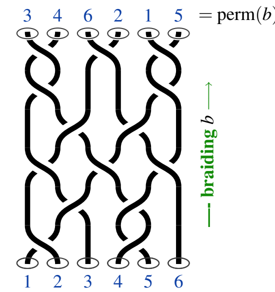
Literature 2.20 (Braiding). Recall (see, e.g., [Mi72]) that by a *group* one means a *group of operations*, on some object, which are associatively composable and invertible (cf. pp. 54). By a *braid group* (due to [Ar25], monographs include [FoN62][Bir75], exposition in [Will20]) one means the group of joint continuous movements of a fixed number $N+1$ of non-coincident points in the plane, from any fixed configuration back to that fixed configuration. The "worldlines" traced out by such points in space-time under such an operation look like a braid with $N+1$ strands, whence the name.
As with actual braids, here it is understood that two such operations are identified if they differ only by continuous deformations of the "strands" without breaking or intersecting these, hence by an *isotopy* in the ambient $\mathbb{R}^3$.



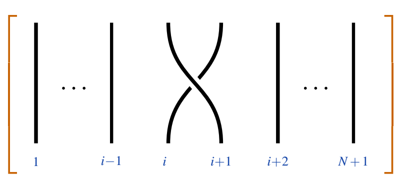
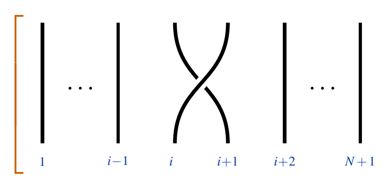
A quick way of saying this with precision (we consider a more explicit description in a moment) is to observe that a braid group is thus the *fundamental group* π_1 (Lit. 2.13) of a configuration space of points in the plane (Lit. 2.18). Here it makes a key difference whether one considers the points in a configuration as *ordered* (labeled by numbers $1, \cdots, N+1$) in which case one speaks of the *pure braid group*, or as indistinguishable (albeit in any case with distinct positions!) in which case one speaks of the *braid group* proper: After traveling along a general braid b the order of the given points may come out permuted by a permutation $\mathrm{perm}(b)$, and the braid is called *pure* precisely if this permutation is trivial:

$$\begin{array}{ccccc}
\underset{\text{pure braid group}}{\mathrm{PBr}(N+1)} & \xhookrightarrow{\mathrm{fib_e(perm)}} & \underset{\text{braid group}}{\mathrm{Br}(N+1)} & \xrightarrow{\mathrm{perm}} & \underset{\text{permutation group ("symmmetric group")}}{\mathrm{Sym}(N+1)} \\
\| & & \| & & \wr \\
\pi_1\Big(\underset{\{1,\cdots,N+1\}}{\mathrm{Conf}}(\mathbb{R}^2)\Big) & \xrightarrow[\text{forget ordering}]{} & \pi_1\Big(\mathrm{Conf}_{N+1}(\mathbb{R}^2)\Big) & \xrightarrow[\text{forget positions}]{} & \pi_1\Big(\mathrm{Conf}_{N+1}(\mathbb{R}^\infty)\Big) \\
\text{configuration space of ordered points in the plane} & & \text{configuration space of un-ordered points in the plane} & & \text{configuration space of un-ordered points in higher dim Eucl. space}
\end{array} \tag{7}$$

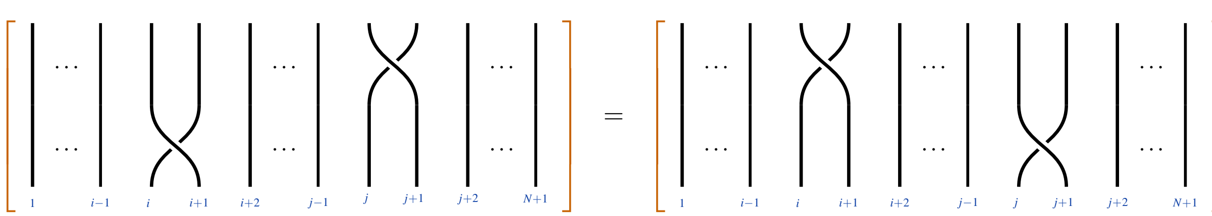
Since these configuration spaces have no other non-trivial homotopy groups (5), the vertical identifications mean equivalently that the homotopy type of these configuration spaces constitute *deloopings* or *classifying spaes* or *Eilenberg-MacLane spaces* in degree 1 (Lit. 2.14) for the braid groups; in particular:

$$\underset{\{1,\cdots,N+1\}}{\mathrm{Conf}}(\mathbb{R}^1) \underset{\text{whe}}{\simeq} B\mathrm{PBr}(N+1) \underset{\text{whe}}{\simeq} K\big(\mathrm{PBr}(N+1), 1\big). \tag{8}$$

A more explicit way to describe the braid group $\mathrm{Br}(N+1)$ is to observe, first, that any braid may, clearly, be obtained as a composition of those elementary braids which do nothing but pass a pair of neighbouring points past each other:

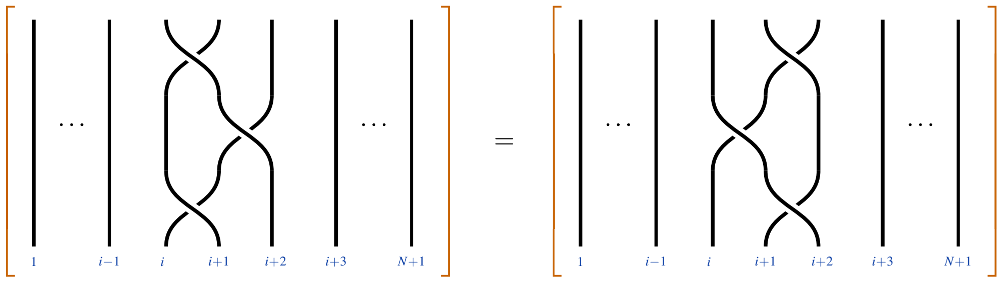
ith generating braid $b_i :\equiv$  and its inverse braid $b_i^{-1} :\equiv$  (9)

While any braid may be obtained as a composition of just these generators, not every pair of such compositions yields distinct braids. For example, if a pair of such elementary braids acts on disjoint strands, then the order in which they are applied does not matter up to the pertinent continuous deformation of braids:



(10)

On the other hand, when consecutive triples of elementary braids do act on the same strands, then an evident continuous deformation relates them as follows:



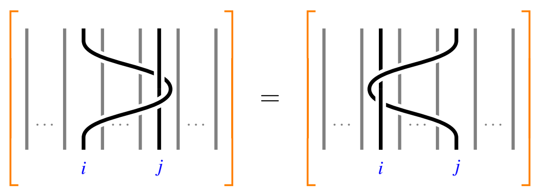
(11)

A classical theorem due to Artin [Ar25, §3] (cf. [FoN62, §7]) says that these are the *only* relations between these generators, in that the braid group is *presented* by these *generators and relations*, in the general sense of group presentations (e.g. [MKS66][Jo90], cf. pp. 65): [3]

$$\underset{\textbf{braid group}}{\mathrm{Br}(N+1)} \;\simeq\; \mathrm{FreeGrp}\big(\underset{\textbf{Artin generators } (9)}{\{\mathrm{e}, b_1, \cdots,, b_n\}}\big) \Big/ \underset{\textbf{Artin braid relations } (10)\,(11)}{\left(\begin{array}{l} b_i \cdot b_j \;=\; b_j \cdot b_i \;\text{ if }\; i+1<j \\ b_i \cdot b_{i+1} \cdot b_i \;=\; b_{i+1} \cdot b_i \cdot b_{i+1} \end{array}\right)} \qquad (12)$$

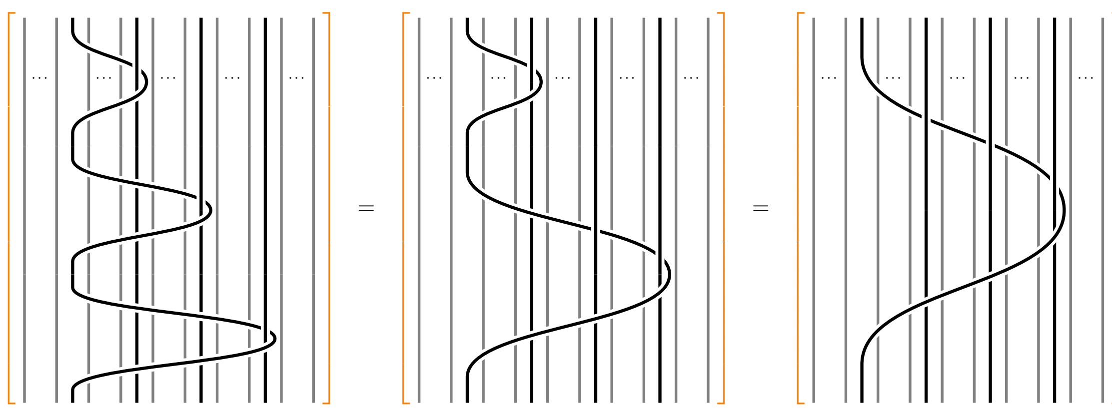
Analogously for the pure braid group (7), it is fairly evident that any pure braid can be obtained by composing "weaves" in which one strand lassoes exactly one other strand:

(i,j)**th generating pure braid** $b_{ij} \;:\equiv$



(13)

For example, consecutive application of such generators for fixed i and decreasing j yields pure braids of the following form:



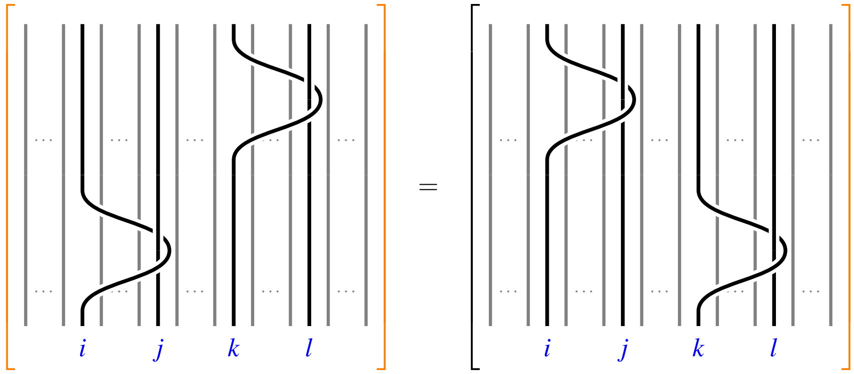
(14)

As before in (12), these pure braid generators (13) constitute a finite presentation, now of the pure braid group (we show the optimized set of pure braid relations due to [Le10, Thm. 1.1, Rem. 3.1]):

$$\underset{\textbf{pure braid group}}{\mathrm{PBr}(N+1)} \;\simeq\; \mathrm{FreeGrp}\big(\underset{\textbf{pure braid generators } (13)}{\{\mathrm{e}\} \sqcup \{b_{ij}\}_{1 \leq i < j \leq N+1}}\big) \Big/ \underset{\textbf{pure braid relations } [\text{Le10, Thm. 1.1}]}{\left(\begin{array}{ll} b_{ij} \cdot b_{rs} \;=\; b_{rs} \cdot b_{ij} & \text{if } r<s<i<j \text{ or } i<r<s<j \\ b_{ji} \cdot b_{ir} \cdot b_{rj} \;=\; b_{ir} \cdot b_{rj} \cdot b_{ji} \;=\; b_{rj} \cdot b_{ji} \cdot b_{ir} & \text{if } r<i<j \\ b_{rs} \cdot \big(b_{jr} \cdot b_{ji} \cdot b_{js}\big) \;=\; \big(b_{jr} \cdot b_{ji} \cdot b_{js}\big) \cdot b_{rs} & \text{if } r<i<s<j \end{array}\right)} \qquad (15)$$

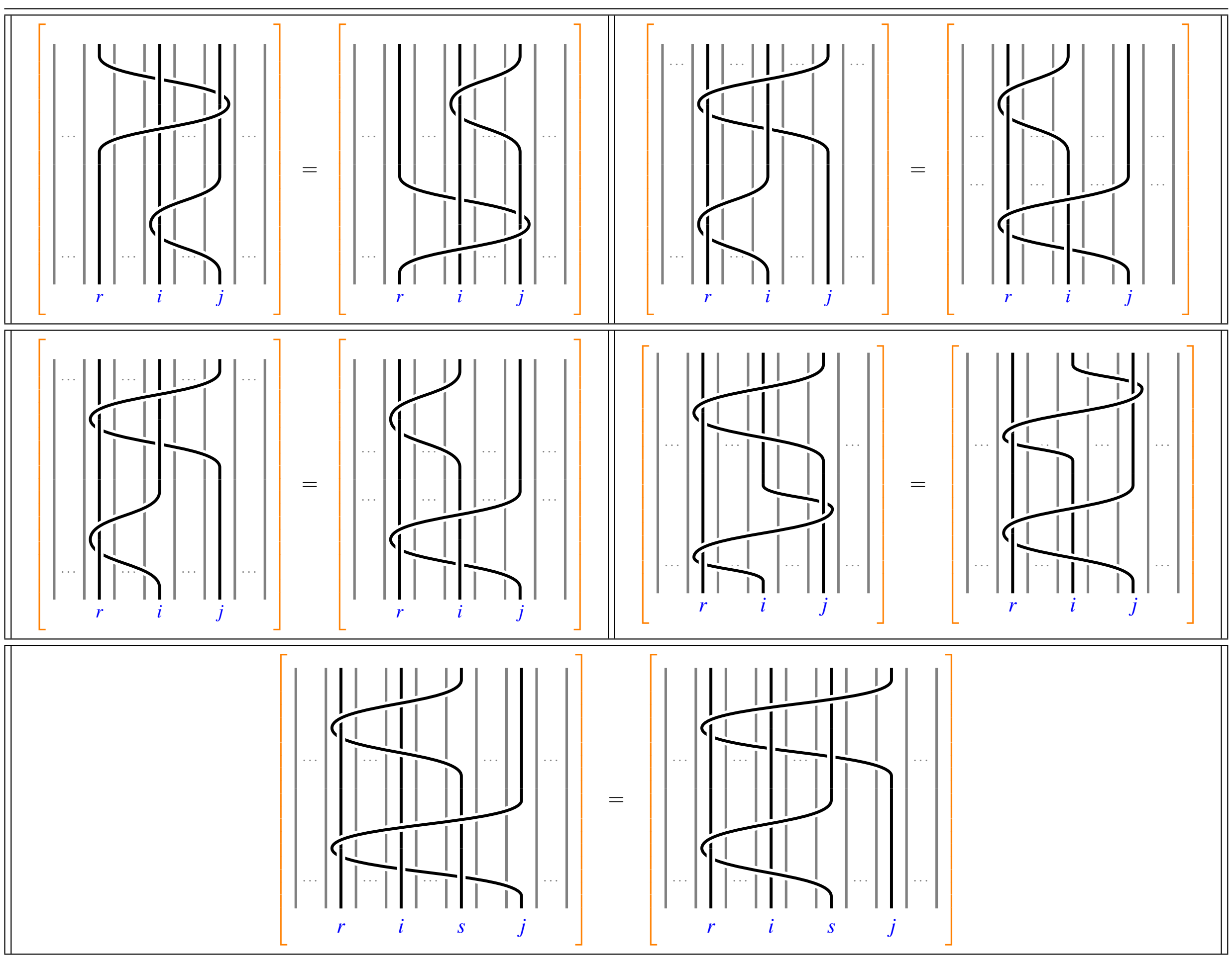
As before in (10), the first of these relations (15) simply say that the order of applying pure braid generators is irrelevant if these act on disjoint intervals of strands:

[3] In (12) we include the neutral element in the set of generators just in order to stick with the convention used in (179) below, where it is most natural to regard the free group-construction as an operation on pointed sets.



(16)

The further relations in the presentation (15) of the pure braid group concern cases where pure braid generators do "overlap", specifically with the products (14) of other generators:



(17)

Notice that all these pure braid relations are *commutator* relations [Le10, Rem. 3.10], saying that one pure braid generator commutes with a product of pure braid generators, such as those in (14). This implies that group homomorphisms out of a pure braid group into an *abelian* group are given by assigning any of the abelian group elements to the pure Artin generators (13) (used in Lem. 6.5 below).