2.4 Quantum Gates & Measurement

We explain how *controlled quantum gates* and *quantum measurement gates* (Lit. 1.1) are naturally represented in the quantum modal logic of §2.3 and give (Prop. 2.38) a formal proof of the *deferred measurement principle* (18).

Data-typing of controlled quantum gates via quantum modal types.

We may observe that, with §2.3, we now have available the natural data-typing of classical/quantum data that is indicated on the right.

Notice how the distinction between classical and quantum data is reflected by the application or not of the (co)monad $\bigcirc$ ($\square$).

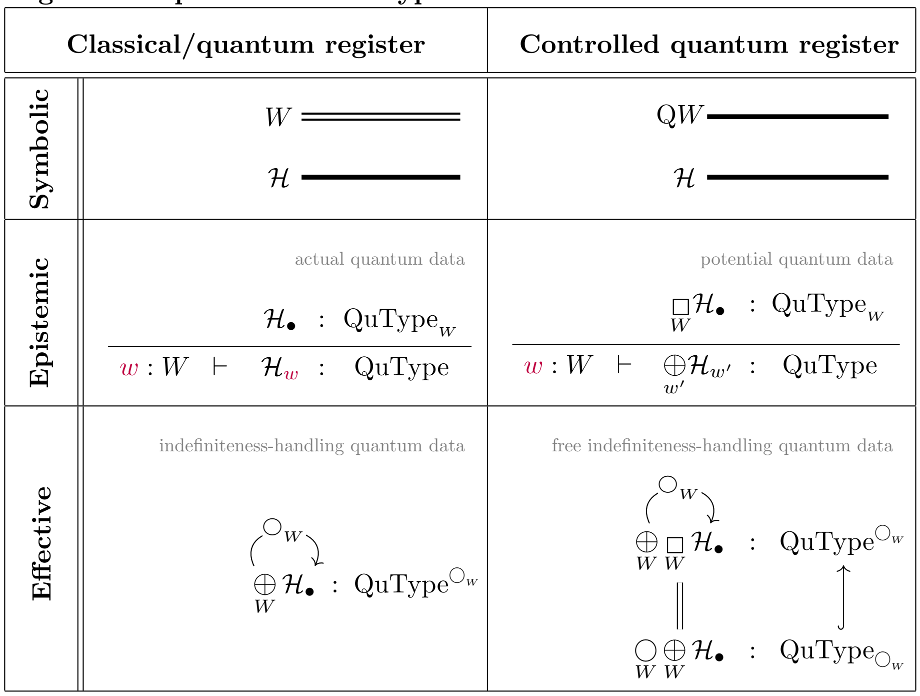
Throughout we use monadicity of $\bigoplus_W$ (Prop. 2.30) to translate (200)

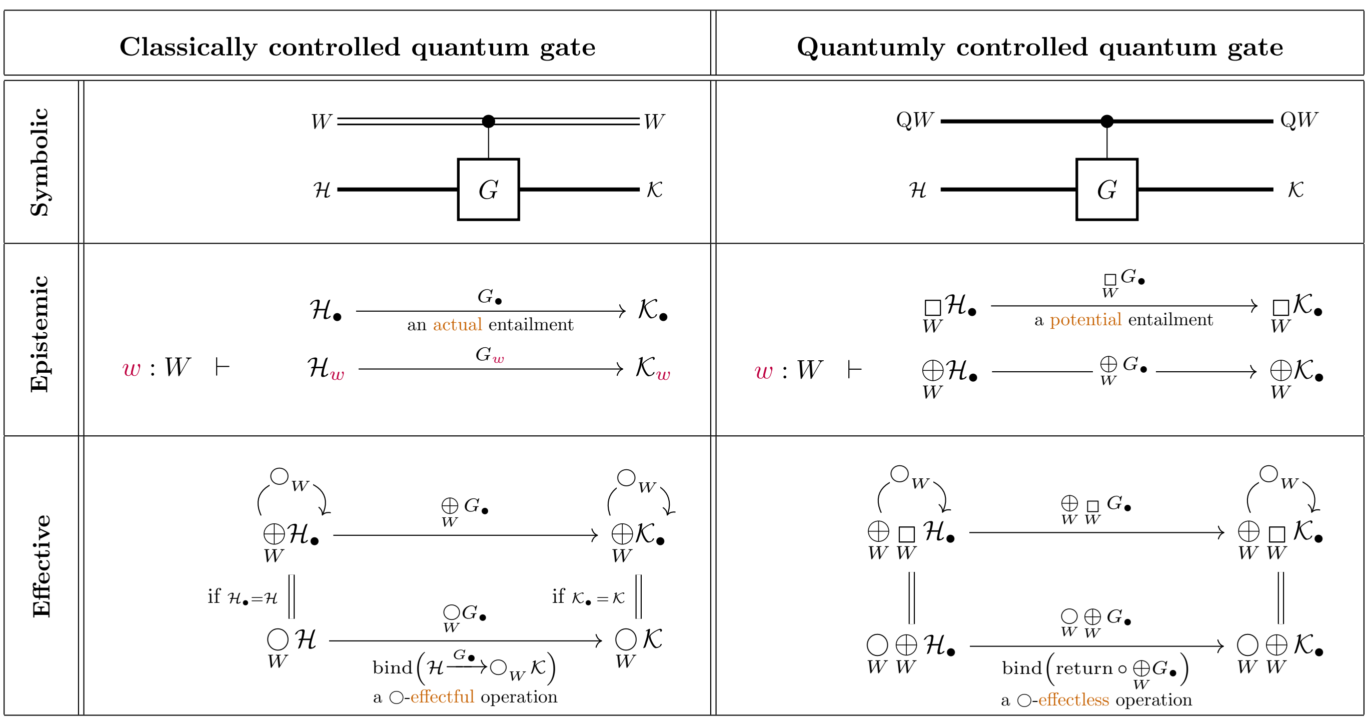
- *epistemic typing* via W-dependent linear types

into

- *effective typing* via $\bigcirc_W$-modal linear types.

Besides the practical utility which we demonstrate in the following, the modal logic of this typing neatly reflects intuition, as shown.





Here the "epistemic"-typing of controlled quantum gates shown in the middle row is manifest: For classical control the quantum gate is a W-dependent linear map, while for quantum control it is a genuine linear map on the W-indexed direct sum. The equivalent (200) "effective" typing in the top line of the bottom row follows by monadicity of $\bigoplus_W$ (see Prop. 2.30). The very last line shows the corresponding Kleisli-triple formulation of "programs with side effects" (66). On the left this requires assuming that the dependent linear type is constant, $\mathcal{H}_\bullet = \mathcal{H}$ (which typically is the case in practice, see the example on p. 80) since that makes it correspond to a free $\bigcirc$-modale. On the right we see the effectless operation (69).

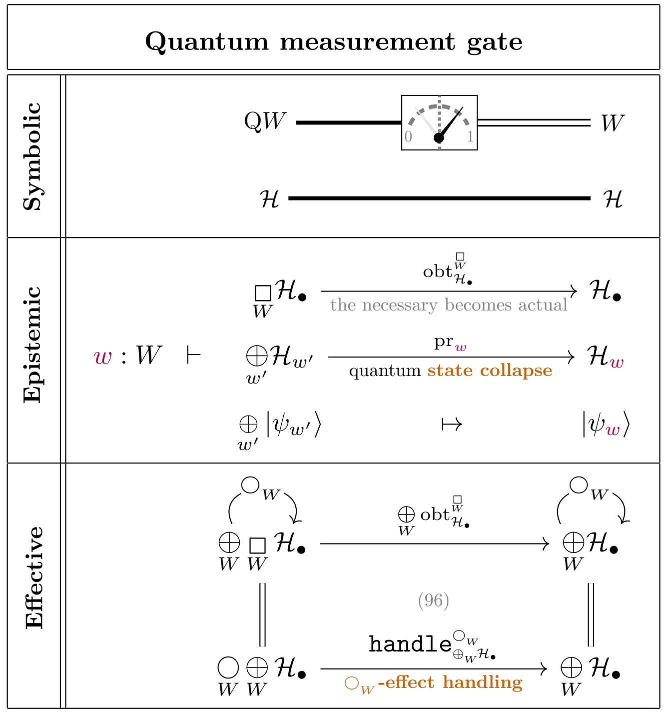
Quantum measurement – Copenhagen-style. Last but not least, we obtain this way a natural typing of the otherwise subtle case of quantum measurement gates: These are now given simply by the $\Box$-counit and, equivalently, by the $\bigcirc$-join (cf. Prop. 2.27), as shown on the right.

Via the language of effectful computation (Lit. 1.17) and with the "reader-monad" $\bigcirc$ modally pronounced as "indefiniteness" (191), this translates to the pleasant statement that:

"For effectively-typed quantum data, quantum measurement is nothing but the *handling of indefiniteness-effects*" (regarded as modale homorphisms via (93)).

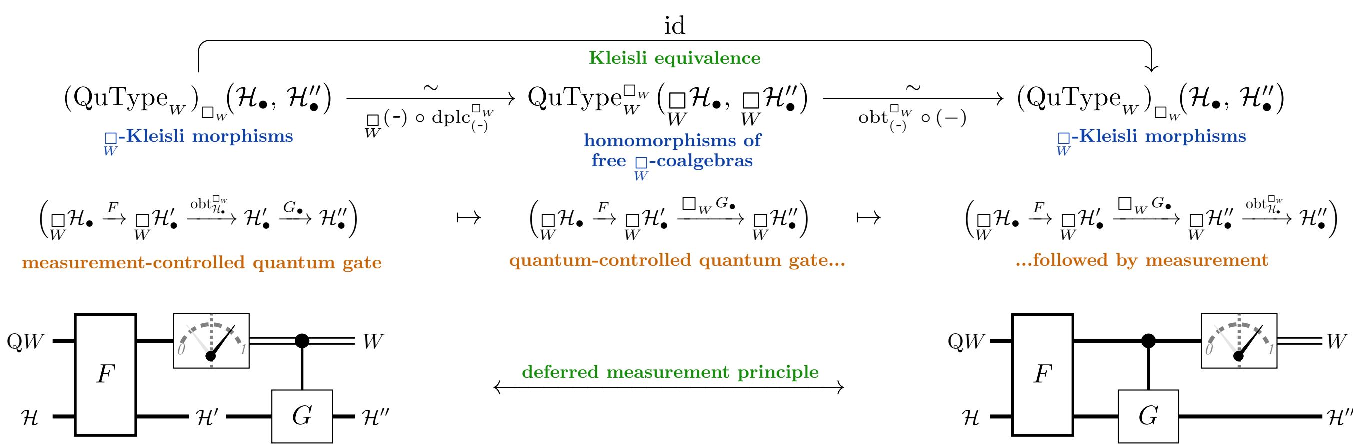
In more detail:
"Before measurement, quantum data is indefinite(-effectful), while quantum measurement actualizes the data by handling of its indefiniteness(-effect)"

This way the puzzlement of the "state collapse" (21) is resolved into an appropriate quantum effect language equivalent (200) to quantum modal logic.



Before looking at examples (p. 80), we record a basic structural result immediately implied by this typing, which may evidently be understood as formalizing the *deferred measurement principle* (18), thus making this principle verifiable in LHoTT as [Sta15] envisioned should be the case for any respectable quantum programming language:

Proposition 2.38 (Deferred measurement principle). *With respect to the above typing of quantum gates, the* $\Box$*-Kleisli equivalence* (94) *is the following transformation of quantum circuits:*



Proof. It just remains to see that the Kleisli equivalence $\underset{W}{\Box}(\text{-}) \circ \mathrm{dplc}^{\Box_W}_{(\text{-})}$ acts in the first step as claimed, hence that the following diagram commutes:

$$
\begin{array}{ccccccc}
\underset{W}{\Box}\mathcal{H}_\bullet & \xrightarrow{\quad F \quad} & \underset{W}{\Box}\mathcal{H}'_\bullet & & & & \\
\Big\downarrow {\scriptstyle \mathrm{dplc}^{\Box_W}_{\mathcal{H}_\bullet}} & & \Big\downarrow {\scriptstyle \mathrm{dplc}^{\Box_W}_{\mathcal{H}'_\bullet}} & \diagdown\!\!\diagdown & & & \\
\underset{W}{\Box}\underset{W}{\Box}\mathcal{H}_\bullet & \xrightarrow[\Box_W F]{} & \underset{W}{\Box}\underset{W}{\Box}\mathcal{H}'_\bullet & \xrightarrow[\underset{W}{\Box}\left(\mathrm{obt}^{\Box_W}_{\mathcal{H}'_\bullet}\right)]{} & \underset{W}{\Box}\mathcal{H}'_\bullet & \xrightarrow[\Box_W G_\bullet]{} & \underset{W}{\Box}\mathcal{H}''_\bullet
\end{array}
$$

But the square commutes since the gate F is independent of the measurement result $w : W$ and hence is a homomorphism of free $\Box$-coalgebras (by Rem. 2.29), while the triangle commutes by the comonad axioms (71). $\Box$