Understanding Topological Quantum Gates in FQH Systems via the Algebraic Topology of exotic Flux Quantization

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Abstract

Fractional quantum Hall (FQH) systems are a main contender for future hardware realizing topologically protected registers ("topological qbits") and protected operations on these ("topological quantum gates"), both plausibly necessary ingredients for future quantum computers at useful scales. But the anyonic braiding "statistics" of FQH quasi-particles that has been experimentally reported is, while necessary and impressive, far from sufficient: What is needed are externally operable and measurable "defect anyons" whose controlled braiding would implement the desired protected gates via adiabatic holonomies.

Here we discuss a novel non-Lagrangian effective description of FQH systems, based on previously elusive proper global quantization of effective topological flux, which directly translates their quantum-observables, -states, -symmetries, and -measurement channels into purely algebro-topological analysis of local systems over the flux moduli spaces. Under the hypothesis — for which we provide evidence — that the appropriate effective flux quantization of FQH systems is in 2-Cohomotopy (a cousin of *Hypothesis H* in high-energy physics), the results here are rigorously derived and as such might usefully inform future laboratory searches for topological quantum hardware. In particular, the theory predicts (i) how defect anyons may arise in FQH systems, (ii) which operable quantum gates and (iii) which measurement readouts these may implement.

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1 Motivation & Introduction

Need for topological quantum protection. The potential promise of *quantum computers* [77][47] is enormous [39][7][80], but their practicability hinges on finding and implementing methods to stabilize quantum registers and gates against decohering noise. Serious arguments [56][23][65][27][28][52][38][113] and practical experience [82] suggest that the currently dominant approach of *quantum error correction* at the software-level (QEC [67][79]) will need to be supplemented [16] [1] by more fundamental physical mechanisms of quantum error *protection* already at the hardware level, in the form of "topological" stabilization of quantum states ("topological qbits") and operations ("topological quantum gates") [58][35][100][99]. While the general idea of topological quantum protection is famous and widely discussed, its fine details have received less attention and are nowhere nearly as well-understood as those of QEC — this in odd contrast to its plausible necessity for scalable quantum computing.

Need for better FQH theory. The main practical contender [2] for the required topological quantum hardware currently are (cf. [3][18][11][5][71]) fractional quantum Hall (FQH) systems ([106], review in [43][110]), where one of the hallmarks of the required *topological order* ([114], cf. [119, §III][88]) has been experimentally reported ([74][75][76]): namely the "anyon statistics" of FQH quasi-particles (topological solitons). However, a clear theoretical concept for how to go from this observation to the implementation of topological quantum gates on FQH hardware appears to have been lacking.

In particular, even if anyonic quasi-particles are detected to be present in the FQH material, their positions or other quantum numbers are not externally controllable parameters, while the operation of topological quantum gates, as commonly understood, requires the externally controlled adiabatic movement, and eventually the measurement, of anyon-like singularities, hence of "defect anyons" instead of "solitonic anyons" (cf. [73, §3]).

Related to this open practical problem is arguably the previous lack of a solid theory/prediction of defect anyons in FQH systems, let alone the discussion of operating quantum (measurement) gates on these.

The problem with effective CS-theory. Experiment shows abundantly that the fractional quantum Hall effect is a *universal* phenomenon in that its characteristic properties are independent of the microscopic nature of the host material and of impurities or deformations of the sample. This suggests the existence of accurate *effective* field theory descriptions whose degrees of freedom reflect not any microscopic host particles but instead the nature of the universally emergent FQH quasi-particles (much like conformal field theory universally serves as effective description of critical phenomena in statistical mechanics, cf. [20, §3.2]). Traditionally, this putative effective FQH theory is sought in the ancient and much-studied realm of *Lagrangian* quantum field theories (cf. [51][40]), where one readily argues ([118][114], cf. [115, §2][110, §5][98, p 5]) that the only candidates are variants of abelian Chern-Simons theory [10][81][68].

However, widely popular as they are, all gauge-field Lagrangians suffer from the deficiency that they are expressed in terms of only the *local* degrees of freedom of the gauge field — the gauge potential forms —, and hence by themselves miss exactly the *global* degrees of freedom that are relevant for topological systems like FQH. While the missing global *flux quantization laws* [1][94] are traditionally tacked onto Lagrangian theories in an afterthought, the effective CS-Lagrangians proposed for FQH systems have the unnerving deficiency that — in their attempt to model the all-important *fractional* quasi-particle current by an effective gauge field —, they appear to be inconsistent with the integrality demanded by ordinary flux-quantization (cf. [115, p 35][110, p 159][98, p 5]).

This issue is an example of the notorious open problem of finding *non-perturbative* quantizations of Lagrangian theories as needed for strongly coupled topological quantum systems [31] (the analog of *mass gap problem* in solid state physics of what in mathematical high energy physics has been pronounced a "Millennium Problem" [13]).

Novel effective FQH theory based on flux quantization. In contrast, we have recently developed a non-Lagrangian theory of topological quantum states in (higher) gauge theories which is compatible with and in fact all based on consistent flux-quantization (survey in [94][98]): The main insight here is that

(a) flux-quantization laws are encoded in *classifying spaces* $\mathcal{A}$ (and consistency requires that their "rationalization" reflects the duality-symmetric form of the gauge-field's Bianchi identities), cf. [94, §3],

[1] [16]: "The qubit systems we have today are a tremendous scientific achievement, but they take us no closer to having a quantum computer that can solve a problem that anybody cares about. [...] What is missing is the breakthrough [...] bypassing quantum error correction by using far-more-stable qubits, in an approach called topological quantum computing."

[2] Much more press coverage has been give to the alternative candidate topological platform of "Majorana zero modes" in nanowires [17]; but even if the persistent doubts about their experimental detection can be dispelled in the future, these topological quantum states would by design be unmovable and hence would not support the hardware-level protected quantum *gates* that we are concerned with here.

(b) the *topological quantum observables on flux* depend only on the homotopy type of this classifying space $\mathcal{A}$, and not on any other (local, microscopic) properties of the theory [92].

The key role of algebraic topology. With this understanding, the question for an effective QFT description of FQH systems is then not answered as traditionally (by choosing a Lagrangian whose EOMs reflect local properties like the Hall current) but instead by finding a classifying space $\mathcal{A}$ whose implied topological quantum observables match the expected global observables, such as, for FQH systems, on the torus the emblematic commutation relation $W_a W_b = \zeta^2 W_b W_a$ (cf. [110, (5.28)] and (33) below).

This construction of topological quantum states of quantized flux proceeds entirely by the analysis of *local systems* on the homotopy type of moduli spaces of flux given by mapping spaces from the spacetime domain into the classifying space for the flux-quantization law (recalled in §2) and as such is squarely a problem in the mathematical subject of homotopy theory and algebraic topology (see §A.1 for pointers).

Novel effective flux quantization for FQH systems. Concretely, a candidate classifying space for the *effective* magnetic flux through FQH systems (as seen by the effective quasi-particles/holes) turns out [95][93][98] [3] to be the 2-sphere $\mathcal{A} \simeq S^2$, modeling effective FQH flux in a variation of the usual classifying space of plain magnetic flux (of which it is the "2-skeleton"):

$$\underset{\text{effective FQH flux}}{\overset{\text{classifying space for}}{}}\quad S^2 \simeq \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty \simeq B\mathrm{U}(1) \quad \underset{\text{ordinary magnetic flux}}{\overset{\text{classifying space for}}{}}$$

In this article, we work out in detail how this classifying space produces quantum effects in FQH systems, in particular how it reproduces quantum phenomena of abelian Chern-Simons theory, but as a quick plausibility check, note that the rationalization of the 2-sphere is encoded by the following differential equations (its "Sullivan minimal model" cf. [94, §3.2]), which are just those equations that characterize the Chern-Simons 3-form H_3 for a gauge field flux density F_2 as it appears in the Lagrangian formulation of Chern-Simons theory:

$$\begin{array}{c}\text{rational model of}\\ \text{classifying space for}\\ \text{effective FQH flux}\end{array}\quad \mathrm{CE}(\mathfrak{l}S^2) \simeq \mathbb{R}_{\mathrm{d}}\begin{bmatrix} F_2 \\ H_3 \end{bmatrix} \Big/ \begin{pmatrix} \mathrm{d}\,F_2 = 0 \\ \mathrm{d}\,H_3 = F_2 F_2 \end{pmatrix} \quad \begin{array}{c}\text{Bianchi identities characterizing}\\ \text{Chern-Simons 3-form /}\\ \text{Green-Schwarz mechanism}\end{array}$$

Incidentally it is in this sense that our effective description of the FQH effect is a mild form of *higher gauge theory* (cf. [97]), since the Chern-Simons 3-form (traditionally understood as a Lagrangian density) here appears as higher degree flux density satisfying a Bianchi identity of the form known from *Green-Schwarz mechanisms.* [4]

Aims. With this novel consistent effective description of FQH systems in hand, our ambition here is to provide previously missing theoretical understanding & prediction of

(i) appearance of defect anyons ⇒ topological qbits,

(ii) operable transformations on these ⇒ topological quantum gates,

(iii) their admissible measurement bases ⇒ topological readout,

in FQH-like systems.

These predictions are solid (we provide rigorous proof) within the effective topological QFT that we use, and hence we begin by recalling the evidence that our topological quantized flux matches the expected properties of FQH systems in previously understood sectors.

While the account here is purely theoretical at this point, we highlight that it does suggest potential experimental pathways to test and eventually implement the above mechanisms (i)-(iii) in FQH systems, notably it predicts that defect anyons should be realizable as defects in the FQH material where the magnetic field is expelled (as could be expected to be the case in type I superconducting impurities in the FQH semi-conductor), cf. Fig. F with §4.3

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[3] As explained in [95][42], following [96], the 2-sphere here is a relative of the 4-sphere which similarly serves as flux quantization of the higher gauge field in 11D supergravity [87, §2.5][32][41] (review in [94]), where its choice as such is referred to as *Hypothesis H* [32]. While this is where our approach to FQH systems here comes from and is informed by [93] the reader here may entirely ignore this, as it were, geometric engineering of FQH systems on M5-probes of 11d SuGra (reviewed in [98, §2-3]).

[4] In the "engineering" of our FQH model on M5-branes referred to in ftn. 3, this 3-form arises as the restriction to an orbi-singularity of the "self-dual" tensor field carried by these branes, which itself is quantized in a higher (and "twistorial") form of Cohomotopy.

2 Flux observables and Classifying spaces

We first recall, from [92], how non-perturbative topological quantum observables on G-Yang-Mills fluxes depend exclusively on (the homotopy type of) the electric/magnetic classifying space $B\big(G \ltimes (\mathfrak{g}/\Lambda)\big)$ (Prop. 2.5 below), and how for general (effective, higher) gauge theories this space is replaced by a classifying space $\mathcal{A}$ for the given flux-quantization law [94] (Rem. 2.7 below). This is the key observation that allows us (in Def. 2.8 below) to get directly at the otherwise elusive non-perturbative topological quantum flux observables of an effective gauge theory, like for FQH systems, from hypothesizing (not an effective Lagrangian density, as done traditionally, but) an effective flux classifying space.

Definition 2.1 (Spacetime). Throughout, we consider

- $X^{1,3} := \mathbb{R}^{1,1} \times \Sigma^2$ a globally hyperbolic 4D spacetime,
- with spatial slices $\mathbb{R}^1 \times \Sigma^2$, to be thought of as a tubular neighborhood of:
- Σ^2, a surface (here: a connected, oriented smooth 2D manifold with boundary) which at times is
- specialized to $\Sigma^2 \equiv \Sigma^2_{g,n,b}$, the unique (up to homeomporphism) surface
  - of genus g,
  - with b boundary components,
  - and n punctures:

$$\underbrace{\Sigma^2_{g,\,b,\,n}}_{\text{genus, boundaries, punctures}} \;\simeq\; \Big(\underbrace{\Sigma^2_{0,0,0}}_{\text{sphere}} \overset{\text{connected sum}}{\#} \underbrace{T^2 \# \cdots \# T^2}_{g \text{ connected summands of tori}}\Big) \overset{\text{complement}}{\setminus} \big\{ \underbrace{D^2 \sqcup \cdots \sqcup D^2}_{b \text{ disjoint summands of open disks}} \overset{\text{disjoint union}}{\sqcup} \underbrace{\overline{D}^2 \sqcup \cdots \sqcup \overline{D}^2}_{n \text{ disjoint summands of closed disks}} \big\}, \tag{1}$$

understood as modeling an effectively 2-dimensional sample Σ^2 of material. [5]

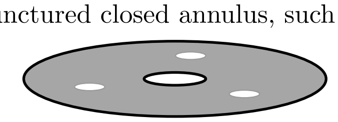
We abbreviate $\Sigma^2_{g,b} := \Sigma^2_{g,b,0}$ and $\Sigma^2_g := \Sigma^2_{g,0} \equiv \Sigma^2_{g,0,0}$.

Example 2.2. We have homoemorphisms as follows:

$$\begin{array}{lll} \Sigma^2_{0,0,0} \simeq S^2 & & \text{sphere} \\ \Sigma^2_{1,0,0} \simeq T^2 & & \text{torus} \\ \Sigma^2_{0,1,0} \simeq D^2 & & \text{disk} \\ \Sigma^2_{0,0,1} \simeq \mathbb{R}^2 & & \text{plane} \\ \Sigma^2_{0,2,0} \simeq A^2 & & \text{(closed) annulus} \\ \Sigma^2_{0,0,2} \simeq \mathbb{R}^2 \setminus \{0\} & & \text{open annulus.} \end{array} \tag{2}$$

Of particular interest to us (in §4.4) is the n-punctured closed annulus, such as

$$\Sigma^2_{0,2,3} \simeq \tag{3}$$



Definition 2.3 (Gauge group). Consider

- G a Lie group,
- with Lie algebra $\mathfrak{g}$,
- among which we choose an Ad-invariant lattice $\Lambda \subset \mathfrak{g}$,
- specialized shortly to $G \equiv \mathbb{R}$,
- and $\Lambda = \mathbb{Z} \subset \mathbb{R}$,

  corresponding to traditional Dirac charge quantization (see Ex. 2.6 below).

[5] Albeit routinely considered in theory, the practicability of direct laboratory realizations of $\Sigma^2_{g,n,b}$ (1) is limited when $g > 0$. The case $g = 1$ (the torus) is readily realized (only) when considering momentum space (the Brillouin torus of a 2D crystal, cf. [88]) instead of position space, but, while noteworthy in itself, this is not the case of FQH systems of concern here. Alternatively, it was argued [4] that suitable defects, called "genons", in a crystal lattice could make a sample of nominal genus $g = 0$ effectively behave like of $g > 0$.
But irrespectively of practicality, the theoretical possibility of $g > 0$ allows to compare our topological quantum flux observables to those of abelian Chern-Simons theory in the case $\Sigma^2_{g>0,0,0}$, and their agreement in this theoretical case supports the validity of our observables also in the more practical cases of $g = 0$, $n, b \neq 0$.

The following theorem 2.4, from [92], is based on well-known ingredients but may have escaped earlier attention in its deliberate disregard of the gauge potentials in favor of focus on the electric/magnetic flux densities — which is what brings out how the topological flux quantum observables are all controlled by maps from Σ^2 to the classifying space $B\big(G \ltimes (\mathfrak{g}/\Lambda)\big)$, cf. Rem. 2.6 below.

Theorem 2.4 (**Yang-Mills G-flux quantum observables** [92, Thm 1])**.** *Non-perturbative quantum observables on the G-Yang-Mills flux-density* [6] *through a closed surface Σ^2 form the group convolution C^*-algebra $\mathbb{C}[-]$ of the Fréchet Lie group of smooth functions $C^\infty(-,-)$ from Σ^2 to the semidirect product of G with the additive group $\mathfrak{g}/\Lambda$*

$$\mathrm{FlxObs}^{\mathrm{ord}}_{\Sigma^2} \;\simeq\; \mathbb{C}\Big[C^\infty\big(\Sigma^2,\, G \ltimes_{\mathrm{Ad}} (\mathfrak{g}/\Lambda)\big)\Big] \;\simeq\; \mathbb{C}\Big[\underbrace{C^\infty\big(\Sigma^2,\, G\big)}_{\substack{\text{electric flux}\\ \text{observables}}} \ltimes_{\mathrm{Ad}} \underbrace{C^\infty\big(\Sigma^2,\, \mathfrak{g}/\Lambda\big)}_{\substack{\text{magnetic flux}\\ \text{observables}}}\Big]\,. \tag{4}$$

Accordingly, we have in this situation that:

Proposition 2.5 (**Topological G-flux quantum observables** [92, §3])**.** *The algebra of topological G-flux quantum observables, hence of the group convolution C^*-algebra on the discrete group of connected components $\pi_0(-)$ of the flux densities, is equivalently the group algebra of the fundamental group of maps into the classifying space:*

$$\begin{aligned} \mathrm{TopFlxObs}^{\mathrm{ord}}_{\Sigma^2} \;&:=\; \mathbb{C}\Big[\pi_0\, C^\infty\big(\Sigma^2,\, G \ltimes_{\mathrm{Ad}} (\mathfrak{g}/\Lambda)\big)\Big] \;\simeq\; \mathbb{C}\Big[\pi_0\, C^\infty\big(\Sigma^2,\, G\big) \ltimes_{\mathrm{Ad}} \pi_0\, C^\infty\big(\Sigma^2,\, \mathfrak{g}/\Lambda\big)\Big] \\ &\simeq\; \mathbb{C}\Big[\pi_1\, \mathrm{Map}_0\big(\Sigma^2,\, B\big(G \ltimes (\mathfrak{g}/\Lambda)\big)\big)\Big]\,. \end{aligned} \tag{5}$$

(See §A.1 for our notation concerning mapping spaces.)

Example 2.6 (**The case of ordinary electromagnetism**)**.** For ordinary electromagnetic flux subject to the usual Dirac charge quantization law (where the magnetic but not explicitly the total electric flux is quantized in integral cohomology, cf. [92, (14)]) the relevant choice in (5) is $G := \mathbb{R}$ and $\Lambda := \mathbb{Z} \hookrightarrow \mathbb{R}$, whence the homotopy type of the classifying space is $\mathcal{A} := B\big(\mathbb{R} \ltimes (\mathbb{R}/\mathbb{Z})\big) \simeq B\mathrm{U}(1)$.

In this case, the algebra (5) of observables on topological flux through a closed surface Σ_g (1) is

$$\begin{aligned} \mathrm{TopFluxObs}^{\mathrm{ord}}_{\Sigma^2_g} \;&\simeq\; \mathbb{C}\big[\pi_0\, C^\infty\big(\Sigma^2,\, \mathrm{U}(1)\big)\big] \;\simeq\; \mathbb{C}\big[\pi_0\, \mathrm{Map}\big(\Sigma^2,\, B\mathbb{Z}\big)\big] \\ &\simeq\; \mathbb{C}\big[H^1\big(\Sigma^2_g;\, \mathbb{Z}\big)\big] \;\simeq\; \mathbb{C}\big[\mathbb{Z}^{2g}\big]\,. \end{aligned} \tag{6}$$

Interestingly, (6) is *not quite* the algebra of observables expected for fractional quantum Hall systems, the latter instead being a non-abelian central extension (see ... below)

Prop. 2.5 is remarkable in how it shows the topological flux quantum observables of ordinary gauge theory to depend exclusively on the classifying space that encodes the flux-quantization law (cf. Ex. 2.6).

A key observation now is the following:

Remark 2.7 (**Proper flux quantization** [91][94])**.** There are different admissible choices for classifying spaces $\mathcal{A}$ of flux-quantization already for ordinary gauge theories and in particular for generalized (higher) gauge theories – each choice defining a global completion of the local field content of the gauge theory.

While usual (perturbative) machinery of constructing quantum field theories based on Lagrangian densities does not capture this global information, since Lagrangian densities do not (being functions only of local gauge potentials but not the global flux-quantized gauge field content), with Prop. 2.5 we have established a direct construction of topological flux quantum observables from the flux-quantization law determined by a classifying space $\mathcal{A}$. We are thus led to the following notions:

Definition 2.8 (**Topological flux sector of flux-quantized quantum gauge theories**)**.** Given a (higher) gauge theory flux-quantized with classifying space $\mathcal{A}$ [94]

- **Generic topological flux observables...**

  The algebra of topological quantum observables on flux through a closed surface Σ^2 is, in view of Prop. 2.5:

$$\mathrm{TopFlxObs}^{\mathcal{A}}_{\Sigma^2} \;=\; \mathbb{C}\big[\pi_1\, \mathrm{Map}_0\big(\Sigma^2,\, \mathcal{A}\big)\big]\,. \tag{7}$$

[6] For the case of abelian G of interest here, these are observables on the *reduced* phase space.

- **...and topological quantum states.**

  In immediate consequence, the (Hilbert) space $\mathcal{H}_{\Sigma^2}$ of topological quantum states must be a module over this algebra, which, being a group algebra, means that it must be a linear representation of this fundamental group:

$$\mathcal{H}_{\Sigma^2} \in \mathrm{Mod}_{\mathbb{C}}\big(\pi_1 \, \mathrm{Map}_0(\Sigma^2, \mathcal{A})\big) .$$

This furnishes previously unavailable access to non-perturbative quantization of exotic (notably: effective) gauge theories in their topological sector: just by changing the choice of classifying space and applying the formula (7). Even before choosing the classifying space $\mathcal{A}$ and hence the flux-quantization law, we see how to characterize topological quantum states in incrementally more general situations:

- **... for solitonic flux.**

  If $\Sigma^2 = \Sigma^2_{g,b,n}$ (1) is possibly non-compact ($n > 0$), then the *solitonic* flux configurations (cf. [94, §2.2][92, §A.2]) are those which are *vanishing at infinity* and thus classified by *pointed* maps on the one-point compactification $(-)_{\cup\{\infty\}}$

$$\mathcal{H}_{(\Sigma^2_{g,b,n})} = \mathrm{Mod}_{\mathbb{C}}\Big(\pi_1 \underbrace{\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}}, \mathcal{A}\big)}_{\substack{\text{moduli space of} \\ \text{solitonic topological flux}}}\Big) . \tag{8}$$

- If the (higher) gauge theory in question is generally covariant (e.g. in that it is topological, as in the case of Chern-Simons theory) then pullback along diffeomorphisms of Σ^2 are meant to constitute gauge transformations of flux configurations, so that the moduli space of topological flux is the homotopy quotient by the diffeomorphism group (see Def. 3.6):

$$\underbrace{\overbrace{\mathcal{H}_{(\Sigma^2_{g,b,n})}}^{\substack{\text{quantum states of} \\ \text{topological flux}}}}_{\substack{\text{generally covariantized} \\ \text{and in all charge sectors}}} \in \overbrace{\underbrace{\mathrm{Mod}_{\mathbb{C}}}_{\substack{\text{reps} \\ \text{of}}}\Big(\underbrace{\pi_1}_{\substack{\text{fundamental} \\ \text{group of}}}}^{\substack{\text{local systems of} \\ \text{state spaces on}}} \overbrace{\Big(\underbrace{\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}}, \mathcal{A}\big)}_{\substack{\text{plain moduli space} \\ \text{of topological flux}}} \underbrace{/\!\!/ \mathrm{Diff}^{+,\partial}(\Sigma^2_{g,b,n})}_{\substack{\text{covariantized} \\ \text{under diffeos}}}\Big)}^{\text{moduli space of topological flux}}\Big) . \tag{9}$$

Remark 2.9. Representations of fundamental groups $\pi_1(\mathcal{M})$ are equivalently known as *local systems* or *flat bundles* of vector spaces over (the given connected component of) the space $\mathcal{M}$ (cf. [73, Lit. 2.22]), and there is deep relevance [90] in identifying such as quantum state spaces subject to symmetries and classical control (cf. [89]).

3 Moduli spaces of Topological Flux

3.1 Plain moduli

(...)

3.2 General covariance

We discuss the action of the diffeomorphism groups on the moduli spaces of solitonic topological flux.

Braid groups and mapping class groups.

Definition 3.1 (**Configuration space and Braid group).**
(i) For Σ a smooth manifold, possibly with boundary, and $n \in \mathbb{N}$, the *configuration space of n points in* Σ is the topological space

$$\mathrm{Conf}_n(\Sigma) \;:=\; \Big\{ (s_1, \cdots, s_n) \,\in\, \Sigma^{\times^n} \,\Big\vert\, \underset{i \neq j}{\forall} s_i \neq s_j \Big\} \Big/ \mathrm{Sym}_n$$

(topologized as the quotient space of a subspace of a product space).
(ii) The fundamental group of this space (assuming now without, substantial restriction, that Σ is connected) is the *braid group on n strands in* Σ (cf. [30, §9]), which as such comes equipped with a forgetful map to the symmetric group:

$$\mathrm{Br}_n(\Sigma) \;:=\; \pi_1 \mathrm{Conf}_n(\Sigma) \longrightarrow\!\!\!\!\to \mathrm{Sym}_n \,. \tag{10}$$

Example 3.2 (**Artin presentation of braid groups**, cf. [34, §7][73, Lit. 2.20]**).** For $n \geq 2$, the surface braid group (10) of the disk (the default case of braid groups) has the following finite presentation:

$$\mathrm{Br}_n \;:=\; \mathrm{Br}_m\big(\Sigma^2_{0,1,n}\big) \;\simeq\; F\langle b_1, \cdots, b_{n-1}\rangle \Big/ \Big(\underset{i+1<j}{\forall} \big(b_i b_j = b_j b_j\big) \,,\, \underset{1 \leq i < n-1}{\forall} \big(\underbrace{b_i\, b_{i+1}\, b_i \;=\; b_{i+1}\, b_i\, b_{i+1}}_{\text{Yang-Baxter relation}} \big) \Big) \,, \tag{11}$$

in terms of which its canonical homomorphism to the symmetric group is the quotient map by one further set of relations:

$$\mathrm{Br}_n \longrightarrow\!\!\!\!\to \mathrm{Sym}_n \;:=\; \mathrm{Br}_n \Big/ \Big(\underset{i}{\forall} \big(b_i b_i = \mathrm{e} \big) \Big) \,.$$

The general surface braid group $\mathrm{Br}_n(\Sigma^2)$ may be presented by adjoining to these *Artin generators* b_i further generators (corresponding to moving single strands along cycles in the surface) and further relations. In each case, there is a projection to the symmetric group by retaining the Artin generators:

$$\mathrm{Br}_n(\Sigma^2) \longrightarrow\!\!\!\!\to \mathrm{Sym}_n$$

Example 3.3 (**Presentation of spherical braid group** [29, p 245,55], cf. [108]**).** The surface braid group (10) of the sphere (often: "spherical braid group") is presented as a quotient of the Artin presentation (11) by one further relation:

$$\mathrm{Br}_n(S^2) \;\simeq\; \mathrm{Br}_n \big/ \big((b_1 \cdots b_{n-1})(b_{n-1} \cdots b_1) \big) \,. \tag{12}$$

Definition 3.4 (**Diffeomorphism Group and Mapping Class Group).** For Σ an oriented manifold, possibly with boundary, we write

$$\begin{array}{ccccc} \mathrm{Homeo}^{+,\partial}(\Sigma) & \hookrightarrow & \mathrm{Homeo}(\Sigma) & \hookrightarrow & \mathrm{Map}(\Sigma,\Sigma) \\ \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} & & \\ \mathrm{Diff}^{+,\partial}(\Sigma) & \hookrightarrow & \mathrm{Diff}(\Sigma) & & \end{array} \tag{13}$$

for its topological groups of homeomorphisms and diffeomorphisms, respectively for the further subgroups of maps preserving the orientation $(+)$ and restricting to the identity on the boundary (∂).

For $\Sigma \equiv \Sigma^2$ a surface, the group of connected components of the latter is known as the *mapping class group* [54, §1][72, §3][30, p. 45]:

$$\mathrm{MCG}(\Sigma^2) \;:=\; \pi_0\big(\mathrm{Diff}^{+,\partial}(\Sigma^2)\big) \,. \tag{14}$$

Example 3.5 (**Mapping class groups of closed oriented surfaces**, cf. [72, §6][30, §6]**).** The mapping class group of the torus is

$$\mathrm{MCG}(\Sigma^2_1) \;\simeq\; \mathrm{Sp}_2(\mathbb{Z}) \;\simeq\; \mathrm{SL}_2(\mathbb{Z}) \,, \tag{15}$$

which is generated by the two elements [103, Thm VII.2 p 78][14, Thm 1.1]

$$S \;:=\; \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} , \qquad T \;:=\; \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{16}$$

and presented subject to the following relations [60, p. 126][9, §2.1]:

$$\mathrm{SL}_2(\mathbb{Z}) \simeq \big\langle S,\, T \,\big|\, S^4 = (TS)^3 = \mathrm{e},\, S^2\,(TS) = (TS)\,S^2 \big\rangle\,. \tag{17}$$

More generally, the mapping class group of Σ^2_g (1), for $g \in \mathbb{N}$, sits in a short exact sequence

$$1 \longrightarrow \underset{\text{Torelli group}}{I_g} \longrightarrow \mathrm{MCG}(\Sigma^2_g) \longrightarrow \underset{\text{symplectic group}}{\mathrm{Sp}_2(\mathbb{Z})} \longrightarrow 1\,, \tag{18}$$

where the action of $\mathrm{MCG}(\Sigma^1_2)$ on $H^2(\Sigma^2_g;\mathbb{Z}) \simeq \mathbb{Z}^g \times \mathbb{Z}^g$ is through the defining action of the integer symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Definition 3.6 (**Moduli spaces of solitonic topological fluxes).** The underlying homeomorphisms of diffeomorphsims (13) of surfaces $\Sigma^2_{g,b,n}$ (1) extend functorially to the one-point compactification (by Prop. A.1) to make a topological group homomorphism

$$\mathrm{Diff}^{(+,\partial)}\big(\Sigma^2_{g,b,n}\big) \xrightarrow{\ \iota\ } \mathrm{Homeo}^{(+,\partial)}\big(\Sigma^2_{g,b,n}\big) \xrightarrow{(-)_{\cup\{\infty\}}} \mathrm{Aut}_{\mathrm{Top}^*}\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}}\big)\,.$$

Via the latter's action (by pre-composition) on pointed mapping spaces (8) we obtain the homotopy quotient (50) of the pointed mapping space [7]

$$\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, \mathcal{A}\big) /\!\!/ \mathrm{Diff}^{+,\partial}\big(\Sigma^2_{g,b,n}\big) \;\in\; \mathrm{Top}^*\,, \tag{19}$$

identified in (9) as the *covariantized moduli space* of $\mathcal{A}$-quantized solitonic topological fluxes on $\Sigma^2_{g,b,n}$.

Proposition 3.7 (**Homotopy type of Diffeomorphism groups).**
(i) *For closed oriented surfaces $\Sigma^2_{g,b,0}$ (1), the homotopy type of their diffeomorphism group (13) is:*

$$\begin{array}{llll}
\int \mathrm{Diff}^+\big(\Sigma^2_{0,\,0,\,0}\big) \simeq \int \mathrm{SO}(3) & \Rightarrow \;\; \mathrm{MCG}(\Sigma^2_{0,\,0,\,0}) \simeq 1 & \text{and} & \pi_1\,\mathrm{Diff}^+\big(\Sigma^2_{0,\,0,\,0}\big) \simeq \mathbb{Z}_2 \\
\int \mathrm{Diff}^+\big(\Sigma^2_{1,\,0,\,0}\big) \simeq \mathrm{SL}_2(\mathbb{Z}) \times \int T^2 & \Rightarrow \;\; \mathrm{MCG}(\Sigma^2_{1,\,0,\,0}) \simeq \mathrm{SL}_2(\mathbb{Z}) & \text{and} & \pi_1\,\mathrm{Diff}^+\big(\Sigma^2_{1,\,0,\,0}\big) \simeq \mathbb{Z}\times\mathbb{Z} \\
\int \mathrm{Diff}^+\big(\Sigma^2_{g\geq 2,\,0,\,0}\big) \simeq * & \Rightarrow \;\; \mathrm{MCG}(\Sigma^2_{g\geq 2,\,0,\,0}) \simeq 1 & \text{and} & \pi_1\,\mathrm{Diff}^+\big(\Sigma^2_{g\geq 2,\,0,\,0}\big) \simeq 1 \\
 & & & \\
\int \mathrm{Diff}^{+,\partial}\big(\Sigma^2_{g,\,b\geq 1,\,0}\big) \simeq * & \Rightarrow \;\; \mathrm{MCG}(\Sigma^2_{g,\,b\geq 1,\,0}) \simeq 1 & \text{and} & \pi_1\,\mathrm{Diff}^{+,\partial}\big(\Sigma^2_{g,\,b\geq 1,\,0}\big) \simeq 1\,.
\end{array} \tag{20}$$

(ii) *For punctured oriented surfaces $\Sigma^2_{g,b,\geq 1}$* except $\Sigma^2_{0,0,<3}$*, their mapping class group is an extension of that of $\Sigma^2_{g,b,0}$ by the surface's braid group (10)*

$$1 \to \mathrm{Br}_{n\geq 1}\big(\Sigma^2_{g,b}\big) \longrightarrow \mathrm{MCG}_n(\Sigma^2_{g,b,n\geq 1}) \longrightarrow \mathrm{MCG}_n(\Sigma^2_{g,b,0}) \to 1\,, \tag{21}$$

which exhausts the homotopy type of their diffeomorphism groups:

$$\int \mathrm{Diff}^{+,\partial}\big(\Sigma^2_{g,b,n\geq 1}\big) \simeq \mathrm{MCG}\big(\Sigma^2_{g,b,n\geq 1}\big) \quad \Rightarrow \quad \pi_1\,\mathrm{Diff}^{+,\partial}\big(\Sigma^2_{g,b,n\geq 1}\big) \simeq 1\,. \tag{22}$$

Proof. In (20), the first statement is due to [105], the first three were proven by [24][25][46], and the fourth is [26, Thm. 1D p 170]. The statement (22) follows with [116][117, Thm. 1.1], [8] which implies the extension (21) by the Birman exact sequence ([8], cf. [69, Thm. 3.13]) as reviewed in [30, Thm 9.1]. □

Example 3.8 (**Mapping class groups of n-punctured disk and sphere).** When the mapping class group of the disk $\Sigma^2_{0,1,0}$ and of the sphere $\Sigma^2_{0,0,0}$ are trivial by (20), the exact sequence (21) shows that the mapping class group of their punctured versions are the braid group (11) and the spherical braid group (12), respectively:

$$\begin{aligned}
\mathrm{MCG}(\Sigma^2_{0,1,n}) &\simeq \mathrm{Br}_n \\
\mathrm{MCG}(\Sigma^2_{0,0,n}) &\simeq \mathrm{Br}_n(S^2)\,.
\end{aligned} \tag{23}$$

Covariant flux monodromy. With all this in hand, we come to the main statement of this section.

---

[7] The connected components of the full mapping space $\pi_0(\mathcal{F}) \equiv \pi_0\Big(\mathrm{Map}^*\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, S^2\big)\Big) \simeq \mathbb{Z}$ are given by the Hopf degree. Since diffeomorphisms have unit Hopf degree, their precomposition preserves the connected components of the mapping space.

[8] The surfaces in [116][117] are assumed without boundary, but equipped with marked closed subcomplexes to be fixed by the diffeomorphisms. Under this definition, a puncture surrounded by a marked circle behaves just as a boundary for the purpose of computing the homotopy type of the diffeomorphism group.

Proposition 3.9 (Extension of mapping class group by flux monodromy). *For every $\Sigma^2_{g,b,n}$ (1) we have a split short exact sequence of groups*

$$1 \longrightarrow \pi_1\Big(\underbrace{\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, \mathcal{A}\big)}_{\text{moduli space}}\Big) \longrightarrow \pi_1\Big(\underbrace{\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, \mathcal{A}\big) /\!\!/ \mathrm{Diff}^{+,\partial}(\Sigma^2_{g,b,n})}_{\text{covariantized moduli space (19)}}\Big) \longrightarrow \underbrace{\mathrm{MCG}(\Sigma^2_{g,b,n})}_{\text{mapping class group (14)}} \longrightarrow 1\,,$$

exhibiting an action of the mapping class group on the fundamental group of the moduli space, and the corresponding semidirect product:

$$\pi_1\Big(\underbrace{\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, \mathcal{A}\big) /\!\!/ \mathrm{Diff}^{+,\partial}(\Sigma^2_{g,b,n})}_{\text{covariantized moduli space (19)}}\Big) \;\simeq\; \underbrace{\mathrm{MCG}(\Sigma^2_{g,b,n})}_{\text{mapping class group (14)}} \ltimes \pi_1\Big(\underbrace{\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, \mathcal{A}\big)}_{\text{moduli space}}\Big)\,. \tag{24}$$

Proof. For notational convenience, we abbreviate

$$\begin{aligned} \mathcal{F} &:= \mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, S^2\big) \\ \mathcal{D} &:= \mathrm{Diff}^{+,\partial}\big(\Sigma^2_{g,b,n}\big)\,, \end{aligned}$$

whence the claim to be proven is split exactness of

$$1 \longrightarrow \pi_1(\mathcal{F}) \longrightarrow \pi_1(\mathcal{F} /\!\!/ \mathcal{D}) \longrightarrow \pi_0(\mathcal{D}) \longrightarrow 1\,. \tag{25}$$

To this end, the Borel homotopy fiber sequence (52)

$$\mathcal{F} \longrightarrow \mathcal{F} /\!\!/ \mathcal{D} \longrightarrow \ast /\!\!/ \mathcal{D}$$

(split by picking the zero-map) induces a long exact sequence of homotopy groups (49) of this form:

$$\begin{array}{l} \overset{\text{by}(51)}{\pi_1(\mathcal{D})} \\ \quad\hookrightarrow \pi_1(\mathcal{F}) \longrightarrow \pi_1(\mathcal{F} /\!\!/ \mathcal{D}) \longrightarrow \pi_0(\mathcal{D}) \\ \quad\hookrightarrow \pi_0(\mathcal{F}) \xrightarrow{\;\sim\;} \pi_0(\mathcal{F} /\!\!/ \mathcal{D})\,. \end{array} \tag{26}$$

Here the last map shown is an isomorphism by (53) (cf. footnote 7), whence the exact sequence truncates to

$$\pi_1(\mathcal{D}) \longrightarrow \pi_1(\mathcal{F}) \longrightarrow \pi_1\big(\mathcal{F} /\!\!/ \mathcal{D}\big) \longrightarrow \pi_0(\mathcal{D}) \longrightarrow 1\,.$$

If, at this point, we invoke Prop. 3.7 then the claim (25) follows for most surfaces, namely those for which $\pi_1(\mathcal{D}) \simeq 1$. But in fact, the claim follows generally by observing that the first connecting map in (26) factors through the trivial group:

$$\pi_1(\mathcal{D}) \;\equiv\; \pi_1\big(\mathrm{Diff}^+(\Sigma^2_{g,b,n})\big) \longrightarrow 1 \longrightarrow \pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, \mathcal{A}\big)\Big) \;\equiv\; \pi_1(\mathcal{F} /\!\!/ \mathcal{F})\,.$$

Namely, by (52) the map is given by taking a given loop of diffeomorphisms to the loop of maps obtained by composing these diffeos the constant map $\Sigma^2_{g,b,n} \to S^2$ – but that gives the constant loop representing the neutral element of π_1. ∎

4 2-Cohomotopical flux through surfaces

We now specify the classifying space $\mathcal{A}$ (9) to the 2-sphere, $\mathcal{A} \equiv S^2$ (so that flux is classified by *2-Cohomotopy*) and work out the resulting covariant topological quantum observables on and quantum states of (according to §2) 2-cohomotopically quantized flux through various surfaces Σ^2, using the results of §3.

Remarkably, in the case of $\Sigma^2 \equiv S^2, T^2$ the sphere or the torus, we find reproduced (in §4.1 and §4.2, respectively) the situation traditionally argued via quantized U(1)-Chern-Simons theory over these surfaces, including fine-print such as regularization of Wilson-loop observables by framings and modular equivariance.

Then, by instead choosing punctured surfaces, we similarly work out the 2-Cohomotopically quantized flux through the punctured sphere (§4.3) and the punctured annulus (§4.4).

4.1 Flux through the plane

We recall here (from [93]) how solitonic flux through the plane $\mathbb{R}^2 \simeq \Sigma^2_{0,0,1}$ (2) quantized in 2-cohomotopy reproduces exactly the Wilson loop observables of anyonic braiding as predicted by abelian Chern-Simons theory (Rem. 4.6 below). First, we briefy recall the Pontrjagin construction that serves for us to relate cohomotopy to solitonic flux density.

The Pontrjagin construction. Among generalized non-abelian cohomology theories, (unstable) Cohomotopy $\widetilde{\pi}^n(-) \equiv \pi_0 \mathrm{Map}^*\big(-, \mathbb{R}^n_{\cup\{\infty\}}\big)$ stands out in that it accurately characterizes the soliton configurations of given charge: This may be understood as the content of the original unstable Pontrjagin theorem (which these days is more famous as the *Pontrjagin-Thom theorem* pertaining only to the *stable* case which is of little concern to us here):

Theorem 4.1 (Pontrjagin theorem – Cohomotopy charge cf. [12, §II.16][62, §IX]**).** *Given a smooth d-manifold Σ^d and $n \in \mathbb{N}$ with $n \leq d$, there is a natural bijection between:*

1. *the reduced n-Cohomotopy of the one-point compactification $\Sigma^d_{\cup\{\infty\}}$,*
2. *the cobordism classes of normally framed submanifolds $Q^{d-n} \hookrightarrow \Sigma^d$ of co-dimension=n*

$$\underset{\text{of 1pt compactification}}{\text{reduced } n\text{-Cohomotopy}} \quad \widetilde{\pi}^n\big(\Sigma^d_{\cup\{\infty\}}\big) \underset{\text{asymptotitic directed distance}}{\overset{\text{regular pre-image of } 0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\sim}}}} \mathrm{Cob}^n_{\mathrm{Fr}}\big(\Sigma^d\big) \quad \begin{array}{c}\text{cobordism classes of}\\ \text{normally framed sub-}\\ \text{manifolds of codim} = n\end{array}$$

where the Cohomotopy charge $[c] \in \widetilde{\pi^n}\big(\Sigma^d\big)$ *of a submanifold $Q^{d-n} \subset \Sigma^d$ with normal framing $NQ \xrightarrow[\sim]{\mathrm{fr}} N \times \mathbb{R}^n \overset{p}{\twoheadrightarrow} \mathbb{R}^n$ is represented for any choice of tubular neighbourhood $NQ \overset{\iota}{\hookrightarrow} \Sigma$ by the "scanning map"*

$$\begin{array}{ccc} \Sigma^d & \xrightarrow{\quad c \quad} & \mathbb{R}^n_{\cup\{\infty\}} \\ s & \longmapsto & \begin{cases} p\big(\mathrm{fr}(s)\big) & | \quad s \in \iota(NQ) \\ \infty & | \quad \text{otherwise}. \end{cases} \end{array}$$

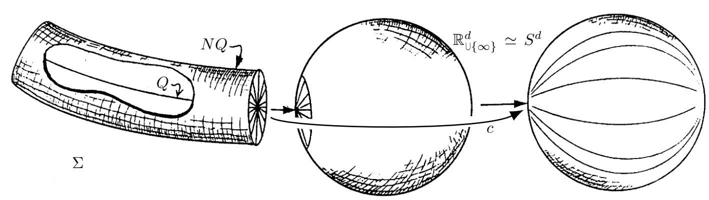


figure adapted from [12, Fig. II-13]

The flux density underlying (sourced by) a given Cohomotopy charge is characterized by the cohomotopical character map (the cohomotopical analog of the Chern-character map on K-cohomology, [33][94]):

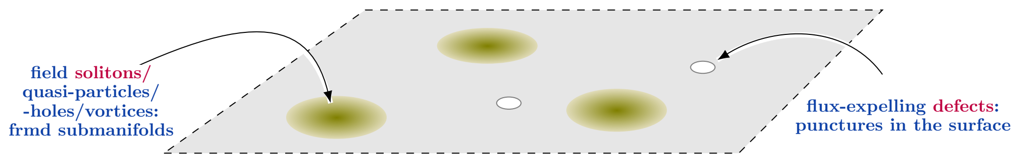
Definition 4.2 (Cohomotopical character map). For $n = d$ the *character map* on cohomotopy

$$\begin{array}{ccc} \widetilde{\pi}^d\big(\Sigma^d\big) & \xrightarrow{\ \mathrm{ch}\ } & H^d_{\mathrm{dR}}\big(\Sigma^d\big)_{\mathrm{cpt}} \\ [c] & \longmapsto & \big[c^* \mathrm{vlm}_n\big] \end{array} \tag{27}$$

takes $[c] \in \widetilde{\pi}^n\big(\Sigma^d_{\cup\{\infty\}}\big)$ — for any representative $c : \Sigma^d_{cpt} \to \mathbb{R}^d_{\cup\{\infty\}}$ which is smooth on $c^{-1}(\mathbb{R}^d)$, such as the scanning maps (27) — to the class in compactly supported de Rham cohomology of the pullback of a d-form $\mathrm{vlm} \in \Omega^n_{\mathrm{dR}}(\mathbb{R}^d)$ compactly supported on a neighborhood of $0 \in \mathbb{R}^d$ and of unit integral.

Remark 4.3 (**Flux density quantized in Cohomotopy**).
(i) In combination, this means that Cohomotopy charge $[c] \in \mathbb{Z} \simeq \widetilde{\pi}^d\big(\Sigma^d_{\cup\{\infty\}}\big) \equiv \pi_0 \mathrm{Map}^*\big(\Sigma^d_{\cup\{\infty\}},\, S^d\big)$ may be understood as sourcing a solitonic flux density $F_d \in \Omega^d_{\mathrm{dR}}\big(\Sigma^d\big)$ (solitonic in that it vanishes at infinity) which is supported with unit weight near $n_+ \in \mathbb{N}$ points in Σ^d (all points outside each other's supporting neighborhoods) and with a negative unit weight near $n_- \in \mathbb{N}$ points (anti-solitons) such that $[c] = n_+ - n_-$.
(ii) For the case $d = 2$ of interest here, this is just the kind of magnetic flux distribution concentrated around solitonic vortex cores as seen in type II superconducting and in fractional quantum Hall semiconducting materials Σ^2, while any punctures in the surface Σ^2 (1) behave as loci where flux is expelled from, as for type I superconducting materials:

Figure F. Via the Pontrjagin theorem, 2-cohomotopical quantization of flux through a surface exhibits N *flux quanta* as a concentration of flux density supported on the tubular neighborhoods of N disjoint points.



However, for the quantum flux observables (8), we need not just the connected components but π_1 of the moduli space of Cohomotopical flux.

By the Pontrjagin theorem, one might naïvely expect that $\mathrm{Map}^*\big((\Sigma^2_{0,0,1})_{\cup\{\infty\}}, S^2\big)$ is the configuration space of *signed* points (hence of $\pm$ unit charged soliton cores) in the plane, topologized to reflect creation/annihilation of oppositely charged pairs — but this is not quite correct as it misses the normal framing on the cobordisms. A correct model [78] is by configurations of *intervals with signed endpoints* (stringy solitons between unit charged "quarks") parallel to one coordinate axis and topologized such as to reflect creation/annihilation of oppositely charged pairs of endpoints. This has the effect that:

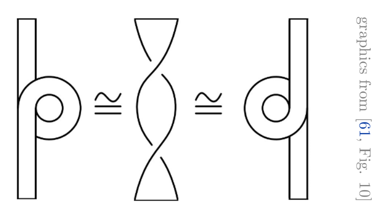
Proposition 4.4 (**Vacuum loops of 2-cohomotopical flux through the plane** [93]).
(i) *Loops of 2-cohomotopical flux moduli on the plane are identified with framed links topologized to reflect link cobordism, whence their homotopy classes is identified with the framed link's total crossing number:*

$$\begin{array}{ccc} \Omega\, \mathrm{Map}^*_0\big(\Sigma^2_{0,0,1},\, S^2\big) & \xrightarrow{\quad [-] \quad} & \pi_1\, \mathrm{Map}^*_0\big(\Sigma^2_{0,0,1},\, S^2\big) \,\simeq\, \mathbb{Z}\,. \\ L & \longmapsto & \#L \\ \text{framed link} & & \text{total crossing number} \end{array}$$

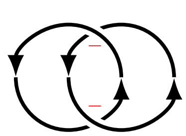
(ii) *Moreover, the pure GNS-states on these observables are labeled by* $|k\rangle$ *and give the expectation values*

$$\langle k | L | k \rangle \;=\; e^{\frac{2\pi \mathrm{i}}{k}\#L}\,.$$

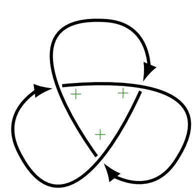
Figure FL. The strands of a framed link may be understood as ribbons that, besides braiding/linking each other may also twist in themselves. If the strands are flattened to always lie in a fixed plane ("blackboard framing") then the twists manifest as self-crossings of strands.



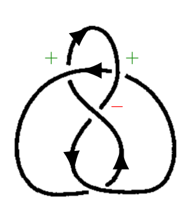
Example 4.5.



$\longmapsto\; -2\,,$



$\longmapsto\; +3\,,$



$\longmapsto\; 0\,.$

Remark 4.6 (**Comparison to anyon vacuum observables of abelian Chern-Simons theory**). In abelian Chern-Simons theory, framed links are exactly the regularized "Wilson loop" observables, and the exponential $e^{2\pi \mathrm{i} \# L/k}$ of the total crossing number (being equal to the linking number plus the framing number) is exactly the value of these observables in the state of level k.

4.2 Flux through closed surfaces

While magnetic flux through *closed* surfaces is not readily realized experimentally (cf. footnote 5), effective field theories of flux on arbitrary surfaces tend to be characterized by their theoretical predictions for closed surfaces (for instance in that the dimension of the Hilbert space of states on Σ^2_g grows with k^g, for k the *level* of the theory). Therefore, a major example of the phenomena in §3 is the following derivation of quantum states of cohomotopically quantized topological flux on closed surfaces, which reproduces the modular data theoretically expected of topologically quantum materials on such surfaces (see Rem. 4.10 below).

Proposition 4.7 (**Monodromy of flux through closed surfaces**). *For the closed oriented surface Σ^2_g (1), $g \in \mathbb{N}$, we have the exact sequence*

$$\begin{array}{ccccccccc} 1 \longrightarrow & \pi_1\Big(\mathrm{Map}^*_0\big(S^2_{\cup\{\infty\}},\, S^2\big)\Big) & \longrightarrow & \pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_g)_{\cup\{\infty\}},\, S^2\big)\Big) & \longrightarrow & \pi_1\Big(\mathrm{Map}^*_0\big(\bigvee_g (S^1_a \vee S^1_b),\, S^2\big)\Big) & \longrightarrow 1 \\ & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & \\ & \mathbb{Z} & \hookrightarrow & \widehat{\mathbb{Z}^{2g}} & \twoheadrightarrow & \mathbb{Z}^{2g}\,, & \end{array}$$

where

$$\widehat{\mathbb{Z}^{2g}} \;:=\; \Big\{ (\vec a, \vec b, n) \,\in\, \mathbb{Z}^g \times \mathbb{Z}^g \times \mathbb{Z}\,, \quad (\vec a, \vec b, n)\cdot(\vec a', \vec b', n') \;:=\; \big(\vec a + \vec a',\, \vec b + \vec b',\, n + n' + \vec a \cdot \vec b' - \vec a' \cdot \vec b\big) \Big\} \tag{28}$$

is the integer Heisenberg group *(cf. [66, p 232]), the extension of $\mathbb{Z}^{2g}$ by $\mathbb{Z}$ (here at level=2).*

Proof. The short exact sequence and identification of the outer groups is due to [48, Thm 1 & p 6]. The identification of the resulting group extension as $2 \in \mathbb{Z} \simeq H^2(T^{2g};\mathbb{Z}) \simeq \mathrm{Ext}\big(\mathbb{Z}^{2g};\mathbb{Z}\big)$ is due to [64, Thm. 1], see also [57, Cor. 7.6]. Observing then that the unit extension $1 \in \mathrm{Ext}(\mathbb{Z}^{2g};\mathbb{Z})$ is given by either of the group cocycles

$$\begin{array}{rcl} (\mathbb{Z}^{2g}) \times (\mathbb{Z}^{2g}) & \longrightarrow & \mathbb{Z} \\ \big((\vec a, \vec b),\, (\vec a', \vec b')\big) & \longmapsto & \vec a \cdot \vec b' \\ \big((\vec a, \vec b),\, (\vec a', \vec b')\big) & \longmapsto & -\vec a' \cdot \vec b \end{array}$$

(which are readily seen to be cohomologous cocycles and evidently indivisible) the claim (28) follows. □

Proposition 4.8 (**Diffeo action over torus is canonical modular action**). *The action (24) of $\mathrm{MCG}(\Sigma^2_1) \simeq \mathrm{SL}_2(\mathbb{Z})$ (15) on $\pi_1 \mathrm{Map}^*_0\big((\Sigma^2_1)_{\cup\{\infty\}},\, S^2\big) \simeq \widehat{\mathbb{Z}^2}$ (28) is the defining action of $\mathrm{Sp}_2(\mathbb{Z}) \simeq \mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$ and trivial on the center, whence the flux monodromy group (24) over the torus is*

$$\mathrm{MCG}(\Sigma^2_1) \;\ltimes\; \pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_1)_{\cup\{\infty\}},\, S^2\big)\Big) \;\simeq\; \mathrm{SL}_2(\mathbb{Z}) \ltimes \widehat{\mathbb{Z}^2}\,. \tag{29}$$

Proof. By the decomposition of Prop. 4.7. □

Proposition 4.9 (**Basic 2-Cohomotopical quantum states over the torus**).
Representations of $\widehat{\mathbb{Z}^2} \rtimes \mathrm{SL}_2(\mathbb{Z})$ (29) — and hence spaces of quantum states (9) for 2-cohomotopical flux over the torus — irreducible already in their restriction to $\widehat{\mathbb{Z}^2}$, are obtained for all even positive integers

$$k \in 2\mathbb{N}_{>0} \qquad \text{with} \qquad \zeta \;:=\; e^{\frac{\pi \mathrm{i}}{k}}\,, \tag{30}$$

by the following formulas:

$$\mathcal{H}_{T^2} \;:=\; \mathrm{Span}\big(|0\rangle,\, |1\rangle,\, \cdots,\, |k-1\rangle\big) \quad \left\{ \begin{array}{lclcccl} \mathrm{SL}_2(\mathbb{Z}) \ltimes \widehat{\mathbb{Z}^2} & \longrightarrow & \mathrm{GL}\big(\mathcal{H}_{T^2}\big) & & & & \\ \Big(\mathrm{I},\, \big(\big[\begin{smallmatrix}1\\0\end{smallmatrix}\big], 0\big)\Big) & \longmapsto & \widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} & : & |n\rangle & \mapsto & \zeta^{2n}\,|n\rangle \\ \Big(\mathrm{I},\, \big(\big[\begin{smallmatrix}0\\1\end{smallmatrix}\big], 0\big)\Big) & \longmapsto & \widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} & : & |n\rangle & \mapsto & \big|(n+1) \,\mathrm{mod}\, k\,\big\rangle \\ \Big(\mathrm{I},\, \big(\big[\begin{smallmatrix}0\\0\end{smallmatrix}\big], 1\big)\Big) & \longmapsto & \widehat{\zeta} & : & |n\rangle & \mapsto & \zeta\,|n\rangle \\ \Big(S,\, \big(\big[\begin{smallmatrix}0\\0\end{smallmatrix}\big], 0\big)\Big) & \longmapsto & \widehat{S} & : & |n\rangle & \mapsto & \frac{1}{\sqrt{|k|}} \sum_{\widehat{n}=0}^{k-1} \zeta^{2n\widehat{n}} \big|\widehat{n}\big\rangle \\ \Big(T,\, \big(\big[\begin{smallmatrix}0\\0\end{smallmatrix}\big], 0\big)\Big) & \longmapsto & \widehat{T} & : & |n\rangle & \mapsto & e^{-\pi \mathrm{i}/12}\,\zeta^{(n^2)}\,|n\rangle\,. \end{array} \right. \tag{31}$$

Here the representations of general group elements follows from applying the group law to the above generators, for instance:

$$\zeta^{-1}\, \widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \;=\; \widehat{W}_{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} \;=\; \zeta^{+1}\, \widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\,. \tag{32}$$

Proof. **(i)** The generating group commutators in $\widehat{\mathbb{Z}^2}$ (28) are evidently respected,

$$\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \circ \widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \;=\; \zeta^2\, \widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \circ \widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\,, \tag{33}$$

and $\mathcal{H}_{T^2}$ is clearly already irreducible as a representation of $\widehat{\mathbb{Z}^2}$.

(ii) To see that we also have a representation of $\mathrm{SL}_2(\mathbb{Z})$ we need to show that the operators $\widehat{S}$ and $\widehat{T}$ respect the relations (17). To that end, it is convenient for the moment to abbreviate the phase factor of $\widehat{T}$ as "c_k":

$$\widehat{T} \;=\; \tfrac{1}{c_k}\, e^{\frac{\pi \mathrm{i}}{k} n^2} \quad \text{with} \quad c_k \;:=\; e^{\pi \mathrm{i}/12}\,. \tag{34}$$

First, we find

$$\begin{aligned} \widehat{S}\widehat{S}\big|n\big\rangle \;&\equiv\; \widehat{S}\Big(\tfrac{1}{\sqrt{|k|}} \textstyle\sum_{\widehat{n}} e^{2\pi \mathrm{i}\frac{\widehat{n}\,n}{k}} \big|\widehat{n}\big\rangle\Big) \\ &\equiv\; \textstyle\sum_{\widehat{\widehat{n}}}\; \underbrace{\tfrac{1}{k}\textstyle\sum_{\widehat{n}} e^{2\pi \mathrm{i}\frac{\widehat{n}\,(n+\widehat{\widehat{n}})}{k}}}_{\delta_0\left(n+\widehat{\widehat{n}} \,\mathrm{mod}\, k\right)} \big|\widehat{\widehat{n}}\big\rangle \\ &=\; \big|-n\big\rangle \qquad \text{by (55)}, \end{aligned}$$

which immediately implies that $\widehat{S}^4 = \mathrm{id}$ and that, with

$$\widehat{T}\,\widehat{S}\big|n\big\rangle \;=\; \tfrac{1}{k^{1/2} c_k} \textstyle\sum_{\widehat{n}} e^{\frac{\pi \mathrm{i}}{k}(\widehat{n}^2 + 2\widehat{n}\,n)} \big|\widehat{n}\big\rangle\,,$$

also $\widehat{S}^2(\widehat{T}\widehat{S}) = (\widehat{T}\widehat{S})\widehat{S}^2$. Hence the only remaining relation to check is $(\widehat{T}\widehat{S})^3 = \mathrm{id}$ or equivalently that

$$\widehat{T}^{-1} \circ \widehat{S}^{-1} \circ \widehat{T}^{-1} \;=\; \widehat{S} \circ \widehat{T} \circ \widehat{S}\,.$$

Unwinding the definitions gives

$$\begin{aligned} &\widehat{T}^{-1}\widehat{S}^{-1}\widehat{T}^{-1}|n\rangle \;=\; \widehat{T}^{-1}\widehat{S}^{-1} e^{-\frac{\pi \mathrm{i}}{k} n^2}|n\rangle \\ &= \widehat{T}^{-1} \tfrac{1}{\sqrt{k}} \textstyle\sum_{\widehat{n}} e^{\frac{\pi \mathrm{i}}{k}(-n^2 - 2\widehat{n}n)}|\widehat{n}\rangle \\ &= \tfrac{1}{\sqrt{k}} \textstyle\sum_{\widehat{n}} e^{\frac{\pi \mathrm{i}}{k}(-n^2 - 2\widehat{n}n - \widehat{n}^2)}|\widehat{n}\rangle \\ &= \tfrac{1}{\sqrt{k}} \textstyle\sum_{\widehat{n}} e^{-\frac{\pi \mathrm{i}}{k}(\widehat{n}+n)^2}|\widehat{n}\rangle \end{aligned} \qquad \text{and} \qquad \begin{aligned} &\widehat{S}\widehat{T}\widehat{S}\big|n\big\rangle \;=\; \widehat{S}\widehat{T} \tfrac{1}{\sqrt{|k|}} \textstyle\sum_{\widehat{n}} e^{2\pi \mathrm{i}\frac{\widehat{n}\,n}{k}} \big|\widehat{n}\big\rangle \\ &= \widehat{S} \tfrac{1}{\sqrt{k}} \textstyle\sum_{\widehat{n}} e^{\frac{\pi \mathrm{i}}{k}(2\widehat{n}\,n + \widehat{n}^2)} \big|\widehat{n}\big\rangle \\ &= \tfrac{1}{k} \textstyle\sum_{\widehat{n},\,\widehat{\widehat{n}}} e^{\frac{\pi \mathrm{i}}{k}(2\widehat{n}\,n + \widehat{n}^2 + 2\widehat{\widehat{n}}\widehat{n})} \big|\widehat{\widehat{n}}\big\rangle \\ &= \tfrac{1}{\sqrt{k}} \textstyle\sum_{\widehat{\widehat{n}}}\, \underbrace{\tfrac{1}{\sqrt{k}} \textstyle\sum_{\widehat{n}} e^{\frac{\pi \mathrm{i}}{k}(\widehat{n} + (n+\widehat{\widehat{n}}))^2}}_{c_k^3}\, e^{-\frac{\pi \mathrm{i}}{k}(n+\widehat{\widehat{n}})^2} \big|\widehat{\widehat{n}}\big\rangle\,. \end{aligned} \tag{35}$$

Here the term over the brace is a constant in n and $\widehat{\widehat{n}}$, by the assumption that k is even [9], whence the relation is satisfied if the normalization factor c_k in (34) is chosen as claimed, because the quadratic Gauss sum here evaluates to

$$c_k \;=\; \Big(\tfrac{1}{\sqrt{k}} \textstyle\sum_{n=0}^{k-1} e^{\frac{\pi \mathrm{i}}{k} n^2}\Big)^{1/3} \underset{(59)}{=} \big(e^{\pi \mathrm{i}/4}\big)^{1/3} \;=\; e^{\pi \mathrm{i}/12}\,. \tag{36}$$

(iii) Finally, we need to see that the semidirect product structure is respected, hence that

$$\widehat{W}_{M\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}\, \widehat{M}|n\rangle \;=\; \widehat{M}\, \widehat{W}_{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}|n\rangle \qquad \forall \begin{cases} M & \in\; \mathrm{SL}_2(\mathbb{Z}) \\ (a,b) & \in\; \mathbb{Z}^2 \\ |n\rangle & \in\; |\mathcal{H}_{T^2}\,. \end{cases}$$

It is sufficient to check this on the generators, where explicit computation yields, indeed:

$$\begin{aligned} \widehat{W}_{S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\, \widehat{S}\big|[n]\big\rangle \;&\equiv\; \widehat{W}^{-1}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \Big(\tfrac{1}{\sqrt{|k|}} \textstyle\sum_{\widehat{n}} e^{2\pi \mathrm{i}\frac{\widehat{n}\,n}{k}} |\widehat{n}\rangle\Big) \\ &=\; \tfrac{1}{\sqrt{|k|}} \textstyle\sum_{\widehat{n}} e^{2\pi \mathrm{i}\frac{(\widehat{n}+1)\,n}{k}} \big|\widehat{n}\big\rangle \\ &=\; e^{2\pi \mathrm{i}\frac{n}{k}}\, \widehat{S}\big|n\big\rangle \\ &=\; \widehat{S}\,\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}|n\rangle\,, \end{aligned} \qquad \begin{aligned} \widehat{W}_{S\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}\, \widehat{S}|n\rangle \;&\equiv\; \widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \Big(\tfrac{1}{\sqrt{|k|}} \textstyle\sum_{\widehat{n}} e^{2\pi \mathrm{i}\frac{\widehat{n}\,n}{k}} |\widehat{n}\rangle\Big) \\ &=\; \tfrac{1}{\sqrt{|k|}} \textstyle\sum_{\widehat{n}} e^{2\pi \mathrm{i}\frac{\widehat{n}}{k}} e^{2\pi \mathrm{i}\frac{\widehat{n}\,n}{k}} |\widehat{n}\rangle \\ &=\; \tfrac{1}{\sqrt{|k|}} \textstyle\sum_{\widehat{n}} e^{2\pi \mathrm{i}\frac{\widehat{n}\,(n+1)}{k}} |\widehat{n}\rangle \\ &=\; \widehat{S}\,\widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}|n\rangle\,, \end{aligned}$$

and

[9] Since the summands in $\sum_{n=0}^{k-1} e^{\frac{\pi \mathrm{i}}{k} n^2}$ are k-periodic for even k, $e^{\frac{\pi \mathrm{i}}{k}(n+k)^2} = e^{\frac{\pi \mathrm{i}}{k} n^2} e^{\pi \mathrm{i}(2n+k)} \underset{k \text{ even}}{=} e^{\frac{\pi \mathrm{i}}{k} n^2}$, the sum is invariant under replacing $n \mapsto n + a$ for $a \in \mathbb{N}$.

$$\widehat{W}_{T\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\,\widehat{T}|n\rangle \;\equiv\; \widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\,\tfrac{1}{c_k}e^{\pi\mathrm{i}\frac{n^2}{k}}|n\rangle \;=\; \tfrac{1}{c_k}e^{2\pi\mathrm{i}\frac{n}{k}}\,e^{\mathrm{i}\pi\frac{n^2}{k}}|n\rangle \;=\; \widehat{T}\,\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}|n\rangle\,,$$

$$\begin{aligned}\widehat{W}_{T\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}\,\widehat{T}|n\rangle &\equiv \tfrac{1}{c_k}\widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}\,\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\,e^{\pi\mathrm{i}\frac{1}{k}}\,e^{\pi\mathrm{i}\frac{n^2}{k}}|n\rangle \\ &= \tfrac{1}{c_k}e^{\pi\mathrm{i}\frac{n^2+2n+1}{k}}|n+1\rangle \\ &= \tfrac{1}{c_k}e^{\pi\mathrm{i}\frac{(n+1)^2}{k}}|n+1\rangle \\ &= \widehat{T}\,\widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}|n\rangle\,,\end{aligned}$$

where in the first step of the last case we used (32). ☐

Remark 4.10 (**Comparison to modular data of abelian Chern-Simons theory on the torus).** The content of Prop. 4.9 captures a good deal of the *modular data* (cf. [37]) expected for FQH systems on the torus:
(i) The algebra (33) of the $\widehat{W}_{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}$ is just that expected [110, (5.28)] of quantum observables for anyonic topological order on the torus as described [10, (17)][81, (32)] by abelian Chern-Simons theory at *level* k and equivalently by U(1)-WZW conformal field theory [112, (4.3-4)].
(ii) Similarly, the operators $\widehat{S}$ and $\widehat{T}$ according to (31) implement the known modular group representation on quantum states of abelian Chern-Simons theory [68, p 65] (following [45][36, (5,7)]) and equivalently of conformal characters of the U(1) 2dCFT [37, Ex. 1]. [10]
(iii) The fact of Prop. 4.9 that, jointly, these operators constitute a representation of the semidirect product of the modular group with the integer Heisenberg group may be regarded as implicit in the literature but does not appear to be easy to cite.
(iv) In any case, it is remarkable that here this structure arises without unpacking any of the usual constructions of CS/WZW theories, but instead as the description of topological flux quantized in 2-Cohomotopy.

While this is already remarkable, it is not sufficient for the description of FQH systems, where (what we may identify as) the anyonic braiding phase ζ (30) may be a more general root of unity and in practice generically is different from (30).

However, while our effective quantum theory of topological flux quantized in 2-Cohomotopy turns out to be so similar to Chern-Simons theory (as per Rem. 4.10 above and the following Rem. 4.22) it actually admits these more general braiding factors:

Lemma 4.11 (**More general representations).** *The representation* (31) *exists more generally for*

$$(k,\,q)\in\mathbb{N}_{>0}\times\mathbb{Z}\quad s.t.\quad \begin{cases} kq\in 2\mathbb{Z},\\ \sum_{n=0}^{k-1} e^{\pi\mathrm{i}\frac{q}{k}n^2}\;\neq\;0\end{cases}\qquad \text{with}\quad \zeta\;:=\;e^{\pi\mathrm{i}\frac{q}{k}}\,. \tag{37}$$

Proof. Inspection shows readily that the proof of Prop. 4.9 goes through verbatim with all factors of $e^{\pi\mathrm{i}/k}$ now generalized to ζ (37) — the only step that needs attention is that from (35) to (36): But for the term over the brace in (35) to be constant in n and $\widehat{\widehat{n}}$ it is clearly sufficient that k *or* q are even, hence that their product kq is even, in which case the normalization factor c_k in (36) can be found unless that term is zero. These are exactly the two conditions assumed in (37). ☐

Lemma 4.12 (**Reducible representation).** *If* $k\in 2\mathbb{N}_{>0}$ *and* $k=\mathrm{ord}(\zeta)$ *then the restriction of the representation* (31) *to* $\widehat{\mathbb{Z}^2}$ *is reducible.*

Proof. Due to the assumption that $k=\mathrm{ord}(\zeta)$, the representation has an alternative linear basis of eigenstates of the original "shift operator":

$$\widetilde{|j\rangle}\;:=\;\Big(\textstyle\sum_{n=0}^{k-1}\zeta^{jn}|n\rangle\Big)\,,\qquad \widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}\widetilde{|j\rangle}\;=\;\zeta^{-j}\widetilde{|j\rangle}\,,\qquad \text{for } j\in\{0,1,\cdots,k-1\}\,.$$

But on this new basis the original multiplication operator acts non-transitively, skipping every second element:

$$\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\widetilde{|j\rangle}\;\equiv\;\zeta^{2n}\widetilde{|j\rangle}\;=\;\widetilde{|[j+2]\rangle}\,,$$

whence we have a $\widehat{\mathbb{Z}^2}$-reduction

$$\mathcal{H}_{T^2}\;\simeq\;\mathbb{C}\big\langle\widetilde{|j\rangle}\big\rangle_{j\in\{0,2,\cdots,k-2\}}\;\oplus\;\mathbb{C}\big\langle\widetilde{|j\rangle}\big\rangle_{j\in\{1,3,\cdots,k-1\}}\,.$$

[10] The exponentiated "central charge" $c_k=e^{2\pi\mathrm{i}/24}$ appearing in (34) and (36) seems to be missed in the earlier literature (cf. [68, p 65][45][36, (5,7)]) but is now well-known to appear, compare [37, (3.1b)][102, (26)].

□

Proposition 4.13 (**General 2-Cohomotopical quantum states over the torus).** *The representation* (31) *exists and is irreducible already when restricted to* $\widehat{\mathbb{Z}^2}$ *iff*

$$\begin{array}{rl} & \big(k \in 2\mathbb{N}_{>0} \quad \text{and} \quad q \in 2\mathbb{N}+1\big) \\ \text{or} & \big(k \in 2\mathbb{N}+1 \quad \text{and} \quad q \in 2\mathbb{N}_{>0}\big) \end{array} \quad \text{and} \quad \mathrm{qcd}(q,k) = 1 \qquad \textit{with} \qquad \zeta := e^{\pi \mathrm{i} \frac{q}{k}} . \tag{38}$$

Proof. To see that these representations exist as claimed, by Lem. 4.11 it just remains to check that the Gauss sum does not vanish: Indeed, for k odd and q even we have

$$\sum_{n=0}^{k-1} e^{\pi \mathrm{i} \frac{q}{k} n^2} = \sum_{n=0}^{k-1} e^{\frac{2\pi \mathrm{i}}{k}(q/2)n^2} \underset{\text{by (57)}}{=} \underbrace{\big(q/2\big|k\big)}_{\neq 0 \text{ by (58)}} \underbrace{\textstyle\sum_{n=0}^{k-1} e^{\frac{2\pi \mathrm{i}}{k} n^2}}_{\neq 0 \text{ by (56)}},$$

while for k even and q odd we have

$$\sum_{n=0}^{k-1} e^{\pi \mathrm{i} \frac{q}{k} b^2} \underset{\text{by (61)}}{=} e^{\pm \pi \mathrm{i}} \sqrt{k} \underbrace{\big(k/2 \,\big|\, q\big)}_{\neq 0 \text{ by (58)}} .$$

The only other choice of k, q compatible with Lem. 4.11 is that k and q are both even. Even when this exists and $\gcd\big(q/2, k/2\big) = 1$, it restricts to an even dimensional $\widehat{\mathbb{Z}^2}$-rep of dimension $k = 2(k/2) = \mathrm{ord}(\zeta)$, which is still reducible, by Lem. 4.12.

Moreover, its reduction does not consist of new irreps: For the Gauss sum not to vanish we need $k \in 4\mathbb{N}$ and hence $q/2$ odd, whence both summands are isomorphic to the abve irrep for $(q/2)$ odd and $(k/2)$ even. □

Remark 4.14 (**Comparison to abelian "spin" Chern-Simons theory).** Our natural number k identifies with *twice* the "level" of Chern-Simons theory: This is apparent from [21, (1.2-3)] which (we are in their special case $N = 1$) says that what is denoted $M = K$ there is *twice* the actual level (denoted "k" there). But since the ground state degeneracy is [21, p 26]

$$\dim\big(\mathcal{H}_{\Sigma^2_g}\big) \;=\; K^g \,, \quad M \in 2\mathbb{N}_{>0} \,,$$

comparison with [68, p 40] shows that the "level" k there, which is our k, is also twice the actual level, $k = K$.

But then with [21] one may allow generalization to odd k in "spin Chern-Simons theory" (corresponding to half-integral "actual level"), which evidently corresponds to our second sequence of irreps in (38).

We conclude that we are seeing here not just the modular data of plain abelian CS theory, but of its "spin"-refinement.

This also means that the anyon braid phases we get are "spinorial" in that they are square roots of the naively expected braid phases of [110, (5.28)], which is also apparent from [112, (4.3)] (where our k is denoted N, see above (4.1) there).

4.3 Flux through n-punctured surface

Here we derive here the observables on 2-cohomotopically quantized topological flux over *n-punctured surfaces*, which in practice will mean: Surfaces of conducting material where magnetic flux is *expelled* from (the vicinity of) n defect points (cf. Rem. 4.3). It is clear (cf. Prop. 4.15) that covariantization of these observables reveals an action of the surface's n-braid group, but we find that the contribution to the observables from the flux monodromy (cf. Prop. 3.9) enhances this to the *framed* (or *ribbon*) braid group (41) as expected in generality for Chern-Simons theories (Rem. 4.22). Or rather, we find that what appears is its subgroup of frmamed braids of vanishing total framing.

Lemma 4.15 (**Homotopy type of compactified n-punctured surface).** *For $n \in \mathbb{N}_{\geq 1}$, the one-point compactification of the n-puncturing of a closed surface $\Sigma^2_{g,b}$ (1) is homotopy equivalent to the wedge sum (46) of that surface with $(n-1)$ circles:*

$$\int\Big(\big(\Sigma^2_{g,b,n}\big)_{\cup\{\infty\}}\Big) \;\simeq\; \int\Big(\Sigma^2_{g,b} \vee \textstyle\bigvee_{n-1} S^1\Big) . \tag{39}$$

Proof. For $n = 1$ the statement is immediate.

For $n = 2$ consider the topological space X obtained by attaching to $\Sigma^2_{g,b}$ an interval with endpoints glued to two distinct points $s_1, s_2 \in \Sigma^2_{g,b}$ (the would-be positions of the punctures), hence consider this pushout of topological spaces:

$$\begin{array}{ccc} S^0 & \xrightarrow{(s_1,s_2)} & \Sigma^2_{g,b} \\ \big\downarrow & & \big\downarrow \\ D^1 & \xrightarrow[\iota_{\mathrm{ext}}]{\text{(po)}} & X\,. \end{array}$$

Moreover, consider another arc *inside* $\Sigma^2_{g,b}$ connecting these two points

$$D^1 \xhookrightarrow{\iota_{\mathrm{int}}} \Sigma^2_{g,b} \hookrightarrow X\,.$$

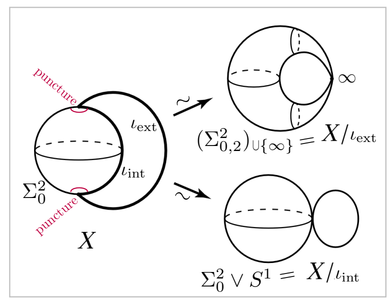
Both of these arcs are evidently contractible sub-complexes of X, and so the quotient projections obtained by identifying either arc with a single point are weak homotopy equivalences (cf. [50, p 11]):

$$\begin{array}{ccc} & \int X & \\ \swarrow{\sim} & & \searrow{\sim} \\ \int\big(X/\iota_{\mathrm{ext}}(D^1)\big) & \overset{\sim}{\dashrightarrow} & \int\big(X/\iota_{\mathrm{int}}(D^1)\big)\,. \end{array}$$

However, as indicated, the "external" quotient on the left is evidently homeomorphic to the desired one-point compactification, while the "internal" quotient on the right is evidently homeomorphic to the claimed wedge sum. This proves the claim for $n = 2$.

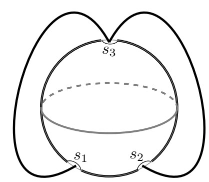
The graphics on the right illustrates the situation for the case $g, b = 0$.

The general statement, including the case $n > 2$, follows analogously by attaching further arcs in this fashion, cf. Fig. A.



□

Figure A. There are several ways to attach arcs for $n > 2$ punctures in the above proof of Lem. 4.15, all equivalent in the result. But for the analysis that follows below it is useful to single out one puncture s_n and take the $n-1$ arcs to connect this one puncture to each of the $n-1$ remaining ones. The case $g, b = 0$ and $n = 3$ is illustrated on the right.



Proposition 4.16 (Monodromy of flux through punctured surface). *For $g, b, \in \mathbb{N}$ and $n \in \mathbb{N}_{>0}$, we have an isomorphism*

$$\pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, S^2\big)\Big) \;\simeq\; \pi_1\Big(\mathrm{Map}^*_0\big(\Sigma^2_{g,b},\, S^2\big)\Big) \;\times\; \mathbb{Z}^{n-1}\,. \tag{40}$$

Proof. We may compute as follows:

$$\begin{aligned} \pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, S^2\big)\Big) &\simeq \pi_1\Big(\mathrm{Map}^*_0\big(\Sigma^2_{g,b} \vee \textstyle\bigvee_{n-1} S^1,\ S^2\big)\Big) && \text{by (39)} \\ &= \pi_1\Big(\mathrm{Map}^*_0\big(\Sigma^2_{g,b} S^2\big) \times \textstyle\prod_{n-1} \mathrm{Map}^*\big(S^1,\, S^2\big)\Big) && \text{by (47)} \\ &= \pi_1\Big(\mathrm{Map}^*_0\big(\Sigma^2_{g,b} S^2\big)\Big) \times \textstyle\prod_{n-1} \pi_1\Big(\mathrm{Map}^*\big(S^1,\, S^2\big)\Big) \\ &= \pi_1\Big(\mathrm{Map}^*_0\big(\Sigma^2_{g,b} S^2\big)\Big) \times \textstyle\prod_{n-1} \pi_2(S^2) && \text{by (44),} \end{aligned}$$

whence the claim follows by $\pi_2(S^2) \simeq \mathbb{Z}$. □

Definition 4.17 (Standard representation of symmetric group). For $n \in \mathbb{N}_{>0}$ the "*standard*" $\mathbb{C}$-linear representation of the symmetric group Sym_n is the complex irrep classified by the partition $(n-1, 1)$, hence the complement of the trivial 1d representation inside the defining permutation representation.

More concretely, with respect to the canonical linear basis

$$\mathbb{C}^n \;\simeq\; \mathbb{C}\langle v_1, v_2, \cdots, v_n\rangle$$

- the defining permutation representation is given, for $\sigma \in \mathrm{Sym}_n$, by $\sigma(v_i) \;:=\; v_{\sigma(i)}$,

- the trivial 1d irrep inside this is
$$\mathbf{1} \simeq \mathbb{C}\langle v_1 + v_2 + \cdots + v_n \rangle \hookrightarrow \mathbb{C}^n ,$$
- and the standard representation is
$$\mathbf{n{-}1} \simeq \mathbb{C}\langle e_1 := v_1 - v_n,\, e_2 := v_2 - v_n,\, \cdots ,\, e_{n-1} := v_{n-1} - v_n \rangle \hookrightarrow \mathbb{C}^n .$$

Hence in the standard representation a transposition among the first $n-1$ basis elements $v_{i<n}$ acts as the same transposition of the corresponding $e_{i<n}$, while the transposition of any $v_{i<n}$ with v_n acts in the standard representation as sign reversal on e_i.

This is clearly the extension of scalars from a $\mathbb{Z}$-linear representation on $\mathbb{Z}^{n-1}$, which we shall hence refer to as the *standard $\mathbb{Z}$-linear representation* of Sym_n.

Proposition 4.18 (**Braid group action on flux monodromy over punctured surface).** *For $n \geq 1$, the action* (24) *of the Artin generators of the braid group* $\mathrm{Br}_n(\Sigma^2_{g,b,n}) \hookrightarrow \mathrm{MCG}(\Sigma^2_{g,b,n})$ (21) *on the flux monodromy* (40) *over an n-punctured surface* (1) *is via the* standard representation (Def. 4.17) *of* Sym_n *on the $\mathbb{Z}^{n-1}$-factor*

Proof. Under the identification of the $\mathbb{Z}^{n-1}$ factor from the proof of Lem. 3.7 and choosing the arc-attachments used there as in Figure A, we see that the ith $\mathbb{Z}$-factor corresponds to the arc from s_n to s_i. Therefore the Artin generators (11) $b_{i<n-1}$ act by transposing the ith with the $i+1$st arc and hence are represented by transposing the corresponding $\mathbb{Z}$-factors, while the Artin generator b_{n-1} reverses the $n-1$st arc and hence is represented by inversion on the corresponding $\mathbb{Z}$-factor. This is precisely the action in the standard representation of Def. 4.17. □

Definition 4.19 (**Framed/ribbon braid group** [70][59], cf. [61, §3.2]**).** The *framed braid group* of *ribbon braid group* of a surface is the wreath product of the ordinary surface braid group (10) with the integers, hence its semidirect product with $\mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z}$ via its defining permutation action on the n factors:
$$\mathrm{FBr}_n(\Sigma^2) := \mathbb{Z} \wr \mathrm{Br}_n(\Sigma^2) \simeq \mathbb{Z}^n \rtimes_{\mathrm{def}} \mathrm{Br}_n(\Sigma^2) . \tag{41}$$

A framed braid in (41) may be understood as a braid of ribbons which, besides braiding with each other, may each twist an integer number of times in themselves, as in Fig. FL: The closure of a framed braid is a framed link.

Similarly, via the integral *standard representation* of Sym_n (Def. 4.17) we may also form the variant $\mathbb{Z}^{n-1} \rtimes_{\mathrm{st}} \mathrm{Br}_n(\Sigma^2)$ of the framed braid group. This is the subgroup on the elements whose *total framing* number vanishes:

Lemma 4.20 (**Framed braids of zero total framing among all framed braids).** *We have an injective group homomorphism*
$$\underset{\text{vanishing total framing}}{\overset{\text{framed braids with}}{}} \quad \underset{\text{standard rep}}{\mathbb{Z}^{n-1} \rtimes_{\mathrm{st}} \mathrm{Br}_n(\Sigma^2)} \longrightarrow \underset{\text{defining rep}}{\mathbb{Z}^{n} \rtimes_{\mathrm{def}} \mathrm{Br}_n(\Sigma^2)} \equiv \mathrm{FBr}_n(\Sigma^2) \quad \underset{\text{framed braids}}{\overset{\text{group of all}}{}}$$
$$\big(e_i,\, b_j\big) \longmapsto \big(v_i - v_n,\, b_j\big) \tag{42}$$

Proof. As in Def. 4.17. □

Proposition 4.21 (**2-Cohomotopical flux monodromy on n-punctured disk).** *For $n \geq 1$, the covariant 2-Cohomotopical fux monodromy on the n-punctured disk $\Sigma^2_{0,1,n}$* (1) *is the subgroup* (42) *of framed braids of vanishing total framing:*
$$\pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_{0,1,n})_{\cup\{\infty\}}\big) /\!\!/ \mathrm{Diff}^{+,\partial}\big(\Sigma^2_{0,1,n}\big)\Big) \simeq \mathbb{Z}^{n-1} \rtimes_{\mathrm{st}} \mathrm{Br}_n \longrightarrow \mathrm{FBr}_n . \tag{43}$$

Proof. We may compute as follows:
$$\begin{aligned}
\pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_{0,1,n})_{\cup\{\infty\}}, S^2\big) /\!\!/ \mathrm{Diff}^{+,\partial}\big(\Sigma^2_{0,1,n}, S^2\big)\Big) &\simeq \pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_{0,1,n})_{\cup\{\infty\}}, S^2\big)\Big) \rtimes \mathrm{MCG}(\Sigma^2_{0,1,n}) && \text{by (24)} \\
&\simeq \Big(\pi_1\Big(\mathrm{Map}^*_0\big(\Sigma^2_{0,1}, S^2\big)\Big) \times \mathbb{Z}^{n-1}\Big) \rtimes \mathrm{MCG}(\Sigma^2_{0,1}) && \text{by (40)} \\
&\simeq \mathbb{Z}^{n-1} \rtimes \mathrm{MCG}(\Sigma^2_{0,1}) && \text{since } \textstyle\int \Sigma^2_{0,1} \simeq * \\
&\simeq \mathbb{Z}^{n-1} \rtimes \mathrm{Br}_n && \text{by (23)} .
\end{aligned}$$
□

Remark 4.22 (**Comparison to Chern-Simons theory on n-punctured surfaces).**
(i) The framed braid group $\mathrm{FBr}_n(\Sigma^2_{0,1})$ (41) of a closed surface $\Sigma^2_{g,b}$ is the expected braid group acting on the quantum states of Chern-Simons theory on $\Sigma^2_{g,b,n}$ as formalized by the Reshetikhin-Turaev construction, cf. [19,

§3.1][109, §3.2.1][84, p 37][85, p 8] — but the framing is expected to act nontrivially only in the generality of the rarely discussed "irregular conformal blocks" [53].

(ii) The intermediate case of framed braids of vanishing total framing that we see appear here (43) from 2-Cohomotopical flux quantization seems not to have been considered elsewhere.

4.4 Flux through n-punctured annulus

The samples Σ^2 attainable in realistic experiments are typically not closed but have a boundary, and in fact the experimental access to the topological quantum properties of the sample are typically only by inspection of effects on its boundary. Therefore we now turn to identifying the topological flux quantum states on surfaces with boundary, and pay special attention to the question of which aspects of the topological quantum states should be realistically measurable via boundary effects.

(...)

A Background

Here we briefly recall and cite some constructions and facts that are referred to in the main text.

A.1 Some algebraic topology

For general background on the algebraic topology and homotopy theory we use see for instance [107] (...).

Topological spaces. We write

- Top for the category of *compactly generated* topological spaces
  with hom-sets of maps denoted $\mathrm{Hom}(-,-)$ and mapping spaces denoted $\mathrm{Map}(-,-)$
- Top^* for pointed such spaces with pointed maps between them
  with hom-sets denoted $\mathrm{Hom}^*(-,-)$ and mapping spaces denoted $\mathrm{Map}^*(-,-)$

The mapping spaces are characterized by natural bijections

$$\begin{aligned} \mathrm{Hom}\big(X \times Y,\, Z\big) &\simeq \mathrm{Hom}\big(X,\, \mathrm{Map}(Y,Z)\big) \\ \mathrm{Hom}^*\big(X \wedge Y,\, Z\big) &\simeq \mathrm{Hom}^*\big(X,\, \mathrm{Map}^*(Y,Z)\big)\,, \end{aligned}$$

where the *smash product* of pointed spaces is

$$X \wedge Y \;:=\; \frac{X \times Y}{\{\infty_Z\} \times Y \;\cup\; X \times \{\infty_Y\}}\,,$$

for instance

$$S^1 \wedge S^n \;\simeq\; S^{n+1}\,, \quad \text{so that} \quad \pi_n \mathrm{Map}^*(S^m, X) \;\simeq\; \pi_0 \mathrm{Map}^*(S^{n+m}, X)\,. \tag{44}$$

Here for $X \in \mathrm{Top}^*$ we generically denote its basepoint by $\infty_X \in X$, also speaking of the "point at infinity", and for $X \in \mathrm{LCHaus}$ we write $X_{\cup\{\infty\}} \in \mathrm{Top}^*$ for its *one-point compactification* (cf. [12, p 199]), thinking of it as *adjoining a point at infinity.*

If a space is already compact, then the adjoined point-at-infinity is disjoint and pointed maps out of the space are identified with plain maps:

$$\left.\begin{array}{r} X \in \mathrm{Top} \\ X \text{ compact} \end{array}\right\} \qquad \text{yields} \qquad X_{\cup\{\infty\}} \simeq X \sqcup \{\infty\} \qquad \text{and} \qquad \mathrm{Maps}^*\big(X_{\cup\{\infty\}},\, Y\big) \;\simeq\; \mathrm{Maps}\big(X,\, Y\big).$$

For $Y \in \mathrm{Top}^*$ we denote the connected component of the map constant on ∞_Y by

$$\mathrm{Map}_0(-,Y) \;\subset\; \mathrm{Map}(-,Y) \qquad \text{and} \qquad \mathrm{Map}^*_0(-,Y) \;\subset\; \mathrm{Map}^*(-,Y)\,.$$

Proposition A.1 (One-point compactification functorial on proper maps [55, p 70][15, Prop. 1.6]**).** *The operation of one-point compactification extends to a functor on the category of locally compact Hausdorff spaces with* proper maps *between them*

$$(-)_{\cup\{\infty\}} \;:\; \mathrm{LCHaus}_{\mathrm{PrpMaps}} \longrightarrow \mathrm{CptHaus}^*\,. \tag{45}$$

Since homeomorphisms are proper, this implies in particular functoriality on homeomorphisms.

The coproduct of $X, Y \in \mathrm{Top}^*$ is the *wedge sum*

$$X \vee Y \;:=\; \frac{X \coprod Y}{\{\infty_X,\, \infty_Y\}}\,, \tag{46}$$

which in particular means that we have a natural bijection

$$\mathrm{Hom}^*\big(X \vee Y,\, Z\big) \;\simeq\; \mathrm{Hom}^*\big(X,\, Z\big) \times \mathrm{Hom}^*\big(Y,\, Z\big) \tag{47}$$

Homotopy. We write Grpd_∞ for the ∞-category of homotopy types and

$$\textstyle\int \;:\; \mathrm{Top} \longrightarrow \mathrm{Grpd}_\infty$$

for the underlying functor. This means that a weak homotopy equivalence between topological spaces is equivalently an equivalence under $\int$:

$$X, Y \in \mathrm{Top} \qquad \vdash \qquad X \xrightarrow[\text{wk hmpt equv}]{f} Y \qquad \Leftrightarrow \qquad \textstyle\int X \xrightarrow[\sim]{\int f} \int Y$$

Given $f : Y \to Z$ a map of pointed topological spaces, with homotopy fiber X

$$X \xrightarrow{\mathrm{hofib}(f)} Y \xrightarrow{\;f\;} Z$$

the resulting long homotopy fiber sequences

$$\begin{array}{ccccc} \Omega X & \longrightarrow & \Omega Y & \longrightarrow & \Omega Z \\ \hookrightarrow X & \longrightarrow & Y & \longrightarrow & X \end{array} \tag{48}$$

pass under $\pi_0(-)$ to long exact sequences of homotopy groups

$$\begin{array}{ccccc} \pi_{n+1}(X) & \longrightarrow & \pi_{n+1}(Y) & \longrightarrow & \pi_{n+1}(Z) \\ \hookrightarrow \pi_n(X) & \longrightarrow & \pi_n(Y) & \longrightarrow & \pi_n(Z) \end{array} \tag{49}$$

Homotopy quotients and Borel construction.

Definition A.2. For $G \circlearrowright X$ a Hausdorff topological group acting continuously on a topological space X, we write

$$X \xrightarrow{\;q\;} X /\!\!/ G \;:=\; X \times_G EG \tag{50}$$

for its Borel construction, and call its homotopy type the *homotopy quotient* of the action.

In the special case when $X = *$ we have

$$* /\!\!/ G \;\simeq\; BG$$

whose homotopy groups are those of G shifted up in degree:

$$\pi_{n+1}(BG) \;\simeq\; \pi_n(G)\,. \tag{51}$$

This makes a long homotopy fiber sequence (48)

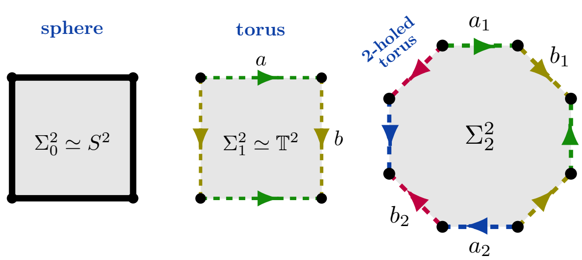
$$G \xrightarrow{\;g \mapsto g(x_0)\;} X \xrightarrow{\;q\;} X /\!\!/ G \longrightarrow BG\,. \tag{52}$$

Hence if G preserves the connected components of X (such as if X only has one connected component), then the long exact sequence of homotopy groups (49) implies that

$$\pi_0(G) \circlearrowright \pi_0(X) \text{ is trivial} \qquad \Rightarrow \qquad \pi_0\big(X\big) \xrightarrow[\sim]{\;\pi_0(q)\;} \pi_0\big(X /\!\!/ G\big). \tag{53}$$

Surfaces. The homotopy type of the closed oriented surface Σ^2_g (1)

$$S^1 \xrightarrow{\;\prod_i [a_i, b_i]\;} \bigvee_g \big(S^1_a \vee S^1_b\big) \longrightarrow \Sigma^2_g \xrightarrow{\;\delta\;} S^2 \tag{54}$$



A.2 Some number theory

Here we briefly compile some facts about Gauss sums used in §4.2, see [6] for more pointers to the literature (see also [22] but beware of typos in (1.1) there).

First, it may be worth recalling the simple cousin of the Gauss sums:

Proposition A.3 (Discrete Fourier transform of Kronecker delta). *For $k \in \mathbb{N}_{>0}$ and $q \in \mathbb{Z}$ we have*

$$\sum_{n=0}^{k-1} e^{\frac{2\pi \mathrm{i}}{k} q n} \;=\; \begin{cases} k & \text{if} \quad q = 0 \\ 0 & \text{if} \quad n \neq 0\,. \end{cases} \tag{55}$$

Proof. The statement for $q = 0$ is immediate. For $q \neq 0$ observe that

$$\Big(1 - e^{\frac{2\pi \mathrm{i}}{k} q}\Big) \sum_{n=0}^{k-1} e^{\frac{2\pi \mathrm{i}}{k} q n} \;=\; 1 - e^{2\pi \mathrm{i}\, n} \;=\; 0\,.$$

□

Now:

Proposition A.4 (**Classical quadratic Gauss sum evaluation**, cf. [63, p 87][83]). *For $k \in \mathbb{N}_{>0}$ we have*

$$\sum_{n=0}^{k-1} e^{\frac{2\pi \mathrm{i}}{k} n^2} \;=\; \begin{cases} (1+\mathrm{i})\sqrt{k} & | \quad k = 0 \operatorname{mod} 4 \\ \sqrt{k} & | \quad k = 1 \operatorname{mod} 4 \\ 0 & | \quad k = 2 \operatorname{mod} 4 \\ \mathrm{i}\sqrt{k} & | \quad k = 3 \operatorname{mod} 4 \,. \end{cases} \tag{56}$$

Proposition A.5 (**Quadratic Gauss sum with multiple exponents** cf. [63, "QS4" p 86]). *For odd $k \in 2\mathbb{N}+1$ we have more generally, for $q \in \mathbb{Z}$,*

$$\sum_{n=0}^{k-1} e^{\frac{2\pi \mathrm{i}}{k} q n^2} \;=\; (q|k) \sum_{n=0}^{k-1} e^{\frac{2\pi \mathrm{i}}{k} n^2} \;=\; \begin{cases} (q|k)\,(1+\mathrm{i})\sqrt{k} & | \quad k = 0 \operatorname{mod} 4 \\ (q|k)\,\sqrt{k} & | \quad k = 1 \operatorname{mod} 4 \\ 0 & | \quad k = 2 \operatorname{mod} 4 \\ (q|k)\,\mathrm{i}\,\sqrt{k} & | \quad k = 3 \operatorname{mod} 4 \,, \end{cases} \tag{57}$$

where

$$(q|k) \;=\; \begin{cases} 0 & \text{if} \quad \gcd(q,k) \neq 1 \\ \pm 1 & \text{if} \quad \gcd(q,k) = 1 \end{cases} \tag{58}$$

is the Jacobi symbol. [11]

In §4.2 we are crucially concerned with the variant of the classical quadratic Gauss sum that has *half* the usual exponents. In its plain form it is elementary to reduce this to the ordinary quadratic Gauss sum:

Proposition A.6 (**Quadratic Gauss sum with halved exponents**). *For $k \in 2\mathbb{N}_{>0}$ we have*

$$\sum_{n=0}^{k-1} e^{\frac{\pi \mathrm{i}}{k} n^2} \;=\; e^{\pi \mathrm{i}/4}\,\sqrt{k}\,. \tag{59}$$

Proof. Setting $r := k/2 \in \mathbb{N}$, we may compute as follows:

$$\begin{aligned} \textstyle\sum_{n=0}^{k-1} e^{\frac{\pi \mathrm{i}}{k} n^2} &= \textstyle\sum_{n=0}^{2r-1} e^{\frac{\pi \mathrm{i}}{2r} n^2} && \text{by def of } r \\ &= \tfrac{1}{2}\Big(\textstyle\sum_{n=0}^{2r-1} + \sum_{n=2r}^{4r-1} \Big) e^{\frac{\pi \mathrm{i}}{2r} n^2} && \text{since summands are } 2r\text{-periodic, cf. ftn. 9} \\ &= \tfrac{1}{2} \textstyle\sum_{n=0}^{4r-1} e^{\frac{2\pi \mathrm{i}}{4r} n^2} \\ &= \tfrac{1}{2}(1+\mathrm{i})\sqrt{4r} && \text{by (56)} \\ &= e^{\pi \mathrm{i}/4}\,\sqrt{2r} \\ &= e^{\pi \mathrm{i}/4}\,\sqrt{k} && \text{by def of } r. \end{aligned}$$

∎

More generally, there is the following reciprocity relation for the parameters of the quadratic Gauss sum with halved exponents, which relates it to the ordinary quadratic Gauss sum:

Proposition A.7 (**Landsberg-Schaar identity** [101], cf. [2][111][49]). *For $k \in 2\mathbb{N}_{>0}$ and $q \in \mathbb{N}_{>0}$ we have*

$$\sum_{n=0}^{k-1} e^{\pi \mathrm{i} \frac{q}{k} n^2} \;=\; \frac{e^{\pi \mathrm{i}/4}}{\sqrt{q/k}} \sum_{n=0}^{q-1} e^{-\pi \mathrm{i} \frac{k}{q} n^2}\,. \tag{60}$$

In summary, this implies the evaluation which we use in the main text:

Proposition A.8 (**Quadratic Gauss sum with multiple halved exponents**). *For $k \in 2\mathbb{N}_{>0}$ and $q \in 2\mathbb{N}+1$ we have:*

$$\sum_{n=1}^{k-1} e^{\pi \mathrm{i} \frac{q}{k} n^2} \underset{(60)}{=} \frac{e^{\pi \mathrm{i}/4}}{\sqrt{q/k}} \sum_{n=0}^{q-1} e^{-2\pi \mathrm{i} \frac{k/2}{q} n^2} \underset{(57)}{=} \begin{cases} e^{\pi \mathrm{i}/4}\sqrt{k}\,\big(k/2\,\big|\,q\big) & | \quad q = 1 \operatorname{mod} 4 \\ e^{-\pi \mathrm{i}/4}\sqrt{k}\,\big(k/2\,\big|\,q\big) & | \quad q = 3 \operatorname{mod} 4\,. \end{cases} \tag{61}$$

[11] The choice of sign in (58) is non-trivial, but in the main text it is of relevance only whether the Jacobi symbol vanishes or not.

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