Topological Quantum Programming via Linear Homotopy Types

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We explain that there is a closer relation between Quantum Computing and Homotopy Type Theory (HoTT) than might superficially be expected.

First, we observe that the specification of topological quantum gates based on anyon braiding is essentially a native construct in HoTT, being given by type transport in the 0-truncation of a certain dependent function type into dependent Eilenberg-MacLane types. This follows by a neat synthetic encoding of "Gauss-Manin connections" into HoTT, combined with a theorem by Feigin, Schechtmann & Varchenko (1991) on conformal quantum field theory. Arguably, this reveals (certified) topological quantum computing as the first use-case of genuinely "homotopical computation".

Moreover, we observe that HoTT admits conservative extensions by dependent linear homotopy types (LHoTT) which constitute genuine quantum data types much as familiar from the `Quipper` quantum programming language. Semantically this reflects the remarkable observation by Biedermann et al. that ∞-categories of parameterized module spectra (interpreting dependent linear homotopy types) are ∞-toposes. Specifically, LHoTT verifies the Grothendieck-Wirthmüller "motivic yoga of operations" on parameterized spectra, and we observe that this implies admissible inference rules for whole a system of (co)monads on linear homotopy types which naturally express quantum effects both on pure as well as on mixed states. In particular this recovers (hence certifies) Coecke et al.'s quantum measurement monads now used in the `zx-Caclulus`, and it provides "dynamic lifting" of measurement results as envisioned by the `Quipper` community.

This talk surveys the work laid out in the circle of articles (joint with Hisham Sati):

- [1] – HoTT as a natural programming/certification language for topological quantum circuits.
- [4] – LHoTT explained as a modal quantum programming/certification language.
- [2] – Parameterized $H\mathbb{C}$-module spectra as semantics for the relevant fragment of LHoTT.
- [3] – Naturally certifying unitarity of quantum gates in LHoTT.

I will first give an exposition of the encoding [1] of topological quantum gates in plain HoTT along the lines of my recent HoTTest talk [12], but with more introduction suitable for the QPL audience, similar to the introduction given at PlanQC 2022 [5].

In the second half I motivate LHoTT and its interpretation as a quantum programming/certification language and showcase (along the lines of the exposition given at QTML 2022 [11]) how the encoding of quantum measurement and the certification of the *deferred measurement principle* is naturally reflected in the system of "quantum modalities" that are defineable just from the type formation rules of dependent linear homotopy types.

We argue that our results show that there is significant potential for (L)HoTT in Quantum Computing/Programming/Information theory and hope that it may serve as advertisement of opportunities for

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energetic young type theorists. The prospect of connecting HoTT with Quantum Systems is substantial, both theoretically as well as practically:

On the theory side, in generalization of how HoTT may be understood as the internal language of ∞-toposes and hence of *unstable homotopy theory*, so LHoTT is to be understood as the internal language of "tangent ∞-toposes" and hence of *parameterized stable homotopy theory*, which is where most of algebraic topology actually takes place (such as all Whitehead-generalized cohohomology theory). The multitude of applications that parameterized stable homotopy theory has to quantum physics in the deeper sense of quantum field theory (including string theory) had been our original motivation towards LHoTT in [10] [9]. The discussion there shows for instance that fiber integration in twisted Whitehead-generalized cohomology theories, which is notably involved in the K-theoretic formulation of geometric quantization of phase spaces, follows essentially the same syntactic rules that make LHoTT suitable as a quantum programming language in the way explained in [4].

On the practical side, it is noteworthy that despite the excitement about quantum computation in general, both in academia but notably also in industry, it is an open secret that there are serious principle questions to be resolved in order to pass beyond the currently available toy applications — and on all of these LHoTT seems to have a say:

- **stabilization** – The main practical obstacle to large-scale quantum computation remains protection against (or else correction of) errors, and topological stabilization remains the grand promise for how to achieve this already at the hardware-level. However, realistic topological quantum gates are expected to be governed by conformal quantum field theory (CFT, specifically: Gauss-Manin / Knizhnik-Zamolodchikov connections on bundles of conformal blocks) describing topologically ordered ground states of quantum materials — which is way outside the expressive scope of existing quantum programming languages, let alone certification languages. And yet, we show that (L)HoTT naturally knows about these structures, in fact it is by work on anyon CFT in quantum materials [6] [7] [5] that we discovered their synthetic encoding in HoTT, in the first place.
- **certification** – But the compilation of large-scale quantum programs to machine-executable circuits is more complex than classically, drastically more so for topological quantum gates, while at the same time traditional debugging by run-time inspection is unavailable for quantum programs (due to state collapse upon register read-out). While it has been argued that this makes formal quantum software verification inevitable for any realistic quantum computation, a fully certifiable typed quantum programming language has remained missing. We explain that LHoTT serves just this purpose, since it features both dependent linear types as well as identity types. An simple example verification (on paper) of `Quipper` code in `LHoTT` is spelled out in [14].

The idea of regarding parameterized spectra as linear homotopy types modelling quantum systems, and the observation that these ought to constitute semantics for a linear extension of HoTT is already due to [10] [9], but a satisfactory syntax for such LHoTT has been given (on paper) only recently in [13]. This extends the syntax of [15] for a "classicality" monad on HoTT (earlier considered as "infinitesimal cohesion" in [8]) by syntax for a non-Cartesian tensor product as needed for genuine quantum data types.

Our results depend only on the fact that LHoTT verifies the "motivic yoga" (of base change of purely linear types along finite intuitionistic type fibers), which is proven syntactically in [13, §2.4] (and modeled semantically in [2]) but might well hold also in other versions of LHoTT syntax yet to be formulated.

References

[1] Myers D., Sati H. & Schreiber U. (2024): *Topological Quantum Gates in Homotopy Type Theory*. *Comm. Math. Phys.* (in print). arXiv:2303.02382.

[2] Sati H. & Schreiber U.: *Entanglement of sections*. arXiv:2309.07245.

[3] Sati H. & Schreiber U.: *Quantum and Reality*. arXiv:2311.11035.

[4] Sati S. & Schreiber U.: *The Quantum Monadology*. arXiv:2310.15735.

[5] H. Sati & U. Schreiber (2022): *Topological Quantum Programming in* `TED-K`. *PlanQC* 33. arXiv:2209.08331. Video: `youtu.be/hDwXQVcloB8`.

[6] H. Sati & U. Schreiber (2023): *Anyonic defect branes and conformal blocks in twisted equivariant differential (TED) K-theory*. *Reviews in Mathematical Physics* 35(06), p. 2350009, doi:10.1142/S0129055X23500095. arXiv:2203.11838.

[7] H. Sati & U. Schreiber (2023): *Anyonic topological order in twisted equivariant differential (TED) K-theory*. *Reviews in Mathematical Physics* 35(03), p. 2350001, doi:10.1142/S0129055X23500010. arXiv:2206.13563.

[8] U. Schreiber: *Differential cohomology in a cohesive topos*. arXiv:1310.7930.

[9] U. Schreiber (2014): *Differential generalized cohomology in Cohesive homotopy type theory*. Talk at *Formalization of Mathematics*, Inst. H. Poincaré, Slides: ncatlab.org/schreiber/files/SchreiberParis2014.pdf.

[10] U. Schreiber (2014): *Quantization via Linear Homotopy Types*. arXiv:1402.7041.

[11] U. Schreiber (2022): *Quantum Data Types via Linear Homotopy Type Theory*. Talk at *Workshop on Quantum Software @ QTML2022*, Slides: `ncatlab.org/schreiber/files/QuantumDataInLHoTT-221117.pdf`.

[12] U. Schreiber (2024): *Topological Quantum Programming in Homotopy Type Theory*. Talk at *Homotopy Type Theory Electronic Seminar Talks* (HoTTEest), Video: youtu.be/Wnm3yCUzNb0.

[13] M. Riley (2022): *A Bunched Homotopy Type Theory for Synthetic Stable Homotopy Theory*. Ph.D. thesis, Wesleyan University, doi:10.14418/wes01.3.139.

[14] M. Riley (2023): *A Linear Dependent Type Theory with Identity Types as a Quantum Verification Language*. Available at `https://mvr.hosting.nyu.edu/pubs/translation.pdf`.

[15] M. Riley, E. Finster & D. Licata: *Synthetic Spectra via a Monadic and Comonadic Modality*. arXiv:2102.04099.