Twisted geometric ∞-bundles

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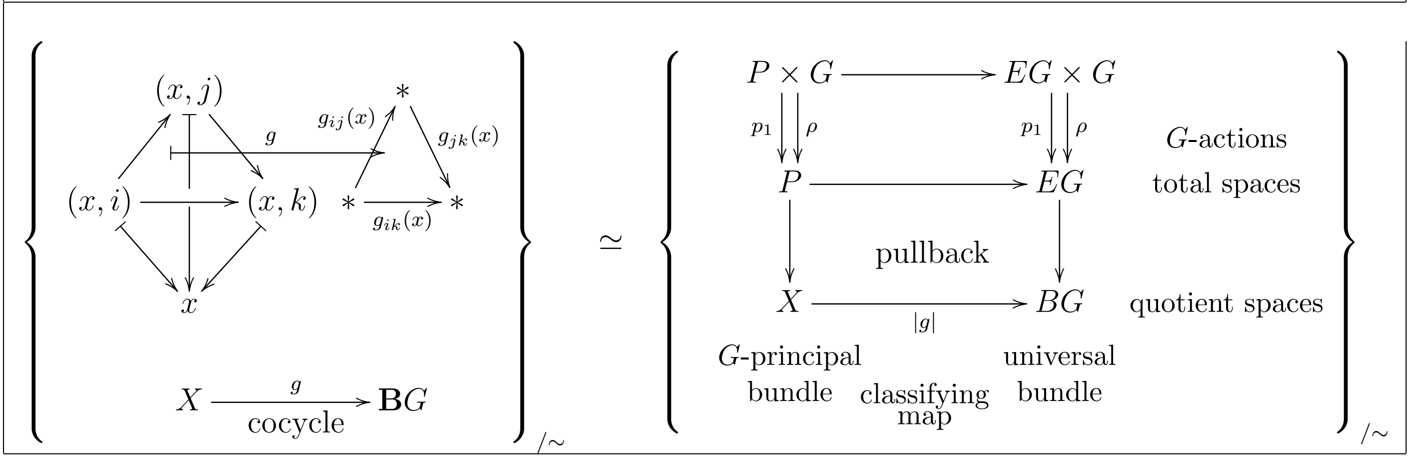
handout[1] for a talk, June 2012
reporting on joint work [NSS] with
Thomas Nikolaus and *Danny Stevenson*
with precursors in [NiWa, RoSt, SSS, S]
and with various applications, indicated in [Lect].

Contents

1 Motivation

Classical fact. For X a manifold and G a topological/Lie group, regarded as a sheaf of groups $C(-, G)$ on X, there is an equivalence:

algebraic data on X		**geometric data on X**
$\left\{ \begin{array}{c} \text{degree-1 nonabelian} \\ \text{sheaf cohomology} \end{array} \right\}$ $H^1(X, G)$	$\simeq$	$\left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{G-principal bundles over } X \end{array} \right\}$ $G\mathrm{Bund}(X)$



Problem. In *higher differential geometry* [S], for instance in *String-geometry* [SSS], geometric groups G are generalized to *geometric grouplike A_∞-spaces*: to *geometric ∞-groups* (examples below in 5). Need to generalize the above classical fact to this case.

[1] Available at `ncatlab.org/schreiber/files/TwistedBundlesTalk.pdf`.

2 Higher geometry

We need

$$\boxed{\text{geometry}} + \boxed{\text{homotopy theory}} = \boxed{\text{higher geometry} \simeq \infty\text{-topos theory}}.$$

Here is a way to think of the above classical fact that will generalize: let

- $C :=$ SmthMfd be the category of all smooth manifolds of finite dimension (or some other site, here, for convenience, assumed to have enough points);
- $\mathrm{gSh}(C)$ be the category of groupoid-valued sheaves over C, for instance $X = \{ X \rightrightarrows X \}, \mathbf{B}G = \{ G \rightrightarrows * \} \in \mathrm{gSh}(C)$;
- $\mathrm{Ho}_{\mathrm{gSh}(C)}$ the *homotopy category* obtained by universally turning $\boxed{\textit{stalkwise groupoid-equivalences}} \mapsto \boxed{\textit{isomorphisms}}$.

Fact. $H^1(X, G) \simeq \mathrm{Ho}_{\mathrm{gSh}(C)}(X, \mathbf{B}G)$ (e.g. using Hollander (2001)).
Definition. To generalize, let

- $$\begin{array}{ccccc} & & \text{Groupoids} & & \\ & \swarrow & & \searrow^{N} & \\ \text{Categories} & & & & \textbf{KanComplexes} \\ & \searrow^{N} & & \swarrow & \\ & & \text{QuasiCategories} & & \\ & & \uparrow & & \\ & & \text{SimplicialSets} & & \end{array}$$ be the Kan complexes, aka ∞-*groupoids* inside all quasi-categories aka ∞-*categories*
- $\mathrm{sSh}(C)_{\mathrm{lfib}} \hookrightarrow \mathrm{Sh}(C, \mathrm{sSet})$ be the (stalkwise Kan) simplicial sheaves;
- $\mathbf{H} := L_W \mathrm{sSh}(C)_{\mathrm{lfib}}$ the *simplicial localization* obtained by universally turning $\boxed{\textit{stalkwise homotopy equivalences}} \mapsto \boxed{\textit{homotopy equivalences}}$.

Fact. (Toën-Vezzosi, Rezk, Lurie) This is the ∞-category theory analog of the sheaf topos: the ∞-*stack* ∞-*topos* over C.
Example. Smooth∞Grpd $:= \mathrm{Sh}_\infty(\mathrm{SmthMfd})$ is the ∞-topos of *smooth* ∞-*groupoids* / *smooth* ∞-*stacks*.
Example. For A a sheaf of abelian groups, $\mathbf{B}^{n+1}A := \mathrm{DoldKan}(A[n+1]) \in \mathrm{sSh}(C)$ is the moduli n-stack of $\mathbf{B}^n A$-principal bundles (details in a moment).
Proposition. Every object in Smooth∞Grpd is presented by a simplicial manifold, but not necessarily by a *locally Kan* simplicial manifold (see below).

Definition A *group* in the ∞-topos is a $G \in \mathbf{H}$ equipped with a groupal A_∞-algebra structure: coherently homotopy associative product with coherent homotopy inverses.
Example. In Smooth∞Grpd this is a *smooth* ∞-*group*: for instance a Lie group, or a Lie 2-group, or a differentiable group stack, or a sheaf of simplicial groups on SmthMfd.
Fact. (classical + Lurie) There is an equivalence

$$\{ \text{ groups in } \mathbf{H} \} \underset{\text{delooping } \mathbf{B}}{\overset{\text{looping } \Omega}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \left\{ \begin{array}{c} \text{pointed connected} \\ \text{objects in } \mathbf{H} \end{array} \right\}$$

Proposition. Let C have a terminal object. For every ∞-group $G \in \mathrm{Grp}(\mathrm{Sh}_\infty(C))$ there is a sheaf of simplicial groups presenting it under $\mathrm{Sh}_\infty(C) \simeq L_W \mathrm{sSh}(C)$; and every ∞-action $\rho : P \times G \to P$ is presented by a corresponding simplicial action.

3 Principal ∞-bundles

Definition. A G-**principal bundle** over $X \in \mathbf{H}$ is

- a morphism $P \to X$; with an ∞-action $\rho : P \times G \to P$;
- such that $P \to X$ is ∞-quotient $P \to P//G \overset{(*)}{\Leftrightarrow}$ principality : $P \times G^n \xrightarrow[\simeq]{(p_1,\rho)} P \times_X \cdots \times_X P$

Theorem. There is equivalence of ∞-groupoids $G\mathrm{Bund}(X) \underset{\underset{\to}{\lim}}{\overset{\mathrm{hofib}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \mathbf{H}(X, \mathbf{B}G)$, where

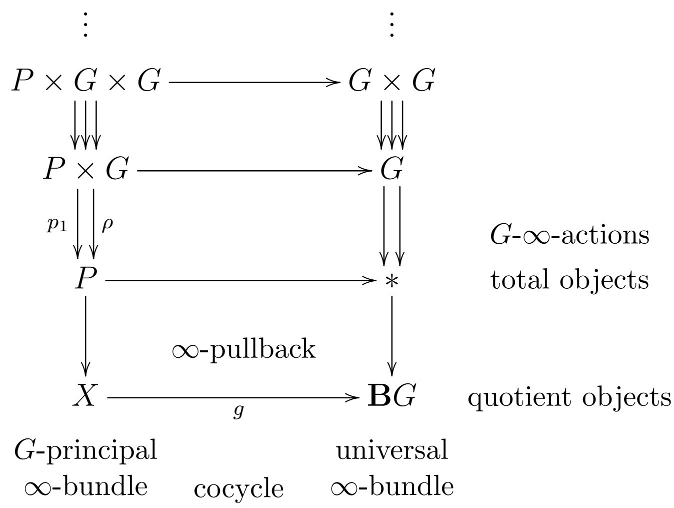
1. hofib sends a cocycle $X \to \mathbf{B}G$ to its homotopy fiber;
2. $\underset{\longrightarrow}{\lim}$ sends an ∞-bundle to the map on ∞-quotients $X \simeq P//G \to *//G \simeq \mathbf{B}G$.

In particular, G-principal ∞-bundles are classified by the intrinsic cohomology of $\mathbf{H}$

$$G\mathrm{Bund}(X)/_\sim \simeq H^1(X, G) := \pi_0 \mathbf{H}(X, \mathbf{B}G)\,.$$

Proof. Repeatedly apply two of the $(*)$ *Giraud-Rezk-Lurie axioms* that characterize ∞-toposes:

1. every ∞-quotient is effective;
2. ∞-colimits are preserved by ∞-pullbacks. $\square$



This gives a general abstract theory of principal ∞-bundles in every ∞-topos. We also have the following *presentations.*

Definition For $G \in \mathrm{Grp}(\mathrm{sSh}(C))$, and $X \in \mathrm{sSh}(C)_{\mathrm{lfib}}$, a *weakly G-principal simplicial bundle* is a G-action ρ over X such that the *principality morphism* $(\rho, p_1) : P \times G \to P \times_X P$ is a stalkwise weak equivalence.

Theorem. Nerve $\left\{ \begin{array}{c} \text{weakly } G\text{-principal simplicial bundles} \\ \text{over } X \end{array} \right\} \simeq G\mathrm{Bund}(X)$.

Example. For X terminal over C and restricted to cohomology classes, this is [JL].

Remark. We need more than that, notably $X = \mathbf{B}G$ itself, see next page.

Example. For $C = *$ we have $\mathrm{sSh}(C)_{\mathrm{lfib}} = \mathrm{KanComplexes}$. Classical theory considers *strictly* principal simplicial bundles [Ma].

Proposition. Strictly principal simplicial bundles over $C = *$ do present the cohomology $H^1(X, G)$, but not in general the full cocycle space $\mathbf{H}(X, \mathbf{B}G)$. For C nontrivial they do in general not even present $H^1(X, G)$.

Theorem. Let $C = \mathrm{SmthMfd}$ or other *cohesive* [S] site. If G is "C-acyclic", then

- $\mathbf{H}(-, \mathbf{B}G)$ is computed by simplicial hyper-Čech cohomology;
- G-principal ∞-bundles over manifolds are presented by locally Kan fibrant simplicial manifolds.

4 Associated and twisted ∞-bundles

Observation. By the above theorem, every G-∞-action $\rho : V \times G \to G$ has a classifying map $\mathbf{c}$:

$$\begin{array}{ccc} V & \longrightarrow & V//G \\ & & \downarrow{\scriptstyle \mathbf{c}} \\ & & \mathbf{B}G \end{array}$$

Proposition. This is the *universal ρ-associated V-bundle.*

Observation. Sections σ of the associated ∞-bundle are *lifts* of the cocycle through $\mathbf{c}$; and these locally factor through V:

$$\left\{ \begin{array}{ccc} P \times_G V & \longrightarrow & V//G \\ {\scriptstyle \sigma}\uparrow\downarrow & & \downarrow{\scriptstyle \mathbf{c}} \\ X & \xrightarrow{g} & \mathbf{B}G \end{array} \right\} \simeq \left\{ \begin{array}{ccc} & & V//G \\ & {\scriptstyle \sigma}\nearrow & \downarrow{\scriptstyle \mathbf{c}} \\ X & \xrightarrow{g} & \mathbf{B}G \end{array} \right\} \quad \begin{array}{ccccc} & & V & \longrightarrow & V//G \\ & {\scriptstyle \sigma|_U}\nearrow & & & \downarrow{\scriptstyle \mathbf{c}} \\ U & \twoheadrightarrow & X & \xrightarrow{g} & \mathbf{B}G \end{array} .$$

Hence sections σ are equivalently

- cocycles in $[g]$-*twisted cohomology*;
- $\mathbf{c}$-valued cocycles in the *slice ∞-topos*: $\quad \Gamma_X(P \times_G V) \simeq \mathbf{H}_{/\mathbf{B}G}(g, \mathbf{c})$

(This is a geometric and unstable variant of the picture in [ABG].)

Theorem. The ∞-bundles classified by $\mathbf{H}_{/\mathbf{B}G}(-, \mathbf{c})$ are *P-twisted ∞-bundles*: twisted G-equivariant ΩV-∞-bundles on P:

$$\begin{array}{ccccc} Q & \longrightarrow & * & & \\ \downarrow & & \downarrow & & \\ P & \longrightarrow & V & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{\sigma} & V//G & \xrightarrow{\mathbf{c}} & \mathbf{B}G \end{array} \qquad (X \xrightarrow{g} \mathbf{B}G)$$

P-twisted ΩV-principal ∞-bundle

G-principal ∞-bundle

section of ρ-associated V-∞-bundle

$$\left\{ \begin{array}{c} \text{sections of} \\ \rho\text{-associated } V\text{-}\infty\text{-bundle} \end{array} \right\} \simeq \left\{ \begin{array}{c} g\text{-twisted } \Omega V\text{-cohomology} \\ \text{relative } \mathbf{c} \end{array} \right\} \simeq \left\{ \begin{array}{c} \Omega V\text{-}\infty\text{-bundles} \\ \text{twisted by } P \end{array} \right\}$$

Example. Connecting homomorphism $\mathbf{c}$ of Lie group $U(1)$-extension

$$\begin{array}{ccccc} \mathbf{B}U(1) & \longrightarrow & \mathbf{B}\hat{G} & \longrightarrow & \mathbf{B}G \\ & & & & \downarrow{\scriptstyle \mathbf{c}} \\ & & & & \mathbf{B}^2U(1) \end{array}$$

induces the $H^3(X, \mathbb{Z})$-twisted smooth $\hat{G}$-bundles known from twisted K-theory.

Example. Associated *connected-fiber* ∞-bundles are *∞-gerbes.*

- A (nonabelian/Giraud-)*gerbe* on X is a connected 1-truncated object in $\mathbf{H}_{/X}$ (a *connected stack* on X).
- A (nonabelian/Giraud-Breen) ∞-*gerbe* over X is a connected object in $\mathbf{H}_{/X}$.
- A G-∞-*gerbe* is an $\mathrm{Aut}(\mathbf{B}G)$-associated ∞-bundle. Its *band* is the underlying $\mathrm{Out}(G)$-principal ∞-bundle.

Observation. G-∞-gerbes *bound by a band* are classified by $(\mathbf{B}\mathrm{Aut}(\mathbf{B}G) \to \mathbf{B}\mathrm{Out}(G))$-twisted cohomology.

5 Selected examples

local coefficient ∞-bundle	**twisting ∞-bundle / twisting cohomology**	**twisted ∞-bundle / twisted cohomology**	see [Lect]
$V \longrightarrow V/\!/G \xrightarrow{\mathbf{c}} \mathbf{B}G$	ρ-associated V-∞-bundle	section	[S]
$\mathbf{B}^2\mathrm{ker}(G) \longrightarrow \mathbf{B}\mathrm{Aut}(\mathbf{B}G) \longrightarrow \mathbf{B}\mathrm{Out}(G)$	band (*lien*)	nonabelian (Giraud-Breen) G-∞-gerbe	[NSS] [S]
$\mathrm{GL}(d)/O(d) \longrightarrow \mathbf{B}O(d) \xrightarrow{\mathbf{orth}} \mathbf{B}\mathrm{GL}(d)$	tangent bundle	orthogonal structure / Riemannian geometry	[S]
$O(d)\backslash O(d,d)/O(d) \to \mathbf{B}(O(d)\times O(d)) \xrightarrow{\mathbf{TypeII}} \mathbf{B}O(d,d)$	generalized tangent bundle	generalized (type II) Riemannian geometry	[S]
$\mathbf{B}U(n) \longrightarrow \mathbf{B}PU(n) \xrightarrow{\mathbf{dd}_n} \mathbf{B}^2U(1)$	circle 2-bundle / bundle gerbe	twisted vector bundle / twisted K-cocycle / bundle gerbe module	[S]
$\mathbf{B}^nU(1) \longrightarrow \mathbf{B}^nU(1)/\!/\mathbb{Z}_2 \xrightarrow{\mathbf{J}_{n-1}} \mathbf{B}\mathbb{Z}_2$	double cover	higher (bosonic) orientifold / $n=2$: Jandl bundle gerbe	[FSSc] [SSW]
$V \longrightarrow \mathbf{B}\mathrm{Spin}^{\nu_{n+1}} \xrightarrow{\nu_{n+1}^{\mathrm{int}}} \mathbf{B}^nU(1)$	circle n-bundle	smooth integral Wu structure	[FSSc]
$\mathbf{B}\mathrm{String} \longrightarrow \mathbf{B}\mathrm{Spin} \xrightarrow{\frac{1}{2}\mathbf{p}_1} \mathbf{B}^3U(1)$	circle 3-bundle / bundle 2-gerbe	twisted String 2-bundle	[SSS] [FSSa]
$V \longrightarrow \mathbf{B}(\mathbb{T}\times\mathbb{T}^*) \xrightarrow{\langle \mathbf{c}_1 \cup \mathbf{c}_1\rangle} \mathbf{B}^3U(1)$	circle 3-bundle / bundle 2-gerbe	T-duality structure	[S]
$\mathbf{B}\mathrm{Fivebrane} \longrightarrow \mathbf{B}\mathrm{String} \xrightarrow{\frac{1}{6}\mathbf{p}_2} \mathbf{B}^7U(1)$	circle 7-bundle	twisted Fivebrane 6-bundle	[SSS] [FSSa]
$\flat\mathbf{B}^nU(1) \longrightarrow \mathbf{B}^nU(1) \xrightarrow{\mathrm{curv}} \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1)$	curvature $(n+1)$-form	circle n-bundle with connection: curvature-twisted flat connection	[S]

Observation. Ordinary cohomology is crucially contravariant: it "pulls back": $\mathbf{H}(-, A) : \mathbf{H}^{\mathrm{op}} \to \infty\mathrm{Grpd}$. Twisted cohomology is not contravariant in $\mathbf{H}$: the information for how to "carry the twists along" is missing. However, by the above it is contravariant in the slice $\mathbf{H}_{/\mathbf{B}G}$ over the moduli of twists.
Example. Cocycles in $\mathbf{H}_{/\mathbf{B}\mathrm{GL}(n)}(TX, \mathbf{orth})$ are metrics on X: these pull back along

$$\mathbf{H}_{/\mathbf{B}\mathrm{GL}(n)}(TY, TX) = \left\{ \begin{array}{ccc} Y & \xrightarrow{f} & X \\ & {\swarrow\!\!\!\simeq} & \\ {\scriptstyle TY}\searrow & & \swarrow{\scriptstyle TX} \\ & \mathbf{B}\mathrm{GL}(n) & \end{array} \right\} = \left\{ \begin{array}{c} \text{local diffeomorphism } f; \\ f^* TX \simeq TY \end{array} \right\}$$

Example. Since $\mathrm{Sh}_\infty(\mathrm{SmthMfd})$ is "cohesive" [S]: there is a canonical notion of *flat* coefficients $\flat \mathbf{B}^n U(1)$ and of flat de Rham coefficients $\flat_{\mathrm{dR}} \mathbf{B}^n U(1)$, and a canonical *curvature* morphism forming a local coefficient bundle

$$\begin{array}{ccc} \flat \mathbf{B}^n U(1) & \longrightarrow & \mathbf{B}^n U(1) \\ & & \downarrow{\scriptstyle \mathrm{curv}} \\ & & \flat_{\mathrm{dR}} \mathbf{B}^{n+1} U(1)_{\mathrm{conn}} \end{array} .$$

The corresponding twisted cohomology is *differential cohomology* with universal coefficient object $\mathbf{B}^n U(1)_{\mathrm{conn}}$, presented by the Deligne complex.
Example. Bosonic string orientifold configurations [SSW], see [Freed]:

$$\begin{array}{cc} \text{worldsheet field} & \text{target space field} \\ (\phi, \nu) \in \mathbf{H}_{/\mathbf{B}\mathbb{Z}_2}(\mathbf{w}_1(T\Sigma), w_X) & (w, \hat{B}) \in \mathbf{H}(X, \mathbf{B}\mathrm{Aut}(U(1))_{\mathrm{conn}}) \end{array}$$

$$\begin{array}{ccccc} \Sigma & \xrightarrow{\phi} & X & \xrightarrow{\hat{B}} & \mathbf{B}\mathrm{Aut}(U(1))_{\mathrm{conn}} \\ & {\swarrow\!\!\!\simeq}^{\nu} & \downarrow{\scriptstyle w_X} & & \\ {\scriptstyle \mathbf{w}_1(T\Sigma)}\searrow & & \mathbf{B}\mathbb{Z}_2 & \swarrow{\scriptstyle \mathbf{J}} & \end{array}$$

Example. There are differential refinements of the first and second fractional Pontryagin classes, of the form [FSSa]:

$$\begin{array}{ccc} \mathbf{B}\mathrm{String}_{\mathrm{conn}} & \longrightarrow & \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \\ & & \downarrow{\scriptstyle \frac{1}{2}\hat{\mathbf{p}}_1} \\ & & \mathbf{B}^3 U(1)_{\mathrm{conn}} \end{array} \qquad \begin{array}{ccc} \mathbf{B}\mathrm{Fivebrane}_{\mathrm{conn}} & \longrightarrow & \mathbf{B}\mathrm{String}_{\mathrm{conn}} \\ & & \downarrow{\scriptstyle \frac{1}{6}\hat{\mathbf{p}}_2} \\ & & \mathbf{B}^7 U(1)_{\mathrm{conn}} \end{array} .$$

The corresponding twisted bundles are *twisted String-principal 2-bundle with 2-connection* and *twisted Fivebrane-principal 6-bundles with 6-connection*: higher analogs of the twisted unitary bundles of twisted K-theory; play a role in the heterotic string [SSS], see [Lect].

Their transgression to codimension 0 is

action functional of level-1 3d Spin-Chern-Simons theory	$[\Sigma_3, \mathbf{B}\mathrm{Spin}_{\mathrm{conn}}] \xrightarrow{[\Sigma_3, \frac{1}{2}\hat{\mathbf{p}}_1]} [\Sigma_3, \mathbf{B}^3 U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_3}(-))} U(1)$;
action functional of a 7d level-1 CS theory of String 2-connections [FSSb]	$[\Sigma_7, \mathbf{B}\mathrm{String}_{\mathrm{conn}}] \xrightarrow{[\Sigma_7, \frac{1}{6}\hat{\mathbf{p}}_2]} [\Sigma_7, \mathbf{B}^7 U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_7}(-))} U(1)$

But before transgression, the action was "localized/extended to the point". See next page...

6 Higher geometric prequantization

It turns out that the differential refineement of smooth twisted cohomology is tightly related to higher notions of *geometric quantization* under the following dictionary.

differential twisted cohomology	**geometric quantization**
twist	extended action functional / prequantum circle n-bundle
twist auto-equivalences	higher Heisenberg group / quantomorphism group
local coefficient bundle	associated prequantum n-bundle

Example. Let $\begin{array}{ccc} \mathbb{C} & \longrightarrow & \mathbb{C}//U(1) \\ & & \downarrow \\ & & \mathbf{B}U(1) \end{array}$ be the canonical complex-linear circle action.

Then

- $\nabla : X \to \mathbf{B}U(1)_{\mathrm{conn}}$ classifies a circle bundle with connection, a *prequantum line bundle* of its curvature 2-form;
- $\Gamma_X(P \times_{U(1)} \mathbb{C})$ is the corresponding space of smooth sections;
- $\mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}}(\nabla, \nabla)_{\simeq}$ is the exp(Poisson bracket)-group action of prequantum operators, containing the Heisenberg group action.

Example. Let $\begin{array}{ccc} \mathbf{B}U(n) & \longrightarrow & \mathbf{B}\mathrm{PU}(n) \\ & & \downarrow{\scriptstyle \mathbf{dd}_n} \\ & & \mathbf{B}^2U(1) \end{array}$ be the canonical 2-circle action.

Then

- $\nabla : X \to \mathbf{B}^2U(1)_{\mathrm{conn}}$ classifies a circle 2-bundle with connection, a *prequantum line 2-bundle* of its curvature 3-form;
- $\Gamma_X(P \times_{\mathbf{B}U(1)} \mathbf{B}U)$ is the corresponding groupoid of smooth sections = twisted bundles;
- $\mathbf{H}_{/\mathbf{B}^2U(1)_{\mathrm{conn}}}(\nabla, \nabla)_{\simeq}$ is the exp(2-plectic bracket)-2-group action of 2-plectic geometry [Rogers], containing the *Heisenberg 2-group* action [RoSc].

Example. ∞-Chern-Simons theory [FS]:

- extended Lagrangian: differential cocycle on moduli ∞-stack of fields $\hat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^nU(1)_{\mathrm{conn}}$ (e.g. $\frac{1}{2}\hat{\mathbf{p}}_1$, $\frac{1}{6}\hat{\mathbf{p}}_2$, $\mathbf{D}\hat{\mathbf{D}}_n \cup \mathbf{D}\hat{\mathbf{D}}_n$,...);
- representation: local coefficient bundle $\begin{array}{ccc} V & \longrightarrow & V//\mathbf{B}^{n-1}U(1) \\ & & \downarrow{\scriptstyle \rho} \\ & & \mathbf{B}^nU(1) \end{array}$;
- extended action in codim k:

$$\exp(2\pi i \textstyle\int_{\Sigma_k} \hat{\mathbf{c}}) : [\Sigma_k, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[\Sigma_k, \hat{\mathbf{c}}]} [\Sigma_k, \mathbf{B}^nU(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_k}(-))} \mathbf{B}^{n-k}U(1)_{\mathrm{conn}}$$

$$\begin{array}{ccccc} [\Sigma_k, \mathbf{B}G_{\mathrm{conn}}] & \xrightarrow{\text{higher Heisenberg operator}} & [\Sigma_k, \mathbf{B}G_{\mathrm{conn}}] & \xrightarrow{\text{higher quantum state}} & V//\mathbf{B}^{n-k-1}U(1)_{\mathrm{conn}} \\ & {\scriptstyle \exp(2\pi i \int_{\Sigma_k} \hat{\mathbf{c}})} \searrow & \Swarrow \;\; \downarrow{\scriptstyle \exp(2\pi i \int_{\Sigma_k} \hat{\mathbf{c}})} & \swarrow {\scriptstyle \rho} & \\ & & \mathbf{B}^{n-k}U(1)_{\mathrm{conn}} & & \end{array}$$

References

[ABG] M. Ando, A. Blumberg, D. Gepner, *Twists of K-theory and TMF*, in R. Doran, G. Friedman, J. Rosenberg, *Superstrings, Geometry, Topology, and C^*-algebras*, Proceedings of Symposia in Pure Mathematics vol 81 `arXiv:1002.3004`

[FSSa] D. Fiorenza, U. Schreiber, J. Stasheff. *Čech-cocycles for differential characteristic classes*, `arXiv:1011.4735`

[FS] D. Fiorenza, U. Schreiber, *∞-Chern-Simons theory*, `ncatlab.org/schreiber/show/infinity-Chern-Simons+theory`

[FSSb] D. Fiorenza, H. Sati, U. Schreiber, *String 2-connections and 7d nonabelian Chern-Simons theory*, `arXiv:1201.5277`

[FSSc] D. Fiorenza, H. Sati, U. Schreiber, *The E_8-moduli 3-stack of the C-field*, `arXiv:1202.2455`

[JL] J.F. Jardine, Z. Luo, *Higher order principal bundles*, K-theory 0681 (2004)

[Ma] P. May, *Simplicial objects in algebraic topology* University of Chicago Press (1967)

[NSS] T. Nikolaus, U. Schreiber, D. Stevenson, *Principal ∞-bundles – I General theory, II Presentations, III Applications* (2012)

[NiWa] T. Nikolaus, K. Waldorf, *Four equivalent versions of non-abelian gerbes*, `arXiv:1103.4815`

[RoSt] D. Roberts, D. Stevenson, *Simplicial principal bundles in parametrized spaces* (2012) `arXiv:1203.2460`
D. Stevenson, *Classifying theory for simplicial parametrized groups* (2012) `arXiv:1203.2461`

[Rogers] C. Rogers, *Higher symplectic geometry*, PhD (2011) `arXiv:1106.4068`

[RoSc] C. Rogers, U. Schreiber, *∞-Geometric prequantization*, `ncatlab.org/schreiber/show/infinity-geometric+prequantization`

[SSS] H. Sati, U. Schreiber, J. Stasheff, *Twisted differential string- and fivebrane structures*, Communications in Mathematical Physics (2012) `arXiv:0910.4001`

[S] U. Schreiber, *Differential cohomology in a cohesive topos* `ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos`

[SSW] U. Schreiber, C. Schweigert, K. Waldorf, *Unoriented WZW models and Holonomy of Bundle Gerbes*, Communications in Mathematical Physics, Volume 274, Issue 1 (2007)

– see also the lectures earlier at this workshop: –

[Freed] D. Freed, *Lectures on twisted K-theory and orientifolds*

[Lect] U. Schreiber, *Twisted differential structures in string theory*, `ncatlab.org/nlab/show/twisted+smooth+cohomology+in+string+theory`