Higher theta functions and Higher CS/WZW holography

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Sketch of a research program.

Abstract

Here we argue that – by invoking previous work we did on formalizing local prequantum field theory in cohesive homotopy theory – we arrive at a mathematical claim, stated as claim 1 below, which establishes an interesting relation between, on the one hand, higher dimensional analogs of the mathematics of theta functions, and on the other hand mathematical formalization of higher dimensional quantum field theory of Chern-Simons type and of Wess-Zumino-Witten type, in particular in dimensions 7 and 6, respectively. We propose that these questions ought to be investigated, and how, and we suggest that we'll do so if we get the chance.

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1 Goal

Quantum field theory and string theory are notorious both for involving structures just outside the reach of available mathematics as well as for providing deep inspiration for new mathematics (e.g. [43]). The special cases of *conformal* and of *topological* quantum field theories have always been more amenable to mathematical investigation [46], but they are often regarded as mere toy examples. Progress in recent years, both in physics and in mathematics, change this picture:

- The recent development of homotopy theory and the formalization and proof of the cobordism hypothesis [35] (see [6] for exposition) shows that the axioms for topological quantum field theory originally found by Atiyah and Segal [46] are, in their refined "local" form, much more deeply rooted in the very bedrock of mathematics than previously manifest. In a sense [35, example 2.4.15] shows that local topological quantum field theory is the unstabilized generalization of the J-homomorphism in stable homotopy theory. This gives the axioms a genuine mathematical relevance quite independent of any physics, witnessed by recent developments such as [11].

- The development of the holographic principle [37] in physics highlights, in view of its analysis in [53, 24], that topological quantum field theories (inside string/supergravity theories) serve to induce actual physical field theory "on their holographic boundaries", in fact that chiral/self-dual higher gauge theories are only definable this way [52, 40] in the first place. In terms of actual mathematics, this is nicely exhibited by the classification result [20] which constructs[1] rational 2d CFTs in a holographic fashion from 3d Chern-Simons theory, see [30] for review amplifying this perspective.

  Moreover, [52, 55, 56, 57] argues that via a chain of reductions and equivalences, there is a whole cascade of non-topological field theories descending from a single topological one. Notably the hallmark of all quantum field theory since the 1950s – 4d Yang-Mills gauge theory (e.g. QED, QCD) – is supposed to be usefully understood as the compactification of a 6-dimensional conformal field theory holographically defined by 7-dimensional Chern-Simons theory (sitting inside some supergravity completion). The quest for understanding this 6d theory is recently attracting much attention [38], but the fact that its only actual definition is via holography involving a 7d Chern-Simons theory [53, 13] deserves reminders (one such reminder is in [39, p. 29]).

On the other hand, while in physics one routinely thinks of quantum field theories as induced from geometric data – namely via *quantization* of Lagrangians – a reflection of this geometric quantization process in the refined local axiomatics of the cobordism hypothesis has been missing. But much of the interesting structure resides precisely here. **In broad terms, the goal of our project** is to find, formulate and study the refinement of the all-important mathematics of geometric quantization of topological field theory to the fully local ("extended", "multi-tiered") setup of the cobordism hypothesis. We have written about this before [45, 47] (following early ideas [44] and their refined formulation in [19, sect. 3]) and recently aspects of this have been further studied in [25, 27], but many open problems remain.

Concretely, our goal here is to focus on just the following aspect, which is still a rather substantial topic in its own right (a concrete technical description of a list of problems is below in 4):

[1] modulo the pertinent identification of spaces of conformal blocks

One may notice that in the mathematical perspective of "geometric quantization" (see [7] for a modern review) of topological field theory, the quantization step is to a large extent a process of attaching analogs of *theta functions* (modular forms or more generally automorphic forms) to given geometric data over a codimension-1 manifold Σ (a construction that Segal essentially summarized in terms of a *modular functor* [46, sect. 5]). Schematically, these are assignments

$$(z, \tau) \mapsto \theta(z|\tau)$$

depending on a point z in the phase space of the Chern-Simons theory – which is a flat connection, hence a "local system" on Σ – and on a point τ in the moduli space of "polarizing" geometric structure (typically: complex structure) on Σ. From the point of view of CS topological field theory such a theta function is a quantum *wavefunction*

$$\theta(z|\tau) = \Psi_\tau(z)$$

with respect to the splitting of phase space into "canonical coordinates" and "canonical momenta" that is encoded by the polarization induced by τ [51, 52]. The space of all theta functions here is hence the space of quantum states of CS topological field theory over Σ. On the other hand, from the point of view of conformal field theory the theta function is the partition function

$$\theta(z|\tau) = \mathrm{Tr}(\exp(-\tau H_z))$$

at "Schwinger parameter" τ and for background source field z [1]. Strictly speaking the traditional theta functions appear here for abelian gauge group, and more generally one speaks of "generalized theta functions" [5]. From this perspective the space of all such functions is then called the space of *conformal blocks*.

Finally, the Mellin transform of the theta function with respect to the Schwinger parameter, hence the expression

$$\zeta_z(s) := \int_0^\infty \tau^{s-1}\, \theta(z|\tau)\, d\tau$$

(defined as indicated where the integrand and its integral exist and analytically continued from there to a meromorphic fucntion on the complex plane) is its *zeta function* ("L-function") which in the special case that Σ is a torus is what in physics is called the "zeta-regularized one-loop vacuum amplitude" [50, 10].

In summary, traditional geometric quantization of Chern-Simons type topological field theories involves the following assignments:

$$\underbrace{L \mapsto \left(\underbrace{\Sigma \mapsto \left(\underbrace{z \mapsto \left(\underbrace{\tau \mapsto \theta(z|\tau)}_{\text{theta function}} \;\in\; \underbrace{\Theta_{z,\tau}}_{\text{theta line}} \right)}_{\text{Jacobi form}} \right)}_{\text{modular functor}} \right)}_{\text{Chern-Simons theory in codimension-1}}$$

L	level / CS Lagrangian
Σ	homotopy type of manifold of codimension 1
z	flat G-connection/local systems on Σ
τ	geometric modulus on Σ

This assignment of zeta-functions (L-functions) ζ_z to local systems z in the quantization of topological/conformal field theory is curiously reminiscent of the Langlands program in number theory (a good review is [22]): the main conjecture here states that there is a natural (Langlands: "functoriality") assignment of L-functions to Galois representations induced via associated automorphic forms. Notice that Galois representations correspond to local systems and are hence the analogs in arithmetic geometry of the concept of flat connections in differential geometry (a standard fact, reviewed for instance in [33]), hence they play the role of the "z" in the above story.

Some aspects of this curious analogy of the Langlands correspondence with structures in conformal field theory had been highlighted in [18, part III]. However, the alternative proposal [31] for a relation of geometric Langlands to physics arguably had the effect of shifting attention away from this analogy. We propose here that, and will explain why, it seems worthwhile to reconsider the analogy with conformal field theory and to consider it in higher dimensional generalization following the cobordism hypothesis. In particular, it seems compelling to study the higher incarnation of these phenomena for the case of 7-dimensional Chern-Simons topological field theory and 6-dimensional conformal field theory.

Further evidence that there is interesting mathematics waiting to be extracted by adopting this perspective is that the celebrated construction of *equivariant elliptic cohomology* in [34] is, essentially, yet another mathematical incarnation of the modular functor assigning spaces of theta-functions. This notable relation is explicitly the content of [34, remark 5.2] but has probably not found due appreciation yet. Of course the point in this perspective is to regard the geometric moduli τ neither in complex analytic geometry nor in arithmetic geometry as in the previous cases, but in "derived arithmetic geometry" modeled on the "brave new rings" of stable homotopy theory [**?**]. This amplifies the need to mathematically understand the process of polarization in geometric quantization in the general modern context of "derived geometry", something we discuss in more detail below in 4. But in addition to all that, [34, section 5.1] hints at a "2-equivariant" refinement of equivariant elliptic cohomology, where the idea is that the (derived) theta-line bundle appears naturally as the "transgression" of a fully local Chern-Simons Lagrangian L to the moduli space of local systems on Σ.

The detailed incarnation of this idea of obtaining the theta-line and many other aspects of Chern-Simons theory via transgression of a fully localized Chern-Simons Lagrangian we have, in the context of higher differential geometry, studied in some detail in [15, 12, 16] and then put in the context of the cobordism hypothesis in [45, sect. 3.9.14] (based on discussion that recently led to [17]). Here the observation is, first, that a Chern-Simons level L may naturally be refined to a "higher line bundle" ("higher gerbe") $\mathbf{L}$ on the universal moduli stack of CS-gauge fields and, second, that this defines a local (in the sense of the cobordism hypothesis) pre-quantum field theory which automatically sends a closed manifold Σ of codimension-1 to the transgression which is the theta-line bundle over the moduli stack of local systems on Σ. This is, in more detail, the content of claim 1 below. Schematically it means that given a fully localized Chern-Simons Lagrangian $\mathbf{L}$ in the guise of a higher line bundle on a higher moduli stack, then the proof of the cobordism-hypothesis induces a natural assignment of the form

$$\underbrace{\mathbf{L} \mapsto \left(\underbrace{\Sigma \mapsto \left(\begin{array}{c} \Theta \\ \downarrow \\ \underbrace{\mathrm{Loc}_G(\Sigma)}_{\text{transgression of } \mathbf{L} \text{ to } \Sigma} \end{array}\right)}_{\text{cobordism hypothesis in codimension 1}} \right)}_{\text{local prequantum field theory}}$$

$\mathbf{L}$	higher CS-line bundle on universal moduli stack
$\mathrm{Loc}_G(\Sigma)$	higher moduli stack of G-local systems on Σ

Here the point is that for suitable choices of G and $\mathbf{L}$ this assignment does reproduce the modular functor of ordinary 3dCS/2dWZW theory (we discuss this below in 4), while on the other hand now this is a special case of something exceedingly more general: this naturally yields higher modular functors – as recently also considered in [17] – for higher dimensional Chern-Simons theories involving higher gauge fields and all that in a range of different flavors of geometry: all we need is that $\mathbf{L}$ lives in "cohesive homotopy theory" [45]. Notably the 7d CS-theories for which we had constructed fully local $\mathbf{L}$ in [13] fit into this construction and induce this way modular functors in dimension 6 which ought to encode partition functions of the mysterious 6d CFT.

It is this new mathematics of local prequantum field theory which we propose to investigate further. There are these four classical topics

1. modular functors;

2. topological modular forms;

3. equivariant elliptic cohomology.

4. geometric Langlands correspondence;

in each of which one has been looking for sensible higher-dimensional generalizations. For instance there are proposals for generalizing topological modular forms to certain topological automorphic forms which were hoped to be related to higher dimensional field theory the way topological modular forms famously are to 2-dimensional field theory. However, these hopes have not quite been satisfied. My claim is that local prequantum field theory in cohesive homotopy theory provides a construction that naturally induces higher dimensional generalizations of all four of these topics, and so our goal is to demonstrate that this is a fruitful way to approach these deep problems, Along the way, there are various small problems to be solved and various concrete spin-offs for the understanding of higher dimensional field theory to be obtained, a detailed description of which we give below in 4.

On a slightly more technical level, the following are objectives to be considered for each of the four intradisciplinary "facets" above:

1. **Higher topological modular forms.**
For the special case that Σ is a torus, the classical theta functions on the moduli of complex structures on Σ (i.e. the moduli stack of elliptic curves) famously are examples of modular forms, which are examples of automorphic forms. By [23, 34] these modular forms have a homotopy-theoretic refinement to "topological modular forms" obtained by refining the formal group underlying an elliptic curve to a complex-oriented ring spectrum, a construction that turns out to reflect deep aspects of 2d CFT [3, 2]. This has caused speculation that higher dimensional quantum field theory should similarly be related to more general "topological automorphic forms" associated with higher dimensional abelian varieties Σ. Attempts to do so consider Shimura varieties of moduli of polarized higher dimensional abelian varieties Σ.

   But the story of the higher CS/WZW-type duality just reviewed predicts a different picture: the abelian varieties that matter are specifically the higher (Griffith/Weil-)Jacobians $\mathrm{Loc}_{\mathbf{B}^{2k+1}U(1)}(\Sigma)$ induced by $(4k+2)$-dimensional Σ (instead of Σ itself) and complex (or, more generally, arithmetic) moduli are supposed to be induced from the complex/arithmetic structures on Σ, see our [48] for review and further pointers. (This distinction in invisible in low dimension due to the coincidence that for Σ a torus/elliptic curve, then it so happens to be canonically equivalent to its own Jacobian!) Accordingly the formal group associated with an elliptic curve, whose identification leads to the concept of topological modular forms, is more properly thought of as the deformation theory of its moduli of line bundles, and so the proper higher analogs of that are the formal groups of deformations of higher Jacobians. These are the Artin-Mazur formal groups [4]. Or rather, Artin-Mazur considered two kinds of formal groups associated with Σ, one given by the deformation theory of $(2k+1)$-line bundles on Σ, the other given by the deformation theory of $(2k+1)$-connections on Σ. (In traditional discussion of topological automorphic forms one tries to split off a 1-dimensional formal group from the formal group of an abelian variety Σ, instead Artin-Mazur give sufficient conditions on Σ (notably: that it is Calabi-Yau) such that the moduli of higher bundles/connections on Σ already have just a 1-dimensional formal group.) But Deligne's theorem (reviewed with an eye towards the present proposal in [48]) says that for describing the higher Jacobians one needs a case in between these two, namely $(2k+1)$-bundles with $k+1$-form connection.

   In conclusion, the physics story predicts that it is the formal groups of moduli of these intermediate structures which ought to give rise to the proper higher dimensional analog of topological modular forms. In particular therefore the formal group of the intermediate Jacobian of a 6-dimensional Σ ought to induce the correct generalization of the formal group of an elliptic curve as one passes from the discussion of 2d CFT to the 6d CFT. This ought to be investigated.

2. **Construction of the 6d CFT from nonabelian 7d CS.**
The key example to apply all these considerations to is the one that comes next after the established 3d/2d theory along the ladder of $(4k+3)d/(4k+2)d$-theories, namely 7d/6d. This case is currently attracting much attention from the theoretical physics community, due to arguments that the 6d theory appearing here provides under KK-reduction the manifestly S-duality equivariant lift of 4d (super-)Yang-Mills theory [55, 56]. While all this is physics speak, it concerns a highly interesting refinement of the classical mathematics of differential geometry of principal bundles and nonabelian cohomology. One expects that all mathematics related to 4d Yang-Mills theory, such as notably Donaldson invariants and

Seiberg-Witten theory, have subtle refinements to the S-duality equivariant 6d theory. Indeed, an argument due to Witten predicts that this S-duality is closely related to geometric Langlands duality (more on Langlands below). This ought to be investigated.

What seems underappreciated however is that the only actual definition of this 6d theory is as the holographic dual of a 7d theory, which in a suitable limit is a nonabelian 7d CS-type theory. In the simple case of an abelian theory this has in fact been established in [52] by analyzing the abelian 7d Chern-Simons theory. In [13, 14] we have discussed the nonabelian 7d Chern-Simons theory generalizing this abelian theory and constructed it in the "fully local" fashion that serves as input for claim 1. In conclusion, this seems to put us in position to systematically derive the nonabelian 6d theory at least in the limit where its holographic dual is dominated by its CS-type subtheory. This ought to investigated.

3. **Higher analogs of the Langlands correspondence.**
While geometric Langlands duality focuses, as the name indicates, on the duality aspect of the theory, the number theoretic Langlands correspondence is primarily more directed: Langlands' crucial "conjecture 3" states that to a suitable Galois representation is naturally (Langlands: "functorially") associated a certain zeta function, and that this may be expressed in terms of an automorphic form in correspondence with the Galois representation. Moreover, these zeta functions appear as Mellin transforms of theta functions (just as is the case for zeta functions arising as one loop vacuum amplitudes in quantum field theory). Keeping in mind that the geometric interpretation of a Galois representation is a flat connection, this means that Langlands' conjecture 3 asserts the natural assignment of a theta function to a flat connection. It is striking that structurally this is precisely what claim 1 produces, with its theta-functions naturally parameterized over the moduli $\mathrm{Loc}_G(\Sigma)$ of flat G-connections. But claim 1 states a considerable generalization of this assignment: on the one hand Σ may be a higher dimensional curve, on the other hand the geometric context may be any suitably "cohesive" context. It is not clear yet if ordinary number theoretic Langlands is indeed a special case of claim 1, and of course this would be a rather bold statement, but at least we have already discussed in [49] (based on [36]) that there is a realization of cohesive homotopy theory ("arithmetic cohesion") which does reproduces quite a few structures that are key to number theoretic Langlands. Hence whether or not claim 1 genuinely generalizes Langland's conjecture, it seems clear that it captures a phenomenon that is at least closely related. This ought to be investigated.

4. **Higher analogs of equivariant elliptic cohomology.**
Arguably the most refined cohomological incarnation of the traditional theory of the 2d WZW model is given by the"2-equivariant elliptic cohomology" indicated in [34]. In outline[2] this is just the kind of construction that takes the 3d Chern-Simons bundle as in claim 1 to the theta line bundles that it induces. Of course the key aspect of [34] is to formulate this in "derived arithmetic geometry", but this is just the context of item 3 above. Therefore it seems plausible that the realization of claim 1 in "arithmetic cohesion" leads to higher analogs of 2-equivariant elliptic cohomology. This will of course be rather sophisticated and is likely out of reach. But somebody should try to investigate it.

[2]See `http://ncatlab.org/nlab/show/equivariant+elliptic+cohomology`

2 Background

To start with, recall the fairly well established and deep relation [1, 51] (review includes [21]) between

- the mathematics of theta functions (Jacobi forms) on moduli of complex curves[3] (and their associated zeta-functions and L-functions);
- the physics of conformal quantum field theory in dimension 2 of Wess-Zumino-Witten type (WZW), as well as that of topological quantum field theory in dimension 3 of Chern-Simons type (CS).

Here the Jacobi theta functions may be understood as local sections of a line bundle which in terms of physics have two interpretations at once: on the one hand they are the *wavefunctions* (quantum states) of the CS-type topological quantum field theory, on the other they are the generating functions for the *correlators* (n-point functions) of the WZW-type conformal quantum field theory. This state of affairs – unifying aspects of the mathematics of *class field theory* and of the physics of *conformal field theory* (CFT) – plays a central role in discussion of geometric Langlands duality.

It is a fact (possibly an underappreciated fact, but see [24]) that this dual interpretation is an example of what in physics is called the *holographic principle*. This principle has become rather famous in a more ambitious but mathematically less well understood incarnation known as *AdS/CFT-duality* [37], where it is argued to holographically relate more general conformal field theories to certain sectors of string theory. However, the central phenomenon of the holographic principle has been manifest in CS/WZW ever since [51]: fields of the higher dimensional theory are identified with sources of the lower dimensional theory, and wavefunctions depending on fields with partition functions depending on sources. Moreover, in a series of articles Edward Witten has argued that

1. in a suitable limit AdS/CFT duality restricts to higher dimensional CS/WZW-type duality [53];
2. this higher dimensional CS/WZW duality serves to holographically *define* what in physics are called self-dual higher gauge theories, by producing higher dimensional analogs of theta functions [51, 52, 54].

This is remarkable, because while large parts of quantum field theory and all the more of string theory are still outside the scope of genuine mathematics, the geometric quantization of Chern-Simons-type field theories is a notable exception and famously does lend itself to mathematical formulation. For instance the seminal work [28] made mathematically precise central aspects of the argument in [52] for the case 7d/6d and did so by initiating a whole new field of mathematics, that of differential generalized cohomology (reviewed e.g. in [8]).

[3] ncatlab.org/nlab/show/theta+function

In previous work [45] we have added to this the following:

1. We have shown that differential generalized cohomology finds a useful axiomatization by "cohesive homotopy types" and specifically [9] by stable cohesive homotopy types, i.e. by "cohesive spectra", in refinement of how the classical Brown representability theorem shows that plain generalized cohomology theories are represented by plain spectra.

2. We have shown that classical (pre-quantum) Hamilton-Lagrange-Jacobi field theory in its refined local deDonder-Weyl formulation and including fine detail such as prequantized Lagrangian correspondences, classical anomalies, and Lie algebras of conserved currents is naturally identified with the study of higher automorphism group in "slices" of a cohesive homotopy theory over a cohesive spectrum.

3. We have shown that in particular $(n + 1)$ dimensional CS-type field theories with their n-dimensional WZW-type "boundary theories" are naturally formalized by cobordism representations taking values in higher correspondences in the "slice" of a cohesive homotopy theory over a cohesive spectrum.

Item 3 here relies crucially on the formalization and proof of the cobordism hypothesis [35], which provides a powerful mathematical axiomatization of topological quantum field theory of the refined form that deserves to be called "fully local". For the toy example of "finite" topological QFTs essentially this has been sketched in [19, sect. 3], but for the discussion of interesting WZW-type theories and of actual theta functions it is crucial to generalize from finite theories to continuous/cohesive theories.

We have, building on [45], arguments that should prove the following claim – which is not yet published, and which hence will be the immediate starting point of the research proposed here:

Claim 1. *Given a cohesive higher universal moduli stack* $\mathbf{B}G$ *of CS-type higher gauge fields, and given an integral characteristic class* $L \in H^{n+1}(\mathbf{B}G, \mathbb{Z})$*,*

1. *there is naturally a fully local n-dimensional CS-type classical (pre-quantum) topological field theory of "level" L;*

2. *to a closed n-dimensional manifold* Σ *this naturally assigns a theta-line bundle on the moduli stack* $\mathrm{Loc}_G(\Sigma)$ *of flat G-connections (G-local systems) on* Σ *(the* phase space *of the CS theory);*

3. *the cobordism hypothesis implies that the quantization of this prequantum line bundle is a higher dimensional analog of what in 3d/2d is famous as the Hitchin connection [26] whose sections are the ordinary Jacobi modular forms.*

4. *connecting this field theory via a "defect" to that induced by a cohesive refinement of the Eilenberg-MacLane spectrum* $H\mathbb{Z}$ *induces the higher analog of the classical algebra of conserved currents and exhibits the higher dimensional analog of what in 2d is famous as the Virasoro algebra of conformal currents.*

As we argue now, this new mathematics of local cobordism representations in slice homotopy theories over cohesive spectra – of which we already have the definition, construction and rudiments of a theory, but which deserves to be developed much further – naturally serves to inform the open problems:

3 Objectives

The ultimate goal of this proposal is to provide an in-depth analysis of the mathematics of the geometric quantization of higher-dimensional Chern-Simons-type topological field theories via cohesive homotopy theory and via the proof of the cobordism hypothesis. This will yield novel higher dimensional analogs of theta functions and of conformal blocks of higher dimensional theories. It will impact on open questions on how to properly lift modular theory (modular functors, modular forms) to higher dimensions and on the construction of conformal field theories in higher dimension, notably in dimension 6.

As I have proven previously, certain higher line bundles (higher gerbes) over higher moduli stacks of higher (gauge-)groups serve as the fully local (fully de-transgressed) incarnation of Chern-Simons-type functionals. In turn, these define local pre-quantum field theories in the sense of the proof of the cobordism hypothesis. As such, they automatically take closed manifolds in codimension one to certain theta-line bundles on higher moduli stacks of local systems. It is this new mathematics of local prequantum field theory which we propose to investigate further.

The impact of this is the following. There are these four classical topics

1. modular functors;
2. topological modular forms;
3. equivariant elliptic cohomology.
4. geometric Langlands correspondence;

in each of which one has been looking for sensible higher-dimensional generalizations. For instance there are proposals for generalizing topological modular forms to certain topological automorphic forms which were hoped to be related to higher dimensional field theory the way topological modular forms famously are to 2-dimensional field theory. However, these hopes have not quite been satisfied. I explain in the remainder that local prequantum field theory in cohesive homotopy theory provides a construction that naturally induces higher dimensional generalizations of all four of these topics, and so my goal is to demonstrate that this is a fruitful way to approach these deep problems,

Each of these areas forms a rather broad and rich subject by itself. Consequently, I do not anticipate discussing exhaustively any of these areas but a bit of funding will enable me to make substantial progress in all of these areas.

Indeed, there are a wealth of smaller interesting problems and phenomena to consider, which I list and describe in detail in 4. As I also explain there, all four areas are reflected in current open questions in mathematical physics, and I will continuously make that connection, which has turned out to be very fruitful both for the mathematics as well as for the physics.

My specific objectives are as follows:

1. The primary objective is to explore the higher theta-line bundles and theta-functions on higher moduli stacks of local systems which appear from considering cobordism representations in cohesive homotopy theory.
   (a) I will relate the traditional modular functor in guise of the traditional Hitchin connection to the modularity as implied by the cobordism hypothesis;
   (b) I will relate to the theta-characteristic square roots known for cobordisms with spin-structure and in higher dimensions for Wu-structure.

2. My secondary objective is to explore the higher analogues of modular functor/modular form theory appearing this way.
    (a) I will study the higher analog of 2-equivariant elliptic cohomolog with formal elliptic curves generalized to, essentially, Artin-Mazur formal groups.
    (b) I will study the zeta functions associated with the given higher theta functions and relate their zeros to the vanishing of vacuum amplitudes of the higher- dimensional field theories.
3. My final objective is to investigate the impact of these insights on the analysis of higher-dimensional topological field theory with their higher holographic boundaries.
    (a) I will work out the modular functor of the de-transgression of the nonabelian 7-dimensional Chern-Simons theory and relate it to the partition function of the 6-dimensional theory on coincident M5-branes.
    (b) I will analyse the full de-transgression of the 11-dimensional Chern-Simons theory of and its higher modular functor valued in K-theory.

4 Methodology

The following is a list of technical subproblems to be considered, breaking up the route to the above objectives into a sequences of plausibly manageable steps marked by clearly identifiable milestones.

1. **higher modular functors**

   (a) Where the traditional concept of modular functor involves complex structure moduli for a Riemann surface Σ, the cobordism hypothesis implies that what fundamentally matters is, generally, the moduli of $(n+1)$-framings on the n-dimensional Σ. Or rather, given a morphism of topological groups $K \to O(n+1)$ and an $(n+1)$-dimensional topological field theory on $(n+1)$-dimeional manifolds with G-structure, then it implies that the relevant moduli are those of once-stabilized K-structures in Σ.

   The first step is hence to determine, given a level $\mathbf{B}G \to B^{n+1}\mathbb{Z}$, for which K-structures this defines a TFT. The conjecture stated in [25] implies, using [35, example], that this are those K for which one has a homotopy K-fixed point structure for the canonical action on $\Omega^n B^{2n+1}\mathbb{Z}$. These conjectures should be worked out and their implications determined.

   (b) There are known fractional level refinements of Chern-Simons action and higher Chern-Simons action functionals on cobordisms with Spin-structure [29] and generally with Wu-structure [28] This needs to be interpreted in terms of the previous point.

   (c) To connect traditional theory to the one encoded by the cobordism hypothesis, the first step is then to show that the classical projectively flat Hitchin connection [26] on the Riemann moduli space of a surface Σ pulls back to a genuinely flat connection on the moduli space of 3-framings of Σ (or to other once-stabilized G-structures, as above). For this one may again use the discussion in [46, sect. 5] which essentially gives this.

   (d) The traditional geometric quantization which leads to the Hitchin connection has a modern interpretation in terms of push-forward in twisted K-theory [42]. Here the traditional Kähler polarization is used to induce a Spin^c-structure, hence an orientation in complex K-theory, and it is this which is seen to fundamentally matter for the quantization process. Hence in summary then the polarization process is one which functorially induces from $(n+1)$-framings on Σ, cohomology orientations on $\mathrm{Loc}_G(\Sigma)$. This needs to be phrased in generality.

   (e) Generally in higher dimensions then, the cobordism hypothesis tells one what variance of the relavant higher topological modular functors/Hitchin connections must be, a point that has been highlighted in [17]. By reasoning as in the previous items, this gives constraints on what the suitable "polarizing moduli" may be, hence which geometric structure one is to put on Σ in order to quantize $\mathrm{Loc}_G(\Sigma)$. This highlights the fact that the traditional prescription to quantize Chern-Simons via Kähler polarizations on $\mathrm{Loc}_G(\Sigma)$ induced from complex structure on Σ is not so much grounded in necessity as in just convenience. The role of arithmetic structure on Σ implicitly appearing in [2] and of "derived arithmetic structure" explicit in [34] ought to be put into this context.

(f) If and when this is achieved, then one has obtained a cohomological formulation of geometric quantization of levels $L \in H^{n+1}(\mathbf{B}G, \mathbb{Z})$ for G a cohesive ∞-group to higher modular functors on once-stabilized K-structure moduli space.

(g) Already in the simplest case, where $K = 1$ (framing structure) one might phrase the open question here as follows: Physics is fairly familiar with n-dimensional field theory whose geometric structure is given by n-framings, these are Einstein's "teleparallel gravity" theories. But the cobordism hypothesis predicts that what matters more fundamentally are $(n+1)$-framings. These are structures in between the very restrictive n-framings of teleparallel theory and the very relaxed stable framining familiar from stable homotopy theory and generalized cohomology theory. Just from the point of view of physics of field theory it would be interesting to investigate geometric theories on $(n+1)$-framed n-manifolds.

2. **Higher topological modular forms.**

- The proposal to associate cohomology theories with Artin-Mazur formal groups exists [**?**] but its impact remains inconclusive. However, from the physics discussion of self-dual higher gauge theory and using Deligne's theorem, it follows that what matters are not exactly what Artin-Mazur considered. They considered formal moduli of bare line n-bundles as well as of n-connections, while for a proper analogy one is to consider $2k+1$-line bundles equipped with $k+1$-connection [48]. For this case the entire analysis of [4] should be redone.
- Of particular interest is to check at which heights such "intermediate" AM formal groups will exist. The general expectation,extrapolating from the known case $n = 1$ and $n = 2$, has at times been that n-dimensional CFT ought to go along with formal groups whose supersingular cases have height n. But this may be too naive and in any case deserves further analysis.
- Assuming this works out and given the previous point, then the conclusion would be that there ought to be a sheaf of spectra on the once-stabilized K-moduli stack of Σ such that ...

3. **Higher analogs of the Langlands correspondence.**

- There exist several proposals to connect the Riemann zeta function to physics, but what is lacking is a systematic look at the fact that physics has its own zeta functions, namely the zeta-regularized traces of Feynman propagators, and that these appear in close conceptual analogy with the situation in number theory. Where Langlands assigns a zeta function to a Galois representation, hence to a flat connection in arithmetic geometry, physics assigns zeta functions to points in the phase space of higher Chern-Simons type field theories, which are higher flat connections.
- The "special values" of the zeta functions appearing in physics include in particular the "one loop vacuum amplitude". Specifically for superstrings, there are deep conjecture about the vanishing of these amplitudes. It seems that *this* is the interesting physics incarnation of the Riemann hypothesis. This needs to be thought about.

4. **Higher analogs of equivariant elliptic cohomology.**

- The “2-equivariance” sketched in [34] says that theta line bundles appear as the transgression of Chern-Simons 3-line bundles. This is morally just as in Claim 1 above, only that some major details need to be looked into...
- In view of the above points one expects the higher analog of this as one passes to cohomology theories controlled by higher “intermediate” Artin-Mazur groups.
- Lots to be thought about here...

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