Understanding Topological Quantum Gates in FQH Systems via the Algebraic Topology of exotic Flux Quantization

April 15, 2025

Abstract

Fractional quantum Hall systems (FQH) are a main contender for future hardware realizing topologically protected registers ("topological qbits") and topologically protected operations on these ("topological quantum gates"), both plausibly necessary ingredients for future quantum computers at useful scale, but both remaining only partially understood.

Here we present a novel non-Lagrangian effective description of FQH systems, based on previously elusive proper global quantization of effective topological flux. This directly translates their quantum-observables, -states, -symmetries, and -measurement channels into purely algebro-topological analysis of flat bundles of Hilbert spaces over the flux moduli spaces. Under the hypothesis — for which we provide evidence — that the appropriate effective flux quantization of FQH systems is in 2-Cohomotopy (a cousin of *Hypothesis H* in high-energy physics), the results here are rigorously derived and as such might usefully inform future laboratory searches for topological quantum hardware.

Contents

[a] Mathematics, Division of Science; and
Center for Quantum and Topological Systems,
NYUAD Research Institute,
New York University Abu Dhabi, UAE.

[b] The Courant Institute for Mathematical Sciences, NYU, NY.

The authors acknowledge the support by *Tamkeen* under the *NYU Abu Dhabi Research Institute grant* `CG008`.

1 Motivation & Introduction

Need for topological quantum protection. The potential promise of *quantum computers* [153][91] is enormous [80][15][163], but their practicability hinges on finding and implementing methods to stabilize quantum registers and gates against decohering noise. Serious arguments [113][47][132][52][53][99][78][207] and practical experience [34] suggest that the currently dominant approach of *quantum error correction* at the software-level (QEC [135][162]) will need to be supplemented [35] [1] by more fundamental physical mechanisms of quantum error *protection* already at the hardware level, in the form of "topological" stabilization of quantum states ("topological qbits") and operations ("topological quantum gates") [121][65][186][185]. While the general idea of topological quantum protection is famous and widely discussed, its fine details have received less attention and are nowhere nearly as well-understood as those of QEC — this in odd contrast to its plausible necessity for scalable quantum computing.

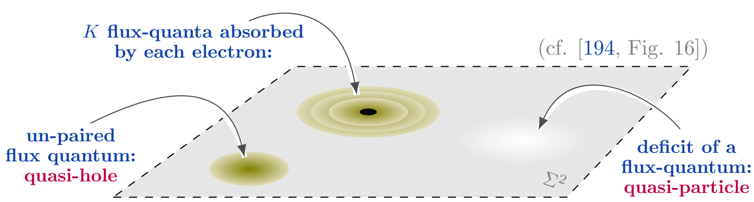
FQH systems: Topological flux quanta. The main practical contender [2] for the required topological quantum hardware currently are (cf. [7][37][22][11][144]) *fractional quantum Hall systems* (FQH, cf. [160][25][194][108][156]):

These are electron gases constrained to an effectively 2-dimensional surface Σ^2 (2DEG, e.g. realized on interfaces between semiconducting materials [156]) at extremely low temperature and penetrated by magnetic flux (cf. [89, §2.2.1][178, §2.1]) so strong that the number of *flux quanta* (cf. [123, (27)]) through the surface Σ^2 is an integer multiple K of the number of electrons confined to Σ^2, called the inverse *filling fraction* $\nu = \frac{1}{K}$ (more generally a rational multiple with $\nu = \frac{p}{K}$, p coprime to K).

In this situation, each electron in Σ^2 appears — which is understood only heuristically, [106][107], cf. [194, pp 882] — to form a "bound state" of sorts with exactly $1/\nu$ flux quanta — which conversely means that any *further* $\pm$ flux quantum inserted into the system, appears like the $\mp\nu$th fraction of an electron and hence as a "quasi-particle" (or "quasi-hole") of charge the $\mp\nu$th fraction of that of an electron:

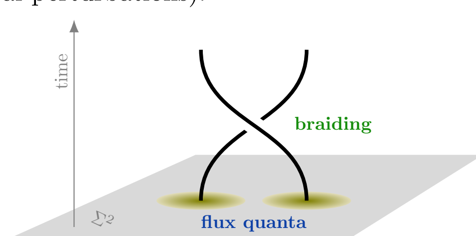
$$\text{FQH system at filling fraction } \nu \;:\quad \pm \text{ flux quanta} \quad \rightsquigarrow \quad \mp\nu \text{ fractional quasi-particles}\,.$$

It is these fractional quasi-particles/holes — hence the flux quanta on top of the exact fractional filling number — that are thought to have the desired topologically protected quantum states (exhibiting "topological order" [210], cf. [221, §III][172]).



(1)

In particular, each "braiding interchange" of worldlines of a pair of such makes their joint quantum state pick up a fixed complex phase factor $\zeta = e^{\pi \mathrm{i} \frac{p}{K}}$ ([197, (3)] following [6]) *independent* of the local details of the braiding process (hence insensitive to local perturbations):



$$\zeta(\nu) \;:=\; e^{\pi \mathrm{i} \nu}\,, \quad \nu := \tfrac{p}{K}\,. \tag{2}$$

Therefore, these FQH flux-quanta/quasi-particles are called "*anyons*" [6][193], in (somewhat inaccurate) reference [215] to *any* possible exchange phase between those of bosons, $\zeta(0) = 1$, and those of fermions, $\zeta(1) = -1$. Experimental observation of this emblematic anyon phase factor in quantum Hall systems has been reported in recent years: [150][151][152][169][86].

[1] [35]: "The qubit systems we have today are a tremendous scientific achievement, but they take us no closer to having a quantum computer that can solve a problem that anybody cares about. [...] What is missing is the breakthrough [...] bypassing quantum error correction by using far-more-stable qubits, in an approach called topological quantum computing."

[2] Much more press coverage has been devoted to the alternative candidate topological platform of "Majorana zero modes" in nanowires [36]; but even if the persistent doubts about their experimental detection can be dispelled in the future, these topological quantum states would by design be unmovable and hence would not support the hardware-level protected quantum *gates* that we are concerned with here.

Now, it is manifest in (2) that a deeper understanding of FQH systems hinges on a deep understanding of their *flux quantization* [178] — and yet just this is a weak spot of existing theory:

The problem of flux quantization in FQH systems. Experiment shows abundantly that the fractional quantum Hall effect is a *universal* phenomenon [110][84, p 1][85] in that its characteristic properties are independent of the microscopic nature of the host material and of impurities and irregularities of the sample. This suggests [66] the existence of accurate *effective* quantum field theoretic descriptions (cf. [63]) whose degrees of freedom reflect not any microscopic host particles but instead the nature of the universally emergent FQH quasi-particles (much like and closely related to how conformal field theory universally serves as effective description of critical phenomena in statistical mechanics, cf. [40, §3.2]).

Traditionally, this putative effective FQH theory is sought in the ancient and much-studied realm of *Lagrangian* quantum field theories (cf. [97][81]), where one argues ([220][210], cf. [63, §13.7][216, §2][202, §5][184, p 5]) that the relevant candidates are variants of abelian Chern-Simons theory [21][157][137], see (120). However, widely popular as they are, all (higher) gauge-field Lagrangians $L = L(A)$ suffer from the deficiency that they are sensitive only to the *local* degrees of freedom of the gauge field $\widehat{A}$ — namely to their underlying "gauge potentials" A on an open cover $\widetilde{X} \xrightarrow{p} X$ of spacetime X —, and hence by themselves miss exactly the *global topological* degrees of freedom (encoded in transition data over the Čech nerve of the cover) that are relevant for topological systems like FQH:

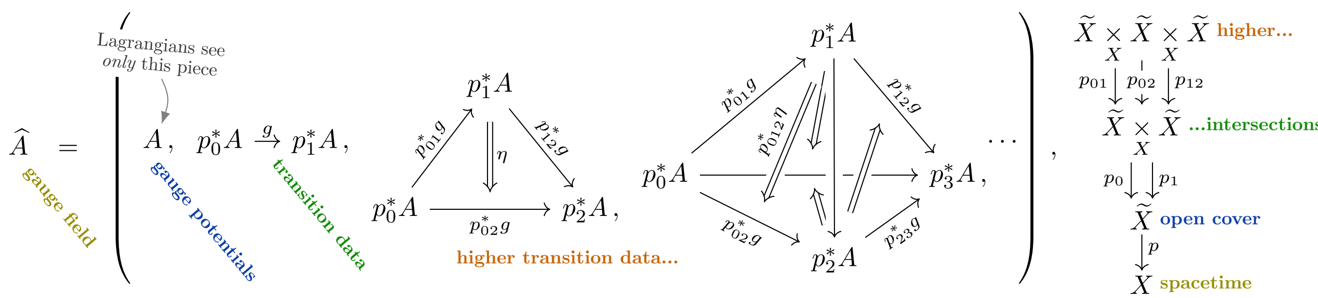


Figure G. The full (non-perturbative) data of a (higher) gauge field configuration $\widehat{A}$ on a spacetime X consists not just of the gauge potentials A, which are only defined locally – namely on an open cover $\widetilde{X}$ of X by charts –, but in "transition data" g which gauge-transforms between coincident gauge potentials on different charts, and further in incrementally higher transition data which higher-gauge transforms between coincident transition data.

It is this (higher) transition data that reflects the *flux quantization law* and thereby captures the topological *charge* or *soliton sector* encoded in the gauge field — and exactly this topological data is lost in Lagrangian formulations, with Lagrangian densities L being dependent only on the local gauge potentials, $L = L(A)$, cf. (120). For further exposition and pointers see [3] in the case of the ordinary electromagnetic field and [178, §3.3] in full generality.

While the missing global *flux quantization laws* [3][178] are traditionally tacked onto Lagrangian theories in an afterthought, the effective CS-Lagrangians proposed for FQH systems have the unnerving deficiency that — in their attempt to model the all-important *fractional* quasi-particle current by an effective gauge field —, they appear to be inconsistent with the integrality demanded by ordinary flux-quantization (cf. [216, p 35][202, p 159] and Rem. A.1 below).

This issue is an example of the notorious open problem of finding *non-perturbative* quantizations of Lagrangian theories as needed for strongly coupled topological quantum systems [56] (the analog in solid state physics of what in mathematical high energy physics is known as the *mass gap problem* which has famously been pronounced a "Millennium Problem" [29]).

Non-Lagrangian effective FQH theory based on flux quantization. In contrast, we have developed a non-Lagrangian theory of topological quantum states in (higher) gauge theories which is compatible with and in fact all based on consistent flux-quantization (survey in [178][184]): The main insight here is (recalled below in §2):

- **(a)** flux-quantization laws are encoded in *classifying spaces* $\mathcal{A}$ (subject to the constraint that their "rationalization" reflects the duality-symmetric form of the gauge-field's Bianchi identities), cf. [178, §3] and (6) below,
- **(b)** the *topological quantum observables on flux* depend only on the homotopy type of this classifying space $\mathcal{A}$, and not on any other (local, microscopic) properties of the theory [176].

A quick way to understand the underlying principle is to recall the classical fact of algebraic topology (cf. [60, p 263][59, Ex. 2.1]) that there exist classifying spaces — here denoted $B^n\mathbb{Z}$ [3], characterized by $\pi_k B^n\mathbb{Z} \simeq \delta_k^n \mathbb{Z}$ — for (integral, reduced) ordinary cohomology

$$\underset{\text{Ordinary cohomology}}{\widetilde{H}^n(X;\mathbb{Z})} \xleftrightarrow{\;\sim\;} \underset{\text{homotopy classes of maps to classifying space}}{\pi_0 \,\mathrm{Map}^*\big(X,\, B^n\mathbb{Z}\big)}\,, \tag{3}$$

and that the usual (Dirac) flux quantization of the electromagnetic field says that its underlying topological charge is a class in $\widetilde{H}^2(X;\mathbb{Z})$, hence represented by a map from spacetime to [4] $B^2\mathbb{Z} \simeq_{\int} BU(1) \simeq_{\int} \mathbb{C}P^\infty$, and that the latter is all it needs to deduce that solitonic magnetic flux through a plane comes in integer units – the *flux quanta* (1):

$$\underset{\text{Maps classifying solitonic magnetic charge}}{\big\{\mathbb{R}^2_{\cup\{\infty\}} \xrightarrow{\;c\;} B^2\mathbb{Z}\big\}}_{\big/\mathrm{hmtp}} \;\simeq\; \pi_0\,\mathrm{Map}^*\big(R^2_{\cup\{\infty\}},\, B^2\mathbb{Z}\big) \;\simeq\; \pi_0\,\mathrm{Map}^*\big(S^2,\, B^2\mathbb{Z}\big) \;\simeq\; \pi_2(B^2\mathbb{Z}) \;\simeq\; \mathbb{Z}\,. \tag{4}$$

In fact, the classifying space $\mathbb{B}^2\mathbb{Z}$ moreover encodes the ordinary topological flux quantum observables through any surface Σ^2, as seen in Ex. 2.8 below.

The key role of algebraic topology. With this understanding, the question for an effective QFT description of FQH systems is then not answered as traditionally (by choosing a Lagrangian whose equations of motion reflect local properties like the Hall current) but instead by finding an effective classifying space $\mathcal{A}$ whose implied topological quantum observables reproduce the expected observations, such as the emblematic (non-)commutation relation of Wilson line operators on the torus shown in (18) below.

It turns out (in §2) that this construction of topological flux quantum states proceeds entirely by the analysis of "*local systems*" (cf. Rem. 2.12 below) on the (generally covariantized) homotopy type of moduli spaces (22) of flux given by mapping spaces from the spacetime domain into the classifying space for the flux-quantization law — and as such is squarely a problem in the mathematical subject of homotopy theory and algebraic topology (for which we have compiled some background in §A.2).

Novel effective flux quantization for FQH systems. Concretely, a candidate classifying space for the *effective* magnetic flux through FQH systems (as seen by the effective quasi-particles/holes) turns out [179][177][184] [5] to be the 2-sphere $\mathcal{A} \simeq S^2$, modeling effective FQH flux in a variation of the ordinary classifying space (4) (of which it is the "2-skeleton"):

$$\underset{\text{Classifying space for effective FQH flux}}{S^2} \simeq \mathbb{C}P^1 \longleftrightarrow \mathbb{C}P^\infty \simeq_{\int} BU(1) \simeq_{\int} \underset{\text{Classifying space for ordinary magnetic flux}}{B^2\mathbb{Z}} \tag{5}$$

In this article, we work out in detail how this classifying space produces quantum effects in FQH systems, in particular how it reproduces quantum phenomena of abelian Chern-Simons theory. As a quick plausibility argument for this claim, note (cf. [58]) that the rationalization of the 2-sphere is encoded by the following differential equations (its "Sullivan minimal model" cf. [178, §3.2]), which are just those equations that characterize the Chern-Simons 3-form H_3 for a gauge field flux density F_2 as it appears in the Lagrangian formulation of Chern-Simons theory (cf. [64, Prop. 1.27(b)] and (120)):

$$\underset{\text{Rational model of classifying space for effective FQH flux}}{\mathrm{CE}(\mathfrak{l}S^2)} \;\simeq\; \mathbb{R}_{\mathrm{d}}\begin{bmatrix} \overset{\text{flux density}}{F_2} \\ \underset{\text{effective higher flux}}{H_3} \end{bmatrix} \Big/ \underset{\text{Bianchi identities characterizing Chern-Simons 3-form / Green-Schwarz mechanism}}{\left(\begin{aligned} \mathrm{d}\,F_2 &= 0 \\ \mathrm{d}\,H_3 &= F_2 F_2 \end{aligned}\right)} \tag{6}$$

Incidentally, it is in this sense that our effective description of the FQH effect is a mild form of *higher gauge theory* (cf. [183]), since the Chern-Simons 3-form (traditionally understood as a Lagrangian density) here appears as higher degree flux density — the 3-form H_3 — satisfying a Bianchi identity of the form known from *Green-Schwarz*

[3] These ordinary classifying spaces are known as *Eilenberg-MacLane spaces* and often denoted "$K(\mathbb{Z}, n)$".

[4] Our notation "$\simeq_{\int}$" stands for *weak homotopy equivalences* (136).

[5] As explained in [179][83], following [180], the 2-sphere here is a relative of the 4-sphere which similarly serves as flux quantization of the higher gauge field in 11D supergravity [170, §2.5][57][82] (review in [178]), where its choice as such is referred to as *Hypothesis H* [57]. While this is where our approach to FQH systems here comes from and is informed by [177], for the present purpose the reader may ignore this geometric engineering of FQH systems on M5-probes of 11d SuGra. But review and relevance for deeper questions of FQH systems (such as their hidden supersymmetry [90][164]) may be found in [184, §2-3].

mechanisms. [6] Moreover, H_3 is the rational image of the *Hopf fibration*, the generator of

$$\mathbb{Z} \;\simeq\; \pi_3(S^2) \;\simeq\; \pi_0 \mathrm{Map}^*(S^3, S^2) \;\simeq\; \pi_1 \mathrm{Map}^*(S^2, S^2)\,. \tag{7}$$

This is the non-torsion class that disappears under passage to the ordinary classifying space (5), and it is this class which we find in §4, Prop. 4.7, to be identified with the observable $\widehat{\zeta}$ of fractional braiding phases (2)!

Aims. With this novel effective theory for FQH systems in hand, our ambition here is to provide previously missing theoretical understanding & prediction of

(i) appearance of defect anyons ⇒ topological qbits,
(ii) operable transformations on these ⇒ topological quantum gates,
(iii) their admissible measurement bases ⇒ topological readout.

§4	Surface Σ^2	Predictions of 2-cohomotopical FQH flux quantization	Relevance
§4.1	The plane $\Sigma^2_{0,0,1}$	Abelian CS Wilson loop observables, pure states ↔ crossing phases	⇒ solitonic anyons
§4.4	The torus $\Sigma^2_{1,0,0}$	Modular data of spin CS/WZW theory (generalized to filling fractions $\nu = q/K$)	⇒ topological order
§4.5 §4.6	Punctured disk $\Sigma^2_{0,0,n}$	Framed braid representation (with conditions on framing)	⇒ defect anyons
§4.8	Punctured annulus $\Sigma^2_{0,2,n}$	Asymptotic boundary symmetry	⇒ edge modes

Comparison to U(1)-Chern-Simons theory. To a large extent, our construction turns out to give a curious and curiously direct (re-)derivation of the fine properties of U(1)-Chern-Simons quantum field theory (cf. [21][157][137][75]) by novel non-Lagrangian means, and as such the result seems of interest in its own right, beyond the topic of FQH systems (see Remarks 4.11 and 4.31 below).

Or rather, we find (in §4.4) that with FQH systems we must be dealing with *"spin" Chern-Simons theory* [41, §5] (a point originally noted for FQH systems in [147, p 381] but usually glossed over), where the filling fraction denominator is identified with *twice* the Chern-Simons level (which is hence half-integral for the common odd FQH denominators):

	Symbol	in $\frac{p}{q}$-FQH	in ordinary CS	in "spin" CS	$\exp\left(\frac{\mathrm{i}}{\hbar} S_{\mathrm{CS}}\right) =$
CS level	k		$\in\ \mathbb{N}_{>0}$	$\in \frac{1}{2}\mathbb{N}_{>0}$	$e^{2\pi \mathrm{i}\, k \int A\, \mathrm{d}A}$
	$K \equiv 2k$	$= q$	$\in 2\mathbb{N}_{>0}$	$\in\ \mathbb{N}_{>0}$	$e^{\pi \mathrm{i}\, K \int A\, \mathrm{d}A}$

(8)

But our theory also differs from usual U(1)-CS-theory in subtle respects:

- On the torus, we find (Prop. 4.33, 4.34) general fractional braiding phases $e^{\pi \mathrm{i} \frac{p}{K}}$, beyond $p = 1$, otherwise only seen for $U(1)^N$ CS-theory with $N > 1$, in "K-matrix formalism" [212][210].
- At the same time, our prediction for the ground state degeneracy on the torus is $\dim(\mathcal{H}_{T^2}) = K$ (Thm. 4.35), independent of p, and hence in general different from the prediction of K-matrix formalism.
- On the other hand, we see (Rem. 4.36) that these K-dimensional state spaces over the torus admit distinct possible flavors of topological order reflected in extra phases picked up under modular transformations.
- On n-punctured disks, where the literature on the Reshetikhin-Turaev construction of CS-theory expects the framed braid group FBr_n to act on the Hilbert space of states, we find subgroups of framed braids with restriction on their total framing number (cf. Rem. 4.46).
- While we recover the fine-print of Wilson loop observables in abelian Chern-Simons theory (Rem. 4.11), we find a clear distinction (cf. Fig. F) between *solitonic anyons* (with abelian braiding not amenable to external control) and *defect anyons* (with possibly non-abelian braiding subject to external control), which does not seem to be clearly expressed in traditional theory.

[6] In the "geometric engineering" of our FQH model on M5-branes referred to in footnote 5, this 3-form arises as the restriction to an orbi-singularity of the "self-dual" tensor field carried by these branes, which itself is quantized in a higher (and "twistorial") form of Cohomotopy, cf. [83][179].

Therefore, while the account here is purely theoretical and largely mathematical, it does make potentially discernible experimental predictions and suggests potential novel pathways to realizing topological quantum gates in FQH systems, notably in predicting that defect anyons may be realized as loci in the FQH material where the magnetic field is expelled, cf. §4.7.

Acknowledgement. We thank Sadok Kallel, Moishe Kohan, and Will Sawin for useful discussions.

2 States of Topological Flux

We first recall (Prop. 2.6 below, from [176]) how non-perturbative topological quantum observables on ordinary G-Yang-Mills flux depend exclusively on the homotopy type of the electric/magnetic *classifying space* $\mathcal{A} \equiv B\big(G \ltimes (\mathfrak{g}/\Lambda)\big)$.

In this algebro-topological formulation, the result has an evident generalization to higher gauge theories with exotic flux quantization laws [178] classified by any other pointed space $\mathcal{A}$, such as the 2-sphere (5) with its higher Chern-Simons flux form (6) in our motivating example of FQH systems.

Since no other established rules for non-perturbative quantization of higher gauge fields exist, we next promote this evident generalization to the missing quantization procedure for higher topological flux, following [176, §4]. This is Def. 2.11 below, where we successively refine the prescription by allowing also non-closed surfaces and bringing in diffeomorphism-equivariance and asymptotic symmetries.

First to set up some notation:

Definition 2.1 (**Spacetime).** Throughout, we consider

- $X^{1,3} := \mathbb{R}^{1,1} \times \Sigma^2$ a globally hyperbolic 4D spacetime,
- with spatial slices $\mathbb{R}^1 \times \Sigma^2$, to be thought of as a tubular neighborhood of:
- Σ^2, a surface (here: a connected, oriented smooth 2D manifold with boundary) which at times is
- specialized to $\Sigma^2 \equiv \Sigma^2_{g,n,b}$, the unique (up to homeomporphism) surface
  - of genus g,
  - with b boundary components,
  - and n punctures:

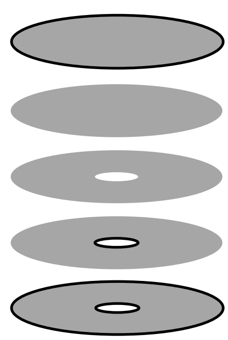
$$\underset{\text{genus, boundaries, punctures}}{\Sigma^2_{g,\,b,\,n}} \;\simeq\; \Big(\underbrace{\Sigma^2_{0,0,0}}_{\text{sphere}} \overset{\text{connected sum}}{\#} \underbrace{T^2 \# \cdots \# T^2}_{g \text{ connected summands of tori}} \Big) \overset{\text{complement}}{\setminus} \Big\{ \underbrace{D^2 \sqcup \cdots \sqcup D^2}_{b \text{ disjoint summands of open disks}} \overset{\text{disjoint union}}{\sqcup} \underbrace{\overline{D}^2 \sqcup \cdots \sqcup \overline{D}^2}_{n \text{ disjoint summands of closed disks}} \Big\}, \tag{9}$$

understood as modeling an effectively 2-dimensional sample of material. [7]

We abbreviate $\Sigma^2_{g,b} := \Sigma^2_{g,b,0}$ and $\Sigma^2_g := \Sigma^2_{g,0} \equiv \Sigma^2_{g,0,0}$.

Example 2.2 (**Some surfaces).** We have homoemorphisms as follows:

$$\begin{array}{rl} \text{sphere} & \Sigma^2_{0,0,0} \simeq S^2 \\ \text{torus} & \Sigma^2_{1,0,0} \simeq T^2 \\ \text{closed disk} & \Sigma^2_{0,1,0} \simeq D^2 \\ \text{open disk} & \Sigma^2_{0,0,1} \simeq \mathbb{R}^2 \\ \text{open annulus / punctured plane} & \Sigma^2_{0,0,2} \simeq \mathbb{R}^2 \setminus \{0\} \\ \text{half-open annulus} & \Sigma^2_{0,1,1} \\ \text{closed annulus} & \Sigma^2_{0,2,0} \simeq A^2 \end{array} \tag{10}$$



Here, the disks and annuli are the surface types readily and commonly realized in laboratory FQH experiments (cf. footnote 7). In order to understand *defect anyons* we will be interested (in §4.8) in the situation where these are further punctured, for example:

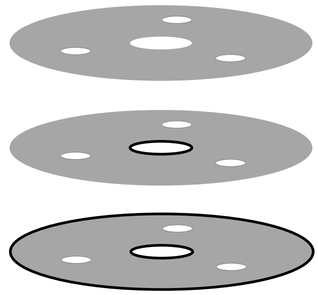
[7] Albeit routinely considered in theory (cf. [211]), the practicability of direct laboratory realizations of $\Sigma^2_{g,n,b}$ (9) with transversal magnetic flux is limited when $g > 0$. The case $g = 1$ (the torus) is readily realized (only) when considering momentum space (the Brillouin torus of a 2D crystal, cf. [172]) instead of position space, but, while noteworthy in itself, this is not the case of FQH systems of concern here. Alternatively, it was argued [10] that suitable defects, called "genons", in a crystal lattice could make a sample of nominal genus $g = 0$ effectively behave like of $g > 0$.

But irrespectively of practicality, the theoretical possibility of $g > 0$ allows to compare our topological quantum flux observables to those of abelian Chern-Simons theory in the case $\Sigma^2_{g>0,0,0}$, and their agreement in this theoretical case supports the validity of our observables also in the more practical cases of $g = 0$, $n, b \neq 0$.

$\Sigma^2_{0,0,3+2} \simeq$ ... $\simeq \mathbb{R}^2 \setminus \{x_1, x_2, x_3, x_4\}$

$\Sigma^2_{0,1,3+1} \simeq$ ... (11)

$\Sigma^2_{0,2,3} \simeq$ ...



Remark 2.3 (Spin structure). In fact, we regard spacetime $X^{1,3}$ (Def. 2.1) — and therefore also the surfaces Σ^2 (9) — to be equipped with spin structure (cf. [143]), but for notational convenience we shall make the choice of spin structure explicit only when it matters, namely below in §4.4 (see Prop. 4.34).

Definition 2.4 (Gauge group). For the following Thm. 2.5 we consider G a Lie group, with Lie algebra $\mathfrak{g}$ and with a choice of Ad-invariant lattice $\Lambda \subset \mathfrak{g}$ (not necessarily full, possibly zero) — but shortly we specify this to $G \equiv \mathbb{R}$ and $\Lambda \equiv \mathbb{Z} \hookrightarrow \mathbb{R}$ (15).

The following theorem 2.5, from [176], is based on well-known ingredients but may have escaped earlier attention in its deliberate disregard of the gauge potentials in favor of focus on the electric/magnetic flux densities — which is what brings out how the topological flux quantum observables are all controlled by maps from Σ^2 to the classifying space $B\big(G \ltimes (\mathfrak{g}/\Lambda)\big)$, cf. Rem. 2.8 below.

Theorem 2.5 (Yang-Mills flux quantum observables [176, Thm 1]). *The non-perturbative quantum observables on the G-Yang-Mills flux-density* [8] *through a closed surface Σ^2_g form the group convolution C^*-algebra $\mathbb{C}[-]$ of the Fréchet-Lie group of smooth functions $C^\infty(-,-)$ from Σ^2 to the semidirect product of G with the additive group $\mathfrak{g}/\Lambda$:*

$$\underset{\text{on YM-flux through }\Sigma^2}{\underset{\text{quantum observables}}{\text{Algebra of}}}\quad \mathrm{FlxObs}_{\Sigma^2} \;\simeq\; \overset{\text{group convolution } C^*\text{-algebra}}{\mathbb{C}}\Big[C^\infty\big(\Sigma^2,\, G \ltimes_{\mathrm{Ad}} (\mathfrak{g}/\Lambda)\big)\Big]$$
$$\simeq\; \mathbb{C}\Big[\underbrace{C^\infty\big(\Sigma^2,\, G\big)}_{\text{electric}} \ltimes_{\mathrm{Ad}} \underbrace{C^\infty\big(\Sigma^2,\, \tfrac{\mathfrak{g}}{\Lambda}\big)}_{\text{magnetic}}\Big]. \tag{12}$$

Here, the left-hand side is defined to be the non-perturbative quantization — Rieffel's C^*-algebraic strict deformation quantization – of the Poisson brackets of electric and magnetic Yang-Mills fluxes. The right-hand side follows by observing that these observables are given by $\mathfrak{g}$-valued smearing functions over Σ and then by computing — with careful attention to the Gauss law constraint — that the Poisson brackets give the Lie algebra of the Fréchet Lie group as shown. The details of this computation are in [176, §A.1]. With this, the conclusion (12) follows by the well-known fact, cf. [176, p. 3], that the non-perturbative quantization of Lie-Poisson phase spaces are the corresponding group convolution algebras.

Accordingly, we have in this situation that (see §A.2 for our notation concerning mapping spaces):

Proposition 2.6 (Topological YM flux quantum observables [176, §3]). *The algebra of* topological *G-flux quantum observables — hence of the group convolution C^*-algebra on the discrete group of connected components $\pi_0(-)$ of the flux densities — through a closed surface Σ^2 is equivalently the group (convolution) algebra* (cf. [70, §3.4][9, p 51][28, §2.4][27, (4)]) *of the fundamental group $\pi_1(-)$* (cf. [2, §2.5]) *of the space of maps* (cf. [2, §1]) *into the classifying space $B(-)$* (140)*:*

$$\underset{\text{flux observables}}{\text{Algebra of topological}}\quad \mathrm{TopFlxObs}_{\Sigma^2} \;:=\; \mathbb{C}\Big[\pi_0\, C^\infty\big(\Sigma^2,\, G \ltimes_{\mathrm{Ad}} \tfrac{\mathfrak{g}}{\Lambda}\big)\Big]$$
$$\simeq\; \mathbb{C}\Big[\pi_0\, \mathrm{Map}\big(\Sigma^2,\, G \ltimes_{\mathrm{Ad}} \tfrac{\mathfrak{g}}{\Lambda}\big)\Big] \;\simeq\; \mathbb{C}\Big[\pi_1\, \mathrm{Map}_0\big(\Sigma^2,\, B\big(G \ltimes \tfrac{\mathfrak{g}}{\Lambda}\big)\big)\Big] \tag{13}$$
$$\simeq\; \mathbb{C}\Big[\mathrm{Map}^*_0\big((\Sigma^2 \times \mathbb{R})_{\cup\{\infty\}},\, B\big(G \ltimes \tfrac{\mathfrak{g}}{\Lambda}\big)\big)\Big].$$

[8] For the case of abelian G of interest here, these are indeed observables on the *reduced* phase space.

Remark 2.7 (Asymptotic boundary localization of topological flux observables).

(i) In the second line of (13), we are showing several isomorphic incarnations of this group algebra of observables (these isomorphisms are explained in [176, §A.2], using basic facts also recalled in §A.2), which each have their use in the following.

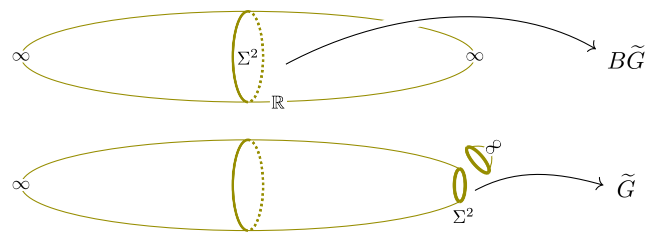
(ii) In particular, the isomorphism between the first and the third one, which for a general group $\tilde{G}$ expresses the statement (recalling that Σ^2 here is assumed closed, hence compact)

$$\pi_0\,\mathrm{Map}^*\big((\Sigma^2\times\mathbb{R})_{\cup\{\infty\}},\, B\widetilde{G}\big) \;\simeq\; \pi_0\,\mathrm{Map}^*\big(S^1\wedge(\Sigma^2)_{\sqcup\{\infty\}},\, B\widetilde{G}\big) \;\simeq\; \pi_0\mathrm{Map}\big(\Sigma^2,\, \widetilde{G}\big) \tag{14}$$

that charge classes of $\widetilde{G}$-gauge fields on 3D space $\Sigma^2\times\mathbb{R}$, which trivialize at infinity, are equivalently such classes on the *suspension* of Σ_2, which in turn are classified by $\widetilde{G}$-valued functions on $\widetilde{G}$.

(iii) This is the homotopy-theoretic incarnation of the *clutching construction* (cf. [102, §7]), according to which the principal $\widetilde{G}$-bundle on a suspension is trivializable on any "hemisphere" (being a cone over Σ^2) and classified by a single $\widetilde{G}$-valued transition function on an "equator', being a copy of Σ^2. Since the actual physical space is $\Sigma^2\times\mathbb{R}$, with its one-point compactificatin to the suspension of Σ^2 only to model the vanishing-at-infinity of solitonic charge, in physics this copy of Σ^2 is naturally identified with the asymptotic boundary of 3D space.

Figure C. Classifying maps for solitonic $\widetilde{G}$-gauge charge on 3D space $\Sigma^2\times\mathbb{R}$ are equivalently (14) maps to $\widetilde{G}$ from a copy of Σ^2 that is naturally thought of as being the asymptotic boundary of space at ∞.



Example 2.8 (The prediction of ordinary electromagnetism...). For ordinary electromagnetic flux, subject to the usual Dirac charge quantization law (where the magnetic but not electric flux is quantized in integral cohomology, cf. [176, (14)]) the relevant choice in Def. 2.4 is:

$$\begin{array}{rcc} & \overset{\text{no electric flux quantization}}{G := \mathbb{R}\,,} & \overset{\text{usual magnetic flux quantization}}{\Lambda := \mathbb{Z}\hookrightarrow\mathbb{R}} \\ \text{classifying space} & \mathcal{A} := B\big(\,\mathbb{R}\,\ltimes\,\tfrac{\mathbb{R}}{\mathbb{Z}}\,\big) \simeq_{\int} B\mathrm{U}(1)\,. & \end{array} \tag{15}$$

In this case, the algebra (13) of observables on topological flux through a closed surface Σ_g (9) is

$$\begin{array}{llll} \text{Algebra of topological flux observables for ordinary electromagnetism} & \mathrm{TopFluxObs}^{B\mathrm{U}(1)}_{\Sigma^2_g} & \simeq & \mathbb{C}\big[\pi_0\,\mathrm{Map}\big(\Sigma^2,\,\mathrm{U}(1)\big)\big] \\ & & \simeq & \mathbb{C}\big[\pi_0\,\mathrm{Map}\big(\Sigma^2,\,B\mathbb{Z}\big)\big] \\ & & \simeq & \mathbb{C}\big[H^1\big(\Sigma^2_g;\,\mathbb{Z}\big)\big] \quad \text{e.g. [60, p 263]} \\ & & \simeq & \mathbb{C}\big[\,\mathbb{Z}^{2g}\,\big] \quad \text{e.g. [79, Thm 6.13],} \end{array} \tag{16}$$

and so a corresponding space of quantum states $\mathcal{H}_{\Sigma^2_g}$ hence carries an action of linear operators $\widehat{W}_{\left[\begin{smallmatrix}\vec a\\ \vec b\end{smallmatrix}\right]}$, for $\vec a,\vec b\in\mathbb{Z}^g$, subject to

$$\widehat{W}_{\left[\begin{smallmatrix}\vec a\\ \vec b\end{smallmatrix}\right]}\circ\widehat{W}_{\left[\begin{smallmatrix}\vec a'\\ \vec b'\end{smallmatrix}\right]} \;=\; \widehat{W}_{\left[\begin{smallmatrix}\vec a+\vec a'\\ \vec b+\vec b'\end{smallmatrix}\right]} \;=\; \widehat{W}_{\left[\begin{smallmatrix}\vec a'\\ \vec b'\end{smallmatrix}\right]}\circ\widehat{W}_{\left[\begin{smallmatrix}\vec a\\ \vec b\end{smallmatrix}\right]}\,. \tag{17}$$

Remark 2.9 (...and its failure to describe FQH systems).

(i) While this algebra of observables (17) is the prediction of the ordinary traditional theory of electromagnetism, it is *not quite* the algebra of observables of magnetic flux actually seen in fractional quantum Hall systems!

(ii) Instead, over the torus ($g=1$) the quantum observables of FQH systems are famously thought to satisfy a *non-commutative deformation* of (17) where the cross-terms pick up the square ζ^2 of the braiding phase factor (2) when commuted past each other [211, (4.9)][63, (4.21)][202, (5.28)]:

$$\widehat{W}_{\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]}\circ\widehat{W}_{\left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]} \;=\; \zeta\,\widehat{W}_{\left[\begin{smallmatrix}1\\ 1\end{smallmatrix}\right]} \;=\; \zeta^2\,\widehat{W}_{\left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]}\circ\widehat{W}_{\left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]}\,. \tag{18}$$

(iii) Hence the correct algebra of observables on FQH flux through a torus must be the group algebra of non-abelian central extension $\widehat{\mathbb{Z}^2}$ of $\mathbb{Z}^2$ by $\mathbb{Z}$ which we may identify as the *integer Heisenberg group* (69).

This motivates looking for coherent generalization of topological flux observables via non-standard flux quantization laws such that they do capture effects like (18) (we find this in §4.4).

Indeed, Prop. 2.6 is remarkable in how it shows the topological flux quantum observables of ordinary gauge theory to depend exclusively on the classifying space that encodes the flux-quantization law (cf. Ex. 2.8). While usual (perturbative) machinery of constructing quantum field theories based on Lagrangian densities does not capture this global information, since Lagrangian densities do not (being functions only of local gauge potentials but not the global flux-quantized gauge field content), with Prop. 2.6 we have established a direct construction of topological flux quantum observables from the flux quantization law determined by a classifying space $\mathcal{A}$.

The topological flux quantization prescription. We are thus led to the following Def. 2.11, which is the foundation of our analysis here. We regard this as a new quantization prescription that pertains to topological flux quantum systems previously inaccessible by established quantization procedures. This prescription is justified, besides the suggestive results it yields below, by how it is a natural generalization of the conclusion of Prop. 2.6. For note that the formula (13) immediately generalizes from the case $\mathcal{A} \equiv B(G \ltimes \frac{\mathfrak{g}}{\Lambda})$ to *any* pointed connected space $\mathcal{A}$:

$$\underset{\text{flux quantized in }\mathcal{A}\text{-cohomology}}{\text{topological flux observables for}} \quad \mathrm{TopFlxObs}^{\mathcal{A}}_{\Sigma^2} \;:=\; \mathbb{C}\big[\pi_1\,\mathrm{Map}_0\big(\Sigma^2,\,\mathcal{A}\big)\big]\,. \tag{19}$$

Remark 2.10 (**Classifying spaces for higher and generalized symmetries).** For any pointed connected space $\mathcal{A}$ its loop space $\Omega\mathcal{A}$ carries a "higher group" structure (cf. [154][181, §2.2][182, §3.1.2]) under concatenation and reversal of loops, and $\mathcal{A}$ is the classifying space for *that* higher group structure (cf. [59, Prop. 2.2]):

$$\mathcal{A} \;\simeq_{\int}\; B(\Omega\mathcal{A})\,.$$

This way, the generalization (19) corresponds to passage to *higher* gauge theories with *higher* (gauge) symmetry. These days in physics the latter is also referred to as "generalized symmetry", see [92] for an account that makes contact with our perspective here.

This algebra (19) of observables being a *group algebra* means that the corresponding spaces of (pure) quantum states — which generally are modules over the observable algebra — are actually modules of a group algebra and, as such, nothing but (unitary) representations of this group (cf. [28, p. 11]):

$$\underset{\text{flux quantized in }\mathcal{A}\text{-cohomology}}{\text{topological quantum states of}} \quad \mathcal{H}^{\mathcal{A}}_{\Sigma^2} \;\in\; \mathrm{URep}\big(\pi_1\,\mathrm{Map}_0(\Sigma^2,\,\mathcal{A})\big)\,. \tag{20}$$

Further generalizations are immediately suggested by these formulas: If $\Sigma^2 = \Sigma^2_{g,b,n}$ (9) is possibly non-compact ($n > 0$), then the *solitonic* flux configurations (cf. [178, §2.2][176, §A.2]) are those which are *vanishing at infinity* and thus classified by *pointed* maps on the one-point compactification $(-)_{\cup\{\infty\}}$ (see §A.2), so that (20) generalizes to:

$$\underset{\text{on non-compact spaces}}{\text{... for solitonic flux}} \quad \overbrace{\mathcal{H}^{\mathcal{A}}_{\Sigma^2}}^{\text{space of quantum states}} \;\in\; \underbrace{\mathrm{URep}}_{\text{unitary representations}}\Big(\underbrace{\pi_1\,\overbrace{\mathrm{Map}^*_0\big(\Sigma^2_{\cup\{\infty\}},\,\mathcal{A}\big)}^{\text{moduli space of solitonic topological flux}}}_{\text{topological flux monodromy group}}\Big)\,. \tag{21}$$

Moreover, if the gauge field is to be regarded as "generally covariant" in the sense of physics, — namely covariant with respect to diffeomorphisms, such as for gravitational and topological systems —, so that a pair of topological flux configurations are to be regarded as gauge equivalent if one is obtained from the other by precomposition with a diffeomorphism, then (cf. [45, Def. 1.1]) the true moduli space is the *homotopy quotient* (139) of the flux moduli space (21) by the action of the diffeomorphism group $\mathrm{Diff}(\Sigma^2)$ (see Def. 3.6). Therefore we set:

Definition 2.11 (**Generally covariant topological flux observables for exotic flux quantization).** The possible spaces of quantum states of generally covariant topological flux quantized in $\mathcal{A}$-cohomology are the irreducible unitary representations of the covariantized flux monodromy group:

$$\underset{\text{topological solitonic flux}}{\underset{\text{quantum states of}}{\text{Generally covariantized}}} \quad \mathcal{H}^{\mathcal{A}}_{\Sigma^2} \;\in\; \overbrace{\underbrace{\mathrm{URep}}_{\text{reps of}}\Big(\underbrace{\pi_1}_{\text{fundamental group of}}}^{\text{local systems of state spaces on}}\overbrace{\Big(\underbrace{\mathrm{Map}^*_0\big(\Sigma^2_{\cup\{\infty\}},\,\mathcal{A}\big)}_{\text{plain moduli space of topological solitonic flux quantized in }\mathcal{A}\text{-cohomology}}\underbrace{/\!\!/\,\mathrm{Diff}^{+,\partial}(\Sigma^2)\,\Big)}_{\text{covariantized under diffeos}}}^{\text{moduli space of topological solitonic flux}}\Big)\,. \tag{22}$$

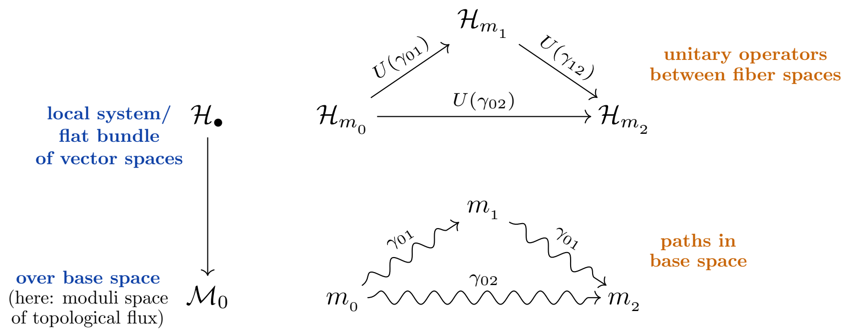
In order to get a handle on these groups (22) of generally covariantized flux monodromy, the first general result we prove below (Prop. 3.9, not surprising, but important) is that they are equivalently the semi-direct product of the plain flux monodromy with the surface's mapping class group:

$$\pi_1\Big(\overbrace{\mathrm{Map}^*_0\big(\Sigma^2_{\cup\{\infty\}},\,\mathcal{A}\big)\;/\!\!/\;\mathrm{Diff}^{+,\partial}(\Sigma^2)}^{\text{covariantized flux moduli space (22)}}\Big)\;\simeq\;\underbrace{\pi_1\Big(\mathrm{Map}^*_0\big(\Sigma^2_{\cup\{\infty\}},\,\mathcal{A}\big)\Big)}_{\text{flux monodromy}}\rtimes\underbrace{\pi_0\Big(\mathrm{Diff}^{+,\partial}(\Sigma^2)\Big)}_{\text{mapping class group (38)}}. \tag{23}$$

Some remarks on the import of this construction (22):

Remark 2.12 (Local systems of vector spaces and quantum state spaces).

(i) Representations of fundamental groups $\pi_1(\mathcal{M}_0)$ as in (22), equivalently (since $\mathcal{M}_0$ is a connected component) of the *fundamental groupoid* $\Pi_1(\mathcal{M}_0)$ (cf. [213, (1.7)]) are equivalently known as *local systems* (cf. [38, §I.1][213, p. 257][42, §2.5][198, §2.6]) or *flat bundles* (cf. [205, §9.2.1]) of vector spaces over $\mathcal{M}_0$ (cf. [149, Lit. 2.22]).



(ii) There is deep relevance [174] in identifying these as quantum state spaces subject to symmetries and classical control in general (cf. [173][27]and the following Prop. 2.16) and specifically so concerning the nature of anyonic topological quantum gates (cf. [149, §3] and the next Rem. 2.13).

(iii) In our situation (22), the base space is the covariantized moduli space of topological fluxes quantized in some $\mathcal{A}$-cohomology, and paths $m \rightsquigarrow m'$ are *combinations* of

(a) gauge transformations of the (higher) gauge field
(from the homotopy quotient's numerator in (22))

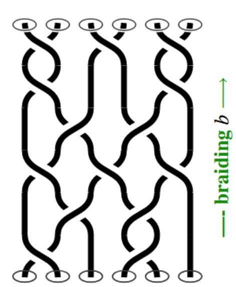
(b) diffeomorphisms, hence gauge transformations of the "gravitational field",
(from the homotopy quotient's denominator in (22)).

The Hilbert space(s) of states $\mathcal{H}^{\mathcal{A}}_{\Sigma^2}$ forming a local system/flat bundle means that both of these kinds of gauge transformations are implemented as unitary quantum operators/observables in a coherent way.

(iv) Some or all of these apparent gauge symmetries may in fact be "asymptotic" and as such become physical observables (this is the content of the next Prop. 2.16) whence the construction (22) encodes flux quantum systems both with their quantum symmetries and their quantum observables.

Note again that the paths here are paths in moduli space, hence not of single flux quanta, but of all flux quanta that are present at once. The same holds for punctures/defects:

Remark 2.13 (Defect anyons and braid representations).



(i) When the surface Σ^2 has (enough) punctures, understood as *defects* in the slab of quantum material (cf. [141]), then its mapping class group appearing in (23) contains a *braid group* (details in §3) whose elements are to be thought of as braided *worldlines* of these defects, illustrated on the right.

(ii) Linear representation of this braid group (*braid representations*, cf. [117][1]), as given by the corresponding flux quantum states according to (22), encode a possibly non-abelian generalization of the braiding phases of solitonic anyons (2), exhibiting the defects as *defect anyons* ([122, p. 4][172]).

(iii) The appearance of such, possibly non-abelian, defect anyons in FQH systems, in explicit contrast to solitonic anyons, has received little attention before, but it is these defects, not the solitons, whose positions are plausibly amenable to external (adiabatic) tuning, through which the braiding of their worldlines — and with that the enaction of the desired topological quantum gates (cf. [117][98][26][118]) — could plausibly be operated.

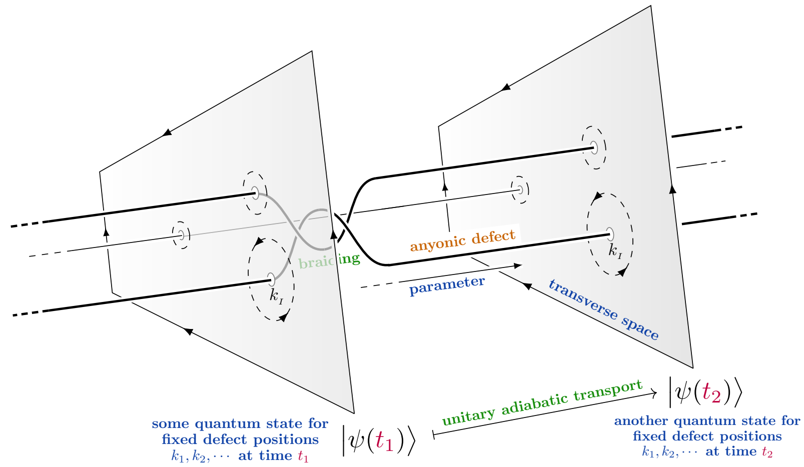


Figure A – Topological braid gates. Covariant flux quantum states on a sufficiently punctured surface, by their diffeomorphism equivariance (22), carry unitary representations of the *braid group* (33) of joint motions of the defect points around each other, exhibiting the punctures as *defect anyons*.
If these braiding processes can be subjected to classical external control, then their adiabatic execution may be expected to result in the flux quantum state to transform according to the corresponding unitary representation operator, thus constituting a programmable *quantum gate*. By the topological nature of the braid group, this gate would be insensitive to isotopy between the anyon worldlines, hence would be topologically protected against noise in the classical control parameters.

Remark 2.14 (**Focus on irreducible representations: Superselection sectors of anyons).**

(i) From the perspective of Def. 2.11, spaces of pure quantum states (of topological flux) are defined as representations of (modules over) the algebra of observables. From this perspective, it is the (unitary) *irreducible* representations that matter in characterizing the actual quantum system, in that reducible representations behave like parallel copies of system, which could just as well be discussed separately: "superselection sectors" (cf. [72, Def. 2.1][8, p. 273]).

(ii) While this may (and should) seem obvious, it is in some contrast to common practice: For punctured surfaces Σ^2 the mapping group appearing (23) contains *braiding* operations (cf. Prop. 3.7) and it is common to consider Yang-Baxter representations for these ("R-matrices", cf. [117][223]), which generally are reducible (cf. [133, p. 15]).

Quantum operations at the boundary. Last not least, we introduce one more aspect to the quantization prescription of Def. 2.11. Namely when Σ^2 has a boundary, then the diffeomorphism action in (22) subsumes two distinct physical aspects

(i) "bulk diffeomorphisms" — those which suitably asymptote to trivial diffeomorphisms on the boundary —, are meant to be *gauge symmetries* of a generally covariant system,

(ii) their cosets, instead, are meant to be *physical observables* that may be measured by observers with access to the boundary.

This is usually discussed in gravity and high energy physics (cf. [24][196, §2.10][20]), but we highlight that grasping this phenomenon here is crucial for understanding FQH systems, since their effective theory is meant to be generally covariant (even topological) and since in practice their observables indeed tend to be boundary observables: "edge modes", cf. [150].

Concretely, with a normal subgroup inclusion ι of bulk gauge/diffeo transformations, hence with a short exact sequence

$$1 \longrightarrow \underset{\text{symmetries}}{\overset{\text{bulk}}{\mathcal{G}_{\mathrm{blk}}}} \xhookrightarrow{\;\;\iota\;\;} \underbrace{\overset{\text{gauge transfors \& diffeos}}{\pi_1\,\mathrm{Map}\big(\Sigma^2_{\cup\{\infty\}},\,\mathcal{A}\big) \rtimes \mathrm{MCG}(\Sigma^2)}}_{=:\,\mathcal{G}} \longrightarrow\!\!\!\!\rightarrow \underset{\text{symmetries}}{\overset{\text{asymptotic}}{\mathcal{G}_{\mathrm{bdr}}}} \longrightarrow 1\,. \tag{24}$$

singled out, we consider as **quantum states equipped their gauge symmetries** the restriction of the representation (22) to the bulk symmetries

$$\underset{\text{(only) their gauge symmetries}}{\text{Quantum states equipped with}} \quad \iota^*_{\mathrm{blk}}\mathcal{H}^{\mathcal{A}}_{\Sigma^2} \;\;\in\;\; \mathrm{URep}\big(\mathcal{G}_{\mathrm{blk}}\big) \tag{25}$$

and we observe, in the next Prop. 2.16, that on these the remaining asymptotic boundary cosets are canonically represented as *twisted intertwining operators*:

Definition 2.15 (**Twisted intertwiners).** For G a group, consider its linear representations $V \in \mathrm{Rep}(G)$, to be denoted $g \in G \;\;\vdash\;\; V_* \xrightarrow{V_g} V_*$.

(i) Given a pair $V^1, V^2 \in \mathrm{Rep}(G)$, a *twisted intertwiner* $V^1 \xrightarrow{(\eta,\alpha)} V^2$ between them is

(**a**) a linear map $\eta : V^1_* \to V^2_*$,

(**b**) an automorphism $\alpha \in \mathrm{Aut}(G)$

such that [9]

$$\underset{g \in G}{\forall} \quad \eta \circ \rho_1(g) \;=\; \rho_2\big(\alpha(g)\big) \circ \eta \,. \tag{26}$$

(ii) Given consecutive twisted intertwiners $V^1 \xrightarrow{(\eta,\alpha)} V^2 \xrightarrow{(\eta',\alpha')} V^3$, their composite is simply componentwise:

$$(\eta', \alpha') \circ (\eta, \alpha) \;=\; \big(\eta' \circ \eta,\, \alpha' \circ \alpha\big)\,. \tag{27}$$

(iii) On the other hand, given a pair of parallel twisted intertwiners $(\eta,\alpha), (\eta',\alpha') : V^1 \rightrightarrows V^2$, we say that a *deformation* $a : (\eta,\alpha) \Rightarrow (\eta',\alpha')$ is $a \in G$ such that

$$\begin{array}{ll} \textbf{(a)} & \eta' = \rho_2(a) \circ \eta \\ \textbf{(b)} & \alpha' = \mathrm{Ad}_a \circ \alpha \end{array} \tag{28}$$

(where "Ad" denotes the adjoint action of the group on itself by inner automorphisms, $\mathrm{Ad}_a(g) := a g a^{-1}$).
(iv) Deformation of twisted intertwiners is an equivalence relation compatible with composition, whence we have a category

$$\mathrm{Rep}^{[\mathrm{tw}]}(G) \;\supset\; \mathrm{Rep}(G)$$

whose objects are G-representations and whose morphisms are deformation classes $[-]$ of twisted intertwiners.
(v) Given a G-representation $V \in \mathrm{Rep}(G) \subset \mathrm{Rep}^{[\mathrm{tw}]}(G)$, we write

$$\mathrm{Aut}^{[\mathrm{tw}]}\big(V\big) \tag{29}$$

for its automorphism group in this category, hence for the group of deformations classes of twisted intertwiners from (ρ, V) to itself.

Proposition 2.16 (Asymptotic boundary observables). *Given a space* $\mathcal{H}^{\mathcal{A}}_{\Sigma^2}$ (22) *of topological flux quantum states* (Def. 2.11)*, there is on its restriction* $\iota^*_{\mathrm{blk}} \mathcal{H}^{\mathcal{A}}_{\Sigma^2}$ (25) *to bulk symmetries a canonical action*
– *of the asymptotic boundary symmetries* $\mathcal{G}_{\mathrm{bdr}}$ (24)*,*
– *via deformation classes of twisted intertwiners* (Def. 2.15).

Concretely — and this holds generally for exact sequences of the form (24) —, the boundary action is given by

$$\mathcal{H} \;\in\; \mathrm{Rep}(\mathcal{G}) \qquad\quad \vdash \qquad\quad \begin{array}{ccc} \mathcal{G}_{\mathrm{bdr}} & \longrightarrow & \mathrm{Aut}^{[\mathrm{tw}]}\big(\iota^* \mathcal{H}\big) \\ {[g]} & \longmapsto & \big[\mathcal{H}_g,\, \mathrm{Ad}_g\big]\,, \end{array} \tag{30}$$

where on the right the notation "[tw]" is from (29) and $\mathrm{Ad}_g : \mathcal{G}_{\mathrm{blk}} \to mathcal G_{\mathrm{blk}}$ is the "external" conjugation action of $\mathcal{G}$ on its normal subgroup $\mathcal{G}_{\mathrm{bdr}}$.

Proof. To be transparent, we write the proof in diagrammatic notation, using the (very large) 2-category structure of the category of (large) categories (cf. [136, §XII.3]).

To begin with, for H a group, write $\mathbf{B}H$ for the groupoid with a single object with automorphism group H, and write Vec for the category of vector spaces. It is standard and readily verified that the category of H-representations is equivalently that of functors $\mathbf{B}H \to \mathrm{Vect}$:

$$\mathrm{Rep}(H) \;\simeq\; \mathrm{Func}\big(\mathbf{B}H,\, \mathrm{Vec}\big)\,,$$

so that ordinary intertwiners (twisted intertwiners with $\alpha = \mathrm{id}$, cf. [120, Rem. 4.2]) are naturally identified with natural transformations of the form

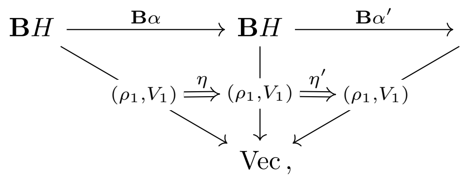
$$\begin{array}{c} \mathbf{B}H \\ {\scriptstyle (\rho_1, V_1)} \Big(\overset{\eta}{\Rightarrow} \Big) {\scriptstyle (\rho_2, V_2)} \\ \mathrm{Vec}\,. \end{array}$$

Along the same lines one finds that general twisted intertwiners (η, α) (26) are identified with natural transformations of this more general form:

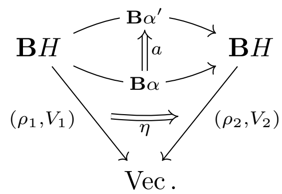
$$\begin{array}{ccc} \mathbf{B}H & \xrightarrow{\;\mathbf{B}\alpha\;} & \mathbf{B}H \\ & {\scriptstyle (\rho_1,V_1)} \searrow \overset{\eta}{\Longrightarrow} \swarrow {\scriptstyle (\rho_2,V_2)} & \\ & \mathrm{Vec}\,, & \end{array} \tag{31}$$

[9] The condition (26) and the terminology "twisted intertwiners" appears in [68, (7.2)][69, (2.2)] (there broadly in a context of 2d coformal field theory), but the concept itself may be older. On the other hand, the concept of deformation of twisted intertwiners in (28) may be new, though it is immediate once one sees the diagrammatic formulation that we give in (31).

that their composition (27) corresponds to the pasting composition of these diagrams



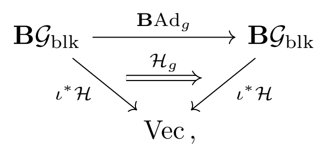
and that their deformations (28) correspond to pasting diagrams of this form:



These diagrams make manifest a 2-category of representations (objects), twisted intertwiners (1-morphisms) and deformations (2-morphisms), and the groups $\mathrm{Aut}^{[\mathrm{tw}]}(V,\rho)$ (29) are equivalently the automorphism groups of the homotopy category of this 2-category.

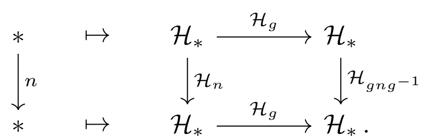
Now in the case at hand, given a normal subgroup inclusion $\mathcal{G}_{\mathrm{blk}} \overset{\iota}{\hookrightarrow} \mathcal{G}$ and $\mathcal{H} \in \mathrm{Rep}(\mathcal{G})$, we observe that

$g \in \mathcal{G} \quad \vdash$



(32)

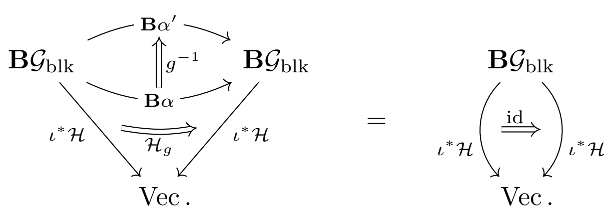
(recall that we write $\mathcal{H}_g : \mathcal{H}_* \to \mathcal{H}_*$ for the given representation operators on the underlying vector space $\mathcal{H}_*$), since the defining commuting squares of this natural transformation commute by the fact that $\mathcal{H}$ actually represents not just $\mathcal{G}_{\mathrm{blk}}$ but all of $\mathcal{G}$ on $\mathcal{H}_*$:



Since the assignment (32) manifestly respects composition, this construction constitutes a group homomorphism from $\mathcal{G}$ into the twisted automorphism 1-group of $\iota^*\mathcal{H}$.

So it remains to check that this construction descends to the quotient by $\mathcal{G}_{\mathrm{blk}}$ on both sides, hence that when $g \in \mathcal{G}_{\mathrm{blk}} \overset{\iota}{\hookrightarrow} \mathcal{G}$ then the above twisted intertwiner is deformable into the identity intertwiner. But such a deformation is evidently given by g^{-1}:

$g \in \mathcal{G}_{\mathrm{blk}} \overset{\iota}{\hookrightarrow} \mathcal{G} \quad \vdash$



This establishes the claimed construction (30). □

Remark 2.17 (Formalizing asymptotic symmetries). In words, our formula (30) for the asymptotic boundary observables says that

– on the *boundary Hilbert space* $\iota^*\mathcal{H}^{\mathcal{A}}_{\Sigma^2}$, which sees all bulk diffeomorphisms as gauge symmetries,

– the asymptotic boundary observables are just the linear operators $\mathcal{H}_g$ that represent diffeomorphisms g,

– except that the action of all purely bulk diffeomorphisms $n \in N \subset G$ is absorbed into the gauge equivariance of the quantum states (accomplished with help of the twisting Ad_g)

This neatly expresses just the kind of statement that is expected for asymptotic symmetries (cf. again [196, §2.10][20]).

Remark 2.18 (Identifying asymptotic symmetries).

(i) Given that we do not start with a Lagrangian density as usual, it just remains to actually specify the normal subgroup $\mathcal{G}_{\mathrm{blk}}$ or equivalently the quotient group $\mathcal{G}_{\mathrm{bdr}}$ over a given surface Σ^2. This specification may have to be regarded as a parameter of the theory.

(ii) However, on surfaces Σ^2 with boundary, the (mapping classes of) *Dehn twist* diffeomorphisms along boundary curves should clearly be asymptotic symmetries *already of the surface* (cf. §4.8).

(iii) Moreover, if we remember, with Def. 2.1, that we are really dealing with quantum states on full 3D space, then Rem. 2.7 suggests that in fact $\mathcal{G}_{\mathrm{blk}} = 1$ in our context, whence all of the covariantized flux monodromy $\mathcal{G}$ (24) should actually be observable in experiment. In this case (at least), our theory predicts non-abelian defect anyons to be observable on punctured disks, see §4.7.

3 Moduli of Topological Flux

We discuss here the effect of the covariantization (22) on the plain moduli spaces of solitonic topological flux. After recalling basics of mapping class groups and braid groups, the main result here is Prop. 3.9 below, which was already announced as (23).

Braid groups and mapping class groups.

Definition 3.1 (**Configuration space and Braid group).**
(i) For Σ a smooth manifold, possibly with boundary, and $n \in \mathbb{N}$, the *configuration space of n points in Σ* is the topological space

$$\mathrm{Conf}_n(\Sigma) \;:=\; \Big\{ (s_1, \cdots, s_n) \,\in\, \Sigma^{\times^n} \,\Big\vert\, \underset{i \neq j}{\forall} s_i \neq s_j \Big\} \Big/ \mathrm{Sym}_n$$

(topologized as the quotient space of a subspace of a product space).
(ii) The fundamental group of this space (assuming now without, substantial restriction, that Σ is connected) is the *braid group on n strands in* Σ (cf. [55, §9]), which as such comes equipped with a forgetful map to the symmetric group:

$$\mathrm{Br}_n(\Sigma) \;:=\; \pi_1 \mathrm{Conf}_n(\Sigma) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Sym}_n\,. \tag{33}$$

Example 3.2 (**Artin presentation of braid groups**, cf. [62, §7][149, Lit. 2.20])**.** For $n \geq 2$, the surface braid group (33) of the disk (the default case of braid groups) has the following finite presentation:

$$\mathrm{Br}_n \;:=\; \mathrm{Br}_m\big(\Sigma^2_{0,1,n}\big) \;\simeq\; F\langle b_1, \cdots, b_{n-1}\rangle \Big/ \Big(\underset{i+1<j}{\forall} \big(b_i b_j = b_j b_j\big)\,,\, \underset{1 \leq i < n-1}{\forall} \underbrace{\big(b_i\, b_{i+1}\, b_i \;=\; b_{i+1}\, b_i\, b_{i+1}\big)}_{\text{Yang-Baxter relation}} \Big)\,, \tag{34}$$

in terms of which its canonical homomorphism to the symmetric group is the quotient map by one further set of relations:

$$\mathrm{Br}_n \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Sym}_n \;:=\; \mathrm{Br}_n \Big/ \Big(\underset{i}{\forall} \big(b_i b_i = \mathrm{e}\big) \Big)\,. \tag{35}$$

The general surface braid group $\mathrm{Br}_n(\Sigma^2)$ may be presented by adjoining to these *Artin generators* b_i further generators (corresponding to moving single strands along cycles in the surface) and further relations. In each case, there is a projection to the symmetric group by retaining the Artin generators:

$$\mathrm{Br}_n(\Sigma^2) \longrightarrow\!\!\!\!\!\rightarrow \mathrm{Sym}_n$$

Example 3.3 (**Presentation of spherical braid group** [54, p 245,55], cf. [199])**.** The surface braid group (33) of the sphere (often: "spherical braid group") is presented as a quotient of the Artin presentation (34) by one further relation:

$$\mathrm{Br}_n(S^2) \;\simeq\; \mathrm{Br}_n \big/ \big((b_1 \cdots b_{n-1})(b_{n-1} \cdots b_1) \big)\,. \tag{36}$$

Definition 3.4 (**Diffeomorphism Group and Mapping Class Group).** For Σ an oriented manifold, possibly with boundary, we write

$$\begin{array}{ccccc} \mathrm{Homeo}^{+,\partial}(\Sigma) & \hookrightarrow & \mathrm{Homeo}(\Sigma) & \hookrightarrow & \mathrm{Map}(\Sigma,\Sigma) \\ \uparrow{\scriptstyle \iota} & & \uparrow{\scriptstyle \iota} & & \\ \mathrm{Diff}^{+,\partial}(\Sigma) & \hookrightarrow & \mathrm{Diff}(\Sigma) & & \end{array} \tag{37}$$

for its topological groups of homeomorphisms and diffeomorphisms, respectively for the further subgroups of maps preserving the orientation (+) and restricting to the identity on the boundary (∂).

For $\Sigma \equiv \Sigma^2$ an orientable surface surface and choosing any one of its orientations, the group of connected components of the latter diffeo group is known as the *mapping class group* [105, §1][148, §3][55, p. 45]:

$$\mathrm{MCG}(\Sigma^2) \;:=\; \pi_0\big(\mathrm{Diff}^{+,\partial}(\Sigma^2)\big)\,. \tag{38}$$

(Ultimately, we are interested in the *spin* mapping class subgroup of diffeomorphisms also preserving a given spin strcture on Σ^2, but we shall make this explicit only where it matters, namely in §4.4, see Prop. 4.34 there.)

Example 3.5 (**Mapping class groups of closed oriented surfaces**, cf. [148, §6][55, §6])**.** The mapping class group of the torus is

$$\mathrm{MCG}(\Sigma^2_1) \;\simeq\; \mathrm{Sp}_2(\mathbb{Z}) \;\simeq\; \mathrm{SL}_2(\mathbb{Z})\,, \tag{39}$$

which is generated by the two elements [189, Thm VII.2 p 78][32, Thm 1.1]

$$S := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\,, \qquad T := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{40}$$

and presented subject to the following relations [126, p. 126][17, §2.1]:

$$\mathrm{SL}_2(\mathbb{Z}) \simeq \big\langle S,\, T \,\big|\, S^4 = (TS)^3 = \mathrm{e},\, S^2\,(TS) = (TS)\,S^2 \big\rangle\,. \tag{41}$$

More generally, the mapping class group of Σ^2_g (9), for $g \in \mathbb{N}$, sits in a short exact sequence (cf. [55, §6])

$$1 \longrightarrow \underset{\text{Torelli group}}{I_g} \hookrightarrow \underset{\text{mapping class group}}{\mathrm{MCG}(\Sigma^2_g)} \twoheadrightarrow \underset{\text{symplectic group}}{\mathrm{Sp}_2(\mathbb{Z})} \longrightarrow 1\,, \tag{42}$$

where the (pre-)composition action of $\mathrm{MCG}(\Sigma^2_1)$ on

$$H_1(\Sigma^2_g;\mathbb{Z}) \simeq H^1(\Sigma^2_g;\mathbb{Z}) \simeq \mathbb{Z}^g \times \mathbb{Z}^g$$

is through the defining action of the integer symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$.

Through this action the modular groups act also on the set of spin structures $H_1(\Sigma^g;\mathbb{Z}_2)$ (cf. [143, p 199]). Concretely, on the torus there are thus 4 distinct spin structures, say

$$\{\mathrm{pp}, \mathrm{aa}, \mathrm{ap}, \mathrm{pa}\} :\simeq \mathbb{Z}_2^2 \simeq H_1\big(\Sigma^2_1;\, \mathbb{Z}_2\big) \tag{43}$$

(with "periodic" or "antiperiodic" boundary conditions for spinors along the two basis 1-cycles, cf. [4, §2]) and $\mathrm{MCG}(\Sigma^2_1) \simeq \mathrm{SL}_2(\mathbb{Z})$ (39) preserves pp and transitively permutes among the other three. On the other hand, the stabilizer subgroup of the aa structure, hence the *spin mapping class group* of diffemorphisms preserving aa, is generated by S and the *square* of T (cf. [19, p 3]) subject to the following relations:

$$\begin{array}{ccc} \mathrm{MCG}(\Sigma^2_1)^{\mathrm{aa}} & \hookrightarrow & \mathrm{MCG}(\Sigma^2_1) \\ \wr\| & & \wr\| \\ \big\langle S, T^2 \,|\, S^4 = [S^2, T^2] = \mathrm{e}\big\rangle & \hookrightarrow & \mathrm{SL}_2(\mathbb{Z})\,. \end{array} \tag{44}$$

Definition 3.6 (Moduli spaces of solitonic topological fluxes). The underlying homeomorphisms of the diffeomorphisms (37) of surfaces $\Sigma^2_{g,b,n}$ (9) extend functorially to the one-point compactification (by Prop. A.2) to make a topological group homomorphism

$$\mathrm{Diff}^{(+,\partial)}\big(\Sigma^2_{g,b,n}\big) \xrightarrow{\ \iota\ } \mathrm{Homeo}^{(+,\partial)}\big(\Sigma^2_{g,b,n}\big) \xrightarrow{(-)_{\cup\{\infty\}}} \mathrm{Aut}_{\mathrm{Top}^*}\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}}\big)\,.$$

Via the latter's action (by pre-composition) on pointed mapping spaces (21) we obtain the homotopy quotient (139) of the pointed mapping space [10]

$$\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, \mathcal{A}\big) /\!\!/ \mathrm{Diff}^{+,\partial}\big(\Sigma^2_{g,b,n}\big) \in \mathrm{Top}^*\,, \tag{45}$$

identified in (22) as the *covariantized moduli space* of $\mathcal{A}$-quantized solitonic topological fluxes on $\Sigma^2_{g,b,n}$.

The following is classical but somewhat scattered in the literature:

Proposition 3.7 (Homotopy type of Diffeomorphism groups).

(i) *For compact oriented surfaces $\Sigma^2_{g,b}$ (9), the homotopy type of their diffeomorphism group (37) is:*

$$\begin{array}{llll} \mathrm{Diff}^+\big(\Sigma^2_{0,0,0}\big) \simeq_{\int} \mathrm{SO}(3) & \Rightarrow & \mathrm{MCG}(\Sigma^2_{0,0,0}) \simeq 1 & \text{and} \quad \pi_1\,\mathrm{Diff}^+\big(\Sigma^2_{0,0,0}\big) \simeq \mathbb{Z}_2 \\ \mathrm{Diff}^+\big(\Sigma^2_{1,0,0}\big) \simeq_{\int} \mathrm{SL}_2(\mathbb{Z}) \times T^2 & \Rightarrow & \mathrm{MCG}(\Sigma^2_{1,0,0}) \simeq \mathrm{SL}_2(\mathbb{Z}) & \text{and} \quad \pi_1\,\mathrm{Diff}^+\big(\Sigma^2_{1,0,0}\big) \simeq \mathbb{Z}\times\mathbb{Z} \\ \mathrm{Diff}^+\big(\Sigma^2_{g\geq 2,0,0}\big) \simeq_{\int} * & \Rightarrow & \mathrm{MCG}(\Sigma^2_{g\geq 2,0,0}) \simeq 1 & \text{and} \quad \pi_1\,\mathrm{Diff}^+\big(\Sigma^2_{g\geq 2,0,0}\big) \simeq 1 \\ & & & \\ \mathrm{Diff}^{+,\partial}\big(\Sigma^2_{g,b\geq 1,0}\big) \simeq_{\int} * & \Rightarrow & \mathrm{MCG}(\Sigma^2_{g,b\geq 1,0}) \simeq 1 & \text{and} \quad \pi_1\,\mathrm{Diff}^{+,\partial}\big(\Sigma^2_{g,b\geq 1,0}\big) \simeq 1\,. \end{array} \tag{46}$$

(ii) *For punctured oriented surfaces $\Sigma^2_{g,b,\geq 1}$, the map from their mapping class group to that of $\Sigma^2_{g,b}$* (by uniquely extending the diffeomorphisms to the punctures) *sits in a long exact sequence* ("generalized Birman sequence") *with the surface's braid group (33), of this form:*

$$\pi_1\,\mathrm{Diff}^{+,\partial}\big(\Sigma^2_{g,b}\big) \longrightarrow \mathrm{Br}_n\big(\Sigma^2_{g,b}\big) \longrightarrow \mathrm{MCG}\big(\Sigma^2_{g,b,n}\big) \longrightarrow \mathrm{MCG}\big(\Sigma^2_{g,b}\big)\,. \tag{47}$$

(a) *Hence when $\pi_1\mathrm{Diff}^{+,\partial}(\Sigma^2_{g,b}) = 1$ the mapping class group sits in a short exact sequence of the form*

$$1 \to \mathrm{Br}_{n\geq 1}\big(\Sigma^2_{g,b}\big) \longrightarrow \mathrm{MCG}(\Sigma^2_{g,b,n\geq 1}) \longrightarrow \mathrm{MCG}(\Sigma^2_{g,b}) \to 1\,, \tag{48}$$

[10] The connected components of the full mapping space $\pi_0(\mathcal{F}) \equiv \pi_0\Big(\mathrm{Map}^*\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, S^2\big)\Big) \simeq \mathbb{Z}$ are given by the Hopf degree (Def. 4.4). Since diffeomorphisms preserve Hopf degree, their precomposition preserves the connected components of the mapping space.

and exhausts the homotopy type of the diffeomorphism group:

$$\mathrm{Diff}^{+,\partial}\big(\Sigma^2_{g,b,n\geq 1}\big) \;\simeq_{\int}\; \mathrm{MCG}\big(\Sigma^2_{g,b,n\geq 1}\big) \qquad \Rightarrow \qquad \pi_1\,\mathrm{Diff}^{+,\partial}\big(\Sigma^2_{g,b,n\geq 1}\big) \;\simeq\; 1\,. \tag{49}$$

(b) *For $g = 0$, $b = 0$ —* where the assumption in (iia) fails by (46) *— the ("spherical") braid group still surjects onto the mapping class group, but with non-trivial kernel $\pi_1\mathrm{Diff}^+(\Sigma^2_0) \simeq \mathbb{Z}_2$* (generated by the "full rotation" braid)

$$1 \longrightarrow \mathbb{Z}_2 \xhookrightarrow{\;\mathrm{rot}\;} \mathrm{Br}_{n\geq 1}(\Sigma^2_0) \longrightarrow\!\!\!\!\rightarrow \mathrm{MCG}(\Sigma^2_{0,0,n}) \longrightarrow 1\,. \tag{50}$$

(c) *Concretely, for $g = b = 0$ we have for the first few n:*

$$\begin{array}{lcl} \mathrm{MGC}(\Sigma^2_{0,1}) & \simeq & 1 \simeq \mathrm{Br}_1(S^2) \\ \mathrm{MGC}(\Sigma^2_{0,2}) & \simeq & \mathbb{Z}_2 \simeq \mathrm{Br}_2(S^2) \\ \mathrm{MCG}(\Sigma^2_{0,3}) & \simeq & \mathrm{Sym}_3 \not\simeq \mathrm{Br}_3(S^2) \\ \mathrm{MCG}(\Sigma^2_{0,4}) & \simeq & \mathrm{PSL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)\,. \end{array} \tag{51}$$

Proof. In (46), the first statement is due to [191], the first three were proven by [48][49][88], and the fourth is [50, Thm. 1D p 170]. The statement (49) follows with [217][218, Thm. 1.1]. [11] The generalized Birman sequence (47) is named in honor of [16], cf. [138, Thm. 3.13]. In its implication of the short exact sequence (48) this is reviewed in [55, Thm 9.1]. The spherical braid group extension (50) is discussed in [55, (9.1)] and the identifications (51) of its quotients are proven, for instance, in [55, Prop. 2.3][18]. □

Example 3.8 (**Mapping class groups of n-punctured disk).** Since the mapping class group of the disk $\Sigma^2_{0,1,0}$ is trivial by (46), the exact sequence (48) shows that the mapping class group of its punctured versions is the plain braid group (34):

$$\mathrm{MCG}(\Sigma^2_{0,1,n}) \;\;\simeq\;\; \mathrm{Br}_n\,. \tag{52}$$

Covariant flux monodromy. With all this in hand, we come to the main statement of this section, announced as (23).

Proposition 3.9 (**Extension of mapping class group by flux monodromy).** *For every $\Sigma^2_{g,b,n}$ (9) we have a split short exact sequence of groups*

$$1 \longrightarrow \pi_1\Big(\underbrace{\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, \mathcal{A}\big)}_{\text{moduli space}}\Big) \longrightarrow \pi_1\Big(\underbrace{\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, \mathcal{A}\big) /\!\!/ \mathrm{Diff}^{+,\partial}(\Sigma^2_{g,b,n})}_{\text{covariantized moduli space (45)}}\Big) \longrightarrow \underbrace{\mathrm{MCG}(\Sigma^2_{g,b,n})}_{\text{mapping class group (38)}} \longrightarrow 1\,,$$

exhibiting an action of the mapping class group on the fundamental group of the moduli space, so that we have the corresponding semidirect product:

$$\pi_1\Big(\underbrace{\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, \mathcal{A}\big) /\!\!/ \mathrm{Diff}^{+,\partial}(\Sigma^2_{g,b,n})}_{\text{covariantized moduli space (45)}}\Big) \;\simeq\; \underbrace{\mathrm{MCG}(\Sigma^2_{g,b,n})}_{\text{mapping class group (38)}} \ltimes \pi_1\Big(\underbrace{\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, \mathcal{A}\big)}_{\text{moduli space}}\Big)\,. \tag{53}$$

Proof. For notational convenience, we abbreviate

$$\begin{aligned} \mathcal{F} &:= \mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, S^2\big) \\ \mathcal{D} &:= \mathrm{Diff}^{+,\partial}\big(\Sigma^2_{g,b,n}\big)\,, \end{aligned}$$

whence the claim to be proven is split exactness of

$$1 \longrightarrow \pi_1(\mathcal{F}) \longrightarrow \pi_1(\mathcal{F} /\!\!/ \mathcal{D}) \longrightarrow \pi_0(\mathcal{D}) \longrightarrow 1\,. \tag{54}$$

To this end, the Borel homotopy fiber sequence (142)

$$\mathcal{F} \longrightarrow \mathcal{F} /\!\!/ \mathcal{D} \longrightarrow * /\!\!/ \mathcal{D}$$

(split by picking the zero-map) induces a long exact sequence of homotopy groups (138) of this form:

[11] The surfaces in [217][218] are assumed without boundary, but equipped with marked closed subcomplexes to be fixed by the diffeomorphisms. Under this definition, a puncture surrounded by a marked circle behaves just as a boundary for the purpose of computing the homotopy type of the diffeomorphism group.

$$
\begin{array}{l}
\hspace{8em}\overset{\text{by}(141)}{\pi_1(\mathcal{D})} \\
\hookrightarrow \pi_1(\mathcal{F}) \longrightarrow \pi_1\big(\mathcal{F} /\!\!/ \mathcal{D}\big) \longrightarrow \pi_0(\mathcal{D}) \\
\hookrightarrow \pi_0(\mathcal{F}) \xrightarrow{\sim} \pi_0(\mathcal{F} /\!\!/ \mathcal{D})\,.
\end{array}
\tag{55}
$$

Here the last map shown is an isomorphism by (143) (cf. footnote 10), whence the exact sequence truncates to

$$
\pi_1(\mathcal{D}) \longrightarrow \pi_1(\mathcal{F}) \longrightarrow \pi_1\big(\mathcal{F} /\!\!/ \mathcal{D}\big) \longrightarrow \pi_0(\mathcal{D}) \longrightarrow 1\,.
$$

If, at this point, we invoke Prop. 3.7 then the claim (54) follows for most surfaces, namely those for which $\pi_1(\mathcal{D}) \simeq 1$. But in fact, the claim follows generally by observing that the first connecting map in (55) factors through the trivial group:

$$
\pi_1(\mathcal{D}) \;\equiv\; \pi_1\big(\mathrm{Diff}^+(\Sigma^2_{g,b,n})\big) \xrightarrow{\quad} \pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\,\mathcal{A}\big)\Big) \;\equiv\; \pi_1(\mathcal{F} /\!\!/ \mathcal{F})\,.
$$
$$
\pi_1\big(\mathrm{Diff}^+(\Sigma^2_{g,b,n})\big) \longrightarrow 1 \longrightarrow \pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\,\mathcal{A}\big)\Big)
$$

Namely, by (142), the map is given by taking a given loop of diffeomorphisms to the loop of maps obtained by composing these diffeos the constant map $\Sigma^2_{g,b,n} \to S^2$ – but that gives the constant loop representing the neutral element of π_1. □

This Proposition 3.9 is our main tool for analyzing the covariantized topological quantum states on $\mathcal{A}$-quantized flux according to (22). In the next section, we specify $\mathcal{A}$ to S^2 and work out the consequences.

4 Flux quantized in 2-Cohomotopy

We now specify the classifying space $\mathcal{A}$ (22) to the 2-sphere, $\mathcal{A} \equiv S^2$ (so that flux is classified by the non-abelian cohomology theory called *2-Cohomotopy*, cf. [192][101, §VII][59, Ex. 2.7]) and work out the resulting covariant topological quantum observables on and quantum states of (according to §2) 2-cohomotopically quantized flux through various surfaces Σ^2, using the results of §3.

Remarkably, in the case of $\Sigma^2 \equiv S^2, T^2$ the sphere or the torus, we find reproduced (in §4.1 and §4.4, respectively) the situation traditionally argued via quantized U(1)-Chern-Simons theory over these surfaces, including fine-print such as regularization of Wilson-loop observables by framings, modular equivariance and refinement to "spin" Chern-Simons theory.

Then, by instead choosing punctured surfaces, we similarly work out the 2-Cohomotopically quantized flux through the punctured sphere (§4.5) and the punctured annulus (§4.8).

Remark 4.1 (**Generalized higher symmetry group of 2-cohomotopical flux).** In view of Rem. 2.10, the choice of classifying space $\mathcal{A} \equiv S^2 \simeq_{\int} B(\Omega S^2)$ corresponds to considering as (homotopy type of the) gauge group the loop group ΩS^2 of the 2-sphere (under concatenation and reversal of loops) which is a "higher group" ("∞-group") exhibiting "generalized symmetry". See also ftn. 12 below.

The looping of the canonical comparison map $1^2 : S^2 \to B^2\mathbb{Z}$ exhibits this generalized symmetry group as a deformation of (the homotopy type of) the standard electromagnetical gauge group U(1):

$$\Omega S^2 \xrightarrow{\Omega 1^2} \Omega B^2\mathbb{Z} \simeq_{\int} B\mathbb{Z} \simeq_{\int} \mathrm{U}(1)\,.$$

In particular, while $\pi_2(\mathrm{U}(1)) \simeq 0$, our deformation on the left has a non-trivial contribution

$$\pi_2\big(\Omega S^2\big) \underset{(124)}{\simeq} \pi_3(S^2) \simeq \mathbb{Z}$$

generated by the Hopf fibration. We already remarked after (6) that this is the homotopical avatar of the Chern-Simons form, and we will see now that it is also the origin of the appearance of anyonic braiding phases of flux solitons quantized in 2-Cohomotopy: This is seen in Prop. 4.7, Prop. 4.15 and Prop. 4.19 below.

4.1 On the plane

We recall here (from [177]) how solitonic flux through the plane $\mathbb{R}^2 \simeq \Sigma^2_{0,0,1}$ (10) quantized in 2-cohomotopy reproduces the Wilson loop link observables of anyonic braiding as predicted by abelian Chern-Simons theory (Rem. 4.11 below).

But to start with, we briefy recall the Pontrjagin construction that serves for us to relate cohomotopy to solitonic flux density.

2-Cohomotopical flux solitons via the Pontrjagin construction. Among generalized non-abelian cohomology theories, (unstable) *Cohomotopy* π^n (cf. [159][192][101, §VII][59, Ex. 2.7]), whose classifying spaces are the n-spheres $S^n \simeq \mathbb{R}^n_{\cup\{\infty\}}$ (128),

$$\widetilde{\pi}^n(-) := \pi_0\,\mathrm{Map}^*\big(-, \mathbb{R}^n_{\cup\{\infty\}}\big)\,,$$

stands out in that it accurately characterizes the solitonic flux configurations of given charge [171][177] — this may be understood as the content of the original unstable Pontrjagin theorem (which these days is more famous as the *Pontrjagin-Thom theorem* pertaining only to the *stable* case which is of little concern to us here):

Proposition 4.2 (**Pontrjagin theorem – Cohomotopy charge**, cf. [23, §II.16][129, §IX]). *Given a smooth d-manifold Σ^d and $n \in \mathbb{N}$ with $n \leq d$, there is a natural bijection between:*

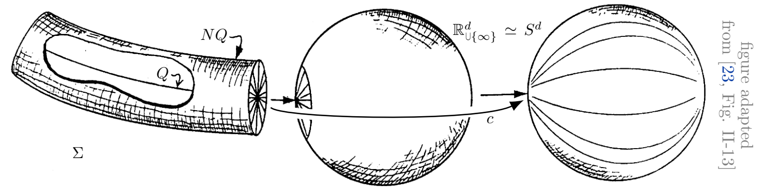
1. *the reduced n-Cohomotopy of the one-point compactification $\Sigma^d_{\cup\{\infty\}}$,*
2. *the cobordism classes of normally framed submanifolds $Q^{d-n} \hookrightarrow \Sigma^d$ of co-dimension=n*

$$\underset{\text{of 1pt compactification}}{\text{Reduced } n\text{-Cohomotopy}}\quad \widetilde{\pi}^n\big(\Sigma^d_{\cup\{\infty\}}\big) \underset{\text{asymptotitic directed distance}}{\overset{\text{regular pre-image of } 0}{\rightleftarrows}} \mathrm{Cob}^n_{\mathrm{Fr}}\big(\Sigma^d\big)\quad \underset{\text{manifolds of codim}=n}{\overset{\text{Cobordism classes of}}{\text{normally framed sub-}}}$$

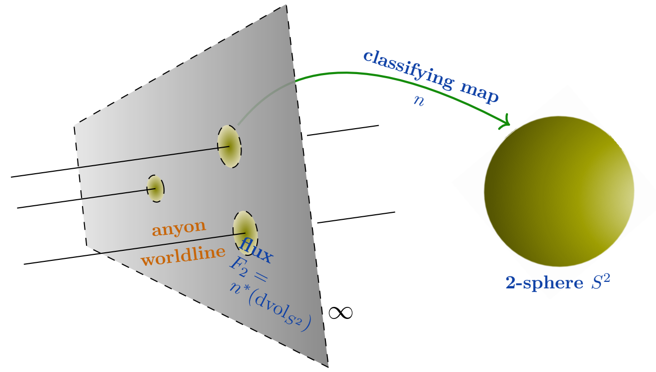
where the Cohomotopy charge $[c] \in \widetilde{\pi^n}\big(\Sigma^d\big)$ *of a submanifold $Q^{d-n} \subset \Sigma^d$ with normal framing $NQ \underset{\sim}{\xrightarrow{\mathrm{fr}}} N \times \mathbb{R}^n \xrightarrow{p} \mathbb{R}^n$*

is represented for any choice of tubular neighborhood $NQ \overset{\iota}{\hookrightarrow} \Sigma$ by the "scanning map"

$$
\begin{array}{ccc}
\Sigma^d & \xrightarrow{\quad c \quad} & \mathbb{R}^n_{\cup\{\infty\}} \\
s & \longmapsto & \begin{cases} p\big(\mathrm{fr}(s)\big) & \mid \; s \in \iota(NQ) \\ \infty & \mid \; \text{otherwise}\,. \end{cases}
\end{array}
$$



Example 4.3 (**2-Cohomotopy of surfaces**)**.** In our situation of surfaces with flux quantized in 2-Cohomotopy, we have $d = n = 2$ in Prop. 4.2, whence the Pontragin theorem identifies solitonic flux concentrated around *points* (soliton cores) with oriented tubular neighbourhoods reflecting either positive or negative units of (magnetic) flux. The *total flux* is the sum of the charges of these solitons (the orientations $\in \{\pm 1\}$ of their tubular neighbourhoods), identified with the Hopf degree (Def. 4.4) of the classifying map.



Definition 4.4 (**Hopf degree**, cf. [129, §IX, Cor 5.8])**.** For $n \in \mathbb{N}$, and

$$S^n \xrightarrow{\;1^n\;} B^n\mathbb{Z} \tag{56}$$

a map representing the generator $1 \in \mathbb{Z} \simeq \pi_n(B^n\mathbb{Z})$, the induced generalized cohomology operation from n-Cohomotopy to ordinary integral n-cohomology

$$\pi^n(X) \;\equiv\; \pi_0\,\mathrm{Map}\big(X,\, S^n\big) \xrightarrow{\;\pi_0(1^n)_*\;} \pi_0\,\mathrm{Map}\big(X,\, B^n\mathbb{Z}\big) \;\simeq\; H^n(X;\, B^n\mathbb{Z}) \tag{57}$$

is a bijection when X is an orientable manifold of dimension n, in which case the operation takes values in integers (generated by the fundamental class of X), called the *Hopf degree* of the maps $X \to S^n$ on the left.

On the other hand, the flux density underlying (sourced by) a given Cohomotopy charge is characterized by the cohomotopical character map (the cohomotopical analog of the Chern-character map on K-cohomology, [59][178]):

Definition 4.5 (**Cohomotopical character map**)**.** For $n = d$ the *character map* on cohomotopy

$$
\begin{array}{ccc}
\widetilde{\pi}^d\big(\Sigma^d\big) & \xrightarrow{\;\mathrm{ch}\;} & H^d_{\mathrm{dR}}\big(\Sigma^d\big)_{\mathrm{cpt}} \\
{[c]} & \longmapsto & \big[c^*\mathrm{vlm}_n\big]
\end{array}
\tag{58}
$$

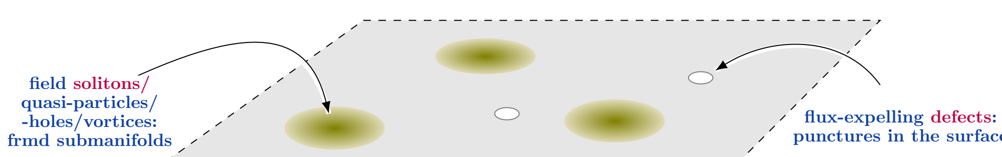
takes $[c] \in \widetilde{\pi}^n\big(\Sigma^d_{\cup\{\infty\}}\big)$ — for any representative $c : \Sigma^d_{cpt} \to \mathbb{R}^d_{\cup\{\infty\}}$ which is smooth on $c^{-1}(\mathbb{R}^d)$, such as the scanning maps (58) — to the class in compactly supported de Rham cohomology of the pullback of a d-form $\mathrm{vlm} \in \Omega^n_{\mathrm{dR}}(\mathbb{R}^d)$ compactly supported on a neighborhood of $0 \in \mathbb{R}^d$ and of unit integral.

Remark 4.6 (**Flux density quantized in Cohomotopy**)**.**

(i) In combination, this means that Cohomotopy charge $[c] \in \mathbb{Z} \simeq \widetilde{\pi}^d\big(\Sigma^d_{\cup\{\infty\}}\big) \equiv \pi_0\mathrm{Map}^*\big(\Sigma^d_{\cup\{\infty\}},\, S^d\big)$ may be understood as sourcing a solitonic flux density $F_d \in \Omega^d_{\mathrm{dR}}\big(\Sigma^d\big)$ (solitonic in that it vanishes at infinity) which is supported with unit weight near $n_+ \in \mathbb{N}$ points in Σ^d (all points outside each other's supporting neighborhoods) and with a negative unit weight near $n_- \in \mathbb{N}$ points (anti-solitons) such that $[c] = n_+ - n_-$.

(ii) For the case $d = 2$ of interest here, this is just the kind of magnetic flux distribution concentrated around solitonic vortex cores as seen in type II superconducting and in fractional quantum Hall semiconducting materials Σ^2, while any punctures in the surface Σ^2 (9) behave as loci where flux is expelled from, as for type I superconducting materials:

Figure F. Via the Pontrjagin theorem, 2-cohomotopical quantization of flux through a surface exhibits N *flux quanta* as a concentration of flux density supported on the tubular neighborhoods of N disjoint points.



2-Cohomotopical flux monodromy. For the quantum flux observables (21), we need not just the connected components π_0 but the fundamental group π_1 of the moduli space of Cohomotopical flux, which we may understand as "*2-Cohomotopy in negative degree 1*", classified by the loop space ΩS^2 of the 2-sphere: [12]

$$\pi_1 \operatorname{Map}^*\big(-,\, S^2\big) \underset{(135)}{\simeq} \pi_0 \operatorname{Map}^*\big(-,\, \Omega S^2\big)\,. \tag{59}$$

Just like the 2-sphere has a canonical comparison map $1^2 : S^2 \to B^2\mathbb{Z}$ (56) whose induced cohomology operation [59, Def. 2.3] extracts ordinary 2-cohomology classes from 2-cohomotopy charges

$$\widetilde{\pi}^2(-) \;\simeq\; \pi_0 \operatorname{Map}^*\big(-,\, S^2\big) \xrightarrow{\;(1^2)_*\;} \pi_0 \operatorname{Map}^*\big(-,\, B^2\mathbb{Z}\big) \;\simeq\; \widetilde{H}^2(-;\,\mathbb{Z})$$

so its loop space has the looped comparison map $\Omega 1^2 : \Omega S^2 \to \Omega B^2\mathbb{Z} \simeq_{\int} B\mathbb{Z}$ inducing the cohomology operation

$$\pi_0 \operatorname{Map}^*\big(-,\, \Omega S^2\big) \xrightarrow{\;(\Omega 1^2)_*\;} \pi_0 \operatorname{Map}^*\big(-,\, B\mathbb{Z}\big) \;\simeq\; \widetilde{H}^1(-;\,\mathbb{Z})$$

which makes precise how 2-cohomotopical flux observables refine ordinary electromagnetic flux observables (16).

It is now immediate to compute the observables on covariantized 2-Cohomotopical solitonic flux on the plane, and there turns out to be essentially a single such observable (Prop. 4.7 below), to be denoted $\widehat{\zeta}$ — but it will take us the better part of the remainder of this section to identify this observable with the braiding phase (2).

Proposition 4.7 (2-Cohomotopical flux monodromy on the plane). *The spaces $\mathcal{H}$ of topological quantum states (22) of solitonic flux quantized in 2-Cohomotopy on the plane, are representations of the group of integers:*

$$\pi_1\Big(\operatorname{Map}^*_0\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big) \,/\!\!/\, \mathrm{Diff}^+(\mathbb{R}^2)\Big) \;\simeq\; \mathbb{Z}\,, \tag{60}$$

hence defined by a single unitary operator $\widehat{\zeta}$:

$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathrm{U}(\mathcal{H}) \\ n &\longmapsto \big(\widehat{\zeta}\big)^n\,. \end{aligned} \tag{61}$$

Proof. The mapping class group of the plane is trivial (Prop. 3.7), so that by Prop. 3.9 the only contribution is from the flux monodromy group itself, which is readily found to be

$$\begin{aligned} \pi_1 \operatorname{Map}^*_0\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big) &\simeq \pi_1 \operatorname{Map}^*_0\big(S^2,\, S^2\big) && \text{by (128)} \\ &\simeq \pi_0 \operatorname{Map}^*_0\big(S^3,\, S^2\big) && \text{by (124)} \\ &\simeq \pi_3(S^2) \\ &\simeq \mathbb{Z}\,, \end{aligned}$$

identifying the observable $\widehat{\zeta}$ with the representation image of the flux monodromy which is classified by the *Hopf fibration.* □

Below in §4.3 we see this same observable $\widehat{\zeta}$ appearing on any closed oriented surface, and further below in §4.4 we prove that on the torus it is identified with the operator of multiplication by a root of unity, $\zeta = e^{\pi \mathrm{i} \frac{p}{K}}$, for $\gcd(p, K) = 1$, as expected for FQH braid phases (2).

However, in the remainder of this subsection here, we work out what the observable $\widehat{\zeta}$ actually observes about 2-cohomotopical flux, and show that these indeed are braiding processes of flux quanta.

Understanding solitonic flux processes. In view of the Pontrjagin construction, we are to regard $\operatorname{Map}^*\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big)$ as the (moduli) space of solitonic flux on the plane, quantized in 2-Cohomotopy, and hence of its loop space $\Omega \operatorname{Map}^*_0\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big)$ — where loops begin and end on the constant map, cf. (131), representing the *flux vacuum* — as the space of "*vacuum scattering processes*", where flux solitons (of positive charge) and anti-solitons (of negative charge) pairwise emerge out of the vacuum, move around, and finally pair-annihilate back into the vacuum.

Of these vacuum processes, our observables (60) detect their homotopy classes $[-]$ (hence their "topological" or "deformation" class in physics jargon), labeled by the integers:

$$\underset{\text{solitonic flux on the plane}}{\overset{\text{space of vacuum processes of}}{}} \quad \Omega \operatorname{Map}^*_0\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big) \xrightarrow{\;[-]\;} \pi_1 \operatorname{Map}^*\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big) \;\simeq\; \mathbb{Z} \quad \underset{\text{classes of these process}}{\overset{\text{topological deformation}}{}}$$

[12] The loop space ΩS^2 (59) of the 2-sphere has received attention as a classifying space also in [145, Def. 1.1], there called the classifying space for "line bundles" (with a Polish "Ɨ"), and related to configuration space in [145], though these discussions seem to remain sketchy. The homotopy groups of ΩS^2 are recognzied as natural sub-quotients of braid groups in [31, p 94], which is a tantalizing observation in our context, whose further relevance however remains unclear to us at this point.

Our task is hence to understand these processes, on the left, and how they are observed, on the right.

To this end, the first step is to better understand the moduli space $\mathrm{Map}^*_0\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big)$ itself:

2-Cohomotopical moduli via the Segal-Okuyama theorem. By the Pontrjagin theorem (Prop. 4.2), one might naïvely expect that $\mathrm{Map}^*\big((\mathbb{R}^2)_{\cup\{\infty\}}, S^2\big)$ is the *configuration space* (cf. [30][115]) of *signed* points (hence of $\pm$ unit charged soliton cores) in the plane, topologized such that continuous curves in the space reflect creation/annihilation of oppositely charged pairs as illustrated on the left of Fig. P. However, this is not quite correct as it misses the normal framing carried also by the cobordisms, according to Pontrjagin's theorem 4.2.

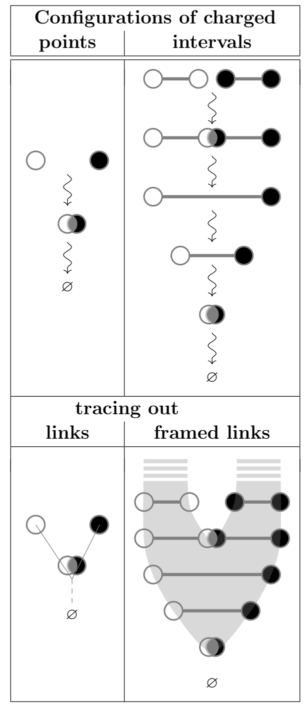
A correct model [155] is by configurations of *intervals with signed endpoints* (stringy solitons between unit charged "quarks") all parallel to one coordinate axis and topologized such as to reflect creation/annihilation of oppositely charged pairs of endpoints.

Figure P – Solitonic flux processes and framed links. Indicated on the left is the naïve process of pair annihiliation of oppositely signed points (of oppositely charged soliton core locations). But the configuration space of signed points [139, p 94] which is topologized to make these processes be continuous paths (with their reverse paths modelling the dual pair-creation processes) turns out [139, p 6] to *not quite* have the correct homotopy type of the 2-Cohomotopical flux moduli space $\mathrm{Map}^*\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big)$.

Indicated on the right is a variant situation where the previous charges are located at the endpoints of intervals, and where their pair annihilation makes the corresponding intervals merge.
The configuration space of such intervals (all parallel to one fixed coordinate axis) topologized to make these processes be continuous paths (cf. [155, Def. 3.1-2]) *does* have the homotopy type of the 2-Cohomotopical flux moduli space! This follows with the result of [155, Thm 1.1].

The upshot is that where a continuous loop in the space on the left is an oriented link, a loop on the space on the right right is an oriented link that is also equipped with a *framing*: a *framed oriented link*.

In traditional Chern-Simons theory, the enhancement of links to framed links is a standard but *ad hoc* way to "regularize" the corresponding *Wilson loop observable*, which at face value actually diverges according to traditional quantization methods. Here with 2-Cohomotopical flux quantization, this framing correction is automatically arises from the moduli space of solitonic flux quantized in 2-Cohomotopy.



Under this identification of (the homotopy type of) our flux moduli space $\mathrm{Map}^*\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big)$ with a configuration space of charged intervals in $\mathbb{R}^2$ parallel to a fixed axis, loops in this space are identified with *framed oriented link diagrams*, or *framed links*, for short — and loop homotopies are identified with "link cobordism":

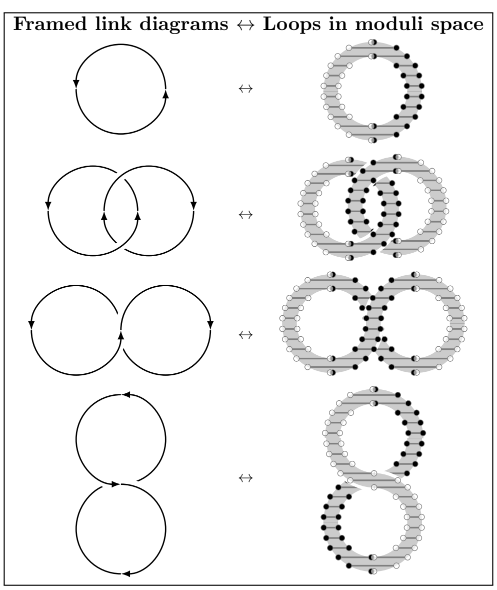
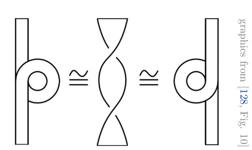
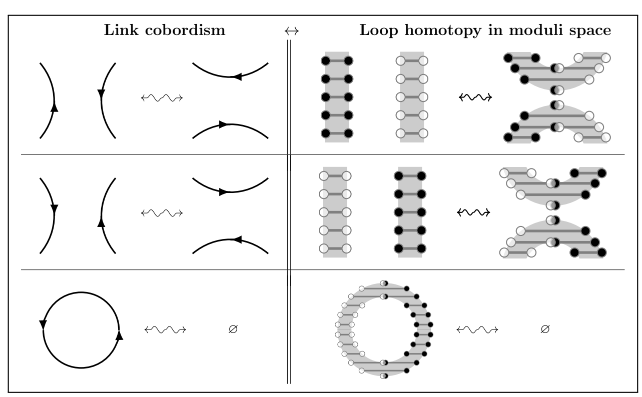


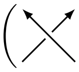
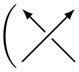
Figure FL. A framed link is a ribbon link, hence a "worldsheet of intervals". When flattened out on a plane ("blackboard farming") the inner twisting of the ribbon is entirely reflected in its self-crossings and thus the interval's extension may be disregarded again.





Definition 4.8 (**Crossing-, Linking- and Framing numbers**).

(i) Any crossing in a framed oriented link diagram L loclly is either of the following, which we assign the *crossing number* ± 1, respectively, as shown:

$\#\Big($  $\Big) \;=\; +1\,,$ $\qquad \#\Big($  $\Big) \;=\; -1\,.$ (62)

(ii) For $(L_i)_{i=1}^N$ the connected components of L, the *linking number* $\mathrm{lnk}(L_i, L_j)$ is *half* the sum of crossing numbers between L_i and L_j (cf. [Oh1, p. 7]).

(iii) The *framing number* $\mathrm{fr}(L_i)$ is the sum of crossing numbers of L_i with itself.

(iv) The sum $\#L$ of the crossing numbers of all crossings of L is hence the sum of all the framing and linking numbers:

$$\#(L) \;:=\; \sum_{\substack{c\,\in \\ \mathrm{crssngs}(L)}} \#(c) \;=\; \sum_i \mathrm{frm}(L_i) + \sum_{i,j} \mathrm{lnk}(L_i, L_j)\,. \tag{63}$$

This has the following effect.

Proposition 4.9 (**Vacuum loops of 2-cohomotopical flux through the plane** [177]).

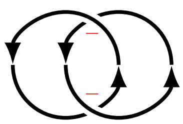
(i) *Loops of 2-cohomotopical flux moduli on the plane are identified with framed links topologized to reflect link cobordism, whence their homotopy class is identified with the framed link's total crossing number:*

$$\begin{array}{ccc} \Omega\,\mathrm{Map}^*_0\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big) & \xrightarrow{\;\;[-]\;\;} & \pi_1\,\mathrm{Map}^*_0\big(\mathbb{R}^2_{\cup\{\infty\}},\, S^2\big) \;\simeq\; \mathbb{Z}\,. \\ L & \longmapsto & \#L \\ \textbf{framed link} & & \textbf{total crossing number} \end{array}$$

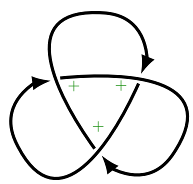
(ii) *Moreover, the pure states on these observables are labeled by $\zeta \in \mathrm{U}(1)$ and give the expectation values*

$$\langle \zeta | L | \zeta \rangle \;=\; \zeta^{\#L}\,. \tag{64}$$

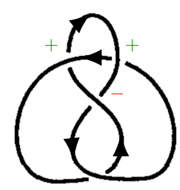
Example 4.10.



$\longmapsto\; -2\,,$



$\longmapsto\; +3\,,$



$\longmapsto\; 0\,.$

Remark 4.11 (**Comparison to Wilson loop link observables of abelian Chern-Simons theory**).

(i) For Chern-Simons theory with abelian gauge group $\mathrm{U}(1)$ it is widely understood by appeal to path-integral arguments ([214, p. 363][67, p. 169] following [Pol88]) that the quantum observables are labeled by framed links L, often considered as equipped with labels (charges) q_i on their ith connected component L_i and the expectation value of these observables in these states is the charge-weighted exponentiated framing- and linking numbers (Def. 4.8) as follows ([214, p. 363], cf. review e.g. in [MPW19, (5.1)]):

$$\mathrm{W}_k(L) \;=\; \exp\Big(\tfrac{2\pi \mathrm{i}}{k} \Big(\textstyle\sum_i q_i^2\, \mathrm{frm}(L_i) + \sum_{i,j} q_i q_j\, \mathrm{lnk}(L_i, L_j) \Big) \Big)\,. \tag{65}$$

(ii) However, with the charges q_i being integers, we may equivalently replace a q_i-charged component L_i with q_i unit-charged parallel copies of L_i, and hence assume without loss of generality that $\forall_i \;\; q_i = 1$. With this, we observe that the Chern-Simons expectation values (65) coincide with our pure topological quantum states (64):

$$W_k(L) \;=\; \exp\Big(\tfrac{2\pi \mathrm{i}}{k} \Big(\sum_i \mathrm{frm}(L_i) + \sum_{i,j} \mathrm{lnk}(L_i, L_j) \Big) \Big) \underset{(63)}{=} \exp\Big(\tfrac{2\pi \mathrm{i}}{k} \#(L) \Big).$$

4.2 On the sphere

We briefly discuss 2-cohomotopical flux on the 2-sphere $\Sigma^2_0 \simeq S^2$, meaning the actual 2-sphere whose point-at-infinity is disjoint, in contrast to the 2-sphere $(\Sigma^2_{0,0,1})_{\cup\{\infty\}} \simeq \mathbb{R}^2_{\cup\{\infty\}}$ that arose as the one-point compactification of the plane in §4.1.

In order for this actual 2-sphere to be realized as an FQH system in the laboratory, one would not only need to produce a 2-dimensional electron gas of spherical topology, but also make it enclose the endpoint of a very long and thin solenoid to approximate a magnetic monopole at its center that would produce magnetic flux going *radially* through the spherical electron gas — which is a tall order. Nevertheless, in FQH theory this situation is often considered as an instructive hypothetical case study.

Also for us here, the following analysis of the 2-sphere case serves in §4.5 as an intermedite step in identifying the braiding phases of solitonic anyons on the plane, that we obtained in the previous §4.1, with corresponding braiding phases on n-punctured disks, hence in experimentally accessible situations.

While it is classical that

$$\pi_1 \,\mathrm{Map}^*\big(\Sigma^2_{0,0,1}),\, S^2\big) \underset{(124)}{\simeq} \pi_3(S^2) \simeq \mathbb{Z}$$

(generated by the Hopf fibration $S^3 \to S^2$), the analogous statement for the un-based sphere needs another argument:

Lemma 4.12 (**Fundamental group of unpointed endomaps of the 2-sphere** [100, Thm. 5.3(1)][127, Lem. 3.1])**.** *The fundamental group of the space of* (un-pointed) *maps $S^2 \to S^2$ is, in the connected component of maps of Hopf degree $k \in \mathbb{Z}$ (Def. 4.4), isomorphic to:*

$$\pi_1\Big(\mathrm{Map}\big(S^2,\, S^2\big),\, \deg = k\Big) \;\simeq\; \mathbb{Z}/(2k)\,.$$

In our notation (9) and (131) this means, in particular, that in the component of vanishing Hopf degree we have:

$$\pi_1 \,\mathrm{Map}\big(\Sigma^2_0,\, S^2\big) \;\simeq\; \mathbb{Z}\,. \tag{66}$$

Lemma 4.13 (**Solitonic 2-cohomotopical flux monodromy on plane and 2-sphere are identified).** *The canonical map*

$$\pi_1 \,\mathrm{Map}^*\big((\Sigma^2_{0,0,1})_{\cup\{\infty\}},\, S^2\big) \xrightarrow[\sim]{\pi_1(p^*)} \pi_1 \,\mathrm{Map}^*\big((\Sigma^2_0)_{\cup\{\infty\}},\, S^2\big) \underset{(129)}{\simeq} \pi_1 \,\mathrm{Map}\big(\Sigma^2_0,\, S^2\big) \tag{67}$$

is an isomorphism.

Proof. The long exact sequence of homotopy groups (138) induced by the evaluation map $\mathrm{Map}(S^2, S^2) \xrightarrow{\mathrm{ev}} S^2$ (125) is, in the relevant part, of the form

$$\begin{array}{ccccccc}
\pi_2(S^2) & \xrightarrow{\;n\;} & \pi_1 \,\mathrm{Map}^*\big(S^2,\, S^2\big) & \xrightarrow{\pi_1(p^*)} & \pi_1 \,\mathrm{Map}\big(S^2,\, S^2\big) & \longrightarrow & \pi_1(S^2) \\
\wr| \text{ Hopf degree} & & \wr| \text{ Hopf fibration} & & \wr| \text{ (66)} & & \wr| \\
\mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & 1\,,
\end{array}$$

where the map on the left must be multiplication by some integer n, by the freeness of $\mathbb{Z}$. But then exactness on the left implies that the middle map must send n to 0, while exactness on the right means that the middle map is surjective hence that $n = 0$, which by exactness on the left implies that the middle map is also injective, hence bijective. □

Remark 4.14 (**Identifying braid phase observable on sphere).** In terms of 2-cohomotopically quantized flux, this says that the algebra of topological flux observables on the plane and on the sphere are both isomorphic to $\mathbb{C}[\mathbb{Z}]$ and canonically identified as such, whence the discussion in §4.1 gives that in an irreducible representation on a (1-dimensional) Hilbert space the generator $1 \in \mathbb{Z}$ acts as multiplication by some phase factor $\zeta \in \mathrm{U}(1) \subset \mathbb{C}$:

$$\begin{array}{ccccc}
\mathbb{Z} & \longrightarrow & \mathrm{U}(\mathcal{H}_{S^2}) & & \\
1 & \longmapsto & \widehat{\zeta} : |\psi\rangle & \mapsto & \zeta|\psi\rangle\,.
\end{array}$$

In the following §4.3 and §4.4 we see that further compatibility of this phase observable $\widehat{\zeta}$ with its incarnation on the torus restricts it to a primitive root of unity, as expected in FQH systems.

4.3 On closed surfaces

While spherical FQH systems as in §4.2 are just barely plausible as having experimental realizations, for closed surfaces of more general genus $g \in \mathbb{N}$ this quickly becomes only less plausible as g increases. Nevertheless, the notoriously rich theoretical predictions for these somewhat hypothetical FQH systems on closed surfaces are crucial intermediate stages in understanding FQH systems in general and hence also on experimentally accessible situations.

The new key we now offer for understanding FQH systems on closed surfaces via topological flux quantization is the following result, whose roots in algebraic topology date back half a century ([93], following [100]), but which gains a substantial new meaning when understood now as being about observables on topopological flux quantized in 2-cohomotopy:

Proposition 4.15 (**Monodromy of 2-cohomotopical flux through closed surfaces).** *The 2-cohomotopical flux monodromy over a closed oriented surface Σ^2_g (9), for any $g \in \mathbb{N}$, forms the $\mathbb{Z}$-extension of the free abelian group $\mathbb{Z}^{2g}$ (16)*

$$\begin{array}{ccccccc} 1 \to \pi_1 \operatorname{Map}^*_0\big(S^2_{\cup\{\infty\}}, S^2\big) & \longrightarrow & \pi_1 \operatorname{Map}^*_0\big((\Sigma^2_g)_{\cup\{\infty\}}, S^2\big) & \longrightarrow & \pi_1 \operatorname{Map}^*_0\big(\bigvee_g (S^1_a \vee S^1_b), S^2\big) \to 1 \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ \mathbb{Z} & \hookrightarrow & \widehat{\mathbb{Z}^{2g}} & \twoheadrightarrow & \mathbb{Z}^{2g}\,, \end{array} \tag{68}$$

that is the integer Heisenberg group *at level* $= 2$ (cf. [75, p 7][76, Def 2.4][77, Def 2.3] [13]) *:*

$$\widehat{\mathbb{Z}^{2g}} \;:=\; \Big\{ \big(\vec a, \vec b, n\big) \,\in\, \mathbb{Z}^g \times \mathbb{Z}^g \times \mathbb{Z}\,, \quad \big(\vec a, \vec b, n\big)\cdot\big(\vec a\,', \vec b\,', n'\big) := \big(\vec a + \vec a\,',\, \vec b + \vec b\,',\, n + n' + \vec a \cdot \vec b\,' - \vec a\,' \cdot \vec b\,\big) \Big\}. \tag{69}$$

Proof. The short exact sequence is due to [93, Thm 1 & p 6], spelled out as Lem. A.8 in the appendix. The identification of the group on the left is by 4.12, whence the resulting group extension must be classified by

$$\begin{aligned} H^2_{\mathrm{grp}}\big(\mathbb{Z}^{2g};\, \mathbb{Z}\big) &\simeq H^2\big(B\mathbb{Z}^{2g};\, \mathbb{Z}\big) \\ &\simeq H^2\big((T^2)^g;\, \mathbb{Z}\big) \\ &\simeq H^2\big(T^2;\, \mathbb{Z}\big)^g \\ &\simeq \mathbb{Z}^g\,. \end{aligned} \tag{70}$$

The identification of the resulting group extension as having class $(2, \cdots, 2) \in \mathbb{Z}^g$ is due to [131, Thm. 1] (cf. the formulas on the previous page there), see also [114, Cor. 7.6]. Observing then that the unit extension class $(1, \cdots, 1) \in \mathbb{Z}^g$ is given by *either* of these two group cocycles:

$$\begin{array}{ccc} (\mathbb{Z}^{2g}) \times (\mathbb{Z}^{2g}) & \longrightarrow & \mathbb{Z} \\ \big((\vec a, \vec b),\, (\vec a\,', \vec b\,')\big) & \longmapsto & +\,\vec a \cdot \vec b\,' \\ \big((\vec a, \vec b),\, (\vec a\,', \vec b\,')\big) & \longmapsto & -\,\vec a\,' \cdot \vec b \end{array}$$

which are readily seen to be cohomologous, it follows that the extension class $(2, \cdots, 2)$ is represented by (69), as claimed. ☐

Remark 4.16 (**Modular equivariance of integer Heisenberg group).** Our way of casting Prop. 4.15 — with the extension cocycle highlighted in (69) identified as the standard symplectic form on $\mathbb{Z}^{2g}$ (instead of the cohomologous $2\,\vec a \cdot \vec b'$ used in the original derivations [131, Thm. 1][114, Cor. 7.6]) — makes manifest that the integral symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$ acts by group automorphisms on $\widehat{\mathbb{Z}^{2g}}$ (covering its defining action $\mathbb{Z}^{2g}$ and necessarily acting trivially on the center $\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}^{2g}}$, cf. [76, (2.6)]):

$$\begin{array}{ccc} & & \operatorname{Aut}\Big(\widehat{\mathbb{Z}^{2g}}\Big) \\ & \overset{\exists !}{\nearrow} & \downarrow \\ \mathrm{Sp}_{2g}(\mathbb{Z}) & \xrightarrow[\text{canonical}]{} & \operatorname{Aut}\big(\mathbb{Z}^{2g}\big) \end{array} \tag{71}$$

Better yet, with (23), this action has a geometrical interpretation as the diffeomorphism action on the 2-cohomotopical flux monodromy over closed surfaces:

Proposition 4.17 (**Mapping class group action on 2-cohomotopical flux monodromy on closed surface).** *Under the identification of Prop. 4.15, the action (53) of the mapping class group of Σ^2_g on the 2-cohomotopical flux*

[13] By the "level" we here mean the extension class in $\mathbb{Z}^g$ (70), and by "level $= n$" we mean the element $(n, \cdots, n) \in \mathbb{Z}^{2g}$. The integer Heisenberg group at at level $= 1$ (cf. [134, p 232][44, p 213]) is isomorphic to groups of certain upper triangular integer matrices (cf. [46, p 35][44, p 299]), and in this form is commonly considered in pure algebra and group theory. In contrast, the of relevance here with level $= 2$ — which is the case of subgroups of the actual eponymous Heisenberg group from quantum mechanics — seems not to have found much attention in the pure algebra/group theory literature.

monodromy over Σ^2_g is via its symplectic representation (42) on the underlying $\mathbb{Z}^{2g}$ extended by the trivial action on the center $\mathbb{Z}$:

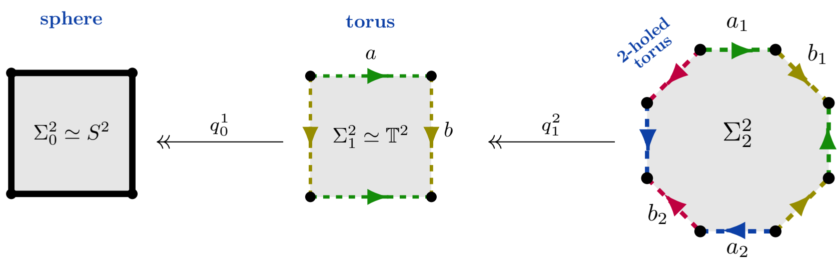
$$
\begin{array}{ccc}
\mathrm{MCG}(\Sigma^2_g) & \xrightarrow{(53)} & \mathrm{Aut}\big(\pi_1\,\mathrm{Map}(\Sigma^2_g,\, S^2)\big) \\
 & & \uparrow\wr\ (68) \\
\Big\downarrow\!\!\downarrow (42) & & \mathrm{Aut}\big(\widehat{\mathbb{Z}^{2g}}\big) \\
 & \nearrow\ (71) & \Big\downarrow\!\!\downarrow \\
\mathrm{Sp}_{2g}(\mathbb{Z}) & \longrightarrow & \mathrm{Aut}\big(\mathbb{Z}^{2g}\big)\,.
\end{array}
\tag{72}
$$

Proof. The first statement follows by inspection of the construction of (68) as spelled out in §A.3: By Lem. A.9 there, the action of $\mathrm{MCG}(\Sigma^2_g)$ on $\mathbb{Z}^{2g} \hookrightarrow \pi_1\,\mathrm{Map}\big(\Sigma^2_g,\, S^2\big)$ is identified with its action on $H^1\big(\Sigma^2_g;\,\mathbb{Z}\big)$, for which it is classical (42) that it is through $\mathrm{Sp}_{2g}(\mathbb{Z})$, as claimed. But then the action on the center of $\widehat{\mathbb{Z}^{2g}}$ is uniquely fixed to be trivial, by (71). □

Functoriality. Next, we observe some form of functoriality in maps between closed surfaces of the result of Prop. 4.15, the crucial implication being the identification of the braid phase observable ζ (2) across all closed surfaces. To this end, write:

$$
\Sigma_g \xleftarrow{\;q_g^{g+1}\;} \Sigma_{g+1}
\tag{73}
$$

for the surjective homeomorphism (146) given by contracting one pair of edges in the standard fundamental polygon (cf. Prop. A.4) of Σ_{g+1}. For instance, for $g \in \{1, 2\}$ the maps



(74)

are given, for q^2_1, by sending the purple and blue colored edges to the point •, and for q^1_2 by sending also the remaining edges to the point.

Lemma 4.18 (**Pullback of 2-cohomotopical flux monodromy on closed surfaces**)**.** *Under the above identification (68), the surjections q (73) map to the canonical inclusion of Heisenberg groups obtained by adjoining the generators corresponding to the contracted edges:*

$$
\begin{array}{ccc}
\pi_1\mathrm{Map}^*\big(\Sigma^2_g,\, S^2\big) & \xrightarrow{(q_g^{g+1})^*} & \pi_1\mathrm{Map}^*\big(\Sigma^2_{g+1},\, S^2\big) \\
\big\downarrow\wr & & \big\downarrow\wr \\
\widehat{\mathbb{Z}^{2g}} & \hookrightarrow & \widehat{\mathbb{Z}^{2g+2}}
\end{array}
\tag{75}
$$

Proof. The proof of Lem. A.8 shows that the short exact sequence (4.15) is natural in Σ^2_g, whence we have a commuting diagram of this form:

$$
\begin{array}{ccccccccc}
 & & \mathbb{Z} & \hookrightarrow & \widehat{\mathbb{Z}^{2g}} & \twoheadrightarrow & \mathbb{Z}^{2g} & & \\
 & & \big\downarrow\wr & & \big\downarrow\wr & & \big\downarrow\wr & & \\
1 & \longrightarrow & \pi_1\,\mathrm{Map}_0\big(S^2,\, S^2\big) & \longrightarrow & \pi_1\,\mathrm{Map}_0\big(\Sigma^2_g,\, S^2\big) & \longrightarrow & \pi_1\,\mathrm{Map}^*_0\big(\bigvee_g (S^1_a \vee S^1_b),\, S^2\big) & \longrightarrow & 1 \\
 & & \big\| & & \big\downarrow (q_g^{g+1})^* & & \big\downarrow & & \\
1 & \longrightarrow & \pi_1\,\mathrm{Map}_0\big(S^2,\, S^2\big) & \longrightarrow & \pi_1\,\mathrm{Map}_0\big(\Sigma^2_{g+1},\, S^2\big) & \longrightarrow & \pi_1\,\mathrm{Map}^*_0\big(\bigvee_{g+1} (S^1_a \vee S^1_b),\, S^2\big) & \longrightarrow & 1 \\
 & & \big\downarrow\wr & & \big\downarrow\wr & & \big\downarrow\wr & & \\
 & & \mathbb{Z} & \hookrightarrow & \widehat{\mathbb{Z}^{2g+2}} & \twoheadrightarrow & \mathbb{Z}^{2g+2}\,. & &
\end{array}
$$

This gives the claim. □

Combination of Lem. 4.13 and (4.18) leads to:

Proposition 4.19 (Identifying central braid phase generator across surfaces). *The canonical comparison map between solitonic flux monodromy on the plane and on Σ_g, $g \in \mathbb{N}$, identifies the central generators as the braiding phase observable of §4.1:*

$$
\begin{array}{ccccc}
\pi_1\,\mathrm{Map}^*\big((\mathbb{R}^2)_{\cup\{\infty\}},\, S^2\big) & \xrightarrow[(67)]{\pi_1(q^*)} & \pi_1\,\mathrm{Map}\big(\Sigma^2_0,\, S^2\big) & \xrightarrow[(75)]{\pi_1\left((q^g_0)^*\right)} & \pi_1\,\mathrm{Map}\big(\Sigma^2_g,\, S^2\big)\,. \\
\big\downarrow{\sim} & & \big\downarrow{\sim} & & \big\downarrow{\sim} \\
\mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} & \hookrightarrow & \widehat{\mathbb{Z}} \\
1 & \longmapsto & 1 & \longmapsto & \left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], 1\right)
\end{array}
$$

Remark 4.20 (Solitonic braiding phases on closed surfaces). In generalization of Rem. 4.14, Prop. 4.19 says that given 2-cohomotopical flux quantum states on Σ^2_g, hence a unitary representation of the 2-cohomotopical flux monodromy on Σ^2_g, hence of the integer Heisenberg group $\widehat{\mathbb{Z}^{2g}}$ (68), for $g \in \mathbb{N}$, then the central observable $\widehat{\zeta}$

$$
\begin{array}{ccccccccc}
\mathbb{Z} & \hookrightarrow & \widehat{\mathbb{Z}^{2g}} & \longrightarrow & \mathrm{U}(\mathcal{H}) & & & & \\
1 & \longmapsto & \left(\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], 1\right) & \longmapsto & \widehat{\zeta} & : & |\psi\rangle & \mapsto & \zeta\,|\psi\rangle\,, \\
 & & \left(\left[\begin{smallmatrix}\vec a\\ \vec b\end{smallmatrix}\right], 0\right) & \longmapsto & \widehat{W}_{\left[\begin{smallmatrix}\vec a\\ \vec b\end{smallmatrix}\right]} & & & &
\end{array}
\tag{76}
$$

is to be understood as observing the braiding phases of solitonic flux according to §4.1, where the braiding happens within an open disk inside the surface.

The next section §4.4 implies that thereby the braiding phase is restricted to primitive roots of unity and then to be identified with the anyon braiding phase $\zeta = e^{\pi \mathrm{i} \frac{p}{K}}$ as seen in FQH systems.

4.4 On the torus

While transverse magnetic flux through a toroidal 2d electron gas is not readily realized experimentally (cf. footnote 7), effective field theories of flux on arbitrary surfaces tend to be characterized by their theoretical predictions for the torus, notably through the dimension and modular transformation properties of the Hilbert space of states (the "topological order" [209][210], cf. Prop. 4.30 below).

Therefore, a major example of the phenomena in §3 is the following derivation of quantum states of 2-cohomotopically quantized topological flux on the torus, which reproduces the *modular data* [74] of U(1)-Chern-Simons theory (Rem. 4.31 below) – in fact of *spin* Chern-Simons theory (8).

Our task in identifying the 2-cohomotopical flux quantum states over the torus is, by Prop. 4.15, to classify the (finite-dimensional unitary) representation of the integer Heisenberg group (69) after covariantization (22):

Proposition 4.21 (Diffeomorphism action over torus is canonical modular action). *The action* (53) *of* $\mathrm{MCG}(\Sigma^2_1) \simeq \mathrm{SL}_2(\mathbb{Z})$ (39) *on* $\pi_1\mathrm{Map}^*_0\big((\Sigma^2_1)_{\cup\{\infty\}},\, S^2\big) \simeq \widehat{\mathbb{Z}^2}$ (69) *is the defining action of* $\mathrm{Sp}_2(\mathbb{Z}) \simeq \mathrm{SL}_2(\mathbb{Z})$ *on* $\mathbb{Z}^2$ *and trivial on the center, whence the flux monodromy group* (53) *over the torus with its* pp*-spin structure* (43) *is*

$$
\mathrm{MCG}(\Sigma^2_1) \ltimes \pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_1)_{\cup\{\infty\}},\, S^2\big)\Big) \;\simeq\; \mathrm{SL}_2(\mathbb{Z}) \ltimes \widehat{\mathbb{Z}^2}
\tag{77}
$$

while for the torus equipped with the aa*-spin structure it is correspondingly the subgroup* $\mathrm{MCG}(\Sigma^2_1)^{\mathrm{aa}} \ltimes \widehat{\mathbb{Z}^2}$ (44).

Proof. This is a special case of Prop. 4.17 □

Representations of $\mathrm{SL}_2(\mathbb{Z})$ and of $\widehat{\mathbb{Z}^2}$ separately are well-studied, but representations of their semidirect product (77) may not have received attention. We next find that its irreps — and hence the topological states of 2-cohomotopically quantized flux on the torus — single out the quantum states of U(1)-Chern-Simons theory generalized them from unit-fractional braiding angles $\pi\frac{1}{K}$ to general braiding angles $\pi\frac{p}{K}$ as expected for FQH systems.

We proceed incrementally, starting with some generalities, then finding the quantum states at $\nu = 1/K$ for even K (Prop. 4.30) as usual in Chern-Simons/CFT theory (Rem. 4.31), and then eventually generalizing this construction to other fractions, ultimately by taking the spin-structures on tori into account (Prop. 4.34 below).

The key novel aspect of our discussion of these integer and cyclic Heisenberg groups is that we consider representations that extend to their semidirect product with a modular group (Prop. 53), hence that admit a *covariantization* in the sense of Def. 2.11. Since it turns out that the key effect of the covariantization is all in the action of the generator S (40) which is shared by both the pp- and the aa-spin mapping class groups we will say for short that:

Definition 4.22 (Covariantizable representations of integer Heisenberg group). A linear representation of $\widehat{\mathbb{Z}^2}$ is *covariantizable* if it extends to a representation of the semidirect product with the subgroup

$$\mathbb{Z}_4 \;\simeq\; \langle S \rangle \;\subset\; \mathrm{SL}_2(\mathbb{Z}) \tag{78}$$

of the modular group (cf. Prop. 4.21) that is generated by S alone.

In order to analyze such extensions, we first note the following elementary facts:

Lemma 4.23 (Extending representations along normal subgroup inclusions, cf. [104, pp 175]**).** *Given $H \hookrightarrow G$ a normal subgroup inclusion and $\rho : H \to \mathrm{GL}(\mathcal{H})$ a $\mathbb{C}$-linear representation, say that $\widehat{\rho} : G \to \mathrm{GL}(\mathcal{H})$ is an* extension *if $\iota^*\widehat{\rho} = \rho$. — We have:*
(i) *Any extension $\widehat{\rho}$ exhibits ρ as isomorphic to its g-translate representations:*

$$\underset{g\in G}{\forall} \;\; \rho \;\simeq\; \rho^g \,, \qquad \text{where } \rho^g := \rho\big(\mathrm{Ad}_g(-)\big) \;. \tag{79}$$

(ii) *If ρ is irreducible then any two extensions $\widehat{\rho}$, $\widehat{\rho}'$ differ at most by tensoring with some multiplicative character of G/N:*

$$\underset{g\in G}{\forall} \;\; \widehat{\rho}'(g) \;=\; d(gN)\cdot \widehat{\rho}(g) \;, \qquad \textit{for some } d : G/N \to \mathbb{C}^\times .$$

Proof. **(i)** We have for $h \in H$:

$$\begin{aligned} \widehat{\rho}(g) \circ \rho(h) \circ \widehat{\rho}^{-1} &= \widehat{\rho}(g) \circ \widehat{\rho}(h) \circ \widehat{\rho}(g^{-1}) \\ &= \widehat{\rho}(ghg^{-1}) \\ &= \rho(ghg^{-1}) \\ &\equiv \rho^g(h)\,, \end{aligned}$$

showing that $\widehat{\rho}(g)$ serves as an intertwiner that exhibits the claimed isomorphism.
(ii) We have, for $h \in H$ and $g \in G$:

$$\begin{aligned} \big(\widehat{\rho}'(g)\circ\widehat{\rho}(g)^{-1}\big)\circ\rho(h)\circ\big(\widehat{\rho}'(g)\circ\widehat{\rho}(g)^{-1}\big)^{-1} &= \widehat{\rho}'(g)\circ\widehat{\rho}(g^{-1})\circ\rho(h)\circ\widehat{\rho}(g)\circ\widehat{\rho}'(g^{-1}) \\ &= \widehat{\rho}'(g)\circ\rho(g^{-1}hg)\circ\widehat{\rho}'(g^{-1}) \\ &= \rho(h)\,, \end{aligned}$$

showing that the difference of the two extensions for any g commutes with all the ρ-operators. Therefore Schur's lemma (cf. [51, Cor. 1.17]) implies that this difference is a multiple $d(g)$ of the identity operator. That this is multiplicative in and depends only on the coset of g follows by the extension properties of $\widehat{\rho}$ and $\widehat{\rho}'$. ☐

Lemma 4.24 (Finite-dimensional Heisenberg irreps and roots of unity). *If a unitary representation of the integer Heisenberg group, $\widehat{(-)} : \widehat{\mathbb{Z}^2} \longrightarrow \mathrm{U}(\mathcal{H})$, is finite-dimensional and irreducible, then the central observable $\widehat{\zeta}$ (76) is given by multiplication with a root of unity: $\widehat{\zeta} = \zeta \cdot \mathrm{id}$ and $\underset{n\in\mathbb{N}_{>0}}{\exists} \; \zeta^n = 1$.*

Proof. First, with the respresentation assumed irreducible and since $\widehat{\zeta}$ commutes with all other representation operators, Schur's lemma (cf. [51, Cor. 1.17]) implies that there is $\zeta \in \mathbb{C}$ with $\widehat{\zeta} = \zeta \cdot \mathrm{id}$. Further, since $\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}$ is unitary we may find an eigenvector, to be denoted $|0\rangle$, with non-vanishing eigenvalue to be denoted ξ:

$$\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}|0\rangle \;=\; \xi|0\rangle\,. \tag{80}$$

Now the commutation relation says that $\widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}$ is a corresponding raising operator, in that the elements

$$|n\rangle \;:=\; \Big(\widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}\Big)^n|0\rangle \;\in\; \mathcal{H} \tag{81}$$

are further eigenvectors of $\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}$ with eigenvalue $\zeta^{2n}\xi$:

$$\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}|n\rangle \;\equiv\; \widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\Big(\widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}\Big)^n|0\rangle \underset{(86)}{=} \zeta^{2n}\Big(\widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}\Big)^n\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}|0\rangle \underset{(80)}{=} \zeta^{2n}\Big(\widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}\Big)^n\xi|0\rangle \;\equiv\; \zeta^{2n}\xi\,|n\rangle\,.$$

But by the assumption of finite-dimensionality there can only be finitely many distinct eigenvalues, which is evidently equivalent to ζ being a root of unity. ☐

Definition 4.25 (Cyclic Heisenberg group). For $o \in \mathbb{N}_{>0}$, we denote the o-cyclic version of the integer Heisenberg group (69) by

$$\widehat{\mathbb{Z}_o^{2g}} \;:=\; \left\{ \big([\vec a],[\vec b],[n]\big) \,\in\, \mathbb{Z}_o^g \times \mathbb{Z}_o^g \times \mathbb{Z}_o \,, \quad \begin{array}{c} \big([\vec a],[\vec b],[n]\big)\cdot\big([\vec a\,'],[\vec b\,'],[n']\big) := \\ \big(\,[\vec a + \vec a\,'],\,[\vec b + \vec b\,'],\,\big[n + n' + \vec a\cdot\vec b\,' - \vec a\,'\cdot\vec b\,\big]\,\big) \end{array} \right\}, \tag{82}$$

where $\mathbb{Z}_o := \mathbb{Z}/o\mathbb{Z}$ denotes the o-cyclic group and $[-] : \mathbb{Z} \twoheadrightarrow \mathbb{Z}_o$ denotes the quotient map.

Lemma 4.26 (Covariant fin-dim reps of integer Heisenberg group that come from cyclic Heisenberg). *If a finite-dimensional irreducible unitary representation* $\widehat{(-)} : \widehat{\mathbb{Z}^2} \longrightarrow \mathrm{U}(\mathcal{H})$ *is covariantizable* (Def. 4.22)*, then it is the pullback of a representation of the o-cyclic Heisenberg group* (82) *along the quotient coprojection* $[-] : \widehat{\mathbb{Z}^2} \twoheadrightarrow \widehat{\mathbb{Z}^2_o}$, *where o may be taken to equal*

$$\mathrm{ord}(\zeta) \;:=\; \min_{n\in\mathbb{N}_{>0}} \big(\zeta^n = 1\big)\,, \tag{83}$$

the order of ζ *in* $\widehat{\zeta} = \zeta\,\mathrm{id}$ (Lem. 4.24).

Proof. The point is that the operator $\widehat{W}_{\left[\begin{smallmatrix} o \\ 0\end{smallmatrix}\right]}$ (84) commutes with all other representation operators, by (86) and (83):

$$\widehat{W}_{\left[\begin{smallmatrix} o \\ 0\end{smallmatrix}\right]} \circ \widehat{W}_{\left[\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right]} \;=\; \underbrace{\zeta^{2o}}_{1}\; \widehat{W}_{\left[\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right]} \circ \widehat{W}_{\left[\begin{smallmatrix} o \\ 0\end{smallmatrix}\right]}\,,$$

and analogously for $\widehat{W}_{\left[\begin{smallmatrix} 0 \\ o\end{smallmatrix}\right]}$. Therefore Schur's lemma implies (cf. [51, Cor. 1.17]) that these operators act as some multiple of the identity operator. We proceed to show that this multiple is unity:

As the operator indices range, the multiples $w_{\left[\begin{smallmatrix} oa \\ ob\end{smallmatrix}\right]} \in \mathbb{C}^\times$, given by $\widehat{W}_{\left[\begin{smallmatrix} oa \\ ob\end{smallmatrix}\right]} = w_{\left[\begin{smallmatrix} oa \\ ob\end{smallmatrix}\right]}\mathrm{id}$, constitute a group homomorphism

$$w_{\left[\begin{smallmatrix} - \\ -\end{smallmatrix}\right]} \;:\; (o\mathbb{Z})^2 \longrightarrow \mathbb{C}^\times\,.$$

As such, this is an invariant of the isomorphism class of the representation (because under isomorphisms the operators get conjugated, whence all these scalar multiplication operators are preserved). But since the isomorphism class of the representation is preserved by $\langle S\rangle$ (by Lem. 4.23, using here the assumption of extension), this means that $w_{\left[\begin{smallmatrix} - \\ -\end{smallmatrix}\right]}$ is preserved by the S-matrix, which implies the desired statement as follows:

$$w_{\left[\begin{smallmatrix} o \\ 0\end{smallmatrix}\right]} \;=\; w_{S\left[\begin{smallmatrix} o \\ 0\end{smallmatrix}\right]} \;=\; w_{\left[\begin{smallmatrix} 0 \\ -o\end{smallmatrix}\right]} \;=\; w^{-1}_{\left[\begin{smallmatrix} 0 \\ o\end{smallmatrix}\right]} \;=\; w^{-1}_{S\left[\begin{smallmatrix} 0 \\ o\end{smallmatrix}\right]} \;=\; w^{-1}_{\left[\begin{smallmatrix} o \\ 0\end{smallmatrix}\right]} \qquad\Rightarrow\qquad w_{\left[\begin{smallmatrix} o \\ 0\end{smallmatrix}\right]} \;=\; 1 \;=\; w_{\left[\begin{smallmatrix} 0 \\ o\end{smallmatrix}\right]}\,.$$

This shows that the representation is pulled back from a representation of the o-cyclic Heisenberg group, as claimed. ☐

Lemma 4.27 (Pullback from cyclic Heisenberg preserves irreducibility). *For* $o \in \mathbb{N}_{>0}$*, given a representation of the cyclic Heisenberg group* $\rho : \widehat{\mathbb{Z}^2_0} \to \mathrm{GL}(\mathcal{H})$*, then its pullback* $p^*\rho : \widehat{\mathbb{Z}^2} \overset{p}{\twoheadrightarrow} \widehat{\mathbb{Z}^2_0} \overset{\rho}{\to} \mathrm{GL}(\mathcal{H})$ *is irreducible iff* ρ *is.*

Proof. In general, a representation ρ is irreducible if its pullback $p^*\rho$ is, since pullback preserves direct sums (so that we would get a contradiction if it were a non-trivial direct sum). The converse does not hold generally but it holds here where the quotient coprojection sends group generators to group generators. ☐

Lemma 4.28 (Some irreps of the integer Heisenberg group). *The following formulas define finite-dimensional irreducible unitary representations of the integer Heisenberg group* (69)*:*

$$\begin{array}{c} \mathcal{H} \;:=\; \mathbb{C}^D \;\simeq\; \mathrm{Span}_{\mathbb{C}}\big(|0\rangle,\,|1\rangle,\,\cdots,\,|D{-}1\rangle\big) \\ \big\langle n_1 \big| n_2 \big\rangle \;:=\; \delta_{n_1 n_2} \end{array} \quad \left\{ \begin{array}{lcllcl} \widehat{\mathbb{Z}^2} & \longrightarrow & \mathrm{U}(\mathcal{H}) & & & \\ \big(\left[\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right],0\big) & \longmapsto & \widehat{W}_{\left[\begin{smallmatrix} 1 \\ 0\end{smallmatrix}\right]} & : |n\rangle & \mapsto & \zeta^{2n}\,|n\rangle \\ \big(\left[\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right],0\big) & \longmapsto & \widehat{W}_{\left[\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right]} & : |n\rangle & \mapsto & |n+1 \,\mathrm{mod}\, D\rangle \\ \big(\left[\begin{smallmatrix} 0 \\ 0\end{smallmatrix}\right],1\big) & \longmapsto & \widehat{\zeta} & : |n\rangle & \mapsto & \zeta\,|n\rangle\,, \end{array} \right. \tag{84}$$

for ζ a root of unity of order $\mathrm{ord}(\zeta)$ (83) *and*

$$D := \begin{cases} \mathrm{ord}(\zeta) & | \quad \mathrm{ord}(\zeta) \in 2\mathbb{N}+1 \\ \mathrm{ord}(\zeta)/2 & | \quad \mathrm{ord}(\zeta) \in 2\mathbb{N}\,. \end{cases}$$

Here the representation of general group elements follows from applying the group law to the above generators, for instance:

$$\zeta^{-1}\,\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \;=\; \widehat{W}_{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} \;=\; \zeta^{+1}\,\widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\,. \tag{85}$$

Proof. First to note that the generating group commutators in $\widehat{\mathbb{Z}^2}$ (69) are evidently respected by the formulas (84), so that they do define a representation of $\widehat{\mathbb{Z}^2}$:

$$\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} \circ \widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \;=\; \zeta^2\,\widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} \circ \widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\,. \tag{86}$$

To see that this is irreducible, by Lem. 4.27 we may equivalently show that these $\widehat{\mathbb{Z}^2}$-representations are irreducible as $\widehat{\mathbb{Z}^2_o}$-representations for $o := \mathrm{ord}(\zeta)$. This being a finite group (of order $|\widehat{\mathbb{Z}^2_0}| = o^3$) we may invoke Schur-orthonormality (cf. [70, Thm 2.12]) of irreps: The $\widehat{\mathbb{Z}^2_o}$ character components of the representations are, for $o = \mathrm{ord}(\zeta) \in \mathbb{N}_{>0}$ and $a, b, c \in \{0, \cdots, o-1\}$:

$$\chi_{\left(\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right],c\right)} \;:=\; \mathrm{tr}\Big(\widehat{W}_{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]} \circ \widehat{\zeta}^{c}\Big) \;=\; \begin{cases} 0 \quad \text{(evidently)} & \big|\; b \neq 0 \,\mathrm{mod}\, D \\ 0 \underset{(149)}{=} \zeta^c \sum_{n=0}^{D-1} \zeta^{2n} & \Big|\; b = 0 \,\mathrm{mod}\, D\,,\; a \neq 0 \;\begin{matrix} \mathrm{mod}\, o & |\; o \in 2\mathbb{N}+1 \\ \mathrm{mod}\, o/2 & |\; o \in 2\mathbb{N} \end{matrix} \\ \zeta^c \cdot D & \big|\; b = 0 \,\mathrm{mod}\, D\,,\; a = 0 \,\mathrm{mod}\, o \\ \zeta^{c+o} \cdot D & \big|\; b = 0 \,\mathrm{mod}\, D\,,\; a = o/2 \end{cases} \tag{87}$$

whose Schur-norm square is found to be unity:

$$\frac{1}{\big|\widehat{\mathbb{Z}^2}\big|} \sum_{a,b,c=0}^{o-1} \Big|\chi_{\left(\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right],c\right)}\Big|^2 \;=\; \left\{ \begin{array}{ll} \frac{1}{o^3} \sum_{a\in\{0\}} \sum_{b\in\{0\}} \sum_{c=0}^{o-1} D^2 \;=\; \frac{D^2}{o^2} & |\; o \in 2\mathbb{N}+1 \\ \frac{1}{o^3} \sum_{a\in\{0,o/2\}} \sum_{b\in\{0,o/2\}} \sum_{c=0}^{o-1} D^2 \;=\; 4\frac{D^2}{o^2} & |\; o \in 2\mathbb{N} \end{array} \right\} = 1\,,$$

signifying irreducible representations. □

Proposition 4.29 (Classification of covariantizable irreps of the integer Heisenberg group). *Any finite-dimensional irreducible unitary representation of $\widehat{\mathbb{Z}^2}$ which is covariantizable* (Def. 4.22) *must be isomorphic to one according to Lem. 4.28.*

Proof. Lem. 4.26 with Lem. 4.27 imply that we are dealing with an irreducible representation of a cyclic Heisenberg group on which the central generator $\widehat{\zeta}$ is given by multiplication with a root of unity. With this, the statement for even $\mathrm{ord}(\zeta) \in 2\mathbb{Z}$ is an instance of the *Stone-von Neumann theorem* in its generalization due to Mackey, as reviewed in [161, §4.1].

For odd $o := \mathrm{ord}(\zeta)$ we give the following elementary argument (which follows an evident proof strategy that however seems to requires the assumption of odd $\mathrm{ord}(\zeta)$ to go through):

Namely, as in the proof of Lem. 4.24 we find elements $|n\rangle \in \mathcal{H}$ (81) with $\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}|n\rangle = \zeta^{2n}\xi|n\rangle$. Now the assumption that $\mathrm{ord}(\zeta)$ is odd, hence that there is no n with $2n = \mathrm{ord}(\zeta)$, implies that the eigenvalues $\zeta^{2n}\xi$ are all distinct for $n \in \{0, 1, \cdots, \mathrm{ord}(\zeta)-1\}$, and hence so must be the corresponding eigenvectors $|n\rangle$. But by Lem. 4.26 we have $|n+o\rangle = |n\rangle$, so that we have constructed a representation of $\widehat{\mathbb{Z}^2_0}$ on the o-dimensional linear span of the $|n\rangle$, $n \in \{0, \cdots, o-1\}$, which:

(i.) must be the whole of the given representation, by the latter's assumed irreducibility,

(ii.) is of the claimed form (4.28) — except possibly for the factor ξ in (80).

Hence to conclude it is now suffient to show that this irrep is isomorphic to that of the same form but with $\xi = 1$. For this we compute the representation character components and observe that these come out as in (87) except for a factor of ξ^o in the third line. But $\widehat{W}_{\left[\begin{smallmatrix}o\\0\end{smallmatrix}\right]} = \mathrm{id}$ implies $\xi^o = 1$, whence our character coincides with and hence (cf. [51, Thm 2.17]) our irrep must be isomorphic to the claimed one. □

We now turn to the actual construction of the modular covariantization of these representations (first in the special case $\nu = 1/K$ for even K, then generalized below).

Proposition 4.30 (Basic 2-Cohomotopical quantum states over the torus).
Unitary representations of $\widehat{\mathbb{Z}^2} \rtimes \mathrm{SL}_2(\mathbb{Z})$ (77) *— and hence spaces of quantum states* (22) *for 2-cohomotopical flux over the torus — irreducible already in their restriction to* $\widehat{\mathbb{Z}^2}$*, are obtained for all even positive integers*

$$K \in 2\mathbb{N}_{>0} \qquad \text{with} \qquad \zeta := e^{\frac{\pi \mathrm{i}}{K}}\,, \tag{88}$$

by the following formulas:

$$\mathcal{H}_{T^2} := \mathbb{C}^K \simeq \mathrm{Span}\big(|0\rangle, |1\rangle, \cdots, |K{-}1\rangle\big) \quad \begin{cases} \mathrm{SL}_2(\mathbb{Z}) \ltimes \widehat{\mathbb{Z}^2} \longrightarrow \mathrm{U}(\mathcal{H}_{T^2}) \\ \Big(\mathrm{I},\, \big(\big[\begin{smallmatrix}1\\0\end{smallmatrix}\big], 0\big)\Big) \longmapsto \widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} : |n\rangle \mapsto \zeta^{2n}\, |n\rangle \\ \Big(\mathrm{I},\, \big(\big[\begin{smallmatrix}0\\1\end{smallmatrix}\big], 0\big)\Big) \longmapsto \widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]} : |n\rangle \mapsto \big|(n+1) \,\mathrm{mod}\, K\,\big\rangle \\ \Big(\mathrm{I},\, \big(\big[\begin{smallmatrix}0\\0\end{smallmatrix}\big], 1\big)\Big) \longmapsto \widehat{\zeta} : |n\rangle \mapsto \zeta\, |n\rangle \\ \Big(S,\, \big(\big[\begin{smallmatrix}0\\0\end{smallmatrix}\big], 0\big)\Big) \longmapsto \widehat{S} : |n\rangle \mapsto \frac{1}{\sqrt{K}} \sum_{\widehat{n}=0}^{k-1} \zeta^{2n\widehat{n}} |\widehat{n}\rangle \\ \Big(T,\, \big(\big[\begin{smallmatrix}0\\0\end{smallmatrix}\big], 0\big)\Big) \longmapsto \widehat{T} : |n\rangle \mapsto e^{-\pi \mathrm{i}/12}\, \zeta^{(n^2)}\, |n\rangle\,. \end{cases} \tag{89}$$

Proof. **(i)** That we have irreducible unitary representation of the subgroup $\widehat{\mathbb{Z}^{2g}}$ is Lem. 4.28, noting that $\mathrm{ord}(\zeta) = 2K$ and $\dim(\mathcal{H}) \simeq K$.
(ii) To see that we also have a representation of the subgroup $\mathrm{SL}_2(\mathbb{Z})$ it is sufficient to show that the operators $\widehat{S}$ and $\widehat{T}$ respect the relations (41). To that end it is useful, for the moment, to abbreviate the phase factor of $\widehat{T}$ as "c_K", hence to write:

$$\widehat{T} = \tfrac{1}{c_K}\, e^{\frac{\pi \mathrm{i}}{K} n^2} \qquad \text{with} \qquad c_K := e^{\pi \mathrm{i}/12}\,. \tag{90}$$

Now first, we find

$$\begin{aligned} \widehat{S}\widehat{S}|n\rangle &\equiv \widehat{S}\Big(\tfrac{1}{\sqrt{K}} \textstyle\sum_{\widehat{n}} e^{\frac{2\pi \mathrm{i}}{K}\widehat{n}\, n} |\widehat{n}\rangle\Big) \\ &\equiv \textstyle\sum_{\widehat{\widehat{n}}} \underbrace{\tfrac{1}{K} \textstyle\sum_{\widehat{n}} e^{\frac{2\pi \mathrm{i}}{K}\widehat{n}\,(n + \widehat{\widehat{n}})}}_{\delta_0\left(n + \widehat{\widehat{n}} \,\mathrm{mod}\, K\right)} |\widehat{\widehat{n}}\rangle \\ &= \big| -n \,\mathrm{mod}\, K\big\rangle \qquad \text{by (149).} \end{aligned} \tag{91}$$

This immediately implies that $\widehat{S}^4 = \mathrm{id}$ and that, with

$$\widehat{T}\,\widehat{S}|n\rangle = \tfrac{1}{k^{1/2} c_K} \textstyle\sum_{\widehat{n}} e^{\frac{\pi \mathrm{i}}{K}(\widehat{n}^2 + 2\widehat{n}\, n)} |\widehat{n}\rangle\,,$$

also $\widehat{S}^2(\widehat{T}\widehat{S}) = (\widehat{T}\widehat{S})\widehat{S}^2$. Hence the only remaining relation to check is $(\widehat{T}\widehat{S})^3 = \mathrm{id}$ or equivalently that

$$\widehat{T}^{-1} \circ \widehat{S}^{-1} \circ \widehat{T}^{-1} = \widehat{S} \circ \widehat{T} \circ \widehat{S}\,.$$

Unwinding the definitions gives

$$\begin{aligned} &\widehat{T}^{-1}\widehat{S}^{-1}\widehat{T}^{-1}|n\rangle = \widehat{T}^{-1}\widehat{S}^{-1} e^{-\frac{\pi \mathrm{i}}{K}n^2} |n\rangle \\ &= \widehat{T}^{-1} \tfrac{1}{\sqrt{K}} \textstyle\sum_{\widehat{n}} e^{\frac{\pi \mathrm{i}}{K}(-n^2 - 2\widehat{n}n)} |\widehat{n}\rangle \\ &= \tfrac{1}{\sqrt{K}} \textstyle\sum_{\widehat{n}} e^{\frac{\pi \mathrm{i}}{K}(-n^2 - 2\widehat{n}n - \widehat{n}^2)} |\widehat{n}\rangle \\ &= \tfrac{1}{\sqrt{K}} \textstyle\sum_{\widehat{n}} e^{-\frac{\pi \mathrm{i}}{K}(\widehat{n} + n)^2} |\widehat{n}\rangle \end{aligned} \quad \text{and} \quad \begin{aligned} &\widehat{S}\widehat{T}\widehat{S}|n\rangle = \widehat{S}\widehat{T} \tfrac{1}{\sqrt{K}} \textstyle\sum_{\widehat{n}} e^{\frac{2\pi \mathrm{i}}{K}\widehat{n}\, n} |\widehat{n}\rangle \\ &= \widehat{S} \tfrac{1}{\sqrt{k}} \textstyle\sum_{\widehat{n}} e^{\frac{\pi \mathrm{i}}{K}(2\widehat{n}\, n + \widehat{n}^2)} |\widehat{n}\rangle \\ &= \tfrac{1}{K} \textstyle\sum_{\widehat{n},\widehat{\widehat{n}}} e^{\frac{\pi \mathrm{i}}{K}(2\widehat{n}\, n + \widehat{n}^2 + 2\widehat{\widehat{n}}\widehat{n})} |\widehat{\widehat{n}}\rangle \\ &= \tfrac{1}{\sqrt{K}} \textstyle\sum_{\widehat{\widehat{n}}} \underbrace{\tfrac{1}{\sqrt{K}} \textstyle\sum_{\widehat{n}} e^{\frac{\pi \mathrm{i}}{K}(\widehat{n} + (n + \widehat{\widehat{n}}))^2}}_{c_K^3} e^{-\frac{\pi \mathrm{i}}{K}(n + \widehat{\widehat{n}})^2} |\widehat{\widehat{n}}\rangle\,. \end{aligned} \tag{92}$$

The term over the brace is a constant in n and $\widehat{\widehat{n}}$, by the assumption that k is even [14], whence the relation is satisfied if the normalization factor c_K in (90) is chosen as claimed, because the quadratic Gauss sum here evaluates to

$$c_K = \Big(\tfrac{1}{\sqrt{K}} \textstyle\sum_{n=0}^{K-1} e^{\frac{\pi \mathrm{i}}{K}n^2}\Big)^{1/3} \underset{(153)}{=} \big(e^{\pi \mathrm{i}/4}\big)^{1/3} = e^{\pi \mathrm{i}/12}\,. \tag{93}$$

[14] Since the summands in $\sum_{n=0}^{K-1} e^{\frac{\pi \mathrm{i}}{K}n^2}$ are K-periodic for even K, $e^{\frac{\pi \mathrm{i}}{K}(n+K)^2} = e^{\frac{\pi \mathrm{i}}{K}n^2} e^{\pi \mathrm{i}(2n+K)} \underset{K \text{ even}}{=} e^{\frac{\pi \mathrm{i}}{K}n^2}$, the sum is invariant under replacing $n \mapsto n + a$ for $a \in \mathbb{N}$.

(iii) Finally, we need to see that the semidirect product structure is respected, hence that

$$\widehat{W}_{M\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}\,\widehat{M}|n\rangle \;=\; \widehat{M}\,\widehat{W}_{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}|n\rangle \qquad \forall \begin{cases} M & \in \;\; \mathrm{SL}_2(\mathbb{Z}) \\ (a,b) & \in \;\; \mathbb{Z}^2 \\ |n\rangle & \in \;\; \mathcal{H}_{T^2}\,. \end{cases}$$

It is sufficient to check this on the generators, where explicit computation yields, indeed:

$$\begin{aligned} \widehat{W}_{S\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\,\widehat{S}\big|[n]\big\rangle &\equiv \widehat{W}^{-1}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}\Big(\tfrac{1}{\sqrt{|K|}}\textstyle\sum_{\widehat{n}} e^{\frac{2\pi \mathrm{i}}{K}\widehat{n}\,n}|\widehat{n}\rangle\Big) \\ &= \tfrac{1}{\sqrt{K}}\textstyle\sum_{\widehat{n}} e^{\frac{2\pi \mathrm{i}}{K}(\widehat{n}+1)\,n}\big|\widehat{n}\big\rangle \\ &= e^{\frac{2\pi \mathrm{i}}{K}n}\,\widehat{S}\big|n\big\rangle \\ &= \widehat{S}\,\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}|n\rangle\,, \end{aligned} \qquad \begin{aligned} \widehat{W}_{S\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}\,\widehat{S}|n\rangle &\equiv \widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\Big(\tfrac{1}{\sqrt{K}}\textstyle\sum_{\widehat{n}} e^{\frac{2\pi \mathrm{i}}{K}\widehat{n}\,n}|\widehat{n}\rangle\Big) \\ &= \tfrac{1}{\sqrt{K}}\textstyle\sum_{\widehat{n}} e^{\frac{2\pi \mathrm{i}}{K}\widehat{n}}e^{\frac{2\pi \mathrm{i}}{K}\widehat{n}\,n}|\widehat{n}\rangle \\ &= \tfrac{1}{\sqrt{K}}\textstyle\sum_{\widehat{n}} e^{\frac{2\pi \mathrm{i}}{K}\widehat{n}\,(n+1)}|\widehat{n}\rangle \\ &= \widehat{S}\,\widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}|n\rangle\,, \end{aligned}$$

and

$$\begin{aligned} \widehat{W}_{T\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\,\widehat{T}|n\rangle &\equiv \widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\,\tfrac{1}{c_k}e^{\frac{\pi \mathrm{i}}{K}n^2}|n\rangle \\ &= \tfrac{1}{c_k}e^{\frac{2\pi \mathrm{i}}{K}n}e^{\frac{\mathrm{i}\pi}{K}n^2}|n\rangle \\ &= \widehat{T}\,\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}|n\rangle\,, \end{aligned} \qquad \begin{aligned} \widehat{W}_{T\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}\,\widehat{T}|n\rangle &\equiv \tfrac{1}{c_K}\widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}\,\widehat{W}_{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}\,e^{\frac{\pi \mathrm{i}}{K}}\,e^{\frac{\pi \mathrm{i}}{K}n^2}|n\rangle \\ &= \tfrac{1}{c_K}e^{\frac{\pi \mathrm{i}}{K}(n^2+2n+1)}|n+1\rangle \\ &= \tfrac{1}{c_K}e^{\frac{\pi \mathrm{i}}{K}(n+1)^2}|n+1\rangle \\ &= \widehat{T}\,\widehat{W}_{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}|n\rangle\,, \end{aligned}$$

where in the first step of the last case, we used (85). □

Remark 4.31 (**Comparison to modular data of abelian Chern-Simons theory on the torus).** The content of Prop. 4.30 captures the *modular data* (cf. [74]) expected for FQH systems on the torus:

(i) The algebra (86) of the $\widehat{W}_{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}$ is just that expected [202, (5.28)] of quantum observables for anyonic topological order on the torus as predicted [21, (17)][157, (32)][75, Prop. 2.2] by abelian Chern-Simons theory at *level* $k = K/2 \in \mathbb{Z}$ (8), and equivalently by U(1)-WZW conformal field theory [204, (4.3-4)].

(ii) Similarly, the operators $\widehat{S}$ and $\widehat{T}$ according to (89) implement the known modular group representation on quantum states of abelian Chern-Simons theory [209, (5.3)][137, p 65] (following [87][71, (5,7)]) and equivalently of conformal characters of the U(1) 2dCFT [74, Ex. 1]. [15]

(iii) The fact of Prop. 4.30 that, jointly, these operators constitute a representation of the semidirect product of the modular group with the integer Heisenberg group is maybe implicit in the literature but does not seem to be citable.

On the other hand, the content of Prop. 4.30 captures only the (experimentally unobserved!) filling factor *unit fractions* $\nu = 1/K$ with even denominator and is hence unsatisfactory by itself. The traditional way to obtain non-unit filling fractions is to generalize to $\mathrm{U}(1)^n$-Chern-Simons theory for $n > 1$ with non-trivial "K-matrices" [210, (2.31)]. But we next see that non-unit fractions and then also odd denominators are already exhibited by 2-cohomotopical flux quanta:

Lemma 4.32 (**More general representations).** *The same formulas* (89) *constitute a representation more generally, for*

$$(K,\,p) \in \mathbb{N}_{>0}\times\mathbb{Z} \quad \text{s.t.} \quad \begin{cases} Kp \in 2\mathbb{Z}, \\ \sum_{n=0}^{K-1} e^{\pi \mathrm{i}\frac{p}{K}n^2} \neq 0 \end{cases} \qquad \text{with} \quad \zeta := e^{\pi\mathrm{i}\frac{p}{K}}\,. \tag{94}$$

Proof. Straightforward inspection shows readily that the proof of Prop. 4.30 goes through verbatim with all factors of $e^{\pi\mathrm{i}/K}$ generalized to ζ (94) — the only step that needs attention is that from (92) to (93): But for the term over the brace in (92) to be constant in n and $\widehat{\widehat{n}}$ it is clearly sufficient that K *or* p are even, hence that their product Kp is even, in which case the normalization factor c_K in (93) can be found unless that term is zero. These are exactly the two conditions assumed in (94). □

[15] The exponentiated "central charge" $c_K = e^{2\pi\mathrm{i}/24}$ appearing in (90) and (93) seems to be missed in the earlier literature [209, (5.3)][137, p 65][87][71, (5,7)] (and also the necessity of K being even, at this point, is not stated by some of these authors) but is now well-known to appear, cf. [74, (3.1b)][188, (26)].

Proposition 4.33 (General 2-cohomotopical quantum states over the torus). *The representation (89) exists and is irreducible already when restricted to $\widehat{\mathbb{Z}^2}$, iff*

$$
\begin{array}{ll}
 & \big(K \in 2\mathbb{N}_{>0} \quad \text{and} \quad p \in 2\mathbb{Z}+1\big) \\
\text{or} & \big(K \in 2\mathbb{N}+1 \quad \text{and} \quad p \in 2\mathbb{Z}_{\neq 0} \;\;\big)
\end{array}
\quad \text{and} \quad \gcd(p,K) = 1 \qquad \textit{with} \qquad \zeta := e^{\pi \mathrm{i} \frac{p}{K}} . \tag{95}
$$

Proof. To see that these representations exist as claimed, by Lem. 4.32 it just remains to check that the Gauss sum does not vanish: Indeed, for K even and p odd we have

$$
\sum_{n=0}^{K-1} e^{\pi \mathrm{i} \frac{p}{K} b^2} \underset{\text{by (155)}}{=} e^{\pm \pi \mathrm{i}} \sqrt{K} \underbrace{\big(K/2 \,\big|\, p\big)}_{\neq 0 \text{ by (152)}} \neq 0 \,,
$$

while for K odd and p even we have

$$
\sum_{n=0}^{K-1} e^{\pi \mathrm{i} \frac{p}{K} n^2} = \sum_{n=0}^{K-1} e^{\frac{2\pi \mathrm{i}}{K}(p/2) n^2} \underset{\text{by (151)}}{=} \underbrace{\big(p/2 \big| K\big)}_{\neq 0 \text{ by (152)}} \underbrace{\textstyle\sum_{n=0}^{K-1} e^{\frac{2\pi \mathrm{i}}{K} n^2}}_{\neq 0 \text{ by (150)}} \neq 0 \,.
$$

Then to see that these representations are irreducible already when restricted to $\widehat{\mathbb{Z}^2}$: By the assumption that $\gcd(p,K) = 1$ we have

$$
\mathrm{ord}(\zeta) \equiv \mathrm{ord}\big(e^{\pi \mathrm{i} p/K}\big) = \left\{ \begin{array}{lcl} 2K & | & K \text{ even (since then } p \text{ odd)} \\ K & | & K \text{ odd (since then } p \text{ even).} \end{array} \right. \tag{96}
$$

Recalling that the dimension of the representation is K in either case, this implies irreducibility by Lem. 4.28. □

With this we have realized all braiding phase fractions which have a factor of 2 either in their numerator or their denominator. To further generalize, we remember that our surfaces carry a spin structure (Rem. 2.3), and that in taking the mapping class group of the torus to be all of $\mathrm{SL}_2(\mathbb{Z})$ we have so far implicitly considered Σ^2_1 as equipped with the "trivial" spin structure "pp" (43). If instead we consider Σ^2_1 equipped with the aa-spin structure, as befits ordinary fermions on the torus, then the mapping class is only the subgroup $\mathrm{MCG}(\Sigma^2_1)^{\mathrm{aa}} \subset \mathrm{SL}_2(\mathbb{Z})$ (44) and we have:

Proposition 4.34 (2-Cohomotopical quantum states over the aa-spin torus). *For all*

$$
(K,p) \in \mathbb{N}_{>0} \times \mathbb{Z} \,, \;\; \gcd(p,K) = 1 \qquad \text{with} \qquad \zeta := e^{\pi \mathrm{i} \frac{p}{K}} ,
$$

the formulas (89) define a representation of the covariantized flux monodromy group on Σ^2_1 equipped with the aa*-spin structure (43), namely a homomorphism*

$$
\mathrm{MCG}(\Sigma^1)^{\mathrm{aa}} \ltimes \widehat{\mathbb{Z}^2} \longrightarrow \mathrm{U}\big(\mathcal{H}_{T^2}\big) \,,
$$

which is irreducible already as a representation of $\widehat{\mathbb{Z}^2}$.

Proof. To see that these representations exist: items (i) and (iii) in the proof of Prop. 4.30 work verbatim as before, it just remains to verify the analog of part (ii) there, namely that restricted to the aa-spin mapping class group the need for K to be even goes away. (In the case $p = 1$ this is asserted in [74, bottom of p 9].) But it is immediate that the presentation (41) is respected

$$
\widehat{S}^2 \circ \widehat{T}^2 = \widehat{T}^2 \circ \widehat{S}^2 \,,
$$

because $\widehat{S}^2 |n\rangle = \big|[-n]\big\rangle$ (91) and because the operator $\widehat{T}^2$ (89) is manifestly even as a function of n (being given by multiplication with ζ^{n_2}). With this, irreduciblility follows exactly as around (96). □

Now we may conclude the situation over the torus:

Theorem 4.35 (Classification of 2-cohomotopical flux quantum states on the torus).

(i) Over the torus with aa*-spin structure, the spaces a 2-cohomotopical flux quantum states (22), which are irreducible already before covariantization, are all isomorphic to the tensor product of 1D rep of $\mathrm{MCG}(\Sigma^2_1)^{\mathrm{aa}}$ with the representation $\mathcal{H}_{T^2}$ (89) for some $\zeta = e^{\pi \mathrm{i} \frac{p}{K}}$ with $\gcd(p,K) = 1$.*

(ii) The same holds over the torus with pp*-spin structure except that here the tensor is with a group character of $\mathrm{SL}_2(\mathbb{Z})$ and the braiding phase must satisfy the further condition that $Kp \in 2\mathbb{Z}$.*

(iii) All of these irreps have dimension ("ground state degeneracy") *equal to K.*

Proof. The claimed covariantizable irreps before covariantization are due to Prop. 4.29, and covariantizations subject to the stated conditions are established by Prop. 4.33 (for the pp-spin structure) and Prop. 4.34 (for the aa-spin structure). Then item (ii) of Lem. 4.23 says that this exhausts the possible covariantizations up to tensoring with an MCG-character, as claimed. □

Remark 4.36 (Fine-structure of topological order of 2-cohomotopical flux on the torus). There are precisely 12 distinct group characters $\mathrm{SL}_2(\mathbb{Z}) \to \mathbb{C}^\times$ (taking values in 12th roots of unity, cf. [32, Cor. 2.4]), so that Thm. 4.35 means that over the torus with pp-spin stucture, there are for every admissible braiding phase $\zeta = e^{\pi \mathrm{i} \frac{p}{K}}$ precisely 12 distinct irreducible spaces of quantum states, all 12 *essentially* as predicted by U(1)-Chern-Simons theory (cf. Rem. 4.31, except for the more general braiding phases ζ), in particular all having the same ground state degeneracy K, but differing subtly in the further fine-print of their "topological order", namely differing in the one of 12 possible sets of extra phases which they pick up under modular transformations.

On the other hand, for the aa-spin structure there are countably many distinct group homomorphisms $\mathrm{MCG}(\Sigma^2_1)^{\mathrm{aa}} \to \mathbb{C}^\times$ so that Thm. means that for each braiding phase in this situation there are these countably many irreducible spaces of quantum states differing by complex phases in their modular transformation property.

4.5 On punctured surfaces

Here we derive the observables on 2-cohomotopically quantized topological flux over *n-punctured surfaces*, which in practice will mean: Surfaces of conducting material where magnetic flux is *expelled* from (the vicinity of) n defect points (cf. Rem. 4.6).

It is clear (cf. Prop. 4.37) that covariantization of these observables reveals an action of the surface's n-braid group, but we find that the contribution to the observables from the flux monodromy (cf. Prop. 3.9) enhances this to the *framed* (or *ribbon*) braid group (111) as expected in generality for Chern-Simons theories (Rem. 4.46). Or rather, we find that what appears is its subgroup of framed braids of vanishing total framing.

Lemma 4.37 (Homotopy type of compactified n-punctured surface). *For $n \in \mathbb{N}_{\geq 1}$, the one-point compactification of the n-puncturing of a closed surface $\Sigma^2_{g,b}$ (9) is homotopy equivalent to the wedge sum (126) of that surface with $(n-1)$ circles:*

$$\left(\Sigma^2_{g,b,n}\right)_{\cup\{\infty\}} \simeq_{\int} \Sigma^2_{g,b} \vee \bigvee_{n-1} S^1 \,. \tag{97}$$

Proof. For $n = 1$ the statement is immediate.

For $n = 2$ consider the topological space X obtained by attaching to $\Sigma^2_{g,b}$ an interval with endpoints glued to two distinct points $s_1, s_2 \in \Sigma^2_{g,b}$ (the would-be positions of the punctures), hence consider this pushout of topological spaces:

$$\begin{array}{ccc} S^0 & \xrightarrow{(s_1,s_2)} & \Sigma^2_{g,b} \\ \downarrow & \text{(po)} & \downarrow \\ D^1 & \xrightarrow[\iota_{\mathrm{ext}}]{} & X \,. \end{array}$$

Moreover, consider another arc *inside* $\Sigma^2_{g,b}$ connecting these two points

$$D^1 \xhookrightarrow{\iota_{\mathrm{int}}} \Sigma^2_{g,b} \hookrightarrow X \,.$$

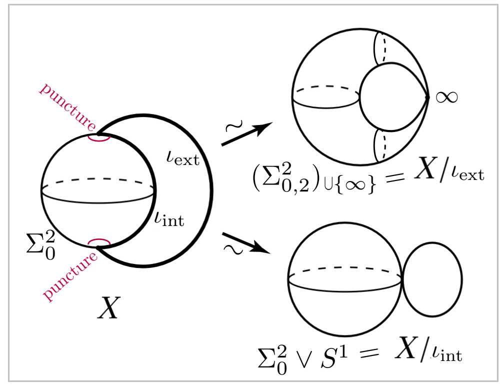
Both of these arcs are evidently contractible sub-complexes of X, and so the quotient projections obtained by identifying either arc with a single point are weak homotopy equivalences (cf. [96, p 11]):

$$\begin{array}{ccccc} & \overset{\simeq_{\int}}{\swarrow} & X & \overset{\simeq_{\int}}{\searrow} & \\ X/\iota_{\mathrm{ext}}(D^1) & & \underset{\simeq_{\int}}{\dashrightarrow} & & X/\iota_{\mathrm{int}}(D^1) \,. \end{array} \tag{98}$$

Now, as indicated in (98), the "external" quotient on the left is evidently homeomorphic to the desired one-point compactification, while the "internal" quotient on the right is evidently homeomorphic to the claimed wedge sum. This proves the claim for $n = 2$.

The graphics on the right illustrates the situation for the case $g, b = 0$.

The general statement, including the case $n > 2$, follows analogously by attaching further arcs in this fashion, cf. Fig. A. □



graphics adapted from [96, p 11]

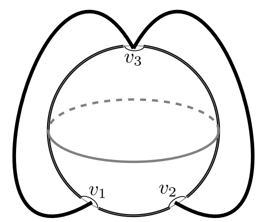


Figure A. There are several ways to attach arcs for $n > 2$ punctures in the above proof of Lem. 4.37, all equivalent in the resulting homotopy type. But for the analysis ofbraiding that follows in Prop. 4.42 it is useful (cf. *Fig. S*) to single out one puncture v_n and take the $n-1$ arcs to connect this one puncture to each of the $n-1$ remaining ones. The case $g, b = 0$ and $n = 3$ is illustrated on the right.

With this result in hand, it is straightforward to compute the solitonic flux monodromy (21) through a punctured surface:

Proposition 4.38 (Flux monodromy through punctured surface). *For $g, b, \in \mathbb{N}$ and $n \in \mathbb{N}_{>0}$, we have an isomorphism*

$$\pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, S^2\big)\Big) \;\simeq\; \pi_1\Big(\mathrm{Map}^*_0\big(\Sigma^2_{g,b},\, S^2\big)\Big) \,\times\, \mathbb{Z}^{n-1}\,. \tag{99}$$

Proof. We may compute as follows:

$$\begin{aligned}
\pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, S^2\big)\Big) &\simeq \pi_1\Big(\mathrm{Map}^*_0\big(\Sigma^2_{g,b} \vee \textstyle\bigvee_{n-1} S^1,\, S^2\big)\Big) && \text{by (97)}\\
&= \pi_1\Big(\mathrm{Map}^*_0\big(\Sigma^2_{g,b} S^2\big) \times \textstyle\prod_{n-1} \mathrm{Map}^*\big(S^1,\, S^2\big)\Big) && \text{by (127)}\\
&= \pi_1\Big(\mathrm{Map}^*_0\big(\Sigma^2_{g,b} S^2\big)\Big) \times \textstyle\prod_{n-1} \pi_1\Big(\mathrm{Map}^*\big(S^1,\, S^2\big)\Big) &&\\
&= \pi_1\Big(\mathrm{Map}^*_0\big(\Sigma^2_{g,b} S^2\big)\Big) \times \textstyle\prod_{n-1} \pi_2(S^2) && \text{by (124),}
\end{aligned}$$

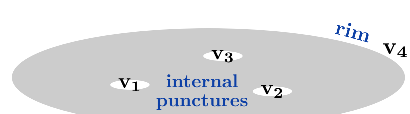
whence the claim follows by $\pi_2(S^2) \simeq \mathbb{Z}$. □

Remark 4.39 (Internal punctures and the rim of the sample).

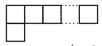
(i) In words, Prop. 4.38 may be understood as saying that the (monodromy of) solitonic flux on an n-punctured surface, for $n \geq 1$, has (not n but) $n-1$ generators associated with $n-1$ of the punctures, while associated with the remaining puncture is the solitonic flux of the surface itself, regarded as containing its point-at-infinity.

(ii) Below in Prop. 4.42 we give further analysis of this situation, showing that it is governed by the $(n-1)$-dimensional "standard irrep" of the symmetric group Sym_n (Def. 4.40 below), but first to note that the discrepancy between n and $n-1$ here has a clear physical meaning at least for $g = 0$, recalling that the n-punctured sphere is homeomorphically the $n-1$-punctured plane, hence the n-punctured open disk $\Sigma^2_{0,0,n} \simeq \mathbb{R}^2 \setminus \{x_1, \cdots, x_{n-1}\}$ (10):

We may think of the nth puncture as modeling the outer rim of the slab of material that is represented by Σ^2, and of the remaining $n-1$ the actual punctures/defects as one will recognize them in the laboratory.



Definition 4.40 (Standard irrep of symmetric group). For $n \in \mathbb{N}_{>0}$ the "*standard*" $\mathbb{C}$-linear representation of the symmetric group Sym_n is the $n-1$-dimensional complex irrep classified by the partition $(n-1, 1)$, hence

by the Young diagram  : this is the quotient of the defining n-dimensional permutation representation by the trivial 1d representation (cf. [70, p 9 & Ex. 4.6][116, Def. 2.5]).

More concretely, with respect to the canonical linear basis

$$\mathbb{C}^n \;\simeq\; \mathbb{C}\langle v_1, v_2, \cdots, v_n \rangle\,, \tag{100}$$

- the *defining permutation representation* of Sym_n is given, for $\sigma \in \mathrm{Sym}_n$, by $\sigma(v_i) := v_{\sigma(i)}$, hence in terms of the Artin generators $(b_i)_{i=1}^{n-1}$ (35) by

$$b_i(v_j) \;=\; \begin{cases} v_{i+1} & \mid \;\; j = i \\ v_i & \mid \;\; j = i+1 \\ v_j & \mid \;\; \text{otherwise}\,, \end{cases}$$

- the trivial 1d irrep inside this is

$$\mathbf{1} \;\simeq\; \mathbb{C}\big\langle \underbrace{v_1 + v_2 + \cdots + v_n}_{=:\, t} \big\rangle \;\hookrightarrow\; \mathbb{C}^n\,, \tag{101}$$

- and the *standard representation* is:

$$\mathbf{n\text{–}1} \;\simeq\; \mathbb{C}\big\langle \underbrace{v_i - v_n}_{=:\, e_i} \big\rangle_{i=1}^{n-1} \;\hookrightarrow\; \mathbb{C}^n\,, \tag{102}$$

  with the Artin generators acting as (cf. *Fig. S*)

$$b_{i<n-1}(e_j) \;=\; \begin{cases} e_{i+1} & \mid \;\; j = i \\ e_i & \mid \;\; j = i+1 \\ e_j & \mid \;\; \text{otherwise} \end{cases} \tag{103}$$

$$b_{n-1}(e_j) \;=\; \begin{cases} e_j - e_{n-1} & \mid \;\; j < n-1 \\ \quad -e_{n-1} & \mid \;\; j = n-1\,. \end{cases} \tag{104}$$

  This is clearly the extension of scalars from a $\mathbb{Z}$-linear representation on $\mathbb{Z}^{n-1}$, which we shall hence refer to as the *standard $\mathbb{Z}$-linear representation* of Sym_n.

Hence over $\mathbb{C}$ we have a reduction of the defining Sym_n-representation explicitly like this:

$$\begin{array}{ccc} t & \longmapsto & v_1 + \cdots + v_n \\ e_i & \longmapsto & v_i - v_n \\ \mathbf{1} \oplus \mathbf{n\!-\!1} & \underset{\sim}{\rightleftarrows} & \mathbb{C}^n_{\mathrm{def}} \qquad \in \mathrm{Rep}_{\mathbb{C}}(\mathrm{Sym}_3) \\ \frac{t - (e_1 + \cdots + e_{n-1})}{n} & \longleftarrow & v_n \\ e_i + \frac{t - (e_1 + \cdots + e_{n-1})}{n} & \longleftarrow & v_{i<n}\,, \end{array}$$

which also shows that over the integers we only have a monomorphism

$$\mathbf{1} \,\oplus\, \mathbf{n\!-\!1} \hookrightarrow \mathbb{Z}^n_{\mathrm{def}} \qquad \in \mathrm{Rep}_{\mathbb{Z}}(\mathrm{Sym}_3) \tag{105}$$

with image the subgroup of n-tuples whose sum is divisble by n.

Example 4.41 (Irreps of Sym_3, cf. [70, §1.3]). For $n = 3$, the sum-of-squares formula (cf. [51, Thm. 3.1(ii)])

$$1^2 + 1^2 + 2^2 \;=\; 6 \;=\; |\mathrm{Sym}_3|\,,$$

implies that the standard representation $\mathbf{2}$ (102) is the only irrep (cf. Rem. 2.14) of Sym_3, besides the 1-dimensional irreps (the trivial one $\mathbf{1}$ and the sign representation $\mathbf{1}_{\mathrm{sgn}}$), hence that this irrep knows all about potential non-abelian defect anyon in situations where their motion group is Sym_3, see Rem. 4.48 and §4.7 below.

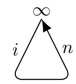
Proposition 4.42 (Braid group action on flux monodromy over punctured surface). *For $n \geq 1$, the action (53) of the Artin generators $b_i \in \mathrm{Br}_n(\Sigma^2_{g,b,n}) \to \mathrm{MCG}(\Sigma^2_{g,b,n})$ (48) on the flux monodromy (99) over an n-punctured surface (9) is via the $\mathbb{Z}$-linear* standard representation (Def. 4.40) *of Sym_n on the $\mathbb{Z}^{n-1}$-factor and the identity on the first factor.*

Proof. See *Fig. S* for illustration of the following ananlysis.

We may without restriction assume the punctures to jointly sit within an open disk inside the surface, $\{v_1, \cdots v_n\} \subset D^2 \subset \Sigma^2_{g,b}$. Then the one-point compactification $(\Sigma^2_{g,b,n})_{\cup\{\infty\}}$ may be obtained by enlarging each puncture to a little missing open disk, erecting a little cone (horn) over the boundary of this disk, and making these cones bend over to make (just) their tips touch – this joint tip is the point ∞. Let then

$$\ell_i \;\in\; \pi_0 \,\mathrm{Map}^*\big(S^1,\, (\Sigma^2_{g,b,n})_{\cup\{\infty\}}\big) \;\simeq\; \pi_1\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}}\big)\,, \qquad i \in \{1, \cdots, n-1\} \tag{106}$$

denote the homotopy class of a loop that starts at ∞, runs down through the ith cone and back through the nth cone

$$\ell_i \;:=$$  .

This is also illustrated by the dashed arrows in the top panel of Fig. S, which, for graphical convenience, shows not the cones themselves, but the arcs whose contraction to $\{\infty\}$ produces the cones, according to the proof of Lem. 3.7.

Now consider a map

$$f \;\in\; \pi_1 \,\mathrm{Map}^*\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, S^2\big) \underset{(135)}{\simeq} \pi_0 \,\mathrm{Map}\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, \Omega S^2\big)$$

with its homotopy class decomposed, according to Prop. 4.38, as

$$\begin{array}{ccc} \pi_0 \,\mathrm{Map}^*\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, \Omega S^2\big) & \simeq & \pi_0 \,\mathrm{Map}^*\big((\Sigma^2_{g,b})_{\cup\{\infty\}},\, \Omega S^2\big) \times \prod_{n-1} \pi_0 \,\mathrm{Map}^*\big(S^1,\, \Omega S^2\big) \\ [f] & \longmapsto & \big([\widetilde{f}], (e_1, \cdots e_{n-1})\big)\,, \end{array} \tag{107}$$

where the integer classes $e_i := [f_* \ell_i] \in \mathbb{Z}$ come from the restriction of f to these loops ℓ_i (106):

$$\begin{array}{ccc} \pi_0 \,\mathrm{Map}^*\big(S^1,\, (\Sigma^2_{g,b,n})_{\cup\{\infty\}}\big) & \xrightarrow{\;f_*\;} & \pi_0 \,\mathrm{Map}^*\big(S^1,\, \Omega S^2\big) \;\simeq\; \mathbb{Z} \\ \ell_i & \longmapsto & e_i\,. \end{array} \tag{108}$$

We need to determine the effect on these components of precomposition with a diffeomorphism of $\Sigma^2_{g,b,n}$ representing the mapping class of the ith Artin generator b_i (34). This diffeo may be chosen such that its unique continuous extension to the one-point compactification,

$$b_i \;\in\; \pi_0 \,\mathrm{Map}^*\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}},\, (\Sigma^2_{g,b,n})_{\cup\{\infty\}}\big)\,,$$

restricts for each $j \in \{1, \cdots, n\}$ to a homeomorphism from the jth cone onto the σ_jth cone, where σ is the permutation underlying the Artin generator. Then direct inspection (illustrated in Fig. S) shows

– for $i < n-1$ that

$$\begin{array}{ccccc} \pi_0 \,\mathrm{Map}^*\big(S^1,\, (\Sigma^2_{g,b,n})_{\cup\{\infty\}}\big) & \xrightarrow{\;b_{i*}\;} & \pi_0 \,\mathrm{Map}^*\big(S^1,\, (\Sigma^2_{g,b,n})_{\cup\{\infty\}}\big) & \xrightarrow{\;f_*\;} & \pi_0 \,\mathrm{Map}^*\big(S^1,\, \Omega S^2\big) \\ \ell_j & \underset{\text{left of Fig. S}}{\longmapsto} & \left\{ \begin{array}{c|l} \ell_{i+1} & j = i \\ \ell_i & j = i+1 \\ \ell_j & \text{otherwise} \end{array} \right\} & \underset{(108)}{\longmapsto} & \left\{ \begin{array}{c|l} e_{i+1} & j = i \\ e_i & j = i+1 \\ e_j & \text{otherwise} \end{array} \right\}, \end{array} \tag{109}$$

– for $i = n-1$ that

$$\begin{array}{ccccc} \pi_0 \,\mathrm{Map}^*\big(S^1,\, (\Sigma^2_{g,b,n})_{\cup\{\infty\}}\big) & \xrightarrow{\;b_{i*}\;} & \pi_0 \,\mathrm{Map}^*\big(S^1,\, (\Sigma^2_{g,b,n})_{\cup\{\infty\}}\big) & \xrightarrow{\;f_*\;} & \pi_0 \,\mathrm{Map}^*\big(S^1,\, \Omega S^2\big) \\ \ell_j & \underset{\text{right of Fig. S}}{\longmapsto} & \left\{ \begin{array}{l|l} \ell_{n-1}^{-1} \circ \ell_j & j < n-1 \\ \ell_{n-1}^{-1} & j = n-1 \end{array} \right\} & \underset{(108)}{\longmapsto} & \left\{ \begin{array}{r|l} e_j - e_{n-1} & j < n-1 \\ -e_{n-1} & j = n-1 \end{array} \right\}. \end{array} \tag{110}$$

Comparison of these formulas with (103) and (104), respectively, identifies the precomposition by Artin generators b_i on maps f to act on their components (107) as

$$b_{i*} \;:\; \big([\widetilde{f}],\, (e_1, \cdots, e_{n-1})\big) \;\longmapsto\; \Big([\widetilde{f}],\, \big(b_i(e_1), \cdots, b_i(e_{n-1})\big)\Big)\,,$$

where on the right $b_i(-) : \mathbb{Z}^{n-1} \to \mathbb{Z}^{n-1}$ is the action of the $\mathbb{Z}$-linear standard representation, as claimed.

Finally, the action on the class $[\widetilde{f}]$ is trivial, as claimed, because this is via the image of b_i under two steps $\mathrm{Br}_n\big(\Sigma^2_{g,b}\big) \to \mathrm{MCG}\big(\Sigma^2_{g,b,n}\big) \to \mathrm{MCG}\big(\Sigma^2_{g,b}\big)$ of the generalized Birman exact sequence (47), trivial by exactness. □

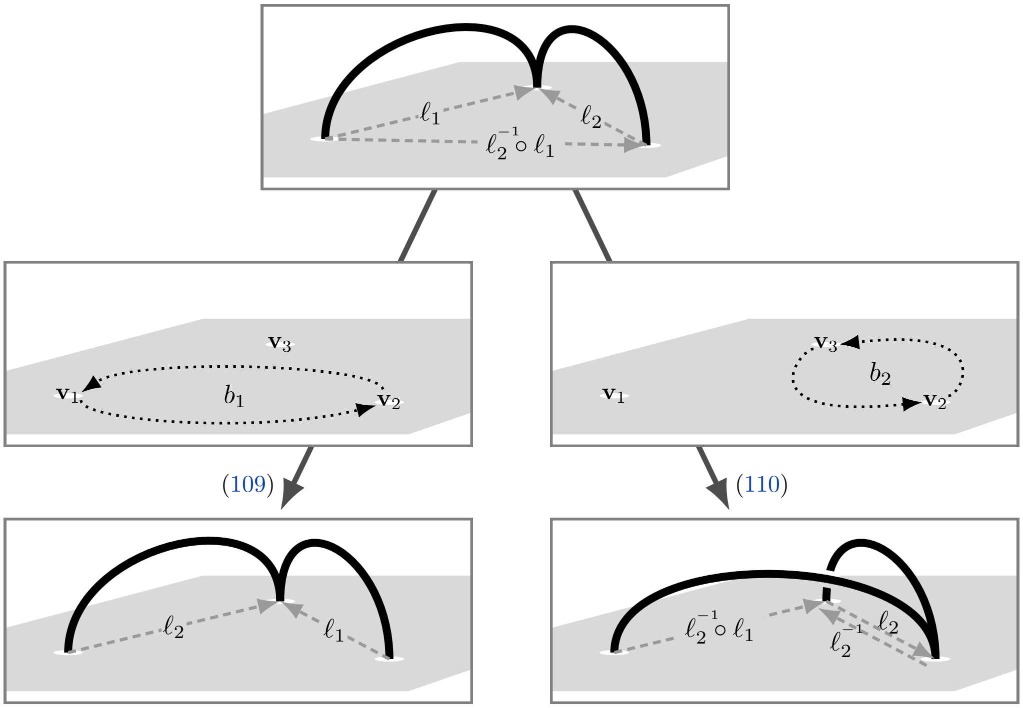


Figure S – Effect of braiding on compactification of punctured surface – proof of Prop. 4.42. Using that the homotopy type of the 1-point compactification $(\Sigma^2_{g,b,n})_{\cup\{\infty\}}$ of a punctured surface is obtained by attaching, for all $j < n$, an arc from the jth to the nth puncture (cf. the proof of Lem. 4.37), the above graphics shows the effect of the Artin generator (35) mapping classes $b_i \in \mathrm{Br}_n(\Sigma^2_{g,b}) \to \mathrm{MCG}\big(\Sigma^2_{g,b,n}\big)$ on these arcs — and on the indicated generators $\ell_j \in \pi_1\big((\Sigma^2_{g,b,n})_{\cup\{\infty\}}, \infty\big)$ obtained after contracting the arcs to $\{\infty\}$ — making manifest that after pushing each loop forward to ΩS^2 this gives the "standard" representation (Def. 4.40) of the symmetric group Sym_n.

4.6 On punctured disks

For recognizing, in terms of known group structures, the covariantized flux observables resulting from Prop. 4.42, recall:

Definition 4.43 (Framed/ribbon braid group [140][125], cf. [128, §3.2]**).** The *framed braid group* or *ribbon braid group* of a surface is the wreath product of the ordinary surface braid group (33) with the integers, hence its semidirect product with $\mathbb{Z}^n = \mathbb{Z} \times \cdots \times \mathbb{Z}$ via its defining permutation action on the n factors:

$$\mathrm{RBr}_n(\Sigma^2) \;:=\; \mathbb{Z} \wr \mathrm{Br}_n(\Sigma^2) \;\simeq\; \mathbb{Z}^n \rtimes_{\mathrm{def}} \mathrm{Br}_n(\Sigma^2)\,. \tag{111}$$

A ribbon braid in (111) may be understood as a braid of ribbons which, besides braiding with each other, may each twist an integer number of times in themselves, as in Fig. FL: The closure of a ribbon braid is a framed link.

Similarly, via the integral *standard representation* of Sym_n (Def. 4.40) we may also form the variant $\mathbb{Z}^{n-1} \rtimes_{\mathrm{st}} \mathrm{Br}_n(\Sigma^2)$ of the framed braid group. This is the subgroup on the elements whose *total framing* number vanishes:

Lemma 4.44 (Framed braids of zero total framing among all framed braids). *We have subgroup inclusions*

$$\begin{array}{ccccc}
\substack{\text{with vanishing}\\ \text{total framing}} & & \substack{\text{with total framing}\\ \text{divisible by } n} & & \substack{\text{with arbitrary}\\ \text{total framing}} \\
\overbrace{\mathbb{Z}^{n-1} \rtimes_{\mathrm{st}} \mathrm{Br}_n(\Sigma^2)}^{\text{group of framed braids}} & \hookrightarrow & \mathbb{Z} \times \big(\mathbb{Z}^{n-1} \rtimes_{\mathrm{st}} \mathrm{Br}_n(\Sigma^2)\big) & \hookrightarrow & \mathbb{Z}^n \rtimes_{\mathrm{def}} \mathrm{Br}_n(\Sigma^2) \equiv \mathrm{FBr}_n(\Sigma^2) \\
\text{standard rep} & & \text{standard rep} & & \text{defining rep} \\
e_i & \longmapsto & e_i & \longmapsto & v_i - v_n \\
 & & t & \longmapsto & v_1 + \cdots + v_n \\
b_i & \longmapsto & b_i & \longmapsto & b_i
\end{array} \tag{112}$$

Proof. By (105). ☐

Proposition 4.45 (**2-Cohomotopical covariant flux monodromy on punctured disks).** *For $n \geq 1$, the 2-Cohomotopical covariant fux monodromy*

(i) *on the n-punctured sphere $\Sigma^2_{0,0,n}$ is to the group of framed braids of with total framing divisible by n (112), quotiented by* rot $\in \mathrm{Br}_n(\Sigma^2) \hookrightarrow \mathrm{FBr}_n(S^2)$ (50)*:*

$$\pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_{0,0,n})_{\cup\{\infty\}},\, S^2\big) \mathbin{/\!\!/} \mathrm{Diff}^{+,\partial}\big(\Sigma^2_{0,0,n}\big)\Big) \quad\hookrightarrow\quad \mathrm{FBr}_n(S^2)/\mathrm{rot}\,; \tag{113}$$

(ii) *on the n-punctured closed disk $\Sigma^2_{0,1,n}$ (9) is the subgroup (112) of framed braids of vanishing total framing (112):*

$$\pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_{0,1,n})_{\cup\{\infty\}},\, S^2\big) \mathbin{/\!\!/} \mathrm{Diff}^{+,\partial}\big(\Sigma^2_{0,0,n}\big)\Big) \;\simeq\; \mathbb{Z}^{n-1} \rtimes_{\mathrm{st}} \mathrm{Br}_n \quad\hookrightarrow\quad \mathrm{FBr}_n\,. \tag{114}$$

Proof. In both cases, we have

$$\begin{aligned}
&\pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_{0,b,n})_{\cup\{\infty\}}, S^2\big) \mathbin{/\!\!/} \mathrm{Diff}^{+,\partial}\big(\Sigma^2_{0,b,n}\big)\Big) \\
&\simeq\; \pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_{0,b,n})_{\cup\{\infty\}},\, S^2\big)\Big) \rtimes \mathrm{MCG}(\Sigma^2_{0,b,n}) && \text{by (53)} \\
&\simeq\; \Big(\pi_1\,\mathrm{Map}^*_0\big((\Sigma^2_{0,b})_{\cup\{\infty\}},\, S^2\big) \times \mathbb{Z}^{n-1}\Big) \rtimes \mathrm{MCG}(\Sigma^2_{0,b,n}) && \text{by (99)} \\
&\simeq\; \pi_1\,\mathrm{Map}^*_0\big((\Sigma^2_{0,b})_{\cup\{\infty\}},\, S^2)\big) \times \big(\mathbb{Z}^{n-1} \rtimes_{\mathrm{st}} \mathrm{MCG}(\Sigma^2_{0,b,n})\big) && \text{by Prop. 4.42}\,,
\end{aligned} \tag{115}$$

where in the last step we used that $\mathrm{MCG}(\Sigma^2_{0,b,n})$ is generated already by the Artin generators alone.

Now for $b = 1$, we have $\Sigma^2_{0,1} \simeq_{\int} *$, so that the first factor in (115) is trivial and the claim (114) follows by (112). On the other hand, for $b = 0$ the first factor in (115) is $\pi_3(S^2) \simeq \mathbb{Z}$ and we are left with $\mathbb{Z} \times \big(\mathbb{Z}^{n-1} \rtimes_{\mathrm{st}} \mathrm{MCG}(\Sigma^2_{0,0,n})\big)$, as claimed in (113). □

Remark 4.46 (**Comparison to Chern-Simons theory on n-punctured surfaces).**

(i) The framed braid group $\mathrm{FBr}_n(\Sigma^2)$ (111) of a closed surface is the expected braid group acting on the quantum states of Chern-Simons theory on $\Sigma^2_{g,b,n}$ as formalized by the Reshetikhin-Turaev construction (cf. [39, §3.1][200, §3.2.1][166, p 37][167, p 8]) — but there the $\mathbb{Z}^n$-factor expected to act nontrivially only in the generality of the rarely discussed "irregular conformal blocks" [103].

(ii) The intermediate cases of framed braids with restriction on their total framing number, that appears in Lem. 4.45 from 2-Cohomotopical flux quantization, seems not to have appeared elsewhere.

Hence 2-Cohomotopical flux quantization predicts that the topological quantum states over the $(n-1)$-punctured closed disk are irreducible unitary representation of the groups appearing in Prop. 4.45.

We next look in more detail at one of the simplest non-trivial cases in some detail.

4.7 On the 2-punctured disk

We analyze in more detail the simple but already remarkble special case (of §4.6) of 2-cohomotopical flux quantum states over the 2-punctured open disk, hence on the 3-punctured sphere (cf. Rem. 4.39), where we find defect anyons whose braiding is controlled by "parastatistic" (Rem. 4.48) topologically realizing, in particular, a non-Clifford qbit-rotation gate (Prop. 4.50).

Example 4.47 (**Flux monodromy over 3-punctured sphere).** For the 3-punctured sphere (2-punctured plane), Prop. 4.34 yields, by (51), the subgroup of the *framed symmetric group* on three framed strands whose total framing is divisible by 3:

$$\pi_1\Big(\mathrm{Map}^*_0\big((\Sigma^2_{0,0,2+1})_{\cup\{\infty\}},\, S^2\big) \mathbin{/\!\!/} \mathrm{Diff}^{+,\partial}\big(\Sigma^2_{0,0,2+1}\big)\Big) \;\simeq\; \mathbb{Z} \times \big(\mathbb{Z}^2 \rtimes_{\mathrm{st}} \mathrm{Sym}_3\big) \quad\hookrightarrow\quad \mathbb{Z}^3 \rtimes_{\mathrm{def}} \mathrm{Sym}_3\,. \tag{116}$$

Remark 4.48 (**Anyons vs. parastatistics).**

(i) Equation (116) may be noteworthy in that it manifestly identifies a group of motions of what must be understood as defect anyons with a *symmetric group*, thus identifying the corresponding topological quantum states as representations of that symmetric group — a situation that is also referred to as *parastatistics*. [16]

[16] Parastatistics has originally been discussed as a speculative statistics of of fundamental particles [95][158], whereas here we see it arise in the form of braiding phases of defect anyons. This may address the concern of [111, p 109], who is the first to propose symmetric irreps as a model for quantum computation (aka *permutational quantum computing* [112]), regarding physical justification for the model.

(ii) In general, it is obvious, but seems underappreciated in the physics literature on anyons, that among all representations of braid groups, hence among all potential "anyon species", there are in particular those arising as pullbacks along the canonical $\mathrm{Br}_n(\Sigma^2) \to \mathrm{Sym}_n$ from such parastatistical representations.

(iii) This traditional disregard is maybe somewhat ironic since, concerning the motivating fault-tolerance of topological quantum gates, such braid representations coming from symmetric group representations are particularly good: They describe quantum gates which are insensitive *not only* to isotopical deformations of the braiding process, as usual anyons, but are insensitive to the process entirely — as they depend only on the process's endpoints. This is, in principle, the ultimate form of fault-tolerance!

(iv) The disregard in the literature for parastatistics as examples of anyon statistics may probably be attributed to the traditional prejudice that anyon species must be identified with simple objects in a unitary braided fusion category. While seemingly natural and oft-repeated, it is worth remembering that this paradigm is an ansatz that is not strictly implied from microscopic analysis. Our analysis here, of topological quantum states of 2-cohomotopical flux, is an example that other species of anyons can plausibly exist and may be worth pursuing in experiment.

Qbit quantum gates operable by defect anyons in 2-cohomotopical flux on 2-punctured open disk. Given that the standard representation $\mathbf{2}$ of Sym_3 is its only irrep of dimension > 1 (Ex. 4.41), it is this irrep that knows everything about potential non-abelian anyon statistics exhibited by 2-cohomotopical flux quanta on the 2-punctured open disk, by (116). If physically realizable, this manifests an interesting set of quantum gates: The irreps of the framed symmetric group $\mathbb{Z}^3 \rtimes \mathrm{Sym}_3$ which factor through some finite quotient $\mathbb{Z}^3_{2K} \rtimes \mathrm{Sym}_3$ are tensor products of irreps of $\mathbb{Z}_{2K}$ and irreps of Sym_3

Focusing on the only non-abelian irrep $\mathbf{2}$ of Sym_3 (Def. 4.40) this means it extends to an irrep of $\mathbb{Z}^3 \rtimes_{\mathrm{def}} \mathrm{Sym}_3$ (116) on which all three central generators act as multiplication with any but the same complex number ξ. To see what this complex number should be in the case of 2-cohomotopical flux observables on the 3-sphere, we restrict this irrep along the inclusion (99)

$$
\begin{array}{ccccc}
\mathbb{Z} \times \big(\mathbb{Z}^2 \rtimes_{\mathrm{st}} \mathrm{Sym}_3\big) & \longrightarrow & \mathbb{Z}^3 \rtimes_{\mathrm{st}} \mathrm{Sym}_3 & \longrightarrow & \mathrm{U}(\mathbf{2}) \\
t & \longmapsto & v_1 + v_2 + v_3 & \longmapsto & \widehat{\xi}^3 : \; |\psi\rangle \mapsto \xi^3 |\psi\rangle \\
e_1 & \longmapsto & v_1 - v_3 & \longmapsto & \mathrm{id} \\
e_2 & \longmapsto & v_2 - v_3 & \longmapsto & \mathrm{id} \\
b_i & \longmapsto & b_i & \longmapsto & \widehat{b}_i \,.
\end{array}
\tag{117}
$$

Comparison with Rem. 4.20 shows that $\xi^3 = \zeta = e^{\pi \mathrm{i} \frac{p}{K}}$ must be the braiding phase of solitonic anyons in the system. So, in some sense, each of the three punctures (the defect anyons) has associated with it 1/3rd of the braiding phase of the solitonic anyons, and yet only the sum of these three contributions is observable. The remaining content of the above representation is the unitarization of the standard representation of $\mathbb{Z}$.

Proposition 4.49 (**Unitarization of linear representations**, cf. [124, Prop. 4.6])**.** *For G a finite group and $\mathcal{H}$ a finite-dimensional complex Hilbert space, every $\mathbb{C}$-linear representation $R : G \to \mathrm{GL}(\mathcal{H})$ on the underlying complex vector space of $\mathcal{H}$ is isomorphic to a unitary representation $U : G \to \mathrm{U}(\mathcal{H}) \hookrightarrow \mathrm{GL}(\mathcal{H})$.*

Proposition 4.50 (**Quantum gates in the unitarization of the standard rep of** Sym_3)**.** *Up to unitary transformation, unitarization* (Prop. 4.49) *of the standard irrep* (Def. 4.40) $\mathbf{2}$ *of* Sym_3 *is generated by*

(i) The Pauli Z-gate (cf. [153, p. xxx])*:*

$$
U(213) \;=\; Z \;:=\; \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
$$

(ii) A rotation gate (cf. [153, (4.4-6)])

$$
U(231) \;=\; R_y(8\pi/3) \;=\; -R_y(2\pi/3) \;:=\; \begin{bmatrix} \cos(4\pi/3) & -\sin(4\pi/3) \\ \sin(4\pi/3) & \cos(4\pi/3) \end{bmatrix} \;=\; -\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}.
\tag{118}
$$

Proof. The defining representation on $\mathbb{C}^3 \simeq \mathrm{Span}_{\mathbb{C}}(v_1, v_2, v_3)$ (100) is evidently unitary with respect to the canonical inner product $\langle v_i | v_j \rangle = \delta_{ij}$, but the basis $\big(e_1 := v_1 - v_3,\, e_2 := v_2 - v_3\big)$, from (102), for the standard sub-representation $\mathbf{2}$, is not orthonormal with respect to this inner product. One choice of orthonormal basis for this subspace is given by

$$
\begin{bmatrix} 1 \\ 0 \end{bmatrix} \;:=\; \tfrac{1}{\sqrt{6}}\big(v_1 + v_2 - 2\,v_3\big)\,, \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \;:=\; \tfrac{1}{\sqrt{2}}\big(v_1 - v_2\big)\,.
$$

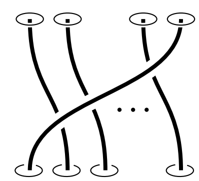
A straightforward computation shows that on this basis the permutations (213) and (231) act as claimed. □

Remark 4.51. **(i)** It is clear from Ex. 4.50 that, in general, the cyclic permutations in $\mathrm{Sym}_n \leftarrow \mathrm{Br}_n$ act in the unitarizations of the corresponding standard reps as rotation gates (on "qdits" for $n > 3$, cf. [219]) by angles which are multiples of $2\pi/n$.

(ii) Together with the phase rotations provided by the solitonic anyon braiding factor (117), such rotation gates are the workhorse of the quantum Fourier transform (cf. [153, §5][208, §3.2.1]) and with it of standard quantum algorithms such as notably Shor's algorithm — while their precision and error protection is a major bottleneck in the implementation of useful quantum algorithms (cf. [61, §III]). Here we find these gates are predicted to have topologically stabilized realizations by braiding of defect anyons in FQH systems (distinct from the usual abelian braiding of the solitonic anyons discussed in §4.1).

(iii) Noteworthy here that the qbit rotation gate (118), and generically also these higher qdit rotation gates, are "non-Clifford" (cf. [201]), which is a crucial but rare feat in currently discussed realizations of topological quantum gates (cf. [142, §D]).

Topological rotation gates, obtained by cyclic braiding of defect anyons, combined with the global phase rotations given by braiding of solitonic anyons, would provide intrinsically exact and topologically protected gates of the kind that make up the quantum Fourier transform (in qdit-bases), and with it many other quantum algorithms.



4.8 On the punctured annulus

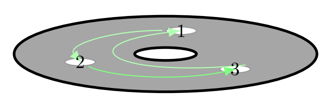
FQH systems, which in realistic experiments are realized on surfaces with boundary, $\Sigma^2_{g,b>0,n}$, famously exhibit "ungapped chiral currents" along their boundaries. Here we observe that the natural candidate observable for these in our theory are Dehn twist diffeomorphism along boundary curves (cf. Rem. 2.17).

Example 4.52. The surface braid group (33) of the n-punctured annulus is [119] the semidirect product of the affine braid group (cf. [73, §1.1]) with a copy of the integers

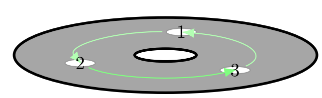
$$\mathrm{Br}_n(\Sigma^2_{0,2}) \;\simeq\; \mathrm{Br}^{\mathrm{aff}}_n \rtimes \underbrace{\langle c \rangle}_{\mathbb{Z}}$$

where the generator c is the braid exhibiting one-step *cyclic permutation* of the punctures *around* the inner boundary.

The braid $b_1 b_2$ (Artin notation) first passes puncture 3 past puncture 2 along the shortest possible path, and then continues to similarly pass the locus 2 (where puncture 3 was just moved to) past 1.



On the other hand, the braid c moves all three punctures one step counterclockwise.
It is manifest that $c \neq b_1 b_2$ on the annulus, but that they become equal when filling in the annulus to become the disk.

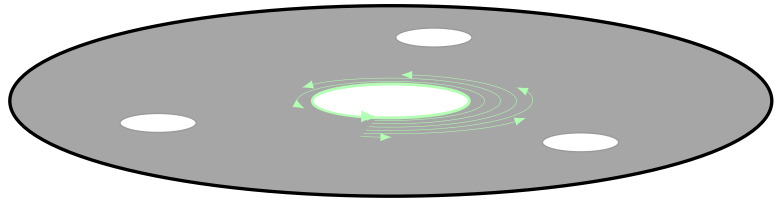


The mapping class group (38) of the n-punctured annulus should (by analysis as in [12]) be the direct extension of that by the integers

$$\begin{array}{ccccccccc}
1 \longrightarrow & \mathrm{Br}_n(\Sigma^2_{0,2}) & \hookrightarrow & \mathrm{MCG}(\Sigma^2_{0,2,n}) & \twoheadrightarrow & \mathrm{MCG}(\Sigma^2_{0,2}) & \longrightarrow 1\,, \\
 & & & \wr\! \mid & & \wr\!\mid & \\
 & & & \mathrm{Br}_n(\Sigma^2) \times \underbrace{\langle d \rangle}_{\mathbb{Z}} & & \langle d \rangle &
\end{array} \tag{119}$$

where the extra generator d (the "Dehn twist", cf. Fig. D) is represented by a diffeomorphism which is the identity everywhere except on tubular neighborhood of the inner boundary, where it radially interpolates between the identity and a full rotation of the inner boundary.

The Dehn twist diffeomorphism d **on the punctured annulus** (119) is the identity far away from the inner boundary, but in a tubular neighborhood of the inner boundary it continuously interpolates radially between the trivial and a full rotation. Hence, while the Dehn twist is the identity on the boundary itself (like all elements in the mapping class group, by definition), it is nevertheless nontrivial "asymptotically towards" the inner boundary.



In this sense the Dehn twist d clearly ought to be regarded as the asymptotic symmetry (24) in the mapping class group of the n-punctured torus.

(...)

5 Conclusion & Outlook

(...)

A Background

Here we briefly recall and cite some background material referred to in the main text.

A.1 Effective CS for FQH

The traditional ansatz for an effective field theory description of fractional quantum Hall systems at unit filling fraction $\nu = 1/K$ postulates that the effective field is a 1-form potential a for the electric current density 2-form J ("statistical gauge field"), itself minimally coupled to the *quasi-hole current* j, and with effective dynamics encoded by the level $= k = K/2$ Chern-Simons (CS) Lagrangian [222][210]:

$$
\begin{array}{llll}
\underset{\text{density 2-form}}{\textbf{Electron current}} & J & = & \overset{\text{current 3-vector}}{\vec{J}} \lrcorner \overset{\text{volume form}}{\mathrm{dvol}} \;\; =: \;\; \mathrm{d}\, a \quad \textbf{Effective gauge field} \\
\underset{\text{density 2-form}}{\textbf{Quasi-particle current}} & j & = & \vec{j} \lrcorner \mathrm{dvol} \\
\underset{\text{density 2-form}}{\textbf{Background flux}} & F & = & \mathrm{d}\, A \quad \textbf{External gauge field} \\
\underset{\text{density 3-form}}{\textbf{Effective Lagrangian}} & L & := & \tfrac{K}{2} \underbrace{a\, \mathrm{d}a}_{\mathrm{CS}(a)} - \underbrace{A\, \mathrm{d}a}_{A\,J} + a\, j \qquad [210, (2.11)]
\end{array}
\tag{120}
$$

This is justified by observing that the Euler-Lagrange equations of this L

$$
\frac{\delta L}{\delta a} = 0 \quad \Leftrightarrow \quad \boxed{J \;=\; \tfrac{1}{K}\big(F - j\big)}
$$

in the relevant case of longitudinal electron current and static quasi-particles

$$
\begin{aligned}
J &\equiv J_0\, \mathrm{d}x\, \mathrm{d}y \; - \; J_x\, \mathrm{d}t\, \mathrm{d}y \\
j &\equiv j_0\, \mathrm{d}x\, \mathrm{d}y \\
F &\equiv B\, \mathrm{d}x\, \mathrm{d}y \; - \; E_y\, \mathrm{d}t\, \mathrm{d}y
\end{aligned}
$$

express just the hallmark properties of the FQHE at filling fraction $\nu = 1/K$:

$$
\Leftrightarrow \left\{
\begin{array}{llll}
J_x & = \tfrac{1}{K} E_y & \Leftrightarrow & \text{Hall conductivity law at } 1/K \text{ filling} \\
J_0 & = \tfrac{1}{K} B & \Leftrightarrow & \text{each electron absorbs } K \text{ flux quanta, but} \\
 & \;\; - \tfrac{1}{K} j_0 & & 1/K\text{th electron missing for each quasi-hole}\,.
\end{array}
\right.
$$

Remark A.1 (Conceptual problem.). This can only be a *local* description on single charts (as is common for Lagrangian field theories, cf. *Fig. G*): Globally, neither J nor F may admit coboundaries a and A, respectively. Instead, both must be subjected to some kind of flux-quantization. For F this must be classical Dirac charge quantization, which however is incompatible with integrality of J in the relevant case of $K > 1$ (cf. [216, p. 35][202, p 159]). The problem is only worsened by the traditional effective ansatz for more general filling fractions ν, which is [210, (2.30-1)] to introduce $n > 1$ copies of the above field and promote the number K in (120) to a matrix (the "K-matrix").

A.2 Some algebraic topology

For general background on the homotopy theory and algebraic topology we use, see for instance [101][79][213][109][2][195].

Topological spaces. We write

- Top for the category of *compactly generated* topological spaces (cf. [182, Ntn. 1.0.16])
  with mapping spaces (cf. [2, §1]) denoted $\mathrm{Map}(-,-)$ and their underlying (hom-)sets denoted $\mathrm{Hom}(-,-)$,
- Top^* for pointed such spaces with pointed maps between them
  with mapping spaces denoted $\mathrm{Map}^*(-,-)$ and their underlying (hom-)sets denoted $\mathrm{Hom}^*(-,-)$.

The mapping spaces are characterized by natural homeomorphisms (cf. [109, (3.98)][195, Thm. 3.47(a)])

$$
\begin{aligned}
\mathrm{Map}\big(X \times Y,\, Z\big) &\simeq \mathrm{Map}\big(X,\, \mathrm{Map}(Y, Z)\big) \\
\mathrm{Map}^*\big(X \wedge Y,\, Z\big) &\simeq \mathrm{Map}^*\big(X,\, \mathrm{Map}^*(Y, Z)\big)\,,
\end{aligned}
\tag{121}
$$

where for a space X pointed by $\infty_X \in X$ with a space Y pointed by $\infty_Y \in Y$

(i) their *smash product* is

$$X \wedge Y \;:=\; \frac{X \times Y}{\{\infty_Z\} \times Y \;\cup\; X \times \{\infty_Y\}}\,, \tag{122}$$

which is symmetric via natural homeomorphisms

$$X \wedge Y \;\simeq\; Y \wedge X\,; \tag{123}$$

for instance:

$$S^1 \wedge S^n \;\simeq\; S^{n+1}\,, \quad \text{so that} \quad \pi_n \mathrm{Map}^*(S^m, X) \;\simeq\; \pi_0 \mathrm{Map}^*(S^{n+m}, X) \;\simeq\; \pi_{n+m}(X)\,; \tag{124}$$

(here the smash product with the circle is called reduced *suspension* and usually denoted $\Sigma := S^1 \wedge (-)$, but we stick with writing "$S^1 \wedge$" in order not to clash with our use of "Σ^2" for the generic surface),

(ii) their pointed mapping space is the fiber over the base point of Y of the map ev that evaluates unpointed maps at the base point of X:

$$\mathrm{Map}^*(X, Y) \xrightarrow{\mathrm{fib}_{(\infty_Y)}} \mathrm{Map}(X, Y) \xrightarrow{\mathrm{ev}_{(\infty_X)}} Z\,. \tag{125}$$

The coproduct of $X, Y \in \mathrm{Top}^*$ is the *wedge sum*

$$X \vee Y \;:=\; \frac{X \coprod Y}{\{\infty_X,\, \infty_Y\}}\,, \tag{126}$$

which in particular means that we have a natural bijection

$$\mathrm{Hom}^*\big(X \vee Y,\, Z\big) \;\simeq\; \mathrm{Hom}^*\big(X,\, Z\big) \times \mathrm{Hom}^*\big(Y,\, Z\big)\,. \tag{127}$$

Incidentally, the smash product (122) naturally distributes over finite wedge sums

$$X \wedge \big(Y \vee Z\big) \;\simeq\; \big(X \wedge Y\big) \;\vee\; \big(X \wedge Z\big)\,.$$

One-point compactification. Here for $X \in \mathrm{Top}^*$ we generically denote its basepoint by $\infty_X \in X$, also speaking of the "point at infinity", and for X a locally compact Hausdorff space we write $X_{\cup\{\infty\}} \in \mathrm{Top}^*$ for its *one-point compactification* (cf. [23, p 199]), thinking of it as *adjoining a point at infinity.*

For example, stereographic projection gives

$$\mathbb{R}^n_{\cup\{\infty\}} \;\simeq\; S^n \tag{128}$$

and if a space is already compact, then the adjoined point-at-infinity is disjoint and pointed maps out of the space are identified with plain maps:

$$X \text{ compact Hausdorff} \qquad \vdash \qquad X_{\cup\{\infty\}} \simeq X \sqcup \{\infty\} \qquad \text{and} \qquad \mathrm{Maps}^*\big(X_{\cup\{\infty\}},\, Y\big) \;\simeq\; \mathrm{Maps}(X,\, Y)\,. \tag{129}$$

We have natural homeomorphisms (cf. [109, Prop. 3.7] [33, Prop. 1.6])

$$(X \sqcup Y)_{\cup\{\infty\}} \;\simeq\; X_{\cup\{\infty\}} \vee Y_{\cup\{\infty\}}\,, \qquad (X \times Y)_{\cup\{\infty\}} \;\simeq\; X_{\cup\{\infty\}} \wedge Y_{\cup\{\infty\}}\,, \tag{130}$$

For $Y \in \mathrm{Top}^*$ we denote the connected component of the map constant on ∞_Y by

$$\mathrm{Map}_0(-, Y) \subset \mathrm{Map}(-, Y) \qquad \text{and} \qquad \mathrm{Map}^*_0(-, Y) \subset \mathrm{Map}^*(-, Y)\,. \tag{131}$$

Proposition A.2 (One-point compactification functorial on proper maps [109, p 70][33, Prop. 1.6]**).** *The operation of one-point compactification extends to a functor on the category of locally compact Hausdorff spaces with* proper maps *between them*

$$(-)_{\cup\{\infty\}} \;:\; \mathrm{LCHaus}_{\mathrm{PrpMaps}} \longrightarrow \mathrm{CptHaus}^*\,. \tag{132}$$

Since homeomorphisms are proper, this implies in particular functoriality on homeomorphisms.

Loop spaces. The *based loop space* of $X \in \mathrm{Top}^*$ is

$$\Omega X \;:=\; \mathrm{Map}^*\big(S^1,\, X\big) \tag{133}$$

whose connected components form the *fundamental group* at the basepoint:

$$\pi_1\, X \;:=\; \pi_0\, \Omega X \tag{134}$$

For example, the fundamental group of pointed mapping spaces $X \to Y$ (based at the map constant on the basepoint of Y) has these alternative expressions:

$$
\begin{array}{rcll}
\pi_1 \operatorname{Map}^*(X,Y) & \equiv & \pi_0\, \Omega \operatorname{Map}^*(X,Y) & \text{by (134)} \\
& \equiv & \pi_0 \operatorname{Map}^*\big(S^1,\, \operatorname{Map}^*(X,Y)\big) & \text{by (133)} \\
& \simeq & \pi_0 \operatorname{Map}^*\big(S^1 \wedge X,\, Y\big) & \text{by (121)} \\
& \simeq & \pi_0 \operatorname{Map}^*\big(X \wedge S^1,\, Y\big) & \text{by (123)} \\
& \simeq & \pi_0 \operatorname{Map}^*\big(X,\, \operatorname{Map}^*(S^1,Y)\big) & \text{by (121)} \\
& \equiv & \pi_0 \operatorname{Map}^*\big(X,\, \Omega Y\big) & \text{by (133).}
\end{array}
\tag{135}
$$

Homotopy. We write Grpd_∞ for the ∞-category of homotopy types and

$$\textstyle\int \;:\; \mathrm{Top} \to \mathrm{Grpd}_\infty$$

for the underlying functor. This means that a *weak homotopy equivalence* between topological spaces is equivalently an equivalence under $\int$:

$$X, Y \in \mathrm{Top} \qquad \vdash \qquad X \simeq_{\int} Y \qquad \Leftrightarrow \qquad \textstyle\int X \xrightarrow[\sim]{\int f} \int Y \tag{136}$$

Given $f : Y \to Z$ a map of pointed topological spaces, with homotopy fiber X

$$X \xrightarrow{\mathrm{hofib}(f)} Y \xrightarrow{\;f\;} Z$$

the resulting long homotopy fiber sequences

$$
\begin{array}{c}
\Omega X \longrightarrow \Omega Y \longrightarrow \Omega Z \\
\hookrightarrow X \longrightarrow Y \longrightarrow X
\end{array}
\tag{137}
$$

pass under $\pi_0(-)$ to long exact sequences of homotopy groups

$$
\begin{array}{c}
\pi_{n+1}(X) \longrightarrow \pi_{n+1}(Y) \longrightarrow \pi_{n+1}(Z) \\
\hookrightarrow \pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \pi_n(Z)
\end{array}
\tag{138}
$$

Homotopy quotients and Borel construction.

Definition A.3. For $G \circlearrowright X$ a Hausdorff topological group acting continuously on a topological space X, we write

$$X \xrightarrow{\;q\;} X /\!\!/ G \;:=\; X \times_G EG \tag{139}$$

for its Borel construction, and call its homotopy type the *homotopy quotient* of the action.

In the special case when $X = *$ we get the traditional *classifying space* (namely of principal G-bundles, cf. [168, Thm. 3.5.1][182, Thm. 4.1.13])

$$* /\!\!/ G \;\simeq\; BG \tag{140}$$

whose loop space recovers G up to weak homotopy equivalence

$$\Omega BG \;\simeq_{\int}\; G\,,$$

hence whose homotopy groups are those of G shifted up in degree:

$$\pi_{n+1}(BG) \;\simeq\; \pi_n(G)\,. \tag{141}$$

This makes a long homotopy fiber sequence (137)

$$G \xrightarrow{\;g \mapsto g(x_0)\;} X \xrightarrow{\;q\;} X /\!\!/ G \longrightarrow BG\,. \tag{142}$$

Hence if G preserves the connected components of X (such as if X only has one connected component), then the long exact sequence of homotopy groups (138) implies that

$$\pi_0(G) \circlearrowright \pi_0(X) \text{ is trivial} \qquad \Rightarrow \qquad \pi_0\big(X\big) \xrightarrow[\sim]{\pi_0(q)} \pi_0\big(X /\!\!/ G\big). \tag{143}$$

A.3 Surfaces & 2-Cohomotopy

We give a streamlined review of the analysis of 2-Cohomotopy moduli of closed surfaces, due to [93], that is used in §4.3, and record some related facts needed there for identifying the modular action on 2-cohomotopical flux monodromy.

Fundamental polygons of closed oriented surfaces. The homeomorphism class of oriented closed surfaces of genus g is represented (cf. [79, Thm 2.8]) by the quotient space of the regular $4g$-gon (called a *fundamental polygon* of the surface) obtained by identifying all boundary vertices with a single point and, going clockwise for $k \in \{0, \cdots, g-1\}$, the $4k+1$st boundary edge with the reverse of the $4k+3$rd, and the $4k+2$nd with the reverse of the $4k+4$th. For small g this is illustrated in (74), cf. [96, p 5]. A more homotopy-theoretic formulation of this statement is as follows.

The fundamental group π_1 of a wedge sum (126) of circles is the free group on the set of summands, whose ith generator is represented by the loop that goes identically through the ith circle summand. For the classification of surfaces of genus g (9) we are concerned with wedge sums of $2g$ circles to be denoted $\bigvee_{i=1}^g \big(S^1_a \vee S^1_b\big)$, whose generators we accordingly denote $(a_i, b_i)_{i=1}^g$.

With this, the classical presentation by fundmental polygons becomes:

Proposition A.4 (**Homotopy type of closed oriented surfaces**, cf. [93, p 151])**.** *The homeomorphism type of the closed oriented surface* Σ^2_g (9) *of genus* $g \in \mathbb{N}$ *is that of the cell attachment* (cf. [2, §3.1]) *shown on the left here:*

$$
\begin{array}{ccccc}
S^1 & \xrightarrow{\;\prod_{i=1}^g [a_i,b_i]\;} & \bigvee_{i=1}^g \big(S^1_a \vee S^1_b\big) & \longrightarrow & \ast \\
\big\downarrow & & \big\downarrow{\scriptstyle i_g} & & \big\downarrow \\
D^2 & \xrightarrow{\qquad\quad (\mathrm{po})} & \Sigma^2_g & \xrightarrow{\;q_0^g \quad (\mathrm{po})\;} & S^2\,,
\end{array}
\tag{144}
$$

whence its homotopy type sits in a long homotopy cofiber sequence of this form:

$$
S^1 \xrightarrow{\;\prod_{i=1}^g [a_i,b_i]\;} \bigvee_{i=1}^g \big(S^1_a \vee S^1_b\big) \xrightarrow{\;\;i_g\;\;} \Sigma^2_g \xrightarrow{\;\;q_0^g\;\;} S^2 \xrightarrow{\;S^1 \wedge \left(\prod_{i=1}^g [a_i,b_1]\right)\;} \bigvee_{i=1}^g \big(S^2_a \vee S^2_b\big)\,.
\tag{145}
$$

In (144) the attaching map $\prod_i [a_i, b_i] \;=\; a_1 b_1 a_1^{-1} b_1^{-1}\, a_2 b_2 a_2^{-1} b_2^{-1} \cdots$, is a representative for the element of $\pi_1\big(\bigvee_i (S^1_a \vee S^1_b)\big)$ of that same name (the consecutive sequence of edges and reverse edges in the boundary of the fundamental polygon), D^2 is the fundamental polygon itself and the pushout enforces the identification of pairs of its boundary edges. Finally, the connecting map q_0^g sends all of these previous boundary edges to the base point.

Remark A.5 (**Compatible surjections of closed surfaces to 2-sphere**)**.** By the pasting law for pushouts, (144) also shows that we have canonical projection maps $q_g^{g+1} : \Sigma^2_{g+1} \to \Sigma^2_g$ (74) compatible with their maps q_0 to S^2 (144), given by sending just the $g+1$st pair of edges to the point:

$$
\begin{array}{ccccccc}
S^1 & \xrightarrow{\;\prod_i [a_i,b_i]\;} & \bigvee_{i=1}^{g+1} \big(S^1_a \vee S^1_b\big) & \twoheadrightarrow & \bigvee_{i=1}^{g} \big(S^1_a \vee S^1_b\big) & \longrightarrow & \ast \\
\big\downarrow & & \big\downarrow{\scriptstyle i_{g+1}} & & \big\downarrow{\scriptstyle i_g} & & \big\downarrow \\
D^2 & \xrightarrow{\qquad (\mathrm{po})} & \Sigma^2_{g+1} & \xrightarrow{\;q_g^{g+1} \quad (\mathrm{po})\;} & \Sigma^2_g & \xrightarrow{\;q_0^g \quad (\mathrm{po})\;} & S^2\,.
\end{array}
\qquad
\Sigma^2_{g+1} \xrightarrow{\;q_0^g\;} S^2
\tag{146}
$$

We also obtain from this the following re-derivation of the integral cohomology of closed surfaces, which is needed in the main text for identifying the modular group action on 2-cohomotopical flux monodromy but also serves as the blueprint for its 2-cohomotopical variant shown further below in Lem. A.7:

Proposition A.6 (**Integral cohomology of closed surfaces**)**.** *For closed oriented surfaces, their ordinary integral cohomology in* $\deg = 1$ *is:*

$$
\widetilde{H}^1(\Sigma^2_g;\, \mathbb{Z}) \;\simeq\; \mathbb{Z}^{2g}.
\tag{147}
$$

Proof. The long exact sequence of homotopy groups (138) which is induced by the homotopy fiber sequence obtained by mapping (145) into the classifying space $B\mathbb{Z}$ (3) is, in the relevant part, of this form:

$$
\begin{array}{ccccccc}
\pi_0\,\mathrm{Map}^\ast\big(S^2,\, B\mathbb{Z}\big) & \xrightarrow{\;(q_0^g)^\ast\;} & \pi_0\,\mathrm{Map}^\ast\big(\Sigma^2_g,\, B\mathbb{Z}\big) & \longrightarrow & \pi_0\,\mathrm{Map}^\ast\big(\bigvee_i (S^1_a \vee S^1_b),\, B\mathbb{Z}\big) & \xrightarrow{\;\left(\prod_i [a_i,b_i]\right)^\ast\;} & \pi_0\,\mathrm{Map}^\ast\big(S^1,\, B\mathbb{Z}\big)\,. \\
\| & & \| & & \| & & \| \\
\underbrace{\widetilde{H}^1\big(S^2;\, \mathbb{Z}\big)}_{0} & & \widetilde{H}^1\big(\Sigma^2_g;\, \mathbb{Z}\big) & & \prod_{i=1}^g \big(\underbrace{\widetilde{H}^1(S^1;\, \mathbb{Z})}_{\mathbb{Z}}\big)^2 & & \underbrace{\widetilde{H}^1\big(S^1;\, \mathbb{Z}\big)}_{\mathbb{Z}}
\end{array}
$$

Hence, to conclude, we need to show that the map on the right — forming the pullback of $B\mathbb{Z}$-valued maps along $\prod_i [a_i, b_0]$ — is the zero-map under π_0. This is the case because the pullback classes are still group commutators and as such they vanish since the π_0 group in question is $\simeq \mathbb{Z}$ and hence abelian:

$$
\begin{array}{ccc}
\underbrace{\pi_0 \, \mathrm{Map}^*\big(\bigvee_i (S^1_a \vee S^1_b),\, B\mathbb{Z}\big)}_{\mathbb{Z}^{2g}} & \xrightarrow{\;\big(\prod_i [a_i, b_1]\big)^*\;} & \underbrace{\pi_0 \, \mathrm{Map}^*\big(S^1,\, B\mathbb{Z}\big)}_{\mathbb{Z}} \,. \\
\big((n_i, m_i)\big)_{i=1}^{g} & \longmapsto & \prod_i \underbrace{[n_i, m_i]}_{0}
\end{array}
$$

□

In the analogous manner, we find:

Lemma A.7 ([93, Prop. 2])**.** *The long exact sequence of homotopy groups* (138) *which is induced by the homotopy fiber sequence obtained by mapping* (145) *into* S^2 *truncates to a short exact sequence:*

$$
1 \longrightarrow \underbrace{\pi_1 \, \mathrm{Map}^*\big(S^2,\, S^2\big)}_{\mathbb{Z}} \xrightarrow{\;(q^g_0)^*\;} \pi_1 \, \mathrm{Map}^*\big(\Sigma^2_g,\, S^2\big) \longrightarrow \prod_{i=1}^{g} \big(\underbrace{\pi_1 \, \mathrm{Map}^*(S^2,\, S^2)}_{\mathbb{Z}}\big)^2 \longrightarrow 1 \,. \qquad (148)
$$

Proof. It is sufficient to see that pullback of S^2-valued maps along $\prod_i [a_i, b_i]$ is the zero map under (suspension and) taking π_1: This is because the pullback classes are still group commutators and as such they vanish since the π_1 groups in question are $\simeq \mathbb{Z}$ and hence abelian:

$$
\begin{array}{ccc}
\underbrace{\pi_1 \, \mathrm{Map}^*\big(\bigvee_i (S^1_a \vee S^1_b),\, S^2\big)}_{\mathbb{Z}^{2g}} & \xrightarrow{\;\big(\prod_i [a_i, b_1]\big)^*\;} & \underbrace{\pi_1 \, \mathrm{Map}^*\big(S^1,\, S^2\big)}_{\mathbb{Z}} \\
\big((n_i, m_i)\big)_{i=1}^{g} & \longmapsto & \prod_i \underbrace{[n_i, m_i]}_{0}
\end{array}
$$

and analogously, under suspension, with all copies of S^1 in this formula replaced by S^2). Here in evaluating these groups we have used (127) and stages of (135), thereby identifying all copies of $\mathbb{Z}$ with $\mathbb{Z} \simeq \pi_2(S^2)$ (and with $\mathbb{Z} \simeq \pi_3(S^2)$ for the case with suspension). □

To see that the statement (148) for the *pointed* mapping space implies the variant statement (68) for the unpointed mapping space:

Lemma A.8 ([93, Thm 1])**.** *For* $g \in \mathbb{N}$, *we have a short exact sequence of this form:*

$$
1 \to \pi_1 \, \mathrm{Map}_0\big(S^2,\, S^2\big) \xrightarrow{\;(q^g_0)^*\;} \pi_1 \, \mathrm{Map}_0\big(\Sigma^2_g,\, S^2\big) \longrightarrow \pi_1 \, \mathrm{Map}^*_0\big(\bigvee_g (S^1_a \vee S^1_b),\, S^2\big) \to 1 \,.
$$

Proof. Consider the long exact sequences of homotopy groups (138) induced by the evaluation sequences (125) on Σ^2_g and on S^2, respectively, with the map between them induced by pullback along q^g_0 (144) extended to the short exact sequence from (148):

$$
\begin{array}{ccccccccc}
 & & \pi_2(S^2) & = & \pi_2(S^2) & & & & \\
 & & \big\downarrow{\scriptstyle \delta^0} & & \big\downarrow{\scriptstyle \delta^g} & & & & \\
1 & \longrightarrow & \pi_1 \, \mathrm{Map}^*\big(S^2,\, S^2\big) & \xrightarrow{\;(q^g_0)^*\;} & \pi_1 \, \mathrm{Map}^*\big(\Sigma^2_g,\, S^2\big) & \xrightarrow{\;(i_g)^*\;} & \pi_1 \, \mathrm{Map}^*\big(\bigvee_i (S^1_a \vee S^1_b),\, S^2\big) & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle \mathrm{ev}^0} & & \big\downarrow{\scriptstyle \mathrm{ev}^g} & \overset{(i_g)^* \circ \overline{\mathrm{ev}^g}}{\dashrightarrow} & & & \\
1 & \dashrightarrow & \pi_1 \, \mathrm{Map}\big(S^2,\, S^2\big) & \overset{(q^g_0)^*}{\dashrightarrow} & \pi_1 \, \mathrm{Map}\big(\Sigma^2_g,\, S^2\big) & & & & \\
 & & \big\downarrow & & \big\downarrow & & & & \\
 & & \underbrace{\pi_1(S^1)}_{1} & = & \underbrace{\pi_1(S^1)}_{1} & & & &
\end{array}
$$

Here all solid sequences are exact, by construction, horizontally as well as vertically. Using this, a routine diagram chase shows [17] that the dashed sequence exists and is exact. □

[17] To spell it out: Since ev is seen to be surjective, we may define $(i_g)^* \circ \overline{\mathrm{ev}}$ on a given element ϕ by choosing any preimage through ev. To see that this is well defined: If $\widehat{\phi}$, $\widehat{\phi}'$ are a pair of preimages, their difference is in the image of $\delta^g = (q^g_0)^* \circ \delta^0$, hence in the image of $(q^g_0)^*$ and hence vanishes under $(i_g)^*$. With the map thus existing, surjectivity is immediate from $(i_g)^*$ being surjective.

As a corollary we note:

Lemma A.9. *The cohomology operation from 2-Cohomotopy in degree -1 to integral cohomology in degree 1, induced by the looping of the unit class* $1^2 : S^2 \to B^2\mathbb{Z}$ (56)*, produces a morphism of short exact sequences:*

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_0\,\mathrm{Map}(S^2,\,\Omega S^2) & \xrightarrow{(q_0^g)^*} & \overbrace{\pi_0\,\mathrm{Map}(\Sigma^2_g,\,\Omega S^2)}^{\pi_1\,\mathrm{Map}\left(\Sigma^2_g,\,S^2\right)} & \longrightarrow & \pi_0\,\mathrm{Map}^*\big(\bigvee_i(S^1_a \vee S^1_b),\,\Omega S^2\big) & \longrightarrow & 1 \\
 & & \Big\downarrow{\scriptstyle (\Omega 1^2)_*} & & \Big\downarrow{\scriptstyle (\Omega 1^2)_*} & & \wr\Big\downarrow{\scriptstyle (\Omega 1^2)_*} & & \\
1 & \longrightarrow & \underbrace{\pi_0\,\mathrm{Map}(S^2,\,B\mathbb{Z})}_{H^1(S^2;\mathbb{Z})\simeq 0} & \xrightarrow{(q_0^g)^*} & \underbrace{\pi_0\,\mathrm{Map}(\Sigma^2_g,\,B\mathbb{Z})}_{H^1(\Sigma^2_g,\mathbb{Z})\simeq\mathbb{Z}^{2g}} & \xrightarrow{\sim} & \underbrace{\pi_0\,\mathrm{Map}^*\big(\bigvee_i(S^1_a \vee S^1_b),\,B\mathbb{Z}\big)}_{\mathbb{Z}^{2g}} & \longrightarrow & 1\,.
\end{array}
$$

Proof. The top exact sequence is from Lem. A.8, and inspection of the proof there shows immediately that it applies verbatim also with the coefficient ΩS^2 replaced by $B\mathbb{Z}$, throughout (for the exactness of the horizontal sequence in the proof, this is in the proof of Prop. A.6). This gives the bottom exact sequence and the compatibility of the two under the cohomology operation, as claimed. □

A.4 Quadratic Gauss sums

Here we briefly compile some facts about Gauss sums used in §4.4, see [14] for more pointers to the literature (see also [43] but beware of typos in (1.1) there).

First, it may be worth recalling the simple cousin of the Gauss sums:

Proposition A.10 (**Discrete Fourier transform of Kronecker delta**)**.** *For* $K \in \mathbb{N}_{>0}$ *and* $p \in \mathbb{Z}$ *we have*

$$\sum_{n=0}^{K-1} e^{\frac{2\pi \mathrm{i}}{K} pn} \;=\; \begin{cases} K & \text{if} \quad p = 0 \\ 0 & \text{if} \quad n \neq 0\,. \end{cases} \tag{149}$$

Proof. The statement for $p = 0$ is immediate. For $p \neq 0$ observe that

$$\Big(1 - e^{\frac{2\pi\mathrm{i}}{K}p}\Big)\sum_{n=0}^{K-1} e^{\frac{2\pi\mathrm{i}}{K}pn} \;=\; 1 - e^{2\pi\mathrm{i}\,n} \;=\; 0\,.$$

□

Now:

Proposition A.11 (**Classical quadratic Gauss sum evaluation**, cf. [130, p 87][165])**.** *For* $K \in \mathbb{N}_{>0}$ *we have*

$$\sum_{n=0}^{K-1} e^{\frac{2\pi\mathrm{i}}{K}n^2} \;=\; \begin{cases} (1+\mathrm{i})\sqrt{K} & \mid \;\; K = 0 \bmod 4 \\ \sqrt{K} & \mid \;\; K = 1 \bmod 4 \\ 0 & \mid \;\; K = 2 \bmod 4 \\ \mathrm{i}\,\sqrt{K} & \mid \;\; K = 3 \bmod 4\,. \end{cases} \tag{150}$$

Proposition A.12 (**Quadratic Gauss sum with multiple exponents**, cf. [130, "QS4" p 86])**.** *For odd* $K \in 2\mathbb{N}+1$ *we have more generally, for* $p \in \mathbb{Z}$*,*

$$\sum_{n=0}^{K-1} e^{\frac{2\pi\mathrm{i}}{K}pn^2} \;=\; (p|K)\sum_{n=0}^{K-1} e^{\frac{2\pi\mathrm{i}}{K}n^2} \;=\; \begin{cases} (p|K)\,(1+\mathrm{i})\sqrt{K} & \mid \;\; K = 0 \bmod 4 \\ (p|K)\,\sqrt{K} & \mid \;\; K = 1 \bmod 4 \\ 0 & \mid \;\; K = 2 \bmod 4 \\ (p|K)\,\mathrm{i}\,\sqrt{K} & \mid \;\; K = 3 \bmod 4\,, \end{cases} \tag{151}$$

To see that the dashed $(q_0^g)^*$ is injective: Consider ϕ, ϕ' a pair of elements in the domain with the same image. Since ev^0 is surjective we may find ev^0-preimages $\widehat{\phi}$, $\widehat{\phi}'$. By commutativity of the middle square we then have $\mathrm{ev}^g \circ (q_0^g)^*(\widehat{\phi}) = \mathrm{ev}^g \circ (q_0^g)^*(\widehat{\phi}')$, and so the difference between $(q_0^g)^*(\widehat{\phi})$ and $(q_0^g)^*(\widehat{\phi}')$ is in the image of $\delta^g = (q_0^g)^* \circ \delta^0$. But since $(q_0^g)^*$ is injective, this means that already the difference between $\widehat{\phi}$ and $\widehat{\phi}'$ is in the image of δ^0, hence vanishes under ev^0, hence $\phi = \phi'$, which was to be seen.

Finally, to see that the dashed sequence is exact in the middle: By the previous construction, the kernel of $(i_g)^* \circ \overline{\mathrm{ev}^g}$ consists exactly of those ϕ whose ev^g-preimage is in the kernel of $(i_g)^*$, hence in the image of $(q_0^g)^*$, hence of those ϕ in the image of $(q_0^g)^* \circ \mathrm{ev}^0$, hence in the image of $(q_0^g)^*$ – which was to be shown.

where

$$(p|K) \;=\; \begin{cases} 0 & \text{if} \quad \gcd(p,\,K) \neq 1 \\ \pm 1 & \text{if} \quad \gcd(p,\,K) = 1 \end{cases} \tag{152}$$

is the Jacobi symbol. [18]

In §4.4 we are crucially concerned with the variant of the classical quadratic Gauss sum that has *half* the usual exponents. In its plain form it is elementary to reduce this to the ordinary quadratic Gauss sum:

Proposition A.13 (Quadratic Gauss sum with halved exponents). *For $k \in 2\mathbb{N}_{>0}$ we have*

$$\sum_{n=0}^{K-1} e^{\frac{\pi \mathrm{i}}{K} n^2} \;=\; e^{\pi \mathrm{i}/4}\sqrt{K}\,. \tag{153}$$

Proof. Setting $r := K/2 \in \mathbb{N}$, we may compute as follows:

$$\begin{aligned}
\textstyle\sum_{n=0}^{K-1} e^{\frac{\pi \mathrm{i}}{K} n^2} \;&=\; \textstyle\sum_{n=0}^{2r-1} e^{\frac{\pi \mathrm{i}}{2r} n^2} && \text{by def of } r \\
&=\; \tfrac{1}{2}\Big(\textstyle\sum_{n=0}^{2r-1} + \sum_{n=2r}^{4r-1}\Big) e^{\frac{\pi \mathrm{i}}{2r} n^2} && \text{since summands are } 2r\text{-periodic, cf. footnote 14} \\
&=\; \tfrac{1}{2}\textstyle\sum_{n=0}^{4r-1} e^{\frac{2\pi \mathrm{i}}{4r} n^2} \\
&=\; \tfrac{1}{2}(1+\mathrm{i})\sqrt{4r} && \text{by (150)} \\
&=\; e^{\pi \mathrm{i}/4}\sqrt{2r} \\
&=\; e^{\pi \mathrm{i}/4}\sqrt{K} && \text{by def of } r.
\end{aligned}$$

□

More generally, there is the following reciprocity relation for the parameters of the quadratic Gauss sum with halved exponents, which relates it to the ordinary quadratic Gauss sum:

Proposition A.14 (Landsberg-Schaar identity [187], cf. [5][203][94]). *For $K \in 2\mathbb{N}_{>0}$ and $p \in \mathbb{N}_{>0}$ we have*

$$\sum_{n=0}^{K-1} e^{\pi \mathrm{i} \frac{p}{K} n^2} \;=\; \frac{e^{\pi \mathrm{i}/4}}{\sqrt{p/K}} \sum_{n=0}^{p-1} e^{-\pi \mathrm{i} \frac{K}{p} n^2}\,. \tag{154}$$

In summary, this implies the evaluation which we use in the main text:

Proposition A.15 (Quadratic Gauss sum with multiple halved exponents). *For $K \in 2\mathbb{N}_{>0}$ and $p \in 2\mathbb{N}+1$ we have:*

$$\sum_{n=1}^{K-1} e^{\pi \mathrm{i} \frac{p}{K} n^2} \underset{(154)}{=} \frac{e^{\pi \mathrm{i}/4}}{\sqrt{p/K}} \sum_{n=0}^{p-1} e^{-2\pi \mathrm{i} \frac{K/2}{p} n^2} \underset{(151)}{=} \begin{cases} e^{\pi \mathrm{i}/4}\sqrt{K}\,\big(K/2\,\big|\,p\big) & \big|\;\; p = 1 \bmod 4 \\ e^{-\pi \mathrm{i}/4}\sqrt{K}\,\big(K/2\,\big|\,p\big) & \big|\;\; p = 3 \bmod 4\,. \end{cases} \tag{155}$$

[18] The sign in (152) is the non-trivial content of the theory of the Jacobi symbol, but for our purposes in the main text it is of relevance only whether the Jacobi symbol vanishes or not.

References

[1] C. A. Abad, *Introduction to representations of braid groups*, Rev. Colomb. Mat. **49** 1 (2015), [doi:10.15446/recolma.v49n1.54160], [arXiv:1404.0724].

[2] M. Aguilar, S. Gitler, and C. Prieto, *Algebraic topology from a homotopical viewpoint*, Springer (2002), [doi:10.1007/b97586].

[3] O. Alvarez, *Topological quantization and cohomology*, Commun. Math. Phys. **100** 2 (1985), 279-309, [euclid:cmp/1103943448].

[4] L. Alvarez-Gaumé, G. Moore, and C. Vafa, *Theta functions, modular invariance, and strings*, Commun. Math. Phys. **106** 1 (1986), 1-4, [doi:10.1007/BF01210925], [euclid:cmp/1104115581].

[5] V. Armitage and A. Rogers, *Gauss Sums and Quantum Mechanics*, J. Phys. A: Math. Gen. **33** (2000) 5993, [doi:10.1088/0305-4470/33/34/305], [arXiv:quant-ph/0003107].

[6] D. P. Arovas, J. R. Schrieffer, and F. Wilczek, *Fractional Statistics and the Quantum Hall Effect*, Phys. Rev. Lett. **53** 7 (1984), 722-723, [doi:10.1103/PhysRevLett.53.722].

[7] D. V. Averin and V. J. Goldman, *Quantum computation with quasiparticles of the fractional quantum Hall effect*, Solid State Commun. **121** 1 (2001), 25-28, [doi:10.1016/S0038-1098(01)00447-1], [arXiv:cond-mat/0110193].

[8] D. J. Baker, *Identity, Superselection Theory, and the Statistical Properties of Quantum Fields*, Philosophy of Science **80** 2 (2013), 262-285, [doi:10.1086/670296].

[9] A. P. Balachandran, S. G. Jo, and G. Marmo, *Group Theory and Hopf Algebras – Lectures for Physicists*, World Scientific (2010), [doi:10.1142/7872].

[10] M. Barkeshli, C.-M. Jian, and X.-L. Qi, *Genons, Twist defects and projective non-Abelian braiding statistics*, Phys. Rev. B **87** (2013) 045130, [doi:10.1103/PhysRevB.87.045130], [arXiv:1302.2673].

[11] M. Barkeshli and X.-L. Qi, *Synthetic Topological Qubits in Conventional Bilayer Quantum Hall Systems*, Phys. Rev. X **4** (2014) 041035, [doi:10.1103/PhysRevX.4.041035], [arXiv:1302.2673].

[12] P. Bellingeri and S. Gervais, *Surface framed braids*, Geom. Dedicata **159** (2012), 51–69, [arXiv:1001.4471] [doi:10.1007/s10711-011-9645-5]

[13] D. Belov and G. W. Moore, *Classification of abelian spin Chern-Simons theories*, [arXiv:hep-th/0505235].

[14] B. C. Berndt and R. J. Evans, *The determination of Gauss sums*, Bull. Amer. Math. Soc. **5** (1981), 107-129, [ams:bull/1981-05-02/S0273-0979-1981-14930-2], [euclid:bams/1183548292].

[15] I. I. Beterov, *Progress and Prospects in the Field of Quantum Computing*, Optoelectron. Instrument. Proc. **60** (2024), 74–83, [doi:10.3103/S8756699024700043].

[16] J. S. Birman, *Mapping class groups and their relationship to braid groups*, Commun. Pure Appl. Math. **22** 2 (1969), 213-238, [doi:10.1002/cpa.3160220206].

[17] J. Blanc and J. Déserti, *Embeddings of* $\mathrm{SL}(2,\mathbb{Z})$ *into the Cremona group*, Transform. Groups **17** 1 (2012), 21-50, [doi:10.1007/s00031-012-9174-9], [arXiv:1103.0114].

[18] W. Bloomquist, *Mapping Class Group Notes* (2024) [ncatlab.org/nlab/files/Bloomquist-MCG.pdf]

[19] P. Bonderson, E. C. Rowell, Q. Zhang, and Z. Wang, *Congruence Subgroups and Super-Modular Categories*, Pacific J. Math. 296 (2018), 257-270, [doi:10.2140/pjm.2018.296.257], [arXiv:1704.02041].

[20] S. Borsboom, H. Posthuma, *Global Gauge Symmetries and Spatial Asymptotic Boundary Conditions in Yang-Mills theory* [arXiv:2502.16151].

[21] M. Bos and V. P. Nair, $U(1)$ *Chern-Simons theory and* $c = 1$ *conformal blocks*, Phys. Lett. B **223** 1 (1989), 61-66, [doi:10.1016/0370-2693(89)90920-9].

[22] S. Bravyi, *Universal Quantum Computation with the* $\nu = 5/2$ *Fractional Quantum Hall State*, Phys. Rev. A **73** (2006) 042313, [doi:10.1103/PhysRevA.73.042313], [arXiv:quant-ph/0511178].

[23] G. Bredon, *Topology and Geometry*, Graduate Texts in Mathematics **139**, Springer (1993), [doi:10.1007/978-1-4757-6848-0].

[24] S. Carlip, *Dynamics of Asymptotic Diffeomorphisms in* $(2+1)$*-Dimensional Gravity*, Class. Quant. Grav. **22** (2005), 3055-3060, [doi:10.1088/0264-9381/22/14/014], [arXiv:gr-qc/0501033].

[25] T. Chakraborty and P. Pietiläinen, *The Quantum Hall Effects – Integral and Fractional*, Springer Series in Solid State Sciences (1995), [doi:10.1007/978-3-642-79319-6].

[26] R. Chen, *Generalized Yang-Baxter Equations and Braiding Quantum Gates*, J. Knot Theory Ramif. **21** 09 (2012) 1250087, [doi:10.1142/S0218216512500873], [arXiv:1108.5215].

[27] F. M. Ciaglia, F. Di Cosmo, A. Ibort, and G. Marmo, *Schwinger's picture of Quantum Mechanics*, Int. J. Geom. Methods Modern Phys. **17** 04 (2020) 2050054, [doi:10.1142/S0219887820500541], [arXiv:2002.09326].

[28] F. M. Ciaglia, A. Ibort, and G. Marmo, *Schwinger's Picture of Quantum Mechanics I: Groupoids*, Int. J. Geom. Methods Modern Phys. **16** 08 (2019) 1950119, [doi:10.1142/S0219887819501196], [arXiv:1905.12274].

[29] Clay Math Institute, *The Millennium Prize Problems*, [www.claymath.org/millennium-problems].

[30] F. R. Cohen, *Introduction to configuration spaces and their applications*, in: *Braids*, Lecture Notes Series **19**, Institute for Mathematical Sciences, National University of Singapore (2009), 183-261, [doi:10.1142/9789814291415_0003]

[31] F. R. Cohen and J. Wu, *On Braid Groups, Free Groups, and the Loop Space of the 2-Sphere*, in: *Categorical Decomposition Techniques in Algebraic Topology*, Progress in Mathematics **215**, Birkhäuser (2003), 93-105, [doi:10.1007/978-3-0348-7863-0_6].

[32] K. Conrad, $\mathrm{SL}_2(\mathbb{Z})$, [kconrad.math.uconn.edu/blurbs/grouptheory/SL(2,Z).pdf]

[33] T. Cutler, *The category of pointed topological spaces* (2020), [ncatlab.org/nlab/files/CutlerPointedTopologicalSpaces.pdf].

[34] DARPA, *Quantum Benchmarking Initiative* (2024), [www.darpa.mil/work-with-us/quantum-benchmarking-initiative].

[35] S. Das Sarma, *Quantum computing has a hype problem*, MIT Tech Review (March 2022), [www.technologyreview.com/2022/03/28/1048355/quantum-computing-has-a-hype-problem/]

[36] S. Das Sarma, *In search of Majorana*, Nature Phys. **19** (2023), 165-170, [arXiv:2210.17365], [doi:10.1038/s41567-022-01900-9].

[37] S. Das Sarma, M. Freedman, and C. Nayak, *Topologically-Protected Qubits from a Possible Non-Abelian Fractional Quantum Hall State*, Phys. Rev. Lett. **94** (2005) 166802, [doi:10.1103/PhysRevLett.94.166802], [arXiv:cond-mat/0412343].

[38] P. Deligne, *Equations différentielles à points singuliers réguliers*, Lecture Notes Math. **163**, Springer (1970), [publications.ias:355].

[39] M. De Renzi, A. M. Gainutdinov, N. Geer, B. Patureau-Mirand, and I. Runkel, *Mapping Class Group Representations From Non-Semisimple TQFTs*, Commun. Contemp. Math. (2021) 2150091, [arXiv:2010.14852], [doi:10.1142/S0219199721500917].

[40] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory*, Springer (1997), [doi:10.1007/978-1-4612-2256-9].

[41] R. Dijkgraaf and E. Witten, *Topological Gauge Theories and Group Cohomology*, Commun. Math. Phys. **129** (1990) 393, [doi:10.1007/BF02096988], [euclid:cmp/1104180750].

[42] A. Dimca, *Sheaves in Topology*, Universitext, Springer (2004), [doi:10.1007/978-3-642-18868-8].

[43] G. Doyle, *Quadratic Form Gauss Sums*, PhD thesis, U. Ottawa (2016), [doi:10.22215/etd/2016-11457]

[44] C. Druţu and M. Kapovich (appendix by B. Nica), *Geometric group theory*, Colloquium Publications **63**, AMS (2018) [ISBN:978-1-4704-1104-6].

[45] F. Dul, *General Covariance from the Viewpoint of Stacks*, Lett. Math. Phys. **113** (2023) 30, [doi:10.1007/s11005-023-01653-3], [arXiv:2112.15473].

[46] D. S. Dummit and R. M. Foote, *Abstract Algebra*, Wiley (2003), [ISBN:978-0-471-43334-7]

[47] M. I. Dyakonov, *Prospects for quantum computing: extremely doubtful*, Int. J. of Modern Phys. Conf. Ser. **33** (2014) 1460357, [doi:10.1142/S2010194514603573], [arXiv:1401.3629].

[48] C. J. Earle and J. Eells, *The diffeomorphism group of a compact Riemann surface*, Bull. Amer. Math. Soc. **73** 4 (1967), 557-559, [euclid:bams/1183528956].

[49] C. J. Earle and J. Eells, *A fibre bundle description of Teichmüller theory*, J. Differential Geom. **3** 1-2 (1969), 19-43, [doi:10.4310/jdg/1214429381].

[50] C. J. Earle and A. Schatz, *Teichmüller theory for surfaces with boundary*, J. Differential Geom. **4** 2 (1970), 169-185, [doi:10.4310/jdg/1214429381].

[51] P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, D. Vaintrob, and E. Yudovina, *Introduction to representation theory*, Student Mathematical Library **59**, AMS (2011), [ams:stml-59], [arXiv:0901.0827].

[52] O. Ezratty, *Where are we heading with NISQ?*, [arXiv:2305.09518].

[53] O. Ezratty, *Where are we heading with NISQ?*, blog post (2023), [www.oezratty.net/wordpress/2023/where-are-we-heading-with-nisq].

[54] E. Fadell and J. Van Buskirk, *On the braid groups of E^2 and S^2*, Bull. Amer. Math. Soc. **67** 2 (1961), 211-213, [euclid:bams/1183524083].

[55] B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton University Press (2012), [ISBN:9780691147949], [jstor:j.ctt7rkjw].

[56] A. Ferraz, K. S. Gupta, G. W. Semenoff, and P. Sodano (eds), *Strongly Coupled Field Theories for Condensed Matter and Quantum Information Theory*, Springer Proceedings in Physics **239**, Springer (2020), [doi:10.1007/978-3-030-35473-2].

[57] D. Fiorenza, H. Sati, and U. Schreiber, *Twisted Cohomotopy implies M-theory anomaly cancellation on 8-manifolds*, Commun. Math. Phys. **377** (2020), 1961-2025, [doi:10.1007/s00220-020-03707-2], [arXiv:1904.10207].

[58] D. Fiorenza, H. Sati, and U. Schreiber, *Twistorial Cohomotopy Implies Green-Schwarz anomaly cancellation*, Rev. Math. Phys. **34** 05 (2022) 2250013, [doi:10.1142/S0129055X22500131], [arXiv:2008.08544].

[59] D. Fiorenza, H. Sati, and U. Schreiber, *The Character map in Nonabelian Cohomology — Twisted, Differential and Generalized*, World Scientific, Singapore (2023), [doi:10.1142/13422], [arXiv:2009.11909]
NB: in the text we refer to the numbering of the published version, differening from that of the preprint version, cf. [ncatlab.org/schreiber/show/The+Character+Map+in+Non-Abelian+Cohomology]

[60] A. Fomenko and D. Fuchs, *Homotopical Topology*, Graduate Texts in Mathematics **273**, Springer (2016), [doi:10.1007/978-3-319-23488-5].

[61] A. G. Fowler and L. C. L. Hollenberg, *Scalability of Shor's algorithm with a limited set of rotation gates*, Phys. Rev. A **70** (2007) 032329, [doi:10.1103/PhysRevA.103.032417].

[62] R. H. Fox, L. Neuwirth, *The braid groups*, Math. Scand. **10** (1962), 119-126, [doi:10.7146/math.scand.a-10518].

[63] E. Fradkin, *Field Theories of Condensed Matter Physics*, Cambridge University Press (2013), [doi:10.1017/CBO9781139015509].

[64] D. Freed, *Classical Chern-Simons theory, 1*, Adv. Math. **113** (1995), 237-303, [arXiv:hep-th/9206021], [doi:10.1006/aima.1995.1039].

[65] M. Freedman, A. Kitaev, M. Larsen, and Z. Wang, *Topological quantum computation*, Bull. Amer. Math. Soc. **40** (2003), 31-38, [doi:10.1090/S0273-0979-02-00964-3], [arXiv:quant-ph/0101025].

[66] J. Fröhlich and T. Kerler, *Universality in quantum Hall systems*, Nucl. Phys. B **354** 2–3 (1991), 369-417, [doi:10.1016/0550-3213(91)90360-A].

[67] J. Fröhlich and C. King, *The Chern-Simons theory and knot polynomials*, Commun. Math. Phys. **126** 1 (1989), 167-199, [doi:10.1007/BF02124336], [euclid:cmp/1104179728].

[68] J. Fuchs and C. Schweigert, *Symmetry breaking boundaries II. More structures; examples*, Nucl. Phys. B **568** (2000), 543-593, [doi:10.1016/S0550-3213(99)00669-0], [arXiv:hep-th/9908025].

[69] J. Fuchs and C. Schweigert, *Lie algebra automorphisms in conformal field theory*, in: *Conference on Infinite Dimensional Lie Theory and Conformal Field Theory* (May 2000), [arXiv:math/0011160].

[70] W. Fulton and J. Harris, *Representation Theory: A First Course*, Springer (1991), [doi:10.1007/978-1-4612-0979-9].

[71] L. Funar, *Theta functions, root systems and 3-manifold invariants*, J. Geom. Phys. **17** 3 (1995), 261-282, [doi:10.1016/0393-0440(94)00050-E].

[72] J. Fröhlich, F. Gabbiani, and P. Marchetti, *Braid statistics in three-dimensional local quantum field theory*, in: *Physics, Geometry and Topology*, NATO ASI Series **238**, Springer (1990), [doi:10.1007/978-1-4615-3802-8_2].

[73] A. Gadbled, A.-L. Thiel, and E. Wagner, *Categorical action of the extended braid group of affine type A*, Commun. Contemp. Math. **19** 03 (2017) 1650024, [doi:10.1142/S0219199716500243], [arXiv:1504.07596].

[74] T. Gannon, *Modular Data: The Algebraic Combinatorics of Conformal Field Theory*, J. Algebr. Comb. **22** (2005), 211–250, [doi:10.1007/s10801-005-2514-2], [arXiv:math/0103044].

[75] R. Gelca and A. Uribe, *From classical theta functions to topological quantum field theory*, in: *The Influence of Solomon Lefschetz in Geometry and Topology*, Contemporary Mathematics **621**, AMS (2014) 35-68 [arXiv:1006.3252] [doi:10.1090/conm/621]

[76] R. Gelca and A. Hamilton, *Classical theta functions from a quantum group perspective*, New York J. Math. **21** (2015) 93–127 [arXiv:1209.1135] [nyjm:j/2015/21-4]

[77] R. Gelca and A. Hamilton, *The topological quantum field theory of Riemann's theta functions*, J. Geometry and Physics **98** (2015) 242-261 [doi:10.1016/j.geomphys.2015.08.008] [arXiv:1406.4269]

[78] E. Gent, *Quantum Computing's Hard, Cold Reality Check*, IEEE Spectrum (Dec. 2023), [spectrum.ieee.org/quantum-computing-skeptics].

[79] P. J. Giblin, *Graphs, Surfaces and Homology – An Introduction to Algebraic Topology*, Chapman and Hall (1977), [doi:10.1007/978-94-009-5953-8].

[80] S. G. Gill et al., *Quantum Computing: Vision and Challenges*, technical report, [arXiv:2403.02240].

[81] G. Giotopoulos and H. Sati, *Field Theory via Higher Geometry I: Smooth Sets of Fields*, J. Geom. Phys. **213** (2025) 105462, [doi:10.1016/j.geomphys.2025.105462], [arXiv:2312.16301].

[82] G. Giotopoulos, H. Sati, and U. Schreiber, *Super-Flux Quantization on 11d Superspace*, J. High Energy Phys. **2024** (2024) 82, [doi;10.1007/JHEP07(2024)082], [arXiv:2403.16456].

[83] G. Giotopoulos, H. Sati, and U. Schreiber, *Flux Quantization on M5-Branes*, J. High Energy Phys. **2024** (2024) 140, [doi;10.1007/JHEP10(2024)140], [arXiv:2406.11304].

[84] S. M. Girvin, *Introduction to the Fractional Quantum Hall Effect*, Séminaire Poincaré **2** (2004) 53–74; reprinted in: *The Quantum Hall Effect*, Progress in Mathematical Physics **45**, Birkhäuser (2005), [doi:10.1007/3-7643-7393-8_4].

[85] A. Giuliani, V. Mastropietro, and M. Porta, *Universality of the Hall conductivity in interacting electron systems*, Commun. Math. Phys. 349 (2017), 1107–1161, [doi:10.1007/s00220-016-2714-8], [arXiv:1511.04047].

[86] P. Glidic et al., *Cross-Correlation Investigation of Anyon Statistics in the $\nu = 1/3$ and $2/5$ Fractional Quantum Hall States*, Phys. Rev. X **13** (2023) 011030, [arXiv:10.1103/PhysRevX.13.011030], [arXiv:2210.01054].

[87] T. Gocho, *The topological invariant of three-manifolds based on the $U(1)$ Gauge theory*, Proc. Japan Acad. Ser. A Math. Sci. **66** 8 (1990), 237-239, [doi:10.3792/pjaa.66.237.full], [dml:1195512360].

[88] A. Gramain, *Le type d'homotopie du groupe des difféomorphismes d'une surface compacte*, Ann. Scient. l'École Normale Sup. **6** 1 (1973), 53-66, [doi:10.24033/asens.1242].

[89] D. Griffiths, *Introduction to Electrodynamics*, Cambridge University Press (2023, 2025), [doi:10.1017/9781009397735].

[90] A. Gromov, E. J. Martinec, and S. Ryu, *Collective excitations at filling factor $5/2$: The view from superspace*, Phys. Rev. Lett. **125** (2020) 077601, [doi:10.1103/PhysRevLett.125.077601], [arXiv:1909.06384].

[91] E. Grumblin and M. Horowitz (eds.), *Quantum Computing: Progress and Prospects*, The National Academies Press (2019), [doi:10.17226/25196], [ISBN:9780309479691].

[92] O. Gwilliam, *Remarks on the locality of generalized global symmetries* [arXiv:2504.05626].

[93] V. L. Hansen, *On the Space of Maps of a Closed Surface into the 2-Sphere*, Math. Scand. **35** (1974), 149-158, [doi:10.7146/math.scand.a-11542], [jstor:24490694].

[94] G. Harcos, *The reciprocity of Gauss sums via the residue theorem*, [ncatlab.org/nlab/files/Harcos-ReciprocityOfGaussSums.pdf].

[95] J. B. Hartle and J. R. Taylor, *Quantum Mechanics of Paraparticles*, Phys. Rev. **178** (1969) 2043, [doi:10.1103/PhysRev.178.2043].

[96] A. Hatcher, *Algebraic Topology*, Cambridge University Press (2002), [ISBN:9780521795401].

[97] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press (1992), [doi:10.2307/j.ctv10crg0r].

[98] C.-L. Ho, A. I. Solomon and C.-H. Oh, *Quantum entanglement, unitary braid representation and Temperley-Lieb algebra*, EPL **92** (2010) 30002, [doi:10.1209/0295-5075/92/30002], [arXiv:1011.6229].

[99] T. Hoefler, T. Haener, and M. Troyer, *Disentangling Hype from Practicality: On Realistically Achieving Quantum Advantage*, Commun. ACM **66** 5 (2023), 82-87, [doi:10.1145/3571725], [arXiv:2307.00523].

[100] S.-T. Hu, *Concerning the homotopy groups of the components of the mapping space Y^{S^p}*, Indagationes Math. **8** (1946), 623-629, [dwc.knaw.nl/DL/publications/PU00018263.pdf]

[101] S.-T. Hu, *Homotopy Theory*, Academic Press (1959), [https://www.maths.ed.ac.uk/~v1ranick/papers/hu2.pdf]

[102] D. Husemöller, *Fibre bundles*, Springer (1966, 1975, 1994), [doi:10.1007/978-1-4757-2261-1].

[103] A. Ikeda, *Homological and Monodromy Representations of Framed Braid Groups*, Commun. Math. Phys. **359** (2018), 1091–1121, [doi:10.1007/s00220-017-3036-1], [arXiv:1702.03918].

[104] I. M. Isaacs, *Character theory of finite groups*, Academic Press, New York (1976), [ISBN:978-0-8218-4229-4].

[105] N. V. Ivanov, *Mapping class groups*, in: *Handbook of Geometric Topology*, North-Holland (2002), 523-633, [doi:10.1016/B978-0-444-82432-5.X5000-8].

[106] J. K. Jain, *Composite-fermion approach for the fractional quantum Hall effect*, Phys. Rev. Lett. **63** (1989) 199, [doi:10.1103/PhysRevLett.63.199].

[107] J. K. Jain, *Microscopic theory of the fractional quantum Hall effect*, Adv. Phys. **41** (1992), 105-146, [doi:10.1080/00018739200101483].

[108] J. K. Jain, *Composite Fermions*, Cambridge University Press (2007), [doi:10.1017/CBO9780511607561].

[109] I. M. James, *General Topology and Homotopy Theory*, Springer (1984), [doi:10.1007/978-1-4613-8283-6].

[110] B. Jeckelmann and B. Jeanneret, *The Quantum Hall Effect as an Electrical Resistance Standard*, in: *The Quantum Hall Effect – Poincaré Seminar 2004*, Progress in Mathematical Physics **45**, Birkhäuser (2005), 55-131, [doi:10.1007/3-7643-7393-8_3].

[111] S. P. Jordan, *Quantum Computation Beyond the Circuit Model*, PhD thesis, MIT (2010), [arXiv:0809.2307].

[112] S. P. Jordan, *Permutational Quantum Computing*, Quantum Information and Computation **10** (2010) 470, [doi:10.26421/QIC10.5-6-7], [arXiv:0906.2508].

[113] S. Kak, *Prospects for Quantum Computing*, talk at CIFAR Nanotechnology program meeting, Halifax (November 2008), [arXiv:0902.4884].

[114] S. Kallel, *Configuration Spaces and the Topology of Curves in Projective Space*, in: *Topology, Geometry, and Algebra: Interactions and new directions*, Contemporary Mathematics **279**, AMS (2001), 151–175, [doi:10.1090/conm/279], [arXiv:math-ph/0003010].

[115] S. Kallel, *Configuration spaces of points: A user's guide*, Encyclopedia of Mathematical Physics 2nd ed. (2024), [arXiv:2407.11092], [ISBN:9780323957038].

[116] D. Kao, *Representations of the Symmetric Group*, VIGRE presentation (2010), [ncatlab.org/nlab/files/Kao-SymRep.pdf]

[117] L. H. Kauffman, S. J. Lomonaco, *Braiding Operators are Universal Quantum Gates*, New J. Phys. **6** (2004), [doi:10.1088/1367-2630/6/1/134] [arXiv:quant-ph/0401090]

[118] L. H. Kauffman, *Majorana Fermions and Representations of the Braid Group*, Int. J. Modern Phys. A **33** 23 (2018) 1830023, [doi:10.1142/S0217751X18300235], [arXiv:1710.04650].

[119] R. P. Kent and D. Pfeifer, *A Geometric and algebraic description of annular braid groups*, Int. J. Algebra Comput. **12** 01n02 (2002), 85-97, [doi:10.1142/S0218196702000997].

[120] A. Kirillov, *An Introduction to Lie Groups and Lie Algebras*, Cambridge University Press (2008), [doi:10.1017/CBO9780511755156].

[121] A. Kitaev, *Fault-tolerant quantum computation by anyons*, Ann. Phys. **303** (2003), 2-30, [doi:10.1016/S0003-4916(02)00018-0], [arXiv:quant-ph/9707021].

[122] A. Kitaev, *Anyons in an exactly solved model and beyond*, Ann. Phys. **321** 1 (2006), 2-111, [doi:10.1016/j.aop.2005.10.005].

[123] C. Kittel, *Introduction to Solid State Physics*, Wiley (1953-), [ISBN:978-0-471-41526-8].

[124] A. W. Knapp, *Lie Groups Beyond an Introduction*, Progress in Mathematics **140** (1996, 2002), [ISBN:9780817642594].

[125] K. H. Ko and L. Smolinsky, *The framed braid group and 3-manifolds*, Proc. Amer. Math. Soc. **115** (1992), 541-551, [doi:10.1090/S0002-9939-1992-1126197-1].

[126] M. Koecher and A. Krieg, *Elliptische Funktionen und Modulformen*, Springer (2007), [doi:10.1007/978-3-540-49325-9].

[127] S. S. Koh, *Note on the properties of the components of the mapping spaces X^{S^p}*, Proc. of the AMS **11** (1960), 896-904, [ncatlab.org/nlab/files/Koh-MappingSpace.pdf]

[128] A. Kokkinakis, *Framed Braid Equivalences*, [arXiv:2503.05342].

[129] A. Kosinski, *Differential manifolds*, Academic Press (1993), [ISBN:978-0-12-421850-5].

[130] S. Lang, *Algebraic number theory*, Graduate Texts in Mathematics **110**, Springer (1970, 1994), [doi:10.1007/978-1-4612-0853-2].

[131] L. L. Larmore and E. Thomas, *On the Fundamental Group of a Space of Sections*, Math. Scand. **47** 2 (1980), 232-246, [jstor:24491393].

[132] J. W. Z. Lau, K. H. Lim, H. Shrotriya, and L. C. Kwek, *NISQ computing: where are we and where do we go?*, AAPPS Bull. **32** (2022) 27, [doi:10.1007/s43673-022-00058-z].

[133] G. Lechner, U. Pennig, and S. Wood, *Yang-Baxter representations of the infinite symmetric group*, Adv. Math. **355** (2019) 106769, [doi:10.1016/j.aim.2019.106769], [arXiv:1707.00196].

[134] S. T. Lee and J. A. Packer, *The Cohomology of the Integer Heisenberg Groups*, J. Algebra **184** 1 (1996), 230-250, [doi:10.1006/jabr.1996.0258].

[135] D. A. Lidar and T. A. Brun (eds.), *Quantum Error Correction*, Cambridge University Press (2013), [ISBN:9780521897877], [doi:10.1017/CBO9781139034807].

[136] S. MacLane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics **5**, second ed., Springer (1997), [doi:10.1007/978-1-4757-4721-8].

[137] M. Manoliu, *Abelian Chern-Simons theory*, J. Math. Phys. **39** (1998), 170-206, [arXiv:dg-ga/9610001], [doi:10.1063/1.532333].

[138] G. Massuyeau, *Lectures on Mapping Class Groups, Braid Groups and Formality*, lecture notes (2021), [massuyea.perso.math.cnrs.fr/notes/formality.pdf]

[139] D. McDuff, *Configuration spaces of positive and negative particles*, Topology **14** 1 (1975), 91-107, [doi:10.1016/0040-9383(75)90038-5].

[140] P. Melvin and N. B. Tufillaro, *Templates and framed braids*, Phys. Rev. A **44** (1991) R3419(R), [doi:10.1103/PhysRevA.44.R3419].

[141] N. D. Mermin, *The topological theory of defects in ordered media*, Rev. Mod. Phys. **51** (1979) 591, [doi:10.1103/RevModPhys.51.591].

[MPW19] M. Mezei, S. S. Pufu, and Y. Wang, *Chern-Simons theory from M5-branes and calibrated M2-branes*, J. High Energ. Phys. **2019** (2019) 165, [doi:10.1007/JHEP08(2019)165], [arXiv:1812.07572].

[142] Microsoft Quantum, *Roadmap to fault tolerant quantum computation using topological qubit arrays*, [arXiv:2502.12252].

[143] J. Milnor: *Spin Structures on Manifolds*, L'Enseignement Mathématique **9** (1963), 198-203, [doi:10.5169/seals-38784].

[144] R. S. K. Mong et al., *Universal Topological Quantum Computation from a Superconductor-Abelian Quantum Hall Heterostructure*, Phys. Rev. X **4** (2014) 011036, [doi:10.1103/PhysRevX.4.011036], [arXiv:1307.4403].

[145] J. Morava, *A homotopy-theoretic context for CKM/Birkhoff renormalization*, [arXiv:2307.10148].

[146] J. Morava, *Some very low-dimensional algebraic topology*, [arXiv:2411.15885].

[147] G. Moore and N. Read, *Nonabelions in the fractional quantum Hall effect*, Nucl. Phys. B **360** (1991) 362, [doi:10.1016/0550-3213(91)90407-O].

[148] S. Morita, *Introduction to mapping class groups of surfaces and related groups*, in: *Handbook of Teichmüller theory, Volume I*, EMS (2007), 353-386, [doi:10.4171/029-1/8].

[149] D. J. Myers, H. Sati, and U. Schreiber, *Topological Quantum Gates in Homotopy Type Theory*, Commun. Math. Phys. **405** (2024) 172, [doi;10.1007/s00220-024-05020-8].

[150] J. Nakamura, S. Fallahi, H. Sahasrabudhe, R. Rahman, S. Liang, G. C. Gardner, and M. J. Manfra, *Aharonov–Bohm interference of fractional quantum Hall edge modes*, Nature Phys. **15** (2019), 563–569, [doi:10.1038/s41567-019-0441-8], [arXiv:1901.08452].

[151] J. Nakamura, S. Liang, G. C. Gardner, and M. J. Manfra, *Direct observation of anyonic braiding statistics*, Nature Phys. **16** (2020), 931–936, [doi:10.1038/s41567-020-1019-1], [arXiv:2006.14115].

[152] J. Nakamura1, S. Liang, G. C. Gardner, and M. J. Manfra, *Fabry-Perot interferometry at the $\nu = 2/5$ fractional quantum Hall state*, Phys. Rev. X **13** (2023) 041012, [doi:10.1103/PhysRevX.13.041012], [arXiv:2304.12415].

[153] M. A. Nielsen, I. L. Chuang, *Quantum computation and quantum information*, Cambridge University Press (2000), [doi:10.1017/CBO9780511976667].

[154] T. Nikolaus, U. Schreiber and D. Stevenson, *Principal ∞-bundles – General Theory* J. Homotopy and Related Structures, **10** 4 (2015) 749-801 [doi:10.1007/s40062-014-0083-6] [arXiv:1207.0248].

[Oh1] T. Ohtsuki, *Quantum Invariants – A Study of Knots, 3-Manifolds, and Their Sets*, World Scientific (2001), [doi:10.1142/4746].

[155] S. Okuyama, *The space of intervals in a Euclidean space*, Algebr. Geom. Topol. **5** (2005), 1555-1572, [arXiv:math/0511645], [doi:10.2140/agt.2005.5.1555].

[156] Z. Papić and A. C. Balram, *Fractional quantum Hall effect in semiconductor systems*, Encyclopedia of Condensed Matter Physics 2nd ed. **1** (2024), 285-307, [doi:10.1016/B978-0-323-90800-9.00007-X], [arXiv:2205.03421].

[Pol88] A. M. Polyakov, *Fermi-Bose transmutation induced by gauge fields*, Mod. Phys. Lett. A **03** 03 (1988), 325-328, [doi:10.1142/S0217732388000398].

[157] A. P. Polychronakos, *Abelian Chern-Simons theories in 2+1 dimensions*, Ann. Phys. **203** 2 (1990), 231-254, [doi:10.1016/0003-4916(90)90171-J].

[158] A. P. Polychronakos, *Path Integrals and Parastatistics*, Nucl. Phys. B **474** (1996), 529-539, [doi:10.1016/0550-3213(96)00277-5], [arXiv:hep-th/9603179].

[159] L. Pontrjagin, *Classification of continuous maps of a complex into a sphere, Communication I*, Doklady Akademii Nauk SSSR **19** 3 (1938), 147-149.

[160] R. E. Prange, S. M. Girvin (eds.), *The Quantum Hall Effect*, Graduate Texts in Contemporary Physics, Springer (1986, 1990), [doi:10.1007/978-1-4612-3350-3].

[161] A. Prasad, *An easy proof of the Stone-von Neumann-Mackey theorem*, Expositiones Mathematicae **29** (2011) 110-118 [arXiv:0912.0574] [doi:10.1016/j.exmath.2010.06.001]

[162] J. Preskill, *Crossing the Quantum Chasm: From NISQ to Fault Tolerance*, talk at Q2B 2023, Silicon Valley (2023), [ncatlab.org/nlab/files/Preskill-Crossing.pdf].

[163] J. Preskill, *Beyond NISQ: The Megaquop Machine*, talk at *Q2B 2024 Silicon Valley* (Dec. 2024), [www.preskill.caltech.edu/talks/Preskill-Q2B-2024.pdf].

[164] S. Pu, A. C. Balram, M. Fremling, A. Gromov, and Z. Papić, *Signatures of Supersymmetry in the $\nu = 5/2$ Fractional Quantum Hall Effect*, Phys. Rev. Lett. **130** (2023) 176501, [arXiv:2301.04169], [doi:10.1103/PhysRevLett.130.176501].

[165] M. Ram Murty and S. Pathak, *Evaluation of the quadratic Gauss sum*, The Math. Student **86** 1-2 (2017), 139-150, [ncatlab.org/nlab/files/RamMurtyPathak-GaussSum.pdf]

[166] I. Romaidis, *Mapping class group actions and their applications to 3D gravity*, PhD thesis, Hamburg (2022), [ediss:9945].

[167] I. Romaidis and I. Runkel, *CFT correlators and mapping class group averages*, Commun. Math. Phys. **405** (2024) 247, [doi:10.1007/s00220-024-05111-6], [arXiv:2309.14000].

[168] G. Rudolph and M. Schmidt, *Differential Geometry and Mathematical Physics Part II. Fibre Bundles, Topology and Gauge Fields*, Springer (2017), [doi:10.1007/978-94-024-0959-8].

[169] M. Ruelle et al., *Comparing fractional quantum Hall Laughlin and Jain topological orders with the anyon collider*, Phys. Rev. X **13** (2023) 011031, [doi:10.1103/PhysRevX.13.011031], [arXiv:2210.01066].

[170] H. Sati, *Framed M-branes, corners, and topological invariants*, J. Math. Phys. **59** (2018) 062304, [doi:10.1063/1.5007185], [arXiv:1310.1060].

[171] H. Sati and U. Schreiber, *Equivariant Cohomotopy implies orientifold tadpole cancellation*, J. Geom. Phys. **156** (2020) 103775, [doi:10.1016/j.geomphys.2020.103775], [arXiv:1909.12277].

[172] H. Sati and U. Schreiber, *Anyonic topological order in TED K-theory*, Rev. Math. Phys. **35** 03 (2023) 2350001, [doi:10.1142/S0129055X23500010], [arXiv:2206.13563].

[173] H. Sati and U. Schreiber, *The Quantum Monadology*, [arXiv:2310.15735].

[174] H. Sati and U. Schreiber, *Entanglement of Sections*, [arXiv:2309.07245].

[175] H. Sati and U. Schreiber, *Flux Quantization on Phase Space*, Ann. Henri Poincaré (2024), [doi:10.1007/s00023-024-01438-x], [arXiv:2312.12517].

[176] H. Sati and U. Schreiber, *Quantum Observables of Quantized Fluxes*, Ann. Henri Poincaré (2024), [doi:10.1007/s00023-024-01517-z], [arXiv:2312.13037].

[177] H. Sati and U. Schreiber, *Abelian Anyons on Flux-Quantized M5-Branes*, [arXiv:2408.11896].

[178] H. Sati and U. Schreiber, *Flux Quantization*, Encyclopedia of Mathematical Physics 2nd ed. **4** Elsevier (2025), 281-324, [doi:10.1016/B978-0-323-95703-8.00078-1].

[179] H. Sati and U. Schreiber, *Anyons on M5-Probes of Seifert 3-Orbifolds via Flux-Quantization*, Lett. Math. Phys. **115** 36 (2025), [doi:10.1007/s11005-025-01918-z], [arXiv:2411.16852].

[180] H. Sati and U. Schreiber, *The character map in Equivariant Twistorial Cohomotopy*, in: *Applied Algebraic Topology*, Beijing J. Pure Appl. Math., special issue (2025, in print), [arXiv:2011.06533].

[181] H. Sati and U. Schreiber: *Proper Orbifold Cohomology* [arXiv:2008.01101]

[182] H. Sati and U. Schreiber, *Equivariant Principal ∞-Bundles*, Cambridge University Press (2025, in print), [arXiv:2112.13654].

[183] H. Sati and U. Schreiber, *Higher Gauge Theory and Nonabelian Differential Cohomology — an exposition*, commissioned for: *Fundamental Structures in Computational and Pure Mathematics*, Trends in Mathematics, Birkhäuser (2025, to appear), [ncatlab.org/schreiber/show/Exposition+of+Higher+Gauge+Theory].

[184] H. Sati and U. Schreiber, *Engineering of Anyons on M5-Probes via Flux Quantization*, Srnií lecture notes (2025), [arXiv:2501.17927].

[185] H. Sati and S. Valera, *Topological Quantum Computing*, Encyclopedia of Mathematical Physics 2nd ed **4** (2025), 325-345, [doi:10.1016/B978-0-323-95703-8.00262-7].

[186] S. Sau, *A Roadmap for a Scalable Topological Quantum Computer*, Physics **10** 68 (2017), [physics.aps.org/articles/v10/68].

[187] M. Schaar, *Mémoire sur la théorie des résidus biquadratiques*, Mémoires de l'Acad'emie Royale des Sciences, des Lettres et des Beaux-Arts de Belgique **24** (1850), [biodiversitylibrary:20728].

[188] C. Schweigert, *Introduction to conformal field theory*, lecture notes (2013/14), [www.math.uni-hamburg.de/home/schweigert/skripten/cftskript.pdf]

[189] J.-P. Serre, *A Course in Arithmetic*, Graduate Texts in Mathematics **7**, Springer (1973), [doi:10.1007/978-1-4684-9884-4].

[190] J.-P. Serre, *Trees*, Springer (1980), [doi:10.1007/978-3-642-61856-7].

[191] S. Smale, *Diffeomorphisms of the 2-sphere*, Proc. Amer. Math. Soc. **10** (1959) 621–626, [jstor:2033664], [doi:10.1090/S0002-9939-1959-0112149-8].

[192] E. Spanier, *Borsuk's Cohomotopy Groups*, Ann. Math. **50** 1 (1949), 203-245, [jstor:1969362].

[193] A. Stern, *Anyons and the quantum Hall effect – A pedagogical review*, Ann. Phys. **323** 1 (2008), 204-249, [doi:10.1016/j.aop.2007.10.008], [arXiv:0711.4697].

[194] H. L. Störmer, *Nobel Lecture: The fractional quantum Hall effect*, Rev. Mod. Phys. **71** (1999) 875, [doi:10.1103/RevModPhys.71.875].

[195] J. Strom, *Modern classical homotopy theory*, Graduate Studies in Mathematics **127**, American Mathematical Society (2011), [ams:gsm/127].

[196] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory*, Princeton University Press (2018), [ISBN:9780691179506], [arXiv:1703.05448]. .

[197] W. P. Su, *Statistics of the fractionally charged excitations in the quantum Hall effect*, Phys. Rev. B **34** (1986) 1031, [doi:10.1103/PhysRevB.34.1031].

[198] T. Szamuely, *Galois groups and fundamental groups*, Cambridge Stud. Adv. Math. **117**, Cambridge University Press, (2009), [doi:10.1017/CBO9780511627064]

[199] C. Tan, *Smallest nonabelian quotients of surface braid groups*, Algebr. Geom. Topol. **24** (2024), 3997-4006, [doi:10.2140/agt.2024.24.3997], [arXiv:2301.01872].

[200] Y. H. Tham, *On the Category of Boundary Values in the Extended Crane-Yetter TQFT*, PhD thesis, Stony Brook (2021), [arXiv:2108.13467].

[201] J. Tolar, *On Clifford groups in quantum computing*, J. Phys.: Conf. Series **1071** (2018) 012022, [doi:10.1088/1742-6596/1071/1/012022], [arXiv:1810.10259].

[202] D. Tong, *The Quantum Hall Effect*, lecture notes (2016), [arXiv:1606.06687], [www.damtp.cam.ac.uk/user/tong/qhe/qhe.pdf].

[203] A. Ustinov, *A Short Proof of the Landsberg–Schaar Identity*, Math Notes **112** (2022), 488–490, [doi:10.1134/S0001434622090188].

[204] E. Verlinde, *Fusion rules and modular transformations in 2D conformal field theory*, Nucl. Phys. B **300** (1988), 360-376, [doi:10.1016/0550-3213(88)90603-7].

[205] C. Voisin (translated by L. Schneps), *Hodge theory and Complex algebraic geometry I*, Cambridge Stud. in Adv. Math. **76** (2002), [doi:10.1017/CBO9780511615344].

[206] K. von Klitzing, *The quantized Hall effect*, Rev. Mod. Phys. **58** 519 (1986), [doi:10.1103/RevModPhys.58.519].

[207] X. Waintal, *The Quantum House Of Cards*, PNAS **121** 1 (2024) e2313269120, [arXiv:2312.17570], [doi:10.1073/pnas.2313269120].

[208] Y. Wang, Z. Hu, B. C. Sanders, and S. Kais, *Qudits and high-dimensional quantum computing*, Front. Phys. **8** 479 (2020), [doi:10.3389/fphy.2020.589504], [arXiv:2008.00959]. .

[209] X.-G. Wen, *Topological Orders in Rigid States*, Int. J. Mod. Phys. B **4** 239 (1990), 239-271, [doi;10.1142/S0217979290000139].

[210] X.-G. Wen, *Topological orders and Edge excitations in FQH states*, Adv. Phys. **44** 5 (1995) 405, [doi:10.1080/00018739500101566], [arXiv:cond-mat/9506066].

[211] X.-G. Wen and Q. Niu, *Ground state degeneracy of the FQH states in presence of random potential and on high genus Riemann surfaces*, Phys. Rev. B **41** (1990) 9377, [doi:10.1103/PhysRevB.41.9377].

[212] X.-G. Wen and A. Zee, *Classification of Abelian quantum Hall states and matrix formulation of topological fluids*, Phys. Rev. B **46** (1992) 2290, [doi:10.1103/PhysRevB.46.2290].

[213] G. W. Whitehead, *Elements of Homotopy Theory*, Springer (1978), [doi:10.1007/978-1-4612-6318-0]

[214] E. Witten, *Quantum Field Theory and the Jones Polynomial*, Commun. Math. Phys. **121** 3 (1989), 351-399, [doi:10.1007/BF01217730].

[215] F. Wilczek, *Quantum Mechanics of Fractional-Spin Particles*, Phys. Rev. Lett. **49** (1982) 957, [doi:10.1103/PhysRevLett.49.957].

[216] E. Witten, *Three Lectures On Topological Phases Of Matter*, Rivista Nuovo Cim. **39** (2016), 313-370, [doi:10.1393/ncr/i2016-10125-3], [arXiv:1510.07698].

[217] T. Yagasaki, *Homotopy types of homeomorphism groups of noncompact 2-manifolds*, Topology Appl. **108** 2 (2000), 123-136, [doi:10.1016/S0166-8641(99)00130-3], [arXiv:math/0010223].

[218] T. Yagasaki, *Homotopy types of Diffeomorphism groups of noncompact 2-manifolds*, [arXiv:math/0109183].

[219] C. Ye, S.-G. Peng, C. Zheng, and G.-L. Long, *Quantum Fourier Transform and Phase Estimation in Qudit System*, Commun. Theor. Phys. **55** 790 (2011), [doi:10.1088/0253-6102/55/5/11].

[220] A. Zee, *Quantum Hall Fluids*, in: *Field Theory, Topology and Condensed Matter Physics*, Lecture Notes in Physics **456**, Springer (1995), [doi:10.1007/BFb0113369], [arXiv:cond-mat/9501022].

[221] B. Zeng, X. Chen, D.-L. Zhou, and X.-G. Wen, *Quantum Information Meets Quantum Matter – From Quantum Entanglement to Topological Phases of Many-Body Systems*, Quantum Science and Technology (QST), Springer (2019), [doi:10.1007/978-1-4939-9084-9], [arXiv:1508.02595].

[222] S. C. Zhang, *The Chern-Simons-Landau-Ginzburg theory of the fractional quantum Hall effect*, Int. J. Mod. Phys. B **06** 01 (1992), 25-58, [doi:10.1142/S0217979292000037].

[223] Y. Zhang, L. H. Kauffman, and M.-L. Ge, *Yang–Baxterizations, Universal Quantum Gates and Hamiltonians*, Quantum Inf. Process. **4** (2005), 159–197, [doi:10.1007/s11128-005-7655-7], [arXiv:quant-ph/0502015].