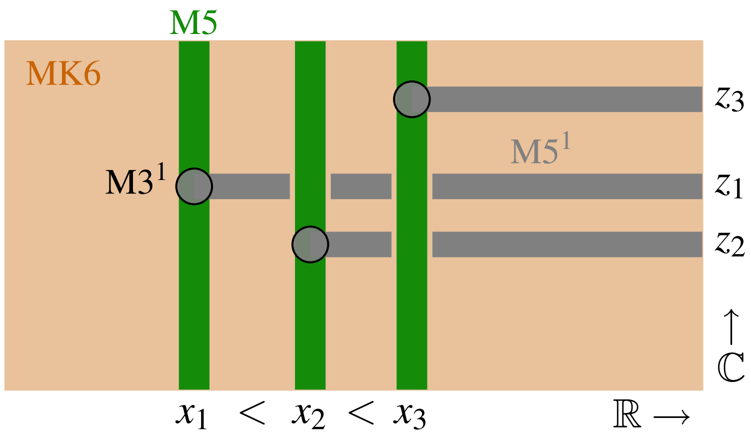
$$\underbrace{\coprod_n \underset{\{1,\cdots,n\}}{\mathrm{Conf}}(\mathbb{C})}_{\substack{\text{Configuration space of}\\ \text{ordered points in the plane}}} \simeq \overbrace{\coprod_n \mathrm{Conf}_n(\mathbb{C};\mathbb{R}_{\mathrm{cpt}})}^{\substack{\text{3-Cohomotopy cocycle space}\\ \text{for codim=1 branes}\\ \mathrm{Map}^*(\mathbb{R}_+\wedge\mathbb{C}_{\mathrm{cpt}},S^3)\,\simeq}} \underset{\substack{\coprod_n \mathrm{Conf}_n(\ast;(\mathbb{R}\times\mathbb{C})_{\mathrm{cpt}})\\ \text{Fiber product of respective configuration spaces}\\ \text{(of un-ordered points escaping to transverse infinity)}\\ \text{reflecting the brane intersections}}}{\times} \overbrace{\coprod_n \mathrm{Conf}_n(\mathbb{R};\mathbb{C}_{\mathrm{cpt}})}^{\substack{\text{3-Cohomotopy cocycle space}\\ \text{for codim-2 branes}\\ \mathrm{Map}^*(\mathbb{R}_{\mathrm{cpt}}\wedge\mathbb{C}_+,S^3)\,\simeq}}$$

e.g.: $\underset{\{1,\cdots,3\}}{\mathrm{Conf}}(\mathbb{C}) \simeq$



The moduli space of flat M3-branes according to *Hypothesis H* is the configuration space of ordered points in their transverse plane.

References

[FSV94] B. Feigin, V. Schechtman, and A. Varchenko:
On algebraic equations satisfied by hypergeometric correlators in WZW models, I
Comm. Math. Phys. **163** (1994), 173–184,
[doi:10.1007/BF02101739]

[SS19] H. Sati and U. Schreiber:
Differential Cohomotopy implies intersecting brane observables
Adv. Theor. Math. Phys. **26** 4 (2022)
[arXiv:1912.10425]

[SS22-Bun] H. Sati and U. Schreiber:
Equivariant principal ∞-bundles,
[arXiv:2112.13654]

[SS22] H. Sati and U. Schreiber:
Anyonic defect branes in TED-K-Theory
[arXiv:2203.11838]

References

[OV96] H. Ooguri and C. Vafa:
Two-Dimensional Black Hole and Singularities of CY Manifolds,
Nucl. Phys. B **463** (1996), 55-72,
[doi:10.1016/0550-3213(96)00008-9], [arXiv:hep-th/9511164]

[LLS02] W. Lerche, A. Lütken, and C. Schweigert:
D-Branes on ALE Spaces and the ADE Classification of CFTs,
Nucl. Phys. B **622** (2002), 269-278,
[doi:10.1016/S0550-3213(01)00613-7], [arXiv:hep-th/0006247]

[Le00] W. Lerche:
On a Boundary CFT Description of Nonperturbative $\mathcal{N} = 2$ Yang-Mills Theory,
[hep-th/0006100].

[dBS13] J. de Boer and M. Shigemori:
Exotic Branes in String Theory,
Phys. Rep. **532** (2013), 65-118,
[arXiv:1209.6056] [doi:10.1016/j.physrep.2013.07.003].

(1) – TED K-Theory
via Cohesive ∞-Topos Theory

(2) – Interacting enhancement
via Hypothesis H

(3) – Anyon braiding
via Cohesive Homotopy Type Theory

Summary

It is largely folklore that:

Topological K-theory

fully Twisted & Equivariant & Differential (TED)

classifies

free topological phases in condensed matter theory		stable D-branes in string theory
	and *some* enhancement to	
interacting phases		non-perturbative effects
	is needed to account for	
topological order		M-branes

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Topological phases	**Topological K theory**	**String/M theory**
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Symmetry protection	Twisted equivariance	Global symmetries

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Symmetry protection	**Twisted equivariance**	**Global symmetries**
CPT symmetry	KR/KU/KO-theory	Type I/IIA/IIB

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This used to be the state of the art.

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What follows are new dictionary entries.

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Gauged internal symmetry	Inner local system-twist	Inside of orbi-singularity

What follows are new dictionary entries.

Single positron state	over Brillouin torus	of codimension d
Bloch-Floquet transform	Hilbert space bundle over Brillouin d-torus	Unstable (tachyonic) $D9/\overline{D9}$-brane state
Dressed Dirac vacuum operator	Family of Fredholm operators	Tachyon field
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Topological order	**Twisted differentiality**	**Gauge symmetries**

[illegible]	over Brillouin d-torus	D9/$\overline{\text{D9}}$-brane state
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Topological order	**Twisted differentiality**	**Gauge symmetries**
Berry connection	Differential K-theory	Chan-Paton gauge field

vacuum operator	Fredholm operators	Tachyon field
Valence bundle of electron/positron states	Virtual bundle of their kernels and cokernels	stable D-brane state after tachyon condensation
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Symmetry protection	**Twisted equivariance**	**Global symmetries**
CPT symmetry	KR/KU/KO-theory	Type I/IIA/IIB
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Topological order	**Twisted differentiality**	**Gauge symmetries**
Berry connection	Differential K-theory	Chan-Paton gauge field
Mass terms	Differential K-LES	Axio-Dilaton RR-field

electron/positron states	kernels and cokernels	after tachyon condensation
Topological phase	K-theory class	Stable D-brane charge

Symmetry protection	**Twisted equivariance**	**Global symmetries**
CPT symmetry	KR/KU/KO-theory	Type I/IIA/IIB
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Anyonic defects	TED-K of Configurations	Defect branes

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Anyonic defects	**TED-K of Configurations**	**Defect branes**
N band nodes	N-punctured Brillouin torus	N defect branes

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Anyonic defects	**TED-K of Configurations**	**Defect branes**
N band nodes	N-punctured Brillouin torus	N defect branes
Interacting n-electron states around N band nodes	Vector bundle over n-point configuration space in N-punctured Brillouin torus	Interacting n probe branes around N defect branes

Gauged internal symmetry	Inner local system-twist	Inside of orbi-singularity

Topological order	**Twisted differentiality**	**Gauge symmetries**
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$\mathfrak{su}_2$-anyon species	Holonomy of inner local system	$\mathrm{SL}(2,\mathbb{Z})$-charges of defect branes

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Anyon braiding	TED-K Gauss-Manin connections	Defect brane monodromy

Topological order	**Twisted differentiality**	**Gauge symmetries**
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TED K-Theory of Cohomotopy Moduli Spaces and Anyonic Topological Order

Urs Schreiber on joint work with Hisham Sati

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New York University, Abu Dhabi

Center for Quantum & Topological Systems

THE END

talk at:

Higher Structures and Field Theory @ ESI Vienna, 25 Aug 2022

slides and pointers at: `ncatlab.org/schreiber/show/TED-K+of+Cohomotopy+and+Anyons`