Twisted cohomology and Gauss-Manin connections. Given $R : \mathrm{Ring}$ (185) and $n : \mathbb{N}$ (111), the **ordinary cohomology** with coefficients R in degree n is (cf. [Ca15, §3. 2][Wä, §4.1], implementated in `Agda`: [BLM22]) the 0-truncation $[-]_0$ (127) of the function type (64) into the n-fold delooping $\mathbf{B}^n(-)$ (194) of the underlying abelian group (187).

$$\left.\begin{array}{l} \overset{\text{coefficients}}{R : \mathrm{Ring}}, \;\overset{\text{degree}}{n : \mathbb{N}}, \\ \underset{\text{domain}}{X : \mathrm{Type}} \end{array}\right\} \;\vdash\; \overset{\text{ordinary cohomology}}{H^n\big(X; R\big)} \;:\equiv\; \big[X \to \mathbf{B}^n R_{\mathrm{udl}}\big]_0 \;:\; \mathrm{Type}$$

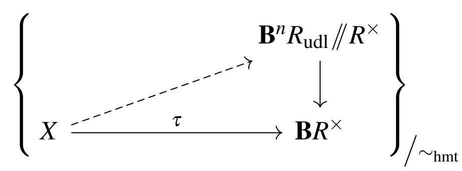
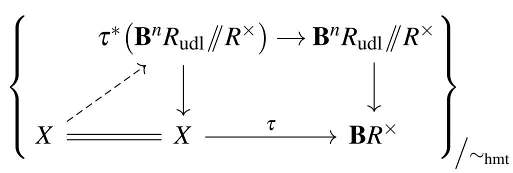
Under the dictionary of §5.1, this type construction clearly interprets as the traditional notion of ordinary cohomology (Lit. 2.15, cf. discussion and references in [FSS20-Cha, Ex. 1.0.2]).

Given furthermore a *twist* $\tau : X \to \mathbf{B}R^\times$ (188), we say (for the second version (198) cf. [vD18, Def. 5.4.2]) that the **τ-twisted ordinary cohomology** of X with coefficients in R is:

$$\left.\begin{array}{l} \overset{\text{coefficients}}{R : \mathrm{Ring}}, \;\overset{\text{degree}}{n : \mathbb{N}}, \\ \underset{\text{domain}}{X : \mathrm{Type}}, \;\; \underset{\text{twist}}{\tau : X \to \mathbf{B}R^\times} \end{array}\right\} \;\vdash\; \overset{\text{twisted cohomology}}{H^{n+\tau}(X;R)} :\equiv \Big[\big(t : \mathbf{B}R^\times\big) \to \Big(\overset{(87)}{\mathrm{fib}_t(\tau)} \to \overset{(195)}{\mathbf{B}^n\big(\circlearrowleft_t R_{\mathrm{udl}}\big)}\Big)\Big]_0 \;:\; \mathrm{Type} \tag{197}$$

$$\simeq \Big[\underbrace{\big(t : \mathbf{B}R^\times\big) \times \big(x_t : \mathrm{fib}_t(\tau)\big)\big)}_{(x:X)\;\;(106)} \to \underbrace{\mathbf{B}^n\big(\circlearrowleft_t R_{\mathrm{udl}}\big)}_{(\tau^*\mathbf{B}(R^\times \circlearrowleft R_{\mathrm{udl}}))(x)}\Big]_0 \;: \mathrm{Type} \tag{198}$$

Under the dictionary of §5.1, this type construction interprets as the traditional notion of twisted ordinary cohomology (Lit. 2.15), see the discussion and references in [NSS12a, Rem. 4.22] and [SS20-Orb, Rem. 2.94][FSS20-Cha, Ex. 2.0.5]:

Twisted ordinary cohomology $H^{n+\tau}\big(X;R\big)$	
Homotopy type theory	**Homotopy theory**
$\Big[\big(t : \mathbf{B}R^\times\big) \to \Big(\mathrm{fib}_t(\tau) \to \mathbf{B}^n\big(\circlearrowleft_t R_{\mathrm{udl}}\big)\Big)\Big]_0$ (197)	
$\Big[\big(x : X\big) \to \big(\tau^*\mathbf{B}(R^\times \circlearrowleft R_{\mathrm{udl}})\big)(x)\Big]_0$ (198)	

(199)

For the construction of Gauss-Manin connections from §4.2, we are to furthermore consider B-indexed families of such twisted cohomology types (199). Remarkably, in the syntax of homotopy types this is a triviality that amounts to introducing a type B into the context and having the terms X and τ depend on it:

Definition 5.16 (Gauss-Manin transport on fibrations of twisted cohomology groups)**.** We say that the type of *fibrations of twisted ordinary cohomology sets* is:

$$\left.\begin{array}{l} \overset{\text{coefficients}}{R : \mathrm{Ring}}, \;\overset{\text{degree}}{n : \mathbb{N}}, \overset{\text{parameter base}}{B : \mathrm{Type}}, \\ \underset{\text{fibration of domains}}{X_{(-)} : B \to \mathrm{Type}}, \;\; \underset{\text{family of twists}}{\tau_{(-)} : (b : B) \to \big(X_b \to \mathbf{B}R^\times\big)} \end{array}\right\} \;\vdash\; \begin{array}{l} \overset{\text{twisted cohomology}}{H^{n+\tau_{(-)}}\big(X_{(-)};R\big)} \;:\equiv \\ \Big[\big(t : \mathbf{B}R^\times\big) \to \Big(\underbrace{\mathrm{fib}_t\big(\tau_{(-)}\big)}_{\mathrm{fib}_{(-,t)}(\mathrm{pr}_X,\tau)\;(204)} \to \mathbf{B}^n\big(\circlearrowleft_t R_{\mathrm{udl}}\big)\Big)\Big]_0 \;:\; \overset{\text{fibered over base}}{B \to \mathrm{Type}} \end{array} \tag{200}$$

Given such, its *Gauss-Manin monodromy* is the corresponding transport (74) over the base type B:

$$\begin{array}{rcl} \mathrm{GMTransport} \;:\; \prod\limits_{b_1, b_2 : B} \Big(\mathrm{Id}_B(b_1, b_2) & \longrightarrow & \Big(H^{n+\tau_{b_1}}\big(X_{b_1};R\big) \xrightarrow{\sim} H^{n+\tau_{b_2}}\big(X_{b_2};R\big)\Big)\Big). \\ \big(b_1 \overset{p_{12}}{\rightsquigarrow} b_2\big) & \longmapsto & (p_{12})_* \end{array} \tag{201}$$

By Thm. 4.13 and under the dictionary in §5.1, this is the type-theoretic construction whose denotational semantics is the parallel transport/monodromy of Gauss-Manin connections on fibrations of twisted ordinary cohomology sets.