

Fractional Quantum Hall Anyons via the Algebraic Topology of Surplus Flux Quanta

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Abstract

Fractional quantum Hall systems (FQH), with their experimentally observed anyonic topological order, are a candidate future hardware platform for intrinsically stable quantum registers (“topological qbits”) and operations (“topological quantum gates”). But theoretical questions remain, notably concerning the desired realization of nonabelian anyon statistics.

Here we analyze a novel non-Lagrangian effective description of FQH anyons based on global quantization of surplus flux in extraordinary non-abelian cohomology theories, specifically in unstable Cohomotopy. Topological quantum order manifests as local systems of gapped Hilbert spaces over the covariantized quantized flux moduli spaces.

Assuming that FQH surplus flux is effectively quantized in 2-Cohomotopy theory (a cousin of *Hypothesis H* in high-energy physics), we rigorously compute covariant flux monodromy and classify its irreducible unitary representations. This reproduces fine details such as of modular topological order on the torus; but it also predicts potential new routes to nonabelian statistics in FQH systems which may inform experimental searches for topological quantum hardware.

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Contents

1	Introduction & Survey	3
2	Exotic Topological Flux Quanta	8
2.1	Topological Quantum States	8
2.2	General Covariance of Flux	13
3	Flux quantized in 2-Cohomotopy	18
3.1	On the plane	19
3.2	On the sphere	24
3.3	On closed surfaces	25
3.3.1	Modular action	25
3.3.2	Functoriality	27
3.4	On the torus	30
3.4.1	Flux monodromy classification	30
3.4.2	Comparison to K-matrix formalism	42
3.5	On punctured surfaces	47
3.6	On punctured disks	52
3.7	On the open annulus	54
3.8	On the 2-punctured disk	55
4	Conclusion & Outlook	58
A	Background	59

1 Introduction & Survey

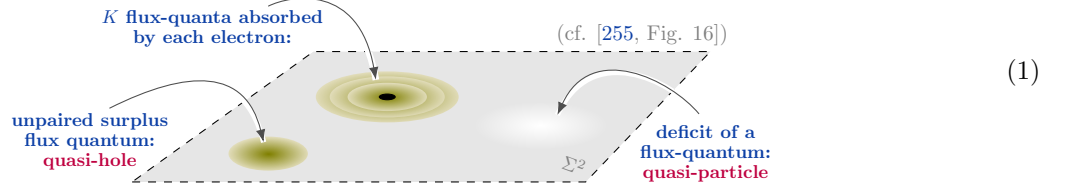
Need for topological quantum protection. The potential promise of *quantum computers* [196][115] is enormous [101][27][209], but their practicability hinges on finding and implementing methods to stabilize quantum registers and gates against decohering noise. Serious arguments [145][64][166][69][70][127][97][266] and practical experience [53] suggest that the currently dominant approach of *quantum error correction* at the software level (QEC [169][208]) will need to be supplemented [54]¹ by more fundamental physical mechanisms of quantum error *protection* already at the hardware level, in the form of “topological” stabilization of quantum states (“topological qubits”) and operations (“topological quantum gates”) [152][86][194][242][190][241]. While the general idea of topological quantum protection has become commonplace, its fine details have received less attention and are nowhere near as well understood as those of QEC — this in contrast to its plausible necessity for scalable quantum computing.

FQH systems: Surplus flux quanta. A practical contender for the required topological quantum hardware is (cf. [10][55][37][16][183]) *fractional quantum Hall systems* (FQH, cf. [206][41][255][138][140][201]). These are predominantly electron gases constrained to an effectively 2-dimensional surface Σ^2 (2DEG, e.g. realized on interfaces between semiconducting materials [201]) at extremely low temperature and penetrated by transverse magnetic flux (cf. [112, §2.2.1][228, §2.1]) so strong that the number of *flux quanta* (cf. [154, (27)]) through the surface Σ^2 is an integer multiple K of the number of electrons confined to Σ^2 , called the inverse of the *filling fraction* $\nu = 1/K$ (more generally a rational number).

At unit-fraction filling $\nu = 1/K$, each electron in Σ^2 appears — in a heuristic picture, [136][137], cf. [255, p. 882] — to form a “bound state” of sorts with K flux quanta. More generally, adiabatic insertion of further *surplus* flux quanta transfers charge of magnitude νe , where e is the magnitude of the electron charge:

FQH system at filling fraction ν : $\pm$ surplus flux quanta $\rightsquigarrow \mp \nu$ units of electron charge.

At Laughlin filling $\nu = 1/K$, this surplus flux insertion produces a fundamental quasi-particle or quasi-hole. At other fillings it may produce a composite: at $\nu = 2/5$, a full flux insertion carries charge of magnitude $2e/5$, whereas a fundamental quasi-particle carries $e/5$ (Rem. 3.50). These surplus flux quasi-particle sectors of FQH systems are candidates for the desired topologically protected quantum states (exhibiting “topological order” [268], cf. [284, §III][223]).



For identical Abelian quasi-particles of a specified species, a “braiding interchange” of their worldlines contributes a statistical phase $\zeta = e^{\pi i \theta}$, *independent* of the local details of the braiding process ([118][7], as reviewed in [138, §9.8]):



The phase depends on the FQH state, the quasi-particle species, and the braid orientation. For fundamental Laughlin quasi-particles at $\nu = 1/K$, it satisfies $\zeta = \pm e^{\pi i \nu}$ in the chosen orientation ([138, §9.8.2]). Interferometry probes the full-braid phase $\zeta^2 = e^{2\pi i \theta}$, without fixing the sign of ζ : the reported phases at $\nu = 1/3$ and $\nu = 2/5$ are consistent with $\theta = 1/3$ and $\theta = -2/5 \equiv 3/5$ modulo 1, respectively ([192, 193]; Ex. 3.49).

These fractional quasi-particles are called “anyons” [7][254][251], in (somewhat inaccurate) reference [274] to *any* possible exchange phase between those of bosons ($\zeta = 1$, $s = 0$) and those of fermions ($\zeta = -1$, $s = \frac{1}{2}$), cf. [119, §11.4]. Experimental evidence for fractional statistics, from interferometry and current-correlation measurements, has been reported by various groups and in different materials: [18][191][192][193][218][107][163][263][219].

¹[54]: “The qubit systems we have today are a tremendous scientific achievement, but they take us no closer to having a quantum computer that can solve a problem that anybody cares about. [...] What is missing is the breakthrough [...] bypassing quantum error correction by using far-more-stable qubits, in an approach called topological quantum computing.”

The braiding phases (2) motivate examining FQH systems directly through their *flux quantization* [228]:

Global flux quantization in FQH systems. Experiment shows abundantly that the fractional quantum Hall effect is a *universal* phenomenon [143][105, p. 1][106] in that its characteristic properties are independent of the microscopic nature of the host material and of impurities and irregularities of the sample. This suggests [87] the existence of accurate *effective* quantum field theoretic descriptions (cf. [81]) whose degrees of freedom reflect not any microscopic host particles but instead the nature of the universally emergent FQH quasi-particles (much like and closely related to how conformal field theory universally serves as an effective description of critical phenomena in statistical mechanics, cf. [57, §3.2]).

Traditionally, this putative effective FQH theory is sought in the ancient and extensively studied realm of *Lagrangian* quantum field theories (cf. [123][102]), where one argues ([283][268], cf. [269, §7.3][81, §13.7][278, §2][261, §5][234, p. 5]) that the relevant candidates are variants of abelian Chern-Simons theory [36][204][173], see §A.1. A local gauge-field Lagrangian density $L = L(A)$ must be accompanied by global field and transition data to specify a quantum gauge theory. These data encode flux quantization and the topological sectors relevant to FQH systems (cf. [228, §3.3] and Fig. G). Chern-Simons theory has such global formulations; our construction instead begins with a proposed flux classifying space.

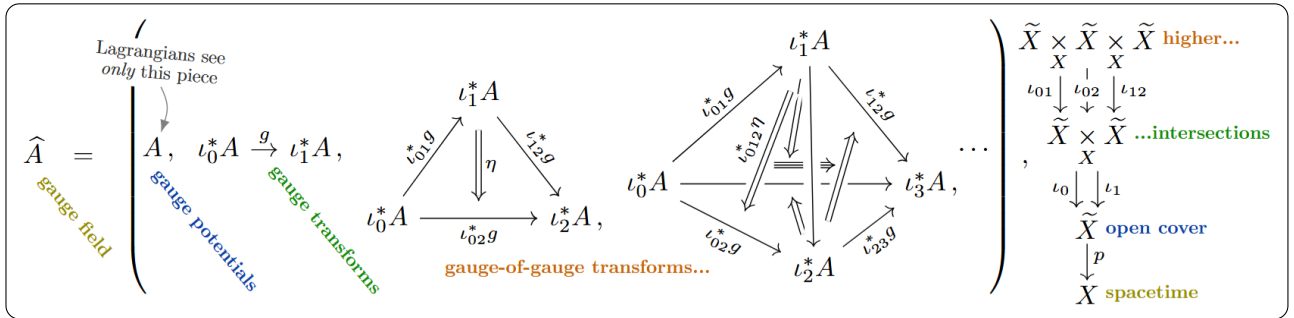


Figure G. The full non-perturbative data of a (higher) gauge field configuration $\hat{A}$ on a spacetime X consists not just of the gauge potentials A , which are only defined locally – namely on an open cover $\tilde{X}$ of X by charts –, but also of “transition data” g which gauge-transforms between coincident gauge potentials on different charts, and further of incrementally higher transition data which higher-gauge transforms between coincident transition data.

This (higher) transition data encodes the *flux quantization law* and the topological *charge* or *soliton sector* of the gauge field. It supplements the local expression $L = L(A)$ in a global formulation of the theory.

For further exposition and pointers see [4] for the case of the ordinary electromagnetic field and [228, §3.3][239] for full generality.

For example, locally eliminating the dynamical field from an FQH Chern-Simons action yields a fractional electromagnetic response that is not, by itself, a globally defined Chern-Simons theory of the probe field. Retaining the dynamical gauge field and its topological sectors resolves this distinction ([278, §2.3]; see §A.1.1). Our alternative construction seeks to derive state spaces and their monodromy directly from the flux-quantization law. Its comparison with Chern-Simons ground-state representations is discussed in §3.4.2.

Non-Lagrangian effective FQH theory based on flux quantization. We have developed a non-Lagrangian theory of topological quantum states in (higher) gauge theories which is compatible with and in fact entirely based on consistent flux-quantization (survey in [228][234]) – the main insight here is (recalled below in §2):

- (a) flux-quantization laws are encoded by extra-ordinary² non-abelian cohomology theories [77, §2][231, §1] with *classifying spaces* $\mathcal{A}$ whose “rationalization” reflects the duality-symmetric form of the gauge-field’s Bianchi identities, cf. [228, §3] and (6) below;
- (b) the *topological quantum observables on flux* depend only on the homotopy type of this classifying space $\mathcal{A}$, and not on any other (local, microscopic) properties of the theory [226].

A quick way to understand the underlying principle is to recall the classical fact (cf. [78, p. 263][77, Ex. 2.1]) of algebraic topology (see §A.3) that there exist classifying spaces — here denoted $B^n \mathbb{Z}$ ³, characterized by $\pi_k B^n \mathbb{Z} \simeq \delta_k^n \mathbb{Z}$ — for (integral, reduced) ordinary cohomology:

²The term *extra-ordinary cohomology theory* is standard (cf. [177][78, §6]) for (Whitehead-)generalized abelian cohomology theories (cf. [77, Ex. 2.10]) represented by spectra of spaces, in contrast to the ordinary cohomology theories represented by (spectra of) Eilenberg-MacLane spaces (3). Here we use the term in the yet greater generality of non-ordinary *and* non-abelian cohomology [77, §2][231, §1], as a more evocative version of the overused and now ambiguous term “generalized cohomology”.

³These ordinary classifying spaces are known as *Eilenberg-MacLane spaces* and traditionally denoted “ $K(\mathbb{Z}, n)$ ”.

Ordinary cohomology

$$\tilde{H}^n(X; \mathbb{Z}) \xleftarrow{\sim} \pi_0 \text{Map}^*(X, B^n \mathbb{Z}), \quad (3)$$

homotopy classes
of maps to
classifying space

and that the usual (Dirac) flux quantization of the electromagnetic field (cf. [4][228, Ex. 3.9]) says that its underlying topological charge is a class in $\tilde{H}^2(X; \mathbb{Z})$, hence represented by a map from spacetime to $B^2 \mathbb{Z} \simeq_f BU(1) \simeq_f \mathbb{C}P^\infty$ ⁴, and that the latter is all that is needed to deduce (cf. [228, Ex. 2.2]) that solitonic magnetic flux through a plane comes in integer units – the *flux quanta* (1):

Homtpy classes of maps classifying
solitonic magnetic flux

number of
flux quanta

$$\{\mathbb{R}_{U\{\infty\}}^2 \xrightarrow{c} B^2 \mathbb{Z}\}_{/\text{hmtpt}} \simeq \pi_0 \text{Map}^*(R_{U\{\infty\}}^2, B^2 \mathbb{Z}) \simeq \pi_0 \text{Map}^*(S^2, B^2 \mathbb{Z}) \simeq \pi_2(B^2 \mathbb{Z}) \simeq \mathbb{Z}. \quad (4)$$

In fact, the classifying space $B^2 \mathbb{Z}$ moreover encodes the ordinary topological flux quantum observables through any surface Σ^2 , as seen in Ex. 2.8 below.

The key role of algebraic topology. With this understanding, the question of an effective QFT description of FQH systems is not answered as traditionally (by choosing a Lagrangian whose equations of motion reflect local properties like the Hall current) but instead by finding an effective classifying space $\mathcal{A}$ whose implied topological quantum observables reproduce the expected observations on global topological states, such as the emblematic (non-)commutation relation of Wilson line operators on the torus, shown in (17) below.

It turns out (in §2) that this construction of topological flux quantum states proceeds entirely by the analysis of “*local systems*” (cf. Rem. 2.12 below) on the (generally covariantized) homotopy type of moduli spaces (21) of flux given by mapping spaces from the spacetime domain into the classifying space for the flux-quantization law — and as such is squarely a problem in the mathematical subject of homotopy theory and algebraic topology (for which we have compiled some background in §A.3).

Novel effective surplus flux quantization for FQH systems. Concretely, a candidate classifying space for the *surplus* magnetic flux through FQH systems (as seen by the effective quasi-particles/holes) turns out [229][227][234]⁵ to be the 2-sphere $\mathcal{A} \simeq S^2$ (see §3), modeling effective FQH flux in a variation of the ordinary classifying space (4) (of which it is the “2-skeleton”):

Classifying space for
surplus FQH flux

$$S^2 \simeq \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty \simeq_f BU(1) \simeq_f B^2 \mathbb{Z}$$

Classifying space for
ordinary magnetic flux

(5)

In this article, we work out the monodromy groups and candidate state spaces determined by this classifying space together with the quantization prescription adopted in Def. 2.11, and compare the resulting data with abelian Chern-Simons theory. Their physical realization requires the additional input discussed in §3.4.2. As motivation for this comparison, note (cf. [76]) that the rationalization of the 2-sphere is encoded by the following differential equations (its “Sullivan minimal model”, cf. [228, §3.2]), which are just those equations that characterize the Chern-Simons 3-form H_3 for a gauge field flux density F_2 as it appears in the Lagrangian formulation of Chern-Simons theory (cf. [82, Prop. 1.27(b)] and (159)):

$$\begin{array}{c} \text{Rational model of} \\ \text{classifying space for} \\ \text{surplus FQH flux} \end{array} \quad \text{CE}(lS^2) \simeq \mathbb{R}_d \left[\begin{array}{c} \text{flux} \\ \text{density} \\ F_2 \\ H_3 \end{array} \right] / \left(\begin{array}{l} dF_2 = 0 \\ dH_3 = F_2 F_2 \end{array} \right) \quad \begin{array}{c} \text{Bianchi identities characterizing} \\ \text{Chern-Simons 3-form /} \\ \text{Green-Schwarz mechanism} \end{array} \quad (6)$$

effective
higher flux

Incidentally, it is in this sense that our effective description of the FQH effect is a mild form of *higher gauge theory* (cf. [239]), since the Chern-Simons 3-form (traditionally understood as a Lagrangian density) here appears as a higher flux density — the 3-form H_3 — satisfying a Bianchi identity of the form known from *Green-Schwarz mechanisms*.⁶ Moreover, H_3 is the rational image of the *Hopf fibration*, the generator of

$$\mathbb{Z} \simeq \pi_3(S^2) \simeq \pi_0 \text{Map}^*(S^3, S^2) \simeq \pi_1 \text{Map}^*(S^2, S^2). \quad (7)$$

⁴Our notation “ $\simeq_f$ ” stands for *weak homotopy equivalences* (217).

⁵As explained in [229][104], following [231], the 2-sphere here is a cousin of the 4-sphere which similarly serves as flux quantization of the higher gauge field in 11D supergravity [220, §2.5][75][103] (review in [228]), where its choice as such is referred to as *Hypothesis H* [75]. While this is where our approach to FQH systems here comes from and is informed by [227], for the present purpose the reader may ignore this geometric engineering of FQH systems on M5-probes of 11D SuGra. A review and discussion of the relevance to deeper questions of FQH systems (such as their hidden supersymmetry [113][211]) may be found in [234, §2-3].

⁶In the “geometric engineering” of our FQH model on M5-branes referred to in footnote 5, this 3-form arises as the restriction to an orbifold singularity of the “self-dual” tensor field carried by these branes, which itself is quantized in a higher (and “twistorial”) form of Cohomotopy, cf. [104][229].

This is the non-torsion class that disappears under passage to the ordinary classifying space (5), and it is this class which we find in §3, Prop. 3.9, to be identified with the observable $\hat{\zeta}$ of fractional braiding phases (2)!

Results. Within this proposed effective description, we derive the monodromy groups and representation-theoretic data over various surface geometries. We compare them with known FQH descriptions; assessing possible experimental distinctions requires the physical realization and stability conditions discussed in §3.4.2:

§3	Surface	Results for 2-Cohomotopical flux quanta	
	Σ^2	broadly	fine-print
§3.1	The plane $\Sigma_{0,0,1}^2$	fractional statistics	framing regularization of link observables appears automatically
§3.4	The torus $\Sigma_{1,0,0}^2$	torus monodromy	classification with irreducible Heisenberg restriction
§3.7	The annulus $\Sigma_{0,0,2}^2$	dihedral representations	irreducible unitary representations of dimensions 1 and 2
§3.8	2-Punctured open disk $\Sigma_{0,0,3}^2$	puncture monodromy	irreducible unitary representations of dimensions 1, 2, 3, and 6

Comparison to U(1)-Chern-Simons theory. To a large extent, our construction turns out to give a curious and curiously direct (re-)derivation of the fine properties of U(1)-Chern-Simons quantum field theory (cf. [36][204][173][96]) by novel non-Lagrangian means, and as such the result seems of interest in its own right, beyond the topic of FQH systems (see Remarks 3.13 and 3.39 below).

For the one-component Laughlin state at filling $\nu = 1/q$, the conventional Chern-Simons description has scalar parameter $K = q = 2k$ in our normalization. Ordinary U(1)-Chern-Simons theory requires even K , while *spin Chern-Simons theory* also permits odd K [58, §5][187, p. 381], as summarized below. The torus representations of §3.4 supply monodromy data for comparison with these theories; identifying a physical Chern-Simons theory further requires matching the spin, charge, and transport data. For multicomponent abelian states the comparison uses a K -matrix, as recalled in § A.1.3.

	Symbol	at $\nu = \frac{1}{q}$	in ordinary CS	in “spin” CS	$\exp(\frac{i}{\hbar} S_{\text{CS}}) =$
CS level	k		$\in \mathbb{N}_{>0}$	$\in \frac{1}{2}\mathbb{N}_{>0}$	$e^{2\pi i k \int A \, dA}$
	$K \equiv 2k$	$= q$	$\in 2\mathbb{N}_{>0}$	$\in \mathbb{N}_{>0}$	$e^{\pi i K \int A \, dA}$

(8)

But our theory also differs from usual U(1)-CS-theory in subtle respects:

- On the torus, we find (Prop. 3.41, 3.43) fractional braiding phases $e^{\pi i \frac{p}{K}}$ for general coprime p and K . In one-component $U(1)_K$ CS-theory, charge q gives the phase $e^{\pi i q^2 / K}$, so nonunit numerators already occur there. Our family allows general coprime numerators, beyond the charge-square numerators available at a fixed one-component level. Comparison with multicomponent K -matrix theory ([272][268], cf. § A.1.3) is discussed in §3.4.2.
- These K -dimensional state spaces admit inequivalent linear extensions distinguished by scalar phases under modular transformations (Rem. 3.45). Character twists of a fixed extension have the same projective action; their physical distinction depends on the scalar-phase data retained in a ground-state realization.
- On n -punctured disks, the covariant flux monodromy group is the quotient of the framed braid group by the common framing shift (Prop. 3.67). The corresponding puncture lattice is $Q_{n-1} = \mathbb{Z}^n / \mathbb{Z}(1, \dots, 1)$, with the dual integral standard action (134). The open annulus and the 3-punctured sphere give explicit representation-theoretic examples (§3.7 and §3.8).
- The construction distinguishes solitonic flux observables from the transformations that move punctures. The action of these transformations on flux monodromy is computed in §3.5; its realization by adiabatic motion of physical defects requires additional input.

Dimensionality. Our field theory is defined in $1+3$ dimensions, with spatial slices $\Sigma^2 \times \mathbb{R}$ (Def. 2.1). The solitonic condition (20) requires the flux to vanish at infinity, in particular towards both ends of the transverse $\mathbb{R}$ -factor.⁷ The resulting topological observables are expressed in terms of Σ^2 by (12). A physical realization as an effectively $(1+2)$ -dimensional system would additionally require dynamics confining the relevant modes to a layer of finite transverse extent. Vanishing at infinity alone fixes neither the location nor the width of such a layer.

Conclusion. In summary, we hypothesize that FQH flux is effectively quantized in 2-Cohomotopy (§3), in particular identifying configurations of anyonic FQH quasi-holes/particles (1) with the configurations of signed points (Ex. 3.5) that the Pontrjagin/Segal theorems associate with charges in 2-Cohomotopy (§3.1)⁸. Our main results are:

- (i) **Consistency checks:** The construction recovers the solitonic braiding phase (2) and framed Wilson loop observables (§3.1), relates states on closed surfaces to Heisenberg representations (§3.3–3.4), and computes mapping class group actions on punctured-surface flux monodromy (§3.5).
- (ii) **Further results:** 2-Cohomotopical flux quantization classifies unitary torus representations whose Heisenberg restriction is irreducible, whose comparison with K-matrix Chern-Simons theory away from unit filling fractions is discussed in §3.4.2. On punctured surfaces, the covariant flux monodromy groups admit non-abelian representations, with explicit classifications for the open annulus and the 3-punctured sphere (§3.7 and §3.8). Identifying these representations with protected states of physical defects requires additional input about their realization.

The punctured-surface calculations therefore supply candidate symmetry actions for defect systems. Establishing an implementation by protected quantum states, and identifying the operations produced by adiabatically moving the defects, requires additional physical input. Related proposals are discussed in [190, §3] and [238].

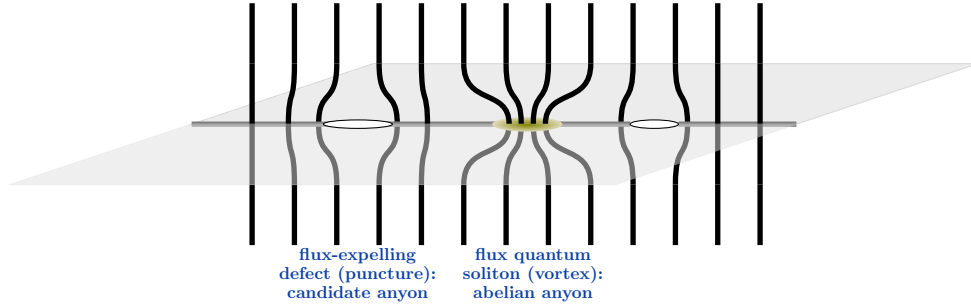


Figure D — Solitonic and defect anyons. Solitonic magnetic flux quanta give the abelian anyons discussed in Rem. 3.13. Punctures model excluded loci of vanishing solitonic flux (cf. Fig. 1). Their covariant flux monodromy has non-abelian representations (§3.7–3.8); realization by physical defect anyons requires additional input.

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⁷In the geometric engineering on M5-branes, this transverse direction is the decompactified “M-theory circle”; cf. footnote 5.

⁸As such, there are tantalizing relations of our results to the proposal [186], see particularly the end of §2 there and footnote 14 below.

2 Exotic Topological Flux Quanta

Here we develop our main Definition 2.11 of quantum states of (ordinary and) extra-ordinary/exotic topological flux quanta.

We begin by adopting a discrete group-algebra construction for ordinary Yang-Mills fluxes (Def. 2.5), motivated by the flux-observable analysis of [226]. Proposition 2.6 identifies this algebra in terms of the electric/magnetic classifying space $\mathcal{A} \equiv B(G \ltimes (\mathfrak{g}/\Lambda))$.

In this algebro-topological formulation (cf. §A.3), the result has an evident generalization to higher gauge theories with extra-ordinary flux quantization laws [228] classified by any other pointed space $\mathcal{A}$, such as the 2-sphere (5) with its higher Chern-Simons flux form (6) in our motivating example of FQH systems, to be treated this way in §3.

Following [226, §4], we adopt this generalization as a quantization prescription for topological flux. This is Def. 2.11 below, where we successively refine the prescription by allowing punctured surfaces and accounting for diffeomorphism-covariance (“general covariance”).

We analyze the effect of general covariance in §2.2, obtaining the monodromy groups whose unitary representations are studied in §3.

This non-Lagrangian prescription is motivated by the topological flux algebra of ordinary Yang-Mills theory discussed in [226], its previous applications in formal high-energy physics (cf. [222][50][226, §4]), and the representation-theoretic structures obtained below for FQH systems. The resulting discrete algebras and their representations are the mathematical objects considered here; their physical interpretation is discussed in §3.4.2.

2.1 Topological Quantum States

First, we set up some notation:

Definition 2.1 (Spacetime). Throughout, we consider

- $X^{1,3} := \mathbb{R}^{1,1} \times \Sigma^2$ a globally hyperbolic 4D spacetime,
- with spatial slices $\mathbb{R}^1 \times \Sigma^2$, to be thought of as a tubular neighborhood of:
- Σ^2 , a surface (here: a connected, oriented smooth 2D manifold with boundary) which at times is
- specialized to $\Sigma^2 \equiv \Sigma_{g,b,n}^2$, the unique (up to diffeomorphism) surface (cf. §A.4):
 - of genus g ,
 - with b boundary components,
 - and n punctures:

$$\Sigma_{g,b,n}^2 \simeq \underbrace{\left(\underbrace{\Sigma_{0,0,0}^2}_{\text{sphere}} \# \underbrace{T^2 \# \dots \# T^2}_{\substack{\text{connected summands} \\ \text{of tori}}} \right)}_{\substack{\text{connected} \\ \text{sum}}} \setminus \underbrace{\left\{ \underbrace{D^2 \sqcup \dots \sqcup D^2}_{\substack{\text{disjoint summands} \\ \text{of open disks}}} \sqcup \underbrace{\overline{D}^2 \sqcup \dots \sqcup \overline{D}^2}_{\substack{\text{disjoint union} \\ \text{of closed disks}}} \right\}}_{\text{complement}} \quad (9)$$

genus g , boundary components b , punctures n

understood as modeling an effectively 2-dimensional sample of material. ⁹

We abbreviate $\Sigma_{g,b}^2 := \Sigma_{g,b,0}^2$ and $\Sigma_g^2 := \Sigma_{g,0}^2 \equiv \Sigma_{g,0,0}^2$.

⁹ Albeit routinely considered in theory (cf. [271]), the practicability of direct laboratory realizations of $\Sigma_{g,b,n}^2$ (9) with transversal magnetic flux is limited when $g > 0$. The case $g = 1$ (the torus) is readily realized (only) when considering momentum space (the Brillouin torus of a 2D crystal, cf. [223]) instead of position space, but, while noteworthy in itself, this is not the case of FQH systems of concern here. Alternatively, it was argued [14] that suitable defects, called “genons”, in a crystal lattice could make a sample of nominal genus $g = 0$ effectively behave like $g > 0$.

But irrespective of practicality, the theoretical possibility of $g > 0$ allows us to compare our topological quantum flux observables to those of abelian Chern-Simons theory in the case $\Sigma_{g>0,0,0}^2$, and their agreement in this theoretical case supports the validity of our observables also in the more practical cases of $g = 0$, $n, b \neq 0$.

Example 2.2 (Some surfaces). We have (boundary-fixing) diffeomorphisms as follows:

$$\begin{array}{llll}
\text{sphere} & \Sigma_{0,0,0}^2 \simeq S^2 & & \\
\text{torus} & \Sigma_{1,0,0}^2 \simeq T^2 & & \\
\text{closed disk} & \Sigma_{0,1,0}^2 \simeq D^2 & \text{[Diagram: Solid gray ellipse]} & \\
\text{open disk} & \Sigma_{0,0,1}^2 \simeq \mathbb{R}^2 & \text{[Diagram: Hollow gray ellipse]} & \\
\text{open annulus} & \Sigma_{0,0,2}^2 \simeq \mathbb{R}^2 \setminus \{0\} & \text{[Diagram: Hollow gray ellipse with a small white circle inside]} & \\
\text{/ punctured plane} & & & \\
\text{half-open annulus} & \Sigma_{0,1,1}^2 & \text{[Diagram: Hollow gray ellipse with a small white circle inside]} & \\
\text{closed annulus} & \Sigma_{0,2,0}^2 \simeq A^2 & \text{[Diagram: Hollow gray ellipse with a small white circle inside]} &
\end{array} \tag{10}$$

Here the disks and annuli are the surface types readily and commonly realized in laboratory FQH experiments (cf. footnote 9).

Remark 2.3 (Spin structure). In fact, we regard spacetime $X^{1,3}$ (Def. 2.1) — and therefore also the surfaces Σ^2 (9) — as equipped with a spin structure (cf. [182]). Covariance for a fixed spin structure uses the subgroup of mapping classes preserving it. We specify the choices on the torus in §3.4 and on punctured genus-zero surfaces in §3.5. For example, of the four spin structures on the 3-punctured sphere, only the one extending over the sphere is invariant under the full Sym_3 ; each of the other three has stabilizer Sym_2 .

Definition 2.4 (Gauge group). For the following Def. 2.5 we consider G a Lie group, with Lie algebra $\mathfrak{g}$ and with a choice of Ad-invariant lattice $\Lambda \subset \mathfrak{g}$ (not necessarily full, possibly zero) — but shortly we specialize this to $G \equiv \mathbb{R}$ and $\Lambda \equiv \mathbb{Z} \hookrightarrow \mathbb{R}$ (14).

The flux-observable analysis of [226, §1 and §A.1] motivates considering the Fréchet Lie group of smooth maps from Σ^2 to $G \ltimes_{\text{Ad}} (\mathfrak{g}/\Lambda)$. Its relation to group convolution and strict deformation quantization provides background motivation. Here we adopt directly the following discrete construction, which is sufficient for the topological sector.

Definition 2.5 (Topological Yang-Mills flux algebra). For a closed surface Σ_g^2 , we define the algebra of topological G -flux observables to be the complex group algebra (240) of the discrete group of connected components of the smooth mapping group:

$$\begin{aligned}
\text{Algebra of topological quantum observables on YM-flux through } \Sigma^2 & \quad \text{TopFlxObs}_{\Sigma^2} \quad \overset{\text{group convolution algebra}}{:=} \quad \mathbb{C} \left[\pi_0 C^\infty(\Sigma^2, G \ltimes_{\text{Ad}} (\mathfrak{g}/\Lambda)) \right] \\
& \simeq \quad \mathbb{C} \left[\underbrace{\pi_0 C^\infty(\Sigma^2, G)}_{\text{electric}} \ltimes_{\text{Ad}} \underbrace{\pi_0 C^\infty(\Sigma^2, \frac{\mathfrak{g}}{\Lambda})}_{\text{magnetic}} \right].
\end{aligned} \tag{11}$$

The notation $\mathbb{C}[-]$ denotes the algebraic group $*$ -algebra, with involution $g^* = g^{-1}$ on group elements and conjugation on coefficients. If a C^* -algebra is desired, we take the full group C^* -completion of this discrete group algebra; its unitary representations extend uniquely to that completion. No convolution algebra or strict deformation quantization of the infinite-dimensional mapping group is needed for this definition. The component group in Def. 2.5 has the following equivalent homotopical descriptions.

Proposition 2.6 (Topological YM flux quantum observables [226, §3]). *The discrete group algebra of Def. 2.5 is equivalently the group algebra (240) of the fundamental group $\pi_1(-)$ (215) of the component of the constant map into the classifying space $B(-)$ (cf. [3, §1]):*

$$\begin{aligned}
\text{Algebra of topological flux observables} \quad \text{TopFlxObs}_{\Sigma^2} & \quad \overset{\text{group convolution algebra}}{:=} \quad \mathbb{C} \left[\pi_0 C^\infty(\Sigma^2, G \ltimes_{\text{Ad}} \frac{\mathfrak{g}}{\Lambda}) \right] \\
& \simeq \mathbb{C} \left[\pi_0 \text{Map}(\Sigma^2, G \ltimes_{\text{Ad}} \frac{\mathfrak{g}}{\Lambda}) \right] \simeq \mathbb{C} \left[\pi_1 \text{Map}_0(\Sigma^2, B(G \ltimes \frac{\mathfrak{g}}{\Lambda})) \right] \\
& \simeq \mathbb{C} \left[\pi_0 \text{Map}^*((\Sigma^2 \times \mathbb{R})_{\cup\{\infty\}}, B(G \ltimes \frac{\mathfrak{g}}{\Lambda})) \right].
\end{aligned} \tag{12}$$

Remark 2.7 (Asymptotic boundary localization of topological flux observables).

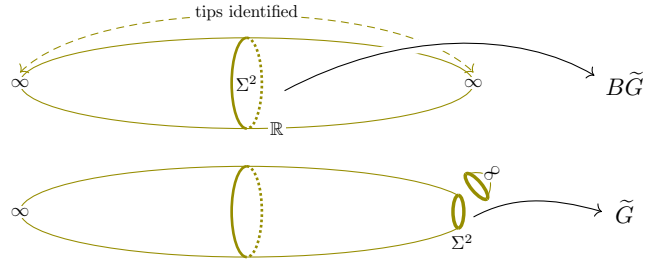
- (i) In the second line of (12), we are showing several isomorphic incarnations of this group algebra of observables (these isomorphisms are explained in [226, §A.2], using basic facts also recalled in §A.3), which each have their use in the following.
- (ii) In particular, for a general topological group $\tilde{G}$, the suspension–loop adjunction gives (recalling that Σ^2 here is closed, hence compact)

$$\pi_0 \text{Map}^*((\Sigma^2 \times \mathbb{R})_{\sqcup \{\infty\}}, B\tilde{G}) \simeq \pi_0 \text{Map}^*(S^1 \wedge (\Sigma^2)_{\sqcup \{\infty\}}, B\tilde{G}) \simeq \pi_0 \text{Map}(\Sigma^2, \tilde{G}) \quad (13)$$

Here $(\Sigma^2 \times \mathbb{R})_{\sqcup \{\infty\}}$ is the *reduced suspension* $S^1 \wedge (\Sigma^2)_{\sqcup \{\infty\}}$: equivalently, the ordinary suspension of Σ^2 with its two tips identified to the single point ∞ . Thus (13) classifies gauge bundles with a chosen trivialization at ∞ by homotopy classes of $\tilde{G}$ -valued maps on Σ^2 .

- (iii) This is a relative form of the *clutching construction* (cf. [131, §7]). The compactified cylinder is the union of two cones over Σ^2 , meeting both along their equator Σ^2 and at their common tip ∞ . Trivialize the bundle on each cone, with both trivializations agreeing with the chosen one at ∞ . The transition function is then the identity at ∞ and a map $\Sigma^2 \rightarrow \tilde{G}$ on the equator. Its homotopy class is the last term of (13).

Figure C. The two ends of $\Sigma^2 \times \mathbb{R}$ are identified to the single point ∞ . With the bundle trivialized there, its clutching data are maps $\Sigma^2 \rightarrow \tilde{G}$ on the equator, up to homotopy (13). The dashed arrow marks the identification of the two displayed tips.



Example 2.8 (The prediction of ordinary electromagnetism...). For ordinary electromagnetic flux, subject to the usual Dirac charge quantization law (where the magnetic but not electric flux is quantized in integral cohomology, cf. [226, (14)]) the relevant choice in Def. 2.4 is:

$$\begin{array}{cc} \text{no electric} & \text{usual magnetic} \\ \text{flux quantization} & \text{flux quantization} \\ G := \mathbb{R}, & \Lambda := \mathbb{Z} \hookrightarrow \mathbb{R} \\ \text{classifying} & \\ \text{space} & \mathcal{A} := B(\mathbb{R} \ltimes \frac{\mathbb{R}}{\mathbb{Z}}) \simeq_f BU(1). \end{array} \quad (14)$$

In this case, the algebra (12) of observables on topological flux through a closed surface Σ_g (9) is

$$\begin{array}{l} \text{Algebra of topological} \\ \text{flux observables for} \\ \text{ordinary electromagnetism} \end{array} \quad \text{TopFluxObs}_{\Sigma_g}^{BU(1)} \simeq \begin{array}{l} \mathbb{C}[\pi_0 \text{Map}(\Sigma^2, U(1))] \\ \simeq \mathbb{C}[\pi_0 \text{Map}(\Sigma^2, B\mathbb{Z})] \\ \simeq \mathbb{C}[H^1(\Sigma_g^2; \mathbb{Z})] \\ \simeq \mathbb{C}[\mathbb{Z}^{2g}] \end{array} \quad \begin{array}{l} \text{e.g. [78, p. 263]} \\ \text{e.g. [100, Thm. 6.13],} \end{array} \quad (15)$$

and so a corresponding space of quantum states $\mathcal{H}_{\Sigma_g^2}$ carries an action of linear operators $\widehat{W}_{\left[\begin{smallmatrix} \vec{a} \\ \vec{b} \end{smallmatrix}\right]}$, for $\vec{a}, \vec{b} \in \mathbb{Z}^g$, subject to

$$\widehat{W}_{\left[\begin{smallmatrix} \vec{a} \\ \vec{b} \end{smallmatrix}\right]} \circ \widehat{W}_{\left[\begin{smallmatrix} \vec{a}' \\ \vec{b}' \end{smallmatrix}\right]} = \widehat{W}_{\left[\begin{smallmatrix} \vec{a} + \vec{a}' \\ \vec{b} + \vec{b}' \end{smallmatrix}\right]} = \widehat{W}_{\left[\begin{smallmatrix} \vec{a}' \\ \vec{b}' \end{smallmatrix}\right]} \circ \widehat{W}_{\left[\begin{smallmatrix} \vec{a} \\ \vec{b} \end{smallmatrix}\right]}. \quad (16)$$

Remark 2.9 (...and its failure to describe FQH systems).

- (i) While this algebra of observables (16) is the prediction of the ordinary traditional theory of electromagnetism, it is *not quite* the algebra of observables of magnetic flux actually seen in fractional quantum Hall systems!
- (ii) Instead, over the torus ($g = 1$) the quantum observables of FQH systems are famously thought to satisfy a *non-commutative deformation* of (16) where the cross-terms pick up the square ζ^2 of the braiding phase factor (2) when commuted past each other [271, (4.9)][133, (4.14)][81, (4.21)][261, (5.28)]:

$$\widehat{W}_{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} \circ \widehat{W}_{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} = \zeta \widehat{W}_{\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]} = \zeta^2 \widehat{W}_{\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]} \circ \widehat{W}_{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}. \quad (17)$$

- (iii) This motivates adopting the universal group algebra of the *integer Heisenberg group* $\widehat{\mathbb{Z}^2}$ (64), a non-abelian central extension of $\mathbb{Z}^2$ by $\mathbb{Z}$. At fixed central phase ζ , representations factor through the quotient by the relation $z = \zeta 1$, where z denotes the central generator; further quotient relations may hold in a particular

representation. Thus (17) motivates this universal algebra, while its identification with the algebra adopted here follows from the cohomotopy calculation in Prop. 3.19.

This motivates looking for a coherent generalization of topological flux observables via non-standard flux quantization laws such that they do capture effects like (17) (we find this in §3.4).

Proposition 2.6 shows that the adopted topological flux algebra depends on the classifying space encoding the flux-quantization law (cf. Ex. 2.8).

This motivates constructing topological flux observables directly from the flux-quantization law specified by a classifying space $\mathcal{A}$.

The topological flux quantization prescription. We are thus led to the following Def. 2.11, which is the quantization prescription adopted in our analysis. Its ordinary gauge-theory case is Def. 2.5 and Prop. 2.6; the same group-algebra construction makes sense for general flux classifying spaces. Note that the formula (12) immediately generalizes from the case $\mathcal{A} \equiv B(G \times \frac{\mathfrak{g}}{\Lambda})$ to *any* pointed connected space $\mathcal{A}$:

$$\text{Topological flux observables for flux quantized in } \mathcal{A}\text{-cohomology} \quad \text{TopFlxObs}_{\Sigma^2}^{\mathcal{A}} := \mathbb{C}[\pi_1 \text{Map}_0(\Sigma^2, \mathcal{A})]. \quad (18)$$

Remark 2.10 (Classifying spaces for higher and generalized symmetries). For any pointed connected space $\mathcal{A}$ its loop space $\Omega\mathcal{A}$ carries a “higher group” structure (cf. [197][232, §2.2][233, §3.1.2]) under concatenation and reversal of loops, and $\mathcal{A}$ is the classifying space for *that* higher group structure (cf. [77, Prop. 2.2]):

$$\mathcal{A} \simeq_{\mathcal{I}} B(\Omega\mathcal{A}).$$

This way, the generalization (18) corresponds to passage to *higher* gauge theories with *higher* (gauge) symmetry. These days in physics the latter is also referred to as “generalized symmetry”, see [116] for an account that makes contact with our perspective here.

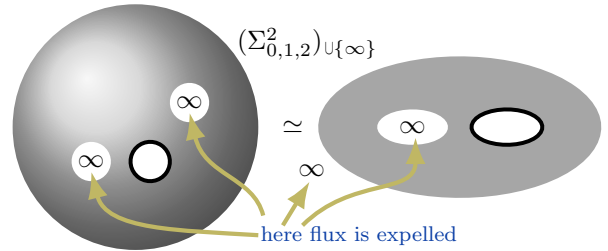
This algebra (18) of observables being a *group algebra* means that the corresponding spaces of (pure) quantum states — which generally are modules over the observable algebra — are actually modules of a group algebra and, as such, nothing but (unitary) representations of this group (cf. [45, p. 11]):

$$\text{Topological quantum states of flux quantized in } \mathcal{A}\text{-cohomology} \quad \mathcal{H}_{\Sigma^2}^{\mathcal{A}} \in \text{URep}(\pi_1 \text{Map}_0(\Sigma^2, \mathcal{A})). \quad (19)$$

Further generalizations are immediately suggested by these formulas: If $\Sigma^2 = \Sigma_{g,b,n}^2$ (9) is possibly non-compact ($n > 0$), then the *solitonic* flux configurations (cf. [228, §2.2][226, §A.2]) are those which are *vanishing at infinity* and thus classified by *pointed* maps on the one-point compactification $(-)\cup\{\infty\}$ (cf. §A.3 and Fig. I), so that (19) generalizes to:

$$\begin{array}{c} \text{space of} \\ \text{quantum states} \end{array} \quad \overbrace{\mathcal{H}_{\Sigma^2}^{\mathcal{A}}}^{\text{... for solitonic flux on non-compact spaces}} \in \underbrace{\text{URep}}_{\text{unitary representations}} \left(\underbrace{\pi_1 \text{Map}_0^*(\Sigma_{\cup\{\infty\}}^2, \mathcal{A})}_{\substack{\text{moduli space of} \\ \text{solitonic topological flux} \\ \text{topological flux} \\ \text{monodromy group}}} \right). \quad (20)$$

Figure I – Solitonic flux and the point at infinity. To “adjoin the point at infinity” – $\Sigma^2 \mapsto \Sigma_{\cup\{\infty\}}^2$ – means to identify all *open ends* of a surface with a single point “ ∞ ”. Regarding this as the basepoint of $\Sigma_{\cup\{\infty\}}^2$ and understanding the basepoint of $\mathcal{A}$ to classify vanishing flux, then taking the classifying maps $\Sigma_{\cup\{\infty\}}^2 \rightarrow \mathcal{A}$ to be *pointed maps* in $\text{Map}_0^*(\Sigma_{\cup\{\infty\}}^2, \mathcal{A})$ (20) literally makes the flux *vanish at infinity*, which is the defining condition for solitonic flux. This way, punctures in Σ^2 represent islands where flux is expelled, cf. Fig. D.



Moreover, if the gauge field is to be regarded as “generally covariant” in the sense of physics — namely covariant with respect to diffeomorphisms, such as for gravitational and topological systems — so that two topological flux configurations are to be regarded as gauge equivalent if one is obtained from the other by precomposition with a diffeomorphism, then (cf. [62, Def. 1.1]) the true moduli space is the *homotopy quotient* (220) of the flux moduli space (20) by the action of the diffeomorphism group $\text{Diff}(\Sigma^2)$ (see Def. 2.20). Therefore we set:

Definition 2.11 (Generally covariant topological quantum states of exotic flux). The possible spaces of quantum states of generally covariant topological flux quantized in $\mathcal{A}$ -cohomology are the irreducible unitary

representations of the covariantized flux monodromy group:

$$\text{Generally covariantized quantum states of topological solitonic flux } \mathcal{H}_{\Sigma^2}^{\mathcal{A}} \in \text{URep} \left(\underbrace{\pi_1}_{\substack{\text{reps of} \\ \text{fundamental} \\ \text{group of}}} \left(\underbrace{\text{Map}_0^*(\Sigma_{\{\infty\}}^2, \mathcal{A})}_{\substack{\text{local systems of} \\ \text{state spaces on} \\ \text{moduli space of topological solitonic flux}}} \parallel \underbrace{\text{Diff}^{+, \partial}(\Sigma^2)}_{\substack{\text{plain moduli space} \\ \text{of topological solitonic flux} \\ \text{quantized in } \mathcal{A}\text{-cohomology}}} \right) \parallel \text{Diff}^{+, \partial}(\Sigma^2) \right). \quad (21)$$

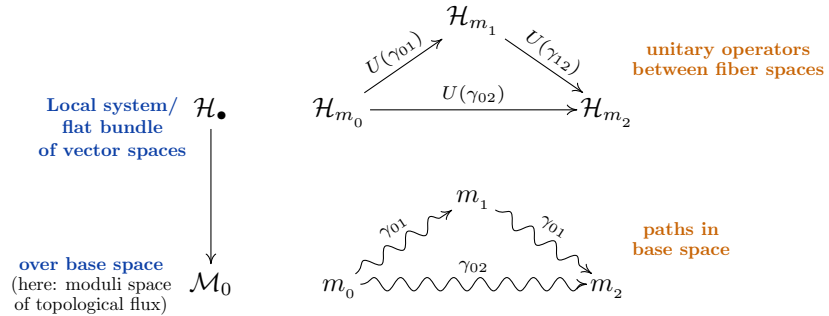
In order to get a handle on these groups (21) of generally covariantized flux monodromy, the first general result we prove below (Prop. 2.24, not surprising, but important) is that they are equivalently the semi-direct product of the plain flux monodromy with the surface's mapping class group:

$$\pi_1 \left(\underbrace{\text{Map}_0^*(\Sigma_{\{\infty\}}^2, \mathcal{A}) \parallel \text{Diff}^{+, \partial}(\Sigma^2)}_{\text{covariantized flux moduli space (21)}} \right) \simeq \underbrace{\pi_1 \left(\text{Map}_0^*(\Sigma_{\{\infty\}}^2, \mathcal{A}) \right)}_{\text{flux monodromy}} \rtimes \underbrace{\pi_0 \left(\text{Diff}^{+, \partial}(\Sigma^2) \right)}_{\text{mapping class group (29)}}. \quad (22)$$

Some remarks on the import of this construction (21):

Remark 2.12 (Local systems of vector spaces as quantum state spaces).

- (i) Representations of fundamental groups $\pi_1(\mathcal{M}_0)$ as in (21), equivalently (since $\mathcal{M}_0$ is a connected component) of the *fundamental groupoid* $\Pi_1(\mathcal{M}_0)$ (187) are also known as *local systems* (cf. [56, §I.1][273, p. 257][59, §2.5][258, §2.6]) or *flat bundles* (cf. [265, §9.2.1]) of vector spaces over $\mathcal{M}_0$ (cf. [190, Lit. 2.22]).



- (ii) There is deep relevance [225] in identifying these as quantum state spaces subject to symmetries and classical control in general (cf. [224][44]) and specifically so concerning the nature of anyonic topological quantum gates (cf. [190, §3] and the next Rem. 2.13).
- (iii) In particular, (higher, cf. [225]) local systems are considered as (relative) spaces of quantum states in generic “pull-push”-constructions of (extended) topological quantum field theories ([83][189][244][85, §A.2]). In this respect the point of (21) is the appropriate identification of the local system’s domain moduli space for the case of topological flux quanta (and the remaining “pull-push”-propagation along cobordisms, receiving so much attention in the topological field theory literature, plays little to no role in laboratory experiment, where the given slab of material commonly just sits there without spontaneously changing its topology!).
- (iv) In our situation (21), the base space is the covariantized moduli space of topological fluxes quantized in some $\mathcal{A}$ -cohomology, and paths $m \rightsquigarrow m'$ are *combinations* of
 - (a) gauge transformations of the (higher) gauge field (from the homotopy quotient’s numerator in (21))
 - (b) diffeomorphisms, hence gauge transformations of the “gravitational field”, (from the homotopy quotient’s denominator in (21)).

The Hilbert space(s) of states $\mathcal{H}_{\Sigma^2}^{\mathcal{A}}$ forming a local system/flat bundle means that both of these kinds of gauge transformations are implemented as unitary quantum operators/observables in a coherent way.

- (v) The local system (21) records the action of these transformations on the flux state spaces. Identifying which transformations are implemented as physical operations requires additional data specifying the physical realization.

Now note that these paths here are paths in moduli space, hence not of single flux quanta but of all flux quanta that are present at once. The same holds for punctures/defects:

Remark 2.13 (Defect anyons and braid representations).

- (i) Punctures of Σ^2 model *defects* in the slab of quantum material (cf. [180]). Their braided *worldlines*, illustrated on the right, define a braid group and a point-pushing homomorphism to the mapping class group in (22). This homomorphism need not be injective (see (39)).
- (ii) Pulling back the mapping-class action on flux quantum states (21) gives linear representations of this braid group (*braid representations*, cf. [150][1]). These encode a possibly non-abelian generalization of the braiding phases of solitonic anyons (2), exhibiting the defects as *defect anyons* ([153, p. 4][223]).
- (iii) Extrinsic defects and their controlled braiding have been studied in FQH-related systems, including genons [14] and parafermion defects [46]. Here we investigate defect monodromy arising from the proposed cohomotopical flux model and its candidate topological quantum gates (cf. [150][126][43][151]). The feasibility of adiabatic control, for either defects or solitonic excitations, depends on a device-specific physical realization.

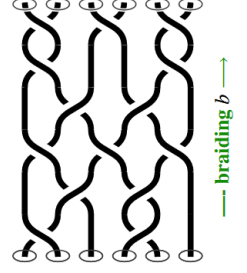
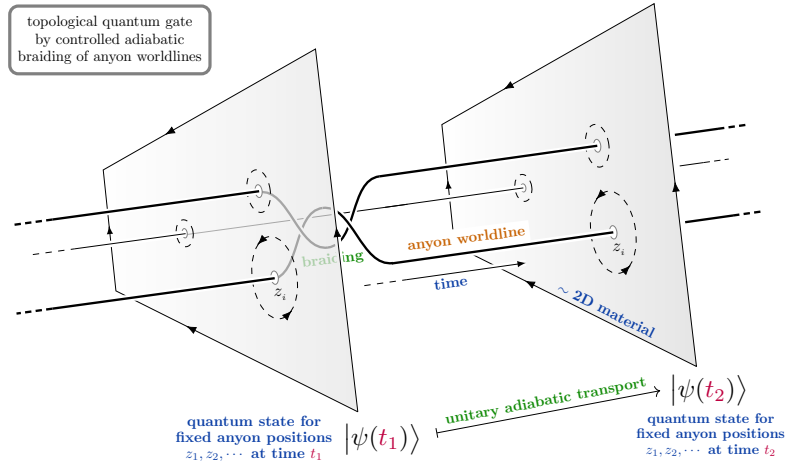


Figure B – Topological braid gates. Covariant flux quantum states on a sufficiently punctured surface, by their diffeomorphism equivariance (21), carry unitary representations of the *braid group* (23) of joint motions of the defects around each other, exhibiting the punctures as *defect anyons*.

If a gapped physical realization has robust adiabatic transport realizing this braid representation, controlled braiding implements the corresponding unitary operators as *quantum gates*. Protection against deformations of the control paths requires flat, or appropriately projectively flat, topological transport that is stable under the allowed perturbations; a spectral gap alone does not ensure this (cf. §3.4.2).



Remark 2.14 (Focus on irreducible representations: Superselection sectors of anyons).

- (i) From the perspective of Def. 2.11, spaces of pure quantum states (of topological flux) are defined as representations of (modules over) the algebra of observables. From this perspective, it is the (unitary) *irreducible* representations that matter in characterizing the actual quantum system, in that reducible representations behave like parallel copies of a system, which could just as well be discussed separately: “superselection sectors” (cf. [91, Def. 2.1][12, p. 273]).
- (ii) While this may (and should) seem obvious, it is in some contrast to common practice: For punctured surfaces Σ^2 the mapping class group appearing in (22) contains *braiding* operations (cf. Prop. 2.21) and it is common to consider Yang-Baxter representations for these (“R-matrices”, cf. [150][288]), which generally are reducible (cf. [167, p. 15]).

2.2 General Covariance of Flux

We discuss here the effect of the covariantization (21) on the plain moduli spaces of solitonic topological flux. After recalling basics of mapping class groups and braid groups, the main result here is Prop. 2.24 below, which was already announced as (22).

Braid groups and mapping class groups.

Definition 2.15 (Configuration spaces and Braid groups, cf. [80][190, Lit. 2.18, 2.20]).

- (i) For Σ a smooth manifold, possibly with boundary, and $n \in \mathbb{N}$, the *configuration space of n points in Σ* is the topological space

$$\text{Conf}_n(\Sigma) := \left\{ (s_1, \dots, s_n) \in \Sigma^{\times n} \mid \forall_{i \neq j} s_i \neq s_j \right\} / \text{Sym}_n$$

(topologized as the quotient space of a subspace of a product space).

- (ii) For Σ connected and of dimension at least 2, the fundamental group of this space is the *braid group on n strands in Σ* (cf. [74, §9]), which as such comes equipped with a forgetful map to the symmetric group:

$$\mathrm{Br}_n(\Sigma) := \pi_1 \mathrm{Conf}_n(\Sigma) \twoheadrightarrow \mathrm{Sym}_n. \quad (23)$$

- (iii) The kernel of (23) is the *pure braid group* PBr :

$$1 \rightarrow \mathrm{PBr}_n(\Sigma) \hookrightarrow \mathrm{Br}_n(\Sigma) \twoheadrightarrow \mathrm{Sym}_n \rightarrow 1 \quad (24)$$

Example 2.16 (Artin presentation of braid groups, cf. [80, §7][190, Lit. 2.20]). For $n \geq 2$, the surface braid group (23) of the disk (the default case of braid groups) has the following finite presentation:

$$\mathrm{Br}_n := \mathrm{Br}_n(\Sigma_{0,1,0}^2) \simeq F\langle b_1, \dots, b_{n-1} \rangle / \left(\bigvee_{i+1 < j} (b_i b_j = b_j b_i), \bigvee_{1 \leq i < n-1} \underset{\text{Yang-Baxter relation}}{(b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1})} \right), \quad (25)$$

in terms of which its canonical homomorphism to the symmetric group is the quotient map by one further set of relations:

$$\mathrm{Br}_n \twoheadrightarrow \mathrm{Sym}_n := \mathrm{Br}_n / \left(\bigvee_i (b_i b_i = e) \right). \quad (26)$$

The general surface braid group $\mathrm{Br}_n(\Sigma^2)$ may be presented by adjoining to these *Artin generators* b_i further generators (corresponding to moving single strands along cycles in the surface) and further relations. In each case, there is a projection to the symmetric group by retaining the Artin generators:

$$\mathrm{Br}_n(\Sigma^2) \twoheadrightarrow \mathrm{Sym}_n$$

Example 2.17 (Presentation of spherical braid group [73], cf. [259]). The surface braid group (23) of the sphere (often: “spherical braid group”) is presented as a quotient of the Artin presentation (25) by one further relation:

$$\mathrm{Br}_n(S^2) \simeq \mathrm{Br}_n / ((b_1 \cdots b_{n-1})(b_{n-1} \cdots b_1)). \quad (27)$$

Definition 2.18 (Diffeomorphism Group and Mapping Class Group). For Σ an oriented manifold, possibly with boundary, we write

$$\begin{array}{ccc} \mathrm{Homeo}^{+, \partial}(\Sigma) & \hookrightarrow & \mathrm{Homeo}(\Sigma) \hookrightarrow \mathrm{Map}(\Sigma, \Sigma) \\ \uparrow \iota & & \uparrow \iota \\ \mathrm{Diff}^{+, \partial}(\Sigma) & \hookrightarrow & \mathrm{Diff}(\Sigma) \end{array} \quad (28)$$

for its topological groups of homeomorphisms and diffeomorphisms, respectively for the further subgroups of maps preserving the orientation (+) and restricting to the identity on the boundary (∂).

For $\Sigma \equiv \Sigma^2$ an orientable surface and choosing any one of its orientations, the group of connected components of the latter diffeo group is known as the *mapping class group* [135, §1][188, §3][74, p. 45]:

$$\mathrm{MCG}(\Sigma^2) := \pi_0(\mathrm{Diff}^{+, \partial}(\Sigma^2)). \quad (29)$$

(For a fixed spin structure, we use the *spin* mapping class subgroup preserving it, with the choices specified in Rem. 2.3.)

Example 2.19 (Mapping class groups of closed oriented surfaces, cf. [188, §6][74, §6]). The mapping class group of the torus is

$$\mathrm{MCG}(\Sigma_1^2) \simeq \mathrm{Sp}_2(\mathbb{Z}) \simeq \mathrm{SL}_2(\mathbb{Z}), \quad (30)$$

which is generated by the two elements [249, Thm VII.2 p. 78][49, Thm. 1.1]:

$$S := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad T := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad (31)$$

and presented subject to the following relations [157, p. 126][29, §2.1]:

$$\mathrm{SL}_2(\mathbb{Z}) \simeq \langle S, T \mid S^4 = (TS)^3 = e, S^2(TS) = (TS)S^2 \rangle. \quad (32)$$

More generally, the mapping class group of Σ_g^2 (9), for $g \in \mathbb{N}$, sits in a short exact sequence (cf. [74, §6])

$$1 \rightarrow \underset{\text{Torelli group}}{I_g} \hookrightarrow \underset{\text{mapping class group}}{\mathrm{MCG}(\Sigma_g^2)} \twoheadrightarrow \underset{\text{symplectic group}}{\mathrm{Sp}_{2g}(\mathbb{Z})} \rightarrow 1, \quad (33)$$

where the (pre-)composition action of $\mathrm{MCG}(\Sigma_g^2)$ on

$$H_1(\Sigma_g^2; \mathbb{Z}) \simeq H^1(\Sigma_g^2; \mathbb{Z}) \simeq \mathbb{Z}^g \times \mathbb{Z}^g$$

is through the defining action of the integer symplectic group $\mathrm{Sp}_{2g}(\mathbb{Z})$.

The set of spin structures is a torsor for $H^1(\Sigma_g^2; \mathbb{Z}_2)$ (cf. [182, p. 199]); equivariantly, its elements are the quadratic refinements q of the mod-2 intersection form, characterized by

$$q(x+y) = q(x) + q(y) + x \cdot y \quad (x, y \in H_1(\Sigma_g^2; \mathbb{Z}_2)).$$

The mapping class action is $q \mapsto q \circ f_*^{-1}$. After choosing an origin in the torsor this is an affine action, whose linear part is the induced action on H^1 ; for $g \geq 2$ there is no origin fixed by the entire mapping class group ([144, §§3–5]). On the torus, the unique odd spin structure pp is invariant. Choosing it as origin, and using Poincaré duality, identifies the four spin structures equivariantly with $H_1(\Sigma_1^2; \mathbb{Z}_2)$:

$$\{\text{pp}, \text{aa}, \text{ap}, \text{pa}\} \simeq \mathbb{Z}_2^2 \simeq H_1(\Sigma_1^2; \mathbb{Z}_2) \quad (34)$$

(with “periodic” or “antiperiodic” boundary conditions for spinors along the two basis 1-cycles, cf. [5, §2]) and $\text{MCG}(\Sigma_1^2) \simeq \text{SL}_2(\mathbb{Z})$ (30) preserves pp and transitively permutes the other three. On the other hand, the stabilizer subgroup of the aa structure, hence the *spin mapping class group* of diffeomorphisms preserving aa , is generated by S and the *square* of T (cf. [34, p. 3]) subject to the following relations:

$$\begin{array}{ccc} \text{MCG}(\Sigma_1^2)^{\text{aa}} & \hookrightarrow & \text{MCG}(\Sigma_1^2) \\ \wr & & \wr \\ \langle S, T^2 \mid S^4 = [S^2, T^2] = e \rangle & \hookrightarrow & \text{SL}_2(\mathbb{Z}). \end{array} \quad (35)$$

Definition 2.20 (Moduli spaces of solitonic topological fluxes). The underlying homeomorphisms of the diffeomorphisms (28) of surfaces $\Sigma_{g,b,n}^2$ (9) extend functorially to the one-point compactification (by Prop. A.6) to make a topological group homomorphism

$$\text{Diff}^{(+,\partial)}(\Sigma_{g,b,n}^2) \xrightarrow{\iota} \text{Homeo}^{(+,\partial)}(\Sigma_{g,b,n}^2) \xrightarrow{(-) \cup \{\infty\}} \text{Aut}_{\text{Top}^*}((\Sigma_{g,b,n}^2) \cup \{\infty\}).$$

Via the latter’s action (by pre-composition) on pointed mapping spaces (20) we obtain the homotopy quotient (220) of the pointed mapping space ¹⁰

$$\text{Map}_0^*((\Sigma_{g,b,n}^2) \cup \{\infty\}, \mathcal{A}) // \text{Diff}^{(+,\partial)}(\Sigma_{g,b,n}^2) \in \text{Top}^*, \quad (36)$$

identified in (21) as the *covariantized moduli space* of $\mathcal{A}$ -quantized solitonic topological fluxes on $\Sigma_{g,b,n}^2$.

The following is classical but somewhat scattered in the literature:

Proposition 2.21 (Homotopy type of Diffeomorphism groups).

(i) For compact oriented surfaces $\Sigma_{g,b}^2$ (9), the homotopy type of their diffeomorphism group (28) is:

$$\begin{array}{lll} \text{Diff}^+(\Sigma_{0,0,0}^2) \simeq_{\mathbb{J}} \text{SO}(3) & \Rightarrow & \text{MCG}(\Sigma_{0,0,0}^2) \simeq 1 \quad \text{and} \quad \pi_1 \text{Diff}^+(\Sigma_{0,0,0}^2) \simeq \mathbb{Z}_2 \\ \text{Diff}^+(\Sigma_{1,0,0}^2) \simeq_{\mathbb{J}} \text{SL}_2(\mathbb{Z}) \times T^2 & \Rightarrow & \text{MCG}(\Sigma_{1,0,0}^2) \simeq \text{SL}_2(\mathbb{Z}) \quad \text{and} \quad \pi_1 \text{Diff}^+(\Sigma_{1,0,0}^2) \simeq \mathbb{Z} \times \mathbb{Z} \\ \text{Diff}_0^+(\Sigma_{g \geq 2,0,0}^2) \simeq_{\mathbb{J}} * & \Rightarrow & \pi_1 \text{Diff}^+(\Sigma_{g \geq 2,0,0}^2) \simeq 1 \\ \text{Diff}_0^{+,\partial}(\Sigma_{g,b \geq 1,0}^2) \simeq_{\mathbb{J}} * & \Rightarrow & \pi_1 \text{Diff}^{+,\partial}(\Sigma_{g,b \geq 1,0}^2) \simeq 1. \end{array} \quad (37)$$

In particular:

$$\pi_1 \text{Diff}^{+,\partial}(\Sigma_{g,b}^2) \simeq 1 \quad \text{except for } (g,b) \in \{(0,0), (1,0)\}. \quad (38)$$

(ii) For punctured oriented surfaces $\Sigma_{g,b \geq 1}^2$, the map from their mapping class group to that of $\Sigma_{g,b}^2$ (obtained by uniquely extending the underlying homeomorphisms over the punctures and using the identification of topological and smooth mapping class groups of surfaces) sits in a long exact sequence (“generalized Birman sequence”) with the surface’s braid group (23), of this form:

$$\pi_1 \text{Diff}^{+,\partial}(\Sigma_{g,b}^2) \longrightarrow \text{Br}_n(\Sigma_{g,b}^2) \longrightarrow \text{MCG}(\Sigma_{g,b,n}^2) \longrightarrow \text{MCG}(\Sigma_{g,b}^2). \quad (39)$$

(a) Hence when $\pi_1 \text{Diff}^{+,\partial}(\Sigma_{g,b}^2) = 1$ the mapping class group sits in a short exact sequence of the form

$$1 \rightarrow \text{Br}_{n \geq 1}(\Sigma_{g,b}^2) \longrightarrow \text{MCG}(\Sigma_{g,b,n \geq 1}^2) \longrightarrow \text{MCG}(\Sigma_{g,b}^2) \rightarrow 1, \quad (40)$$

and exhausts the homotopy type of the diffeomorphism group:

$$\text{Diff}^{+,\partial}(\Sigma_{g,b,n \geq 1}^2) \simeq_{\mathbb{J}} \text{MCG}(\Sigma_{g,b,n \geq 1}^2) \Rightarrow \pi_1 \text{Diff}^{+,\partial}(\Sigma_{g,b,n \geq 1}^2) \simeq 1. \quad (41)$$

(b) For $g = 0$, $b = 0$, and $n \geq 3$ — where the assumption in (iia) fails by (37) — the spherical braid group surjects onto the mapping class group with kernel $\mathbb{Z}_2$, the image of $\pi_1 \text{Diff}^+(S^2)$ (generated by the full rotation braid rot):

$$1 \rightarrow \mathbb{Z}_2 \xhookrightarrow{\text{rot}} \text{Br}_{n \geq 3}(\Sigma_0^2) \twoheadrightarrow \text{MCG}(\Sigma_{0,0,n}^2) \rightarrow 1. \quad (42)$$

¹⁰ For every pointed coefficient space $\mathcal{A}$, precomposition fixes the constant map to its basepoint and hence preserves the connected component $\text{Map}_0^*((\Sigma_{g,b,n}^2) \cup \{\infty\}, \mathcal{A})$. This applies also to surfaces with boundary and requires no degree classification of the other components.

- (c) For $n = 1, 2$, the map from the spherical braid group to the mapping class group is an isomorphism; hence the kernel in (42) is trivial in these exceptional cases. Concretely, for $g = b = 0$ we have:

$$\begin{aligned} \text{MCG}(\Sigma_{0,0,1}^2) &\simeq 1 \simeq \text{Br}_1(S^2) \\ \text{MCG}(\Sigma_{0,0,2}^2) &\simeq \mathbb{Z}_2 \simeq \text{Br}_2(S^2) \\ \text{MCG}(\Sigma_{0,0,3}^2) &\simeq \text{Sym}_3 \not\simeq \text{Br}_3(S^2) \\ \text{MCG}(\Sigma_{0,0,4}^2) &\simeq \text{PSL}_2(\mathbb{Z}) \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2). \end{aligned} \quad (43)$$

Proof. In (37) the first statement is due to [252], the first three were proven by [65][66][111], and the fourth is [67, Thm. 1D p. 170]. The statement (41) follows with [280][281, Thm. 1.1].¹¹ The generalized Birman sequence (39) is named in honor of [28], cf. [176, Thm. 3.13]. In its implication of the short exact sequence (40) this is reviewed in [74, Thm. 9.1]. The spherical braid group extension (42) is discussed in [74, (9.1)] and the identifications (43) of its quotients are proven, for instance, in [74, Prop. 2.3][33]. $\square$

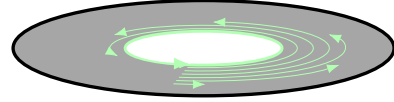
Example 2.22 (Mapping class groups of the n -punctured disk). Since the mapping class group of the disk $\Sigma_{0,1,0}^2$ is trivial by (37), the exact sequence (40) shows that the mapping class group of its punctured versions is the plain braid group (25):

$$\text{MCG}(\Sigma_{0,1,n}^2) \simeq \text{Br}_n. \quad (44)$$

Example 2.23 (Mapping class group of the closed annulus). The mapping class group (29) of the closed annulus (10) is the integers (cf. [74, Prop. 2.4]), generated by the *boundary Dehn twist* indicated in Fig. DT.

$$\text{MCG}(\Sigma_{0,2}^2) \simeq \mathbb{Z}. \quad (45)$$

Figure DT – The mapping class group of the closed annulus is freely generated by a single element, called the *boundary Dehn twist*, which is, up to deformation, the diffeomorphism that at each radius is a circular rotation, radially interpolating between the zero-rotation on one boundary and the unit rotation on the other.



Covariant flux monodromy. With all this in hand, we come to the main statement to be proven in this section, announced as (22).

Proposition 2.24 (Extension of mapping class group by flux monodromy). For every $\Sigma_{g,b,n}^2$ (9) we have a split short exact sequence of groups

$$1 \longrightarrow \underbrace{\pi_1\left(\text{Map}_0^*\left((\Sigma_{g,b,n}^2) \cup \{\infty\}, \mathcal{A}\right)\right)}_{\text{moduli space}} \longrightarrow \underbrace{\pi_1\left(\text{Map}_0^*\left((\Sigma_{g,b,n}^2) \cup \{\infty\}, \mathcal{A}\right) // \text{Diff}^{+, \partial}(\Sigma_{g,b,n}^2)\right)}_{\text{covariantized moduli space (36)}} \xrightarrow{\quad} \underbrace{\text{MCG}(\Sigma_{g,b,n}^2)}_{\text{mapping class group (29)}} \longrightarrow 1,$$

exhibiting an action of the mapping class group on the fundamental group of the moduli space, so that we have the corresponding semidirect product:

$$\underbrace{\pi_1\left(\text{Map}_0^*\left((\Sigma_{g,b,n}^2) \cup \{\infty\}, \mathcal{A}\right) // \text{Diff}^{+, \partial}(\Sigma_{g,b,n}^2)\right)}_{\text{covariantized moduli space (36)}} \simeq \underbrace{\text{MCG}(\Sigma_{g,b,n}^2)}_{\text{mapping class group (29)}} \ltimes \underbrace{\pi_1\left(\text{Map}_0^*\left((\Sigma_{g,b,n}^2) \cup \{\infty\}, \mathcal{A}\right)\right)}_{\text{moduli space}}. \quad (46)$$

Proof. For notational convenience, we abbreviate

$$\begin{aligned} \mathcal{F} &:= \text{Map}_0^*\left((\Sigma_{g,b,n}^2) \cup \{\infty\}, \mathcal{A}\right) \\ \mathcal{D} &:= \text{Diff}^{+, \partial}(\Sigma_{g,b,n}^2), \end{aligned}$$

whence the claim to be proven is split exactness of

$$1 \longrightarrow \pi_1(\mathcal{F}) \longrightarrow \pi_1(\mathcal{F} // \mathcal{D}) \xrightarrow{\quad} \pi_0(\mathcal{D}) \longrightarrow 1. \quad (47)$$

To this end, the Borel homotopy fiber sequence (223)

$$\mathcal{F} \longrightarrow \mathcal{F} // \mathcal{D} \xleftarrow{\quad} * // \mathcal{D}$$

¹¹The surfaces in [280][281] are assumed to be without boundary, but equipped with marked closed subcomplexes to be fixed by the diffeomorphisms. Under this definition, a puncture surrounded by a marked circle behaves just as a boundary for the purpose of computing the homotopy type of the diffeomorphism group.

(split by picking the zero-map) induces a long exact sequence of homotopy groups (219) of this form:

$$\begin{array}{c}
 \xrightarrow{\text{by (222)}} \pi_1(\mathcal{D}) \xrightarrow{\quad} \\
 \left\{ \begin{array}{c} \xrightarrow{\quad} \pi_1(\mathcal{F}) \longrightarrow \pi_1(\mathcal{F} \parallel \mathcal{D}) \longrightarrow \pi_0(\mathcal{D}) \end{array} \right\} \\
 \xrightarrow{\quad} \pi_0(\mathcal{F}) \xrightarrow{\sim} \pi_0(\mathcal{F} \parallel \mathcal{D}).
 \end{array} \tag{48}$$

Here the last map shown is an isomorphism by (224) (cf. footnote 10), whence the exact sequence truncates to

$$\pi_1(\mathcal{D}) \longrightarrow \pi_1(\mathcal{F}) \longrightarrow \pi_1(\mathcal{F} \parallel \mathcal{D}) \xleftarrow{\quad} \pi_0(\mathcal{D}) \longrightarrow 1.$$

If, at this point, we invoke Prop. 2.21 then the claim (47) follows for most surfaces, namely those for which $\pi_1(\mathcal{D}) \simeq 1$. But in fact, the claim follows generally by observing that the first connecting map in (48) factors through the trivial group:

$$\begin{array}{ccc}
 \pi_1(\mathcal{D}) \equiv \pi_1(\text{Diff}^+(\Sigma_{g,b,n}^2)) & \xrightarrow{\quad} & \pi_1(\text{Map}_0^*((\Sigma_{g,b,n}^2) \cup \{\infty\}, \mathcal{A})) \equiv \pi_1(\mathcal{F}). \\
 & \searrow \quad \nearrow & \\
 & 1 &
 \end{array}$$

Namely, by (223), the map is given by taking a given loop of diffeomorphisms to the loop of maps obtained by composing these diffeos with the constant map $(\Sigma_{g,b,n}^2) \cup \{\infty\} \rightarrow \mathcal{A}$ — but that gives the constant loop representing the neutral element of π_1 . $\square$

This Proposition 2.24 is our main tool for analyzing the covariantized topological quantum states on $\mathcal{A}$ -quantized flux according to (21). In the next section, we specialize $\mathcal{A}$ to S^2 and work out the consequences.

3 Flux quantized in 2-Cohomotopy

We now specify the classifying space $\mathcal{A}$ (21) to the 2-sphere, $\mathcal{A} \equiv S^2$ so that flux is classified by the non-abelian cohomology theory called *2-Cohomotopy*¹² (49), and we work out (according to §2) the resulting covariant topological quantum observables on and quantum states of 2-cohomotopically quantized flux through various surfaces Σ^2 , using the results of §2.2.

Remarkably, in the case of $\Sigma^2 \equiv S^2$ the sphere or $\Sigma^2 \equiv T^2$ the torus, we find reproduced (in §3.1 and §3.4, respectively) the situation traditionally argued via quantized U(1)-Chern-Simons theory over these surfaces, including fine-print such as regularization of Wilson-loop observables by framings, modular equivariance and refinement to “spin” Chern-Simons theory.

Then, by instead choosing punctured surfaces, we similarly work out the 2-Cohomotopically quantized flux through the annulus (§3.7) and the 2-punctured disk (§3.8).

Definition 3.1 (Cohomotopy, cf. [253][130, §VII][77, Ex. 2.7]). The generalized non-abelian cohomology theory [77, §2] whose classifying spaces are the n -spheres S^n is called *Cohomotopy*¹³, denoted

$$\tilde{\pi}^n(X) := \pi_0 \text{Map}^*(X, S^n), \quad \text{for } X \in \text{Top}^*. \quad (49)$$

Here the terminology and notation indicate the “duality” with the *homotopy* groups $\pi_n(X) \simeq \pi_0 \text{Map}^*(S^n, X)$.

Remark 3.2 (Relation to the O(3)/CP¹ sigma-model). Taking flux-quantization (18) to be in 2-Cohomotopy, $\mathcal{A} \equiv S^2$ (Def. 3.1), so that topological charges are represented by continuous maps $\Sigma^2 \rightarrow S^2$, has some similarity with the classification of solitons (or *lumps*) in the CP¹ model, equivalently the O(3) nonlinear sigma-model with target S^2 . Its fields are *smooth* maps to S^2 (cf. [79, §VI.A][174, §6.1]). Planar solitons with a fixed value at spatial infinity carry the degree charge in $\pi_2(S^2) \simeq \mathbb{Z}$. The “Hopf term” (cf. [275, (6)][79, §II.C] [40]), essentially the abelian Chern-Simons term (159), instead measures topology of their spacetime histories, as discussed in Rem. 3.14 below. See also the discussion of continuous 2-Cohomotopy in the Skyrme-Faddeev model by [8, 9].

However, our re-identification of such maps $\Sigma^2 \rightarrow S^2$ as classifying exotically quantized flux according to §2, [228], puts these arguments in a coherent perspective concerning flux quanta in FQH systems, cf. the following Ex. 3.5.

Remark 3.3 (Generalized higher symmetry group of 2-cohomotopical flux).

- (i) In view of Rem. 2.10, the choice of classifying space $\mathcal{A} \equiv S^2 \simeq_f B(\Omega S^2)$ corresponds to considering as (homotopy type of the) gauge group the loop group ΩS^2 of the 2-sphere (under concatenation and reversal of loops) which is a “higher group” (“ ∞ -group”) exhibiting “generalized symmetry”. See also footnote 14 below.
- (ii) The looping of the canonical comparison map $1^2 : S^2 \rightarrow B^2\mathbb{Z}$ exhibits this generalized symmetry group as a deformation of (the homotopy type of) the standard electromagnetic gauge group U(1):

$$\Omega S^2 \xrightarrow{\Omega 1^2} \Omega B^2\mathbb{Z} \simeq_f B\mathbb{Z} \simeq_f \text{U}(1).$$

This map induces an isomorphism on π_1 , but while $\pi_{>1}(\text{U}(1)) \simeq 0$, the deformation ΩS^2 on the left has non-trivial homotopy groups in arbitrarily high degree, in particular a non-finite contribution

$$\pi_2(\Omega S^2) \underset{(205)}{\simeq} \pi_3(S^2) \simeq \mathbb{Z}$$

generated by the Hopf fibration. We already remarked after (6) that this is the homotopical avatar of the Chern-Simons form, and we will see now that it is also the origin of the appearance of anyonic braiding phases of flux solitons quantized in 2-Cohomotopy: This is seen in Prop. 3.9, Prop. 3.19 and Prop. 3.23 below.

¹² More precisely, we expect that flux in FQH systems is classified by the *tangentially twisted* version of 2-cohomotopy according to [77, Ex. 3.8], but in the present context the tangential twisting is trivial in the cases of main interest, namely on the plane and on the torus. The surfaces on which tangential twisting would be relevant are just those on which experimental realizability of FQH systems is dubious. If and when this experimental situation changes, a dedicated discussion of the tangentially twisted analogues of the results in §3.3 would be called for. It is fairly clear how this will proceed: The relevant tangentially twisted version of the Pontrjagin theorem (recalled here as Prop. 3.4) is given in [75, §2] and the tangentially twisted version of the May-Segal theorem is reviewed as [147, Thm. 4.2].

¹³ The original literature [20][253] on Cohomotopy (also [205], without using that terminology) is focused on equipping the Cohomotopy sets $\pi^n(X)$ with group structure, which is possible when X is a CW-complex of dimension $\leq 2n - 2$, and hence speaks of *cohomotopy groups* in these special situations. For the present purpose this group structure plays no role but instead the perspective of generalized non-abelian cohomology is the natural one: Also ordinary non-abelian cohomology [114] $\tilde{H}^1(X; G) \simeq \pi_0 \text{Map}^*(X, BG)$ (cf. [77, Ex. 2.2][233, Thm. 4.1.13]) has no group structure (unless G happens to be abelian).

3.1 On the plane

We recall here (from [227]) how solitonic flux through the plane $\mathbb{R}^2 \simeq \Sigma_{0,0,1}^2$ (10) quantized in 2-cohomotopy reproduces the Wilson loop observables of anyonic links as predicted by abelian Chern-Simons theory (Rem. 3.13 below). But to start with, we briefly recall the Pontrjagin construction that serves to relate Cohomotopy charge to solitonic flux density in general.

2-Cohomotopical flux solitons via the Pontrjagin construction. Among generalized non-abelian cohomology theories, (unstable) *Cohomotopy* π^n (cf. [205][253][130, §VII][77, Ex. 2.7]), whose classifying spaces are the n -spheres $S^n \simeq \mathbb{R}_{\cup\{\infty\}}^n$ (209),

$$\tilde{\pi}^n(-) := \pi_0 \text{Map}^*(-, \mathbb{R}_{\cup\{\infty\}}^n),$$

stands out in that it accurately characterizes the solitonic flux configurations of given charge [221][227] — this may be understood as the content of the original unstable Pontrjagin theorem (which these days is more famous as the *Pontrjagin-Thom theorem* pertaining only to the *stable* case which is of little concern to us here):

Proposition 3.4 (Pontrjagin theorem – Cohomotopy charge, cf. [38, §II.16][161, §IX]). *Given a smooth d -manifold Σ^d without boundary and $n \in \mathbb{N}_{>0}$ with $n \leq d$, there is a natural bijection between:*

1. *the reduced n -Cohomotopy of the one-point compactification $\Sigma_{\cup\{\infty\}}^d$,*
2. *the classes of compact normally framed submanifolds $Q^{d-n} \hookrightarrow \Sigma^d$ of co-dimension= n , under compact normally framed cobordisms in $\Sigma^d \times [0, 1]$*

$$\begin{array}{ccc} \text{Reduced } n\text{-Cohomotopy} & \tilde{\pi}^n(\Sigma_{\cup\{\infty\}}^d) & \xrightleftharpoons[\text{cohomotopy charge}]{\text{regular pre-image of } 0} \text{Cob}_{\text{Fr}}^d(\Sigma^d) & \text{Cobordism classes of} \\ \text{of 1pt compactification} & & & \text{normally framed sub-} \\ & & & \text{manifolds of codim} = n, \end{array} \quad (50)$$

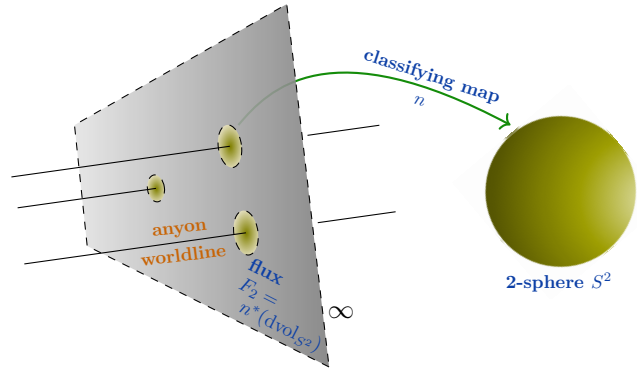
Here the Cohomotopy charge $[c] \in \tilde{\pi}^n(\Sigma_{\cup\{\infty\}}^d)$ of $Q^{d-n} \subset \Sigma^d$ with normal framing $NQ \xrightarrow{\text{fr}} Q \times \mathbb{R}^n \xrightarrow{p} \mathbb{R}^n$ is represented for any choice of tubular neighborhood $NQ \hookrightarrow \Sigma$ by the “scanning map”

$$\begin{array}{l} \Sigma^d \xrightarrow{c} \mathbb{R}_{\cup\{\infty\}}^n \\ s \mapsto \begin{cases} p(\text{fr}(s)) & | \quad s \in \iota(NQ) \\ \infty & | \quad \text{otherwise.} \end{cases} \end{array}$$

Example 3.5 (2-Cohomotopy of surfaces).

For surfaces without boundary, with flux quantized in 2-Cohomotopy, we have $d = n = 2$ in Prop. 3.4, whence the Pontrjagin theorem identifies solitonic flux concentrated around *points* (soliton cores) with oriented tubular neighborhoods reflecting either positive or negative units of (magnetic) flux.

The *total flux* is the sum of the charges of these solitons (the orientations $\in \{\pm 1\}$ of their tubular neighborhoods), identified with the *Hopf degree* (Def. 3.6) of the classifying map.



Definition 3.6 (Hopf degree, cf. [161, §IX, Cor 5.8]). For $n \in \mathbb{N}_{>0}$, and

$$S^n \xrightarrow{1^n} B^n\mathbb{Z} \quad (51)$$

a map representing the generator $1 \in \mathbb{Z} \simeq \pi_n(B^n\mathbb{Z})$, the induced generalized cohomology operation from n -Cohomotopy to ordinary integral n -cohomology

$$\pi^n(X) \equiv \pi_0 \text{Map}(X, S^n) \xrightarrow{\pi_0(1^n)_*} \pi_0 \text{Map}(X, B^n\mathbb{Z}) \simeq H^n(X; \mathbb{Z}) \quad (52)$$

is a bijection when X is a connected closed oriented manifold of dimension n . Evaluation on its fundamental class identifies $H^n(X; \mathbb{Z})$ with $\mathbb{Z}$; the resulting integer is the *Hopf degree* of a map $X \rightarrow S^n$.

For a connected oriented n -manifold X without boundary, the corresponding compactly supported degree uses the one-point compactification:

$$\tilde{\pi}^n(X_{\cup\{\infty\}}) \simeq \tilde{H}^n(X_{\cup\{\infty\}}; \mathbb{Z}) \simeq H_{\text{cpt}}^n(X; \mathbb{Z}) \simeq \mathbb{Z}.$$

Here the spaces considered are of finite type, as in (9). For example, maps $\mathbb{R}^2 \rightarrow S^2$ are null-homotopic; the integer degree arises after compactification.

For a surface with boundary, compactification at infinity does not fix the field on that boundary. Thus the compact disk has $\tilde{\pi}^2((D^2)_{\cup\{\infty\}}) = 0$. Fixing the boundary value instead gives $[D^2/\partial D^2, S^2]_* \simeq \mathbb{Z}$, the relative degree. The punctured closed disks in §3.6 retain the former boundary convention: the field is unrestricted on the physical boundary.

On the other hand, the flux density underlying (sourced by) a given Cohomotopy charge is characterized by the cohomotopical character map (the cohomotopical analog of the Chern-character map on K-cohomology, [77][228]):

Definition 3.7 (Cohomotopical character map). For $n = d$ the *character map* on cohomotopy

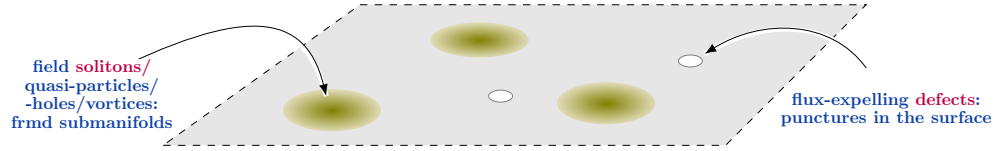
$$\begin{aligned} \tilde{\pi}^d(\Sigma_{\cup\{\infty\}}^d) &\xrightarrow{\text{ch}} H_{\text{dR}}^d(\Sigma^d)_{\text{cpt}} \\ [c] &\longmapsto [c^* \text{vol}_n] \end{aligned} \quad (53)$$

takes $[c] \in \tilde{\pi}^n(\Sigma_{\cup\{\infty\}}^d)$ — for any representative $c : \Sigma_{\cup\{\infty\}}^d \rightarrow \mathbb{R}^d_{\cup\{\infty\}}$ which is smooth on $c^{-1}(\mathbb{R}^d)$, such as the scanning maps in Prop. 3.4 — to the class in compactly supported de Rham cohomology of the pullback of a d -form $\text{vol} \in \Omega_{\text{dR}}^n(\mathbb{R}^d)$ compactly supported on a neighborhood of $0 \in \mathbb{R}^d$ and of unit integral.

Remark 3.8 (Flux density quantized in Cohomotopy).

- (i) For a connected oriented d -manifold Σ^d without boundary, this means that Cohomotopy charge $[c] \in \mathbb{Z} \simeq \tilde{\pi}^d(\Sigma_{\cup\{\infty\}}^d) \equiv \pi_0 \text{Map}^*(\Sigma_{\cup\{\infty\}}^d, S^d)$ may be understood as sourcing a solitonic flux density $F_d \in \Omega_{\text{dR}}^d(\Sigma^d)$ (solitonic in that it vanishes at infinity) which is supported with unit weight near $n_+ \in \mathbb{N}$ points in Σ^d (all points outside each other’s supporting neighborhoods) and with a negative unit weight near $n_- \in \mathbb{N}$ points (anti-solitons) such that $[c] = n_+ - n_-$.
- (ii) For the case $d = 2$ of interest here, this is just the kind of magnetic flux distribution concentrated around solitonic vortex cores as seen in type II superconducting and in fractional quantum Hall semiconducting materials Σ^2 , while any punctures in the surface Σ^2 (9) behave as loci from which flux is expelled (cf. Fig. I):

Figure F. Via the Pontrjagin theorem, 2-cohomotopical quantization of flux through a surface exhibits N flux quanta as a concentration of flux density supported on the tubular neighborhoods of N disjoint points.



2-Cohomotopical flux monodromy. For the quantum flux observables (20), we need not just the connected components π_0 but the fundamental group π_1 of the moduli space of Cohomotopical flux, which we may understand as “2-Cohomotopy in negative degree 1”, classified by the loop space ΩS^2 of the 2-sphere: ¹⁴

$$\pi_1 \text{Map}^*(-, S^2) \underset{(216)}{\simeq} \pi_0 \text{Map}^*(-, \Omega S^2). \quad (54)$$

Just like the 2-sphere has a canonical comparison map $1^2 : S^2 \rightarrow B^2\mathbb{Z}$ (51) whose induced cohomology operation [77, Def. 2.3] extracts ordinary 2-cohomology classes from 2-cohomotopy charges

$$\tilde{\pi}^2(-) \simeq \pi_0 \text{Map}^*(-, S^2) \xrightarrow{(1^2)_*} \pi_0 \text{Map}^*(-, B^2\mathbb{Z}) \simeq \tilde{H}^2(-; \mathbb{Z}),$$

so its loop space has the looped comparison map $\Omega 1^2 : \Omega S^2 \rightarrow \Omega B^2\mathbb{Z} \simeq B\mathbb{Z}$ inducing the cohomology operation

$$\pi_0 \text{Map}^*(-, \Omega S^2) \xrightarrow{(\Omega 1^2)_*} \pi_0 \text{Map}^*(-, B\mathbb{Z}) \simeq \tilde{H}^1(-; \mathbb{Z})$$

which makes precise how 2-cohomotopical flux observables refine ordinary electromagnetic flux observables (15).

It is now immediate to compute the observables on covariantized 2-Cohomotopical solitonic flux on the plane, and there turns out to be essentially a single such observable (Prop. 3.9 below), to be denoted $\hat{\zeta}$ — but it will take us the better part of the remainder of this section to identify this observable with the braiding phase (2).

¹⁴ The loop space ΩS^2 (54) of the 2-sphere has received attention as a classifying space also in [185, Def. 1.1], there called the classifying space for “line bundles” (with a Polish “P”), and has been related to configuration spaces of points in [186][185], reminiscent of the role they play for us in relating to group-completed configuration spaces in §3.1 below.

In [48, p. 94] the homotopy groups of ΩS^2 are recognized as natural sub-quotients of braid groups, which is a tantalizing observation in our context, whose further relevance however remains unclear to us at this point.

Proposition 3.9 (2-Cohomotopical flux monodromy on the plane). *The spaces $\mathcal{H}$ of topological quantum states (21) of solitonic flux quantized in 2-Cohomotopy on the plane are representations of the group of integers:*

$$\pi_1 \left(\text{Map}_0^*(\mathbb{R}_{\cup\{\infty\}}^2, S^2) // \text{Diff}^+(\mathbb{R}^2) \right) \simeq \mathbb{Z}, \quad (55)$$

$$\begin{aligned} \text{hence defined by a single unitary operator } \widehat{\zeta}: \quad \mathbb{Z} &\longrightarrow \text{U}(\mathcal{H}) \\ n &\longmapsto (\widehat{\zeta})^n. \end{aligned} \quad (56)$$

Proof. The mapping class group of the plane is trivial (Prop. 2.21), so that by Prop. 2.24 the only contribution is from the flux monodromy group itself, which is readily found to be

$$\begin{aligned} \pi_1 \text{Map}_0^*(\mathbb{R}_{\cup\{\infty\}}^2, S^2) &\simeq \pi_1 \text{Map}_0^*(S^2, S^2) && \text{by (209)} \\ &\simeq \pi_0 \text{Map}^*(S^3, S^2) && \text{by (205)} \\ &\simeq \pi_3(S^2) \\ &\simeq \mathbb{Z} && \text{Hopf fibration,} \end{aligned}$$

identifying the observable $\widehat{\zeta}$ with the representation image of the flux monodromy which is classified by the *Hopf fibration*. $\square$

Below in §3.3 we identify this same observable $\widehat{\zeta}$ on every closed oriented surface. On the torus, Lem. 3.32 shows that in a finite-dimensional irreducible Heisenberg representation it acts by a root of unity, $\zeta = e^{\pi i p/K}$ with $\gcd(p, K) = 1$. Requiring a plane sector to occur in such a torus representation therefore restricts its phase to these values, as expected for FQH braiding phases (2).

However, in the remainder of this subsection here, we work out what the observable $\widehat{\zeta}$ actually observes about 2-cohomotopical flux, and show that these indeed are braiding processes of flux quanta.

Understanding solitonic flux processes. In view of the Pontrjagin construction, we regard $\text{Map}^*(\mathbb{R}_{\cup\{\infty\}}^2, S^2)$ as the (moduli) space of solitonic flux on the plane, quantized in 2-Cohomotopy, and hence its loop space $\Omega \text{Map}_0^*(\mathbb{R}_{\cup\{\infty\}}^2, S^2)$ — where loops begin and end on the constant map, cf. (212), representing the *flux vacuum* — as the space of “*vacuum scattering processes*”, where flux solitons (of positive charge) and anti-solitons (of negative charge) pairwise emerge out of the vacuum, move around, and finally pair-annihilate back into the vacuum.

Of these vacuum processes, our observables (55) detect their homotopy classes $[-]$ (hence their “topological” or “deformation” class in physics jargon), labeled by the integers:

$$\begin{array}{c} \text{space of vacuum processes of} \\ \text{solitonic flux on the plane} \end{array} \quad \Omega \text{Map}_0^*(\mathbb{R}_{\cup\{\infty\}}^2, S^2) \xrightarrow{[-]} \pi_1 \text{Map}^*(\mathbb{R}_{\cup\{\infty\}}^2, S^2) \simeq \mathbb{Z} \quad \begin{array}{c} \text{topological deformation} \\ \text{classes of these processes} \end{array}$$

Our task is hence to understand these processes, on the left, and how they are observed, on the right.

To this end, the first step is to better understand the moduli space $\text{Map}_0^*(\mathbb{R}_{\cup\{\infty\}}^2, S^2)$ itself:

2-Cohomotopical moduli via the Segal-Okuyama theorem. By the Pontrjagin theorem (Prop. 3.4), one might naïvely expect that $\text{Map}^*((\mathbb{R}^2)_{\cup\{\infty\}}, S^2)$ is the *configuration space* (cf. [47][147]) of *signed* points (hence of $\pm$ unit charged soliton cores) in the plane, topologized such that continuous curves in the space reflect creation/annihilation of oppositely charged pairs as illustrated on the left of Fig. P. However, this is not quite correct as it misses the normal framing carried also by the cobordisms, according to Pontrjagin’s theorem 3.4.

A correct model [200] is by configurations of *intervals with signed endpoints* (stringy solitons between unit charged “quarks”) all parallel to one coordinate axis and topologized so as to reflect creation/annihilation of oppositely charged pairs of endpoints.

Figure P – Solitonic flux processes and framed links.

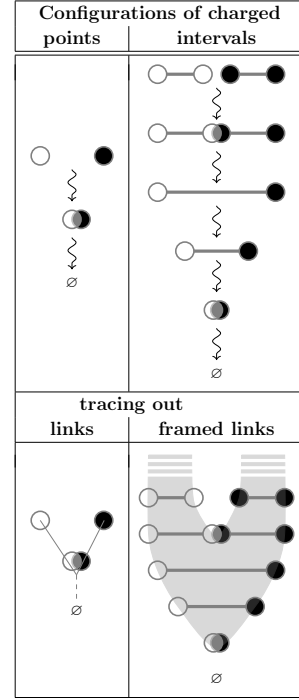
Indicated on the left is the naïve process of pair annihilation of oppositely signed points (of oppositely charged soliton core locations). But the configuration space of signed points [178, p. 94] which is topologized to make these processes continuous paths (with their reverse paths modelling the dual pair-creation processes) turns out [178, p. 6] to *not quite* have the correct homotopy type of the 2-Cohomotopical flux moduli space $\text{Map}^*(\mathbb{R}_{\cup\{\infty\}}^2, S^2)$.

Indicated on the right is a variant situation where the previous charges are located at the endpoints of intervals, and where their pair annihilation makes the corresponding intervals merge.

The configuration space of such intervals (all parallel to one fixed coordinate axis) topologized to make these processes continuous paths (cf. [200, Def. 3.1-2]) *does* have the homotopy type of the 2-Cohomotopical flux moduli space! This follows from the result of [200, Thm. 1.1].

The upshot is that where a continuous loop in the space on the left is an oriented link, a loop in the space on the right is an oriented link that is also equipped with a *framing*: a *framed oriented link*.

In traditional Chern-Simons theory, the enhancement of links to framed links is a standard but *ad hoc* way to “regularize” the corresponding *Wilson loop observable*, which at face value actually diverges according to traditional quantization methods. Here with 2-Cohomotopical flux quantization, this framing correction automatically arises from the moduli space of solitonic flux quantized in 2-Cohomotopy (a point that is further expanded on in [237]).



Under this identification of (the homotopy type of) our flux moduli space $\text{Map}^*(\mathbb{R}_{\cup\{\infty\}}^2, S^2)$ with a configuration space of charged intervals in $\mathbb{R}^2$ parallel to a fixed axis, loops in this space are identified with *framed oriented link diagrams*, or *framed links*, for short — and loop homotopies are identified with “link cobordism”:

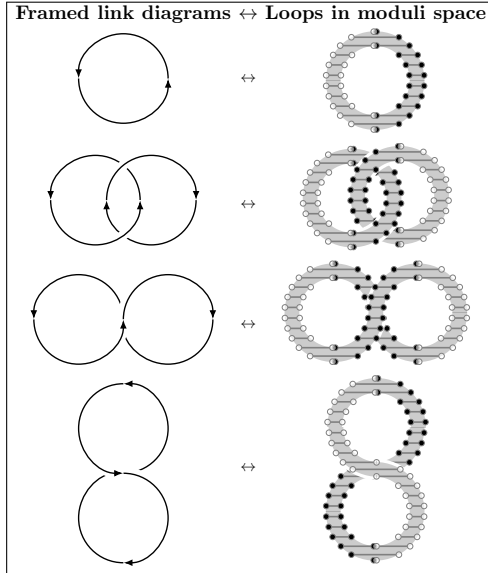
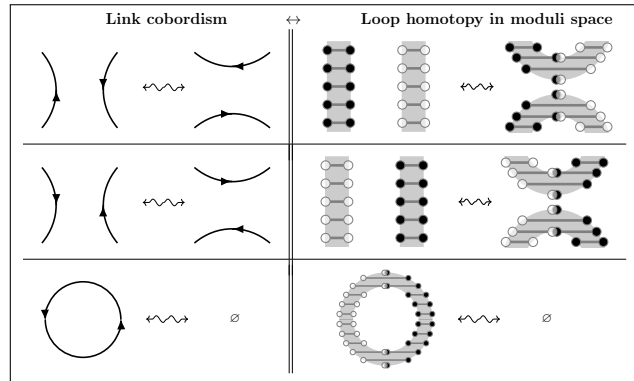
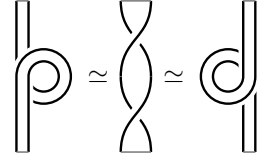


Figure FL. A framed link is a ribbon link, hence a “worldsheet of intervals”. When flattened out on a plane (“blackboard framing”) the inner twisting of the ribbon is entirely reflected in its self-crossings and thus the interval’s extension may be disregarded again.



Definition 3.10 (Crossing-, Linking- and Framing numbers).

- (i) Any crossing in a framed oriented link diagram L locally has one of the following forms, to which we assign the *crossing number* ± 1 , respectively, as shown:

$$\# \left(\begin{array}{c} \nearrow \\ \searrow \end{array} \right) = +1, \quad \# \left(\begin{array}{c} \nwarrow \\ \searrow \end{array} \right) = -1. \quad (57)$$

- (ii) For $(L_i)_{i=1}^N$ the connected components of L , the *linking number* $\text{lnk}(L_i, L_j)$ is *half* the sum of crossing numbers between L_i and L_j (cf. [199, p. 7]).
- (iii) The *framing number* $\text{fr}(L_i)$ is the sum of crossing numbers of L_i with itself.
- (iv) The sum $\#L$ of the crossing numbers of all crossings of L is hence the sum of all the framing and linking numbers:

$$\#(L) := \sum_{\substack{c \in \\ \text{crssngs}(L)}} \#(c) = \sum_i \text{frm}(L_i) + \sum_{i \neq j} \text{lnk}(L_i, L_j). \quad (58)$$

This has the following effect.

Proposition 3.11 (Vacuum loops of cohomotopical flux through the plane [227, Thm. 3.18, Rem. 4.4]).

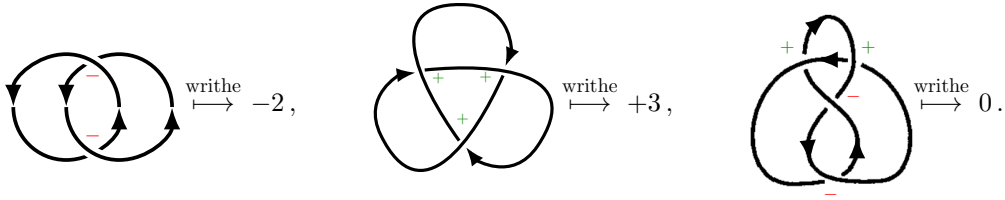
- (i) *Loops of 2-cohomotopical flux moduli on the plane are identified with framed links topologized to reflect link cobordism, whence their homotopy class is identified with the framed link's total crossing number $\#L$ (its “writhe”, cf. [199, p. 523]):*

$$\begin{array}{ccc} \Omega \text{Map}_0^*(\mathbb{R}_{\cup\{\infty\}}^2, S^2) & \xrightarrow{[-]} & \pi_1 \text{Map}_0^*(\mathbb{R}_{\cup\{\infty\}}^2, S^2) \simeq \mathbb{Z}. \\ L & \longmapsto & \#L. \\ \text{framed link} & & \text{writhe} \end{array} \quad (59)$$

- (ii) *Moreover, the pure states on these observables are labeled by $\zeta \in \text{U}(1)$ and give the expectation values*

$$\langle \zeta | L | \zeta \rangle = \zeta^{\#L}. \quad (60)$$

Example 3.12 (Writhe for basic knots and links).



Remark 3.13 (Comparison to Wilson loop link observables of abelian Chern-Simons theory).

- (i) For Chern-Simons theory with abelian gauge group $\text{U}(1)$ it is widely understood by appeal to path-integral arguments ([277, p. 363][88, p. 169] following [203]) that the quantum observables are labeled by framed links L , often considered as equipped with labels (charges) q_i on their i th connected component L_i and the expectation value of these observables in these states is the charge-weighted exponentiated framing- and linking numbers (Def. 3.10) as follows ([277, p. 363], cf. review in [181, (5.1)]):

$$W_k(L) = \exp \left(\frac{\pi i}{K} \left(\sum_i q_i^2 \text{frm}(L_i) + \sum_{i \neq j} q_i q_j \text{lnk}(L_i, L_j) \right) \right). \quad (61)$$

- (ii) For integer charges, replace each component L_i with $|q_i|$ unit-charged parallel copies using its given framing, reversing their orientations when $q_i < 0$ and omitting the component when $q_i = 0$. With the induced framings, this preserves both the $q_i q_j$ linking contributions and the q_i^2 self-framing contribution in (61). Denoting the resulting unit-charged framed link again by L , we find that its Chern-Simons expectation value coincides with our pure topological quantum state (60):

$$W_k(L) = \exp \left(\frac{\pi i}{K} \left(\sum_i \text{frm}(L_i) + \sum_{i \neq j} \text{lnk}(L_i, L_j) \right) \right) \stackrel{(58)}{=} \exp \left(\frac{\pi i}{K} \#(L) \right).$$

Remark 3.14 (Comparison to the literature on Hopfions). The Hopf invariant in Prop. 3.9 also classifies static “Hopf-charged solitons” (or *Hopfions*) on three-dimensional space, with fixed value at spatial infinity: their configurations define maps $S^3 \rightarrow S^2$ and hence elements of $\pi_3(S^2) \simeq \mathbb{Z}$. Such textures occur in the Skyrme-Faddeev model ([72], review in [71][174, §9.11]) and in three-dimensional $\mathbb{CP}^1$ models [213]. In the planar setting of Rem. 3.2, it is instead suitable spacetime histories of degree-charged solitons that carry this Hopf invariant. For the vacuum loops considered here, the adjunction in Prop. 3.9 identifies these histories with based maps $S^3 \rightarrow S^2$. The analysis of [275] (review in [79, §II.C]) relates the Hopf term for planar soliton histories to fractional spin and anyonic statistics in 2+1 dimensions. What seems not to have been noticed before in previous literature is the finer identification of Prop. 3.11 using the finer theorems by Segal-Okuyama [200][227] and the resulting identification with Chern-Simons Wilson loop observables (Rem. 3.13), which, we suggest, fully nails down this identification on Euclidean space.

Moreover, what seems to have received no attention before is that this identification of anyons further refines to more general 3-spaces of the form $\mathbb{R}^1 \times \Sigma^2$ (Def. 2.1), which is what we proceed to discuss.

3.2 On the sphere

We briefly discuss 2-cohomotopical flux on the 2-sphere $\Sigma_0^2 \simeq S^2$, meaning the actual 2-sphere whose point-at-infinity is disjoint, in contrast to the 2-sphere $(\Sigma_{0,0,1}^2)_{\cup\{\infty\}} \simeq \mathbb{R}_{\cup\{\infty\}}^2$ that arose as the one-point compactification of the plane in §3.1.

In order for this actual 2-sphere to be realized as an FQH system in the laboratory, one would not only need to produce a 2-dimensional electron gas of spherical topology, but also make it enclose the endpoint of a very long and thin solenoid to approximate a magnetic monopole at its center that would produce magnetic flux going *radially* through the spherical electron gas — which is a tall order. Nevertheless, in FQH theory this situation is often considered as an instructive hypothetical case study.

Also for us here, the following analysis of the 2-sphere case serves in §3.5 as an intermediate step in identifying the braiding phases of solitonic anyons on the plane, that we obtained in the previous §3.1, with corresponding braiding phases on n -punctured disks, hence in experimentally accessible situations.

While it is classical that

$$\pi_1 \text{Map}^*((\Sigma_{0,0,1}^2)_{\cup\{\infty\}}, S^2) \underset{(205)}{\simeq} \pi_3(S^2) \simeq \mathbb{Z}$$

(generated by the Hopf fibration $S^3 \rightarrow S^2$), the analogous statement for the un-based sphere needs another argument:

Lemma 3.15 (Fundamental group of unpointed endomaps of the 2-sphere [129, Thm. 5.3(1)][158, Lem. 3.1]). *The fundamental group of the space of (un-pointed) maps $S^2 \rightarrow S^2$ is, in the connected component of maps of Hopf degree $k \in \mathbb{Z}$ (Def. 3.6), isomorphic to:*

$$\pi_1(\text{Map}(S^2, S^2), \deg = k) \simeq \mathbb{Z}/(2k).$$

In our notation (9) and (212) this means, in particular, that in the component of vanishing Hopf degree we have:

$$\pi_1 \text{Map}(\Sigma_0^2, S^2) \simeq \mathbb{Z}. \quad (62)$$

Lemma 3.16 (Solitonic 2-cohomotopical flux monodromy on plane and 2-sphere are identified). *The canonical map*

$$\pi_1 \text{Map}^*((\Sigma_{0,0,1}^2)_{\cup\{\infty\}}, S^2) \xrightarrow[\sim]{\pi_1(p^*)} \pi_1 \text{Map}^*((\Sigma_0^2)_{\cup\{\infty\}}, S^2) \underset{(210)}{\simeq} \pi_1 \text{Map}(\Sigma_0^2, S^2) \quad (63)$$

is an isomorphism.

Proof. The long exact sequence of homotopy groups (219) induced by the evaluation map $\text{Map}(S^2, S^2) \xrightarrow{\text{ev}} S^2$ (206) is, in the relevant part, of the form

$$\begin{array}{ccccccc} \pi_2(S^2) & \xrightarrow{n} & \pi_1 \text{Map}^*(S^2, S^2) & \xrightarrow{\pi_1(p^*)} & \pi_1 \text{Map}(S^2, S^2) & \longrightarrow & \pi_1(S^2) \\ \wr | \text{ Hopf degree} & & \wr | \text{ Hopf fibration} & & \wr | (62) & & \wr | \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & 1, \end{array}$$

where the map on the left must be multiplication by some integer n , by the freeness of $\mathbb{Z}$. But then exactness on the left implies that the middle map must send n to 0, while exactness on the right means that the middle map is surjective hence that $n = 0$, which by exactness on the left implies that the middle map is also injective, hence bijective. $\square$

Remark 3.17 (Identifying braid phase observable on sphere). In terms of 2-cohomotopically quantized flux, this says that the algebras of topological flux observables on the plane and on the sphere are both isomorphic to $\mathbb{C}[\mathbb{Z}]$ and canonically identified as such, whence the discussion in §3.1 gives that in an irreducible representation on a (1-dimensional) Hilbert space the generator $1 \in \mathbb{Z}$ acts as multiplication by some phase factor $\zeta \in U(1) \subset \mathbb{C}$:

$$\begin{aligned} \mathbb{Z} &\longrightarrow U(\mathcal{H}_{S^2}) \\ 1 &\longmapsto \widehat{\zeta} : |\psi\rangle \mapsto \zeta |\psi\rangle. \end{aligned}$$

In §3.3 and §3.4 we identify this central observable across surfaces. If the sphere sector is required to occur in a finite-dimensional torus representation, its phase must be a root of unity, by Lem. 3.32, as expected in FQH systems.

3.3 On closed surfaces

While spherical FQH systems as in §3.2 are just barely plausible as having experimental realizations, for closed surfaces of more general genus $g \in \mathbb{N}$ this quickly becomes even less plausible as g increases. Nevertheless, the notoriously rich theoretical predictions for these somewhat hypothetical FQH systems on closed surfaces are crucial intermediate stages in understanding FQH systems in general and hence also in experimentally accessible situations. In particular, requiring a sector on the open disk from §3.1 to occur in a finite-dimensional torus representation constrains its braiding phase ζ to a root of unity. Proposition 3.23 identifies the central generators, while Lem. 3.32 supplies the finite-dimensionality constraint. This is the compatibility condition we impose when comparing these sectors across surfaces, as expected for FQH systems (2).

The new key we now offer for understanding FQH systems on closed surfaces via topological flux quantization is the following Prop. 3.19, whose roots in algebraic topology date back half a century ([120], following [129]), but which gains new meaning when understood now as being about observables on topological flux quantized in 2-cohomotopy:

Definition 3.18 (Integer Heisenberg group, level 2, (cf. [96, p. 7][94, Def. 2.4][95, Def. 2.3][30, (8)][31, (1.2)]). **By the integer Heisenberg group at level=2**¹⁵, to be denoted $\widehat{\mathbb{Z}^{2g}}$ for $g \in \mathbb{N}$, we refer to the group

$$\widehat{\mathbb{Z}^{2g}} := \left\{ (\vec{a}, \vec{b}, n) \in \mathbb{Z}^g \times \mathbb{Z}^g \times \mathbb{Z}, \quad (\vec{a}, \vec{b}, n) \cdot (\vec{a}', \vec{b}', n') := (\vec{a} + \vec{a}', \vec{b} + \vec{b}', n + n' + \vec{a} \cdot \vec{b}' - \vec{a}' \cdot \vec{b}) \right\}. \quad (64)$$

which is the central extension of the free abelian group $\mathbb{Z}^{2g}$ by (cf. [39, §IV]) the 2-cocycle shown shaded in (64), hence by the restriction of the canonical symplectic form on $\mathbb{R}^{2g}$ along $\mathbb{Z}^{2g} \hookrightarrow \mathbb{R}^{2g}$, hence is the subgroup $\widehat{\mathbb{Z}^{2g}} \hookrightarrow \widehat{\mathbb{R}^{2g}}$ of the ordinary Heisenberg group (cf. [210, §9.5]) on integer-valued elements.

Proposition 3.19 (Monodromy of 2-cohomotopical flux through closed surfaces). *The 2-cohomotopical flux monodromy (20) over a closed oriented surface Σ_g^2 (9), for $g \in \mathbb{N}$, forms the $\mathbb{Z}$ -extension of the free abelian group $\mathbb{Z}^{2g}$ (15)*

$$\begin{array}{ccccc} 1 \rightarrow \pi_1 \text{Map}_0^*(S_{\cup\{\infty\}}^2, S^2) & \longrightarrow & \pi_1 \text{Map}_0^*((\Sigma_g^2)_{\cup\{\infty\}}, S^2) & \longrightarrow & \pi_1 \text{Map}_0^*(V_g(S_a^1 \vee S_b^1), S^2) \rightarrow 1 \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ \mathbb{Z} & \hookrightarrow & \widehat{\mathbb{Z}^{2g}} & \twoheadrightarrow & \mathbb{Z}^{2g}, \end{array} \quad (65)$$

that is the integer Heisenberg group at level=2 (64).

Proof. The top short exact sequence is due to [120, Thm. 1 & p. 6], recalled as Lem. A.12 in the appendix. The identification of the group on the left is by Lem. 3.15. Now [165, Thm. 1] shows that the central extension in question has as its only non-trivial group commutators $[a_i, b_i] = 2$, for $i \in \{1, \dots, g\}$, in accordance with (64). $\square$

A more elegant proof of Prop. 3.19, for the case of the torus ($g = 1$), was recently found in [148, §3].

3.3.1 Modular action

Our way of casting Prop. 3.19 — with the extension cocycle highlighted in (64) identified as the standard symplectic form on $\mathbb{Z}^{2g}$ (instead of the cohomologous $2\vec{a} \cdot \vec{b}'$ used in the original derivations [165, Thm. 1][146, Cor. 7.6]) — makes manifest that the integral symplectic group $\text{Sp}_{2g}(\mathbb{Z})$ acts by group automorphisms on $\widehat{\mathbb{Z}^{2g}}$ (covering its defining action on $\mathbb{Z}^{2g}$ and acting trivially on the center $\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}^{2g}}$, cf. [94, (2.6)]):

$$\begin{array}{ccc} & & \text{Aut}(\widehat{\mathbb{Z}^{2g}}) \\ & \nearrow \exists \hat{\rho} & \downarrow \\ \text{Sp}_{2g}(\mathbb{Z}) & \xrightarrow[\text{canonical}]{\rho} & \text{Aut}(\mathbb{Z}^{2g}). \end{array} \quad (66)$$

Better yet, with (22), one such lifted action is singled out by the geometrical interpretation as the diffeomorphism action on the 2-cohomotopical flux monodromy over closed surfaces. This we identify now:

¹⁵By the “level” we here mean the integer n in the nontrivial group commutators $[a_i, b_i] = n$. The integer Heisenberg group at level = 1 (cf. [168, p. 232][61, p. 213]) is isomorphic to groups of certain upper triangular integer matrices (cf. [63, p. 35][61, p. 299]), and in this form is commonly considered in pure algebra and group theory (cf. [168, (1.1)]). In contrast, the case of relevance here, with level = 2 — which is the case of subgroups of the actual eponymous Heisenberg group from quantum mechanics — seems not to have found much attention in the pure algebra/group theory literature.

Proposition 3.20 (Mapping class group action on 2-cohomotopical flux monodromy on a closed surface). *Under the identification of Prop. 3.19, the action (46) of the mapping class group of Σ_g^2 on the 2-cohomotopical flux monodromy over Σ_g^2 is via its symplectic representation (33) on the underlying $\mathbb{Z}^{2g}$*

$$\begin{array}{ccc}
\text{MCG}(\Sigma_g^2) & \xrightarrow{(46)} & \text{Aut}(\pi_1 \text{Map}_0(\Sigma_g^2, S^2)) \\
\downarrow (33) & & \uparrow (65) \\
& & \text{Aut}(\widehat{\mathbb{Z}^{2g}}) \\
& \nearrow \exists! \hat{\rho} & \downarrow \\
\text{Sp}_{2g}(\mathbb{Z}) & \xrightarrow{\rho} & \text{Aut}(\mathbb{Z}^{2g}).
\end{array} \tag{67}$$

Moreover, there is a choice of the top right isomorphism $\pi_1 \text{Map}_0(\Sigma_g^2, S^2) \simeq \widehat{\mathbb{Z}^{2g}}$ (65) under which

$$\hat{\rho}(M) : (v, n) \mapsto (Mv, n). \tag{68}$$

Proof. Commutativity of the outer diagram follows by inspection of the construction of (65) as spelled out in §A.4: By Lem. A.13 there, the induced action of $\text{MCG}(\Sigma_g^2)$ on $\pi_1 \text{Map}(\Sigma_g^2, S^2) \twoheadrightarrow \mathbb{Z}^{2g}$ is identified with its action on $H^1(\Sigma_g^2; \mathbb{Z})$, for which it is classical (33) that it is through $\text{Sp}_{2g}(\mathbb{Z})$, as claimed.

We proceed to analyze the nature of $\hat{\rho}$. Here the case $g = 0$ is immediate, therefore we assume now that $g \geq 1$. Here we proceed by comparing the MCG-action by precomposition with the postcomposition action by an orientation-reversing involutive diffeomorphism fixing the base point of the 2-sphere:

$$S^2 \xrightarrow{r} S^2. \tag{69}$$

Postcomposition with r induces an automorphism of $\pi_1 \text{Map}(\Sigma_g^2, S^2)$ which acts as -1 on $\mathbb{Z}^{2g}$, hence in particular preserves the symplectic form, and which commutes with the precomposition action of any $[f] \in \text{MCG}(\Sigma_g^2)$.

(i) To see that $\hat{\rho}$ exists uniquely:

Observe that every automorphism Φ of $\widehat{\mathbb{Z}^{2g}}$ preserving the symplectic form must be the identity on the center: First, being an automorphism, Φ must restrict to an automorphism of the center, $\mathbb{Z} \simeq \langle \zeta \rangle$, hence must act on the center as $\zeta \mapsto \zeta^\epsilon$ for $\epsilon \in \{+1, -1\}$. But furthermore, with Φ also being a symplectomorphism, we have

$$\zeta^{2\epsilon} = \Phi([a_i, b_i]) = [\Phi(a_i), \Phi(b_i)] = \zeta^{2\omega(\Phi(a_i), \Phi(b_i))} = \zeta^2, \tag{70}$$

which implies $\epsilon = 1$, as claimed.

With this in hand, Lem. 3.21 gives

$$r \circ [v, n] = (-v, n + \lambda(v)) \tag{71a}$$

$$[v, n] \circ [f]^{-1} = (M_f v, n + \ell_f(v)), \tag{71b}$$

whence we deduce

$$r \circ ([v, n] \circ [f]) = (r \circ [v, n]) \circ [f] \tag{72a}$$

$$\Rightarrow \ell_f(v) + \lambda(M_f v) = \lambda(v) - \ell_f(v) \tag{72b}$$

$$\Leftrightarrow \lambda(M_f v) - \lambda(v) = -2\ell_f(v). \tag{72c}$$

This shows that all ℓ_f are fixed by λ as functions of f and as such only depend on M_f . This uniquely determines, through (71b), the action $\hat{\rho}$, as claimed.

(ii) To see that the shear ℓ_f may be taken to vanish upon another choice of isomorphism, as claimed in (68):

The result (72c) also shows that λ is invariant modulo 2 under symplectic transformations. But such invariance under the specific symplectomorphisms

$$\begin{array}{ccc}
\mathbb{Z}^{2g} & \longrightarrow & \mathbb{Z}^{2g} \\
a_i & \mapsto & a_i + b_i \\
b_i & \mapsto & b_i
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\mathbb{Z}^{2g} & \longrightarrow & \mathbb{Z}^{2g} \\
a_i & \mapsto & a_i \\
b_i & \mapsto & b_i + a_i
\end{array} \tag{73}$$

implies, by linearity, that λ already vanishes modulo 2, hence that

$$\lambda = 2\mu, \quad \text{for some linear } \mathbb{Z}^{2g} \xrightarrow{\mu} \mathbb{Z}. \tag{74}$$

Using this we may absorb the shears in (71) in an automorphism

$$\begin{aligned} \widehat{\mathbb{Z}^{2g}} &\xrightarrow{U} \widehat{\mathbb{Z}^{2g}} \\ (v, n) &\mapsto (v, n + \mu(v)), \end{aligned} \quad (75)$$

in that

$$\begin{aligned} \widehat{\mathbb{Z}^{2g}} &\xrightarrow{U((-)\circ[f]^{-1})U^{-1}} \widehat{\mathbb{Z}^{2g}} \\ (v, n) &\mapsto (M_f v, (n - \mu(v) + \ell_f(v) + \mu(M_f v))) \\ &= (M_f v, n) \end{aligned} \quad (76)$$

as claimed in (68). □

In this proof of Prop. 3.20 we used the following independent observation:

Lemma 3.21.

(i) The automorphisms Φ of $\widehat{\mathbb{Z}^{2g}}$ which are the identity on the center, $\Phi(0, n) = (0, n)$, are of the form

$$\Phi(v, n) = (Mv, n + \ell(v)), \quad (77)$$

for a symplectic matrix M and any linear map ℓ , the shear, hence this subgroup of the automorphism group is:

$$\text{Aut}_{\text{ctr}}(\widehat{\mathbb{Z}^{2g}}) \simeq \text{Hom}(\mathbb{Z}^{2g}, \mathbb{Z}) \rtimes \text{Sp}_{2g}(\mathbb{Z}). \quad (78)$$

(ii) Such an automorphism is inner precisely if $M = 1$ and $\ell \in 2\text{Hom}(\mathbb{Z}^{2g}, \mathbb{Z})$.

Proof.

(i) Given such an automorphism Φ , on the quotient by the center, it induces an automorphism $M \in \text{Aut}(\mathbb{Z}^{2g})$ (not *a priori* symplectic) and hence $\Phi(v, 0) = (Mv, \ell(v))$ for some function ℓ (not *a priori* linear). But the homomorphism property $\Phi((u, 0) \cdot (v, 0)) = \Phi(u, 0) \cdot \Phi(v, 0)$ demands that

$$\begin{aligned} \ell(u + v) + \omega(u, v) &= \ell(u) + \ell(v) + \omega(Mu, Mv) \\ \Leftrightarrow \ell(u + v) - \ell(u) - \ell(v) &= \omega(Mu, Mv) - \omega(u, v). \end{aligned}$$

Here either side is a $\mathbb{Z}$ -valued function that is both symmetric and skew-symmetric in its arguments (u, v) . This implies that both sides vanish for all arguments, which says that ℓ is linear and that M is symplectic.

With this, the homomorphism property on general elements, $\Phi(v, n) \cdot \Phi(v', n') = \Phi((v, n) \cdot (v', n'))$, is already implied:

$$\begin{aligned} &(Mv + Mv', n + n' + \ell(v) + \ell(v') + \omega(Mv, Mv')) \\ &= (M(v + v'), n + n' + \omega(v, v') + \ell(v + v')), \end{aligned}$$

and hence the claim follows.

(ii) The inner automorphism induced by (u, m) acts as

$$(v, n) \mapsto (u, m) \cdot (v, n) \cdot (-u, -m) = (v, n + 2\omega(u, v)).$$

Hence the claim follows by observing that every linear $\ell(-)$ as above is of the form $\omega(u, -)$ for

$$u := \sum_i (\ell(b_i) a_i - \ell(a_i) b_i). \quad \square$$

3.3.2 Functoriality

Next, we observe some form of functoriality in maps between closed surfaces of the result of Prop. 3.19, the crucial implication being the identification of the braid phase observable ζ (2) across all closed surfaces. To this end, write:

$$\Sigma_g \xleftarrow{q_g^{g+1}} \Sigma_{g+1} \quad (79)$$

for the surjective continuous map (227) given by contracting one pair of edges in the standard fundamental polygon (cf. Prop. A.8) of Σ_{g+1} . For instance, for $g \in \{0, 1\}$ the maps

$$\text{sphere } \Sigma_0^2 \simeq S^2 \xleftarrow{q_0^1} \text{torus } \Sigma_1^2 \simeq T^2 \xleftarrow{q_1^2} \text{2-handled torus } \Sigma_2^2 \quad (80)$$

are given, for q_1^2 , by sending the purple and blue colored edges to the point $\bullet$, and for q_0^1 by sending also the remaining edges to the point.

Lemma 3.22 (Pullback of 2-cohomotopical flux monodromy on closed surfaces). *Under the above identification (65), the surjections q (79) map to the canonical inclusion of Heisenberg groups obtained by adjoining the generators corresponding to the contracted edges:*

$$\begin{array}{ccc} \pi_1 \text{Map}^*(\Sigma_g^2, S^2) & \xrightarrow{(q_g^{g+1})^*} & \pi_1 \text{Map}^*(\Sigma_{g+1}^2, S^2) \\ \downarrow \wr & & \downarrow \wr \\ \widehat{\mathbb{Z}^{2g}} & \hookrightarrow & \widehat{\mathbb{Z}^{2g+2}} \end{array} \quad (81)$$

Proof. The proof of Lem. A.12 shows that the short exact sequence (65) is natural in Σ_g^2 , whence we have a commuting diagram of this form:

$$\begin{array}{ccccccc} \mathbb{Z} & \hookrightarrow & \widehat{\mathbb{Z}^{2g}} & \twoheadrightarrow & \mathbb{Z}^{2g} & & \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ 1 \rightarrow \pi_1 \text{Map}_0(S^2, S^2) & \rightarrow & \pi_1 \text{Map}_0(\Sigma_g^2, S^2) & \rightarrow & \pi_1 \text{Map}_0^*(\bigvee_g(S_a^1 \vee S_b^1), S^2) & \rightarrow & 1 \\ \parallel & & \downarrow (q_g^{g+1})^* & & \downarrow & & \\ 1 \rightarrow \pi_1 \text{Map}_0(S^2, S^2) & \rightarrow & \pi_1 \text{Map}_0(\Sigma_{g+1}^2, S^2) & \rightarrow & \pi_1 \text{Map}_0^*(\bigvee_{g+1}(S_a^1 \vee S_b^1), S^2) & \rightarrow & 1 \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\ \mathbb{Z} & \hookrightarrow & \widehat{\mathbb{Z}^{2g+2}} & \twoheadrightarrow & \mathbb{Z}^{2g+2} & & \end{array}$$

This gives the claim. $\square$

Combination of Lem. 3.16 and Lem. 3.22 leads to:

Proposition 3.23 (Identifying central braid phase generator across surfaces). *The canonical comparison map between solitonic flux monodromy on the plane and on Σ_g , $g \in \mathbb{N}$, identifies the central generators as the braiding phase observable of §3.1:*

$$\begin{array}{ccccc} \pi_1 \text{Map}^*((\mathbb{R}^2)_{\cup\{\infty\}}, S^2) & \xrightarrow[\text{(63)}]{\pi_1(q^*)} & \pi_1 \text{Map}(\Sigma_0^2, S^2) & \xrightarrow[\text{(81)}]{\pi_1((q_0^g)^*)} & \pi_1 \text{Map}(\Sigma_g^2, S^2) \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ \mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z} & \hookrightarrow & \widehat{\mathbb{Z}^{2g}} \\ 1 & \mapsto & 1 & \mapsto & \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 1 \right) \end{array}$$

Remark 3.24 (Braiding phases of solitonic anyons on closed surfaces). In generalization of Rem. 3.17, Prop. 3.23 says that given 2-cohomotopical flux quantum states on Σ_g^2 , hence a unitary representation of the 2-cohomotopical flux monodromy on Σ_g^2 , hence of the integer Heisenberg group $\widehat{\mathbb{Z}^{2g}}$ (65), for $g \in \mathbb{N}$, then the central observable $\widehat{\zeta}$

$$\begin{array}{lcl} \mathbb{Z} & \hookrightarrow & \widehat{\mathbb{Z}^{2g}} \longrightarrow \text{U}(\mathcal{H}) \\ 1 & \mapsto & \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 1 \right) \mapsto \widehat{\zeta} : |\psi\rangle \mapsto \zeta |\psi\rangle, \\ & & \left(\begin{bmatrix} a \\ b \end{bmatrix}, 0 \right) \mapsto \widehat{W}_{\begin{bmatrix} a \\ b \end{bmatrix}} \end{array} \quad (82)$$

is to be understood as observing the braiding phases of solitonic flux according to §3.1, where the braiding happens within an open disk inside the surface.

For $g \geq 1$, finite-dimensional irreducible representations restrict this central phase to a root of unity: apply Lem. 3.32 to an irreducible summand of the restriction to one handle's Heisenberg subgroup. The torus case is analyzed in §3.4, with $\zeta = e^{\pi i p/K}$ as in the FQH braiding phase (2).

Remark 3.25 (Braiding phases of defect anyons on closed surfaces). There is the following alternative way to exhibit the central generator of (65) as witnessing braiding processes: For $g \geq 1$ and $n \geq 3$, let $\widehat{\mathbb{Z}^{2g}}$ denote the level=2 extension of $\mathbb{Z}^{2g}$ as in (64) but by the finite cyclic group $\mathbb{Z}_{2(n+g-1)}$. This is evidently the quotient of the integer Heisenberg group (64) by the $2(n+g-1)$ st power of the central element (whence its irreducible unitary representations ρ force the braiding phase ζ (82) to be a $(2(n+g-1))$ st root of unity), but it turns out to also be [24, (10)] (cf. also [30, Cor. 8][31, Prop. 1.1]) the quotient of the surface braid group $\text{Br}_n(\Sigma_g^2)$ (23) which identifies each Artin braid generator b_i (26) with this central generator:

$$\begin{aligned} \text{Br}_n(\Sigma_g^2) &\longrightarrow \widehat{\mathbb{Z}^{2g}} \xrightarrow{\rho} \text{U}(\mathcal{H}) \\ b_i &\longmapsto \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, 1 \right) \longmapsto \hat{\zeta}. \end{aligned}$$

The surface braid group describes motions of *defect anyons* (cf. Rem. 2.13 and §3.8). The displayed quotient identifies a common algebraic description with solitonic flux, subject to the additional central relation: writing z for the central generator and $m := 2(n+g-1)$, we have

$$\mathbb{C}[\widehat{\mathbb{Z}^{2g}}] \simeq \mathbb{C}[\widehat{\mathbb{Z}^{2g}}]/(z^m - 1).$$

Thus a solitonic representation descends to this quotient precisely when z^m acts as the identity. At a fixed central phase this is the condition $\zeta^m = 1$; the resulting surface-braid representation sends every Artin generator to the scalar ζ . This compares the corresponding representations of the quotient observable algebra; their realization by physical defect motion requires additional input.

Remark 3.26 (Comparison to expectations in FQH systems). It is folklore in the literature on anyons that the phase ζ (17) in the quantum algebra of observables on the torus may be understood as witnessing anyon braiding phases in FQH systems (cf. [194, (30)][261, (3.33)][251, §4.3]). The above results substantiate this intuition by a rigorous derivation of the statement from 2-cohomotopical flux quantization and as such lend support to our hypothesis that 2-Cohomotopy is the correct effective flux-quantization law for FQH systems.

Remark 3.27 (Instability of the Hopf monodromy under target enlargement). Comparison of (15) with (65) amplifies how the anyon braiding phase $\hat{\zeta}$ (82) appears due to the exotic flux quantization in $S^2 \simeq \mathbb{C}P^1$ and disappears with the ordinary flux quantization in $\mathbb{C}P^\infty \simeq \text{BU}(1)$ along the inclusion $\iota : S^2 \simeq \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^\infty \simeq \text{BU}(1)$ (5):

$$\begin{aligned} \pi_1\left(\text{Map}(\Sigma_g^2, \mathbb{C}P^1)\right) &\xrightarrow{\iota_*} \pi_1\left(\text{Map}(\Sigma_g^2, \mathbb{C}P^\infty)\right) \\ \widehat{\mathbb{Z}^{2g}} &\longrightarrow \mathbb{Z}^{2g} \\ \widehat{W}_{\begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}} &\longmapsto \widehat{W}_{\begin{bmatrix} \vec{a} \\ \vec{b} \end{bmatrix}} \\ \hat{\zeta} &\longmapsto 0. \end{aligned}$$

In the Hall-dual situation of anomalous Hall effects in Chern insulators [235], maps to $\mathbb{C}P^1$ describe a single occupied band in a two-band system. Enlarging the target to $\mathbb{C}P^\infty$ gives the classifying space for complex line bundles and retains the stable first Chern class. In particular, the inclusion induces an isomorphism $\pi_2(\mathbb{C}P^1) \xrightarrow{\simeq} \pi_2(\mathbb{C}P^2) \simeq \mathbb{Z}$, so a nonzero Chern number survives this enlargement. It is the additional Hopf invariant, and hence the central monodromy generator displayed above, that disappears: the map $\pi_3(\mathbb{C}P^1) \simeq \mathbb{Z} \longrightarrow \pi_3(\mathbb{C}P^2) = 0$ is zero.

This is an instance of unstable two-band topology. The band-theory distinction between *fragile* topology (cf. [21]) and *delicate* topology specifies which occupied and unoccupied bands may be added [22]; the two-band Hopf invariant is an example of the latter [195].

The next proposition describes the resulting monodromy after target enlargement: the central Hopf contribution disappears already at $S^2 \simeq \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2$, while the $\mathbb{Z}^{2g}$ contribution remains:

Proposition 3.28 (Effect of intermediate flux-quantization laws). *For $g \in \mathbb{N}$ and $N \in \mathbb{Z}$ we have:*

$$N \geq 2 \quad \Rightarrow \quad \pi_1\left(\text{Map}_0(\Sigma_g^2, \mathbb{C}P^N)\right) \simeq \mathbb{Z}^{2g}. \quad (83)$$

Proof. This is [146, Lem. 7.5]. □

3.4 On the torus

While transverse magnetic flux through a toroidal 2D electron gas is not readily realized experimentally (cf. footnote 9), effective field theories of flux on arbitrary surfaces tend to be characterized by their theoretical predictions for the torus, notably through the dimension and modular transformation properties of the Hilbert space of states (transport data characterizing topological order [267][268], cf. Prop. 3.38 below and the qualifications in §3.4.2).

Therefore, a major example of the phenomena in §2.2 is the following derivation of quantum states of 2-cohomotopically quantized topological flux on the torus, which reproduces the *modular data* [93] of U(1)-Chern-Simons theory (Rem. 3.39 below) – in fact of *spin* Chern-Simons theory (8).

3.4.1 Flux monodromy classification

Our task in identifying the 2-cohomotopical flux quantum states over the torus is, by Prop. 3.19, to study finite-dimensional unitary representations of the integer Heisenberg group (64) after covariantization (21). In this subsection we classify those representations which remain irreducible upon restriction to the Heisenberg subgroup; the scope of this assumption is discussed in Rem. 3.45 (iii).

Proposition 3.29 (Diffeomorphism action over the torus is the canonical modular action). *The action (46) of $\text{MCG}(\Sigma_1^2) \simeq \text{SL}_2(\mathbb{Z})$ (30) on $\pi_1 \text{Map}_0^*((\Sigma_1^2)_{\cup\{\infty\}}, S^2) \simeq \widehat{\mathbb{Z}^2}$ (64) is the defining action of $\text{Sp}_2(\mathbb{Z}) \simeq \text{SL}_2(\mathbb{Z})$ on $\mathbb{Z}^2$ and trivial on the center, whence the flux monodromy group (46) over the torus with its pp-spin structure (34) is*

$$\text{MCG}(\Sigma_1^2) \ltimes \pi_1 \left(\text{Map}_0^*((\Sigma_1^2)_{\cup\{\infty\}}, S^2) \right) \simeq \text{SL}_2(\mathbb{Z}) \ltimes \widehat{\mathbb{Z}^2} \quad (84)$$

while for the torus equipped with the aa-spin structure it is correspondingly the subgroup $\text{MCG}(\Sigma_1^2)^{\text{aa}} \ltimes \widehat{\mathbb{Z}^2}$ (35).

Proof. This is a special case of Prop. 3.20. □

Representations of $\text{SL}_2(\mathbb{Z})$ and of $\widehat{\mathbb{Z}^2}$ separately are well-studied. Here we consider their compatibility in the semidirect product (84). We find representations which recover the familiar U(1)-Chern-Simons modular matrices at unit-fractional braiding angles π/K and extend the construction to general rational braiding angles $\pi p/K$.

We proceed incrementally: after some generalities, we construct the states at $\theta = 1/K$ for even K (Prop. 3.38) and compare them with Chern-Simons/CFT theory (Rem. 3.39). We then treat all rational braiding angles for the pp-spin structure (Prop. 3.41) and both allowed scalar sectors for the aa-spin structure (Prop. 3.43). The conclusion is Thm. 3.44.

The condition organizing this discussion is that a Heisenberg representation extend to its semidirect product with the relevant mapping class group (46), hence admit a *covariantization* in the sense of Def. 2.11:

Definition 3.30 (Covariantizable representations of the integer Heisenberg group). We say that a linear representation of the integer Heisenberg group $\widehat{\mathbb{Z}^2}$ is *covariantizable* for a given spin structure if it extends to a representation of the semidirect product with (cf. Prop. 3.29)

- (i) for the pp-spin structure:
the torus mapping class group (30) generated by S and T (31),
- (ii) for the aa-spin structure:
the spin mapping class subgroup (35) generated only by S and T^2 .

In order to analyze such extended representations, we first note the following elementary facts:

Lemma 3.31 (Extending representations along normal subgroup inclusions, cf. [134, p. 175]). *Given $\iota : H \hookrightarrow G$ a normal subgroup inclusion and $\rho : H \rightarrow \text{GL}(\mathcal{H})$ a finite-dimensional $\mathbb{C}$ -linear representation, say that $\widehat{\rho} : G \rightarrow \text{GL}(\mathcal{H})$ is an extension if $\iota^* \widehat{\rho} = \rho$. — We have:*

- (i) Any extension $\widehat{\rho}$ exhibits ρ as isomorphic to its g -translate representations:

$$\forall_{g \in G} \quad \rho \simeq \rho^g, \quad \text{where } \rho^g := \rho(\text{Ad}_g(-)) . \quad (85)$$

- (ii) If ρ is irreducible then any two extensions $\widehat{\rho}, \widehat{\rho}'$ differ at most by tensoring with some multiplicative character of G/H :

$$\forall_{g \in G} \quad \widehat{\rho}'(g) = d(gH) \cdot \widehat{\rho}(g) , \quad \text{for some } d : G/H \rightarrow \mathbb{C}^\times .$$

Proof. (i) We have for $h \in H$:

$$\begin{aligned}\widehat{\rho}(g) \circ \rho(h) \circ \widehat{\rho}(g)^{-1} &= \widehat{\rho}(g) \circ \widehat{\rho}(h) \circ \widehat{\rho}(g^{-1}) \\ &= \widehat{\rho}(ghg^{-1}) \\ &= \rho(ghg^{-1}) \\ &\equiv \rho^g(h),\end{aligned}$$

showing that $\widehat{\rho}(g)$ serves as an intertwiner that exhibits the claimed isomorphism.

(ii) We have, for $h \in H$ and $g \in G$:

$$\begin{aligned}(\widehat{\rho}'(g) \circ \widehat{\rho}(g)^{-1}) \circ \rho(h) \circ (\widehat{\rho}'(g) \circ \widehat{\rho}(g)^{-1})^{-1} &= \widehat{\rho}'(g) \circ \widehat{\rho}(g^{-1}) \circ \rho(h) \circ \widehat{\rho}(g) \circ \widehat{\rho}'(g^{-1}) \\ &= \widehat{\rho}'(g) \circ \rho(g^{-1}hg) \circ \widehat{\rho}'(g^{-1}) \\ &= \rho(h),\end{aligned}$$

showing that the ratio of the two extensions for any g commutes with all the ρ -operators. Therefore Schur's lemma (231) implies that this ratio is a scalar $d(g)$ times the identity operator. The representation laws imply that d is multiplicative, and agreement of the extensions on H implies that d is trivial there. Thus d factors through G/H . If both extensions are unitary, then d takes values in $U(1)$. $\square$

Lemma 3.32 (Finite-dimensional Heisenberg irreps and roots of unity). *If a unitary representation of the integer Heisenberg group, $(-): \widehat{\mathbb{Z}^2} \rightarrow U(\mathcal{H})$, is finite-dimensional and irreducible, then the central observable $\widehat{\zeta}$ (82) is given by multiplication with a root of unity: $\widehat{\zeta} = \zeta \cdot \text{id}$ and $\exists_{n \in \mathbb{N}_{>0}} \zeta^n = 1$.*

Proof. First, with the representation assumed irreducible and since $\widehat{\zeta}$ commutes with all other representation operators, Schur's lemma (231) implies that there is $\zeta \in \mathbb{C}$ with $\widehat{\zeta} = \zeta \cdot \text{id}$. Further, since $\widehat{W}_{[1]}$ is unitary we may find an eigenvector, to be denoted $|0\rangle$, with non-vanishing eigenvalue to be denoted ξ :

$$\widehat{W}_{[1]}|0\rangle = \xi|0\rangle. \quad (86)$$

Now the commutation relation says that $\widehat{W}_{[0]}$ is a corresponding raising operator, in that the elements

$$|n\rangle := (\widehat{W}_{[0]})^n|0\rangle \in \mathcal{H} \quad (87)$$

are further eigenvectors of $\widehat{W}_{[1]}$ with eigenvalue $\zeta^{2n}\xi$:

$$\widehat{W}_{[1]}|n\rangle \equiv \widehat{W}_{[1]}(\widehat{W}_{[0]})^n|0\rangle \stackrel{(94)}{=} \zeta^{2n}(\widehat{W}_{[0]})^n\widehat{W}_{[1]}|0\rangle \stackrel{(86)}{=} \zeta^{2n}(\widehat{W}_{[0]})^n\xi|0\rangle \equiv \zeta^{2n}\xi|n\rangle.$$

But by the assumption of finite-dimensionality there can only be finitely many distinct eigenvalues, which is evidently equivalent to ζ being a root of unity. $\square$

Definition 3.33 (Cyclic Heisenberg group). For $g, o \in \mathbb{N}_{>0}$, we denote the o -cyclic version of the integer Heisenberg group (64) by

$$\widehat{\mathbb{Z}_o^{2g}} := \left\{ ([\vec{a}], [\vec{b}], [n]) \in \mathbb{Z}_o^g \times \mathbb{Z}_o^g \times \mathbb{Z}_o, \quad ([\vec{a}'], [\vec{b}'], [n']) := ([\vec{a} + \vec{a}'], [\vec{b} + \vec{b}'], [n + n' + \vec{a} \cdot \vec{b}' - \vec{a}' \cdot \vec{b}]) \right\}, \quad (88)$$

where $\mathbb{Z}_o := \mathbb{Z}/o\mathbb{Z}$ denotes the o -cyclic group and $[-]: \mathbb{Z} \rightarrow \mathbb{Z}_o$ denotes the quotient map.

Lemma 3.34 (Covariantizable irreps factor through cyclic Heisenberg groups). *If a finite-dimensional irreducible unitary representation $(-): \widehat{\mathbb{Z}^2} \rightarrow U(\mathcal{H})$ is covariantizable (Def. 3.30) for the pp-spin structure, then it is the pullback of a representation of the o -cyclic Heisenberg group (88) along the quotient coprojection $[-]: \widehat{\mathbb{Z}^2} \twoheadrightarrow \widehat{\mathbb{Z}_o^2}$, where o may be taken to equal $\text{ord}(\zeta) := \min\{n \in \mathbb{N}_{>0} \mid \zeta^n = 1\}$,*

$$(89)$$

the order of ζ in $\widehat{\zeta} = \zeta \text{id}$ (Lem. 3.32).

Moreover, for the aa-spin structure the analogous statement holds with o replaced by $2o$.

Proof. Put $U := \widehat{W}_{[1]}$ and $V := \widehat{W}_{[0]}$. The Heisenberg group law gives $UV = \zeta^2 VU$. Since $\zeta^o = 1$, the operators U^o and V^o commute with the entire Heisenberg representation. By Schur's lemma (231), write

$$U^o = \alpha \text{id}, \quad V^o = \beta \text{id}, \quad \alpha, \beta \in U(1).$$

For any $a, b \in \mathbb{Z}$, the group law then gives

$$\widehat{W}_{[ob]} = \zeta^{-o^2 ab} U^{oa} V^{ob} = \alpha^a \beta^b \text{id}.$$

Thus these scalar values form a character of $(o\mathbb{Z})^2$. Covariance preserves them under the relevant modular action, by Lem. 3.31 (i). For both spin structures, S -covariance gives

$$\alpha = \beta^{-1}, \quad \beta = \alpha \quad \implies \quad \alpha = \beta \in \{+1, -1\}. \quad (90)$$

For the pp-spin structure, T sends $\widehat{W}_{[0]}$ to $\widehat{W}_{[o]}$, so that

$$\beta = \alpha\beta \quad \implies \quad \alpha = \beta = 1. \quad (91)$$

Hence $U^o = V^o = \widehat{\zeta}^o = \text{id}$, and the representation factors through $\widehat{\mathbb{Z}}_o^2$. For the aa-spin structure, T^2 only imposes $\beta = \alpha^2 \beta$, which already follows from (90). In this case $U^{2o} = V^{2o} = \widehat{\zeta}^{2o} = \text{id}$, so the representation factors through $\widehat{\mathbb{Z}}_{2o}^2$. $\square$

Lemma 3.35 (Pullback from cyclic Heisenberg preserves irreducibility). *For $o \in \mathbb{N}_{>0}$, given a representation of the cyclic Heisenberg group $\rho : \widehat{\mathbb{Z}}_o^2 \rightarrow \text{GL}(\mathcal{H})$, its pullback $p^* \rho : \widehat{\mathbb{Z}}^2 \xrightarrow{p} \widehat{\mathbb{Z}}_o^2 \xrightarrow{\rho} \text{GL}(\mathcal{H})$ is irreducible iff ρ is.*

Proof. This holds generally for pullback along surjective group homomorphisms. $\square$

Lemma 3.36 (Some irreps of the integer Heisenberg group). *Let $\zeta \in \text{U}(1)$ be any root of unity and put*

$$o := \text{ord}(\zeta), \quad D := \text{ord}(\zeta^2) = \frac{o}{\gcd(o, 2)}.$$

For every $s, t \in \{0, 1\}$, the following formulas define a D -dimensional irreducible unitary representation of the integer Heisenberg group (64):

$$\begin{aligned} \mathcal{H} &:= \mathbb{C}^D \\ &\simeq \text{Span}_{\mathbb{C}}(|0\rangle, |1\rangle, \dots, |D-1\rangle) \\ \langle n_1 | n_2 \rangle &:= \delta_{n_1 n_2} \end{aligned} \quad \left\{ \begin{array}{l} \widehat{\mathbb{Z}}^2 \longrightarrow \text{U}(\mathcal{H}) \\ \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 0 \right) \mapsto \widehat{W}_{[1]} : |n\rangle \mapsto e^{s\pi i/D} \zeta^{2n} |n\rangle \\ \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 0 \right) \mapsto \widehat{W}_{[0]} : |n\rangle \mapsto \begin{cases} |n+1\rangle & \text{if } n < D-1 \\ (-1)^{s+t} |0\rangle & \text{if } n = D-1 \end{cases} \\ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 1 \right) \mapsto \widehat{\zeta} : |n\rangle \mapsto \zeta |n\rangle, \end{array} \right. \quad (92)$$

Here $n \in \{0, \dots, D-1\}$, and the scalar powers of the two standard generators are

$$(\widehat{W}_{[1]})^D = (-1)^s \text{id}, \quad (\widehat{W}_{[0]})^D = (-1)^{s+t} \text{id}.$$

Here, the representation of general group elements follows from applying the group law to the above generators, for instance:

$$\zeta^{-1} \widehat{W}_{[0]} \widehat{W}_{[1]} = \widehat{W}_{[1]} = \zeta^{+1} \widehat{W}_{[0]} \widehat{W}_{[1]}. \quad (93)$$

Proof. First note that the operators are unitary and that the generating group commutators in $\widehat{\mathbb{Z}}^2$ (64) are evidently respected by the formulas (92), so that they do define a representation of $\widehat{\mathbb{Z}}^2$:

$$\widehat{W}_{[1]} \circ \widehat{W}_{[0]} = \zeta^2 \widehat{W}_{[0]} \circ \widehat{W}_{[1]}. \quad (94)$$

To see that this is irreducible, put

$$U := \widehat{W}_{[1]}, \quad V := \widehat{W}_{[0]}, \quad \tilde{o} := 2D.$$

Since

$$U^D = (-1)^s \text{id}, \quad V^D = (-1)^{s+t} \text{id}, \quad \zeta^{2D} = 1,$$

these representations factor through $\widehat{\mathbb{Z}}_o^2$. Hence, by Lem. 3.35, we may equivalently show that they are irreducible as $\widehat{\mathbb{Z}}_o^2$ -representations. This being a finite group (of order $|\widehat{\mathbb{Z}}_o^2| = \tilde{o}^3$), we may invoke Schur-orthonormality (236) of irreps.

By the group law, we have

$$\widehat{W}_{[a]} \circ \widehat{\zeta}^c = \zeta^{c-ab} U^a V^b.$$

If $b \neq 0 \pmod D$, this operator has vanishing diagonal entries, hence vanishing trace. Otherwise,

$$V^b = (-1)^{(s+t)b/D} \text{id},$$

and therefore

$$\text{tr}\left(\widehat{W}_{[a]} \circ \widehat{\zeta}^c\right) = \zeta^{c-ab} e^{s\pi ia/D} (-1)^{(s+t)b/D} \sum_{n=0}^{D-1} \zeta^{2na}.$$

Since $\text{ord}(\zeta^2) = D$, the sum on the right is D if $a = 0 \pmod D$ and vanishes otherwise, by (251). Thus the $\widehat{\mathbb{Z}}_{\tilde{o}}^2$ character components, for $a, b, c \in \{0, \dots, \tilde{o}-1\}$, are:

$$\chi_{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}, c\right)} := \text{tr}\left(\widehat{W}_{[a]} \circ \widehat{\zeta}^c\right) = \begin{cases} D \zeta^{c-ab} (-1)^{s\frac{a}{D} + (s+t)\frac{b}{D}} & \text{if } a = b = 0 \pmod D \\ 0 & \text{otherwise.} \end{cases} \quad (95)$$

There are exactly $\tilde{o}/D$ choices each of a and b for which this character is nonzero. On these elements its absolute value is D . Hence its Schur-norm square is unity:

$$\frac{1}{|\widehat{\mathbb{Z}}_{\tilde{o}}^2|} \sum_{a,b,c=0}^{\tilde{o}-1} \left| \chi_{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}, c\right)} \right|^2 = \frac{1}{\tilde{o}^3} \sum_{\substack{0 \leq a,b < \tilde{o} \\ a=b=0 \pmod D}} \sum_{c=0}^{\tilde{o}-1} D^2 = \frac{1}{\tilde{o}^3} \left(\frac{\tilde{o}}{D}\right)^2 \tilde{o} D^2 = 1,$$

which proves irreducibility. $\square$

Proposition 3.37 (Classification of covariantizable irreps of the integer Heisenberg group). *Any finite-dimensional irreducible unitary representation of $\widehat{\mathbb{Z}}^2$ which is covariantizable (Def. 3.30) is isomorphic to one from Lem. 3.36. In the notation of that lemma, $t = 0$ for both spin structures, and*

- for the pp-spin structure,

$$(-1)^s = \zeta^{D^2}, \quad \text{equivalently} \quad s = \begin{cases} 1 & \text{if } o \equiv 2 \pmod{4}, \\ 0 & \text{otherwise;} \end{cases}$$

- for the aa-spin structure, either $s = 0$ or $s = 1$.

Proof. By Lem. 3.32, the central generator $\widehat{\zeta}$ acts by multiplication with a root of unity ζ . Put

$$U := \widehat{W}_{[1]}, \quad V := \widehat{W}_{[0]}, \quad D := \text{ord}(\zeta^2).$$

Since $UV = \zeta^2 VU$, the operators U^D and V^D commute with all Heisenberg representation operators. Hence Schur's lemma implies that

$$U^D = \alpha \text{id}, \quad V^D = \beta \text{id}, \quad \alpha, \beta \in \mathbb{U}(1).$$

Next we see that S -covariance forces $t = 0$ for both spin structures. Namely, the modular generator S acts by

$$U \mapsto V^{-1}, \quad V \mapsto U.$$

Covariance therefore exhibits the given representation as isomorphic to its translate under this action. Since scalar operators are preserved under conjugation, taking D th powers gives $\alpha = \beta^{-1}$ and $\beta = \alpha$, and hence

$$\alpha = \beta \in \{+1, -1\}.$$

Writing these signs in the (s, t) -parameterization of Lem. 3.36,

$$\alpha = (-1)^s, \quad \beta = (-1)^{s+t}, \quad s, t \in \{0, 1\},$$

the equality $\alpha = \beta$ says precisely that

$$(-1)^t = 1, \quad \text{hence} \quad t = 0.$$

Thus S -covariance leaves only the two possibilities

$$(\alpha, \beta) = ((-1)^s, (-1)^s), \quad s \in \{0, 1\}.$$

We now identify the given representation with the corresponding representation of Lem. 3.36. Since

$$U^{2D} = V^{2D} = \text{id}, \quad \zeta^{2D} = 1,$$

the representation factors through the cyclic Heisenberg group $\widehat{\mathbb{Z}}_{\tilde{o}}^2$ with $\tilde{o} := 2D$. By Lem. 3.35, the resulting representation of this finite group is irreducible.

The full center of $\widehat{\mathbb{Z}}_{\tilde{o}}^2$ is generated by the distinguished central generator and by the D th powers of the two standard generators. Its character in the given representation is therefore specified by

$$(\zeta, \alpha, \beta) = (\zeta, (-1)^s, (-1)^s).$$

Moreover,

$$\widehat{\mathbb{Z}}_o^2 / Z(\widehat{\mathbb{Z}}_o^2) \simeq (\mathbb{Z}/D\mathbb{Z})^2,$$

and the commutator pairing induced by this central character is

$$((a, b), (a', b')) \mapsto \zeta^{2(ab' - a'b)}.$$

This pairing is nondegenerate because ζ^2 has order D . Hence the finite *Stone-von Neumann–Mackey theorem*, applied with this full central character (cf. [207, §4.1]), gives uniqueness of the irreducible representation with these central-character data. The representation of Lem. 3.36 with parameters $(s, t) = (s, 0)$ has precisely these data, so the given representation is isomorphic to it.

Finally, T -covariance has the effect of fixing the remaining parameter s for the pp-spin structure. Namely, the modular generator T acts by

$$U \mapsto U, \quad V \mapsto \widehat{W}_{[1]} = \zeta^{-1}UV.$$

Using the commutation relation, we obtain

$$(\zeta^{-1}UV)^D = \zeta^{-D^2}U^DV^D = \zeta^{-D^2}\alpha\beta \text{ id}.$$

Since covariance preserves the scalar operator V^D , this implies

$$\beta = \zeta^{-D^2}\alpha\beta, \quad \text{hence} \quad (-1)^s = \alpha = \zeta^{D^2}.$$

If $o = \text{ord}(\zeta)$ is odd, then $D = o$, so that $\zeta^{D^2} = 1$. If o is even, then $o = 2D$ and $\zeta^D = -1$, so that

$$\zeta^{D^2} = (-1)^D.$$

Therefore, in the pp case, $s = 1$ precisely when $o = 2 \bmod 4$, and $s = 0$ otherwise.

For the aa-spin structure, only T^2 -covariance is required. Its corresponding scalar condition is

$$\beta = \zeta^{-2D^2}\alpha^2\beta = \beta,$$

which imposes no further restriction on s . □

We now turn to the actual construction of the modular covariantization of these representations.

Covariantization for pp spin structure. We begin with the pp-spin structure and the case $(s, t) = (0, 0)$ in (92), taking $p = 1$ and even K .

Proposition 3.38 (Basic 2-Cohomotopical quantum states over the torus).

Unitary representations of $\text{SL}_2(\mathbb{Z}) \ltimes \widehat{\mathbb{Z}}^2$ (84) — and hence spaces of quantum states (21) for 2-cohomotopical flux over the torus — irreducible already in their restriction to $\widehat{\mathbb{Z}}^2$, are obtained for all even positive integers

$$K \in 2\mathbb{N}_{>0} \quad \text{with} \quad \zeta := e^{\frac{\pi i}{K}}, \quad (96)$$

by the following formulas:

$$\begin{aligned} \mathcal{H}_{T^2} &:= \mathbb{C}^K \\ &\simeq \text{Span}(|0\rangle, |1\rangle, \dots, |K-1\rangle) \end{aligned} \quad \left\{ \begin{array}{l} \text{SL}_2(\mathbb{Z}) \ltimes \widehat{\mathbb{Z}}^2 \longrightarrow \text{U}(\mathcal{H}_{T^2}) \\ \left(\text{I}, \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 0 \right) \right) \mapsto \widehat{W}_{[0]} : |n\rangle \mapsto \zeta^{2n} |n\rangle \\ \left(\text{I}, \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 0 \right) \right) \mapsto \widehat{W}_{[1]} : |n\rangle \mapsto |(n+1) \bmod K\rangle \\ \left(\text{I}, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 1 \right) \right) \mapsto \widehat{\zeta} : |n\rangle \mapsto \zeta |n\rangle \\ \left(S, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0 \right) \right) \mapsto \widehat{S} : |n\rangle \mapsto \frac{1}{\sqrt{K}} \sum_{\widehat{n}=0}^{K-1} \zeta^{2n\widehat{n}} |\widehat{n}\rangle \\ \left(T, \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0 \right) \right) \mapsto \widehat{T} : |n\rangle \mapsto \frac{1}{c_\zeta} \zeta^{(n^2)} |n\rangle, \end{array} \right. \quad (97)$$

where the normalization factor $c_\zeta \in \mathbb{C}$ in the last line is any cube root

$$c_\zeta := \left(\frac{1}{\sqrt{K}} \sum_{n=0}^{K-1} \zeta^{(n^2)} \right)^{1/3}, \quad (98)$$

which with (96) and by (255) may here be taken to be $(e^{\pi i/4})^{1/3} = e^{\pi i/12}$.

Proof. (i) The restriction to $\widehat{\mathbb{Z}}^2$ is the irreducible unitary representation of Lem. 3.36 with $D = K$ and $(s, t) = (0, 0)$. Indeed, $\text{ord}(\zeta^2) = K$ and $\dim(\mathcal{H}_{T^2}) = K$; the scalar sector is the one required by Prop. 3.37 for these parameters.

(ii) The Fourier operator $\hat{S}$ is unitary by (251), and $\hat{T}$ is unitary because $|c_\zeta| = 1$ for the normalization above. To see that we also have a representation of the subgroup $\text{SL}_2(\mathbb{Z})$, it is sufficient to show that the operators $\hat{S}$ and $\hat{T}$ respect the relations (32).

Now first, we find

$$\begin{aligned}\hat{S}\hat{S}|n\rangle &\equiv \hat{S}\left(\frac{1}{\sqrt{K}}\sum_{\hat{n}}e^{\frac{2\pi i}{K}\hat{n}n}|\hat{n}\rangle\right) \\ &\equiv \sum_{\hat{n}}\underbrace{\frac{1}{K}\sum_{\hat{n}}e^{\frac{2\pi i}{K}\hat{n}(n+\hat{n})}}_{\delta_0(n+\hat{n} \bmod K)}|\hat{n}\rangle \\ &= |-n \bmod K\rangle \quad \text{by (251).}\end{aligned}\tag{99}$$

This immediately implies that $\hat{S}^4 = \text{id}$ and that, with (98),

$$\hat{T}\hat{S}|n\rangle = \frac{1}{K^{1/2}c_\zeta}\sum_{\hat{n}}e^{\frac{\pi i}{K}(\hat{n}^2+2\hat{n}n)}|\hat{n}\rangle,$$

also $\hat{S}^2(\hat{T}\hat{S}) = (\hat{T}\hat{S})\hat{S}^2$: the diagonal phase of $\hat{T}$ is even and K -periodic since K is even. Hence the only remaining relation to check is $(\hat{T}\hat{S})^3 = \text{id}$ or equivalently that

$$\hat{T}^{-1} \circ \hat{S}^{-1} \circ \hat{T}^{-1} = \hat{S} \circ \hat{T} \circ \hat{S}.$$

Unwinding the definitions gives

$$\begin{aligned}\hat{T}^{-1}\hat{S}^{-1}\hat{T}^{-1}|n\rangle &= c_\zeta\hat{T}^{-1}\hat{S}^{-1}e^{-\frac{\pi i}{K}n^2}|n\rangle \\ &= \hat{T}^{-1}\frac{c_\zeta}{\sqrt{K}}\sum_{\hat{n}}e^{\frac{\pi i}{K}(-n^2-2\hat{n}n)}|\hat{n}\rangle \\ &= \frac{c_\zeta^2}{\sqrt{K}}\sum_{\hat{n}}e^{\frac{\pi i}{K}(-n^2-2\hat{n}n-\hat{n}^2)}|\hat{n}\rangle \\ &= \frac{c_\zeta^2}{\sqrt{K}}\sum_{\hat{n}}e^{-\frac{\pi i}{K}(\hat{n}+n)^2}|\hat{n}\rangle\end{aligned}\quad \text{and} \quad \begin{aligned}\hat{S}\hat{T}\hat{S}|n\rangle &= \hat{S}\hat{T}\frac{1}{\sqrt{K}}\sum_{\hat{n}}e^{\frac{2\pi i}{K}\hat{n}n}|\hat{n}\rangle \\ &= \hat{S}\frac{c_\zeta^{-1}}{\sqrt{K}}\sum_{\hat{n}}e^{\frac{\pi i}{K}(2\hat{n}n+\hat{n}^2)}|\hat{n}\rangle \\ &= \frac{c_\zeta^{-1}}{K}\sum_{\hat{n},\hat{\hat{n}}}e^{\frac{\pi i}{K}(2\hat{n}n+\hat{n}^2+2\hat{\hat{n}}\hat{n})}|\hat{\hat{n}}\rangle \\ &= \frac{c_\zeta^{-1}}{\sqrt{K}}\sum_{\hat{n}}\underbrace{\frac{1}{\sqrt{K}}\sum_{\hat{n}}e^{\frac{\pi i}{K}(\hat{n}+(n+\hat{n}))^2}}_{c_\zeta^3}e^{-\frac{\pi i}{K}(n+\hat{n})^2}|\hat{\hat{n}}\rangle.\end{aligned}\tag{100}$$

The term over the brace is a constant in n and $\hat{n}$, by the assumption that K is even,¹⁶ whence the relation is satisfied if the normalization factor c_ζ is chosen as claimed in (98).

(iii) Finally, we need to see that the semidirect product structure is respected, hence that

$$\widehat{W}_{M[a]_b}\widehat{M}|n\rangle = \widehat{M}\widehat{W}_{[a]_b}|n\rangle \quad \forall \begin{cases} M & \in \text{SL}_2(\mathbb{Z}) \\ (a,b) & \in \mathbb{Z}^2 \\ |n\rangle & \in \mathcal{H}_{T^2}. \end{cases}$$

It is sufficient to check this on the generators; the central generator is fixed because its operator is scalar. In the computations below, all summation and state indices are taken modulo K . Explicitly:

$$\begin{aligned}\widehat{W}_{S[0]}\hat{S}|[n]\rangle &\equiv \widehat{W}_{[0]}^{-1}\left(\frac{1}{\sqrt{|K|}}\sum_{\hat{n}}e^{\frac{2\pi i}{K}\hat{n}n}|\hat{n}\rangle\right) & \widehat{W}_{S[1]}\hat{S}|n\rangle &\equiv \widehat{W}_{[0]}\left(\frac{1}{\sqrt{K}}\sum_{\hat{n}}e^{\frac{2\pi i}{K}\hat{n}n}|\hat{n}\rangle\right) \\ &= \frac{1}{\sqrt{K}}\sum_{\hat{n}}e^{\frac{2\pi i}{K}(\hat{n}+1)n}|\hat{n}\rangle & &= \frac{1}{\sqrt{K}}\sum_{\hat{n}}e^{\frac{2\pi i}{K}\hat{n}}e^{\frac{2\pi i}{K}\hat{n}n}|\hat{n}\rangle \\ &= e^{\frac{2\pi i}{K}n}\hat{S}|n\rangle & &= \frac{1}{\sqrt{K}}\sum_{\hat{n}}e^{\frac{2\pi i}{K}\hat{n}(n+1)}|\hat{n}\rangle \\ &= \hat{S}\widehat{W}_{[1]}|n\rangle, & &= \hat{S}\widehat{W}_{[1]}|n\rangle,\end{aligned}$$

and

$$\begin{aligned}\widehat{W}_{T[0]}\hat{T}|n\rangle &\equiv \widehat{W}_{[0]}\frac{1}{c_\zeta}e^{\frac{\pi i}{K}n^2}|n\rangle & \widehat{W}_{T[1]}\hat{T}|n\rangle &\equiv \frac{1}{c_\zeta}\widehat{W}_{[0]}\widehat{W}_{[1]}e^{\frac{\pi i}{K}}e^{\frac{\pi i}{K}n^2}|n\rangle \\ &= \frac{1}{c_\zeta}e^{\frac{2\pi i}{K}n}e^{\frac{i\pi}{K}n^2}|n\rangle & &= \frac{1}{c_\zeta}e^{\frac{\pi i}{K}(n^2+2n+1)}|n+1\rangle \\ &= \hat{T}\widehat{W}_{[1]}|n\rangle, & &= \frac{1}{c_\zeta}e^{\frac{\pi i}{K}(n+1)^2}|n+1\rangle \\ & & &= \hat{T}\widehat{W}_{[1]}|n\rangle,\end{aligned}$$

where in the first step of the last case we used (93). □

¹⁶ Since the summands in $\sum_{n=0}^{K-1}e^{\frac{\pi i}{K}n^2}$ are K -periodic for even K , $e^{\frac{\pi i}{K}(n+K)^2} = e^{\frac{\pi i}{K}n^2}e^{\pi i(2n+K)} \underset{K \text{ even}}{=} e^{\frac{\pi i}{K}n^2}$, the sum is invariant under replacing $n \mapsto n+a$ for $a \in \mathbb{N}$.

Remark 3.39 (Comparison to modular data of abelian Chern-Simons theory on the torus). The content of Prop. 3.38 captures the *modular data* (cf. [93]) of abelian Chern-Simons theory:

- (i) The algebra (94) of the $\widehat{W}_{[a]}^{[b]}$ is just that expected [261, (5.28)] of quantum observables for anyonic topological order on the torus as predicted [36, (17)][204, (32)][96, Prop. 2.2] by abelian Chern-Simons theory at *level* $k = K/2 \in \mathbb{Z}$ (8), and equivalently by U(1)-WZW conformal field theory [264, (4.3-4)].
- (ii) Similarly, the operators $\widehat{S}$ and $\widehat{T}$ according to (97) implement a linear normalization of the known modular action on quantum states of abelian Chern-Simons theory [267, (5.3)][173, pp. 65–67, arXiv version] (following [108][90, (5,7)]) and equivalently of conformal characters of the U(1) 2D CFT [93, Ex. 1].¹⁷
- (iii) Prop. 3.38 also records the compatibility of these two actions: jointly, the operators give a representation of the semidirect product of the modular group with the integer Heisenberg group.

The basic construction of Prop. 3.38 captures the braiding angle *unit fractions* $\theta = 1/K = 1/(2k)$ with even denominator. Further irreps of the covariantized flux monodromy group yield general rational braiding angles, including non-unit numerators and odd denominators. These are statements about braiding phases; assigning a filling fraction requires additional physical data (cf. Rem. 3.50). The comparison with K -matrix predictions is developed below in Props. 3.47 and 3.59. The more general representations are as follows:

Lemma 3.40 (More general representations). *For*

$$\zeta := e^{\pi i \frac{p}{K}}, \text{ where} \quad (101a)$$

$$(K, p) \in \mathbb{N}_{>0} \times \mathbb{Z}, \text{ gcd}(p, K) = 1, \quad (101b)$$

- (i) *For $Kp \in 2\mathbb{Z}$ the same formulas (97) constitute a unitary representation $\rho_\zeta^{(0)}$ of $\text{SL}_2(\mathbb{Z}) \ltimes \widehat{\mathbb{Z}^2}$.*
- (ii) *Generally, writing*

$$\varepsilon := Kp \bmod 2 \in \{0, 1\}, \quad (102a)$$

$$\eta := (-1)^\varepsilon \zeta = e^{\pi i \frac{p+\varepsilon K}{K}}, \quad (102b)$$

$$\begin{aligned} \chi : \text{SL}_2(\mathbb{Z}) \ltimes \widehat{\mathbb{Z}^2} &\longrightarrow \{+1, -1\}, \\ \left(M, \left(\begin{bmatrix} a \\ b \end{bmatrix}, c\right)\right) &\longmapsto (-1)^{ab+a+b+c}, \end{aligned} \quad (102c)$$

where $M \in \text{SL}_2(\mathbb{Z})$ is arbitrary, the representation $\rho_\eta^{(0)}$ of part (i) exists, and its tensor product with the character (102c),

$$\rho_\zeta := \chi^\varepsilon \otimes \rho_\eta^{(0)}, \quad (103)$$

is a unitary representation of $\text{SL}_2(\mathbb{Z}) \ltimes \widehat{\mathbb{Z}^2}$ with central phase ζ and satisfying

$$\widehat{W}_{[0]}^{[K]} = \widehat{W}_{[K]}^{[0]} = (-1)^\varepsilon \text{id},$$

corresponding to the scalar sector $(s, t) = (\varepsilon, 0)$ in Lem. 3.36.

Proof. (i) The proof of Prop. 3.38 extends with all factors of $e^{\pi i/K}$ replaced by ζ (101). Fourier orthogonality still applies because ζ^2 has order K . Both the Gauss-sum manipulation in (100) and the boundary case of T -covariance use the K -periodicity of the quadratic phase, which follows from $Kp \in 2\mathbb{Z}$:

$$\zeta^{(n+K)^2} = \zeta^{n^2} e^{2\pi i p n} e^{\pi i p K} = \zeta^{n^2}. \quad (104)$$

It just remains to see that the normalization factor c_ζ (98) has unit norm, so that $\widehat{T}$ is unitary. For this, observe

¹⁷The canonical choice $c_\zeta = e^{2\pi i/24}$ in (98) supplies the scalar factor c_ζ^{-1} in $\widehat{T}$ for our linear normalization (cf. [93, (3.1b)][247, (26)]). Projective treatments such as [173, pp. 65–67, arXiv version] encode the phase anomaly through a projective multiplier or a central extension. The absence of this scalar from the projective matrices therefore does not imply that the anomaly was missed.

that

$$\begin{aligned}
\left| \sum_{m=0}^{K-1} \zeta^{m^2} \right|^2 &= \sum_{m=0}^{K-1} \sum_{n=0}^{K-1} \zeta^{(n+m)^2} \zeta^{-n^2} && \text{by (104)} \\
&= \sum_{m=0}^{K-1} \zeta^{m^2} \sum_{n=0}^{K-1} e^{2\pi i \frac{p}{K} mn} \\
&= \sum_{m=0}^{K-1} \zeta^{m^2} K \delta_{0,m} && \text{by } \gcd(p, K) = 1 \\
&= K.
\end{aligned} \tag{105}$$

This shows that $|c_\zeta| = 1$ in (98).

(ii) First we verify the character (102c). The function

$$(a, b, c) \mapsto (-1)^{ab+a+b+c}$$

is multiplicative by the group law of $\widehat{\mathbb{Z}^2}$. Indeed, the exponent evaluated on the product of (a, b, c) and (a', b', c') is, modulo 2,

$$\begin{aligned}
&(a + a')(b + b') + (a + a') + (b + b') + c + c' + ab' - a'b \\
&\equiv (ab + a + b + c) + (a'b' + a' + b' + c').
\end{aligned}$$

Moreover, this character is invariant under both modular generators

$$S : (a, b, c) \mapsto (b, -a, c), \quad T : (a, b, c) \mapsto (a + b, b, c),$$

since

$$ab + a + b \equiv -ab + b - a \equiv (a + b)b + (a + b) + b \pmod{2}.$$

Since S and T generate $\text{SL}_2(\mathbb{Z})$, the character therefore extends to the semidirect product by being trivial on the modular group, as asserted in (102c).

Next, if $\varepsilon = 0$, then Kp is even and $\eta = \zeta$. If $\varepsilon = 1$, then both K and p are odd, so that $p + K$ is even. In either case,

$$\gcd(p + \varepsilon K, K) = 1, \quad K(p + \varepsilon K) \in 2\mathbb{Z}.$$

Hence part (i) supplies the representation $\rho_\eta^{(0)}$.

Tensoring with the unitary character χ^ε preserves unitarity and all representation relations. Since χ takes the value -1 on the distinguished central generator, the central phase of (103) is

$$(-1)^\varepsilon \eta = \zeta.$$

Likewise, χ takes the value -1 on each of the two standard Heisenberg generators. Thus, if U_η, V_η denote their operators in $\rho_\eta^{(0)}$, their operators in ρ_ζ are

$$U_\zeta = (-1)^\varepsilon U_\eta, \quad V_\zeta = (-1)^\varepsilon V_\eta.$$

The formulas (97) give

$$U_\eta^K = V_\eta^K = \text{id},$$

and therefore

$$U_\zeta^K = V_\zeta^K = (-1)^{\varepsilon K} \text{id} = (-1)^\varepsilon \text{id}.$$

The last equality holds because $\varepsilon = 1$ implies that K is odd. This is precisely the asserted scalar sector. For $\varepsilon = 1$, this sign twist changes the Heisenberg central phase from η to ζ . By contrast, the mapping class group characters used below are trivial on the Heisenberg subgroup. $\square$

Proposition 3.41 (General 2-Cohomotopical quantum states over the pp-torus). *For*

$$(K, p) \in \mathbb{N}_{>0} \times \mathbb{Z}, \quad \gcd(p, K) = 1, \quad \zeta := e^{\pi i \frac{p}{K}},$$

let ρ_ζ be the representation constructed in Lem. 3.40 (ii). Then:

- (i) The representation ρ_ζ is unitary, has dimension K , and is irreducible already when restricted to $\widehat{\mathbb{Z}^2}$. Its scalar sector in Lem. 3.36 is

$$(s, t) = (\varepsilon, 0), \quad \varepsilon := Kp \pmod{2},$$

so that

$$\widehat{W}_{\begin{bmatrix} K \\ 0 \end{bmatrix}} = \widehat{W}_{\begin{bmatrix} 0 \\ K \end{bmatrix}} = (-1)^{Kp} \text{id}.$$

- (ii) Every finite-dimensional unitary representation of $\text{SL}_2(\mathbb{Z}) \ltimes \widehat{\mathbb{Z}^2}$ which is irreducible already upon restriction to $\widehat{\mathbb{Z}^2}$ is isomorphic to one of the form

$$(\lambda \circ \text{pr}) \otimes \rho_\zeta,$$

where

$$\text{pr} : \text{SL}_2(\mathbb{Z}) \ltimes \widehat{\mathbb{Z}^2} \longrightarrow \text{SL}_2(\mathbb{Z})$$

is the quotient projection and

$$\lambda : \text{SL}_2(\mathbb{Z}) \longrightarrow \text{U}(1)$$

is a unitary character. Conversely, every such tensor product has these properties. For each fixed central phase ζ and fixed choice of ρ_ζ , distinct characters λ give non-isomorphic representations.

Proof. (i) Existence and unitarity follow from Lem. 3.40. To see irreducibility, recall that its construction is

$$\rho_\zeta = \chi^\varepsilon \otimes \rho_\eta^{(0)}, \quad \eta = (-1)^\varepsilon \zeta.$$

Since

$$\eta^2 = \zeta^2 = e^{2\pi i \frac{p}{K}}, \quad \gcd(p, K) = 1,$$

we have

$$\text{ord}(\eta^2) = K.$$

The restriction of $\rho_\eta^{(0)}$ to $\widehat{\mathbb{Z}^2}$ is therefore the irreducible representation of Lem. 3.36 with central phase η , dimension $D = K$, and parameters $(s, t) = (0, 0)$. Tensoring with the one-dimensional character χ^ε preserves its invariant subspaces and its dimension. Hence ρ_ζ is likewise irreducible upon restriction to $\widehat{\mathbb{Z}^2}$ and has dimension K . The asserted scalar sector is established in Lem. 3.40 (ii).

(ii) Let $\tilde{\rho}$ be a representation satisfying the stated hypotheses. By Lem. 3.32, its central generator acts by a root of unity ζ . Writing

$$K := \text{ord}(\zeta^2),$$

we may express this phase as

$$\zeta = e^{\pi i \frac{p}{K}} \quad \text{with} \quad \gcd(p, K) = 1.$$

By Prop. 3.37, its Heisenberg restriction is isomorphic to the representation of Lem. 3.36 whose parameters satisfy

$$t = 0, \quad (-1)^s = \zeta^{K^2} = (-1)^{Kp}.$$

These are precisely the central phase and scalar sector of the restriction of ρ_ζ from part (i). Hence the Heisenberg restrictions of $\tilde{\rho}$ and ρ_ζ are isomorphic.

After conjugating by a unitary intertwiner, we may assume that these restrictions agree. For any

$$g \in \text{SL}_2(\mathbb{Z}) \ltimes \widehat{\mathbb{Z}^2},$$

the operator

$$\tilde{\rho}(g)\rho_\zeta(g)^{-1}$$

then commutes with their common Heisenberg representation. By irreducibility and Schur's lemma, it is a scalar $d(g)\text{id}$ with $d(g) \in \text{U}(1)$. The representation laws imply that d is multiplicative, and agreement on the Heisenberg subgroup implies that d is trivial there. Thus $d = \lambda \circ \text{pr}$ for a unitary character $\lambda : \text{SL}_2(\mathbb{Z}) \rightarrow \text{U}(1)$, which proves the asserted form.

Conversely, tensoring ρ_ζ with $\lambda \circ \text{pr}$ preserves unitarity and leaves its Heisenberg restriction unchanged, hence irreducible.

Finally, an intertwiner between two such tensor products at fixed ζ and fixed ρ_ζ commutes with their common irreducible Heisenberg restriction, so is scalar. If it is invertible, the intertwining condition on the modular subgroup forces the two characters λ to coincide. $\square$

Thus every root of unity occurs as a central phase already for the pp-spin structure. The even-product subfamily of Lem. 3.40 (i) remains useful for comparison with familiar bosonic examples:

Remark 3.42 (Relation to bosonic FQH systems). Besides the standard FQH effect for two-dimensional electron gases, rapidly rotating Bose-Einstein condensates with repulsive contact interactions provide models of bosonic FQH states [276]. A basic example is the bosonic Laughlin state at $\nu = 1/2$ [51, (79)]. Exact-diagonalization studies give evidence for states in the Jain sequences $\nu = r/(r \pm 1)$, including $\nu = 2/3, 3/4, 4/3, 5/4$ [215][216] (see also [51, p. 31]). Each of these reduced fractions has an even numerator or denominator, as in the condition $Kp \in 2\mathbb{Z}$ of Lem. 3.40 (i).

This arithmetic comparison concerns that subfamily. Prop. 3.41 also admits odd Kp , with scalar sector $(s, t) = (1, 0)$. Thus the full pp classification does not by itself distinguish bosonic from fermionic constituents, nor identify a filling fraction with a braiding angle. The occurrence of bosonic Laughlin correlations in a rotating system of two fermions [171], and the effective bosonic behavior of flux-coupled quasiparticles (cf. [269, §7.3.3]), further illustrate why these physical identifications require care.

Our surfaces carry a spin structure (Rem. 2.3). Using all of $\text{SL}_2(\mathbb{Z})$ corresponds to the pp-spin structure in (34). We now consider the different covariance condition for the aa-spin structure.

Covariantization for the aa-spin structure. For the aa-spin structure, the mapping class group is the subgroup $\text{MCG}(\Sigma_1^{\text{aa}}) \subset \text{SL}_2(\mathbb{Z})$ generated by S and T^2 (35). Both scalar sectors $(s, t) = (0, 0)$ and $(1, 0)$ then admit extensions for every central phase, as follows:

Proposition 3.43 (2-Cohomotopical quantum states over the aa-spin torus). *For*

$$(K, p) \in \mathbb{N}_{>0} \times \mathbb{Z}, \quad \gcd(p, K) = 1, \quad \zeta := e^{\pi i p / K},$$

define

$$\tilde{p} := \begin{cases} p & | p \in 2\mathbb{Z} + 1, \\ p + K & | p \in 2\mathbb{Z}, \end{cases} \quad \xi := e^{\pi i \tilde{p} / K}.$$

Here and below, fractional powers of ξ mean

$$\xi^x := e^{\pi i \tilde{p} x / K}.$$

Then, for either $s \in \{0, 1\}$, the following operators on

$$\mathcal{H}_{T^2} := \mathbb{C}^K \simeq \text{Span}(|0\rangle, \dots, |K-1\rangle)$$

with the displayed basis orthonormal, define a unitary representation $\rho_{\zeta, s}^{\text{aa}}$ of $\text{MCG}(\Sigma_1^{\text{aa}}) \ltimes \widehat{\mathbb{Z}^2}$ by:

$$\begin{aligned} \widehat{W}_{[1]}|n\rangle &= \xi^{2n+s}|n\rangle, \\ \widehat{W}_{[0]}|n\rangle &= \begin{cases} |n+1\rangle & | n < K-1, \\ (-1)^s|0\rangle & | n = K-1, \end{cases} \\ \widehat{\zeta}|n\rangle &= \zeta|n\rangle, \\ \widehat{S}|n\rangle &= \frac{1}{\sqrt{K}} \sum_{m=0}^{K-1} \xi^{2(n+s/2)(m+s/2)}|m\rangle, \\ \widehat{T^2}|n\rangle &= e^{-\pi i/6} \xi^{2(n+s/2)^2}|n\rangle. \end{aligned} \tag{106}$$

Here $n \in \{0, \dots, K-1\}$ and $\widehat{T^2}$ denotes the operator assigned to the generator T^2 . These representations are irreducible already upon restriction to $\widehat{\mathbb{Z}^2}$, and satisfy

$$(\widehat{W}_{[1]})^K = (\widehat{W}_{[0]})^K = (-1)^s \text{id}.$$

Hence, for each ζ , their restrictions realize the two isomorphism classes with $(s, t) = (0, 0)$ and $(s, t) = (1, 0)$ in Lem. 3.36.

Proof. For orientation, in the case $p = 1$ and $s = 0$, the displayed modular matrices recover the rank-one lattice formulas of [93, §3, Example 1, p. 221], with lattice $\Lambda = \sqrt{K}\mathbb{Z}$. There it is observed that, for integral lattices which are not even, the lattice construction still gives a representation of the index-three subgroup $\langle S, T^2 \rangle \subset \text{SL}_2(\mathbb{Z})$. For this rank-one lattice, this includes precisely the additional case of odd K . We verify below the full semidirect product representation for arbitrary coprime (K, p) and both choices of s .

First observe that, if p is even, coprimality forces K to be odd. Hence $\tilde{p}$ is always odd, with

$$\gcd(\tilde{p}, K) = 1, \quad \xi^2 = \zeta^2, \quad \xi^K = -1, \quad \text{ord}(\xi^2) = K.$$

For brevity, write

$$U := \widehat{W}_{[1]}, \quad V := \widehat{W}_{[0]}, \quad Q := \widehat{T^2}.$$

The operators U and V are unitary and satisfy

$$UV = \xi^2 VU = \zeta^2 VU, \quad U^K = V^K = (-1)^s \text{id}.$$

Together with $\widehat{\zeta} = \zeta \text{id}$, these give a unitary representation of $\widehat{\mathbb{Z}^2}$. Moreover, the eigenvalues ξ^{2n+s} of U are pairwise distinct, while V permutes their one-dimensional eigenspaces transitively. Therefore this representation is irreducible.

Next we verify the mapping class group relations. The operator $\widehat{S}$ is unitary by finite Fourier orthogonality for the primitive K -th root ξ^2 , while Q is manifestly unitary. The presentation (35) of the aa-spin mapping class group is

$$\text{MCG}(\Sigma_1^{\text{aa}}) = \langle S, T^2 \mid S^4 = 1, S^2 T^2 = T^2 S^2 \rangle.$$

Writing $[r]_K$ for the representative of r modulo K in $\{0, \dots, K-1\}$, Fourier orthogonality gives

$$\widehat{S}^2|n\rangle = \begin{cases} |[-n]_K\rangle & | s = 0, \\ -|K-1-n\rangle & | s = 1. \end{cases}$$

Indeed, the coefficient of $|j\rangle$ in $\widehat{S}^2|n\rangle$ is

$$\frac{1}{K} \sum_{m=0}^{K-1} \xi^{2(m+s/2)(j+n+s)}.$$

For $s = 0$ this gives the first case above. For $s = 1$ it vanishes unless $j + n + 1 = K$, in which case its value is $\xi^K = -1$. In particular, $\widehat{S}^4 = \text{id}$.

Furthermore, the diagonal coefficient of Q is K -periodic in n , since

$$\frac{\xi^{2(n+K+s/2)^2}}{\xi^{2(n+s/2)^2}} = \xi^{4Kn+2Ks+2K^2} = 1.$$

It is also invariant under $n \mapsto -n - s$. The displayed formula for $\widehat{S}^2$ therefore implies

$$\widehat{S}^2 Q = Q \widehat{S}^2.$$

Thus the mapping class group relations hold.

Finally, we check covariance with the Heisenberg operators. Substitution in the formulas gives

$$\widehat{S}U = V^{-1}\widehat{S}, \quad \widehat{S}V = U\widehat{S}.$$

Here the boundary terms agree because

$$\xi^{2K(n+s/2)} = (-1)^s.$$

Likewise, Q commutes with U , and the ratio of its successive diagonal coefficients is

$$\frac{\xi^{2(n+1+s/2)^2}}{\xi^{2(n+s/2)^2}} = \xi^{4n+2s+2}.$$

Using the periodicity just established, this also applies at the boundary and yields

$$QUQ^{-1} = U, \quad QVQ^{-1} = \xi^{-2}U^2V = \zeta^{-2}U^2V.$$

These are exactly the covariance relations for the actions of S and T^2 on $\widehat{\mathbb{Z}^2}$. The central operator $\widehat{\zeta}$ is scalar and hence fixed under conjugation. Consequently the operators define the claimed unitary representation of the semidirect product.

Since $\text{ord}(\zeta^2) = K$, the scalar identities $U^K = V^K = (-1)^s \text{id}$ identify its Heisenberg restriction with the $(s, t) = (s, 0)$ case of Lem. 3.36, by Prop. 3.37. The two choices of s give inequivalent restrictions, as their scalar values of U^K differ. $\square$

Combining the classification of Heisenberg restrictions with these extension constructions gives the result over the torus:

Theorem 3.44 (Classification of 2-cohomotopical flux quantum states on the torus).

The finite-dimensional unitary spaces of 2-cohomotopical flux quantum states on the torus whose Heisenberg restriction is irreducible are classified as follows.

Let ζ range over all roots of unity, and write

$$\zeta = e^{\pi i p/K}, \quad (K, p) \in \mathbb{N}_{>0} \times \mathbb{Z}, \quad \gcd(p, K) = 1.$$

Then necessarily $K = \text{ord}(\zeta^2)$. In the following, pr denotes the projection from the respective semidirect product onto its mapping class group.

- (i) *Over the torus with aa-spin structure, these representations are precisely*

$$(\lambda \circ \text{pr}) \otimes \rho_{\zeta, s}^{\text{aa}}, \quad s \in \{0, 1\},$$

where $\rho_{\zeta, s}^{\text{aa}}$ is the representation constructed in Prop. 3.43 and

$$\lambda : \text{MCG}(\Sigma_1^2)^{\text{aa}} \longrightarrow \text{U}(1)$$

is any unitary group character. Their restrictions to $\widehat{\mathbb{Z}^2}$ have parameters $(s, t) = (s, 0)$ in Lem. 3.36, with

$$(\widehat{W}_{[1]}^K)^K = (\widehat{W}_{[0]}^K)^K = (-1)^s \text{id}.$$

- (ii) *Over the torus with pp-spin structure, these representations are precisely*

$$(\lambda \circ \text{pr}) \otimes \rho_{\zeta}^{\text{pp}},$$

where $\rho_{\zeta}^{\text{pp}} := \rho_{\zeta}$ is the representation constructed in Lem. 3.40(ii) and classified in Prop. 3.41, and

$$\lambda : \text{SL}_2(\mathbb{Z}) \longrightarrow \text{U}(1)$$

is any unitary group character. Its restriction to $\widehat{\mathbb{Z}^2}$ has parameters

$$t = 0, \quad s \equiv Kp \pmod{2},$$

so that

$$(\widehat{W}_{[1]})^K = (\widehat{W}_{[0]})^K = \zeta^{K^2} \text{id} = (-1)^{Kp} \text{id}.$$

In particular, no parity restriction on Kp is imposed on the braiding phase.

- (iii) All these representations have dimension (“ground state degeneracy”)

$$\dim \mathcal{H}_{T^2} = K = \text{ord}(\zeta^2) = \frac{\text{ord}(\zeta)}{\gcd(2, \text{ord}(\zeta))}.$$

For each ζ , fix the reference representations appearing above. Then the parameters (s, λ) in the aa case, and λ in the pp case, are unique up to isomorphism of the resulting representations. Representations with different ζ are inequivalent.

Proof. Let a finite-dimensional unitary representation of the respective semidirect product be given, whose restriction to $\widehat{\mathbb{Z}^2}$ is irreducible. By Lem. 3.32, the central generator acts as $\widehat{\zeta} = \zeta \text{id}$ for some root of unity ζ . Writing $K = \text{ord}(\zeta^2)$, we may express $\zeta = e^{\pi i p / K}$ with $\gcd(p, K) = 1$.

By Prop. 3.37, the Heisenberg restriction must be one of the $(s, t) = (s, 0)$ cases of Lem. 3.36, where s is arbitrary for the aa-spin structure and is constrained by

$$(-1)^s = \zeta^{K^2} = (-1)^{Kp}$$

for the pp-spin structure. Their dimensions are K .

The existence of extensions for every one of these Heisenberg representations is established by Prop. 3.43 in the aa case and by Prop. 3.41 in the pp case. Tensoring any such extension with a unitary character of the mapping class group preserves its Heisenberg restriction and hence its irreducibility.

Conversely, after unitarily identifying the Heisenberg restriction with that of the corresponding reference representation, Lem. 3.31 (ii) implies that any other extension differs from it by tensoring with a character of the quotient, which is the respective mapping class group. Since both extensions are unitary, this character takes values in $\text{U}(1)$. This proves that the displayed families exhaust the representations under consideration.

Finally, different values of ζ are distinguished by the action of the central generator, and the two aa cases are distinguished by the scalar $(-1)^s$ in $(\widehat{W}_{[0]})^K$. For a fixed Heisenberg restriction, any intertwiner between two character twists of the chosen reference extension must be scalar, by Schur’s lemma. It can therefore intertwine the full representations only if the two characters agree. This proves the asserted uniqueness. $\square$

Remark 3.45 (Linear modular extensions on the torus).

- (i) The abelianization of $\text{SL}_2(\mathbb{Z})$ is $\mathbb{Z}/12\mathbb{Z}$ (cf. [49, Cor. 2.4]). Hence there are precisely twelve unitary characters, all taking values in twelfth roots of unity. By Thm. 3.44, for every root of unity ζ there are precisely twelve isomorphism classes in the pp case under the theorem’s irreducibility assumption. All have dimension $K = \text{ord}(\zeta^2)$ and the same Heisenberg restriction; they differ by their scalar phases under modular transformations.

- (ii) For the aa-spin structure, abelianizing the presentation (35) gives

$$(\text{MCG}(\Sigma_1^2)^{\text{aa}})_{\text{ab}} \simeq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}.$$

Indeed, the relation $S^4 = 1$ remains, while the commutation relation between S^2 and T^2 imposes no further condition in the abelianization. A unitary character is therefore specified independently by

$$\lambda(S) \in \mu_4, \quad \lambda(T^2) \in \text{U}(1), \quad \mu_4 := \{1, i, -1, -i\}.$$

Thus, for each ζ and each $s \in \{0, 1\}$, the extensions form a family parameterized by $\mu_4 \times \text{U}(1)$ once a reference extension is fixed. Including both Heisenberg sectors gives two disjoint copies of this family, all of dimension K . In particular, there are uncountably many isomorphism classes at each central phase, since no finite-order condition is imposed on the value of $\lambda(T^2)$.

- (iii) The classification assumes irreducibility upon restriction to $\widehat{\mathbb{Z}^2}$. Dropping this assumption allows further irreducible representations. For example, let ρ be one of the extensions above, let Γ be the corresponding mapping class group, and let τ be a finite-dimensional irreducible unitary representation of Γ . Then

$$\rho \otimes (\tau \circ \text{pr})$$

is irreducible: any commuting operator has the form $\text{id} \otimes A$ by irreducibility of the Heisenberg restriction of ρ , and A must be scalar by irreducibility of τ . If $\dim \tau > 1$, its Heisenberg restriction is a direct sum of $\dim \tau$ copies of that of ρ . Thus the modular group may act nontrivially on a multiplicity space without changing the central braiding phase.

Over punctured surfaces, considered next, defect motions determine mapping classes through the point-pushing homomorphism (39). Pulling back a mapping-class representation gives a braid representation, which is trivial on the kernel of that homomorphism. Its image may describe non-abelian transformations of defect states. For the sphere with at least three punctures, the kernel is specified in (42); it is trivial for one or two punctures.

- (iv) The distinctions in (i) and (ii) concern isomorphism classes of linear unitary representations. For a fixed reference extension, scalar character twists have the same projective action:

$$[\lambda(M)\rho(M)] = [\rho(M)] \quad \text{in } \text{PU}(K).$$

Thus the scalar phases are additional data of a linear lift. If this lift is part of the specified monodromy data, the character twists remain distinct as proved above. Their identification with distinct physical topological orders requires specifying a realization by ground-state transport and its stability under the allowed perturbations; see §3.4.2. A representation of the mapping class group alone does not establish the additional fusion and compatibility conditions of modular data (cf. [93, §2]).

3.4.2 Comparison to K-matrix formalism

We compare the representation-theoretic analysis of the 2-cohomotopic surplus flux monodromy over the torus from above (§ 3.4.1) with existing descriptions of topological order in FQH systems, notably with K-matrix Chern-Simons theory (§ A.1.3).

Ground-state monodromy and topological order. Our construction supplies candidate local systems directly from the flux moduli space. Over a fixed spin-structured torus, the monodromy group is the semidirect product in Prop. 3.29. Its irreducible unitary representations define the algebraic superselection sectors of the adopted observable algebra. For those whose Heisenberg restriction is irreducible, Thm. 3.44 and Rem. 3.45 classify the resulting linear monodromy data.

For a fixed connected parameter space $\mathcal{M}_0$, these data consist of the isomorphism class of a finite-dimensional unitary local system (21), equivalently of a representation

$$\rho: \pi_1(\mathcal{M}_0, m) \longrightarrow \text{U}(\mathcal{H}_m). \quad (107)$$

This specifies how quantum states transform along parameter cycles, including the scalar phases of that transformation.

When realized by robust adiabatic transport of gapped ground states, such local systems encode a core signature of topological order. This transport signature is part of the characterization of a physical phase. Relating it to a complete characterization involves the following further data and questions:

- (i) **Ground-state realization.** Specify a smooth gapped family of Hamiltonians over $\mathcal{M}_0$ whose ground-state spaces and adiabatic transport realize the candidate local system. This supplies the physical meaning of its fibers and monodromy operators.
- (ii) **Topological transport.** The ground-state spaces carry the Berry connection. Its holonomy gives (107) when the connection is flat, or when a flat topological part has been specified. Projectively flat transport gives projective monodromy, with a linear lift, when it exists, as additional data (cf. [270, §V]). Flatness is a separate condition from the spectral gap that permits adiabatic transport.
- (iii) **Stability and scalar phases.** Identify the transport data that are robust under the allowed local perturbations and specify how auxiliary gapped degrees of freedom are treated. Even an auxiliary system with a one-dimensional ground space can contribute geometric phases. Thus the character twists of Rem. 3.45 (iv) are to be compared using the specified stability criterion and scalar-phase data. Scalar phases belonging to the robust topological transport remain part of this comparison, including those invisible in the projective representation.
- (iv) **Completeness.** The choice of $\mathcal{M}_0$ determines which transport processes are probed by the monodromy. A full classification of the intended physical topological orders requires establishing which candidate local

systems are realized and whether their isomorphism classes distinguish these orders, or which additional invariants are needed.

Relation to Experimental Topological Order. Toroidal FQH systems offer a natural setting in which to investigate ground-state realizations of the classified local systems and compare their monodromy operators with adiabatic quantum-state transport.

Fractional Chern insulators are of particular interest in this connection. The “quantum anomalous” version of the fractional quantum Hall effect (FQAH effect) has been experimentally established in these crystalline materials (cf. [42, 282, 184]). Their intrinsic *Berry curvature* plays the role of external magnetic flux in the momentum-space description (cf. [235, Fig. 3]). The periodicity of quasi-momenta makes the effective momentum space a *Brillouin torus* ([260, §2.1][84, p. 52]), providing a link to the torus geometry considered here (cf. [202]).

There is a further link in the two-band setting (cf. [42, §II]): the complex Grassmannian $\text{Gr}_2^1 \simeq \mathbb{C}P^1 \simeq S^2$ classifies the occupied line of a gapped two-band Bloch Hamiltonian (cf. [232, §3.1.2]). Such Bloch data are therefore described by maps from the Brillouin torus to S^2 ([248, (8.3-4)][235, (4)][236, §I.A]). This places two-band Bloch data in the same mapping-space framework as the cohomotopical description of FQH flux (cf. [148, §5.1]). Here the connected component matters: with a compatible orientation of S^2 , the degree d of the Bloch map is the occupied band’s Chern number. The torus classification in §3.4 uses the degree-zero component, while a nonzero-Chern band lies in $\text{Map}_d(T^2, S^2)$ with $d \neq 0$. In this component, [146, Prop. 1.5][148, Thm. 3.5] give

$$\pi_1 \text{Map}_d(T^2, S^2) \simeq \widehat{\mathbb{Z}^2} / \langle z^{2|d|} \rangle, \quad d \neq 0,$$

where z is the central generator of the integer Heisenberg group (64). The constant-map section used in Prop. 2.24 is unavailable in these components, so their covariant monodromy requires a separate analysis before applying the representation classification. This motivates the momentum-space investigations in [235, 236, 240].

Comparison to effective K-Matrix CS-Theory. The K -matrix Chern-Simons formalism also constructs ground-state representations of modular transformations, including scalar signature factors and spin-structure dependence ([25, §5.6]; see § A.1.3). Wen derives the corresponding transformations for abelian FQH wavefunctions, retaining scalar information about the chiral central charge [270, §§V–VI]. Thus the filling fraction (178) and degeneracy (179) are only part of its predictions.

A comparison with Thm. 3.44 must match the Heisenberg and modular actions, the spin structure, and the choice of linear lift. The basic even-order example agrees with the one-component Chern-Simons operators as explained after (183); a systematic comparison of all extensions remains to be established. The following calculations compare the specified braiding and dimension formulas. They do not by themselves establish an equivalence or a distinction of physical topological orders.

To this end, recall (Thm. 3.44 iii) that a finite-dimensional irreducible Heisenberg representation with braiding phase $\zeta = e^{\pi i p/K}$, where $\gcd(p, K) = 1$, has dimension K . If such a representation is realized as the full torus ground-state space, this gives the conditional prediction:

$$\begin{array}{ll} \text{braiding angle } \theta: & p/K \\ \text{ground state degeneracy:} & K \end{array} \quad (\text{irreducible Heisenberg restriction; } \gcd(p, K) = 1) \quad (108)$$

(Our approach does not explicitly refer to the filling fraction ν , but traditional theory relates this to the braiding angle, cf. Rem. 3.50 below.)

Remark 3.46. For a fixed quasi-particle species, the exchange phase $\zeta = e^{\pi i \theta}$ determines θ modulo 2 (2). The interferometry discussed in Ex. 3.49 probes the full-braid phase $\zeta^2 = e^{2\pi i \theta}$, hence determines only the residue class

$$\theta \bmod 1 \in \mathbb{Q}/\mathbb{Z}. \quad (109)$$

We call (109) the *full-braid parameter* and use it in the following denominator comparisons. This does not identify ζ with $-\zeta$: an odd integer shift of the total exchange parameter changes the exchange phase. In the convention of [109, (10)], a compensating fermionic exchange factor keeps the total statistics unchanged. The classification in Thm. 3.44 retains ζ itself and distinguishes these signs.

First we will show:

Proposition 3.47 (Comparing to K -matrix predictions in Jain states). *The K -matrix formalism verifies the dimension–denominator relation (108) for all fundamental quasi-particles (114) in all Jain states (110), using the full-braid parameter (Rem. 3.46): the torus ground state degeneracy equals the denominator of θ modulo 1.*

We prove this by the following Lemmas (esp. 3.54 and 3.55), recalling the relevant K -matrix predictions for Jain states and deriving the formulas needed for the comparison.

Beyond the Jain states (110), we consider nonsingular symmetric two-component K -matrices with coprime entries, including the Halperin regime specified in Ex. 3.58. Proposition 3.59 verifies the same numerical relation for their fundamental quasi-particles. Example 3.57 explains why the dimension of a full multi-component state space need not equal the denominator of the braiding angle of one component.

These comparisons establish agreement of the specified dimension and denominator formulas. They do not identify irreducibility for a multi-component observable algebra with irreducibility upon restriction to one integer Heisenberg group, nor do they extend Thm. 3.44 to all covariant representations.

We proceed to establish these results.

Remark 3.48 (Jain sequence of filling fractions). We recall that the *Jain sequence* ([136, 137, 138]) of FQH filling fractions consists of those of the form

$$\nu = \frac{n}{2pn \pm 1}, \quad n, p \in \mathbb{N}_{\geq 1}. \quad (110)$$

It is filling fractions of this Jain form which currently dominate the experimentally observed situations and which are generally expected to be thermodynamically robust, whereas non-Jain FQH states are generally expected to be fragile.

For comparison with the Jain predictions, we recall two Abelian cases studied by interferometry:

Example 3.49 (Experimental full-braid phases). The interferometry results below are consistent with the indicated full-braid phases $e^{2\pi i\theta}$. We use the orientation convention of (180); θ is specified modulo 1.

Filling fraction	Full-braid parameter	Interferometry
$\nu = 1/3$	$\theta \equiv 1/3$	[192]
$\nu = 2/5$	$\theta \equiv -2/5 \equiv 3/5$	[193]

In particular, [193] extracts the full-braid angle $\theta_a = (-0.43 \pm 0.05) 2\pi$, consistent with $-4\pi/5$. This determines ζ^2 , not the choice of exchange phase $\zeta = \pm e^{\pi i\theta}$. Current-correlation experiments provide further evidence for fractional statistics [18, 218, 107]; this should be distinguished from extracting a numerical braid phase. In particular, [107] explicitly qualifies the quantitative connection between the measured correlations and a specific exchange phase.

Remark 3.50 (Braiding phases in the Jain sequence). For the positive Jain branch $\nu = n/(2pn + 1)$, the quasi-particle statistics parameter in [109, (3)] is $(2p(n - 1) + 1)/(2pn + 1)$. The same paper uses the parameter $-2p/(2pn + 1)$ after separating off fermionic exchange statistics [109, (10)]. These differ by an integer and hence give the same full-braid phase $e^{2\pi i\theta}$, as relevant to (109) (cf. [138, §9.8]).¹⁸

In the notation for both Jain branches, and with the braiding convention of (180), the fundamental quasi-particle has

$$\nu = \frac{n}{2pn + s} \implies \theta_{\text{Jain}} \equiv -\frac{2ps}{2pn + s} \pmod{1}, \quad p, n \in \mathbb{N}_{\geq 1}, \quad s \in \{+1, -1\}. \quad (111)$$

We recover this formula directly from the Jain K -matrices below, in Lem. 3.55 with $r = 1$. For example, it gives $1/3$ modulo 1 at $\nu = 1/3$ and $3/5 \equiv -2/5$ modulo 1 at $\nu = 2/5$, agreeing with Ex. 3.49 in the stated orientation. Reversing a braid conjugates its phase, replacing θ by $-\theta$; this differs from an integer shift of the statistics parameter.

The quasi-particle species matters independently of orientation. At $\nu = 2/5$, the matrix (112) is $K = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ with $Q = (1, 1)^t$. The fundamental weight $W = (1, 0)^t$ has charge magnitude $e|Q^t K^{-1} W| = e/5$ and statistics parameter $W^t K^{-1} W = 3/5$. A full electromagnetic flux insertion instead has weight $W = Q$, charge magnitude $2e/5$, and parameter $Q^t K^{-1} Q = 2/5$. It is a composite of fundamental quasi-particles; its phase is not the fundamental quasi-particle phase measured in [193].

The denominator comparison in Prop. 3.47 is insensitive to both integer shifts and orientation reversal. It uses (111) for a fundamental quasi-particle; the weight $W = Q$ instead describes the full multifundamental combination (114).

¹⁸For earlier work on fractional statistics and filling fractions, see [257]. We do not use a universal identification of the braiding angle with $\pm\nu$: a comparison must specify the quasi-particle species and the orientation convention for braiding.

Example 3.51 (Symmetric K -Matrix formulation for Jain states). FQH liquids at Jain filling fractions $\nu = \frac{n}{2pn \pm 1}$ (110) are meant to be described in the K -matrix formalism (§ A.1.3, here in the “symmetric basis”) by $n \times n$ K -matrices and an n -component charge vector Q of the following form (slightly generalizing [272, (5.9)][268, (4.53)] to include the negative sign branch):

$$K := \begin{pmatrix} 2p \pm 1 & 2p & 2p & 2p & \cdots \\ 2p & 2p \pm 1 & 2p & 2p & \cdots \\ 2p & 2p & 2p \pm 1 & 2p & \cdots \\ 2p & 2p & 2p & 2p \pm 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad Q = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}. \quad (112)$$

Hence

$$K = 2pQQ^t \pm I. \quad (113)$$

Moreover, the weight vector W for a “fundamental quasi-particle” is meant [268, p. 32] to have a single entry 1 and all other entries 0, while the “dominating” quasi-particle has a weight vector with all entries 1. Interpolating between these two cases, and since the ordering of the fundamental quasi-particle species is irrelevant, we will consider weight vectors:

$$W_{1\dots r} := \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \text{ } r\text{th entry} \quad \text{for } 1 \leq r \leq n. \quad (114)$$

In this notation, the fundamental quasi-particle is $W_1 := W_{1\dots 1}$.

Lemma 3.52 (cf. [99, §2]). *The inverse of the K -matrix (112) exists and is given by:*

$$K^{-1} = \pm I - \frac{2p}{1 \pm 2pn} QQ^t. \quad (115)$$

Proof. We make the ansatz:

$$K^{-1} = aI + bQQ^t$$

and solve $KK^{-1} = I$ for $a, b \in \mathbb{R}$. The condition is that:

$$(\pm I + 2pQQ^t)(aI + bQQ^t) = I.$$

Observing that $(QQ^t)^2 = nQQ^t$, this is equivalent to the matrix equation

$$\pm aI + (\pm b + 2pa + 2pnb)QQ^t = I.$$

Taking $a = \pm 1$ and $b = -2p/(1 \pm 2pn)$ makes the first coefficient equal to 1 and the second equal to 0, proving the displayed inverse directly. This verification also applies when $n = 1$, where I and QQ^t are not linearly independent. $\square$

A special case of the following result is claimed in [272, below (5.9)], but it is worth making this explicit:

Lemma 3.53. *For the matrices in (112), the relation (178) for the filling fraction holds, in that:*

$$Q^t K^{-1} Q = \frac{n}{2pn \pm 1}. \quad (116)$$

Proof. From (115), and observing that $Q^t Q = n$, we obtain:

$$Q^t K^{-1} Q = \pm n - \frac{2p}{1 \pm 2pn} n^2 = \frac{\pm n(1 \pm 2pn) - 2pn^2}{1 \pm 2pn} = \frac{n}{2pn \pm 1}, \quad (117)$$

as claimed. $\square$

For completeness, we derive the familiar Jain torus degeneracy (179) directly from the symmetric K -matrix (112); the formula is stated explicitly in [170, §II, following (2.4)]:

Lemma 3.54. *The K -matrix ground state degeneracy (179) for the Jain state K -matrix (112) equals the denominator of the Jain filling fraction, in that:*

$$|\det(K)| = 2pn \pm 1. \quad (118)$$

Proof. Using that determinants remain unchanged when adding one column to another, consider adding to the first column all other columns. This makes all entries in the first column equal to $2pn \pm 1$. Therefore:

$$\det(K) = (2pn \pm 1) \det(K'),$$

where K' is like K except that all entries in its first column equal 1.

Using next that determinants are also unchanged when subtracting one row from another, consider subtracting the first row of K' from each of the other rows. This results in a matrix K'' as shown in parentheses in the following equation:

$$\det(K) = (2pn \pm 1) \det(K'') = (2pn \pm 1) \det \begin{pmatrix} 1 & 2p & 2p & 2p & \cdots \\ 0 & \pm 1 & 0 & 0 & \cdots \\ 0 & 0 & \pm 1 & 0 & \cdots \\ 0 & 0 & 0 & \pm 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

With K'' being upper triangular, its determinant equals the product of its diagonal entries and hence has unit absolute value. This proves the claim. $\square$

Next, we evaluate the braiding parameters for the weight vectors (114):

Lemma 3.55. *The K -matrix braiding angles (180) corresponding to the weight vectors $W_{1\dots r}$ (114) are, modulo $\mathbb{Z}$ (109):*

$$\theta_{w_{1\dots r}} \bmod 1 = \mp \frac{2pr^2}{2pn \pm 1} \bmod 1. \quad (119)$$

Proof. Using that $W_{1\dots r}^t W_{1\dots r} = r = W_{1\dots r}^t Q$ and $W_{1\dots n} = Q$, we find from (115) and in generalization of (117):

$$\begin{aligned} W_{1\dots r}^t K^{-1} W_{1\dots r} &= \pm r - \frac{2p}{1 \pm 2pn} r^2 = \frac{\pm r(1 \pm 2pn) - 2pr^2}{1 \pm 2pn} = \frac{\pm r + 2pr(n - r)}{1 \pm 2pn} \\ &= \frac{r \pm 2pr(n - r)}{2pn \pm 1} = \frac{\pm r(2pn \pm 1) \mp 2pr^2}{2pn \pm 1} = \pm r \mp \frac{2pr^2}{2pn \pm 1}. \quad \square \end{aligned}$$

This essentially completes the proof of the claim of Prop. 3.47; we just need to isolate which of the fractions in (119) are actually reduced.

Proof of Prop. 3.47. By Lems. 3.54 and 3.55 the claim holds for all multifundamental quasi-particle species $W_{1\dots r}$ (114) for which the fraction in (119) is reduced, hence for which

$$\begin{aligned} \gcd(2pr^2, 2pn \pm 1) &= 1 \\ \Leftrightarrow \gcd(r^2, 2pn \pm 1) &= 1 \\ \Leftrightarrow \gcd(r, 2pn \pm 1) &= 1. \end{aligned} \quad (120)$$

This clearly subsumes the fundamental quasi-particle $W_{1\dots 1}$, as claimed. (It also works for the full multicomponent quasi-particle $W_{1\dots n}$.) $\square$

Remark 3.56. For each specified root ζ , Thm. 3.44 supplies extensions of the indicated Heisenberg representations for both the pp and aa spin structures. The even numerator in (119) is not an existence criterion for these extensions: that equation specifies an angle modulo 1, and a choice of representative fixes the sign of $\zeta = e^{\pi i \theta}$, while ζ^2 is independent of it. Comparison with a spin Chern-Simons theory requires matching the spin structure, quadratic refinement, and scalar phases of a linear lift. Such spin-dependent modular data are available in [25, §5.6.1]; see also § A.1.3.

Of course, it is easy to find K -matrix scenarios that violate the relation (108):

Example 3.57. K -matrix theory predicts that two FQH liquids of filling fraction $1/p$, for $p \in \mathbb{N}_{>0}$, may inhabit the same physical space *without* interacting, as such described by

$$K := \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}, \quad Q := \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (121)$$

The predicted ground state degeneracy (179) would be $|\det(K)| = p^2$, while the braiding angles (180) of the fundamental quasi-particles would have denominator just p .

The dimension p^2 in (121) is the dimension of the tensor product of the two p -dimensional component Hilbert spaces. Such a tensor product is irreducible for the product of the two component observable algebras when each factor is irreducible. Thus its dimension does not contradict (108), which assumes an irreducible representation of a single integer Heisenberg group. For example, restricting to the first component alone gives p copies of its p -dimensional representation; this restriction does not describe the full two-component observable algebra.

For the following symmetric two-component matrices, the same numerical relation does hold when the entries are coprime and the matrix is nonsingular. This is an arithmetic statement and does not assert that nonzero coupling alone forces an indecomposable topological order.

Example 3.58 (Symmetric two-component K -matrices and the Halperin regime). Consider the following family of symmetric two-component K -matrices and charge vectors, which includes the coprime Halperin cases (cf. [272, p. 2299][92, §IV.A]):

$$K = \begin{pmatrix} p & l \\ l & p \end{pmatrix}, \quad Q := \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{for} \quad \begin{cases} p \in \mathbb{N}_{\geq 0}, l \in \mathbb{Z} \setminus \{0\} \\ \gcd(p, l) = 1, \quad p^2 \neq l^2. \end{cases} \quad (122)$$

The conventional homogeneous holomorphic Halperin (p, p, l) regime has $p > l \geq 0$, with p odd for fermionic constituents and even for bosonic constituents. In the nonzero-coupling family displayed here this means $p > l > 0$. In particular, the eigenvalues $p \pm l$ are positive; the plasma analysis in [92, §IV.A, (14)] explains the phase-separation instability for $l > p$. The following arithmetic result holds throughout the larger nonsingular family (122), without requiring a homogeneous Halperin-wavefunction realization.

Proposition 3.59. *For every symmetric two-component K -matrix and charge vector in (122), the torus ground state degeneracy predicted by (179) coincides with the denominator of the full-braid parameter (109) computed from (180) for either fundamental quasi-particle.*

Proof. This is a straightforward computation: The ground state degeneracy (179) is:

$$|\det(K)| = |p^2 - l^2|. \quad (123)$$

Moreover, observing that the inverse K -matrix is

$$K^{-1} = \frac{1}{p^2 - l^2} \begin{pmatrix} p & -l \\ -l & p \end{pmatrix},$$

we find the braiding angle (180) of the fundamental quasi-particles W_i (114) to be

$$W_i^t K^{-1} W_i = \frac{p}{p^2 - l^2}.$$

This fraction has nonzero denominator and is in lowest terms, since $\gcd(p, p^2 - l^2) = \gcd(p, l^2) = 1$. Hence the absolute value of its denominator equals $|p^2 - l^2|$, as claimed. $\square$

3.5 On punctured surfaces

Here we derive the observables on 2-cohomotopically quantized topological flux over n -punctured surfaces, which in practice will mean: Surfaces of conducting material where magnetic flux is *expelled* from (the vicinity of) n defects (cf. Rem. 3.8).

Covariantization of these observables reveals an action of the surface's n -braid group (cf. Prop. 2.24). We determine here the action of the Artin half-twists supported in a disk containing the punctures. The puncture contribution is the lattice $\mathbb{Z}^n / \mathbb{Z}(1, \dots, 1)$: integer labels at the punctures, considered modulo a common shift. This is the dual of the sum-zero lattice, and distinguishing these two integral forms of the standard representation is essential for interpreting the resulting groups in terms of framed braids (140), as considered in §3.6.

The group calculations in this subsection use orientation-preserving diffeomorphisms. For a fixed spin structure, their covariant versions are restricted to its stabilizer, as in Rem. 2.3. In §3.6–3.8 we choose the spin structure extending across the punctures from the unique spin structure on the sphere or disk. Every oriented mapping class in those calculations preserves this choice.

Lemma 3.60 (Homotopy type of compactified n -punctured surface). *For $n \in \mathbb{N}_{\geq 1}$, the one-point compactification of the n -puncturing of a compact surface (possibly with boundary) $\Sigma_{g,b}^2$ (9) is homotopy equivalent to the wedge sum (207) of that surface with $(n-1)$ circles:*

$$(\Sigma_{g,b,n}^2)_{\cup\{\infty\}} \simeq \Sigma_{g,b}^2 \vee \bigvee_{n-1} S^1. \quad (124)$$

Proof. For $n = 1$ the statement is immediate.

For $n = 2$ consider the topological space X obtained by attaching to $\Sigma_{g,b}^2$ an interval with endpoints glued to two distinct points $s_1, s_2 \in \Sigma_{g,b}^2$ (the would-be positions of the punctures), hence consider this pushout of topological spaces:

$$\begin{array}{ccc} S^0 & \xrightarrow{(s_1, s_2)} & \Sigma_{g,b}^2 \\ \downarrow & & \downarrow \\ D^1 & \xrightarrow[\iota_{\text{ext}}]{(\text{po})} & X. \end{array}$$

Moreover, consider another arc *inside* $\Sigma_{g,b}^2$ connecting these two points

$$D^1 \xrightarrow{\iota_{\text{int}}} \Sigma_{g,b}^2 \hookrightarrow X.$$

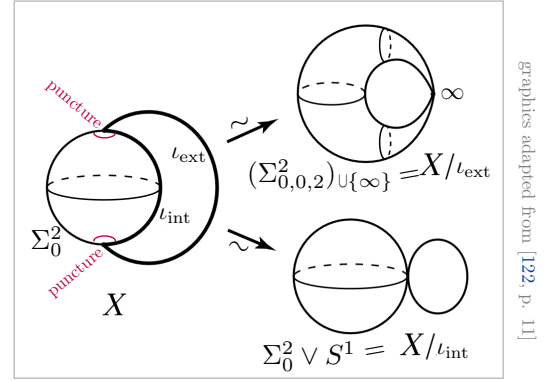
Both of these arcs are evidently contractible sub-complexes of X , and so the quotient projections obtained by identifying either arc with a single point are weak homotopy equivalences (cf. [122, p. 11]):

$$\begin{array}{ccc} & \xrightarrow{\simeq_{\mathcal{J}}} X \xrightarrow{\simeq_{\mathcal{J}}} & \\ X/\iota_{\text{ext}}(D^1) & \xrightarrow[\simeq_{\mathcal{J}}]{\text{dashed}} & X/\iota_{\text{int}}(D^1). \end{array} \quad (125)$$

Now, as indicated in (125), the “external” quotient on the left is evidently homeomorphic to the desired one-point compactification, while the “internal” quotient on the right is evidently homeomorphic to the claimed wedge sum. This proves the claim for $n = 2$.

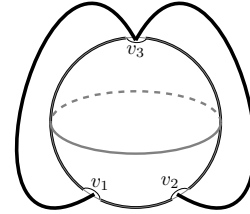
The graphic on the right illustrates the situation for the case $g, b = 0$.

The general statement, including the case $n > 2$, follows analogously by attaching further arcs in this fashion, cf. Fig. A. $\square$



graphics adapted from [122, p. 11]

Figure A. There are several ways to attach arcs for $n > 2$ punctures in the above proof of Lem. 3.60, all equivalent in the resulting homotopy type. But for the analysis of braiding that follows in Thm. 3.64 it is useful (cf. Fig. S) to single out one puncture v_n and take the $n - 1$ arcs to connect this one puncture to each of the $n - 1$ remaining ones. The case $g, b = 0$ and $n = 3$ is illustrated on the right.



With this result in hand, it is straightforward to compute the solitonic flux monodromy (20) through a punctured surface:

Proposition 3.61 (Flux monodromy through punctured surface). *For $g, b \in \mathbb{N}$ and $n \in \mathbb{N}_{>0}$, we have an isomorphism*

$$\pi_1\left(\text{Map}_0^*\left((\Sigma_{g,b,n}^2) \cup \{\infty\}, S^2\right)\right) \simeq \pi_1\left(\text{Map}_0^*\left(\Sigma_{g,b}^2, S^2\right)\right) \times \mathbb{Z}^{n-1}. \quad (126)$$

Proof. We may compute as follows:

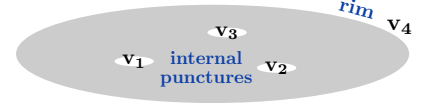
$$\begin{aligned} \pi_1\left(\text{Map}_0^*\left((\Sigma_{g,b,n}^2) \cup \{\infty\}, S^2\right)\right) &\simeq \pi_1\left(\text{Map}_0^*\left(\Sigma_{g,b}^2 \vee \bigvee_{n-1} S^1, S^2\right)\right) && \text{by (124)} \\ &= \pi_1\left(\text{Map}_0^*\left(\Sigma_{g,b}^2, S^2\right) \times \prod_{n-1} \text{Map}^*(S^1, S^2)\right) && \text{by (208)} \\ &= \pi_1\left(\text{Map}_0^*\left(\Sigma_{g,b}^2, S^2\right)\right) \times \prod_{n-1} \pi_1\left(\text{Map}^*(S^1, S^2)\right) \\ &= \pi_1\left(\text{Map}_0^*\left(\Sigma_{g,b}^2, S^2\right)\right) \times \prod_{n-1} \pi_2(S^2) && \text{by (205),} \end{aligned}$$

whence the claim follows by $\pi_2(S^2) \simeq \mathbb{Z}$. $\square$

Remark 3.62 (Internal punctures and the rim of the sample).

- (i) In words, Prop. 3.61 may be understood as saying that the (monodromy of) solitonic flux on an n -punctured surface, for $n \geq 1$, has (not n but) $n-1$ generators associated with $n-1$ of the punctures, while associated with the remaining puncture is the solitonic flux of the surface itself, regarded as containing its point-at-infinity.
- (ii) Below in Thm. 3.64 we give further analysis of this situation, showing that the Artin half-twists act on the puncture factor by the dual integral standard representation of the symmetric group Sym_n (Def. 3.63 below), but first note that the discrepancy between n and $n-1$ here has a clear physical meaning at least for $g = 0$, recalling that the n -punctured sphere is homeomorphic to the $(n-1)$ -punctured plane, hence the $(n-1)$ -punctured open disk $\Sigma_{0,0,n}^2 \simeq \mathbb{R}^2 \setminus \{x_1, \dots, x_{n-1}\}$ (10):

We may think of the n th puncture as modeling the outer rim of the slab of material that is represented by Σ^2 , and of the remaining $n-1$ punctures as the actual material defects as one will recognize them in the laboratory.



- (iii) With this physical interpretation, the rim puncture is distinguished: motions of the internal defects preserve it. The full mapping class group of the punctured sphere also allows permutations that move this puncture. The examples in §3.7 and §3.8 first treat that full group and then describe the restriction preserving the rim. A disk with its actual boundary fixed pointwise is the separate case of Prop. 3.67(ii).

Definition 3.63 (Standard representation and its integral lattices). For $n \geq 2$ the “standard” $\mathbb{C}$ -linear representation of the symmetric group Sym_n is the $n-1$ -dimensional complex irrep classified by the partition $(n-1, 1)$, hence by the Young diagram $\begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \square & & & \end{array}$: this is the quotient of the defining n -dimensional permutation representation by the trivial 1-dimensional representation (cf. [89, p. 9 & Ex. 4.6][149, Def. 2.5]).

More concretely, with respect to the canonical linear basis

$$\mathbb{C}^n \simeq \mathbb{C}\langle v_1, v_2, \dots, v_n \rangle, \quad (127)$$

- the *defining permutation representation* of Sym_n is given, for $\sigma \in \text{Sym}_n$, by $\sigma(v_i) := v_{\sigma(i)}$, hence in terms of the Artin generators $(b_i)_{i=1}^{n-1}$ (26) by

$$b_i(v_j) = \begin{cases} v_{i+1} & | \quad j = i \\ v_i & | \quad j = i+1 \\ v_j & | \quad \text{otherwise,} \end{cases}$$

- the trivial 1-dimensional irrep inside this is

$$\mathbf{1} \simeq \mathbb{C}\langle \underbrace{v_1 + v_2 + \dots + v_n}_{=: t} \rangle \hookrightarrow \mathbb{C}^n, \quad (128)$$

- and the *standard representation* is:

$$\mathbf{n-1} \simeq \mathbb{C}\langle \underbrace{v_i - v_n}_{=: e_i} \rangle_{i=1}^{n-1} \hookrightarrow \mathbb{C}^n, \quad (129)$$

with the adjacent transpositions acting on these basis vectors as follows (in the first line, $1 \leq i < n-1$):

$$b_i(e_j) = \begin{cases} e_{i+1} & | \quad j = i \\ e_i & | \quad j = i+1 \\ e_j & | \quad \text{otherwise} \end{cases} \quad (130)$$

$$b_{n-1}(e_j) = \begin{cases} e_j - e_{n-1} & | \quad j < n-1 \\ -e_{n-1} & | \quad j = n-1. \end{cases} \quad (131)$$

The $\mathbb{Z}$ -span of these basis vectors is the sum-zero lattice

$$A_{n-1} := \left\{ (a_1, \dots, a_n) \in \mathbb{Z}^n \mid \sum_{i=1}^n a_i = 0 \right\} = \mathbb{Z}\langle e_1, \dots, e_{n-1} \rangle.$$

We denote its action by st and call it the *integral standard representation*:

$$\begin{aligned} \text{st} : \text{Sym}_n &\longrightarrow \text{Aut}_{\mathbb{Z}}(A_{n-1}), \\ \text{st}(\sigma) \left(\sum_{i=1}^n a_i v_i \right) &:= \sum_{i=1}^n a_i v_{\sigma(i)}. \end{aligned} \quad (132)$$

Hence over $\mathbb{C}$ we have a reduction of the defining Sym_n -representation explicitly like this:

$$\begin{array}{ccc}
t & \mapsto & v_1 + \cdots + v_n \\
e_i & \mapsto & v_i - v_n \\
\mathbf{1} \oplus \mathbf{n-1} & \xrightarrow{\sim} & \mathbb{C}_{\text{def}}^n \in \text{Rep}_{\mathbb{C}}(\text{Sym}_n) \\
\frac{t-(e_1+\cdots+e_{n-1})}{n} & \longleftarrow & v_n \\
e_i + \frac{t-(e_1+\cdots+e_{n-1})}{n} & \longleftarrow & v_{i < n},
\end{array}$$

which also shows that over the integers we only have a monomorphism

$$\mathbb{Z} \oplus A_{n-1} \hookrightarrow \mathbb{Z}_{\text{def}}^n \in \text{Rep}_{\mathbb{Z}}(\text{Sym}_n) \quad (133)$$

with image the subgroup of n -tuples whose sum is divisible by n .

The *dual integral standard representation*, denoted $\text{st}^\vee$, is

$$\begin{aligned}
\text{st}^\vee : \text{Sym}_n &\longrightarrow \text{Aut}_{\mathbb{Z}}(A_{n-1}^\vee), \\
(\text{st}^\vee(\sigma)\varphi)(x) &:= \varphi(\text{st}(\sigma^{-1})x).
\end{aligned} \quad (134)$$

Here st is defined in (132), and the dual lattice is

$$A_{n-1}^\vee := \text{Hom}_{\mathbb{Z}}(A_{n-1}, \mathbb{Z}) \simeq \mathbb{Z}^n / \mathbb{Z}(1, \dots, 1).$$

Here the isomorphism is induced by the map from $\mathbb{Z}^n$ sending $(a_1, \dots, a_n)$ to the functional $x \mapsto \sum_{i=1}^n a_i x_i$ on A_{n-1} . This map is surjective since the values on $e_i = v_i - v_n$ can be prescribed arbitrarily, and its kernel consists exactly of the constant tuples. The permutation action on the quotient agrees with the dual action (134). In particular, the evaluation coordinates on this dual lattice are

$$\begin{aligned}
A_{n-1}^\vee &\simeq \mathbb{Z}^{n-1} \\
[(a_1, \dots, a_n)] &\longmapsto (q_1, \dots, q_{n-1}) = (a_1 - a_n, \dots, a_{n-1} - a_n).
\end{aligned}$$

Under $\text{st}^\vee$ (134), the adjacent transposition b_i exchanges q_i and q_{i+1} for $i < n-1$, while the last one acts by

$$\text{st}^\vee(b_{n-1})(q_1, \dots, q_{n-1}) = (q_1 - q_{n-1}, \dots, q_{n-2} - q_{n-1}, -q_{n-1}).$$

These are formulas for the values of a functional on the basis e_i ; (130) and (131) instead give the images of the basis vectors themselves.

The two lattices have isomorphic complexifications, but for $n \geq 3$ they are not isomorphic as integral Sym_n -representations. For example, their coinvariants (the quotients by all elements $\sigma x - x$) are

$$(A_{n-1})_{\text{Sym}_n} = 0, \quad (A_{n-1}^\vee)_{\text{Sym}_n} \simeq \mathbb{Z}/n\mathbb{Z} \quad (n \geq 3).$$

Indeed, all differences $v_i - v_j$ with $i \neq j$ have the same class in the first quotient, and $(v_1 - v_2) + (v_2 - v_3) = v_1 - v_3$ forces this class to vanish. In the second quotient all v_i have the same class, with the sole remaining relation $n[v_1] = 0$. For $n = 2$, both lattices carry the sign representation; for $n = 1$, we set $A_0 = A_0^\vee = 0$.

Now, the main result of this section:

Theorem 3.64 (Braid group action on flux monodromy over punctured surface). *For $n \geq 1$, choose a disk in $\Sigma_{g,b}^2$ containing all punctures and use arcs in this disk for the decomposition (126). The action (46) of the Artin half-twists $b_i \in \text{Br}_n(\Sigma_{g,b}^2) \rightarrow \text{MCG}(\Sigma_{g,b,n}^2)$ (40), represented by diffeomorphisms supported in this disk, is via the dual integral standard representation $\text{st}^\vee$ (134) on the puncture factor*

$$\mathbb{Z}^{n-1} \simeq A_{n-1}^\vee \simeq \mathbb{Z}^n / \mathbb{Z}(1, \dots, 1)$$

and the identity on the unpunctured-surface factor. In particular, this action of the disk braid group factors through Sym_n .

Proof. See Fig. S for an illustration of the following analysis.

We may, without restriction, assume the punctures to jointly sit within an open disk inside the surface, $\{v_1, \dots, v_n\} \subset D^2 \subset \Sigma_{g,b}^2$. Then the one-point compactification $(\Sigma_{g,b,n}^2)_{\cup\{\infty\}}$ may be obtained by enlarging each puncture to a little missing open disk, erecting a little cone (horn) over the boundary of this disk, and making these cones bend over to make (just) their tips touch – this joint tip is the point ∞ . Let then

$$\ell_i \in \pi_0 \text{Map}^*(S^1, (\Sigma_{g,b,n}^2)_{\cup\{\infty\}}) \simeq \pi_1((\Sigma_{g,b,n}^2)_{\cup\{\infty\}}), \quad i \in \{1, \dots, n-1\} \quad (135)$$

denote the homotopy class of a loop that starts at ∞ , runs down through the i th cone and back through the n th cone

$$\ell_i := \begin{array}{c} \infty \\ \nearrow \quad \searrow \\ i \quad \quad n \end{array}.$$

This is also illustrated by the dashed arrows in the top panel of Fig. S, which, for graphical convenience, shows not the cones themselves, but the arcs whose contraction to $\{\infty\}$ produces the cones, according to the proof of Lem. 3.60.

Now consider a based map $f : (\Sigma_{g,b,n}^2)_{\cup\{\infty\}} \rightarrow \Omega S^2$, representing a class

$$[f] \in \pi_1 \text{Map}^*((\Sigma_{g,b,n}^2)_{\cup\{\infty\}}, S^2) \underset{(216)}{\simeq} \pi_0 \text{Map}^*((\Sigma_{g,b,n}^2)_{\cup\{\infty\}}, \Omega S^2)$$

with its homotopy class decomposed, according to Prop. 3.61, as

$$\begin{aligned} \pi_0 \text{Map}^*((\Sigma_{g,b,n}^2)_{\cup\{\infty\}}, \Omega S^2) &\simeq \pi_0 \text{Map}^*(\Sigma_{g,b}^2, \Omega S^2) \times \prod_{n-1} \pi_0 \text{Map}^*(S^1, \Omega S^2) \\ [f] &\longmapsto ([\tilde{f}], (q_1, \dots, q_{n-1})), \end{aligned} \quad (136)$$

where the integer classes $q_i := [f_* \ell_i] \in \mathbb{Z}$ come from the restriction of f to these loops ℓ_i (135):

$$\begin{aligned} \pi_0 \text{Map}^*(S^1, (\Sigma_{g,b,n}^2)_{\cup\{\infty\}}) &\xrightarrow{f_*} \pi_0 \text{Map}^*(S^1, \Omega S^2) \simeq \mathbb{Z} \\ \ell_i &\longmapsto q_i. \end{aligned} \quad (137)$$

We need to determine the effect on these components of precomposition with a diffeomorphism of $\Sigma_{g,b,n}^2$ representing the mapping class of the i th Artin generator b_i (25). This diffeo may be chosen such that its unique continuous extension to the one-point compactification,

$$b_i \in \pi_0 \text{Map}^*((\Sigma_{g,b,n}^2)_{\cup\{\infty\}}, (\Sigma_{g,b,n}^2)_{\cup\{\infty\}}),$$

restricts for each $j \in \{1, \dots, n\}$ to a homeomorphism from the j th cone onto the $\sigma(j)$ th cone, where σ is the permutation underlying the Artin generator. Then direct inspection (illustrated in Fig. S) shows

- for $i < n - 1$ that

$$\begin{aligned} \pi_0 \text{Map}^*(S^1, (\Sigma_{g,b,n}^2)_{\cup\{\infty\}}) &\xrightarrow{b_{i*}} \pi_0 \text{Map}^*(S^1, (\Sigma_{g,b,n}^2)_{\cup\{\infty\}}) \xrightarrow{f_*} \pi_0 \text{Map}^*(S^1, \Omega S^2) \\ \ell_j &\xrightarrow[\text{left of Fig. S}]{\quad} \left\{ \begin{array}{l|l} \ell_{i+1} & j = i \\ \ell_i & j = i + 1 \\ \ell_j & \text{otherwise} \end{array} \right\} \xrightarrow[(137)]{\quad} \left\{ \begin{array}{l|l} q_{i+1} & j = i \\ q_i & j = i + 1 \\ q_j & \text{otherwise} \end{array} \right\}, \end{aligned} \quad (138)$$

- for $i = n - 1$ that

$$\begin{aligned} \pi_0 \text{Map}^*(S^1, (\Sigma_{g,b,n}^2)_{\cup\{\infty\}}) &\xrightarrow{b_{i*}} \pi_0 \text{Map}^*(S^1, (\Sigma_{g,b,n}^2)_{\cup\{\infty\}}) \xrightarrow{f_*} \pi_0 \text{Map}^*(S^1, \Omega S^2) \\ \ell_j &\xrightarrow[\text{right of Fig. S}]{\quad} \left\{ \begin{array}{l|l} \ell_{n-1}^{-1} \circ \ell_j & j < n - 1 \\ \ell_{n-1}^{-1} & j = n - 1 \end{array} \right\} \xrightarrow[(137)]{\quad} \left\{ \begin{array}{l|l} q_j - q_{n-1} & j < n - 1 \\ -q_{n-1} & j = n - 1 \end{array} \right\}. \end{aligned} \quad (139)$$

These formulas describe the action on the integer values q_j , hence on functionals on the puncture-loop lattice. More precisely, identify the abelianized puncture loops $[\ell_j]$ with the basis $e_j = v_j - v_n$ of A_{n-1} . A tuple $(q_1, \dots, q_{n-1})$ specifies the functional $\varphi \in A_{n-1}^\vee$ with $\varphi(e_j) = q_j$. The transformations (138) and (139) are exactly the dual action $\text{st}^\vee$ (134): for an adjacent transposition, its inverse is itself, so by (132) we have $\varphi \circ \text{st}(b_i) = \text{st}^\vee(b_i)\varphi$. This identifies the puncture contribution with $A_{n-1}^\vee \simeq \mathbb{Z}^n / \mathbb{Z}(1, \dots, 1)$.

To see also that the first component is unchanged, let $P := \{v_1, \dots, v_n\}$ and write

$$\pi : \Sigma_{g,b}^2 \longrightarrow \Sigma_{g,b}^2 / P \simeq (\Sigma_{g,b,n}^2)_{\cup\{\infty\}}$$

for the quotient map. Under the decomposition (136), the unpunctured-surface component is $[\tilde{f}] = [f \circ \pi]$, with v_n as basepoint on $\Sigma_{g,b}^2$. If h_i represents b_i on $\Sigma_{g,b}^2$ and $\bar{h}_i$ is its induced map on the quotient, then $\bar{h}_i \circ \pi = \pi \circ h_i$. After forgetting the punctures, h_i is isotopic to the identity: its mapping class is the trivial image under the composite

$$\text{Br}_n(\Sigma_{g,b}^2) \rightarrow \text{MCG}(\Sigma_{g,b,n}^2) \rightarrow \text{MCG}(\Sigma_{g,b}^2).$$

This is also the exactness assertion in the generalized Birman sequence (39). Consequently, $f \circ \bar{h}_i \circ \pi$ and $f \circ \pi$ are freely homotopic. Since ΩS^2 is a connected H -space, the action of its fundamental group on based homotopy classes is trivial, so their based homotopy classes agree as well [122, Prop. 4A.2 & Ex. 4A.3].

Altogether, using $\text{st}^\vee$ (134), the action on the components (136) is

$$b_{i*} : ([\tilde{f}], (q_1, \dots, q_{n-1})) \longmapsto ([\tilde{f}], \text{st}^\vee(b_i)(q_1, \dots, q_{n-1})).$$

The calculation concerns the Artin half-twists in the chosen disk; additional surface-braid generators, such as motions around handles, are not included in this assertion. $\square$

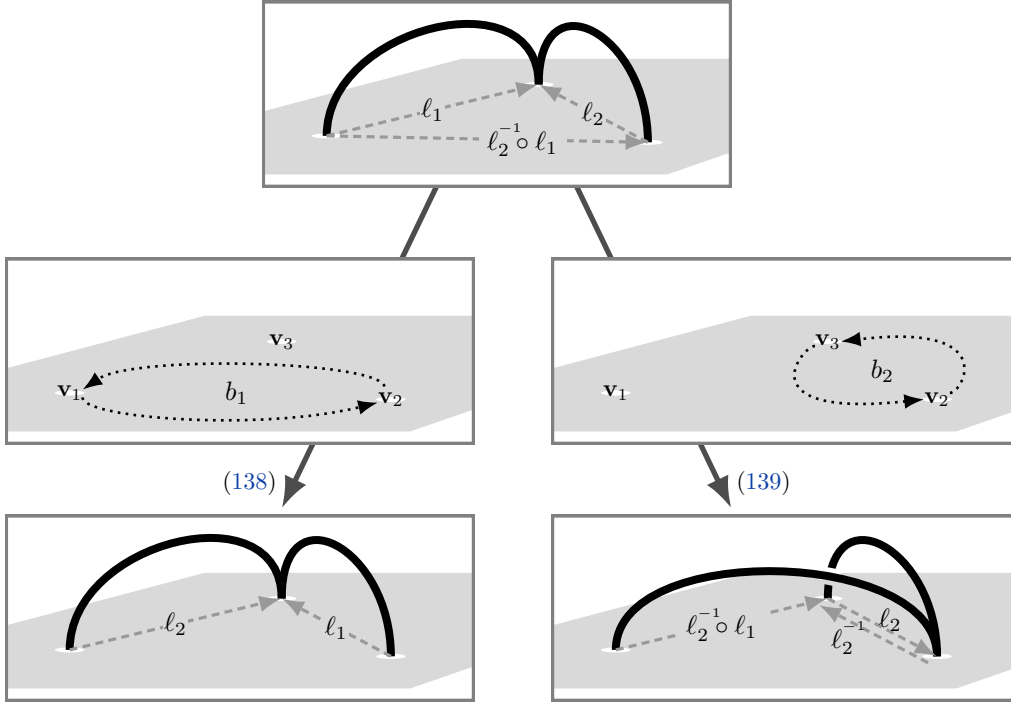


Figure S – Effect of braiding on compactification of punctured surface – proof of Thm. 3.64. Using that the homotopy type of the 1-point compactification $(\Sigma_{g,b,n}^2) \cup \{\infty\}$ of a punctured surface is obtained by attaching, for all $j < n$, an arc from the j th to the n th puncture (cf. the proof of Lem. 3.60), the above graphic shows the effect of the Artin generator (26) mapping classes $b_i \in \text{Br}_n(\Sigma_{g,b}^2) \rightarrow \text{MCG}(\Sigma_{g,b,n}^2)$ on these arcs — and on the indicated generators $\ell_j \in \pi_1((\Sigma_{g,b,n}^2) \cup \{\infty\}, \infty)$ obtained after contracting the arcs to $\{\infty\}$ — making manifest that the induced action on the integer evaluations $q_j = [f_* \ell_j]$ is the dual integral standard representation $\text{st}^\vee$ (134) of the symmetric group Sym_n .

3.6 On punctured disks

We now apply Thm. 3.64 to punctured disks and spheres. To describe the resulting groups, we distinguish the sum-zero framing lattice A_{n-1} from the quotient lattice

$$Q_{n-1} := A_{n-1}^\vee \simeq \mathbb{Z}^n / \mathbb{Z}(1, \dots, 1).$$

Their permutation actions are st (132) and $\text{st}^\vee$ (134), respectively. It is the latter lattice that occurs in the flux monodromy.

Definition 3.65 (Framed braids and algebraic framing groups [179][156], cf. [159, §3.2]). For a surface Σ^2 , define the *algebraic framing group* by the permutation action of $\text{Br}_n(\Sigma^2) \twoheadrightarrow \text{Sym}_n$ (23):

$$\text{FBr}_n^{\text{alg}}(\Sigma^2) := \mathbb{Z} \wr \text{Br}_n(\Sigma^2) \simeq \mathbb{Z}^n \rtimes_{\text{def}} \text{Br}_n(\Sigma^2). \quad (140)$$

On the disk, a choice of trivialization of the tangent bundle identifies this with the usual *framed* or *ribbon braid group*, denoted

$$\text{FBr}_n := \text{FBr}_n^{\text{alg}}(D^2) \simeq \mathbb{Z}^n \rtimes_{\text{def}} \text{Br}_n.$$

Each integer records a full twist of one ribbon, as in Fig. FL; the closure is a framed link.

For a general closed surface, (140) is an algebraic construction. Geometric framed surface braids are instead defined using configurations of nonzero tangent vectors, and the resulting framing extension need not split [23, Def. 1 & Thm. 5]. Already for a single strand on the sphere, the geometric group is $\pi_1(US^2) \simeq \pi_1(\text{SO}(3)) \simeq \mathbb{Z}_2$, where US^2 denotes the unit tangent bundle, whereas $\text{FBr}_1^{\text{alg}}(S^2) \simeq \mathbb{Z}$.

The quotient framing lattice and the sum-zero lattice enter two different exact sequences:

Lemma 3.66 (Framing quotients and sum-zero subgroups). For $n \geq 1$, let H be a group equipped with a homomorphism $H \rightarrow \text{Sym}_n$, and put $\mathbf{1} := (1, \dots, 1) \in \mathbb{Z}^n$. Then we have a central exact sequence

$$\begin{aligned} 1 \rightarrow \mathbb{Z} \xrightarrow{\delta} \mathbb{Z}^n \rtimes_{\text{def}} H \xrightarrow{p} Q_{n-1} \rtimes_{\text{st}^\vee} H \rightarrow 1 \\ k \mapsto (k\mathbf{1}, 1_H) \\ (a, h) \mapsto ([a], h). \end{aligned} \quad (141)$$

Thus its quotient identifies framing tuples that differ by a common integer shift. There is also an exact sequence of Sym_n -modules (with trivial action on the last term)

$$\begin{aligned} 0 \rightarrow A_{n-1} \xrightarrow{j} Q_{n-1} \xrightarrow{\bar{\Sigma}} \mathbb{Z}/n\mathbb{Z} \rightarrow 0 \\ a \mapsto [a] \\ [b] \mapsto \sum_{i=1}^n b_i \bmod n. \end{aligned} \quad (142)$$

Here j intertwines st (132) with $\text{st}^\vee$ (134).

For comparison, the sum-zero lattice itself gives the following subgroup inclusions, using st (132):

$$\begin{array}{ccccc} \text{algebraically} & \text{with vanishing} & \text{with total framing} & \text{with arbitrary} & \\ \text{framed braids} & \text{total framing} & \text{divisible by } n & \text{total framing} & \\ \mathbb{Z}^{n-1} \rtimes_{\text{st}} \text{Br}_n(\Sigma^2) & \hookrightarrow & \mathbb{Z} \times (\mathbb{Z}^{n-1} \rtimes_{\text{st}} \text{Br}_n(\Sigma^2)) & \hookrightarrow & \mathbb{Z}^n \rtimes_{\text{def}} \text{Br}_n(\Sigma^2) \equiv \text{FBr}_n^{\text{alg}}(\Sigma^2) \\ \text{standard rep} & & \text{standard rep} & & \text{defining rep} \\ e_i & \mapsto & e_i & \mapsto & v_i - v_n \\ & & t & \mapsto & v_1 + \dots + v_n \\ b_i & \mapsto & b_i & \mapsto & b_i \end{array} \quad (143)$$

These inclusions concern A_{n-1} ; the quotient in (141) instead uses Q_{n-1} and $\text{st}^\vee$ (134).

Proof. Since H permutes coordinates, it fixes $\mathbf{1}$, so $\delta(\mathbb{Z})$ is central. The map p is a surjective homomorphism with precisely this kernel, proving (141).

In (142), the map j is injective since $A_{n-1} \cap \mathbb{Z}\mathbf{1} = 0$. The map $\bar{\Sigma}$ is well-defined because replacing a by $a + k\mathbf{1}$ changes its coordinate sum by nk . It is surjective, and if $\sum_i a_i = nk$, then $a - k\mathbf{1} \in A_{n-1}$ represents the same class in Q_{n-1} . This proves exactness. Both maps commute with coordinate permutations.

Finally, the inclusions (143) follow from (133): the first lattice is A_{n-1} , and the image of $\mathbb{Z} \oplus A_{n-1}$ in $\mathbb{Z}^n$ consists of the tuples whose sum is divisible by n . $\square$

Proposition 3.67 (2-Cohomotopical covariant flux monodromy on punctured disks). Equip the punctured sphere or disk with the spin structure extending across the punctures from the sphere or disk. For $n \geq 1$, put $H_n := \text{MCG}(\Sigma_{0,0,n}^2)$. The 2-Cohomotopical covariant flux monodromy is as follows:

- (i) On the n -punctured sphere, it is the product of the unpunctured-sphere contribution with the semidirect product for $\text{st}^\vee$ (134):

$$\begin{aligned} G_n^{\text{sph}} &:= \pi_1 \left(\text{Map}_0^*((\Sigma_{0,0,n}^2) \cup \{\infty\}, S^2) // \text{Diff}^{+, \partial}(\Sigma_{0,0,n}^2) \right) \\ &\simeq \mathbb{Z} \times (Q_{n-1} \rtimes_{\text{st}^\vee} H_n). \end{aligned} \quad (144)$$

The first $\mathbb{Z}$ is the factor $\pi_3(S^2)$.

- (ii) On the n -punctured closed disk, it is the quotient of the usual framed braid group by the simultaneous unit framing shift:

$$\begin{aligned} G_n^{\text{disk}} &:= \pi_1 \left(\text{Map}_0^*((\Sigma_{0,1,n}^2) \cup \{\infty\}, S^2) // \text{Diff}^{+, \partial}(\Sigma_{0,1,n}^2) \right) \\ &\simeq Q_{n-1} \rtimes_{\text{st}^\vee} \text{Br}_n \\ &\simeq \text{FBr}_n / \langle (1, 1) \rangle. \end{aligned} \quad (145)$$

The action here is again (134); the element $(1, 1)$ changes every framing by one and has trivial underlying braid.

For the spherical mapping class group, the small cases and the spherical braid description are

$$H_n \simeq \begin{cases} 1 & | \quad n = 1 \\ \mathbb{Z}_2 & | \quad n = 2 \\ \text{Br}_n(S^2)/\langle \text{rot} \rangle & | \quad n \geq 3. \end{cases}$$

In the last line, rot is the central full rotation of order two from (42); it induces the trivial permutation and hence acts trivially on Q_{n-1} . In particular,

$$G_1^{\text{sph}} \simeq \mathbb{Z}, \quad G_2^{\text{sph}} \simeq \mathbb{Z} \times (\mathbb{Z} \rtimes_{-1} \mathbb{Z}_2), \quad G_1^{\text{disk}} = 1.$$

Proof. The constant map is fixed by every diffeomorphism of the punctured surface. It therefore gives a section of the homotopy-quotient fibration over $B\text{Diff}^{+, \partial}(\Sigma_{0,b,n}^2)$. Its homotopy exact sequence consequently yields a split extension of the flux-monodromy group by $\pi_0 \text{Diff}^{+, \partial}(\Sigma_{0,b,n}^2) = \text{MCG}(\Sigma_{0,b,n}^2)$. This argument also applies for $n = 1, 2$; it does not require the identity component of the diffeomorphism group to be contractible.

For $b \in \{0, 1\}$, we thus have

$$\begin{aligned} & \pi_1 \left(\text{Map}_0^*((\Sigma_{0,b,n}^2) \cup \{\infty\}, S^2) // \text{Diff}^{+, \partial}(\Sigma_{0,b,n}^2) \right) \\ & \simeq \pi_1 \left(\text{Map}_0^*((\Sigma_{0,b,n}^2) \cup \{\infty\}, S^2) \right) \rtimes \text{MCG}(\Sigma_{0,b,n}^2) \quad \text{by the fixed constant map} \\ & \simeq \left(\pi_1 \left(\text{Map}_0^*(\Sigma_{0,b}^2, S^2) \right) \times Q_{n-1} \right) \rtimes \text{MCG}(\Sigma_{0,b,n}^2) \quad \text{by (126)} \\ & \simeq \pi_1 \left(\text{Map}_0^*(\Sigma_{0,b}^2, S^2) \right) \times \left(Q_{n-1} \rtimes_{\text{st}^\vee} \text{MCG}(\Sigma_{0,b,n}^2) \right) \quad \text{by Thm. 3.64.} \end{aligned} \tag{146}$$

Here $\text{st}^\vee$ is (134). For these genus-zero surfaces, the mapping class group is generated by the Artin half-twists from a disk containing the punctures, so the cited theorem determines its entire action.

For $b = 1$, the surface $\Sigma_{0,1}^2 = D^2$ is contractible, so the first factor in (146) is trivial. Its mapping class group with n punctures is Br_n , and (141) gives (145).

For $b = 0$, the first factor is $\pi_1 \text{Map}_0^*(S^2, S^2) \simeq \pi_3(S^2) \simeq \mathbb{Z}$, giving (144). The descriptions of H_n follow from (43) and (42), with $H_1 = 1$ and $H_2 = \text{Sym}_2$ treated separately. For $n = 2$, the dual integral standard representation is the sign action on $Q_1 \simeq \mathbb{Z}$; for $n = 1$, $Q_0 = 0$. $\square$

3.7 On the open annulus

On the open annulus $\Sigma_{0,0,2}^2$, hence on the 2-punctured sphere (10), choose the spin structure extending across both punctures. Prop. 3.67 gives the covariantized flux monodromy group

$$\mathbb{Z} \times (Q_1 \rtimes_{\text{st}^\vee} \text{Sym}_2) \simeq \mathbb{Z} \times (\mathbb{Z} \rtimes_{-1} \mathbb{Z}_2). \tag{147}$$

Here $Q_1 = \mathbb{Z}^2/\mathbb{Z}(1,1) \simeq \mathbb{Z}$, and the dual integral standard representation $\text{st}^\vee$ (134) is the sign action. The nontrivial mapping class interchanges the two punctures, hence the two ends of the open annulus. The corresponding semidirect product is the *infinite dihedral group*:

$$\mathbb{Z} \rtimes \mathbb{Z}_2 \simeq \left\{ (n, s \bmod 2) \in \mathbb{Z} \times \mathbb{Z}_2, \quad (n, s \bmod 2) \cdot (n', s' \bmod 2) = (n + (-1)^s n', (s + s') \bmod 2) \right\}. \tag{148}$$

Lemma 3.68 (Finite-dimensional unitary irreps over the open annulus). *The finite-dimensional irreducible unitary representations of $\mathbb{Z} \rtimes_{-1} \mathbb{Z}_2$ (148) are, up to unitary isomorphism, the four 1-dimensional representations*

$$\begin{aligned} & \mathbb{Z} \rtimes \mathbb{Z}_2 \longrightarrow \text{U}(\mathbb{C}) \\ \chi_{++} & : (n, s \bmod 2) \longmapsto 1 \\ \chi_{+-} & : (n, s \bmod 2) \longmapsto e^{\pi i s} \\ \chi_{-+} & : (n, s \bmod 2) \longmapsto e^{\pi i n} \\ \chi_{--} & : (n, s \bmod 2) \longmapsto e^{\pi i (n+s)} \end{aligned} \tag{149}$$

and, for $\lambda \in \text{U}(1) \setminus \{+1, -1\}$, the 2-dimensional representations

$$\begin{aligned} \rho_\lambda & : \mathbb{Z} \rtimes \mathbb{Z}_2 \longrightarrow \text{U}(\mathbb{C}^2) \\ (1, 0 \bmod 2) & \longmapsto D_\lambda := \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix} \\ (0, 1 \bmod 2) & \longmapsto X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned} \tag{150}$$

Here $\rho_\lambda \simeq \rho_{\lambda'}$ precisely when $\lambda' \in \{\lambda, \lambda^{-1}\}$.

Every finite-dimensional irreducible unitary representation of the full group (147) is uniquely obtained by choosing one of these irreps ρ , up to the stated equivalence, and a phase $\eta \in \mathbb{U}(1)$, and setting

$$\begin{aligned} \mathbb{Z} \times (\mathbb{Z} \rtimes_{-1} \mathbb{Z}_2) &\longrightarrow \mathbb{U}(\mathcal{H}_\rho) \\ (m, (n, s \bmod 2)) &\longmapsto \eta^m \rho(n, s \bmod 2). \end{aligned} \quad (151)$$

Proof. Write $r = (1, 0 \bmod 2)$ and $s = (0, 1 \bmod 2)$ for the generators of (148), so that $s^2 = 1$ and $srs = r^{-1}$. In a finite-dimensional unitary irrep, choose a unit eigenvector v of $\rho(r)$ with eigenvalue $\lambda^{-1} \in \mathbb{U}(1)$. Then $\rho(s)v$ is an eigenvector with eigenvalue λ , and the span of v and $\rho(s)v$ is invariant under both generators. By irreducibility this is the whole representation space.

If $\lambda \neq \lambda^{-1}$, these two unit eigenvectors are orthogonal, and on this basis the generators act as in (150). Conversely, the two distinct eigenspaces of D_λ are interchanged by X , so these representations are irreducible. Their spectra on r show that the only possible equivalence of parameters is $\lambda \leftrightarrow \lambda^{-1}$, which is implemented by X .

If $\lambda = \lambda^{-1}$, then $\rho(r) = \pm \text{id}$ on the above span. An eigenvector of the unitary involution $\rho(s)$ spans an invariant line, which by irreducibility is the whole representation. The two independent sign choices give (149).

Finally, in an irrep of (147), the generator of the central $\mathbb{Z}$ -factor acts by a scalar $\eta \in \mathbb{U}(1)$, by Schur's lemma. The restriction to the dihedral factor is therefore irreducible, giving (151) and its uniqueness. $\square$

In particular, for any $o \in \mathbb{N}_{>0}$, the irreps that factor through the finite dihedral quotient

$$\begin{aligned} \mathbb{Z} \rtimes \mathbb{Z}_2 &\twoheadrightarrow \mathbb{Z}_o \rtimes \mathbb{Z}_2 \\ (n, s \bmod 2) &\longmapsto (n \bmod o, s \bmod 2) \end{aligned} \quad (152)$$

are exactly those for which $\rho(r)^o = \text{id}$. Thus (149) gives χ_{++} and χ_{+-} for every o , and also χ_{-+} and χ_{--} when o is even. The 2-dimensional irreps are (150) with

$$\lambda = e^{2\pi i \frac{k}{o}}, \quad k \in \{1, \dots, \lfloor (o-1)/2 \rfloor\}, \quad D_\lambda = R_z(4\pi \frac{k}{o}).$$

The parameter η in (151) remains independent of this finite-quotient condition on the dihedral factor. In particular, the two puncture classes in Q_1 sum to zero, so their operators are inverses; their product is the identity, independently of the character of the Hopf $\mathbb{Z}$ -factor.

If one end is distinguished as the outer rim and must be preserved, the end-interchanging mapping class is excluded. The mapping class group of the open annulus preserving each end is trivial, so the covariant flux monodromy reduces to $\mathbb{Z} \times Q_1 \simeq \mathbb{Z}^2$ and its finite-dimensional unitary irreps are all one-dimensional. Thus the two-dimensional sectors above use the symmetry interchanging the two ends.

3.8 On the 2-punctured disk

We now specialize §3.6 to the 2-punctured open disk, hence to the 3-punctured sphere (cf. Rem. 3.62). We first allow all permutations of the three punctures. For the spin structure extending over the sphere, the full mapping class group is Sym_3 , acting on the puncture lattice by $\text{st}^\vee$ (134). This yields irreducible unitary representations of dimensions 1, 2, 3, and 6, as follows. The restriction preserving the rim puncture is described at the end of this subsection.

Example 3.69 (Flux monodromy over the 3-punctured sphere). Prop. 3.67, with (43), gives the group

$$\pi_1\left(\text{Map}_0^*((\Sigma_{0,0,2+1}^2) \cup \{\infty\}, S^2) // \text{Diff}^{+, \partial}(\Sigma_{0,0,2+1}^2)\right) \simeq \mathbb{Z} \times (Q_2 \rtimes_{\text{st}^\vee} \text{Sym}_3) =: G_3. \quad (153)$$

Here $Q_2 = \mathbb{Z}^3 / \mathbb{Z}(1, 1, 1)$, with Sym_3 permuting the classes $[v_1], [v_2], [v_3]$, subject to $[v_1] + [v_2] + [v_3] = 0$. By (141), the puncture factor $Q_2 \rtimes \text{Sym}_3$ is the quotient of $\mathbb{Z}^3 \rtimes \text{Sym}_3$ by the common framing shift. The first $\mathbb{Z}$ -factor in (153) is independently the Hopf factor $\pi_3(S^2)$.

Proposition 3.70 (Finite-dimensional unitary irreps over the 3-punctured sphere). *The unitary characters of Q_2 are parametrized by*

$$\widehat{Q_2} \simeq \left\{ \mathbf{z} = (z_1, z_2, z_3) \in \mathbb{U}(1)^3 \mid z_1 z_2 z_3 = 1 \right\}.$$

Explicitly, the corresponding characters are

$$\begin{aligned} \chi_{\mathbf{z}} : Q_2 &\longrightarrow \mathbb{U}(\mathbb{C}) \\ [(a_1, a_2, a_3)] &\longmapsto z_1^{a_1} z_2^{a_2} z_3^{a_3}. \end{aligned} \quad (154)$$

Choose one representative $\mathbf{z}$ of each Sym_3 -orbit under permutation of coordinates, and write $H_{\mathbf{z}}$ for its stabilizer. The finite-dimensional irreducible unitary representations of G_3 (153) are, up to unitary isomorphism,

$$\rho_{\eta, \mathbf{z}, \sigma}(m, g) := \eta^m \text{Ind}_{Q_2 \rtimes H_{\mathbf{z}}}^{Q_2 \rtimes \text{Sym}_3} (\chi_{\mathbf{z}} \otimes \sigma)(g). \quad (155)$$

Here $\eta \in \text{U}(1)$, and σ runs over the irreducible unitary representations of $H_{\mathbf{z}}$; the representation being induced is $(q, h) \mapsto \chi_{\mathbf{z}}(q)\sigma(h)$. For the chosen orbit representatives, these parameters are unique up to unitary equivalence of σ . Their dimensions are given by

$\mathbf{z}$	$H_{\mathbf{z}}$	$[\text{Sym}_3 : H_{\mathbf{z}}]$	$\dim(\rho_{\eta, \mathbf{z}, \sigma})$
$(z, z, z), z^3 = 1$	Sym_3	1	1, 1, 2
$(z, z, z^{-2}), z^3 \neq 1$	Sym_2	3	3, 3
three distinct coordinates	$\{1\}$	6	6

(156)

In the last column, each entry lists the dimensions obtained from the inequivalent irreps of the indicated stabilizer, for fixed η and fixed orbit of $\mathbf{z}$.

Proof. A character of $\mathbb{Z}^3$ descends to Q_2 precisely when it is trivial on $(1, 1, 1)$, giving (154). For $h \in H_{\mathbf{z}}$, the identity $\chi_{\mathbf{z}}(hq) = \chi_{\mathbf{z}}(q)$ makes $(q, h) \mapsto \chi_{\mathbf{z}}(q)\sigma(h)$ a representation of $Q_2 \rtimes H_{\mathbf{z}}$.

Now the Mackey algorithm (Prop. A.17) gives (155). Explicitly, in a finite-dimensional unitary representation the commuting operators from Q_2 admit a simultaneous orthogonal decomposition into character spaces. The symmetric group permutes these spaces, and irreducibility forces their characters to form a single orbit. The stabilizer acts irreducibly on any one character space, and induction reconstructs the representation from this action. Conversely, inducing an irreducible stabilizer representation in this way is irreducible: every invariant subspace decomposes into character spaces, its intersection with one such space is stabilizer-invariant, and the symmetric group acts transitively on them. The independent central $\mathbb{Z}$ acts by $m \mapsto \eta^m$, by Schur's lemma.

The stabilizers are Sym_3 , Sym_2 , or the trivial group according as the three coordinates are all equal, exactly two are equal, or all are distinct. Multiplying their irreducible representation dimensions by the corresponding orbit sizes gives (156). $\square$

For example, if $\omega = e^{2\pi i/3}$, then $\mathbf{z} = (1, \omega, \omega^2)$ has trivial stabilizer and yields a 6-dimensional irrep. Its $Q_2 \rtimes \text{Sym}_3$ -representation already factors through the finite quotient $(Q_2/3Q_2) \rtimes \text{Sym}_3$. Thus even among representations of finite quotients, the 2-dimensional standard representation of Sym_3 does not exhaust the irreducibles of dimension greater than one.

The 2-dimensional irreps in (156) have a particularly simple description. Write t for the generator of the Hopf $\mathbb{Z}$ -factor, and use $[v_i]$ and $s \in \text{Sym}_3$ also for their canonical inclusions in G_3 . For $\eta \in \text{U}(1)$ and $\xi \in \text{U}(1)$ with $\xi^3 = 1$, these irreps are

$$\begin{aligned} \rho_{\eta, \xi} : \mathbb{Z} \times (Q_2 \rtimes_{\text{st}\vee} \text{Sym}_3) &\longrightarrow \text{U}(\mathbb{C}^2) \\ t &\longmapsto \eta \text{id} \\ [v_i] &\longmapsto \xi \text{id} \\ s &\longmapsto U(s), \end{aligned} \quad (i = 1, 2, 3) \quad (157)$$

where U is the unitarization of the standard representation $\mathbf{2}$ of Sym_3 . The condition $\xi^3 = 1$ is exactly the relation $[v_1] + [v_2] + [v_3] = 0$; the Hopf phase η is independent of ξ . For $\xi = 1$, (157) factors through the projection $G_3 \rightarrow \mathbb{Z} \times \text{Sym}_3$.

Proposition 3.71 (Unitarization of the standard representation of Sym_3). *The standard irrep $\mathbf{2}$ of Sym_3 (Def. 3.63) admits an orthonormal basis in which the generating permutations, written in one-line notation, act by the following matrices:*

(i) *The Pauli Z-matrix:*

$$U(213) = Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(ii) *A rotation matrix (cf. [196, (4.4-6)])*

$$U(231) = R_y(8\pi/3) = -R_y(2\pi/3) := \begin{bmatrix} \cos(4\pi/3) & -\sin(4\pi/3) \\ \sin(4\pi/3) & \cos(4\pi/3) \end{bmatrix} = -\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}. \quad (158)$$

Proof. The defining representation on $\mathbb{C}^3 \simeq \text{Span}_{\mathbb{C}}(v_1, v_2, v_3)$ (127) is evidently unitary with respect to the canonical inner product $\langle v_i | v_j \rangle = \delta_{ij}$, but the basis $(e_1 := v_1 - v_3, e_2 := v_2 - v_3)$, from (129), for the standard subrepresentation $\mathbf{2}$, is not orthonormal with respect to this inner product. One choice of orthonormal basis for this

subspace is given by

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} := \frac{1}{\sqrt{6}}(v_1 + v_2 - 2v_3), \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} := \frac{1}{\sqrt{2}}(v_1 - v_2).$$

A straightforward computation shows that on this basis the permutations (213) and (231) act as claimed. $\square$

Remark 3.72 (Finite projective image). Writing $C := U(231)$ in (158), the displayed matrices satisfy

$$Z^2 = C^3 = \text{id}, \quad ZCZ = C^{-1}.$$

They generate the six distinct matrices $C^j Z^\epsilon$, for $j \in \{0, 1, 2\}$ and $\epsilon \in \{0, 1\}$, giving a faithful copy of Sym_3 . None of its nonidentity elements is scalar. Hence the projective image of every irrep (157) is also Sym_3 : both the Hopf factor and the puncture lattice act by scalar phases. These phases do not supply a relative phase between the two basis states. In particular, these representations by themselves do not provide a universal set of single-qubit gates.

Preserving the outer rim. If v_3 represents the outer rim, the permitted mapping classes lie in its stabilizer $\text{Sym}_2 = \langle (213) \rangle$ in Sym_3 , where we retain the one-line permutation notation. The corresponding subgroup of (153) is

$$G_{3,\text{rim}} := \mathbb{Z} \times (Q_2 \rtimes_{\text{st}\vee} \text{Sym}_2) \subset G_3.$$

The representations (157) restrict reducibly to this subgroup: the Hopf factor and Q_2 still act by scalars, while $\mathbf{2}|_{\text{Sym}_2} \simeq \mathbf{1} \oplus \text{sgn}$. In the basis of Prop. 3.71, the two coordinate lines are invariant and the internal-defect exchange acts by Z . The three-cycle represented by $U(231)$ moves v_3 and hence does not preserve the rim. Thus the projective image of these restricted representations is $\{[\text{id}], [Z]\}$.

Other two-dimensional irreps of $G_{3,\text{rim}}$ can occur: a character of Q_2 not fixed by the transposition has an orbit of size two and induces an irreducible representation. More generally, the same character-space argument as above shows that the irreducible dimensions for this subgroup are 1 and 2. For an actual closed disk with two internal punctures and boundary fixed pointwise, Prop. 3.67 (ii) instead gives $Q_1 \rtimes \text{Br}_2$. This is a different surface and boundary condition; its braid-group factor is retained, rather than replaced by its permutation image.

4 Conclusion & Outlook

Seeking further theoretical understanding of topological quantum order, specifically in experimentally promising FQH systems, we analyzed a novel non-Lagrangian description of FQH anyons, arising by taking seriously the fact that these are bound states of surplus *flux quanta* seen against the backdrop of an exact rational filling fraction. This suggests that effective exotic surplus *flux quantization* may go to the heart of the nature of FQH anyons. We began by recalling motivation for such surplus flux quantization in the nonabelian cohomology theory known as 2-Cohomotopy, a deformation of standard Dirac flux quantization, and then provided a thorough algebro-topological and representation-theoretic analysis of the topological quantum order predicted thereby.

At the heart of this analysis is the proof that the monodromy of 2-cohomotopical flux:

- (i) on the plane witnesses abelian anyon braiding, in fact reproduces abelian Chern-Simons Wilson loop quantum observables, directly equipped with their traditional renormalization choices (which the CS-Lagrangian alone does not provide, cf. [237]),
- (ii) on the torus witnesses in addition the integer Heisenberg group algebra of torus anyon observables, including their modular equivariance depending on the choice of spin structure, and enforcing in superselection sectors the quantization of the braiding phase to a root of unity (cf. § 3.4).

This shows that in familiar situations and for abelian anyons, surplus flux quantization in 2-Cohomotopy matches in fine detail the expected properties of abelian anyons seen in experiment and via effective abelian Chern-Simons theory.

But 2-Cohomotopical flux quantization also makes predictions in further situations that have not been described by other means:

- (i) The same general covariance that on the torus implies the expected modular data predicts on punctured surfaces the monodromy action of nonabelian braid groups describing the braiding of defect anyons attached to the punctures. Physically the punctures here describe defects where (surplus) flux is expelled, suggesting that doping FQH liquids by superconducting islands may be a route towards stable nonabelian anyons in FQH systems (cf. Fig. D).

This prediction is further discussed in [238].

- (ii) The same 2-sphere coefficient that we have analyzed here as classifying exotic effective surplus FQH flux quantization appears naturally as the (Bloch sphere) coefficient of Bloch Hamiltonians in 2-band Fractional Quantum *Anomalous* Hall systems (FQAH). Indeed, under position/momentum space duality the discussion of FQH systems here maps to an analogous discussion of FQAH systems, where it first of all predicts a novel route towards realizing (abelian) anyons in fractional Chern insulators, in the guise of Hopf solitons in the Bloch Hamiltonian, localized in momentum space (cf. Ex. 3.5).

This prediction is further discussed in [235].

- (iii) With the mapping space to the 2-sphere thus understood as the space of (spectrally flattened) Bloch Hamiltonians in 2-band fractional Chern insulators, one may consider imposing further natural physical conditions. One such condition is a bound on the averaged *quantum metric*, reflecting that deformations of realistic systems will not explore the full abstract space of Bloch Hamiltonians. We find that with such a constraint imposed, the situation in item (ii) decomposes effectively into a cyclic layer of $|C|$ FQH bands (C the system's Chern class), permuted under lattice translation, just as previously argued by [212, §I][15, §2] but now exhibiting nonabelian anyonic topological order.

This prediction will be further discussed in [110].

These results support the suggestion that the non-Lagrangian 2-cohomotopical quantization of surplus FQH flux may be a valuable theoretical alternative for the description and understanding of anyonic topological order in fractional quantum (anomalous) Hall systems, especially in the search for new experimental pathways to stable nonabelian anyons desired for serious application in quantum computing technologies.

A Background

Here we briefly recall and cite some background material referred to in the main text:

- §A.1 – Effective CS for FQH
- §A.2 – Some category theory
- §A.3 – Some algebraic topology
- §A.4 – Surfaces & 2-Cohomotopy
- §A.5 – Some representation theory
- §A.6 – Quadratic Gauss sums

A.1 Effective CS for FQH

We recall effective Chern-Simons descriptions of FQH systems, distinguishing their local response formulas, global quantum theories, and phenomenological interpretation. In particular, their ground-state monodromy representations are part of the comparison with our construction.

A.1.1 Traditional Effective Chern-Simons for FQH systems at unit-fraction fillings

The traditional ansatz for an effective field theory description of fractional quantum Hall systems at unit-fraction filling $\nu = 1/K$ postulates that the effective field is a 1-form potential a for the electric current density 2-form J (“statistical gauge field”), itself minimally coupled to the *quasi-hole current* j , and with effective dynamics encoded by the level $= k = K/2$ Chern-Simons (CS) Lagrangian [287][268]:

$$\begin{aligned}
 \text{Electron current density 2-form } J &= \overset{\text{current 3-vector}}{\vec{J}} \lrcorner \text{dvol} =: \text{d} a \quad \text{Effective gauge field} \\
 \text{Quasi-particle current density 2-form } j &= \vec{j} \lrcorner \text{dvol} \\
 \text{Background flux density 2-form } F &= \text{d} A \quad \text{External gauge field} \\
 \text{Effective Lagrangian density 3-form } L &:= \frac{K}{2} \underbrace{a \text{d} a}_{\text{CS}(a)} - \underbrace{A \text{d} a}_{A J} - a j \quad [268, (2.11)], \text{ cf. } [269, (7.3.10)]
 \end{aligned} \tag{159}$$

This is motivated by observing that the Euler-Lagrange equations of this L ,

$$\frac{\delta L}{\delta a} = 0 \quad \Leftrightarrow \quad J = \frac{1}{K} (F + j), \tag{160}$$

in the relevant case of longitudinal electron current and static quasi-particles

$$\begin{aligned}
 J &\equiv J_0 \text{d} x \text{d} y - J_x \text{d} t \text{d} y \\
 j &\equiv j_0 \text{d} x \text{d} y \\
 F &\equiv B \text{d} x \text{d} y - E_y \text{d} t \text{d} y
 \end{aligned} \tag{161}$$

express just the hallmark properties of the FQHE at filling fraction $\nu = 1/K$:

$$(160) \Leftrightarrow \begin{cases} J_x = \frac{1}{K} E_y & \Leftrightarrow \text{Hall conductivity law at } 1/K \text{ filling} \\ J_0 = \frac{1}{K} B + \frac{1}{K} j_0 & \Leftrightarrow \text{each electron absorbs } K \text{ flux quanta, but} \\ & \quad 1/K\text{th surplus electron for each quasi-particle.} \end{cases} \tag{162}$$

Proceeding from this Lagrangian density, it is furthermore traditional (cf. [278, p. 35][261, p. 159]) to consider “integrating out” the effective gauge field a (159), by solving its equations of motion (160) in the case $j = 0$, which is

$$J = \frac{1}{K} F, \tag{163}$$

locally via

$$a := \frac{1}{K} A, \tag{164}$$

and plugging this expression for a back into the Lagrangian density (159) to obtain:

$$\text{Induced CS-Lagrangian for anyonic flux quanta } L' := L|_{\substack{j=0, \\ a=A/K}} = -\frac{1}{2K} A \text{d} A. \tag{165}$$

Expression (165) gives the local Hall response. It does not replace the full theory globally: for a general background connection A , division by K in (164) need not define a connection on a line bundle. As emphasized in [278, p. 35], the dynamical field a must then be retained to describe the topological sectors and fractionally charged excitations.

Setting the external probe $A = 0$ and omitting source insertions j instead isolates the intrinsic Chern-Simons sector ([278, p. 50][261, p. 160]):

$$\text{Intrinsic CS sector for anyonic flux quanta} \quad L'' := L|_{j=0, A=0} = \frac{K}{2} a \, da. \quad (166)$$

The action associated with (166) is globally defined with the appropriate level quantization and, for odd K , spin structure. This specialization permits computing intrinsic ground-state and braiding data; the coupling to A is retained when computing electromagnetic response.

A.1.2 A background choice for surplus flux

To relate the effective fields to surplus flux, we now make a restricted background choice within ordinary integral cohomology. This is a choice of variables and background, rather than a prerequisite for the existence of the quantum Chern-Simons theory (cf. also [279]).

For smooth field configurations solving (160), the corresponding necessary conditions on flux classes are:

- (i) $[J]$ must be integral — the number of electrons,
- (ii) $[F + j]$ must be integrally divisible by K — this characterizes the number of quasi-particles (whose presence reflects the failure of each electron to be bound to K flux quanta).

For $j = 0$, suppose that the background connection A is supplied with a K th root a_{bnd} . At the level of curvatures this gives

$$F \equiv K J_{\text{bnd}} \quad (167)$$

(read: K flux quanta per number of *bound* electrons, the defining property of exact filling fraction $\nu = 1/K$), for J_{bnd} a flux density with local potential

$$\begin{aligned} J_{\text{bnd}} &= da_{\text{bnd}} \\ A &= K a_{\text{bnd}}, \end{aligned} \quad (168)$$

with the integral flux classes satisfying:

$$\begin{aligned} [J_{\text{bnd}}] &\in H^2(-; \mathbb{Z}) \\ [F] &= K [J_{\text{bnd}}]. \end{aligned} \quad (169)$$

This restricts the background to one with a chosen root; it is not a condition imposed on every electromagnetic probe of the theory. On this background, (165) becomes:

$$L' \equiv -\frac{1}{2K} A \, dA = -\frac{K}{2} a_{\text{bnd}} \, da_{\text{bnd}}, \quad (170)$$

which has the usual integral or spin Chern-Simons level quantization.

Before integrating out a , the chosen root permits completing the square in the parent Lagrangian (159):

$$\begin{aligned} L &= \frac{K}{2} a \, da - K a_{\text{bnd}} \, da - a j \\ &\simeq \frac{K}{2} (a - a_{\text{bnd}}) \, d(a - a_{\text{bnd}}) - \frac{K}{2} a_{\text{bnd}} \, da_{\text{bnd}} - a j \\ &= \underbrace{\left(\frac{K}{2} a_{\text{sur}} \, da_{\text{sur}} - a_{\text{sur}} j \right)}_{L_{\text{dyn}}} - \underbrace{\left(\frac{K}{2} a_{\text{bnd}} \, da_{\text{bnd}} + a_{\text{bnd}} j \right)}_{L_{\text{bckg}}}, \end{aligned} \quad (171)$$

where “ $\simeq$ ” means equality up to boundary terms, and where $a_{\text{sur}} := a - a_{\text{bnd}}$ is the potential for the number of *surplus* electrons, hence where we have split

$$\begin{aligned} J &= J_{\text{bnd}} + J_{\text{sur}} \\ a &= a_{\text{bnd}} + a_{\text{sur}}, \end{aligned} \quad (172)$$

where, for smooth fields, a , a_{bnd} , and a_{sur} are connections on line bundles. Against this fixed background, the equation of motion (160) reads:

$$J_{\text{sur}} = \frac{1}{K} j. \quad (173)$$

For smooth integral flux classes this imposes a global compatibility condition on the total source class $[j]$. It does not prohibit individual anyon insertions: a Wilson line is a defect, and the associated gauge field is considered on its complement with prescribed holonomy. Conditions on total charge on a closed surface must be distinguished

from the existence of a local anyon species (cf. [278, §3.1]). The dynamical and background terms are:

$$L_{\text{dyn}} = \frac{K}{2} a_{\text{sur}} da_{\text{sur}} - a_{\text{sur}} j \quad (174a)$$

$$L_{\text{bckg}} = \frac{K}{2} a_{\text{bnd}} da_{\text{bnd}} + a_{\text{bnd}} j \quad (174b)$$

$$L \simeq L_{\text{dyn}} - L_{\text{bckg}}. \quad (174c)$$

When j here is taken to be the distributional current concentrated on a knot/link, the path integral for the dynamical Lagrangian (174a) computes (after the usual renormalization choices, cf. [237, §2.2]) the Wilson loop observable of this link, which expresses the corresponding anyon braiding phase.

We learn here that:

With the chosen background root, the dynamical field may be written as the surplus current potential $a_{\text{sur}} = a - a_{\text{bnd}}$ in (174). This is the background-dependent rewriting of the original effective field a that motivates our comparison with surplus flux quantization.

This finally relates to our conceptual starting point (5), where Lagrangian Chern-Simons theory for surplus flux quanta is replaced by surplus flux quantization in 2-Cohomotopy.

A.1.3 Hierarchical K-Matrix Chern-Simons for FQH at other filling fractions

The argument for the effective Lagrangian (159), which applies only to unit-fraction fillings $\nu = 1/K$, has been generalized ([32][272], review in [268, §2.1][269, §7.3.3][261, (5.31)]) by appeal to the *hierarchical* picture of FQH quasi-particles [117][118], which envisions that the quasi-particles appearing at some unit-fraction filling $1/K_1$ experience a secondary K_2 -fractional quantum Hall effect, now with respect not to the actual magnetic gauge field but the effective gauge field $a^{(1)} := a$ (159).

Repeating the above assumptions (159) on effective FQH fields, this picture motivates the introduction of a *secondary* effective gauge field $a^{(2)}$, for the quasi-particle current j , with its own Chern-Simons dynamics, and hence the generalization of the Lagrangian density (159) to ([268, p. 11][269, p. 300]):

$$L := \frac{K_1}{2} \underbrace{a^{(1)} da^{(1)}}_{\text{CS}(a^{(1)})} - \underbrace{A da^{(1)}}_{AJ} - \underbrace{a^{(1)} da^{(2)}}_{aj} + \frac{K_2}{2} \underbrace{a^{(2)} da^{(2)}}_{\text{CS}(a^{(2)})} \quad (175)$$

$$\simeq K_{ij} \frac{1}{2} a^{(i)} da^{(j)} - Q_i A da^{(i)}, \quad (176)$$

where $\simeq$ denotes equality modulo a total derivative: the density in (175) equals the expression in (176) plus $\frac{1}{2} d(a^{(1)} \wedge a^{(2)})$. On a spacetime with boundary, this contributes the corresponding boundary term to the action. The *K-matrix* $K \equiv (K_{ij})$ and *charge vector* $Q \equiv (Q_i)$ are

$$K := \begin{bmatrix} K_1 & -1 \\ -1 & K_2 \end{bmatrix}, \quad Q := \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (177)$$

This is argued to be an effective description for the filling fraction

$$\nu \equiv \frac{1}{K_1 - \frac{1}{K_2}} = Q^t K^{-1} Q.$$

Proceeding in this manner it is envisioned that n th-order FQH systems are effectively described by a nonsingular symmetric integer $n \times n$ matrix K and an integer charge n -vector Q via a Lagrangian density (175),

- whose **filling fraction** is (cf. [268, (2.31)]):

$$\nu = Q^t K^{-1} Q, \quad (178)$$

- whose **ground state degeneracy** on the closed surface of genus g is argued to be (cf. [272, (1.2)]):

$$\dim(\mathcal{H}_{\Sigma_g^2}) = |\det(K)|^g \quad (179)$$

- which host anyons that are labeled by weight vectors $W \in \mathbb{Z}^n$ and exhibit **braiding phases** (cf. [268, (2.30)]):

$$\zeta_W = e^{\pi i (W^t K^{-1} W)}. \quad (180)$$

Here the *fundamental* anyon species are represented by the integer basis vectors W_i (hence those with components $(W_i)_j = \delta_{ij}$), whose braiding phase among themselves is hence

$$\zeta_{W_i} = e^{\pi i (K^{-1})_{ii}}. \quad (181)$$

General weight vectors $W = \sum_i a_i W_i$ are thought to describe some kind of bound states of the fundamental anyon species W_i , with a_i -fold admixture of the species W_i .

Ground-state monodromy. For a nonsingular symmetric integral K , the torus state space has a basis indexed by $D_K := \mathbb{Z}^n / K\mathbb{Z}^n$, with a spin structure fixed when required. If K has even diagonal entries, its discriminant pairing and quadratic form are

$$b_K([u], [v]) = u^t K^{-1} v \bmod 1, \quad q_K([u]) = \frac{1}{2} u^t K^{-1} u \bmod 1. \quad (182)$$

Writing $\sigma(K)$ for the signature, the generators (31) act, in one linear normalization, by

$$\begin{aligned} \hat{S}|[u]\rangle &= \frac{1}{\sqrt{|D_K|}} \sum_{[v] \in D_K} e^{2\pi i b_K([u], [v])} |[v]\rangle, \\ \hat{T}|[u]\rangle &= e^{-2\pi i \sigma(K)/24} e^{2\pi i q_K([u])} |[u]\rangle. \end{aligned} \quad (183)$$

Fourier orthogonality and the Gauss–Milgram identity give $\hat{S}^4 = 1$ and $(\hat{T}\hat{S})^3 = 1$ in this generator convention; moreover, $\hat{S}^2\hat{T} = \hat{T}\hat{S}^2$. For odd lattices, spin structures and their quadratic refinements enter the corresponding formulas ([25, §§5.6–6]). Thus Chern–Simons theory provides linear transformation data, including scalar phases, in addition to dimensions and braiding angles.

For $K = (K)$ with $K > 0$ even, (183) reduces to the modular operators in (97) with $\zeta = e^{\pi i/K}$ and $c_\zeta = e^{\pi i/12}$. Matching more general cases requires fixing the spin structure and linear normalization. Ground-state transport for abelian FQH wavefunctions, including scalar phases, is treated in [270, §V].

Remark A.1 (Phenomenology of the K-matrix formulation).

- (i) The microscopic hierarchy picture and the composite-fermion picture have distinct motivations and make different microscopic claims ([138, §12.1][139]). Such questions concern the realization of a proposed effective theory; they do not invalidate the mathematical construction of its ground-state representations.
- (ii) The rank of K counts effective gauge fields and need not equal the number of microscopic electron species or physical layers. The formalism describes abelian orders in single-component systems as well as in systems with several layers or spin components (cf. [270, §II.E–G][17][285][128][286]).
- (iii) A proposed K and charge vector Q specify an effective abelian theory. Establishing which material realizes that theory, or deriving it from a microscopic Hamiltonian, is an additional physical task. Our classification of flux-monodromy representations likewise requires a ground-state realization for comparison with experiment (§3.4.2).

A.2 Some category theory

Category theory is the *algebra* in *algebraic topology* (cf. [162, §3][175, §2] and §A.3). We need only basics, but we do need the actual theory (namely adjunctions, hence universal constructions) and not just monoidal (braided-, fusion-, ...) categories regarded as algebraic structures themselves, as is now common in (topological) quantum theory (cf. [124][160]). Broader motivation of category theory for mathematical physicists is in [98]; detailed lecture notes tailored towards our needs here are provided in [245]; for a further introduction we recommend [2][11]; standard monographs are [35][172].

A *category* $\mathcal{C}$ is a class of *objects* $X, Y, \dots$ with prescribed sets $\text{Hom}(X, Y)$ of (homo-)morphisms (“homs”) between them, regarded abstractly as maps $f : X \rightarrow Y$ and ultimately defined by their composition law,

$$(-) \circ (-) : \text{Hom}(Y, Z) \times \text{Hom}(X, Y) \rightarrow \text{Hom}(X, Z), \quad X \xrightarrow{f} Y \xrightarrow{g} Z, \quad \underbrace{\qquad\qquad\qquad}_{g \circ f} \uparrow \quad (184)$$

which is required to be associative and unital in the evident sense.

The archetypical example is the category **Set** of sets, but also each individual set S may be equivalently regarded as a category whose only homs happen to be identities:

$$\text{Set} = \left\{ \begin{array}{l} \text{objects: sets} \\ \text{homs: functions} \end{array} \right\}, \quad \text{Set} \ni S = \left\{ \begin{array}{l} \text{objects: elements} \\ \text{homs: trivial} \end{array} \right\}. \quad (185)$$

We use this occasion to highlight for lay readers the distinction between the notation “ $\rightarrow$ ” for maps (and generally for homs) between sets/objects, in contrast to the notation “ $\mapsto$ ” for the corresponding *assignments* between (generalized) elements:

$$S \xrightarrow{f} S' \\ S \ni s \mapsto f(s) \in S'.$$

More generally, there are the “concrete” categories (cf. [2, §I.5]) of sets with algebraic structure (vector spaces, groups, algebras, representations, modules, ...), with their structure-preserving functions between them (the concrete *homomorphisms*), e.g.:

$$\text{Vec} = \left\{ \begin{array}{l} \text{objects: vector spaces} \\ \text{homs: linear maps} \end{array} \right\}, \quad \text{Grp} = \left\{ \begin{array}{l} \text{objects: groups} \\ \text{homs: homomorphisms} \end{array} \right\}, \quad \text{GRep} = \left\{ \begin{array}{l} \text{objects: representations} \\ \text{homs: intertwiners} \end{array} \right\}. \quad (186)$$

But general categories may have homs without underlying functions of sets. For instance, given a group G , its *delooping groupoid* is the category **BG** with a single object $*$, with homs $* \xrightarrow{g} *$ labeled by group elements, and composition being the group operation (187); while for X a topological space, there is the category called its *fundamental groupoid* $\Pi_1(X)$ (cf. [273, (1.7)]) whose homs are the homotopy classes (fixing endpoints) of continuous paths in X with composition the concatenation of paths:

$$\mathbf{BG} = \left\{ \begin{array}{l} \text{objects: single one} \\ \text{homs: group elements} \end{array} \right\}, \quad \Pi_1(X) = \left\{ \begin{array}{l} \text{objects: points of } X \\ \text{homs: htmpy classes of paths in } X \end{array} \right\}. \quad (187)$$

Generally, a (small) category all of whose homs are invertible is called a *groupoid* (cf. [132]). Here invertible homs are like gauge transformations in that they identify the objects they relate, while retaining the information of *how* the identification happens, whence it is fruitful to think of groupoids as sets of *elements with gauge transformations* between them (cf. [246, pp. 6]):

$$\text{generic groupoid} = \left\{ \begin{array}{l} \text{objects: elements} \\ \text{homs: gauge transformations} \end{array} \right\}. \quad (188)$$

Next, a *functor* $F : \mathcal{C} \rightarrow \mathcal{D}$ between a pair of categories is a function between objects and between sets of morphisms which respects this compositional structure:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \left[\begin{array}{ccc} X & \mapsto & F(X) \\ \downarrow f & & \downarrow F(f) \\ Y & \mapsto & F(Y) \\ \downarrow g & & \downarrow F(g) \\ Z & \mapsto & F(Z) \end{array} \right] & & \left[\begin{array}{ccc} & & \\ & & \\ & & \\ & & \\ & & \end{array} \right] \end{array}$$

There is an evident composition of functors, which is evidently associative and unital, whence categories with

functors between them form a (“very large”) category:

$$\mathbf{CAT} = \left\{ \begin{array}{l} \text{objects: categories} \\ \text{homs: functors} \end{array} \right\}. \quad (189)$$

A *natural transformation* $\alpha : F \Rightarrow G$ between a pair of parallel functors $F, G : \mathcal{C} \Rightarrow \mathcal{D}$ is for each object X of $\mathcal{C}$ a morphism $F(X) \xrightarrow{\alpha(X)} G(X)$ of $\mathcal{D}$ such that the following squares “commute”, meaning that the two possible diagonal composites of morphisms coincide:

$$\begin{array}{ccc} \mathcal{C} & \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \end{array} & \mathcal{D} \\ X \mapsto F(X) & \xrightarrow{\alpha(X)} & G(X) \\ \downarrow f & \downarrow F(f) & \downarrow G(f) \\ Y \mapsto F(Y) & \xrightarrow{\alpha(Y)} & G(Y). \end{array} \quad (190)$$

There is an evident “vertical” composition of such natural transformations $F \xRightarrow{\alpha} G \xRightarrow{\beta} H$ which is evidently associative and unital, and hence functors $\mathcal{C} \rightarrow \mathcal{D}$ with natural transformations between them form a category, called the *functor category*:

$$\mathcal{D}^{\mathcal{C}} = \left\{ \begin{array}{l} \text{objects: functors } \mathcal{C} \rightarrow \mathcal{D} \\ \text{homs: natural transformations} \end{array} \right\}. \quad (191)$$

A good example of the interplay of these notions is the observation that the functor category from a delooping groupoid (187) to the category of vector spaces (186) is the category of group representations (cf. §A.5):

$$G\mathbf{Rep} \simeq \mathbf{Func}(\mathbf{BG}, \mathbf{Vec}). \quad (192)$$

Adjunctions. These notions (categories, functors and natural transformations) famously constitute the substrate of category theory — but what makes it a theory, with non-trivial theorems, is the further notion of *adjunctions*.

First, note that there is also a “horizontal” composition of natural transformations by functors:

$$\begin{array}{ccc} \mathcal{C}' \xrightarrow{L} \mathcal{C} & \begin{array}{c} \xrightarrow{F} \\ \Downarrow \alpha \\ \xrightarrow{G} \end{array} & \mathcal{D} \xrightarrow{R} \mathcal{D}' \\ X' & \longmapsto & R \circ F \circ L(X') \xrightarrow{R \circ \alpha \circ L(X')} R \circ G \circ L(X'). \end{array}$$

Now, an *adjunction* $L \dashv R$ between a pair of back-and-forth functors, $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$, is a pair of natural transformations

$$\begin{array}{ccc} \text{ret} : \text{id}_{\mathcal{C}} \Rightarrow R \circ L & \text{and} & \text{obt} : L \circ R \Rightarrow \text{id}_{\mathcal{D}} \\ \text{“the unit”} & & \text{“the co-unit”} \end{array} \quad (193)$$

such that:

$$\begin{array}{ccc} \mathcal{C} & \begin{array}{c} \xrightarrow{\text{id}_{\mathcal{C}}} \\ \Downarrow \text{ret} \\ \xrightarrow{L} \end{array} & \mathcal{D} \\ \mathcal{C} \xrightarrow{L} \mathcal{D} \xrightarrow{R} \mathcal{C} & \begin{array}{c} \Downarrow \text{obt} \\ \xrightarrow{L} \end{array} & \mathcal{D} \\ & \text{id}_{\mathcal{D}} & \end{array} = \mathcal{C} \begin{array}{c} \xrightarrow{L} \\ \Downarrow \text{id}_L \\ \xrightarrow{L} \end{array} \mathcal{D} \quad (194)$$

and

$$\begin{array}{ccc} \mathcal{D} & \begin{array}{c} \xrightarrow{\text{id}_{\mathcal{D}}} \\ \Downarrow \text{ret} \\ \xrightarrow{R} \end{array} & \mathcal{C} \\ \mathcal{D} \xrightarrow{R} \mathcal{C} \xrightarrow{L} \mathcal{D} & \begin{array}{c} \Downarrow \text{obt} \\ \xrightarrow{R} \end{array} & \mathcal{C} \\ & \text{id}_{\mathcal{C}} & \end{array} = \mathcal{D} \begin{array}{c} \xrightarrow{R} \\ \Downarrow \text{id}_R \\ \xrightarrow{R} \end{array} \mathcal{C}. \quad (195)$$

Remarkably, if an adjunction exists then it is essentially unique and equivalent to there being a natural bijection $\widetilde{(-)}$ (“forming adjoints”) between hom-sets, of this form:

$$\begin{array}{ccc} X & Y & \mapsto \\ \uparrow g & \downarrow f & \\ X' & Y' & \mapsto \end{array} \quad \begin{array}{ccc} \text{Hom}(L(X), Y) & \xleftarrow{\widetilde{(-)}_{X,Y}} & \text{Hom}(X, R(Y)) \\ \downarrow f \circ (-) \circ L(g) & & \downarrow R(f) \circ (-) \circ g \\ \text{Hom}(L(X'), Y') & \xleftarrow{\widetilde{(-)}_{X',Y'}} & \text{Hom}(X', R(Y')) \end{array}$$

(whence the terminological allusion to adjoint operators), and under this identification the (co)unit is the adjoint

of the identity:

$$\text{ret}_X = \widetilde{\text{id}_{L(X)}}, \quad \text{obt}_Y = \widetilde{\text{id}_{R(Y)}}.$$

Finally, an (adjoint-)equivalence between categories, $\mathcal{C} \simeq \mathcal{D}$ is an adjunction where both the unit and the counit are invertible.

Example A.2 (Homotopy fibers and cosets, [77, Def. 1.14, Ex. 1.12]). For $F : \mathcal{X} \rightarrow \mathcal{Y}$ a functor between (small) groupoids and y an object of $\mathcal{Y}$, the *homotopy fiber* of F at y is the groupoid $\text{hofib}_y(F)$ whose objects are pairs (x, c_x) with $x \in \mathcal{X}$ and $c_x : F(x) \rightarrow y$, and whose morphisms from (x, c_x) to $(x', c_{x'})$ are morphisms $f : x \rightarrow x'$ satisfying $c_x = c_{x'} \circ F(f)$. Thus its morphisms are the following commuting triangles:

$$\text{hofib}_y(F) = \left\{ \begin{array}{ccc} x & \xrightarrow{f} & x' \\ F(x) & \xrightarrow{F(f)} & F(x') \\ & \searrow c_x & \swarrow c_{x'} \\ & y & \end{array} \right\}. \quad (196)$$

For $\iota : H \hookrightarrow G$ a subgroup inclusion, the homotopy fiber of $\mathbf{B}\iota : \mathbf{B}H \rightarrow \mathbf{B}G$ (187) is equivalent to the quotient set G/H :

$$\text{hofib}_*(\mathbf{B}\iota) \equiv \left\{ * \xrightarrow{h} * \mid \begin{array}{c} g, g' \in G \\ h \in H \end{array} \right\} \simeq \{g \cdot H \mid g \in G\} = G/H. \quad (197)$$

The key example of adjunctions appearing in the main text is the following classical phenomenon of representation theory (cf. §A.5), there also known as *Frobenius reciprocity*:

Proposition A.3 (Induced representations, cf. [125][198]). Here $G\text{Rep}_{\mathbb{C}}$ denotes all algebraic complex representations, allowing vector spaces of arbitrary dimension. For finite-index subgroup inclusions, induction and coinduction preserve the finite-dimensional subcategories.

(i) For $H \hookrightarrow G$ a subgroup inclusion, the restriction functor $\iota^* : G\text{Rep}_{\mathbb{C}} \rightarrow H\text{Rep}_{\mathbb{C}}$ has

– a left adjoint given by

$$\mathcal{V} \in H\text{Rep}_{\mathbb{C}} \quad \vdash \quad \iota_! \mathcal{V} := \mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathcal{V} \quad (198)$$

where on the right we have the tensor product of (left-with-right) $\mathbb{C}[H]$ -modules equipped with the G -action given by

$$\left. \begin{array}{c} [a, v] \in \mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathcal{V} \\ g \in G \end{array} \right\} \quad \vdash \quad g \cdot [a, v] := [g \cdot a, v].$$

– a right adjoint given by

$$\mathcal{V} \in H\text{Rep}_{\mathbb{C}} \quad \vdash \quad \iota_* \mathcal{V} := \text{hom}_{\mathbb{C}[H]}(\mathbb{C}[G], \mathcal{V}), \quad (199)$$

where on the right we have the vector space of left $\mathbb{C}[H]$ -module homomorphisms equipped with the action

$$\left. \begin{array}{c} f \in \text{hom}_{\mathbb{C}[H]}(\mathbb{C}[G], \mathcal{V}) \\ g \in G \\ a \in \mathbb{C}[G] \end{array} \right\} \quad \vdash \quad (g \cdot f)(a) := f(a \cdot g).$$

making an adjoint triple of functors

$$\begin{array}{ccc} & \xrightarrow{\iota_!} & \\ & \perp & \\ H\text{Rep}_{\mathbb{C}} & \xleftarrow{\iota^*} & G\text{Rep}_{\mathbb{C}} \\ & \perp & \\ & \xrightarrow{\iota_*} & \end{array} \quad (200)$$

(ii) Moreover, if H has finite index in G , then the left and right adjoints are naturally isomorphic:

$$[G : H] := |G/H| < \infty \quad \vdash \quad \iota_! \simeq \iota_*.$$

Remark A.4 (CoUnit of induced representations). In more detail, inspection shows (cf. [198]) that the unit of the left induced representation (198) is given by “inserting” the neutral element

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\text{ret}_y} & \mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathcal{V} \\ v & \longmapsto & [e, v]. \end{array}$$

and the counit of the right induced representation (199) is given by evaluating at the neutral element:

$$\begin{array}{ccc} \text{hom}_{\mathbb{C}[H]}(\mathbb{C}[G], \mathcal{V}) & \xrightarrow{\text{obt}_y} & \mathcal{V} \\ f & \longmapsto & f(e). \end{array}$$

But under the equivalence (192), this adjoint triple (200) is of the form $\text{Func}(\mathbf{B}H, \text{Vec}) \xrightleftharpoons[\perp]{\perp} \text{Func}(\mathbf{B}G, \text{Vec})$ and this is an example of an extremely general kind of adjunctions known as (left and right) *Kan extensions* or *base change* between categories of (higher) local systems, which we just state in the form needed in the main text:

Proposition A.5 (Base change for local systems, cf. [225, Ex. A.18]). *For $F : \mathcal{X} \rightarrow \mathcal{Y}$ a functor between (small) groupoids, its induced precomposition functor on functor categories (191)*

$$F^* : (\mathcal{Y} \rightarrow \text{Vec}_{\mathbb{C}}) \mapsto (\mathcal{X} \xrightarrow{F} \mathcal{Y} \rightarrow \text{Vec}_{\mathbb{C}})$$

has both a left and a right adjoint:

$$\begin{array}{ccc} & \xrightarrow{F_!} & \\ \text{Vec}_{\mathbb{C}}^{\mathcal{X}} & \xleftarrow{F^*} & \text{Vec}_{\mathbb{C}}^{\mathcal{Y}} \\ & \xrightarrow{F_*} & \end{array} \quad (201)$$

which agree when the homotopy fibers (196) of F are finite sets.

A.3 Some algebraic topology

Algebraic topology is the study of topological spaces *up to homotopy* by algebraic means, namely with tools of category theory (cf. [162, §3][175, §2] and §A.2). General background on algebraic topology and homotopy theory may be found in [130][100][273][141][3] [256][77, §1].

Topological spaces. We write

- Top for the category of *compactly generated* topological spaces (cf. [233, Ntn. 1.0.16]) with mapping spaces (cf. [3, §1]) denoted $\text{Map}(-, -)$ and their underlying (hom-)sets denoted $\text{Hom}(-, -)$,
- Top^* for pointed such spaces with pointed maps between them with mapping spaces denoted $\text{Map}^*(-, -)$ and their underlying (hom-)sets denoted $\text{Hom}^*(-, -)$.

The mapping spaces are characterized by natural homeomorphisms (cf. [141, (3.98)][256, Thm. 3.47(a)])

$$\begin{aligned} \text{Map}(X \times Y, Z) &\simeq \text{Map}(X, \text{Map}(Y, Z)) \\ \text{Map}^*(X \wedge Y, Z) &\simeq \text{Map}^*(X, \text{Map}^*(Y, Z)), \end{aligned} \quad (202)$$

where for a space X pointed by $\infty_X \in X$ and a space Y pointed by $\infty_Y \in Y$

- (i) their *smash product* is

$$X \wedge Y := \frac{X \times Y}{\{\infty_X\} \times Y \cup X \times \{\infty_Y\}}, \quad (203)$$

which is symmetric via natural homeomorphisms

$$X \wedge Y \simeq Y \wedge X; \quad (204)$$

for instance:

$$S^1 \wedge S^n \simeq S^{n+1}, \quad \text{so that} \quad \pi_n \text{Map}^*(S^m, X) \simeq \pi_0 \text{Map}^*(S^{n+m}, X) \simeq \pi_{n+m}(X) \quad (205)$$

(here the smash product with the circle is called reduced *suspension* and usually denoted $\Sigma := S^1 \wedge (-)$, but we stick with writing “ $S^1 \wedge$ ” in order not to clash with our use of “ Σ^2 ” for the generic surface);

- (ii) their pointed mapping space is the fiber over the base point of Y of the map ev that evaluates unpointed maps at the base point of X :

$$\text{Map}^*(X, Y) \xrightarrow{\text{fib}_{(\infty_Y)}} \text{Map}(X, Y) \xrightarrow{\text{ev}_{(\infty_X)}} Y. \quad (206)$$

The coproduct of $X, Y \in \text{Top}^*$ is the *wedge sum*

$$X \vee Y := \frac{X \amalg Y}{\{\infty_X, \infty_Y\}}, \quad (207)$$

which in particular means that we have a natural bijection

$$\text{Hom}^*(X \vee Y, Z) \simeq \text{Hom}^*(X, Z) \times \text{Hom}^*(Y, Z). \quad (208)$$

Incidentally, the smash product (203) naturally distributes over finite wedge sums

$$X \wedge (Y \vee Z) \simeq (X \wedge Y) \vee (X \wedge Z).$$

One-point compactification. Here for $X \in \text{Top}^*$ we generically denote its basepoint by $\infty_X \in X$, also speaking of the “point at infinity”, and for X a locally compact Hausdorff space we write $X_{\sqcup\{\infty\}} \in \text{Top}^*$ for its *one-point compactification* (cf. [38, p. 199]), thinking of it as *adjoining a point at infinity*.

For example, stereographic projection gives

$$\mathbb{R}_{\sqcup\{\infty\}}^n \simeq S^n \quad (209)$$

and if a space is already compact, then the adjoined point-at-infinity is disjoint and pointed maps out of the space are identified with plain maps:

$$X \text{ compact Hausdorff} \quad \vdash \quad X_{\sqcup\{\infty\}} \simeq X \sqcup \{\infty\} \quad \text{and} \quad \text{Maps}^*(X_{\sqcup\{\infty\}}, Y) \simeq \text{Maps}(X, Y). \quad (210)$$

We have natural homeomorphisms (cf. [141, Prop. 3.7] [52, Prop. 1.6])

$$(X \sqcup Y)_{\sqcup\{\infty\}} \simeq X_{\sqcup\{\infty\}} \vee Y_{\sqcup\{\infty\}}, \quad (X \times Y)_{\sqcup\{\infty\}} \simeq X_{\sqcup\{\infty\}} \wedge Y_{\sqcup\{\infty\}}, \quad (211)$$

For $Y \in \text{Top}^*$ we denote the connected component of the map constant on ∞_Y by

$$\text{Map}_0(-, Y) \subset \text{Map}(-, Y) \quad \text{and} \quad \text{Map}_0^*(-, Y) \subset \text{Map}^*(-, Y). \quad (212)$$

Proposition A.6 (One-point compactification functorial on proper maps [141, p. 70][52, Prop. 1.6]). *The operation of one-point compactification extends to a functor on the category of locally compact Hausdorff spaces with proper maps between them*

$$(-)_{\sqcup\{\infty\}} : \text{LCHaus}_{\text{PrpMaps}} \longrightarrow \text{CptHaus}^*. \quad (213)$$

Since homeomorphisms are proper, this implies in particular functoriality on homeomorphisms.

Loop spaces. The *based loop space* of $X \in \text{Top}^*$ is

$$\Omega X := \text{Map}^*(S^1, X) \quad (214)$$

whose connected components form the *fundamental group* at the basepoint (cf. [3, §2.5]):

$$\pi_1 X := \pi_0 \Omega X \quad (215)$$

For example, the fundamental group of pointed mapping spaces $X \rightarrow Y$ (based at the map constant on the basepoint of Y) has these alternative expressions:

$$\begin{aligned} \pi_1 \text{Map}^*(X, Y) &\equiv \pi_0 \Omega \text{Map}^*(X, Y) && \text{by (215)} \\ &\equiv \pi_0 \text{Map}^*(S^1, \text{Map}^*(X, Y)) && \text{by (214)} \\ &\simeq \pi_0 \text{Map}^*(S^1 \wedge X, Y) && \text{by (202)} \\ &\simeq \pi_0 \text{Map}^*(X \wedge S^1, Y) && \text{by (204)} \\ &\simeq \pi_0 \text{Map}^*(X, \text{Map}^*(S^1, Y)) && \text{by (202)} \\ &\equiv \pi_0 \text{Map}^*(X, \Omega Y) && \text{by (214)}. \end{aligned} \quad (216)$$

Homotopy. We write Grpd_∞ for the ∞ -category of homotopy types and

$$\mathbb{f} : \text{Top} \longrightarrow \text{Grpd}_\infty$$

for the underlying functor. This means that a *weak homotopy equivalence* between topological spaces is equivalently an equivalence under $\mathbb{f}$:

$$X, Y \in \text{Top} \quad \vdash \quad X \simeq_{\mathbb{f}} Y \quad \Leftrightarrow \quad \mathbb{f}X \xrightarrow[\sim]{\mathbb{f}f} \mathbb{f}Y. \quad (217)$$

Given $f : Y \rightarrow Z$ a map of pointed topological spaces, with homotopy fiber X

$$X \xrightarrow{\text{hofib}(f)} Y \xrightarrow{f} Z$$

the resulting long homotopy fiber sequences

$$\begin{array}{c} \Omega X \longrightarrow \Omega Y \longrightarrow \Omega Z \\ \hookrightarrow X \longrightarrow Y \longrightarrow Z \end{array} \quad (218)$$

pass under $\pi_0(-)$ to long exact sequences of homotopy groups

$$\begin{array}{c} \pi_{n+1}(X) \longrightarrow \pi_{n+1}(Y) \longrightarrow \pi_{n+1}(Z) \\ \searrow \hspace{10em} \nearrow \\ \pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \pi_n(Z) \end{array} \quad (219)$$

Homotopy quotients and Borel construction.

Definition A.7. For $G \curvearrowright X$ a Hausdorff topological group acting continuously on a topological space X , we write

$$X \xrightarrow{q} X // G := X \times_G EG \quad (220)$$

for its *Borel construction* (cf. [233, Ex. 2.3.5]), and call its homotopy type the *homotopy quotient* of the action.

In the special case when $X = *$ we get the traditional *classifying space* (namely of principal G -bundles, cf. [217, Thm. 3.5.1][233, Thm. 4.1.13])

$$* // G \simeq BG \quad (221)$$

whose loop space recovers G up to weak homotopy equivalence

$$\Omega BG \simeq G,$$

hence whose homotopy groups are those of G shifted up in degree:

$$\pi_{n+1}(BG) \simeq \pi_n(G). \quad (222)$$

This makes a long homotopy fiber sequence (218)

$$G \xrightarrow{g \mapsto g(x_0)} X \xrightarrow{q} X // G \longrightarrow BG. \quad (223)$$

Hence if G preserves the connected components of X (such as if X only has one connected component), then the long exact sequence of homotopy groups (219) implies that

$$\pi_0(G) \curvearrowright \pi_0(X) \text{ is trivial} \quad \Rightarrow \quad \pi_0(X) \xrightarrow[\sim]{\pi_0(q)} \pi_0(X // G). \quad (224)$$

A.4 Surfaces & 2-Cohomotopy

We give a streamlined review of the analysis of 2-Cohomotopy moduli of closed surfaces, due to [120], that is used in §3.3, and record some related facts needed there for identifying the modular action on 2-cohomotopical flux monodromy.

Fundamental polygons of closed oriented surfaces. The homeomorphism class of oriented closed surfaces of genus g is represented (cf. [100, Thm. 2.8]) by the quotient space of the regular $4g$ -gon (called a *fundamental polygon* of the surface) obtained by identifying all boundary vertices with a single point and, going clockwise for $k \in \{0, \dots, g-1\}$, the $4k+1$ st boundary edge with the reverse of the $4k+3$ rd, and the $4k+2$ nd with the reverse of the $4k+4$ th. For small g this is illustrated in (80), cf. [122, p. 5]. A more homotopy-theoretic formulation of this statement is as follows.

The fundamental group π_1 of a wedge sum (207) of circles is the free group on the set of summands, whose i th generator is represented by the loop that goes identically through the i th circle summand. For the classification of surfaces of genus g (9) we are concerned with wedge sums of $2g$ circles to be denoted $\bigvee_{i=1}^g (S_a^1 \vee S_b^1)$, whose generators we accordingly denote $(a_i, b_i)_{i=1}^g$.

With this, the classical presentation by fundamental polygons becomes:

Proposition A.8 (Homotopy type of closed oriented surfaces, cf. [120, p. 151]). *The homeomorphism type of the closed oriented surface Σ_g^2 (9) of genus $g \in \mathbb{N}$ is that of the cell attachment (cf. [3, §3.1]) shown on the left here:*

$$\begin{array}{ccccc} S^1 & \xrightarrow{\Pi_{i=1}^g [a_i, b_i]} & \bigvee_{i=1}^g (S_a^1 \vee S_b^1) & \longrightarrow & * \\ \downarrow & & \downarrow i_g & & \downarrow \\ D^2 & \xrightarrow{\quad (po) \quad} & \Sigma_g^2 & \xrightarrow{q_0^g} & S^2, \end{array} \quad (225)$$

whence its homotopy type sits in a long homotopy cofiber sequence of this form:

$$S^1 \xrightarrow{\Pi_{i=1}^g [a_i, b_i]} \bigvee_{i=1}^g (S_a^1 \vee S_b^1) \xrightarrow{i_g} \Sigma_g^2 \xrightarrow{q_0^g} S^2 \xrightarrow{S^1 \wedge (\Pi_{i=1}^g [a_i, b_i])} \bigvee_{i=1}^g (S_a^2 \vee S_b^2). \quad (226)$$

In (225) the attaching map $\prod_i [a_i, b_i] = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots$, is a representative for the element of $\pi_1(\bigvee_i (S_a^1 \vee S_b^1))$ of that same name (the consecutive sequence of edges and reverse edges in the boundary of the fundamental polygon), D^2 is the fundamental polygon itself and the pushout enforces the identification of pairs of its boundary edges. Finally, the connecting map q_0^g sends all of these previous boundary edges to the base point.

Remark A.9 (Compatible surjections of closed surfaces to 2-sphere). By the pasting law for pushouts, (225) also shows that we have canonical projection maps $q_g^{g+1} : \Sigma_{g+1}^2 \rightarrow \Sigma_g^2$ (80) compatible with their maps q_0 to S^2 (225), given by sending just the $g + 1$ st pair of edges to the point:

$$\begin{array}{ccccccc}
 S^1 & \xrightarrow{\prod_i [a_i, b_i]} & \bigvee_{i=1}^{g+1} (S_a^1 \vee S_b^1) & \twoheadrightarrow & \bigvee_{i=1}^g (S_a^1 \vee S_b^1) & \longrightarrow & * \\
 \downarrow & & \downarrow i_{g+1} & & \downarrow i_g & & \downarrow \\
 D^2 & \xrightarrow{\quad (po) \quad} & \Sigma_{g+1}^2 & \xrightarrow{q_g^{g+1}} & \Sigma_g^2 & \xrightarrow{q_0^g} & S^2 \\
 & & \downarrow q_0^g & & \downarrow q_0^g & & \uparrow \\
 & & \Sigma_{g+1}^2 & \xrightarrow{q_g^{g+1}} & \Sigma_g^2 & \xrightarrow{q_0^g} & S^2
 \end{array} \quad (227)$$

We also obtain from this the following re-derivation of the integral cohomology of closed surfaces, which is needed in the main text for identifying the modular group action on 2-cohomotopical flux monodromy but also serves as the blueprint for its 2-cohomotopical variant shown further below in Lem. A.11:

Proposition A.10 (Integral cohomology of closed surfaces). *For closed oriented surfaces, their ordinary integral cohomology in $\deg = 1$ is:*

$$\tilde{H}^1(\Sigma_g^2; \mathbb{Z}) \simeq \mathbb{Z}^{2g}. \quad (228)$$

Proof. The long exact sequence of homotopy groups (219) which is induced by the homotopy fiber sequence obtained by mapping (226) into the classifying space $B\mathbb{Z}$ (3) is, in the relevant part, of this form:

$$\begin{array}{ccccccc}
 \pi_0 \text{Map}^*(S^2, B\mathbb{Z}) & \xrightarrow{(q_0^g)^*} & \pi_0 \text{Map}^*(\Sigma_g^2, B\mathbb{Z}) & \longrightarrow & \pi_0 \text{Map}^*(\bigvee_i (S_a^1 \vee S_b^1), B\mathbb{Z}) & \xrightarrow{(\prod_i [a_i, b_i])^*} & \pi_0 \text{Map}^*(S^1, B\mathbb{Z}) \\
 \parallel & & \parallel & & \parallel & & \parallel \\
 \underbrace{\tilde{H}^1(S^2; \mathbb{Z})}_0 & & \tilde{H}^1(\Sigma_g^2; \mathbb{Z}) & & \prod_{i=1}^g (\underbrace{\tilde{H}^1(S^1; \mathbb{Z})}_{\mathbb{Z}})^2 & & \underbrace{\tilde{H}^1(S^1; \mathbb{Z})}_{\mathbb{Z}}
 \end{array}$$

Hence, to conclude, we need to show that the map on the right — forming the pullback of $B\mathbb{Z}$ -valued maps along $\prod_i [a_i, b_i]$ — is the zero-map under π_0 . This is the case because the pullback classes are still group commutators and as such they vanish since the π_0 group in question is $\simeq \mathbb{Z}$ and hence abelian:

$$\begin{array}{ccc}
 \underbrace{\pi_0 \text{Map}^*(\bigvee_i (S_a^1 \vee S_b^1), B\mathbb{Z})}_{\mathbb{Z}^{2g}} & \xrightarrow{(\prod_i [a_i, b_i])^*} & \underbrace{\pi_0 \text{Map}^*(S^1, B\mathbb{Z})}_{\mathbb{Z}} \\
 ((n_i, m_i))_{i=1}^g & \mapsto & \prod_i \underbrace{[n_i, m_i]}_0
 \end{array} \quad \square$$

In an analogous manner, we find:

Lemma A.11 ([120, Prop. 2]). *The long exact sequence of homotopy groups (219) which is induced by the homotopy fiber sequence obtained by mapping (226) into S^2 truncates to a short exact sequence:*

$$1 \longrightarrow \underbrace{\pi_1 \text{Map}^*(S^2, S^2)}_{\mathbb{Z}} \xrightarrow{(q_0^g)^*} \pi_1 \text{Map}^*(\Sigma_g^2, S^2) \xrightarrow{(i_g)^*} \prod_{i=1}^g (\underbrace{\pi_1 \text{Map}^*(S^1, S^2)}_{\mathbb{Z}})^2 \longrightarrow 1. \quad (229)$$

Proof. It is sufficient to see that pullback of S^2 -valued maps along $\prod_i [a_i, b_i]$ is the zero map under (suspension and) taking π_1 : This is because the pullback classes are still group commutators and as such they vanish since the π_1 groups in question are $\simeq \mathbb{Z}$ and hence abelian:

$$\begin{array}{ccc}
 \underbrace{\pi_1 \text{Map}^*(\bigvee_i (S_a^1 \vee S_b^1), S^2)}_{\mathbb{Z}^{2g}} & \xrightarrow{(\prod_i [a_i, b_i])^*} & \underbrace{\pi_1 \text{Map}^*(S^1, S^2)}_{\mathbb{Z}} \\
 ((n_i, m_i))_{i=1}^g & \mapsto & \prod_i \underbrace{[n_i, m_i]}_0
 \end{array}$$

(and analogously, under suspension, with all copies of S^1 in this formula replaced by S^2). Here in evaluating these groups we have used (208) and stages of (216), thereby identifying all copies of $\mathbb{Z}$ with $\mathbb{Z} \simeq \pi_2(S^2)$ (and with $\mathbb{Z} \simeq \pi_3(S^2)$ for the case with suspension). $\square$

Finally, to see that the statement (229) for the *pointed* mapping space implies the variant statement (65) for the unpointed mapping space which is of our actual interest:

Lemma A.12 ([120, Thm. 1]). *For $g \in \mathbb{N}$, we have a short exact sequence of this form:*

$$1 \rightarrow \pi_1 \operatorname{Map}_0(S^2, S^2) \xrightarrow{(q_0^g)^*} \pi_1 \operatorname{Map}_0(\Sigma_g^2, S^2) \rightarrow \pi_1 \operatorname{Map}_0^*(\bigvee_g(S_a^1 \vee S_b^1), S^2) \rightarrow 1.$$

Proof. Consider the long exact sequences of homotopy groups (219) induced by the evaluation sequences (206) on Σ_g^2 and on S^2 , respectively, with the map between them induced by pullback along q_0^g (225) extended to the short exact sequence from (229):

$$\begin{array}{ccccccc} \pi_2(S^2) & \xlongequal{\quad} & \pi_2(S^2) & & & & \\ \downarrow \delta^0 & & \downarrow \delta^g & & & & \\ 1 \longrightarrow \pi_1 \operatorname{Map}^*(S^2, S^2) & \xrightarrow{(q_0^g)^*} & \pi_1 \operatorname{Map}^*(\Sigma_g^2, S^2) & \xrightarrow{(i_g)^*} & \pi_1 \operatorname{Map}^*(\bigvee_i(S_a^1 \vee S_b^1), S^2) & \longrightarrow & 1 \\ \downarrow j_*^0 & & \downarrow j_*^g & & \nearrow (i_g)^* \circ \overline{j_*^g} & & \\ 1 \dashrightarrow \pi_1 \operatorname{Map}(S^2, S^2) & \dashrightarrow^{(q_0^g)^*} & \pi_1 \operatorname{Map}(\Sigma_g^2, S^2) & & & & \\ \downarrow & & \downarrow & & & & \\ \underbrace{\pi_1(S^2)}_1 & \xlongequal{\quad} & \underbrace{\pi_1(S^2)}_1 & & & & \end{array}$$

Here j_*^0 and j_*^g denote the maps induced by inclusion of pointed maps into unpointed maps. All solid sequences are exact, by construction, horizontally as well as vertically. Using this, a routine diagram chase shows ¹⁹ that the dashed sequence exists and is exact. $\square$

As a corollary, we note:

Lemma A.13 (Flux monodromy mapping to ordinary cohomology). *The cohomology operation from 2-Cohomotopy in degree -1 to integral cohomology in degree 1, induced by the looping of the unit class $1^2 : S^2 \rightarrow B^2\mathbb{Z}$ (51), produces a morphism of short exact sequences:*

$$\begin{array}{ccccccc} 1 \longrightarrow \pi_0 \operatorname{Map}(S^2, \Omega S^2) & \xrightarrow{(q_0^g)^*} & \overbrace{\pi_0 \operatorname{Map}(\Sigma_g^2, \Omega S^2)}^{\pi_1 \operatorname{Map}(\Sigma_g^2, S^2)} & \longrightarrow & \pi_0 \operatorname{Map}^*(\bigvee_i(S_a^1 \vee S_b^1), \Omega S^2) & \longrightarrow & 1 \\ \downarrow (\Omega 1^2)_* & & \downarrow (\Omega 1^2)_* & & \downarrow (\Omega 1^2)_* & & \\ 1 \longrightarrow \underbrace{\pi_0 \operatorname{Map}(S^2, B\mathbb{Z})}_{H^1(S^2; \mathbb{Z}) \simeq 0} & \xrightarrow{(q_0^g)^*} & \underbrace{\pi_0 \operatorname{Map}(\Sigma_g^2, B\mathbb{Z})}_{H^1(\Sigma_g^2; \mathbb{Z}) \simeq \mathbb{Z}^{2g}} & \xrightarrow{\sim} & \underbrace{\pi_0 \operatorname{Map}^*(\bigvee_i(S_a^1 \vee S_b^1), B\mathbb{Z})}_{\mathbb{Z}^{2g}} & \longrightarrow & 1. \end{array}$$

Proof. The top exact sequence is from Lem. A.12, and inspection of the proof there shows immediately that it applies verbatim also with the coefficient ΩS^2 replaced by $B\mathbb{Z}$, throughout (for the exactness of the horizontal sequence in the proof, this is in the proof of Prop. A.10). This gives the bottom exact sequence and the compatibility of the two under the cohomology operation, as claimed. $\square$

A.5 Some Representation Theory

We need only basic representation theory (cf. [250][89][68]) and the Mackey classification of irreps of finite semidirect product groups.

¹⁹To spell it out: Since j_*^g is seen to be surjective, we may define $(i_g)^* \circ \overline{j_*^g}$ on a given element ϕ by choosing any preimage through j_*^g . To see that this is well defined: If $\hat{\phi}, \hat{\phi}'$ are a pair of preimages, their difference is in the image of $\delta^g = (q_0^g)^* \circ \delta^0$, hence in the image of $(q_0^g)^*$ and hence vanishes under $(i_g)^*$. With the map thus existing, surjectivity is immediate from $(i_g)^*$ being surjective.

To see that the dashed $(q_0^g)^*$ is injective: Consider ϕ, ϕ' a pair of elements in the domain with the same image. Since j_*^0 is surjective we may find j_*^0 -preimages $\hat{\phi}, \hat{\phi}'$. By commutativity of the middle square we then have $j_*^g \circ (q_0^g)^*(\hat{\phi}) = j_*^g \circ (q_0^g)^*(\hat{\phi}')$, and so the difference between $(q_0^g)^*(\hat{\phi})$ and $(q_0^g)^*(\hat{\phi}')$ is in the image of $\delta^g = (q_0^g)^* \circ \delta^0$. But since $(q_0^g)^*$ is injective, this means that already the difference between $\hat{\phi}$ and $\hat{\phi}'$ is in the image of δ^0 , hence vanishes under j_*^0 , hence $\phi = \phi'$, which was to be seen.

Finally, to see that the dashed sequence is exact in the middle: By the previous construction, the kernel of $(i_g)^* \circ \overline{j_*^g}$ consists exactly of those ϕ whose j_*^g -preimage is in the kernel of $(i_g)^*$, hence in the image of $(q_0^g)^*$, hence of those ϕ in the image of $(q_0^g)^* \circ j_*^0$, hence in the image of $(q_0^g)^*$ – which was to be shown.

Linear representations. Consider G a group. The cardinality of its underlying set, hence the *order* of G , is denoted $|G|$. A finite-dimensional $\mathbb{C}$ -linear *representation* of G is a group homomorphism $\rho : G \rightarrow \mathrm{GL}(V)$ for V a finite-dimensional vector space, also to be denoted $G \curvearrowright_\rho V$.

- Given a pair (ρ, ρ') of representations, an isomorphism $\eta : \rho \xrightarrow{\sim} \rho'$ is a linear isomorphism of the underlying vector spaces, $\eta : V \xrightarrow{\sim} V'$ such that $\eta \circ \rho = \rho' \circ \eta$ (an “intertwiner”). In particular, with any choice of linear basis $V \xrightarrow{\sim} \mathbb{C}^{\dim(V)}$ a representation ρ is isomorphic to a matrix representation $G \rightarrow \mathrm{GL}_n(\mathbb{C})$ for $n = \dim(V)$.
- Given ρ, ρ' a pair of (matrix) representations, their direct sum $\rho \oplus \rho' : G \rightarrow \mathrm{GL}_{n+n'}(\mathbb{C})$ is represented by the corresponding block-diagonal matrices. A representation ρ is called *irreducible* (“irrep”) if $V \neq 0$ and its only G -invariant linear subspaces are 0 and V . A representation is *semisimple* if it is a direct sum of irreducibles. Finite-dimensional complex representations of finite groups are semisimple by Maschke’s theorem; finite-dimensional unitary representations of any group are semisimple because invariant subspaces have invariant orthogonal complements. In these cases irreducibility is equivalent to having no nontrivial direct-sum decomposition. For general representations the latter condition only defines *indecomposability*.

- For any group G , we denote the set of isomorphism classes $[-]$ of irreducible $\mathbb{C}$ -linear representations by
$$\mathrm{Irr}(G) \simeq \{[\rho_i]\}_{i \in I}. \quad (230)$$

- *Schur’s Lemma* says, in particular, that every intertwining operator η from an irreducible representation to itself (hence every linear operator that commutes with all the representation operators of an irreducible representation) is a scalar multiple of the identity (cf. [68, Cor. 1.17]):

$$\eta \in \mathrm{End}_{\mathbb{C}}(V), \quad n = \dim V, \quad \forall_{g \in G} \quad \eta \circ \rho(g) = \rho(g) \circ \eta \quad \Rightarrow \quad \exists_{z \in \mathbb{C}} \quad \eta = z \cdot \mathrm{id}_n. \quad (231)$$

- For finite G , the *sum-of-squares formula* (cf. [68, Thm. 3.1(ii)]) says that the sum of the squared dimensions of the distinct irreps equals the order of the group:

$$\sum_{i \in I} (\dim(\rho_i))^2 = |G|. \quad (232)$$

Example A.14 (Irreps of $\mathbb{Z}_d$). For $d \in \mathbb{N}_{>0}$, the $\mathbb{C}$ -linear irreps of $\mathbb{Z}_d$, to be denoted $\mathbf{1}_{[n]} \in \mathrm{Rep}_{\mathbb{C}}(\mathbb{Z}_d)$, are all 1-dimensional and labeled by $[n] \in \mathbb{Z}_d$, where $[1] \in \mathbb{Z}_d$ acts on $\mathbf{1}_{[n]}$ by multiplication with $e^{2\pi i \frac{n}{d}}$.

Example A.15 (Irreps of Sym_3 , cf. [89, §1.3]). Irreps of Sym_3 include the trivial 1-dimensional representation $\mathbf{1}$, the sign representation $\mathbf{1}_{\mathrm{sgn}}$, and the standard representation $\mathbf{2}$ (129). Since $1^2 + 1^2 + 2^2 = 6 = |\mathrm{Sym}_3|$, the sum-of-squares formula (232) implies that there are no further distinct irreps.

Proposition A.16 (Unitarization of linear representations, cf. [155, Prop. 4.6]). *For G a finite group and $\mathcal{H}$ a finite-dimensional complex Hilbert space, every $\mathbb{C}$ -linear representation $R : G \rightarrow \mathrm{GL}(\mathcal{H})$ on the underlying complex vector space of $\mathcal{H}$ is isomorphic to a unitary representation $U : G \rightarrow \mathrm{U}(\mathcal{H}) \hookrightarrow \mathrm{GL}(\mathcal{H})$.*

Representation characters. The *character* χ^ρ of a finite-dimensional representation ρ is the function on G assigning the traces of the representation matrices

$$\begin{aligned} G &\xrightarrow{\chi^\rho} \mathbb{C} \\ g &\longmapsto \mathrm{tr}(\rho(g)). \end{aligned} \quad (233)$$

- Passage to characters is evidently linear in direct sums of representations, in that $\chi^{\rho \oplus \rho'} = \chi^\rho + \chi^{\rho'}$. On finite-dimensional semisimple representations it is injective on isomorphism classes (cf. [68, Thm. 2.17]):

$$\begin{aligned} \mathrm{Rep}_{\mathrm{ss}}(G)/\sim &\hookrightarrow \mathbb{C}^G \\ [\rho] &\longmapsto \chi^\rho. \end{aligned} \quad (234)$$

Here $\mathrm{Rep}_{\mathrm{ss}}(G)$ denotes the finite-dimensional semisimple complex representations. Thus (234) applies to all finite-dimensional unitary representations, and to all complex representations of finite groups in the present finite-dimensional setting. Without semisimplicity a character determines only the semisimplification.

- For finite G , $|G| < \infty$, the evident normalized inner product on functions $\chi : G \rightarrow \mathbb{C}$

$$\langle \chi, \chi' \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi'(g)} \quad (235)$$

exhibits *Schur orthonormality*, which is the statement that for the characters of irreducible representations ρ_i, ρ_j (230) we have (cf. [89, Thm. 2.12][68, Thm. 3.8])

$$\langle \chi^{\rho_i}, \chi^{\rho_j} \rangle = \delta_{ij}. \quad (236)$$

Tensor products of representations. Given a pair of linear representations (ρ, ρ') over, respectively, a pair of groups (G, G') , their *external tensor product* $\rho \boxtimes \rho'$ (cf. [89, Ex. 2.36]) is the representation of the direct product group $G \times G'$ given by

$$\rho \boxtimes \rho' : G \times G' \xrightarrow{\rho \times \rho'} \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_{n'}(\mathbb{C}) \longrightarrow \mathrm{GL}_{nn'}(\mathbb{C}). \quad (237)$$

- If ρ and ρ' are irreducible then so is $\rho \boxtimes \rho'$ and all irreps of $G \times G'$ arise this way, hence (cf. still [89, Ex. 2.36]):

$$\begin{aligned} \mathrm{Irr}(G) \times \mathrm{Irr}(G') &\xrightarrow{\sim} \mathrm{Irr}(G \times G') \\ ([\rho], [\rho']) &\mapsto [\rho \boxtimes \rho']. \end{aligned} \quad (238)$$

- Given a pair ρ and ρ' of linear representations of the same group G , their plain tensor product is the restriction of their external tensor product (237) along the diagonal of G :

$$\rho \otimes \rho' : G \xrightarrow{\mathrm{diag}} G \times G \xrightarrow{\rho \boxtimes \rho'} \mathrm{GL}_{nn'}(\mathbb{C}). \quad (239)$$

Group algebra. For G a group, its *group convolution algebra* $\mathbb{C}[G]$ (or just *group algebra* for short, cf. [89, §3.4][13, p. 51][45, §2.4][44, (4)]) is the $\mathbb{C}$ -linear span of G equipped with the algebra structure which on its canonical basis elements $\vec{e}_g$ is the group product ($\vec{e}_g \cdot \vec{e}_{g'} := \vec{e}_{gg'}$):

$$\mathbb{C}[G] := \left\{ \sum_{g \in G} c_g \cdot \vec{e}_g \in \mathrm{Span}_{\mathbb{C}}(G), \left(\sum_{g \in G} c_g \cdot \vec{e}_g \right) \cdot \left(\sum_{g' \in G} c'_{g'} \cdot \vec{e}_{g'} \right) := \sum_{g \in G} \left(\sum_{h \in G} c_h c'_{h^{-1}g} \right) \cdot \vec{e}_g \right\}. \quad (240)$$

- Hence group representations $G \rightarrow \mathrm{GL}(V)$ are equivalently algebra homomorphisms $\mathbb{C}[G] \rightarrow \mathrm{End}(V)$, hence are equivalently *modules* over the group algebra.
- For example, the group algebra understood as a module over itself is the *regular representation* $G \curvearrowright \mathbb{C}[G]$.

Induced representations. Here $\mathrm{Rep}(G)$ includes representations of arbitrary dimension, as in Prop. A.3. For $H \xrightarrow{i} G$ a subgroup inclusion, and $\rho : H \rightarrow \mathrm{GL}(V)$ a representation of H , its left or right *induced G -representation along i* , to be denoted $i_! \rho$ or $i_* \rho$, respectively, is (cf. [68, §4.8])

$$\left. \begin{aligned} i_! \rho &:= G \curvearrowright \left(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V \right) \\ i_* \rho &:= G \curvearrowright \left(\mathrm{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], V) \right) \end{aligned} \right\} \in \mathrm{Rep}(G), \quad (241)$$

where the constructions on the right are the tensor product and hom-space of modules over the group algebra $\mathbb{C}[H]$ (240), understood as equipped with their residual $\mathbb{C}[G]$ -module structure given by left multiplication (for $i_! \rho$) or by right multiplication (for $i_* \rho$) of the group algebra $\mathbb{C}[G]$ on itself.

- If G is finite, $|G| < \infty$, left and right inductions are naturally isomorphic (and generally when $H \subset G$ is of finite index) and one writes

$$\mathrm{Ind}_H^G \rho := \iota_! \rho \simeq \iota_* \rho : \mathrm{Rep}(H) \longrightarrow \mathrm{Rep}(G). \quad (242)$$

Mackey theory. Consider $G = A \rtimes H$ a finite semidirect product group where the normal subgroup A is abelian. The following Prop. A.17 classifies its irreducible representations from knowledge of the irreps of A and of certain subgroups of G . The statement involves the following notions:

- $\tilde{A} := \mathrm{Hom}(A, \mathbb{C}^\times)$ the dual group of irreps of A ,
- $\{\chi_i\}_{i \in \tilde{A}/H}$ a set of representatives of orbits in this group under the action of H ,
- $H_i := \mathrm{Stab}_H(\chi_i)$ the corresponding stabilizer subgroups,
- $\hat{\chi}_i$ the rep of $A \rtimes H_i$ given by $\hat{\chi}_i(a, h) := \chi_i(a)$ for the chosen orbit representative $\chi_i \in \tilde{A}$,
- $\hat{\rho}$ the pullback of any irrep ρ along $A \rtimes H_i \rightarrow H_i$.

Proposition A.17 (Mackey algorithm, cf. [250, Prop. 25 p. 62]). *The irreps of $A \rtimes H$ are, up to isomorphism, exactly the induced representations (242) of tensor products (239) of the form*

$$\text{Ind}_{A \rtimes H_i}^{A \rtimes H} (\widehat{\chi}_i \otimes \widehat{\rho}),$$

for $i \in \widetilde{A}/H$ and $\rho \in \text{Irr}(H_i)_{/\sim}$; and two such are isomorphic iff they come from the same index i and isomorphic irreps ρ .

Wreath product groups. Given G a group and $H \xrightarrow{s} \text{Sym}_n$ a group equipped with a homomorphism to a symmetric group, their *wreath product* is the semidirect product of G^n with H acting by permutation of factors (cf. [19, Def. 8.1][142, §4.1])

$$G \wr_s H := G^n \rtimes_s H. \quad (243)$$

- For G a finite group with set I of classes $[\rho^i]$ of irreducible representations (230), say that an I -partition of some $n \in \mathbb{N}$ is a tuple

$$(n) := (n_i \in \mathbb{N})_{i \in I} \quad \text{with} \quad \sum_i n_i = n. \quad (244)$$

- Given such, write

$$\text{Sym}_{(n)} := \prod_{i \in I} \text{Sym}_{n_i} \hookrightarrow \text{Sym}_n$$

for the subgroup of permutations of n elements that permute the n_i elements among each other, for all $i \in I$. There is an evident linear representation of the corresponding wreath product (243) on

$$\bigotimes_{i \in I} \rho_i^{\boxtimes n_i} \in \text{Rep}(G \wr \text{Sym}_{(n)}), \quad (245)$$

where the subgroup G^n is represented by the given external tensor product (237) of its irreps (230) and the subgroup $\text{Sym}_{(n)}$ acts by permutation of tensor factors. At the same time, an irrep σ of $\text{Sym}_{(n)}$ is of the form (238)

$$\sigma \simeq \bigotimes_{i \in I} \sigma_{j(i)}, \quad [\sigma_{j(i)}] \in \text{Irr}(\text{Sym}_{n_i}), \quad (246)$$

and we may regard this as a representation of $G \wr \text{Sym}_{(n)} \twoheadrightarrow \text{Sym}_{(n)}$.

Proposition A.18 (Irreps of finite wreath product groups, [142, Thm. 4.3.34]). *For finite G , the irreducible representations of $G \wr \text{Sym}_n$ (243) are, up to isomorphism, exactly the induced representations (242) along the inclusion*

$$\left(\prod_{i \in I} (G \wr \text{Sym}_{n_i}) \right) \simeq G \wr \text{Sym}_{(n)} \hookrightarrow G \wr \text{Sym}_n$$

of the representations which are tensor products (239) of the representations (245) corresponding to I -partitions (244) with irreps σ (246) of $\text{Sym}_{(n)}$:

$$\text{Ind}_{G \wr \text{Sym}_{(n)}}^{G \wr \text{Sym}_n} \left(\left(\bigotimes_{i \in I} \rho_i^{\boxtimes n_i} \right) \otimes \sigma \right) \in \text{Rep}(G \wr \text{Sym}_n). \quad (247)$$

Example A.19 (Irreps of $\mathbb{Z}_d \wr \text{Sym}_3$). The $\mathbb{C}$ -linear irreps of $\mathbb{Z}_d \wr \text{Sym}_3$ according to Prop. A.18 may be organized into three classes, depending on the nature of the partition (244) — recall for the following the irreps $\mathbf{1}_{[n]}$ of $\mathbb{Z}_d$ from Ex. A.14 and the irreps $\mathbf{1}$, $\mathbf{1}_{\text{sgn}}$, and $\mathbf{2}$ of Sym_3 from Ex. A.15.

Case 1: Partition involves a single irrep $\mathbf{1}_n$. In this case $\text{Sym}_{(3)} \simeq \text{Sym}_3$ and the induction functor is the identity, so that the corresponding irreps of $G \wr \text{Sym}_3$ according to (247) are of the form

$$\mathbf{1}_{[n]}^{\boxtimes 3} \otimes \sigma \in \text{Rep}_{\mathbb{C}}(\mathbb{Z}_d \wr \text{Sym}_3) \quad (248)$$

for $[n] \in \mathbb{Z}_d$ and $\sigma \in \{\mathbf{1}, \mathbf{1}_{\text{sgn}}, \mathbf{2}\}$.

Case 2: Partition involves two distinct irreps $\mathbf{1}_{[n]}$, $\mathbf{1}_{[n']}$. In this case $\text{Sym}_{(3)} \simeq 1 \times \text{Sym}_2 \simeq \text{Sym}_2$, so that the corresponding irreps of $G \wr \text{Sym}_3$ according to (247) are of the form

$$\mathbb{C}[\mathbb{Z}_d \wr \text{Sym}_3] \otimes_{\mathbb{C}[\mathbb{Z}_d \times (\mathbb{Z}_d \wr \text{Sym}_2)]} \left(\left(\mathbf{1}_{[n]} \boxtimes \mathbf{1}_{[n']}^{\boxtimes 2} \right) \otimes \sigma \right) \quad (249)$$

for $[n] \neq [n'] \in \mathbb{Z}_d$ and $\sigma \in \{\mathbf{1}, \mathbf{1}_{\text{sgn}}\}$

Case 3: Partition involves three distinct irreps $\mathbf{1}_{[n_1]}$, $\mathbf{1}_{[n_2]}$, $\mathbf{1}_{[n_3]}$. In this case $\text{Sym}_{(3)} \simeq 1 \times 1 \times 1 \simeq 1$, so that the corresponding irreps of $G \wr \text{Sym}_3$ according to (247) are of the form

$$\mathbb{C}[\mathbb{Z}_d \wr \text{Sym}_3] \otimes_{\mathbb{C}[\mathbb{Z}_d^3]} (\mathbf{1}_{[n_1]} \boxtimes \mathbf{1}_{[n_2]} \boxtimes \mathbf{1}_{[n_3]}) \quad (250)$$

for $[n_i] \in \mathbb{Z}_d$ pairwise distinct.

A.6 Quadratic Gauss sums

Here we briefly compile some facts about Gauss sums used in §3.4, see [26] for more pointers to the literature (see also [60] but beware of typos in (1.1) there).

First, it may be worth recalling the simple cousin of the Gauss sums:

Proposition A.20 (Discrete Fourier transform of Kronecker delta). *For $K \in \mathbb{N}_{>0}$ and $p \in \mathbb{Z}$ we have*

$$\sum_{n=0}^{K-1} e^{2\pi i \frac{p}{K} n} = \begin{cases} K & \text{if } p = 0 \pmod{K} \\ 0 & \text{if } p \neq 0 \pmod{K}. \end{cases} \quad (251)$$

Proof. The statement for $p = 0 \pmod{K}$ is immediate. For $p \neq 0 \pmod{K}$ observe that

$$\left(1 - e^{2\pi i \frac{p}{K}}\right) \sum_{n=0}^{K-1} e^{2\pi i \frac{p}{K} n} = 1 - e^{2\pi i p} = 0.$$

□

Now:

Proposition A.21 (Classical quadratic Gauss sum evaluation, cf. [164, p. 87][214]). *For $K \in \mathbb{N}_{>0}$ we have*

$$\sum_{n=0}^{K-1} e^{2\pi i \frac{1}{K} n^2} = \begin{cases} (1+i)\sqrt{K} & \text{if } K = 0 \pmod{4} \\ \sqrt{K} & \text{if } K = 1 \pmod{4} \\ 0 & \text{if } K = 2 \pmod{4} \\ i\sqrt{K} & \text{if } K = 3 \pmod{4}. \end{cases} \quad (252)$$

Proposition A.22 (Quadratic Gauss sum with multiple exponents, cf. [164, “QS4” p. 86][214, Lem. 5.1]). *For odd $K \in 2\mathbb{N} + 1$ we have more generally,²⁰ for $p \in \mathbb{Z}$:*

(i) *if $\gcd(p, K) = 1$:*

$$\sum_{n=0}^{K-1} e^{2\pi i \frac{p}{K} n^2} = \pm \begin{cases} \sqrt{K} & \text{if } K = 1 \pmod{4} \\ i\sqrt{K} & \text{if } K = 3 \pmod{4}, \end{cases} \quad (253)$$

(ii) *for arbitrary p with $d := \gcd(p, K)$:*

$$\sum_{n=0}^{K-1} e^{2\pi i \frac{p}{K} n^2} = d \sum_{n=0}^{K/d-1} e^{2\pi i \frac{p/d}{K/d} n^2}. \quad (254)$$

In §3.4 we are crucially concerned with the variant of the classical quadratic Gauss sum that has *half* the usual exponents. In its plain form it is elementary to reduce this to the ordinary quadratic Gauss sum:

Proposition A.23 (Quadratic Gauss sum with halved exponents). *For $K \in 2\mathbb{N}_{>0}$ we have*

$$\sum_{n=0}^{K-1} e^{\frac{\pi i}{K} n^2} = e^{\pi i/4} \sqrt{K}. \quad (255)$$

Proof. Setting $r := K/2 \in \mathbb{N}$, we may compute as follows:

$$\begin{aligned} \sum_{n=0}^{K-1} e^{\frac{\pi i}{K} n^2} &= \sum_{n=0}^{2r-1} e^{\frac{\pi i}{2r} n^2} && \text{by definition of } r \\ &= \frac{1}{2} \left(\sum_{n=0}^{2r-1} + \sum_{n=2r}^{4r-1} \right) e^{\frac{\pi i}{2r} n^2} && \text{since summands are } 2r\text{-periodic, cf. ftn. 16} \\ &= \frac{1}{2} \sum_{n=0}^{4r-1} e^{\frac{2\pi i}{4r} n^2} \\ &= \frac{1}{2} (1+i) \sqrt{4r} && \text{by (252)} \\ &= e^{\pi i/4} \sqrt{2r} \\ &= e^{\pi i/4} \sqrt{K} && \text{by definition of } r. \end{aligned}$$

□

²⁰The sign in (253) is called the *Jacobi symbol*; we leave it unevaluated here.

More generally, there is the following reciprocity relation for the parameters of the quadratic Gauss sum with halved exponents, which relates it to the ordinary quadratic Gauss sum:

Proposition A.24 (Landsberg-Schaar identity [243], cf. [6][262][121]). *For $K \in 2\mathbb{N}_{>0}$ and $p \in \mathbb{N}_{>0}$ we have*

$$\sum_{n=0}^{K-1} e^{\pi i \frac{p}{K} n^2} = \frac{e^{\pi i/4}}{\sqrt{p/K}} \sum_{n=0}^{p-1} e^{-\pi i \frac{K}{p} n^2}. \quad (256)$$

In summary, this implies the evaluation which we use in the main text:

Proposition A.25 (Quadratic Gauss sum with multiple halved exponents). *For $K \in 2\mathbb{N}_{>0}$ and $p \in 2\mathbb{N}+1$ we have:*

(i) *if $\gcd(p, K) = 1$:*

$$\sum_{n=0}^{K-1} e^{\pi i \frac{p}{K} n^2} \stackrel{(256)}{=} \frac{e^{\pi i/4}}{\sqrt{p/K}} \sum_{n=0}^{p-1} e^{-2\pi i \frac{K/2}{p} n^2} \stackrel{(253)}{=} \pm \begin{cases} e^{+\pi i/4} \sqrt{K} & \text{if } p \equiv 1 \pmod{4} \\ e^{-\pi i/4} \sqrt{K} & \text{if } p \equiv 3 \pmod{4} \end{cases}. \quad (257)$$

(ii) *for arbitrary odd p with $d := \gcd(p, K)$:*

$$\sum_{n=0}^{K-1} e^{\pi i \frac{p}{K} n^2} = d \sum_{n=0}^{K/d-1} e^{\pi i \frac{p/d}{K/d} n^2}, \quad (258)$$

hence

$$\left| \sum_{n=0}^{K-1} e^{\pi i \frac{p}{K} n^2} \right| = \sqrt{K \gcd(p, K)}. \quad (259)$$

These relations are used in §3.4 to classify torus monodromy representations with irreducible Heisenberg restriction.

Declarations.

The authors declare that no data were generated or analyzed in this theoretical research.

The authors declare that they have no conflict of interest.

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