Lightcone quantization of topological flux observables
for flux-quantization in $\mathcal{A}$-theory [SS23-Qnt].

Nonprtrbtv BRST complex of topological fields. The observables on charge (super-selection) sectors are evidently the linear combinations of $\pi_0 \mathrm{Maps}\big(X^d,\, \mathcal{A}\big)$.

$$\mathrm{Obs}_0(X^d, \mathcal{A}) \;=\; H_0\big(\mathrm{Maps}\big(X^d,\, \mathcal{A}\big);\, \mathbb{C}\big)\,.$$

The higher homotopies $\pi_n \mathrm{Maps}\big(X^d,\, \mathcal{A}\big)$ are higher gauge transformations, whence higher chains are (topological) higher "BRST-ghost" field observables.

$$\mathrm{BRST}_\bullet(X^d, \mathcal{A}) \;=\; C_\bullet\big(\mathrm{Maps}\big(X^d,\, \mathcal{A}\big);\, \mathbb{C}\big)$$

Therefore the gauge invariant observables are the chain-homology of this BRST complex, hence are the complex homology of the topological phase space.

$$\mathrm{Obs}_\bullet(X^d, \mathcal{A}) \;=\; H_\bullet\big(\mathrm{Maps}\big(X^d,\, \mathcal{A}\big);\, \mathbb{C}\big)$$

Non-perturbative M-theory spacetime domain. In lifting a type IIA spacetime domain. X^9_{IIA} (a pointed space) to fully non-perturbative M-theory, the IIA-circle fiber S^1_A is meant to appear decompactified as $\mathbb{R}^1$. But assuming that fluxes vanish at infinity along this direction, the corresponding fiber domain is $\mathbb{R}^1_{\cup\{\infty\}}$ and hence the M-theory spacetime domain is $\mathbb{R}^1_{\cup\{\infty\}} \wedge X^9_{\mathrm{IIA}}$.

Makes topological phase space a loop space. This implies that the phase space of flux-quantized topological fields in M-theory is a based loop space:

$$\begin{aligned} \mathrm{Maps}^{*/}\big(X^{10}_{\mathrm{M}},\, \mathcal{A}\big) \;&\equiv\; \mathrm{Maps}^{*/}\big(\mathbb{R}^1_{\cup\{\infty\}} \wedge X^9_{\mathrm{IIA}},\, \mathcal{A}\big) \\ &\simeq \Omega\, \mathrm{Maps}^{*/}\big(X^9_{\mathrm{IIA}},\, \mathcal{A}\big) \end{aligned}$$

M-theory quantizes itself. From this, the topological observables inherit a non-commutative *Pontrjagin-Hopf algebra* structure, which makes them be quantum observables [CSS23-Qnt]:

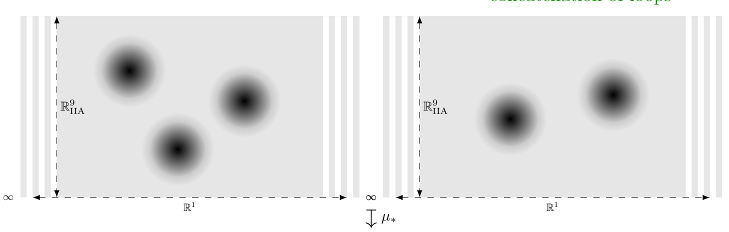
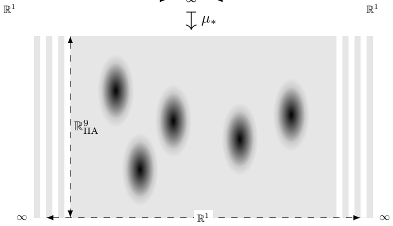
$$\mathrm{QObs}_\bullet\big(X^{10}_{\mathrm{M}},\, \mathcal{A}\big) \;=\; H_\bullet\Big(\Omega\, \mathrm{Maps}\big(X^9_{\mathrm{IIA}},\, \mathcal{A}\big)\Big)$$

whose operator product is given by by translation followed by "fusion" of solitons in the M-theory circle-direction.

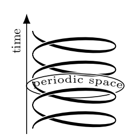
Pontrjagin product:

$$H_\bullet(\Omega Y) \otimes H_\bullet(\Omega Y) \xrightarrow[\text{Künneth}]{\sim} H_\bullet(\Omega Y \times \Omega Y) \xrightarrow[\text{pushforward in homology}]{(\text{-})\cdot(\text{-}) \,:=\, \mu_*} H_\bullet(\Omega Y)$$

$$\Omega Y \times \Omega Y \xrightarrow[\text{concatenation of loops}]{\mu} \Omega Y$$

The Discrete lightcone emerges. But, generally, the operator product of quantum observables reflects *temporal* order (originally observed by [Fey42, p. 35][Fey48, p. 381], cf. [Ong]), whence we are faced with a topological version of "discretized light cone" quantization (cf. [BFSS97][Susk97]).

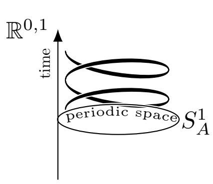


The star-involution of Light-cone time-reversal must hence be the combination of the Pontrjagin antipode (spatial inversion) with complex conjugation (plain temporal inversion), which together makes the quantum observables into a complex Hopf algebra.

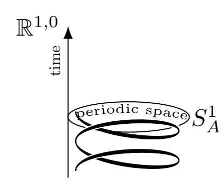
Pontrjagin antipode:

$$H_\bullet(\Omega Y) \xrightarrow{\ \mathrm{inv}_*\ } H_\bullet(\Omega Y)$$

$$\Omega Y \xrightarrow[\text{reversal of loops}]{\mathrm{inv}} \Omega Y$$



$$\begin{array}{ccccc} & & H_\bullet(\Omega Y;\mathbb{C}) & & \\ & \nearrow^{\text{Pontrjagin-antipode}\; H_\bullet(\mathrm{inv};\mathbb{C})\; S^1_A\text{-inversion}} & & \searrow^{\text{complex conjugation}\; H_\bullet(\Omega Y; (\text{-})^*)\; \text{time inversion}} & \\ H_\bullet(\Omega Y;\mathbb{C}) & & \xrightarrow[\text{light-cone parameter inversion}]{(\text{-})^*} & & H_\bullet(\Omega Y;\mathbb{C}) \\ & \searrow_{H_\bullet(\Omega Y; (\text{-})^*)} & & \nearrow_{H_\bullet(\mathrm{inv}; (\text{-})^*)} & \\ & & H_\bullet(\Omega Y;\mathbb{C}) & & \end{array}$$



Quantum states of topological fields are therefore the positive linear functionals on this complex Pontrjagin-Hopf homology algebra:

$$\mathrm{QStates}\big(X^{10}_{\mathrm{M}}, \mathcal{A}\big) \;=\; \Big\{\rho \;:\; \mathrm{QObs}_\bullet\big(X^{10}_{\mathrm{M}}, \mathcal{A}\big) \xrightarrow{\text{linear}} \mathbb{C} \;\Big|\; \underset{\mathcal{O}}{\forall}\;\; \rho\big(\mathcal{O}^* \cdot \mathcal{O}\big) \;\in\; \mathbb{R}_{\geq 0} \subset \mathbb{C}\Big\}$$