**The idea of adiabatic quantum transport.** A traditional computing process of the form indicated in *Figure P* exchanges energy between the quantum system and its control environment. In fact, common NISQ architectures (Lit. 2.3) are designed to encode qbit states as energy eigenstates of anharmonic quantum oscillators, so that passing between their energy levels (the notorious *quantum jumps*) is what it means to execute computations on these systems in the first place. At the same time, these energetic interaction channels with their environment is what makes these NISQ machines suffer from noise and decoherence.

In contrast, a topological quantum process (Lit. 2.4) is, first of all, to take place entirely on the (topologically ordered) *ground states* of a topological quantum material (Lit. 2.7), hence on their lowest energy states, without absorbing any energy from the environment: The notorious *energy gap* which measures the fidelity of topological phases of matter (Lit. 2.7), separating their topological ground states from their ordinary excited states, is the room within which the control environment may shed energy without disturbing the coherent quantum phase.

This state of affairs is neatly captured by one of the classical theorems of mathematical quantum mechanics: The *Quantum Adiabatic Theorem* (Lit. 2.6) says (as nicely brought out for quantum computation already in [ZR99]) that in the asymptotic limit of sufficiently gentle (= "adiabatic") movement of external classical parameters, the induced quantum system's evolution asymptotically preserves gapped energy eigen-states, hence in particular preserves gapped ground states, and hence acts on the Hilbert space $\mathcal{H}$ of gapped ground states by unitary operators $U_p$ that vary continuously with the parameter path $p$.

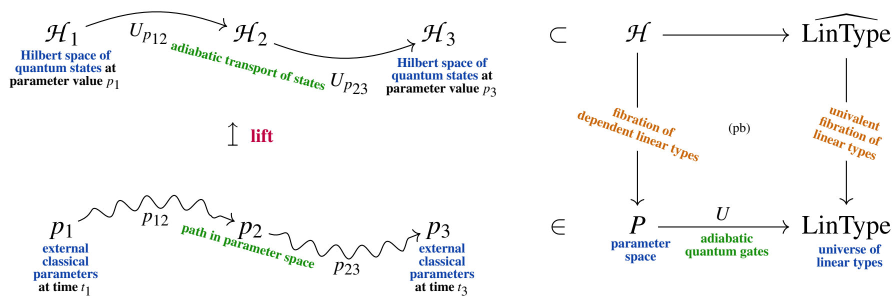
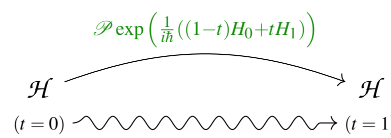



**Figure T.** Schematically shown on the left is the "adiabatic" (Lit. 2.6) transport of quantum states along linear maps depending on continuous paths in a classical parameter space.
The diagram on the right indicates our description of such situations by (linear) homotopy type families depending on a base homotopy type, as explained in §5 below (see (74) and (106) below, noticing that we relegate discussion of *linearity* of quantum types to [SS23-QM][SS23-EoS]).

**The idea of quantum annealing.** For example, a widely-known implementation of the above *Quantum Adiabatic Theorem* in quantum computation is the paradigm of "quantum annealing" [KN98][FGGS00] (review in [RSDC22]). Here one considers a single-parameter path linearly interpolating between two given Hamiltonians, $H_0, H_1$ on a fixed Hilbert space $\mathcal{H} :\equiv \mathcal{H}_1 = \mathcal{H}_2$:

If one can arrange for both Hamiltonians to have unique gapped ground states and for the ground state of the first Hamiltonian to be preparable, while that of the second Hamiltonian is unknown but identifiable with the answer to a given computational problem, then the corresponding adiabatic quantum transport effectively computes that answer.




Due to its restriction to finding a unique ground state, quantum annealing as such is not a universally programmable form of computation: It is guaranteed to discover the ground state of $H_1$ and does nothing else. However, in this restrictivity annealing does foreshadow a key aspect of topological quantum computation in degenerate form (cf. Lit. 2.6): Since the ground state of $H_1$ is unique, the annealing process does not actually depend on the exact parameter path chosen to arrive there, it is *robust against perturbations of the computational path* (cf. [CFP02, p. 2]).

**The idea of topological quantum computation.** Generally, the profound practical problem with implementing the theoretically straightforward idea of programmable quantum processes (Lit. 2.1) is that real quantum machines are not in idealized isolation but are coupled to their environment, which necessarily acts like a "thermal bath": Inevitable noise in the environment causes perturbations that tend to de-cohere the machine's quantum state and thus tend to destroy its intended quantum computation (cf. Lit. 2.5).

Concretely, parameter paths realizable in real laboratories are noisy (cf. Lit. 2.3), hence are drawn randomly from an ensemble of small perturbations of the intended path. The result of transporting a pure quantum state along such a noisy ensemble is in general a decohered mixture of pure states which may no longer support the quantum interference effects on which quantum algorithms crucially rely; unless, that is, one could somehow guarantee that the quantum transport depending on these paths is actually *in*dependent of their small perturbations, really depending only on the global properties of these paths.

This is the idea of topological quantum computation (Lit. 2.4): to ensure that the quantum adiabatic process depends only on the *topological homotopy classes* (Lit. 2.12, 2.13) of the parameter paths (relative to their endpoints).

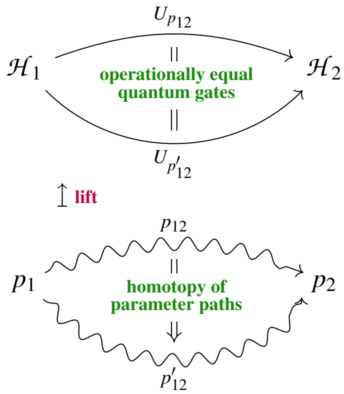
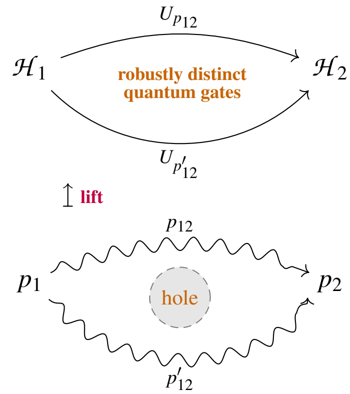



**Figure H.** An adiabatic quantum transport (Fig. T) is *topological* (or rather: *homotopical*, cf. Lit. 2.12) if it depends on the parameter path between fixed endpoints only up to small continuous deformations, namely up to homotopy (indicated on the left).
When this is the case, then quantum transport depends *robustly* on "global" properties of parameter paths, such as their winding number (cf. Lit. 2.13) around "holes" in parameter space (schematically indicated on the right) and hence constitutes a form of *topological quantum computation* (Lit. 2.4).




The topological quantum computer scientist is thus led to search for topological quantum materials which are dependent on classical parameter spaces that have a rich structure of "holes" in them, namely with a rich *fundamental group* (Lit. 2.13).

**The idea of anyon braid grates.** There could be several possible choices for such topological quantum systems (cf. [ZR99]), but the original proposal by Kitaev (Lit. 2.1) may be the most promising and has come to often be treated as synonymous with topological quantum computation as such. Here one imagines that a quantum material's gapped and topologically ordered ground state (Lit. 2.7) depends topologically on tuples $(z_1, \cdots, z_n)$ of pairwise distinct positions of *defect points* ("anyons", Lit. 2.17) which are effectively constrained to move inside a surface $\Sigma$ (such as for a crystalline material consisting of a few monolayers of atoms).

For example, much attention has been focused on the idea that such defects might be realized by quantum vortices in the surfaces of quantum fluids, such as certain Bose-Einstein condensates (e.g. [MPSS19]). The defect parameters could also be more abstract, such as being the critical "nodal" values (not of positions but) of *momenta* of electrons in topological semimetals (Lit. 2.10). Such nodal momentum values typically vary with fairly easily controllable external parameters such as external strain exerted on the material's crystal structure.

In any case, in such a situation the classical parameter space $P$ is effectively the *configuration space of points* (Lit. 2.18) in the surface $\Sigma$

$$\underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\Sigma) \;=\; \left\{ (z_1, \cdots, z_N) \in \Sigma \,\middle|\, \underset{I \neq J}{\forall}\, z_I \neq z_J \right\}.$$

A *path* in such a space is an $n$-tuple of "worldlines" of defects which may move around each other but never coincide (at any given instant of time), thus forming the appearance of a "braid" of $n$ strands in 3d space (Lit. 2.20).

This implies that even if $\Sigma$ is taken to be topologically trivial (e.g. $\Sigma$ could be the disk through the equator of a Bose-Einstein condensate) there are still plenty of distinct homotopy classes of paths in $\underset{\{1,\cdots,N\}}{\mathrm{Conf}}(\Sigma)$, corresponding to all those braids which cannot be untied. If a quantum material can be found whose degenerate ground states are transported topologically but non-trivially along such braidings of defect points, then this would realize topological quantum computation by *anyon braid gates* in the original sense of Freedman and Kitaev (Lit. 2.4).

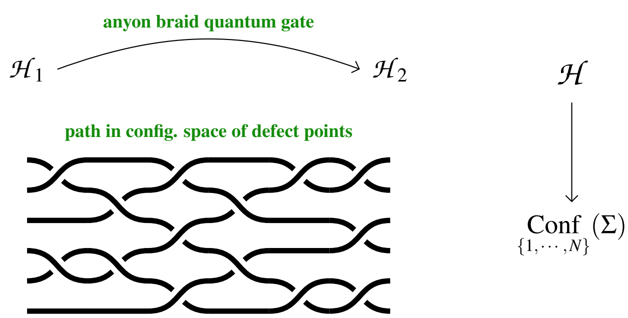



**Figure A.** If the classical parameter space of a dependent quantum system (*Figure T*) is a configuration space (Lit. 2.18) of (anyonic) defect points (Lit. 2.17) in a plane, then a parameter path is a *braid* (Lit. 2.20).
If the (degenerate) ground state of a topological quantum system depends topologically on the defect poisitions (Lit. 2.7), then their adiabatic transport along such braid paths realizes quantum gates that form a linear representation of the braid group.
In the technologically viable situation of $\mathfrak{su}_2$-anyons, this is the *monodromy representation* of the canonical flat KZ-connection on the *bundle of conformal block* over configuration (Lit. 2.24).

**The idea of certified braid gate types.** But none of this intricate internal structure of topological braid quantum gates is visible to existing quantum programming languages; and any traditional implementation of this information (via conformal quantum field theory methods) would be formidable to construct and then inefficient to use. Yet, detailed verification (Lit. 2.26) of the operation and compilation of these braid gates will arguably be crucial for practical scalable quantum computation (Lit. 2.5), and will serve for topological quantum simulation already now.

Our claim here is that this problem finds an elegant solution by regarding it through the novel lens of homotopically-typed programming languages (Lit. 2.27), where the construction of types certifying braid quantum gate operation magically turns out to be essentially a native language construct (Thm. 6.8 below). This we explain now.