Jan 30, 2009 (On the train somewhere between Hamburg and Cologne)

Dear Todd,

here is a brief note whose main purpose is to order my own thoughts about a question I have been thinking about – so please feel free to ignore this – but which I am thinking that possibly you might enjoy making one or the other comment about, or might even find an aspect of that is of interest to you in its own right.

It appears to me that the following situation in the overlap of

- groupoidification;
- Trimblean weak ω-categories

would be interesting to consider in more detail.

The general context of a closed monoidal homotopical category $\mathcal{V}$ with interval object $\mathcal{I}$, the way we talked about on the nLab and elsewhere, is one in which we have

- a notion of fundamental Trimblean ω-category $\Pi_\omega(B)$ for each object B of $\mathcal{V}$;
- a notion of B-bundle for each pointed object $\mathrm{pt} \xrightarrow{\mathrm{pt}_B} B$ of $\mathcal{V}$ such that the trivial "rank-1" B-bundle over an object X is $X \times \Omega_{\mathrm{pt}} B \xrightarrow{p_1} X$;

where the loop monoid $\Omega_{\mathrm{pt}} B$ of B is the pullback

$$\begin{array}{ccc} \Omega_{\mathrm{pt}} B & \longrightarrow & [\mathcal{I}, B] \\ \downarrow & \lrcorner & \downarrow{\scriptstyle d_0 \times d_1} \\ \mathrm{pt} & \xrightarrow{\mathrm{pt}_B} & B \end{array} \quad .$$

I am calling this the "loop monoid" since it should naturally carry the structure of a *monoidal* Trimblean ω-category in that we should be entitled to think of this as $\mathrm{End}_{\Pi_\omega B}(\mathrm{pt}_B)$. Notice that in the case that $\mathcal{V}$ is a *category of fibrant objects* in the sense of [1] and $\mathcal{I}$ is compatible with this in that $[\mathcal{I}, B]$ is a path object for B, then the image of $\Omega_{\mathrm{pt}} B$ in the homotopy category $\mathrm{Ho}_{\mathcal{V}}$ is discussed in [1] as a group object: the group of based loops in B. I am suggesting here to refine this by remembering a full monoidal ω-category structure and by allowing B to be non-groupoidal, i.e. directed.

For instance

- if we choose $\mathcal{V} = \mathsf{Groupoids}$ with the standard folk structure of a closed monoidal homotopical category and with the standard interval object and let $B = \mathbf{B}G$ be any one-object groupoid, then $\Omega \mathbf{B} G = G$ and we recover the ordinary theory of G-principal bundles (once we pass to groupoids modeled on topological or smooth spaces, which I'll suppress for the purpose of this letter here).
- But we can take $\mathcal{V} = \mathsf{Categories}$ with its standard directed interval object and consider $B = \mathrm{Vect}_k$ with point $k \in \mathrm{Vect}_k$. Then $\Omega_{\mathrm{pt}} B = k$ is a monoid that is not a group and we obtain the notion of k-vector bundles.
- And passing to higher degrees take next $\mathcal{V} = 2\mathsf{Categories}$ with the standard extra structure and let $B = \mathbf{B}\mathrm{Vect}_k$ be the bicategory corresponding to the monoidal category Vect_k, then the loop monoid $\Omega_{\mathrm{pt}} B = \mathrm{Vect}_k$ is a monoidal category and we get a theory of 2-vector bundles.
- *And so on...* where we want to keep in mind setups that lead to loop monoids such as $\Omega_{\mathrm{pt}} B = \mathrm{Ch}_+(\mathrm{Vect}_k)$. For the present purpose I don't want to try to say more about technical details on this but rather leave it at mentioning this natural possibility and continue with describing what the natural question is that this letter is supposed to be about.

The point now is that in rather general contexts – but in particular in the context of topological quantum field theory – we are dealing with *spans of B-bundles* in the above general sense. For $\mathcal{V} = \mathsf{Groupoids}$ these are spans as familiar from *groupoidification*, but for more general $\mathcal{V}$ they will be more general, and that's what I'll be interested in here.

But to immediately put the plug back on the bottle I'll concentrate attention now on a very specific but probably noteworthy special case, namely that where all our B-bundles involved are "trivial, rank-1" in the above sense.

For instance a span

$$\begin{array}{ccc} & \Psi & \\ \swarrow & & \searrow \\ \Omega_{\mathrm{pt}}B & & X \times \Omega_{\mathrm{pt}}B \end{array}$$

should be interpreted as a generalized *section* of the trivial bundle $X \times \Omega_{\mathrm{pt}}B \to X$, which is nothing but a generalized function with values in the monoid of loops $\Omega_{\mathrm{pt}}B$. Indeed, every ordinary section gives rise to such a span, but the general span of this kind may contain sections which are more "distributional". For instance we could have $\Psi = \mathrm{pt}$ so that $\Psi \to X \times \Omega_{\mathrm{pt}}B$ is just a point in X with the choice of a point in the fiber $\Omega_{\mathrm{pt}}B$ over it.

I shouldn't be boring you with this, were it not for the fact that I want to draw attention to this situation for the last cases in the above list of examples, where $\Omega_{\mathrm{pt}}B = \mathrm{Vect}_k$ or more generally $\Omega_{\mathrm{pt}}B = \mathrm{Ch}_+(\mathrm{Vect}_k)$. In these cases a span as above with $\Psi = \mathrm{pt}$ looks very much like what in different contexts would be modeled as a coherent *skyscraper sheaf* of sections of a (complex of) "vector bundle"(s) with fiber concentrated over a single point.

Possibly it's trivial and boring, in any case it is not a deep statement, but I nevertheless find it noteworthy here, if only for myself, that the crucial reason for considering (complexes of) coherent sheaves is that these can be pushed forward, which is precisely what also the generalized span-incarnation of sections of B-bundles allows us to do.

Looking at what I have written so far I see that this is getting more long-winded than indended. So let me come to the main point:

it is crucial now that with the monoidal structure on $\Omega_{\mathrm{pt}}B$ also the spans as above should inherit a monoidal structure of sorts, reflecting their interpretation as generalized $\Omega_{\mathrm{pt}}B$-valued functions. But this means that we now have to expect an interesting *interplay* between

- the span pull-push operations on Ψ;
- the algebraic operations on Ψ.

You may recognize that this is conceptually precisely the central theme of David Ben-Zvi's work which we talked about at the Café – the only difference being a different attitude towards the concrete realization of the structures under consideration. I am thinking that the above kind of approach with its emphasis on Trimblean A_∞-structures naturally induced from the presence of an interval object that at the same time controls the notion of fiber bundles, might provide a useful perspective on the general problem at hand here.

A concrete question therefore would be the following, taken directly from B.-Z.'s work but stated here in the above languge:

Given a pullback diagram

$$\begin{array}{ccccc} & & (X \times_Y X') \times \Omega_{\mathrm{pt}}B & & \\ & \swarrow & & \searrow & \\ X \times \Omega_{\mathrm{pt}}B & & & & X' \times \Omega_{\mathrm{pt}}B \\ & \searrow & & \swarrow & \\ & & Y\Omega_{\mathrm{pt}}B & & \end{array}$$

of trivial bundles, characterize generalized sections of (i.e. spans into) the correspondence object $(X \times_Y X') \times \Omega_{\mathrm{pt}}B$ by an *algebraic* quotient of the cartesian product of generalized sections of $X \times \Omega_{\mathrm{pt}}B$ and $X' \times \Omega_{\mathrm{pt}}B$ in terms of the monoidal structure of monoidal Trimblean ω-categories expected on both.

Notice that in B.-Z.'s context it is proven that the former is the algebraic tensor product of the latter two over the algebraic item assigned to Y, all in some ∞-ized sense which I won't recall here. This statement then leads to a series of important consequences, notably on various Hochschild (co)homologies. Which is all very nice. But here I would like to see if one can understand all that from the above kind of perspective.

I'll stop here for the time being. Hopefully the above is making some sense. If you find some thought in here which you have a comment on I'd be very interested. I was thinking that this notion of interval object in a general homotopical context with its joint implication on Trimblean ω-cateories and higher bundles should be something the two of us might fruitfully chat about a bit more. But if you feel you'd rather not be distracted at the moment by trying to make sense of my ramblings here, please just ignore all this.

All the best,
Urs

References

[1] K. Brown, *Abstract Homotopy Theory and Generalized Sheaf Cohomology*, Transactions of the American Mathematical Society, Vol. 186 (1973), 419-458