

Twisted Cohomotopy implies M-theory anomaly cancellation on 8-manifolds

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To Mike Duff on the occasion of his 70th birthday

Abstract

We consider the hypothesis that the C-field 4-flux and 7-flux forms in M-theory are in the image of the non-abelian Chern character map from the non-abelian generalized cohomology theory called J-twisted Cohomotopy theory. We prove for M2-brane backgrounds in M-theory on 8-manifolds that such charge quantization of the C-field in Cohomotopy theory implies a list of expected anomaly cancellation conditions, including: shifted C-field flux quantization and C-field tadpole cancellation, but also the DMW anomaly cancellation and the C-field's integral equation of motion.

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1 Introduction and survey

We consider the following hypothesis, which we make precise as Def. 3.5 below, based on details developed in §2, see §4 for background, motivation and outlook:

Hypothesis H: *The C-field 4-flux and 7-flux forms in M-theory are subject to charge quantization in J-twisted Cohomotopy cohomology theory in that they are in the image of the non-abelian Chern character map from J-twisted Cohomotopy theory.*

In support of Hypothesis H, we prove in §3 that it implies the following phenomena, expected for M2-brane backgrounds in M-theory on 8-manifolds (recalled in Remark 3.1 below):

Cohomotopy theory		Expression	M-theory
§3.2	Compatible twisting on 4- & 7-Cohomotopy theory	$W_7(TX) = 0$ (13)	DMW anomaly cancellation condition [DMW03a][DMW03b, 6]
		$\frac{1}{24}\chi_8(TX) = I_8(TX)$ $:= \frac{1}{48}(p_2(TX) - \frac{1}{4}(p_1(TX))^2)$ (13)	one-loop anomaly polynomial [DLM95][Wi96b, (5.5)]
§2.4	Any cocycle in J-twisted 7-Cohomotopy	Spin(7)-structure g (14)	$\geq 1/8$ BPS M2-brane background [IP88][IPW88][Ts06]
§2.3	Any cocycle in compatibly twisted 4&7-Cohomotopy	Sp(1) · Sp(1)-structure τ (15)	$4/8$ BPS M2-brane background [MF10, 7.3]
§3.3	Chern character of rationally twisted 4-Cohomotopy	$dG_4 = 0$ $dG_7 = -\frac{1}{2}G_4 \wedge G_4 + L_8$ (19)	C-field Bianchi identity with generic higher curvature correction [ST16]
§3.3	Chern character of compatibly rationally twisted 4&7-Cohomotopy	$d\tilde{G}_4 = 0$ $dG_7 = -\frac{1}{2}(\tilde{G}_4 - \frac{1}{4}P_4) \wedge \tilde{G}_4 + K_8$ (20)	Shifted C-field Bianchi identity with generic higher curvature correction [Ts04]
§3.4	Chern character 4-form of Sp(2)-twisted 4-Cohomotopy	$\tilde{G}_4 = G_4 + \frac{1}{4}p_1(\nabla)$ (21)	C-field shift [Wi96a][Wi96b][Ts04]
		$[\tilde{G}_4] \in H^4(X, \mathbb{Z})$ (22)	Shifted C-field flux quantization [Wi96a][Wi96b][DFM03][HS05]
§3.5		$(G_4)_0 = \frac{1}{2}p_1(\nabla)$ (24)	Background charge [Fr09, p. 11][Fr00]
§3.6		$\text{Sq}^3([\tilde{G}_4]) = 0$ (23)	Integral equation of motion [DMW03a][DMW03b, 5]
§3.7	Chern character 7-form of compatibly Sp(2)-twisted 4&7-Cohomotopy	$\tilde{G}_7 = G_7 + \frac{1}{2}H_3 \wedge \tilde{G}_4$ (27)	Page charge [Pa83, (8)][DS91, (43)][Mo05]
		$d\tilde{G}_7 = -\frac{1}{2}\chi_8(\nabla)$ (28)	
		$2 \int_{S^7} i^* \tilde{G}_7 \in \mathbb{Z}$ (29)	Level quantization of Hopf-WZ term [In00]
§3.8	Integrated Chern character of compatibly Sp(2)-twisted 4&7-Cohomotopy	$N_{M2} = -I_8$ (33)	C-field tadpole cancellation [SVW96]

Table 1 –Implications of C-field charge quantization in J-twisted Cohomotopy.

Organization of the paper.

- In §1 we survey our constructions and results.
- In §2 we introduce twisted Cohomotopy theory, and prove some fundamental facts about it.
- In §3 we use these results to explain and prove the statements in *Table 1*.
- In §4 we comment on background and implications.

Generalized abelian cohomology. Before we start, we briefly say a word on “generalized” cohomology theories, recalling some basics, but in a broader perspective: The *ordinary cohomology groups* $X \mapsto H^\bullet(X, \mathbb{Z})$ famously satisfy a list of nice properties, called the *Eilenberg-Steenrod axioms*. Dropping just one of these axioms (the *dimension axiom*) yields a larger class of possible abelian group assignments $X \mapsto E^\bullet(X)$, often called *generalized cohomology theories*. One example are the *complex topological K-theory groups* $X \mapsto \mathrm{KU}^\bullet(X)$.

By the *Brown representability theorem*, every generalized cohomology theory in this sense has a *classifying space* E_n for each degree, such that the n -th cohomology group is equivalently the set of homotopy classes of maps into this space:¹

$$\text{Generalized abelian cohomology theory} \quad E^n(X) \simeq \left\{ X \overset{\text{Brown's representability theorem}}{\dashrightarrow} \left\{ \begin{array}{l} \text{continuous function} \\ = \text{cocycle in } E\text{-theory} \end{array} \right\} \rightarrow E_n \right\} / \sim_{\text{homotopy}}. \quad (1)$$

For example, ordinary cohomology theory has as classifying spaces the *Eilenberg-MacLane spaces* $K(\mathbb{Z}, n)$, while complex topological K-theory in degree 1 is classified by the space underlying the stable unitary group.

For generalized cohomology theories in this sense of Eilenberg-Steenrod, Brown’s representability theorem translates the *suspension axiom* into the statement that the classifying spaces E_n in (1) are loop spaces of each other, $E_n \simeq \Omega E_{n+1}$, and thus organize into a sequence of classifying spaces $(E_n)_{n \in \mathbb{N}}$ called a *spectrum*. The fact that each space in a spectrum is thereby an infinite loop space makes it behave like a homotopical *abelian* group (since higher-dimensional loops may be homotoped and hence commuted around each other, by the Eckmann-Hilton argument).

Generalized non-abelian cohomology. We highlight the fact that not all cohomology theories are abelian. The classical example, for G any non-abelian Lie group, is the *first non-abelian cohomology* $X \mapsto H^1(X, G)$, defined on any manifold X as the first Čech cohomology of X with coefficients in the sheaf of G -valued functions. Nevertheless, this non-abelian cohomology theory also has a classifying space, called BG , and in terms of this it is given exactly as the abelian generalized cohomology theories in (1):

$$\text{Degree-1 non-abelian cohomology theory} \quad H^1(X, G) \overset{\text{principal bundle theory}}{\simeq} \left\{ X \overset{\text{continuous function}}{\dashrightarrow} \left\{ \begin{array}{l} \text{cocycle} \end{array} \right\} \rightarrow BG \right\} / \sim_{\text{homotopy}}. \quad (2)$$

Hence the joint generalization of generalized abelian cohomology theory (1) and non-abelian 1-cohomology theories (2) are assignments of homotopy classes of maps into *any* coefficient space A

$$\text{Non-abelian generalized cohomology theory} \quad H(X, A) := \left\{ X \overset{\text{continuous function}}{\dashrightarrow} \left\{ \begin{array}{l} \text{cocycle} \end{array} \right\} \rightarrow A \right\} / \sim_{\text{homotopy}}. \quad (3)$$

All this may naturally be further generalized from topological spaces to higher stacks. In the literature of this broader context the perspective of non-abelian generalized cohomology is more familiar. But it applies to the topological situation as the easiest special case, and this is the case with which we are concerned for the present purpose.

Higher principal bundles. This way, the classical statement (2) of principal bundle theory finds the following elegant homotopy-theoretic generalization. For every *connected* space A , its based loop space $G := \Omega A$ is a higher

¹Here and in the following, a dashed arrow indicates a map representing a cocycle that can be freely chosen, as opposed to solid arrows indicating fixed structure maps.

homotopical group under concatenation of loops (an “ ∞ -group”). Moreover, A itself is equivalently the classifying space for that higher group:

$$A \simeq B \overbrace{\Omega A}^G \quad (4)$$

Every connected space... is the classifying space... of its loop group.

in that non-abelian G -cohomology in degree 1 classifies higher homotopical G -principal bundles:

$$H(X, BG) = H^1(X, G) \xrightarrow{\simeq} GBundles(X)/\sim \quad (5)$$

$$\left[\begin{array}{ccc} \begin{array}{c} \text{non-abelian } G\text{-cohomology} \\ H^1(X, G) \\ \text{cocycle} \\ X \xrightarrow{c} BG \end{array} & \xrightarrow{\simeq} & \begin{array}{c} \text{higher homotopical } G\text{-principal bundles} \\ GBundles(X)/\sim \\ \text{universal } G\text{-principal bundle} \\ G//G \\ \text{homotopy pullback} \\ \begin{array}{ccc} \begin{array}{c} G\text{-principal bundle classified by } c \\ P \\ \downarrow c^*(p_{BG}) \\ X \end{array} & \xrightarrow{\quad} & \begin{array}{c} G//G \\ \downarrow p_{BG} \\ BG \end{array} \\ \text{classifying map for } P & & \end{array} \end{array} \right] \quad (5)$$

Cohomotopy cohomology theory. The primordial example of a non-abelian generalized cohomology theory (3) is *Cohomotopy cohomology theory*, denoted π^\bullet . By definition, its classifying spaces are simply the n -spheres S^n :

$$\pi^n(X) := \left\{ X \xrightarrow[\text{cocycle}]{\text{continuous function}} S^n \right\} / \sim_{\text{homotopy}} \quad (6)$$

Since the $(n \geq 1)$ -spheres are connected, the equivalence (4) applies and says that Cohomotopy theory is equivalently non-abelian 1-cohomology for the loop groups of spheres $G := \Omega S^n$:

$$\pi^n(X) \simeq H^1(X, \Omega S^n)$$

Cohomotopy theory non-abelian 1-cohomology for sphere loop group

Evaluated on spaces which are themselves spheres, Cohomotopy cohomology theory gives the (unstable!) *homotopy groups of spheres*, the “vanishing point” of algebraic topology:

$$\pi^n(S^k) \simeq \left\{ S^k \xrightarrow{\quad} S^n \right\} / \sim \simeq \pi_k(S^n) \quad (7)$$

n -cohomotopy groups of k -sphere k -homotopy groups of n -sphere

A whole range of classical theorems in differential topology all revolve around characterizations of Cohomotopy sets, even if this is not often fully brought out in the terminology.

The quaternionic Hopf fibration. A notable example, for the following purpose, of a class in the Cohomotopy group of spheres, is given by the *quaternionic Hopf fibration*

$$S^7 \xrightarrow[\simeq]{h_{\mathbb{H}}} S(\mathbb{H}^2) \xrightarrow[(q_1:q_2) \mapsto [q_1:q_2]]{h_{\mathbb{H}}} \mathbb{H}P^1 \xrightarrow[\simeq]{} S^4, \quad (7)$$

quaternionic Hopf fibration $h_{\mathbb{H}}$
 unit sphere in quaternionic 2-space quaternionic projective 1-space

which represents a generator of the non-torsion subgroup in the 4-Cohomotopy of the 7-sphere, as shown on the left here:

$$\begin{array}{ccccc} \text{quaternionic Hopf fibration} & [S^7 \xrightarrow{h_{\mathbb{H}}} S^4] \in \pi^4(S^7) & \xrightarrow[\Sigma^\infty]{} & S^4(S^7) \ni \Sigma^\infty [S^7 \xrightarrow{h_{\mathbb{H}}} S^4] & \text{stabilized quaternionic Hopf fibration} \\ \downarrow & \downarrow & & \downarrow & \\ \text{non-torsion generator} & (1, 0) \in \mathbb{Z} \times \mathbb{Z}_{12} & \xrightarrow[(n,a) \mapsto (n \bmod 24)]{} & \mathbb{Z}_{24} \ni 1 & \text{torsion generator} \end{array} \quad (8)$$

Shown on the right is the abelian approximation to non-abelian Cohomotopy cohomology theory, called *stable Cohomotopy theory* and represented, via (1), by the *sphere spectrum* \mathbb{S} , whose component spaces are the infinite-loop space completions of the n -spheres: $\mathbb{S}_n \simeq \Omega^\infty \Sigma^\infty S^n$. Crucially, in this approximation, the quaternionic Hopf fibration becomes a torsion generator: non-abelian 4-Cohomotopy witnesses integer cohomology groups not only in degree 4, but also in degree 7; but when seen in the abelian/stable approximation this “extra degree” fades away and leaves only a torsion shadow behind. From the perspective, composition with the quaternionic Hopf fibration can be viewed as a transformation that translates classes in degree-7 Cohomotopy to classes in degree-4 Cohomotopy:

$$\begin{array}{ccc}
 & S^7 & 7\text{-Cohomotopy} & \pi^7(X) \\
 & \downarrow h_{\mathbb{H}} & \text{reflects into} & \downarrow (h_{\mathbb{H}})_* \\
 X \xrightarrow{c} & S^4 & 4\text{-Cohomotopy} & \pi^4(X) \\
 & \xrightarrow{(h_{\mathbb{H}})_*(c)} & &
 \end{array} \tag{9}$$

Twisted non-abelian generalized cohomology. Regarding generalized cohomology theory as homotopy theory of classifying spaces (3) makes transparent the concept of *twistings* in cohomology theory: Instead of mapping into a fixed classifying spaces, a *twisted cocycle* maps into a varying classifying space that may twist and turn as one moves in the domain space. In other words, a *twisting* τ of A -cohomology theory on some X is a bundle over X with typical fiber A , and a τ -twisted cocycle is a *section* of that bundle:

$$\begin{array}{l}
 \tau\text{-twisted} \\
 \text{non-abelian generalized} \\
 A\text{-cohomology theory}
 \end{array}
 A^\tau(X) := \left\{ \begin{array}{ccc}
 \text{continuous section} \\
 = \text{twisted cocycle} & \begin{array}{c} A\text{-fiber bundle} \\ P \\ \downarrow p \\ X \end{array} & \begin{array}{c} \text{universal} \\ A\text{-fiber bundle} \\ A//\text{Aut}(A) \\ \downarrow \\ B\text{Aut}(A) \end{array} \\
 X \xrightarrow{\tau} & X & \xrightarrow{\tau} B\text{Aut}(A) \\
 & \text{classifying map for } P &
 \end{array} \right\} / \sim_{\text{homotopy } B\text{Aut}(A)} \tag{10}$$

$$\simeq \left\{ \begin{array}{ccc}
 X \xrightarrow{\text{continuous function}} & A//\text{Aut}(A) \\
 \text{twist } \tau & \swarrow \text{homotopy} & \searrow \\
 & B\text{Aut}(A) &
 \end{array} \right\} / \sim_{\text{homotopy } B\text{Aut}(A)}$$

Here the equivalent formulation shown in the second line follows because A -fiber bundles are themselves classified by nonabelian $\text{Aut}(A)$ -cohomology, as shown on the right of the first line (due to (5)).

Twisted Cohomotopy theory. For the example (6) of Cohomotopy cohomology theory in degree $d - 1$ there is a canonical twisting on Riemannian d -manifolds, given by the unit sphere bundle in the orthogonal tangent bundle:

$$\begin{array}{l}
 J\text{-twisted} \\
 \text{Cohomotopy theory}
 \end{array}
 \pi^{TX^d}(X^d) := \left\{ \begin{array}{ccc}
 \text{continuous section} \\
 = \text{twisted cocycle} & \begin{array}{c} \text{tangent} \\ \text{unit sphere bundle} \\ S(TX^d) \\ \downarrow p \\ X \end{array} & \begin{array}{c} \text{universal tangent} \\ \text{unit sphere bundle} \\ S^{d-1}//O(d) \\ \downarrow \\ BO(d) \end{array} \\
 X \xrightarrow{TX^d} & X & \xrightarrow{TX^d} BO(d) \\
 & \text{classifying map of} \\
 & \text{tangent/frame bundle} &
 \end{array} \right\} / \sim_{\text{homotopy } BO(d)} \tag{11}$$

$$\simeq \left\{ \begin{array}{ccc}
 X \xrightarrow{\text{continuous function}} & S^{d-1}//O(d) \\
 TX^d \text{ twist} & \swarrow \text{homotopy} & \searrow \\
 & BO(d) &
 \end{array} \right\} / \sim_{\text{homotopy } BO(d)}$$

Since the canonical morphism $O(d) \rightarrow \text{Aut}(S^{d-1})$ is known as the *J-homomorphism*, we may call this *J-twisted Cohomotopy theory*, for short.

Compatibly J-twisted Cohomotopy in degrees 4 & 7. In view of (9) it is natural to ask for the maximal subgroup $G \subset O(8)$ for which the quaternionic Hopf fibration is equivariant, so that its homotopy quotient $h_{\mathbb{H}} // G$ exists and serves as a map of *G-twisted Cohomotopy theories* (11) from degree 7 and 4. This subgroup turns out to be the central product of the quaternion unitary groups $\text{Sp}(n)$ for $n = 1, 2$:

$$\begin{array}{ccc}
 & S^7 // \text{Sp}(2) \cdot \text{Sp}(1) & \\
 & \swarrow \quad \downarrow & \\
 \text{Sp}(2) \cdot \text{Sp}(1) \subset O(8) \text{ is maximal subgroup s.t.} & B(\text{Sp}(2) \cdot \text{Sp}(1)) & \xrightarrow{h_{\mathbb{H}} // \text{Sp}(2) \cdot \text{Sp}(1)} \text{universally twisted quaternionic Hopf fibration} \\
 \text{central product of quaternion-unitary groups} & & \\
 & \swarrow \quad \downarrow & \\
 & S^4 // \text{Sp}(2) \cdot \text{Sp}(1) &
 \end{array} \quad (12)$$

In other words, J-twisted Cohomotopy (11) exists compatibly in degrees 4 & 7 precisely on those 8-manifolds which carry topological $\text{Sp}(2) \cdot \text{Sp}(1)$ -structure, i.e., whose structure group of the tangent bundle is equipped with a reduction along $\text{Sp}(2) \cdot \text{Sp}(1) \hookrightarrow O(8)$. This reduction is equivalent to a factorization of the classifying map as shown on the left below, with some cohomological consequences shown on the right:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 X^8 & \xrightarrow{\tau} & B(\text{Sp}(2) \cdot \text{Sp}(1)) \\
 \downarrow TX^8 & \searrow \text{Sp}(2) \cdot \text{Sp}(1)\text{-structure} & \\
 \text{tangent bundle} & & \\
 BO(8) & \longleftarrow & B(\text{Sp}(2) \cdot \text{Sp}(1)) \\
 \text{classifying space of orthogonal structure} & & \text{classifying space of Sp}(2) \cdot \text{Sp}(1)\text{-twists}
 \end{array} & \implies & \left\{ \begin{array}{l}
 \text{Euler class} \qquad \qquad \text{Pontrjagin classes} \\
 \frac{1}{24} \chi_8 = I_8 := \frac{1}{48} (p_2 - \frac{1}{4} (p_1)^2) \\
 (H^2(X^8, \mathbb{Z}_2) = 0) \implies (w_6 = 0) \implies (W_7 = 0) \\
 \text{Stiefel-Whitney class} \qquad \qquad \text{integral Stiefel-Whitney class}
 \end{array} \right. \quad (13)
 \end{array}$$

J-Twisted Cohomotopy and Topological G-Structure. For every topological coset space realization G/H of an n -sphere, there is a canonical homotopy equivalence between the classifying spaces for G -twisted Cohomotopy and for topological H -structure (i.e., reduction of the structure group to H), as follows:

$$\begin{array}{ccc}
 \text{coset space structure on topological } n\text{-sphere} & & G\text{-twisted Cohomotopy / topological } H\text{-structure} \\
 S^n \simeq_{\text{homeo}} G/H & \implies & S^n // G \simeq_{\text{htpy}} BH .
 \end{array}$$

(One may think of this as “moving G from numerator on the right to denominator on the left”.)

In particular, on Spin 8-manifolds we have the following equivalences between J-twisted Cohomotopy cocycles (11) and topological G -structures:

$$\begin{array}{ccc}
 S^7 // \text{Spin}(8) \simeq B\text{Spin}(7) & \implies & \left\{ \begin{array}{l}
 \text{classifying space for J-twisted Cohomotopy theory} \\
 \text{cocycle in J-twisted Cohomotopy } c \\
 X^8 \xrightarrow{TX^8} B\text{Spin}(8) \xrightarrow{\text{homotopy}} S^7 // \text{Spin}(8) \\
 \text{tangent spin structure}
 \end{array} \right\} \simeq \left\{ \begin{array}{l}
 \text{classifying space for topological Spin}(7)\text{-structure} \\
 \text{topological Spin}(7)\text{-structure } g \\
 X^8 \xrightarrow{TX^8} B\text{Spin}(8) \xrightarrow{\text{homotopy}} B\text{Spin}(7) \\
 \text{tangent spin structure}
 \end{array} \right. \quad (14)
 \end{array}$$

Triality between Sp(2)-structure and Spin(5)-structure. While the group $\text{Sp}(2) \cdot \text{Sp}(1)$ (12) is abstractly isomorphic to the central product of Spin-groups $\text{Spin}(5) \cdot \text{Spin}(3)$, the two are *distinct* as subgroups of $\text{Spin}(8)$, and not conjugate to each other. But as subgroups they are turned into each other by the ambient action of triality:

$$\begin{array}{ccccc}
 & \text{central product of} & & \text{central product of} & \\
 & \text{quaternion-unitary groups} & & \text{Spin-groups} & \\
 \text{Sp}(2) \hookrightarrow & \text{Sp}(2) \cdot \text{Sp}(1) & \xrightarrow{\cong} & \text{Spin}(5) \cdot \text{Spin}(3) & \hookrightarrow \text{Spin}(5) \\
 \downarrow & & & & \downarrow \\
 & \text{Spin}(8) & \xrightarrow[\text{triality automorphism}]{\cong} & \text{Spin}(8) & \\
 \end{array}$$

While $\text{Spin}(5)$ on the right is the structure group of normal bundles to M5-branes, acting on fibers of 4-spherical fibrations around 5-branes through its vector representation, $\text{Sp}(2)$ on the left is the structure group of normal bundles to M2-branes, acting on the 7-spherical fibrations around 2-branes via its defining left action on quaternionic 2-space $\mathbb{H}^2 \simeq_{\mathbb{R}} \mathbb{R}^8$ ([MFGM09][MF10]):

$$\begin{array}{ccc}
 \text{left quaternion} & \text{Sp}(2) & \\
 \text{multiplication} & \downarrow & \\
 S(\mathbb{H}^2) = S^7 & & \\
 \end{array}
 \qquad
 \begin{array}{ccc}
 \text{Spin}(5) & \text{vector} & \\
 & \text{representation} & \\
 S^4 = S(\mathbb{R}^5) & & \\
 \end{array}$$

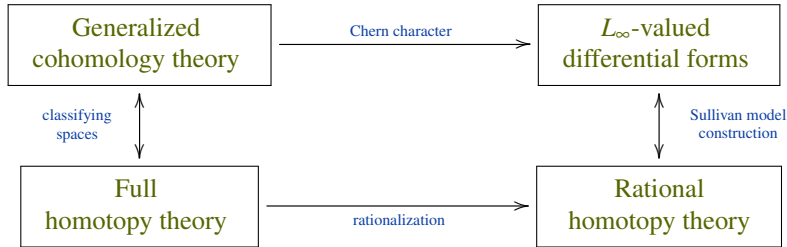
In this article we consider only the M2-case. But all formulas we derive translate to the M5 case via triality.

Generalized Chern characters. Since generalized cohomology theory is rich, one needs tools to break it down. The first and foremost of these is the *generalized Chern character* map. This extracts differential form data underlying a cocycle in nonabelian generalized cohomology. The Chern character is familiar in twisted K-theory (see [GS19a][GS19c]), as shown in the top half of the following:

$$\begin{array}{ccc}
 \boxed{\text{Torsionful generalized cohomology theory}} & \xrightarrow{\text{approximation by generalized Chern character}} & \boxed{L_{\infty}\text{-valued de Rham cohomology theory}} & (18) \\
 \hline
 \boxed{\text{Chern character on ordinary integral cohomology}} & \begin{array}{ccc} \text{ordinary integral cohomology} & \xrightarrow{\text{extension of scalars \& de Rham theorem}} & \text{de Rham cohomology} \\ H^3(X, \mathbb{Z}) & \longrightarrow & H_{\text{dR}}^3(X) \\ \tau & \longmapsto & [H_3] \\ \text{bundle gerbe} & & \text{closed 3-form} \end{array} \\
 \boxed{\text{Chern character on B-field-twisted K-theory}} & \begin{array}{ccc} \tau\text{-twisted complex K-theory} & \xrightarrow{\tau\text{-twisted Chern character } \text{ch}^{\tau}} & H_3\text{-twisted de Rham cohomology} \\ \text{KU}^{\tau}(X) & \longrightarrow & H_{\text{dR}}^{[H_3]}(X) \\ V & \longmapsto & [\text{tr}(\exp(F))] \\ \text{virtual twisted vector bundle} & & \text{exponentiated curvature form} \end{array} \\
 \boxed{\text{Chern character on non-abelian } O(n)\text{-cohomology}} & \begin{array}{ccc} \text{non-abelian } O(n)\text{-cohomology} & \xrightarrow{\text{characteristic forms}} & \text{de Rham cohomology tensor invariant polynomials on } \mathfrak{o}(n) \\ H^1(X, O(n)) & \longrightarrow & H_{\text{dR}}(X) \otimes \text{inv}(\mathfrak{o}(n)) \\ \tau & \longmapsto & \tau_{\mathbb{R}} \in \mathbb{R} \left[[W_i(\nabla \tau)], [P_k(\nabla \tau)] \right]_{i,k} \\ \text{vector bundle} & & \begin{array}{l} \text{Stiefel-Whitney} \\ \text{forms} \end{array} \quad \begin{array}{l} \text{Pontrjagin} \\ \text{forms} \end{array} \end{array} \\
 \boxed{\text{Chern character on J-twisted } n\text{-Cohomotopy}} & \begin{array}{ccc} \tau\text{-twisted Cohomotopy theory} & \xrightarrow{\text{cohomotopical Chern character}} & \tau_{\mathbb{R}}\text{-twisted rational Cohomotopy theory} \\ \pi^{\tau}(X) & \longrightarrow & \pi^{\tau_{\mathbb{R}}}(X)_{\mathbb{R}} \end{array}
 \end{array}$$

In order to see what the *cohomotopical Chern character* in the last line is, we need some general theory of generalized Chern characters. This is *rational homotopy theory*:

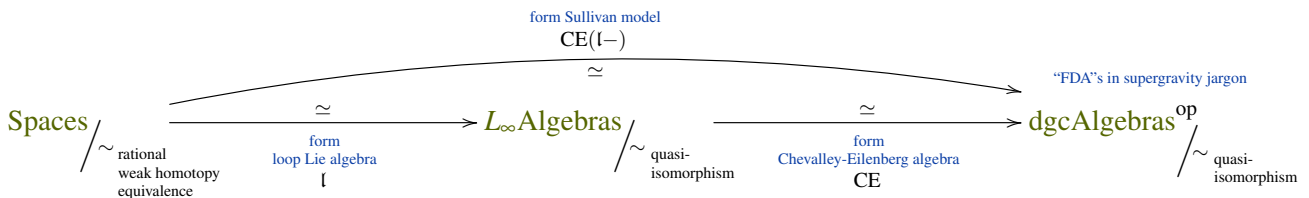
Rational homotopy theory. In the language of homotopy theory, generalized Chern character maps are examples of *rationalization*, whereby the homotopy type of a topological space (here: the classifying space of a generalized cohomology theory) is approximated by tensoring all its homotopy groups with the rational numbers (equivalently: the real numbers), thereby disregarding all torsion subgroups in homotopy groups and in cohomology groups.



What makes rational homotopy theory amenable to computations is the existence of *Sullivan models*. These are differential graded-commutative algebras (dgc-algebras) on a finite number of generating elements (spanning the rational homotopy groups) subject to differential relations (enforcing the intended rational cohomology groups). In the supergravity literature Sullivan models are also known as “FDA”s. Here are some basic examples (see [FSS16b][FSS18a][FSS18b][FSS19d], review in [FSS19a]):

	Rational super space	Loop super L_∞-algebra	Chevalley-Eilenberg super dgc-algebras (“Sullivan models”, “FDA”s)
General	X	$\mathfrak{L}X$	$CE(\mathfrak{L}X)$
Super spacetime	$\mathbb{T}^{d,1 \mathbb{N}}$	$\mathbb{R}^{d,1 \mathbb{N}}$	$\mathbb{R}[\{\psi^\alpha\}_{\alpha=1}^N, \{e^a\}_{a=0}^d] / \left(\begin{array}{l} d\psi^\alpha = 0 \\ de^a = \bar{\psi}\Gamma^a\psi \end{array} \right)$
Eilenberg-MacLane space	$K(\mathbb{R}, p+2) \simeq_{\mathbb{R}} B^{p+1}S^1$	$\mathbb{R}[p+1]$	$\mathbb{R}[c_{p+2}] / (dc_{p+2} = 0)$
Odd-dimensional sphere	$S_{\mathbb{R}}^{2k+1}$	$\mathfrak{l}(S^{2k+1})$	$\mathbb{R}[\omega_{2k+1}] / (d\omega_{2k+1} = 0)$
Even-dimensional sphere	$S_{\mathbb{R}}^{2k}$	$\mathfrak{l}(S^{2k})$	$\mathbb{R}[\omega_{2k}, \omega_{4k-1}] / \left(\begin{array}{l} d\omega_{2k} = 0 \\ d\omega_{4k-1} = -\omega_{2k} \wedge \omega_{2k} \end{array} \right)$
M2-extended super spacetime	$\widehat{\mathbb{T}^{10,1 32}}$	m2brane	$\mathbb{R}[\{\psi^\alpha\}_{\alpha=1}^{32}, \{e^a\}_{a=0}^{10}, h_3] / \left(\begin{array}{l} d\psi^\alpha = 0 \\ de^a = \bar{\psi}\Gamma^a\psi \\ dh_3 = \frac{i}{2}(\bar{\psi}\Gamma_{ab}\psi) \wedge e^a \wedge e^b \end{array} \right)$

Under *Sullivan’s theorem* the rational homotopy type of well-behaved spaces are equivalently encoded in their Sullivan model dgc-algebras. For spaces and algebras which are nilpotent and of finite type one has:



When applying the rational approximation to twisted generalized cohomology theory, the order matters: There are in general more *rational twists* $X \xrightarrow{\tau} B\text{Aut}(A_{\mathbb{R}})$ for *twisted rational cohomology* than there are rationalizations $\tau_{\mathbb{R}}$ of full twists $X \xrightarrow{\tau} B\text{Aut}(A)$ for *rational twisted cohomology*.² We consider first the general rational twists:

Rationally twisted rational Cohomotopy. We find that the *rationally twisted rational Cohomotopy* sets in degrees 4 and 7 are equivalently characterized by cohomotopical Chern character forms as follows:

	rational twist	rational twisted Cohomotopy	cohomotopical Chern characters
7-Cohomotopy	$X \xrightarrow{\tau^7} B\text{Aut}(S_{\mathbb{R}}^7)$	$\pi^{(\tau^7)}(X) \simeq$	$\left\{ \begin{array}{l} \overset{7\text{-form}}{\tilde{G}_7} \mid d\tilde{G}_7 = K_8 \end{array} \right\} / \sim$ <p style="text-align: right; font-size: small;">characteristic form of twist τ^7</p>
4-Cohomotopy	$X \xrightarrow{\tau^4} B\text{Aut}(S_{\mathbb{R}}^4)$	$\pi^{(\tau^4)}(X) \simeq$	$\left\{ \begin{array}{l} \overset{4\text{-form}}{\text{\& 7-form}} \\ (G_4, G_7) \mid \begin{array}{l} dG_4 = 0 \\ dG_7 = -\frac{1}{2}G_4 \wedge G_4 + L_8 \end{array} \end{array} \right\} / \sim$ <p style="text-align: right; font-size: small;">characteristic form of twist τ^4</p>

Here *all* real 8-classes $[K_8], [L_8] \in H^8(X, \mathbb{R})$ may appear, for *some* rational twists $\tau^{4,7}$. Constraints on these characteristic forms appear when we consider more than rational twisted structure:

Compatibly rationally twisted rational Cohomotopy. We may ask that the rational twists $\tau^{4,7}$ in (19) are related analogously to how the twisted parametrized Hopf fibration (12) relates the full (non-rational) twists, through (16). We find that this happens precisely when the difference of the characteristic 8-classes in (19) is a complete square

$$L_8 = K_8 + \left(\frac{1}{4}P_4\right) \wedge \left(\frac{1}{4}P_4\right)$$

and in that case the situation of (19) becomes the following:

	Compatible rational twists	Rational compatibly twisted Cohomotopy	Cohomotopical Chern characters
7-Cohomotopy	$X \xrightarrow{\tau^7} B\text{Aut}(S_{\mathbb{R}}^7)$	$\pi^{(\tau^7)}(X) \simeq$	$\left\{ \begin{array}{l} \tilde{G}_7 \mid d\tilde{G}_7 = K_8 \end{array} \right\} / \sim$ <p style="text-align: right; font-size: small;">characteristic form of twist τ^7</p>
	<div style="border: 1px solid black; padding: 5px; width: fit-content; margin: auto;"> $\begin{array}{l} \overset{\text{shifted 4-form}}{\tilde{G}_4 := G_4 + \frac{1}{4}P_4} \\ \tilde{G}_7 := G_7 + \frac{1}{2}H_3 \wedge \tilde{G}_4 \\ \text{shifted 7-form} \end{array}$ </div>	\simeq	$\left\{ \begin{array}{l} \left(\begin{array}{l} H_3, \\ \tilde{G}_4, G_7 \end{array} \right) \mid \begin{array}{l} dH_3 = \tilde{G}_4 - \frac{1}{2}P_4 \\ d\tilde{G}_4 = 0 \\ dG_7 = -\frac{1}{2}dH_3 \wedge \tilde{G}_4 + K_8 \end{array} \end{array} \right\} / \sim$
4-Cohomotopy	$X \xrightarrow{\tau^4} B\text{Aut}(S_{\mathbb{R}}^4)$	$\pi^{(\tau^4)}(X) \simeq$	$\left\{ \begin{array}{l} (\tilde{G}_4, G_7) \mid \begin{array}{l} d\tilde{G}_4 = 0 \\ dG_7 = -\frac{1}{2}(\tilde{G}_4 - \frac{1}{2}P_4) \wedge \tilde{G}_4 + K_8 \end{array} \end{array} \right\} / \sim$

Here still *all* real 8-classes and 4-classes $[K_8] \in H^8(X, \mathbb{R})$, $[P_4] \in H^4(X, \mathbb{R})$ may appear, for *some* pair of compatible rational twists.

²This is in contrast with twisting vs. differential refinement where the order does not matter – see [GS19a][GS19b].

- (ii) Such that the corresponding extended spacetime fibration $\widehat{X}^8 \rightarrow X^8$ (25) admits a global section; hence, equivalently, such that the given cocycle in twisted 4-Cohomotopy lifts through the quaternionic Hopf fibration to a cocycle in twisted 7-Cohomotopy:

$$\begin{array}{ccc}
 \begin{array}{c} \text{classifying space of} \\ \text{compatible 3-flux} \\ \widehat{X}^8 \\ \downarrow p := c^*(h) \\ X^8 \\ \leftarrow i \\ X^8 \end{array} & \Leftrightarrow & \begin{array}{c} \text{lift to cocycle in} \\ \text{J-twisted 7-Cohomotopy} \\ S^7 // \text{Sp}(2) \\ \downarrow h := h_{\mathbb{H}} // \text{Sp}(2) \\ S^4 // \text{Sp}(2) \\ \leftarrow c \\ X^8 \end{array} \\
 \text{global section of} \\ \text{3-spherical fibration} & & \text{homotopy} \\
 \text{induced} \\ \text{3-spherical} \\ \text{fibration} & & \text{cocycle in} \\
 & & \text{J-twisted 4-Cohomotopy} \\
 & & \text{Sp}(2)\text{-parametrized} \\
 & & \text{quaternionic Hopf} \\
 & & \text{fibration}
 \end{array} \tag{31}$$

Here the choice of points in (30) matters only in so far as a sufficient number of points has to be removed for a lifted cocycle \hat{c} (31) to exist at all.

By (26) this lift exhibits a **4-fluxless** background at least at the level of integral cohomology. In order to refine this to 4-fluxlessness at the finer level of (stable) Cohomotopy, we observe the following:

- (i) Since the 7-sphere is parallelizable, upon restriction of \hat{c} (31) to the boundary $\partial X^8 \xrightarrow{i} X^8$ (30) the twist vanishes, and we are left with a pair of compatible cocycles in plain Cohomotopy theory as in (9):

$$\begin{array}{ccc}
 \text{boundary restriction of} \\ \text{twisted 7-Cohomotopy cocycle} \\ \hat{c}|_{\partial X^8} \\ \downarrow \\ (h_{\mathbb{H}})_* \hat{c}|_{\partial X^8} \\ \text{underlying boundary} \\ \text{4-Cohomotopy cocycle} \\ \begin{array}{c} S^7 \\ \downarrow h_{\mathbb{H}} \\ S^4 \end{array} \\
 \text{plain} \\ \text{quaternionic} \\ \text{Hopf fibration} \\ \sqcup_{\{x_1, x_2, \dots\}} S^7 \simeq \partial X^8 \xrightarrow{\quad} S^4
 \end{array}$$

- (ii) By (8), cocycles in stable 7-Cohomotopy have no side-effect in stable 4-Cohomotopy, hence remain stably **cohomotopically 4-fluxless** precisely if they are multiples of **24**:

$$\text{For } [c_1], [c_2] \in \pi^7(\partial X^8) \text{ we have } \left\{ \begin{array}{l} (h_{\mathbb{H}})_*[c_1] = (h_{\mathbb{H}})_*[c_2] \in \mathbb{S}^4(\partial X^8) \\ [c_1] =_{\text{mod } 24} [c_2] \in \mathbb{S}^7(\partial X^8) \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \text{stable 4-Cohomotopy} \\ \text{stable 7-Cohomotopy} \end{array} \right.$$

This means that the **unit charge** of a lift \hat{c} in (31), as seen by stable Cohomotopy, is **24**. In view of (29) this says that the **cohomotopically normalized 7-flux** of X^8 is

$$N_{M2} := \frac{-1}{12} \int_{X^8} i^* d\tilde{G}_7 = \frac{-1}{12} \int_{\partial X^8} i^* \tilde{G}_7. \tag{32}$$

Our final result is that:
this **equals the I_8 -number** (13) of the manifold:

$$N_{M2} = I_8[X^8]. \tag{33}$$

2 J-Twisted Cohomotopy theory

We now introduce our twisted generalized cohomology theory, J-twisted Cohomotopy theory, and discuss some general properties.

2.1 Twisted Cohomotopy

The non-abelian cohomology theory (see [NSS12], following [SSS12]) represented by the n -spheres is called *Cohomotopy*, going back to [Bo36][Sp49]. Hence for X a topological space, its *Cohomotopy set* in degree n is

$$\pi^n(X) = \pi_0 \text{Maps}(X, S^n) = \left\{ X \overset{\text{cocycle in Cohomotopy}}{\dashrightarrow} S^n \right\} / \sim. \quad (34)$$

A basic class of examples is Cohomotopy of a manifold X in the same degree as the dimension $\dim(X)$ of that manifold. The classical *Hopf degree theorem* (see, e.g., [Kos93, IX (5.8)], [Kob16, 7.5]) says that for X connected, orientable and closed, this is canonically identified with the integral cohomology of X , and hence with the integers

$$\pi^n(X) \xrightarrow[\simeq]{S^n \rightarrow K(\mathbb{Z}, n)} H^n(X; \mathbb{Z}) \simeq \mathbb{Z}, \quad \text{for } n = \dim(X). \quad (35)$$

In its generalization to the *equivariant Hopf degree theorem* this becomes a powerful statement about equivariant Cohomotopy theory and thus, via *Hypothesis H*, about brane charges at orbifold singularities [HSS18]. We discuss this in detail elsewhere [SS19a][BSS19b].

Here we generalize ordinary Cohomotopy (34) to *twisted Cohomotopy* (Def. 2.1 below), following the general theory of non-abelian (unstable) twisted cohomology theory [NSS12, Sec. 4].³ Generally, Cohomotopy in degree n may be twisted by $\text{Aut}(S^n)$ -principal ∞ -bundles, for $\text{Aut}(S^n) \subset \text{Maps}(S^n, S^n)$ the automorphism ∞ -group of S^n inside the mapping space from S^n to itself.

A well-behaved subspace of twists comes from $O(n+1)$ -principal bundles, or their associated real vector bundles of rank $n+1$, under the inclusion

$$\widehat{J}_n : O(n+1) \hookrightarrow \text{Aut}(S^n) \hookrightarrow \text{Maps}(S^n, S^n), \quad (36)$$

which witnesses the canonical action of orthogonal transformations in Euclidean space \mathbb{R}^{n+1} on the unit sphere $S^n = S(\mathbb{R}^{n+1})$. The restriction of these to $O(n)$ -actions

$$J_n : O(n) \hookrightarrow O(n+1) \xrightarrow{\widehat{J}_n} \text{Maps}(S^n, S^n)$$

are known as the unstable *J-homomorphisms* [Wh42] (see [Kos93][Ma12] for expositions). By general principles [NSS12], the homotopy quotient $S^n // O(n+1)$ of S^n by the action via \widehat{J}_n is canonically equipped with a map \tilde{J}_n to the classifying space $BO(n+1)$, such that the homotopy fiber is S^n :

$$\begin{array}{ccc} S^n & \longrightarrow & S^n // O(n+1) \\ & & \downarrow \\ & & BO(n+1). \end{array}$$

One may think of this as the universal spherical fibration which is the S^n -fiber ∞ -bundle associated to the universal $O(n+1)$ -principal bundle via the homotopy action \widehat{J}_n .

³All constructions here are homotopical, in particular all group actions, principal bundles, etc. are “higher structures up to coherent homotopy”, in a sense that has been made completely rigorous via the notion of ∞ -groups, and their ∞ -actions on ∞ -principal bundles [NSS12]. But the pleasant upshot of this theory is that when homotopy coherence is systematically accounted for, then higher structures behave in all general ways as ordinary structures, for instance in that homotopy pullbacks satisfy the same structural pasting laws as ordinary pullbacks. Beware, this means in particular that all our equivalences are weak homotopy equivalences (even when we denote them as equalities), and that all our commutative diagrams are commutative up to specified homotopies (even when we do not display these).

Definition 2.1 (Twisted Cohomotopy). Given a map $\tau : X \rightarrow BO(n+1)$, we define the τ -twisted cohomotopy set of X to be

$$\begin{aligned} \pi^\tau(X) &:= \left\{ \begin{array}{ccc} & S^n // O(n+1) & \\ \text{cocycle in} & \nearrow & \downarrow \tilde{J}_{O(n+1)} \\ \text{twisted} & & \\ \text{Cohomotopy} & & \\ X & \xrightarrow[\text{twist}]{\tau} & BO(n+1) \end{array} \right\} / \sim \\ &= \left\{ \begin{array}{ccc} & E & \longrightarrow S^n // O(n+1) \\ \text{cocycle in} & \nearrow & \downarrow \tilde{J}_{O(n+1)} \\ \text{twisted} & & \\ \text{Cohomotopy} & & \\ X & \xrightarrow[\text{twist}]{\tau} & BO(n+1) \end{array} \right\} / \sim \end{aligned} \quad (37)$$

Here in the second line, $E \rightarrow X$ denotes the n -spherical fibration classified by τ and the universal property of the homotopy pullback shows that cocycles in τ -twisted equivariant Cohomotopy are equivalently sections of this n -spherical fibration.

Remark 2.2 (Notation). Here the notation $\pi^\tau(X)$ is motivated, as usual in twisted cohomology, from thinking of the map τ as encoding, in particular, also the degree $n \in \mathbb{N}$.

Remark 2.3 (Cohomotopy twist by Spin structure). In applications, the twisting map τ is often equipped with a lift through some stage of the Whitehead tower of $BO(n+1)$, notably with a lift through $BSpin(n+1)$ or further to $BSpin(n+1)$

$$\begin{array}{ccccc} X & \xrightarrow{\hat{\tau}} & BSpin(n+1) & \longrightarrow & BO(n+1) \\ & & \searrow & \nearrow & \\ & & & \tau & \end{array}$$

In this case, due to the homotopy pullback diagram

$$\begin{array}{ccc} S^n // Spin(n+1) & \longrightarrow & S^n // O(n+1) \\ \downarrow & \text{(pb)} & \downarrow \tilde{J}_{O(n+1)} \\ BSpin(n+1) & \longrightarrow & BO(n+1) \end{array}$$

the twisted cohomotopy set from Def. 2.1 is equivalently given by

$$\pi^\tau(X) \simeq \left\{ \begin{array}{ccc} & S^n // Spin(n+1) & \\ \text{cocycle in} & \nearrow & \downarrow \tilde{J}_{Spin(n+1)} \\ \text{twisted} & & \\ \text{Cohomotopy} & & \\ X & \xrightarrow[\text{twist}]{\hat{\tau}} & BSpin(n+1) \end{array} \right\} / \sim \quad (38)$$

Most of the examples in §2.3 and §3 arise in this form.

In order to extract differential form data (“flux densities”) from cocycles in twisted Cohomotopy, in Prop. 2.5 below, we consider rational twisted Cohomotopy (Def. 2.4) below. A standard reference on the rational homotopy theory involved is [FHT00]. Reviews streamlined to our context can be found in [FSS16a, Appendix A][BSS18].

Definition 2.4 (Chern character on twisted Cohomotopy). We write $\pi^\tau(X) \xrightarrow{L_{\mathbb{R}}} \pi^\tau(X)_{\mathbb{R}}$ for the rationalization of twisted Cohomotopy to *rational twisted Cohomotopy*, given by applying rationalization to all spaces and maps involved in a twisted Cohomotopy cocycle.

We now characterize cocycles in rational twisted Cohomotopy in terms of differential form data (which will be the corresponding “flux density” in §3).

Proposition 2.5 (Differential form data underlying twisted Cohomotopy). *Let X be a smooth manifold which is simply connected (see Remark 2.6 below) and $\tau : X \rightarrow BSO(n+1)$ a twisting for Cohomotopy in degree n , according to Def. 2.1. Let ∇_τ be any connection on the real vector bundle V classified by τ with Euler form $\mathcal{X}_{2k+2}(\nabla_\tau)$ (see [MQ86, below (7.3)][Wu06, 2.2]).*

(i) **If $n = 2k + 1$ is odd, $n \geq 3$,** a cocycle defining a class in the rational τ -twisted Cohomotopy of X (Def. 2.4) is equivalently given by a differential $2k + 1$ -form $G_{2k+1} \in \Omega^{2k+1}(X)$ on X which trivializes the negative of the Euler form

$$\pi^\tau(X)_\mathbb{R} \simeq \{G_{2k+1} \mid dG_{2k+1} = -\mathcal{X}_{2k+2}(\nabla_\tau)\} / \sim. \quad (39)$$

(ii) **If $n = 2k$ is even, $n \geq 2$,** a cocycle defining a class in the rational τ -twisted Cohomotopy of X (Def. 2.4) is given by a pair of differential forms $G_{2k} \in \Omega^{2k}(X)$ and $G_{4k-1} \in \Omega^{4k-1}(X)$ such that

$$dG_{2k} = 0; \quad \pi^*G_{2k} = \frac{1}{2}\mathcal{X}_{2k}(\nabla_{\hat{\tau}}) \quad (40)$$

$$d2G_{4k-1} = -G_{2k} \wedge G_{2k} + \frac{1}{4}p_k(\nabla_\tau), \quad (41)$$

where $p_k(\nabla_\tau)$ is the k -th Pontrjagin form of ∇_τ , $\pi : E \rightarrow X$ is the unit sphere bundle over X associated with τ , $\hat{\tau} : E \rightarrow BSO(n)$ classifies the vector bundle \hat{V} on E defined by the splitting $\pi^*V = \mathbb{R}_E \oplus \hat{V}$ associated with the tautological section of π^*V over E , and $\nabla_{\hat{\tau}}$ is the induced connection on \hat{V} . That is,

$$\pi^\tau(X)_\mathbb{R} \simeq \left\{ (G_{2k}, 2G_{4k-1}) \mid \begin{array}{l} dG_{2k} = 0, \quad \pi^*G_{2k} = \frac{1}{2}\mathcal{X}_{2k}(\nabla_{\hat{\tau}}) \\ d2G_{4k-1} = -G_{2k} \wedge G_{2k} + \frac{1}{4}p_k(\nabla_\tau) \end{array} \right\} / \sim. \quad (42)$$

Proof. By the assumption that the smooth manifold X is simply connected, it has a Sullivan model dgc-algebra $\text{CE}(IX)$ and we may compute the rational twisted Cohomotopy by choosing a Sullivan model $\mathcal{I}E$ for the spherical fibration classified by τ . By definition of rational twisted Cohomotopy, we are interested in the set of homotopy equivalence classes of dgca morphisms $\text{CE}(\mathcal{I}E) \rightarrow \text{CE}(IX)$ that are sections of the morphism $\text{CE}(IX) \rightarrow \text{CE}(\mathcal{I}E)$ corresponding to the projection $E \rightarrow X$. The Sullivan model model for E is well known. We recall from [FHT00, Sec. 15, Example 4]:

(I). The Sullivan model for the total space of a $2k + 1$ -spherical fibration $E \rightarrow X$ is of the form

$$\text{CE}(\mathcal{I}E) = \text{CE}(IX) \otimes \mathbb{R}[\omega_{2k+1}] / (d\omega_{2k+1} = -c_{2k+2}), \quad (43)$$

where

(a) $c_{2k+2} \in \text{CE}(IX)$ is some element in the base algebra, which by (43) is closed and so it represents a rational cohomology class

$$[c_{2k+2}] = H^{2k+2}(X; \mathbb{R}).$$

This class classifies the spherical fibration, rationally. Moreover, if the spherical fibration $E \rightarrow X$ happens to be the unit sphere bundle $E = S(V)$ of a real vector bundle $V \rightarrow X$, then the class of c_{2k+2} is the rationalized Euler class $\mathcal{X}_{2k+2}(V)$ of V :

$$[c_{2k+2}] = \mathcal{X}_{2k+2}(V) \in H^{2k+2}(X; \mathbb{R}). \quad (44)$$

(b) and in this case, under the quasi-isomorphism $\text{CE}(\mathcal{I}E) \rightarrow \Omega_{\text{dR}}^\bullet(E)$ the new generator ω_{2k+1} corresponds to a differential form that evaluates to the unit volume on each $(2k + 1)$ -sphere fiber:

$$\langle \omega_{2k+1}, [S^{2k+1}] \rangle = 1. \quad (45)$$

(This is not stated in [FHT00, Sec. 15, Example 4], but follows with [Che44], see [Wa04, Ch. 6.6, Thm. 6.1].)

The morphism $\text{CE}(\mathcal{I}X) \rightarrow \text{CE}(\mathcal{I}E)$ is the obvious inclusion, so a section is completely defined by the image of ω_{2k+1} in $\text{CE}(\mathcal{I}X)$. This image will be an element $g_{2k+1} \in \text{CE}(\mathcal{I}X)$ such that $dg_{2k+1} = c_{2k+2}$, and every such element defines a section $\text{CE}(\mathcal{I}E) \rightarrow \text{CE}(\mathcal{I}X)$ and so a cocycle in rational twisted cohomotopy. Under the quasi-isomorphism $\text{CE}(\mathcal{I}X) \rightarrow \Omega_{\text{dR}}^\bullet(X)$ defining $\text{CE}(\mathcal{I}X)$ as a Sullivan model of X , the element c_{2k+2} is mapped to a closed differential form $\chi_{2k+2}(\nabla_\tau)$ representing the Euler class $\chi_{2k+2}(V)$ of V , and so g_{2k+1} corresponds to a differential form G_{2k+1} on X with $dG_{2k+1} = \chi_{2k+2}(\nabla_\tau)$.

(II). The Sullivan model for the total space of $2k$ -spherical fibration $E \rightarrow X$ is of the form⁴

$$\text{CE}(\mathcal{I}E) = \text{CE}(\mathcal{I}X) \otimes \mathbb{R}[\omega_{2k}, \omega_{4k-1}] / \left(\begin{array}{l} d\omega_{2k} = 0 \\ d2\omega_{4k-1} = -\omega_{2k} \wedge \omega_{2k} + c_{4k} \end{array} \right), \quad (46)$$

where

- (a) $c_{4k} \in \text{CE}(\mathcal{I}X)$ is some element in the base algebra, which by (46) is closed and represents the rational cohomology class of the cup square of the class of ω_{4k} :

$$[c_{4k}] = [\omega_{2k}]^2 \in H^{4k}(X; \mathbb{R}).$$

This class classifies the spherical fibration, rationally.

- (b) under the quasi-isomorphism $\text{CE}(\mathcal{I}E) \rightarrow \Omega_{\text{dR}}^\bullet(E)$ the new generator ω_{2k} corresponds to a closed differential form that restricts to the volume form on the $2k$ -sphere fibers $S^{2k} \simeq E_x \hookrightarrow E$ over each point $x \in X$:

$$\langle \omega_{2k}, [S^{2k}] \rangle = 1. \quad (47)$$

Note that the element $[\omega_{2k}]^2$ is a priori an element in $H^{4k}(E, \mathbb{R})$. By writing $[c_{4k}] = [\omega_{2k}]^2 \in H^{4k}(X; \mathbb{R})$ we mean that $[\omega_{2k}]^2$ is actually the pullback of the element $[c_{4k}]$ via the projection $\pi: E \rightarrow X$.

Moreover, if the spherical fibration $\pi: E \rightarrow X$ happens to be the unit sphere bundle $E = S(V)$ of a real vector bundle $V \rightarrow X$, then the tautological section of π^*V defines a splitting $\pi^*V = \mathbb{R}_E \oplus \widehat{V}$ and

- (a) the class of ω_{2k} is half the rationalized Euler class $\chi_{2k}(\widehat{V})$ of \widehat{V} :

$$[\omega_{2k}] = \frac{1}{2}\chi(\widehat{V}) \in H^{2k}(E; \mathbb{R}). \quad (48)$$

- (b) the class of c_{4k} is one fourth the rationalized k -th Pontrjagin class $p_k(V)$ of V :

$$[c_{4k}] = \frac{1}{4}p_k(V) \in H^{4k}(X; \mathbb{R}). \quad (49)$$

The second equation is actually a consequence of the first one and of the naturality and multiplicativity of the total rational Pontrjagin class:

$$\pi^*p_k(V) = p_k(\mathbb{R}_E \oplus \widehat{V}) = p_k(\widehat{V}) = \chi_{2k}(\widehat{V})^2.$$

Reasoning as in the odd sphere bundles case, a section of $\text{CE}(\mathcal{I}X) \rightarrow \text{CE}(\mathcal{I}E)$, and so a cocycle in rational twisted cohomotopy, is the datum of elements $g_{2k}, g_{4k-1} \in \text{CE}(\mathcal{I}X)$ such that $dg_{2k} = 0$ and $d2g_{4k-1} = -g_{2k} \wedge g_{2k} + c_{4k}$. Under the quasi-isomorphism $\text{CE}(\mathcal{I}E) \rightarrow \Omega_{\text{dR}}^\bullet(E)$, the element g_{2k} , seen as an element in $\text{CE}(\mathcal{I}E)$, is mapped to a closed differential form $\frac{1}{2}\chi_{2k}(\nabla_{\hat{\tau}})$ representing $1/2$ the Euler class $\chi_{2k}(\widehat{V})$ of \widehat{V} , while under the quasi-isomorphism $\text{CE}(\mathcal{I}X) \rightarrow \Omega_{\text{dR}}^\bullet(X)$ the element c_{4k} is mapped to a closed differential form $\frac{1}{4}p_k(\nabla_{\hat{\tau}})$ representing $1/4$ the k -th Pontrjagin class $\frac{1}{4}p_k(V)$ of V . Therefore, the quasi-isomorphism $\text{CE}(\mathcal{I}X) \rightarrow \Omega_{\text{dR}}^\bullet(X)$ turns the elements g_{2k} and g_{4k-1} into differential forms G_{2k} and G_{4k-1} on X , subject to the identities $dG_{2k} = 0$, $\pi^*G_{2k} = \frac{1}{2}\chi_{2k}(\nabla_{\hat{\tau}})$, and $d2G_{4k-1} = -G_{2k} \wedge G_{2k} + \frac{1}{4}p_k(\nabla_{\hat{\tau}})$. \square

⁴There is an evident sign typo in the statement (but not in the proof) of [FHT00, Sec. 15, Example 4] with respect to equation (43): The standard fact that the Euler class squares to the top Pontrjagin class means that there must be the relative minus sign in (43), which is exactly what the proof of [FHT00, Sec. 15, Example 4] actually concludes.

Remark 2.6 (Simply-connectedness assumption). The assumption in Prop. 2.5 that X be simply connected is just to ensure the existence of a Sullivan model for X , as used in the proof. (It would be sufficient to assume, for that purpose, that the fundamental group is nilpotent). If X is not simply connected and not even nilpotent, then a similar statement about differential form data underlying twisted Cohomotopy cocycles on X will still hold, but statement and proof will be much more involved. Hence we assume simply connected X here only for convenience, not for fundamental reasons. A direct consequence of this assumption, which will play a role in §3, is that, by the Hurewicz theorem and the universal coefficient theorem, the degree 2 cohomology of X with coefficients in \mathbb{Z}_2 is given by:

$$H^2(X; \mathbb{Z}_2) \simeq \text{Hom}_{\text{Ab}}(H_2(X, \mathbb{Z}), \mathbb{Z}_2). \quad (50)$$

2.2 Twisted Cohomotopy via topological G -structure

We discuss how cocycles in J -twisted Cohomotopy are equivalent to choice of certain topological G -structures (Prop. 2.8 below).

The following fact plays a crucial role throughout:

Lemma 2.7 (Homotopy actions and reduction of structure group). *Let G be a topological group and V any topological space.*

(i) *Then for every homotopy-coherent action of G on V , the corresponding homotopy quotient $V // G$ forms a homotopy fiber sequence of the form*

$$V \longrightarrow V // G \longrightarrow BG$$

and, in fact, this association establishes an equivalence between homotopy V -fibrations over BG and homotopy coherent actions of V on G .

(ii) *In particular, if $\iota : H \hookrightarrow G$ is an inclusion of topological groups, then the homotopy fiber of the induced map $B\iota$ of classifying spaces is the coset space G/H :*

$$G/H \xrightarrow{\text{fib}} BH \xrightarrow{B\iota} BG$$

thus exhibiting the weak homotopy equivalence $(G/H) // G \simeq BH$.

Proof. This equivalence goes back to [DDK80]. A modern account which generalizes to geometric situations (relevant for refinement of all constructions here to differential cohomology) is in [NSS12, Sec. 4]. When the given homotopy-coherent action of the topological group G on V happens to be given by an actual topological action we may use the Borel construction to represent the homotopy quotient. For the case of $H \hookrightarrow G$ a topological subgroup inclusion, we may compute as follows:

$$\begin{aligned} BH &\simeq * \times_H EH \\ &\simeq * \times_H EG \\ &\simeq * \times_H (G \times_G EG) \\ &\simeq (* \times_H G) \times_G EG \\ &\simeq (G/H) \times_G EG \\ &\simeq (G/H) // G. \end{aligned}$$

Here the first weak equivalence is the usual definition of the classifying space, while the second uses that one may take a universal H -bundle EH , up to weak homotopy equivalence, any contractible space with free H -action, hence in particular EG . The third line uses that G is the identity under Cartesian product followed by the quotient by the diagonal G -action. \square

Proposition 2.8 (Twisted cohomotopy cocycle is reduction of structure group). *Cocycles in twisted Cohomotopy (Def. 2.1) are equivalent to choices of topological G -structure for $G = O(n) \hookrightarrow O(n+1)$:*

$$\pi^\tau(X) = \left\{ \begin{array}{ccc} & \text{cocycle in} & BO(n) \\ & \text{twisted} & \downarrow \\ & \text{Cohomotopy} & BO(n+1) \\ X \xrightarrow[\text{twist}]{\tau} & & \end{array} \right\} / \sim$$

Moreover, if the twist is itself factored through $B\text{Spin}(n+1)$ as in Remark 2.3, then τ -twisted Cohomotopy is equivalent to reduction along $\text{Spin}(n) \hookrightarrow \text{Spin}(n+1)$:

$$\pi^\tau(X) = \left\{ \begin{array}{ccc} & \text{cocycle in} & B\text{Spin}(n) \\ & \text{twisted} & \downarrow \\ & \text{Cohomotopy} & B\text{Spin}(n+1) \\ X \xrightarrow[\text{twist}]{\hat{\tau}} & & \end{array} \right\} / \sim$$

Generally, if there is a coset realization of an n -sphere $S^n \simeq G/H$ and the twist is factored through G -structure, then τ -twisted Cohomotopy is further reduction to topological H -structure:

$$\pi^\tau(X) = \left\{ \begin{array}{ccc} & \text{cocycle in} & BH \\ & \text{twisted} & \downarrow \\ & \text{Cohomotopy} & BG \\ X \xrightarrow[\text{twist}]{\hat{\tau}} & & \end{array} \right\} / \sim$$

Proof. This follows by applying Lemma 2.7 and using the fact that $S^n \simeq O(n+1)/O(n)$. □

Remark 2.9 (Cohomotopy twists from coset space structures on spheres).

(i) Prop. 2.8 say that for each topological coset space structure on an n -sphere $S^n \simeq G/H$ the corresponding G -twisted Cohomotopy (Def. 2.1) classifies reduction to topological H -structure.

(ii) Coset space structures on n -spheres come in three infinite series and a few exceptional cases:

Spherical coset spaces	[MS43], see [GG70, p.2]
$S^{n-1} \simeq \text{Spin}(n)/\text{Spin}(n-1)$	standard, e.g. [BS53, 17.1]
$S^{2n-1} \simeq \text{SU}(n)/\text{SU}(n-1)$	
$S^{4n-1} \simeq \text{Sp}(n)/\text{Sp}(n-1)$	
$S^7 \simeq \text{Spin}(7)/G_2$	[Va01, Thm. 3]
$S^7 \simeq \text{Spin}(6)/\text{SU}(3)$	by $\text{Spin}(6) \simeq \text{SU}(4)$
$S^7 \simeq \text{Spin}(5)/\text{SU}(2)$	by $\text{Spin}(5) \simeq \text{Sp}(2)$ and $\text{SU}(2) \simeq \text{Sp}(1)$ [ADP83][DNP83]
$S^6 \simeq G_2/\text{SU}(3)$	[FI55]

Table S. Coset space structures on topological n -spheres.

(iii) Assembling these for the case of the 7-sphere, we interpret the result in terms of special holonomy and G -structures corresponding to consecutive reductions.

Using this, the following construction is a rich source of twisted Cohomotopy cocycles:

Lemma 2.10 (Classifying maps to normal bundles via twisted Cohomotopy). *Let Y be a manifold, $n \in \mathbb{N}$ a natural number and $Y \xrightarrow{N} BO(n+1)$ a classifying map.*

(i) *If $X \xrightarrow{\pi} Y$ denotes the n -spherical fibration classified by N , hence, by Remark 2.9, the homotopy pullback in the following diagram:*

$$\begin{array}{ccccc}
 & & T_Y X & & \\
 & \xrightarrow{\quad} & & \xrightarrow{\quad} & \\
 X & \xrightarrow{c} & S^n // O(n+1) & \xrightarrow{\simeq} & BO(n) \\
 \pi \downarrow & & \downarrow & \swarrow & \\
 Y & \xrightarrow{N} & BO(n+1) & &
 \end{array} \tag{51}$$

then the total top horizontal map is equivalently the classifying map for the vertical tangent bundle $T_Y X$, of X over Y , as shown.

(ii) *In particular, if $Y := \Sigma \times \mathbb{R}_{>0}$ is the Cartesian product of some manifold Σ with a real ray, so that each fiber of π over Σ is identified with the Cartesian space \mathbb{R}^{n+1} with the origin removed*

$$S^n \times \mathbb{R}_{>0} \simeq \mathbb{R}^{n+1} \setminus \{0\}$$

then the pullback map c in (51) is a cocycle in $N \circ \pi$ -twisted Cohomotopy on X , according to Def. 2.1:

$$\begin{array}{ccc}
 & S^n // O(n+1) & \\
 & \downarrow & \\
 X & \xrightarrow{c} & BO(n+1) \\
 & \downarrow & \\
 & BO(\dim(\Sigma) + n + 1) &
 \end{array}
 \begin{array}{l}
 \swarrow \\
 \xrightarrow{N \circ \pi} \\
 \searrow TX
 \end{array}
 \tag{52}$$

2.3 Twisted Cohomotopy in degrees 4 and 7 combined

We discuss here twisted Cohomotopy in degree 4 and 7 jointly, related by the quaternionic Hopf fibration $h_{\mathbb{H}}$. This requires first determining the space of twists that are compatible with $h_{\mathbb{H}}$, which is the content of Prop. 2.20 and Prop. 2.22 below. This yields the scenario of incremental G -structures shown in *Figure T*. The twists that appear are subgroups of $\text{Spin}(8)$ related by triality (Prop. 2.17 below), and in fact the classifying space for the C-field implied by *Hypothesis H* comes out to be the homotopy-fixed locus of triality.

It will be useful to have the following notation for a basic but crucial operation on Spin groups:

Definition 2.11 (Central product of groups). *Given a tuple of groups G_1, G_2, \dots, G_n , each equipped with a central \mathbb{Z}_2 -subgroup inclusion $\mathbb{Z}_2 \simeq \{1, -1\} \subset Z(G_i) \subset G_i$, we write*

$$G_1 \cdot G_2 \cdots G_{n-1} \cdot G_n := (G_1 \times G_2 \times \cdots \times G_n) / \text{diag } \mathbb{Z}_2 \tag{53}$$

for the quotient group of their direct product group by the corresponding diagonal \mathbb{Z}_2 -subgroup:

$$\{(1, 1, \dots, 1), (-1, -1, \dots, -1)\} \hookrightarrow G_1 \times G_2 \times \cdots \times G_n .$$

Just to save space we will sometimes suppress the dots and write $G_1 G_2 := G_1 \cdot G_2$, etc.

Example 2.12 (Central product of symplectic groups). The notation in Def. 2.11 originates in [Ale68][Gra69] for the examples

$$\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) := (\mathrm{Sp}(n) \times \mathrm{Sp}(1)) / \{(1, 1), (-1, -1)\}. \quad (54)$$

For $n \geq 2$ this is such that a $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ -structure on a $4n$ -dimensional manifold is equivalently a quaternion-Kähler structure [Sal82]. Specifically, for $n = 2$ there is a canonical subgroup inclusion

$$\begin{array}{ccc} & & \mathrm{Spin}(8) \\ & \nearrow & \downarrow \\ \mathrm{Sp}(2) \cdot \mathrm{Sp}(1) & \hookrightarrow & \mathrm{SO}(8) \simeq \mathrm{SO}(\mathbb{H}^2) \\ (A, q) & \longmapsto & (x \mapsto A \cdot x \cdot \bar{q}) \end{array} \quad (55)$$

given by identifying elements of $\mathrm{Sp}(2)$ as quaternion-unitary 2×2 -matrices A , elements of $\mathrm{Sp}(1)$ as multiples of the 2×2 identity matrix by unit quaternions q , and acting with such pairs by quaternionic matrix conjugation on elements $x \in \mathbb{H}^2 \simeq_{\mathbb{R}} \mathbb{R}^8$ as indicated. This lifts to an inclusion into $\mathrm{Spin}(8)$ through the defining double-covering map (see [CV97, 2.]). Notice that reversing the Sp -factors gives an isomorphic group, but a different subgroup inclusion

$$\begin{array}{ccc} & & \mathrm{Spin}(8) \\ & \nearrow & \downarrow \\ \mathrm{Sp}(1) \cdot \mathrm{Sp}(2) & \hookrightarrow & \mathrm{SO}(8) \simeq \mathrm{SO}(\mathbb{H}^2) \\ (q, A) & \longmapsto & (x \mapsto q \cdot x \cdot \bar{A}) \end{array} \quad (56)$$

For more on this see Prop. 2.17 below.

Example 2.13 (Central product of Spin groups). For $n_1, n_2 \in \mathbb{N}$, we have the central product (Def. 2.11) of the corresponding Spin groups

$$\mathrm{Spin}(n_1) \cdot \mathrm{Spin}(n_2) := (\mathrm{Spin}(n_1) \times \mathrm{Spin}(n_2)) / \{(1, 1), (-1, -1)\}. \quad (57)$$

(This notation is used for instance in [McI99, p. 9] [HN12, Prop. 17.13.1].) Here the canonical subgroup inclusions of Spin groups $\mathrm{Spin}(n) \xrightarrow{\iota_n} \mathrm{Spin}(n+k)$ induce a canonical subgroup inclusion of (57) into $\mathrm{Spin}(n_1 + n_2)$:

$$\begin{array}{ccccc} & & (\alpha, \beta) & \longmapsto & \iota_{n_1}(\alpha) \cdot \iota_{n_2}(\beta) \\ & & \downarrow & & \downarrow \\ (\mathbb{Z}_2)_{\mathrm{diag}} & \xrightarrow{\ker} & \mathrm{Spin}(n_1) \times \mathrm{Spin}(n_2) & \longrightarrow & \mathrm{Spin}(n_1 + n_2) \\ & & \downarrow \mathrm{quot} & \nearrow & \\ & & \mathrm{Spin}(n_1) \cdot \mathrm{Spin}(n_2) & & \end{array} \quad (58)$$

Notice that these groups sit in short exact sequences as follows:

$$1 \longrightarrow \mathrm{Spin}(n_1) \xrightarrow{\iota_{n_1}} \mathrm{Spin}(n_1) \cdot \mathrm{Spin}(n_2) \xrightarrow{\mathrm{pr}_{n_2}} \mathrm{SO}(n_2) \longrightarrow 1. \quad (59)$$

For low values of n_1, n_2 there are exceptional isomorphisms between the groups (54) and (57) as abstract groups, but as subgroups under the inclusions (55) and (58) these are different. This is the content of Prop. 2.17 below. First we record the following, for later use:

Definition 2.14 (Universal class of central products). For $n_1, n_2 \in \mathbb{N}$, write

$$\varpi \in H^2(B(\mathrm{Spin}(n_1) \cdot \mathrm{Spin}(n_2)); \mathbb{Z}_2)$$

for the *universal characteristic class* on the classifying space of the central product Spin group (Def. 2.13) which is the pullback of the second Stiefel-Whitney class $w_2 \in H^2(B\mathrm{SO}(n_2), \mathbb{Z}_2)$ from the classifying space of the underlying $\mathrm{SO}(n_2)$ -bundles, via the projection (59):

$$\varpi := (B\mathrm{pr}_{n_2})^*(w_2). \quad (60)$$

See also [Sal82, Def. 2.1], following [MR76].

Lemma 2.15 (Obstruction to direct product structure). *For $n_1, n_2 \in \mathbb{N}$, let $X \xrightarrow{\tau} B(\text{Spin}(n_1) \cdot \text{Spin}(n_2))$ be a classifying map for a central product Spin structure (Def. 2.13). Then the following are equivalent:*

(i) *The class ϖ from Def. 2.14 vanishes:*

$$\varpi(\tau) = 0 \in H^2(X; \mathbb{Z}_2).$$

(ii) *The classifying map τ has a lift to the direct product Spin structure:*

$$\begin{array}{ccc} & & B(\text{Spin}(n_1) \times \text{Spin}(n_2)) \\ & \nearrow \hat{\tau} & \downarrow \\ X & \xrightarrow{\tau} & B(\text{Spin}(n_1) \cdot \text{Spin}(n_2)). \end{array}$$

(iii) *The underlying $\text{SO}(n_2)$ -bundle admits Spin structure:*

$$\begin{array}{ccc} & & B\text{Spin}(n_2) \\ & \nearrow \widehat{B\text{pr}_{n_2} \circ \tau} & \downarrow \\ X & \xrightarrow{B\text{pr}_{n_2} \circ \tau} & B\text{SO}(n_2). \end{array}$$

Proof. By (57) and (59) we have the following short exact sequence of short exact sequences of groups:

$$\begin{array}{ccccc} 1 & \hookrightarrow & \text{Spin}(5) & \xlongequal{\quad} & \text{Spin}(5) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}_2 & \hookrightarrow & \text{Spin}(5) \times \text{Spin}(3) & \twoheadrightarrow & \text{Spin}(5) \cdot \text{Spin}(3) \\ \parallel & & \downarrow & & \downarrow \text{pr}_3 \\ \mathbb{Z}_2 & \hookrightarrow & \text{Spin}(3) & \twoheadrightarrow & \text{SO}(3) \end{array}$$

Since the bottom left morphism is an identity, it follows that also after passing to classifying spaces and forming connecting homomorphisms, the corresponding morphism on the bottom right in the following diagram is a weak homotopy equivalence:

$$\begin{array}{ccccccc} B\mathbb{Z}_2 & \longrightarrow & B(\text{Spin}(5) \times \text{Spin}(3)) & \longrightarrow & B(\text{Spin}(5) \cdot \text{Spin}(3)) & \xrightarrow{\varpi} & B^2\mathbb{Z} \\ \parallel & & \downarrow & & \downarrow B\text{pr}_3 & & \parallel \\ B\mathbb{Z}_2 & \longrightarrow & B\text{Spin}(3) & \longrightarrow & B\text{SO}(3) & \xrightarrow{w_2} & B^2\mathbb{Z} \end{array}$$

By the top homotopy fiber sequence, this exhibits ϖ as the obstruction to the lift from central product Spin structure to direct product Spin structure. \square

Example 2.16. Applying Def. 2.11 to three copies of $\text{Sp}(1)$ yields the group

$$\text{Sp}(1) \cdot \text{Sp}(1) \cdot \text{Sp}(1) := (\text{Sp}(1) \times \text{Sp}(1) \times \text{Sp}(1)) / \{(1, 1, 1), (-1, -1, -1)\}. \quad (61)$$

The notation appears for instance in [OP01][BM14].

- Observe that, due to the exceptional isomorphisms $\text{Spin}(3) \simeq \text{Sp}(1)$ and $\text{Spin}(4) \simeq \text{Spin}(3) \times \text{Spin}(3)$ there are isomorphisms

$$\text{Spin}(4) \cdot \text{Spin}(3) \simeq \text{Spin}(3) \cdot \text{Spin}(3) \cdot \text{Spin}(3) \simeq \text{Sp}(1) \cdot \text{Sp}(1) \cdot \text{Sp}(1). \quad (62)$$

- The group (61) is acted upon via automorphisms interchange the three dot-factors by the symmetric group on three elements:

$$s_3 \curvearrowright (\mathrm{Sp}(1) \cdot \mathrm{Sp}(1) \cdot \mathrm{Sp}(1)) \quad (63)$$

- Beware that the central product of groups with central \mathbb{Z}_2 -subgroup (Def. 2.11) is not a binary associative operation: for instance, we have

$$\mathrm{Sp}(1) \cdot \mathrm{Sp}(1) \simeq \mathrm{Spin}(3) \cdot \mathrm{Spin}(3) \simeq \mathrm{SO}(4), \quad (64)$$

which does not even contain the \mathbb{Z}_2 -subgroup anymore that one would diagonally quotient out in (62), hence the would-be iterated binary expression “ $(\mathrm{Sp}(1) \cdot \mathrm{Sp}(1)) \cdot \mathrm{Sp}(1)$ ” does not even make sense. Instead we have

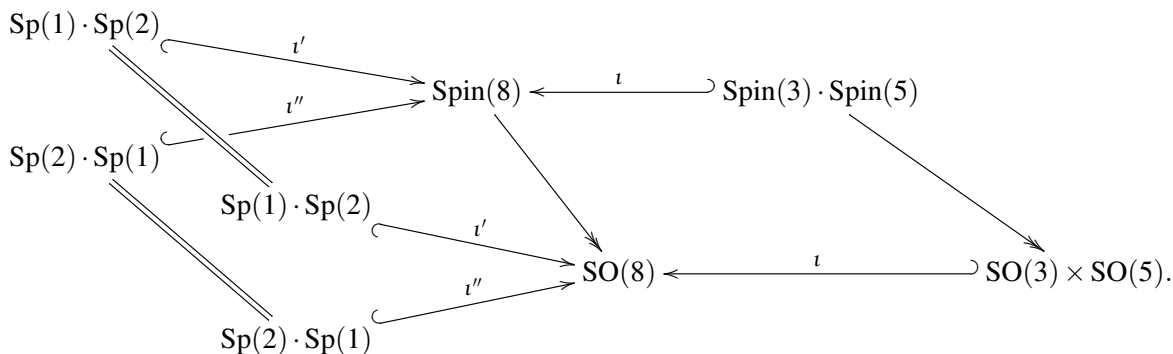
$$\mathrm{Sp}(1) \cdot \mathrm{Sp}(1) \cdot \mathrm{Sp}(1) \simeq (\mathrm{Sp}(1) \times \mathrm{Sp}(1)) \cdot \mathrm{Sp}(1). \quad (65)$$

But it is useful to observe that

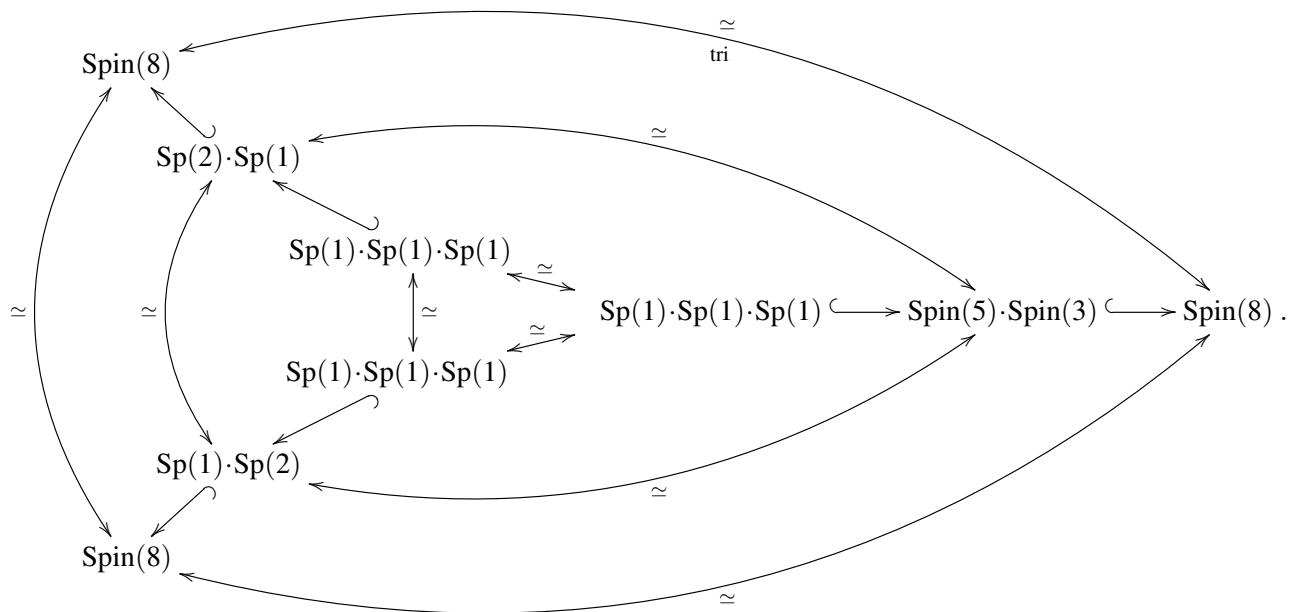
$$\mathrm{Sp}(1) \simeq \mathrm{Sp}(1) \cdot \mathbb{Z}_2 \quad \text{and} \quad \mathrm{Sp}(1) \times \mathrm{Sp}(1) \simeq \mathrm{Sp}(1) \cdot \mathbb{Z}_2 \cdot \mathrm{Sp}(1). \quad (66)$$

All of the above will play a role in Prop. 2.22 below.

Proposition 2.17 (Triality of quaternionic subgroups of $\mathrm{Spin}(8)$). *The subgroup inclusions into $\mathrm{Spin}(8)$ of $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ via (55), $\mathrm{Sp}(1) \cdot \mathrm{Sp}(2)$ via (56), and $\mathrm{Spin}(5) \cdot \mathrm{Spin}(3)$ via (58), represent three distinct conjugacy classes of subgroups, and under the defining projection to $\mathrm{SO}(8)$ they map to subgroups of $\mathrm{SO}(8)$ as follows:*



Moreover, the triality group $\mathrm{Out}(\mathrm{Spin}(8))$ acts transitively by permutation on the set of these three conjugacy classes.



Proof. This follows by analysis of the action of triality on the corresponding Lie algebras; see [CV97, Sec. 2], [Kol02, Prop. 3.3 (3)]. \square

Remark 2.18 (Subgroups). **(i)** For emphasis, notice that the subgroups appearing in Prop. 2.17 are all isomorphic as abstract groups

$$\mathrm{Sp}(1) \cdot \mathrm{Sp}(2) \simeq \mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \simeq \mathrm{Spin}(5) \cdot \mathrm{Spin}(3) \simeq \mathrm{Spin}(3) \cdot \mathrm{Spin}(5)$$

due to the classical exceptional isomorphisms

$$\mathrm{Sp}(1) \simeq \mathrm{Spin}(3), \quad \mathrm{Sp}(2) \simeq \mathrm{Spin}(5)$$

and via the evident automorphisms that permutes central product factors. However, when each is equipped with *its* canonical subgroup inclusion into $\mathrm{Spin}(8)$, via (55), (56) and (58), then these are distinct subgroups. Moreover, Prop. 2.17 says that the first three of these are even in distinct conjugacy classes of subgroups, while the two $\mathrm{Spin}(3) \cdot \mathrm{Spin}(5)$ and $\mathrm{Spin}(5) \cdot \mathrm{Spin}(3)$ are in the same conjugacy class.

(ii) In the following, when considering these subgroup inclusions and their induced morphisms on classifying spaces, we will always mean that *canonical* inclusion of the subgroup of that name. When we need to refer to another, non-canonical embedding of any of these groups G , then we will always make this explicit as a triality automorphism $G \xrightarrow{\simeq} G'$ followed by the canonical inclusion of G' . See for instance (111) below for an example.

For the development in §3 we need to know in particular how universal characteristic classes behave under the triality automorphisms:

Lemma 2.19 (Pullback of classes along triality). *The integral cohomology ring of $B\mathrm{Spin}(8)$ is*

$$H^*(B\mathrm{Spin}(8); \mathbb{Z}) \simeq \mathbb{Z}[\tfrac{1}{2}p_1, \tfrac{1}{4}(p_2 - (\tfrac{1}{2}p_1)^2) - \tfrac{1}{2}\mathcal{X}_8, \mathcal{X}_8, \beta(w_6)] / (2\beta(w_6)), \quad (67)$$

where p_k are Pontrjagin classes, \mathcal{X}_8 is the Euler class, w_6 is a Stiefel-Whitney class, β is the Bockstein homomorphism, so that $W_7 := \beta(w_6)$ is an integral Stiefel-Whitney class.

(i) Under the delooping of the triality automorphism from Prop. 2.17 to classifying spaces

$$\begin{array}{ccc} B(\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)) & \xrightarrow{\simeq} & B(\mathrm{Spin}(5) \cdot \mathrm{Spin}(3)) \\ \downarrow & & \downarrow \\ B\mathrm{Spin}(8) & \xrightarrow[\mathrm{Btri}]{\simeq} & B\mathrm{Spin}(8) \end{array} \quad (68)$$

these classes pull back as follows:

$$\begin{array}{ccc} & \tfrac{1}{2}p_1 & \longmapsto \tfrac{1}{2}p_1 \\ (\mathrm{Btri})^* : & \mathcal{X}_8 & \longmapsto -\tfrac{1}{4}(p_2 - (\tfrac{1}{2}p_1)^2) + \tfrac{1}{2}\mathcal{X}_8 \\ & \tfrac{1}{4}(p_2 - (\tfrac{1}{2}p_1)^2) - \tfrac{1}{2}\mathcal{X}_8 & \longmapsto -\mathcal{X}_8 \end{array} \quad (69)$$

(ii) Notice that, in particular,

$$((\mathrm{Btri})^*)^{-1} = (\mathrm{Btri})^*.$$

and

$$(\mathrm{Btri})^* : \tfrac{1}{4}p_2 \longmapsto -\mathcal{X}_8 + \left(\tfrac{1}{4}p_1\right)^2 - \tfrac{1}{2}\left(\tfrac{1}{4}(p_2 - (\tfrac{1}{2}p_1)^2) - \tfrac{1}{2}\mathcal{X}_8\right). \quad (70)$$

Proof. This follows by combining [CV97, Lemmas 2.5, 4.1, 4.2], following [GG70, Thm. 2.1], and using the property $(\mathrm{tri}^*)^{-1} = \mathrm{tri}^*$, recalled in [CV97, 2.]. \square

Now we may have a closer look at the quaternionic Hopf fibration $S^7 \simeq S(\mathbb{H}^2) \xrightarrow{h_{\mathbb{H}}} \mathbb{H}P^1 \simeq S^4$:

Proposition 2.20 (Symmetries of the quaternionic Hopf fibration).

(i) *The symmetry group of $h_{\mathbb{H}}$ and hence the group of twists for Cohomotopy jointly in degrees 4 and 7, is the group (54),*

$$\mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \hookrightarrow \mathrm{O}(8) , \quad (71)$$

with its canonical action (55), in that this is the largest subgroup of $\mathrm{O}(8) \simeq \mathrm{O}(\mathbb{H}^2)$ under which $h_{\mathbb{H}}$ is equivariant.

(ii) *The corresponding action on the codomain 4-sphere $S^4 \simeq S(\mathbb{R}^5)$ is via the canonical projection (59) to $\mathrm{SO}(5)$*

$$\mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \xrightarrow{\simeq} \mathrm{Spin}(5) \cdot \mathrm{Spin}(3) \xrightarrow{\mathrm{pr}_5} \mathrm{SO}(5) . \quad (72)$$

Proof. This statement essentially appears as [GWZ86, Prop. 4.1] and also, somewhat more implicitly, in [Po95, p. 263]. To make this more explicit, we may observe, with Table S, that the quaternionic Hopf fibration has the following coset space description:

$$\begin{array}{ccccc} S^3 & \xrightarrow{\mathrm{fib}(h_{\mathbb{H}})} & S^7 & \xrightarrow{h_{\mathbb{H}}} & S^4 \\ \parallel & & \parallel & & \parallel \\ \mathrm{Spin}(4) & \xrightarrow{\iota_4} & \mathrm{Sp}(2) & \xrightarrow{\mathrm{id}} & \mathrm{Sp}(2) \\ \mathrm{Spin}(3) & \xrightarrow{\mathrm{id}} & \mathrm{Sp}(1) & \xrightarrow{q \mapsto (q,1)} & \mathrm{Sp}(1) \times \mathrm{Sp}(1) \end{array} \quad (73)$$

where $\iota_4 : \mathrm{Spin}(4) \hookrightarrow \mathrm{Spin}(5) \simeq \mathrm{Sp}(2)$ denotes the canonical inclusion. This can also be deduced from [HaTo09, Table 1]. In the octonionic case the analogous statement is noticed in [OPPV12, p. 7]. \square

The following Prop. 2.22 gives the homotopy-theoretic version of Prop. 2.20, which is the key for the discussion in §3 below. In order to clearly bring out all subtleties, we first recall the following fact:

Lemma 2.21 ($\mathrm{Spin}(4)$ -action on quaternions). *Under the exceptional isomorphism*

$$\begin{array}{ccccc} \mathrm{Sp}(1) \times \mathrm{Sp}(1) & \xrightarrow{\simeq} & \mathrm{Spin}(4) & \longrightarrow & \mathrm{SO}(4) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sp}(2) & \xrightarrow{\simeq} & \mathrm{Spin}(5) & \longrightarrow & \mathrm{SO}(5) \end{array}$$

the action of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ on $\mathbb{R}^4 \simeq_{\mathbb{R}} \mathbb{H}$ is the conjugation action of pairs (q_1, q_2) of unit quaternions on any quaternion x :

$$\begin{array}{ccc} \mathrm{Spin}(4) \times \mathbb{R}^4 & \longrightarrow & \mathbb{R}^4 \\ \simeq \downarrow & & \downarrow \simeq \\ (\mathrm{Sp}(1) \times \mathrm{Sp}(1)) \times \mathbb{H} & \xrightarrow{\mathrm{conj}(-, -)(-)} & \mathbb{H} \\ ((q_1, q_2), x) & \longmapsto & q_1 \cdot x \cdot \bar{q}_2 \end{array} \quad (74)$$

Proposition 2.22 (The $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ -parametrized quaternionic Hopf fibration). *The homotopy quotient of the quaternionic Hopf fibration $h_{\mathbb{H}}$ by its equivariance group (Prop. 2.20) is equivalently the map of classifying spaces*

$$\begin{array}{ccc} S^7 // \mathrm{Sp}(2) \cdot \mathrm{Sp}(1) & \xleftarrow{\simeq} & B(\mathrm{Sp}(1) \cdot \mathrm{Sp}(1)) \\ \downarrow h_{\mathbb{H}} // \mathrm{Sp}(2) \cdot \mathrm{Sp}(1) & & \downarrow B([q_1, q_2] \mapsto [q_1, q_2, q_2]) \\ S^4 // \mathrm{Sp}(2) \cdot \mathrm{Sp}(1) & \xleftarrow{\simeq} & B(\mathrm{Sp}(1) \cdot \mathrm{Sp}(1) \cdot \mathrm{Sp}(1)) \end{array}$$

which is induced by the following inclusion of central product groups from Example 2.16:

$$\begin{array}{ccc} \mathrm{Sp}(1) \cdot \mathrm{Sp}(1) & \hookrightarrow & \mathrm{Sp}(1) \cdot \mathrm{Sp}(1) \cdot \mathrm{Sp}(1) \\ [q_1, q_2] & \longmapsto & [q_1, q_2, q_2] \end{array} \quad (75)$$

Proposition 2.23 (G-structures induced by Cohomotomy in degree 7). *We have the following sequence of homotopy pullbacks of universal 7-spherical fibrations, hence of twists for Cohomotomy in degree 7 (see Figure D):*

$$\begin{array}{ccccccc}
 S^7 & \xrightarrow{\text{fib}} & BSU(2) & \longrightarrow & BSpin(5) & & \\
 \parallel & & \downarrow & & \downarrow & \text{(pb)} & \\
 S^7 & \xrightarrow{\text{fib}} & BSU(3) & \longrightarrow & BSpin(6) & & \\
 \parallel & & \downarrow & & \downarrow & \text{(pb)} & \\
 S^7 & \xrightarrow{\text{fib}} & BG_2 & \longrightarrow & BSpin(7) & & \\
 \parallel & & \downarrow & & \downarrow & \text{(pb)} & \\
 S^7 & \xrightarrow{\text{fib}} & BSpin(7) & \xrightarrow{B\iota'} & BSpin(8) & &
 \end{array}$$

Proof. First, observe that there is the following analogous commuting diagram of Lie groups:

$$\begin{array}{ccccccc}
 SU(2) & \hookrightarrow & SU(3) & \hookrightarrow & G_2 & \hookrightarrow & Spin(7) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \iota' \\
 Spin(5) & \hookrightarrow & Spin(6) & \hookrightarrow & Spin(7) & \hookrightarrow & Spin(8) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 SO(5) & \hookrightarrow & SO(6) & \hookrightarrow & SO(7) & \hookrightarrow & SO(8) .
 \end{array} \tag{78}$$

Here the bottom squares evidently commute and are pullback squares by the definition of Spin groups, while the three total vertical rectangles commute and are pullback squares by [On93, Table 2, p. 144]. By the pasting law,⁵ this implies that also the top squares are pullbacks, hence exhibiting intersections of subgroup inclusions. Notice that the top right vertical inclusion ι' is *not* the canonical inclusion of Spin(7) in Spin(8), but is a subgroup inclusion in a distinct Spin(7)-conjugacy class, of which there are three [Va01, Thm. 5 on p. 6]. The intersection in the top right square is also proven in [Va01, Thm. 5 on p. 13], and that of the middle square in [Va01, Lem. 9 on p. 10]. Again, by the pasting law, this implies that also the top squares are pullbacks, hence exhibiting intersections of subgroup inclusions.

Applying delooping (passage to classifying spaces) to these top squares, this shows that we have a homotopy commuting diagram as follows:

$$\begin{array}{ccccccc}
 S^7 & \xrightarrow{\text{fib}} & S^7 & \xrightarrow{\text{fib}} & S^7 & \xrightarrow{\text{fib}} & S^7 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 S^3 & \xrightarrow{\text{fib}} & S^5 & \xrightarrow{\text{fib}} & S^6 & \xrightarrow{\text{fib}} & S^7 \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 * & \longrightarrow & BSU(2) & \longrightarrow & BSU(3) & \longrightarrow & BG_2 & \longrightarrow & BSpin(7) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 BSpin(4) & \longrightarrow & BSpin(5) & \longrightarrow & BSpin(6) & \longrightarrow & BSpin(7) & \longrightarrow & BSpin(8) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 S^4 & \xrightarrow{\text{fib}} & S^5 & \xrightarrow{\text{fib}} & S^6 & \xrightarrow{\text{fib}} & S^7 & \xrightarrow{\text{fib}} &
 \end{array} \tag{79}$$

⁵ Recall that this says that if

$$\begin{array}{ccccc}
 A & \longrightarrow & B & \longrightarrow & C \\
 \downarrow & & \downarrow & \text{(pb)} & \downarrow \\
 D & \longrightarrow & E & \longrightarrow & F
 \end{array}$$

is a commuting diagram, where the right square is a pullback, then the left square is a pullback precisely if the full outer rectangle is a pullback. The same holds for homotopy-commutative diagrams and homotopy-pullback squares.

The spherical homotopy fibers shown in this diagram follow by using Lemma 2.7 with classical results about coset space structures of topological spheres, as summarized in Table S.

In order to see that each square in the diagram of classifying spaces is a homotopy pullback, we now use the following basic fact from homotopy theory (see e.g. [CPS05, 5.2]): Assume that Y_1, Y_2 are connected spaces, and we are given a homotopy-commutative square as on the right in the following diagram

$$\begin{array}{ccccc} \text{fib}(f_1) & \longrightarrow & X_1 & \xrightarrow{f_1} & Y_1 \\ \simeq \downarrow & & \downarrow & \text{(pb)} & \downarrow \\ \text{fib}(f_2) & \longrightarrow & X_2 & \xrightarrow{f_2} & Y_2 . \end{array}$$

Then the square is a homotopy pullback square if and only if the induced left vertical morphism between horizontal homotopy fibers is a weak homotopy equivalence; as indicated. To see that in our case these induced left vertical morphisms are indeed weak homotopy equivalences, we first observe that for each of the squares above the horizontal homotopy fibers are n -spheres of the same dimension n :

$$\begin{array}{ccccc} S^7 \simeq \frac{\text{Spin}(7)}{G_2} & \longrightarrow & BG_2 & \longrightarrow & B\text{Spin}(7) \\ \simeq \downarrow & & \downarrow & & \downarrow \\ S^7 \simeq \frac{\text{Spin}(8)}{\text{Spin}(7)} & \longrightarrow & B\text{Spin}(7) & \longrightarrow & B\text{Spin}(8) \end{array}$$

and

$$\begin{array}{ccccc} S^6 \simeq \frac{G_2}{\text{SU}(3)} & \longrightarrow & BSU(3) & \longrightarrow & BG_2 \\ \simeq \downarrow & & \downarrow & & \downarrow \\ S^6 \simeq \frac{\text{Spin}(7)}{\text{Spin}(6)} & \longrightarrow & B\text{Spin}(6) & \longrightarrow & B\text{Spin}(7) \end{array}$$

(for the coset realization of S^6 on the top left see [FI55]) and

$$\begin{array}{ccccc} S^5 \simeq \frac{\text{SU}(3)}{\text{SU}(2)} & \longrightarrow & BSU(2) & \longrightarrow & BSU(3) \\ \simeq \downarrow & & \downarrow & & \downarrow \\ S^5 \simeq \frac{\text{Spin}(6)}{\text{Spin}(5)} & \longrightarrow & B\text{Spin}(5) & \longrightarrow & B\text{Spin}(6) . \end{array}$$

To see in detail that the homotopy fibers on the left are not only pairwise weakly homotopy equivalent, but that the universally induced dashed morphism exhibits such a weak homotopy equivalence, we proceed as follows. For $G := \text{Spin}(n)$ one of the Spin groups appearing above, pick any one topological space EG modelling the total space of the universal G bundle (hence any weakly contractible topological space equipped with a free continuous G -action). Then for $G' \xrightarrow{\iota} G$ any subgroup, we have that the projection $(EG)/G' \rightarrow (EG)/G$ is a Serre fibration modelling $BG' \xrightarrow{B\iota} BG$ (e.g. [Mi11, 11.4]). Since ordinary pullbacks of Serre fibrations are already homotopy pullbacks, this means that the above homotopy pullback squares are represented by actual pullback squares of topological spaces in the following diagram:

$$\begin{array}{ccccc} S^n \simeq \frac{G'}{G' \cap G''} & \longrightarrow & (EG)/(G' \cap G'') & \longrightarrow & (EG)/G' \\ \simeq \downarrow & \text{(pb)} & \downarrow & \text{(pb)} & \downarrow \\ S^n \simeq \frac{G}{G''} & \longrightarrow & (EG)/G'' & \longrightarrow & (EG)/G . \end{array}$$

Here the dashed morphism is the canonical continuous function induced by the given group inclusions, so that it is now sufficient to observe that this is a homeomorphism.

While this does not follow for general subgroup intersections, it does follow as soon as the given coset spaces are homeomorphic, as is the case here. Namely, pick any point $x \in S^n$ and observe that we have a commuting square of continuous functions as follows.

$$\begin{array}{ccc} S^n & \xleftarrow[\simeq_{\text{homeo}}]{[g'] \mapsto g'(x)} & \frac{G'}{G' \cap G''} \\ \parallel & & \downarrow \\ S^n & \xleftarrow[\simeq_{\text{homeo}}]{[g] \mapsto g(x)} & \frac{G}{G''} \end{array}$$

Since in this diagram the top, bottom and left maps are homeomorphisms, it follows that the right map is also a homeomorphism. \square

Remark 2.24 (Twisted generalized cohomotopy). One may also consider twisted Cohomotopy with coefficients in fibrations of *pairs* of spheres:

$$\left\{ \begin{array}{ccc} & & (S^p \times S^q) // (\mathrm{O}(p) \times \mathrm{O}(q)) \\ & \nearrow \text{dashed} & \downarrow \\ X & \xrightarrow{TX} & \mathrm{BO}(n) \end{array} \right\} / \sim$$

(i) Corresponding twists arise from “doubly exceptional geometry”, in that we have the following pasting diagram of homotopy pullbacks, further refining those of Prop. 2.23:

$$\begin{array}{ccc} S^7 \times S^7 & \longrightarrow & BG_2 \\ \downarrow & \text{(pb)} & \downarrow i_{G_2} \\ S^7 & \longrightarrow & B\mathrm{Spin}(7) \\ \downarrow & \text{(pb)} & \downarrow i_{\mathrm{Spin}(7)} \\ * & \longrightarrow & B\mathrm{Spin}(8) \end{array} \quad \text{equivalently} \quad \begin{array}{ccc} S^7 \times S^7 & \longrightarrow & S^7 // \mathrm{Spin}(7) \\ \downarrow & \text{(pb)} & \downarrow i_{G_2} \\ S^7 & \longrightarrow & S^7 // \mathrm{Spin}(8) \\ \downarrow & \text{(pb)} & \downarrow i_{\mathrm{Spin}(7)} \\ * & \longrightarrow & B\mathrm{Spin}(8) \end{array}$$

This follows analogously as in Prop. 2.23, with [On93, p. 146].

(ii) Further twists for Cohomotopy with coefficients in $S^p \times S^p$ arise from topological G -structure for rotation groups $\mathrm{O}(p, p)$ in split signature, and hence from *generalized geometry* (e.g. [Hul07]). This is because indefinite orthogonal groups are homotopy equivalent to their maximal compact subgroups via the polar decomposition

$$\mathrm{O}(p, p) \simeq_{\text{wh}} \mathrm{O}(p) \times \mathrm{O}(p)$$

(see, e.g., [HN12, Sec. 17.2]) and similarly for higher connected covers (see [SS19]). Therefore, we might call Cohomotopy with coefficients in $S^p \times S^p$, and twisted by generalized geometry, *generalized Cohomotopy* (not to be confused with older terminology [Ja62]). We will discuss the details elsewhere.

2.5 Twisted Cohomotopy via Poincaré-Hopf

We characterize here the TX -twisted Cohomotopy of compact orientable smooth manifolds X in terms of the “Cohomotopy charge” carried by a finite number of point singularities in X . This is the content of Prop. 2.25 below. The proof is a cohomotopical restatement of the classical Poincaré-Hopf (PH) theorem (see e.g. [DNF85, Sec. 15.2]), but the perspective of twisted Cohomotopy is noteworthy in itself and is crucial for the discussion of M2-brane tadpole cancellation in §3.8 below.

Proposition 2.25 (Twisted cohomotopy and the Euler characteristic). *Let X be an orientable compact smooth manifold. Then:*

(i) *A cocycle in the TX -twisted Cohomotopy of X (Def. 2.1) exists if and only if the Euler characteristic of X vanishes:*

$$\pi^{TX}(X) \neq \emptyset \iff \chi[X] \neq 0.$$

(ii) *Generally, there exists a finite set of points $\{x_i \in X\}$ such that the operation of restriction to open neighbourhoods of these points exhibits an injection of the TX -twisted Cohomotopy of their complement $\pi^{TX}(X \setminus \coprod_i \{x_i\})$ (Def. 2.1) into the product of untwisted Cohomotopy sets (34) $\pi^{\dim(X)}(U_{x_i} \setminus \{x_i\})$ of these pointed neighborhoods. Moreover, the latter are integers which sum to the Euler characteristic $\chi[X]$ of X :*

$$\begin{array}{ccc} \pi^{TX}(X \setminus \coprod_i \{x_i\}) & \xrightarrow{\text{restr.}} & \prod_i \pi^{\dim(X)-1}(U_{x_i} \setminus \{x_i\}) \xrightarrow{\simeq} \prod_i \mathbb{Z} \\ \downarrow & & \downarrow \Sigma_i \\ * & \xrightarrow{\chi[X]} & \mathbb{Z} \end{array} \quad (80)$$

Proof. This follows with the classical Poincaré-Hopf theorem, (83) below. We recall the relevant terminology:

- (i) For v a vector field on X , a point $x \in X$ is called an *isolated zero* of v if there exists an open contractible neighborhood $U_x \subset X$ such that the restriction $v|_{U_x}$ of v to this neighborhood vanishes at x and only at x .
- (ii) This means that on $U_x \setminus \{x\}$ the vector field v induces a map to the $(\dim(X) - 1)$ -sphere

$$v/|v| : U_x \setminus \{x\} \xrightarrow{v/|v|} S(T_x X) \simeq S^{\dim(X)-1}. \quad (81)$$

Here the equivalence on the right is to highlight that the sphere arises as the fiber of the unit sphere bundle of the tangent bundle TU_x , which may be identified with the unit sphere in $T_x X$, by the assumed contractibility of U_x .

- (iii) Given an isolated zero x , the *Poincaré-Hopf index* of v at that point is the degree of the associated map (81) to the sphere, for any choice of local chart:

$$\text{index}_x(v) := \text{deg}(U_x \setminus \{x\} \xrightarrow{v/|v|} S(T_x X) \simeq S^{\dim(X)-1}). \quad (82)$$

Now for X orientable and compact, the *Poincaré-Hopf theorem* (e.g. [DNF85, Sec. 15.2]) says that for any vector field $v \in \Gamma(TX)$ with a finite set $\{x_i \in X\}$ of isolated zeros, the sum of the indices (82) of v equals the Euler characteristic $\chi[X]$ of X :

$$\sum_{\substack{\text{isolated zero} \\ x_i \in X}} \text{index}_{x_i}(v) = \chi[X]. \quad (83)$$

To conclude, observe that the maps to spheres in (81) are but the restriction of the corresponding cocycle in the TX -Cohomotopy of $X \setminus \coprod_i \{x_i\}$:

$$\begin{array}{ccc} & & S^{\dim(X)} // \text{SO}(\dim(X)) \\ & \nearrow^{v/|v|} & \downarrow \\ X \setminus \coprod_i \{x_i\} & \xrightarrow{TX} & B\text{SO}(\dim(X)) \end{array}$$

Finally, the identification of the PH-index with an integer is via the Hopf degree theorem (35), now understood as the characterization of untwisted Cohomotopy in (35). \square

We may equivalently use the differential form data that underlies a cocycle in twisted Cohomotopy, by Prop. 2.5, to re-express the cohomotopical PH-theorem, Prop. 2.25, via Stokes' theorem. Let X be an orientable compact smooth manifold of even dimension $\dim(X) = 2n + 2$, for $n \in \mathbb{N}$ and let $v \in TX$ be a vector field with isolated zeros $\{x_i \in X\}$. For any fixed choice of Riemannian metric on X and any small enough positive real number ε , write

$$D_{x_i}^\varepsilon := \{x \in X \mid d(x, x_i) < \varepsilon\} \subset X$$

for the open ball of radius ε around x_i . The complement of these open balls is hence a manifold with boundary a disjoint union of $(2n + 1)$ -spheres:

$$\partial(X \setminus \coprod_i \{x_i\}) \simeq \coprod_i S^{2n+1}.$$

Then, by Prop. 2.5, the cocycle in twisted Cohomotopy on $X \setminus \coprod_i \{x_i\}$ which corresponds to the vector field v has underlying it a differential $(2n + 1)$ -form G_{2n+1} which satisfies

$$dG_{2n+1} = -\chi_{2n+2}(\nabla).$$

By Stokes' theorem we thus have

$$\begin{aligned} \chi[X] &= \lim_{\varepsilon \rightarrow 0} \int_{X \setminus \coprod_i D_{x_i}^\varepsilon} \chi \\ &= -\lim_{\varepsilon \rightarrow 0} \sum_i \int_{\partial D_{x_i}^\varepsilon} G_{2n+1} \end{aligned}$$

We may summarize the above by the following.

Lemma 2.26 (Cohomological PH-theorem). *In the above setting, the Euler characteristic is given by the integral of $-G_{2n+1}$ over the boundary components around the zeros of v :*

$$-\sum_i \int_{S_i^{2n+1}} G_{2n+1} = \chi[X]. \quad (84)$$

2.6 Twisted Cohomotopy via Pontrjagin-Thom

We recall the unstable Pontrjagin-Thom theorem relating untwisted Cohomotopy to normally framed submanifolds, (85) below. Then we show that twisted Cohomotopy jointly in degrees 4 and 7 (as per §2.3) knows about *calibrated submanifolds* in 8-manifolds, Prop. 2.27 below. Finally we observe that in this case vanishing submanifolds under a twisted Pontrjagin-Thom construction means, equivalently, a factorization through the quaternionic Hopf fibration, (91) below.

Framed submanifolds from untwisted Cohomotopy. One striking aspect of *Hypothesis H*, is that unstable Cohomotopy of a manifold X is exactly the cohomology theory which classifies (cobordism classes of) *submanifolds* $\Sigma \subset X$, subject to constraints on the normal bundle $N_X \Sigma$ of the embedding.

In the case of vanishing twist, this is the statement of the classical *unstable Pontrjagin-Thom isomorphism* (e.g. [Kos93, IX.5])

$$\pi^n(X) \begin{array}{c} \xleftarrow{\text{PT}^n} \\ \xrightarrow[\text{fib}_0 \circ \text{reg}]{\simeq} \end{array} \text{FrSubMfd}^{\text{codim}=n}(X) / \sim_{\text{bord}}. \quad (85)$$

For a closed smooth manifold X and any degree $n \in \mathbb{N}$, this identifies degree n cocycles

$$[X \xrightarrow{c} S^n] \in \pi^n(X)$$

in the untwisted unstable Cohomotopy (34) of X with the cobordism classes of normally framed submanifolds Σ of codimension n

$$(\Sigma \hookrightarrow X, N_X \Sigma \xrightarrow[\simeq]{\text{fr}} \Sigma \times \mathbb{R}^n, \dim(\Sigma) = \dim(X) - n)$$

given as the preimage of a chosen base point

$$\text{pt} \in S^n \tag{86}$$

under a smooth function representative c of $[c]$ for which pt is a regular value $c^{-1}(\{\text{pt}\}) =: \Sigma \subset X$.

As advocated in [Sa13], we may naturally think of the submanifolds $\Sigma \subset X$ appearing in the unstable Pontrjagin-Thom isomorphism (85) as *branes* whose charge is given by the Cohomotopy class $[c]$. This reveals Cohomotopy as the canonical cohomology theory for measuring charges of branes given as (cobordism classes of) submanifolds. To see this in full detail one needs to consider the refinement of (85) to twisted and *equivariant* Cohomotopy. In the rational approximation this is discussed in [HSS18], the full non-rational theory of M-branes at singularities classified by equivariant Cohomotopy will be discussed elsewhere [SS19a][BSS19b].

Here we content ourselves with highlighting two related facts, which are needed for the discussion in §3.

Calibrated submanifolds from twisted Cohomotopy. The manifold \mathbb{R}^8 carries an exceptional *calibration* by the *Cayley 4-form* $\Phi \in \Omega^4(\mathbb{R}^8)$ [HL82], which singles out 4-dimensional submanifold embeddings $\Sigma_4 \hookrightarrow \mathbb{R}^8$ as the corresponding *calibrated submanifolds*. The space of all such *Cayley 4-planes*, canonically a subspace of the Grassmannian space $\text{Gr}(4, 8)$ of *all* 4-planes in 8 dimensions, is denoted

$$\text{CAY} \subset \text{Gr}(4, 8) \tag{87}$$

in [BH89, (2.19)][GMM95, (5.20)]. We will write

$$\text{CAY}_{\text{sL}} \subset \text{CAY} \subset \text{Gr}(4, 8) \tag{88}$$

for the further subspace of those Cayley 4-planes which are also special Lagrangian submanifolds. There are canonical symmetry actions of $\text{Spin}(7)$ and of $\text{Spin}(6)$, respectively, on these spaces [HL82, Prop. 1.36]:

$$\begin{array}{ccc} \text{Spin}(7) & & \text{Spin}(6) \\ \curvearrowright & \text{and} & \curvearrowright \\ \text{CAY} & & \text{CAY}_{\text{sL}} \end{array} \tag{89}$$

Hence the corresponding homotopy quotients

$$\text{CAY} // \text{Spin}(7) \quad \text{and} \quad \text{CAY}_{\text{sL}} // \text{Spin}(6) \tag{90}$$

are the *moduli spaces* for Cayley 4-planes and for special Lagrangian Cayley 4-planes, respectively: for X a $\text{Spin}(7)$ -manifold, a dashed lift in

$$\begin{array}{ccc} & \text{CAY} // \text{Spin}(7) & \\ \text{X} \dashrightarrow & \downarrow & \\ & \text{BSpin}(7) & \end{array} \quad \begin{array}{ccc} & \text{CAY}_{\text{sL}} // \text{Spin}(6) & \\ \text{X} \dashrightarrow & \downarrow & \\ & \text{BSpin}(6) & \end{array}$$

is a distribution on X by tangent spaces to (special Lagrangian) calibrated submanifolds.

Proposition 2.27 (Calibrations from twisted cohomotopy). *The moduli spaces of (special Lagrangian) Cayley 4-planes (90) are compatibly weakly homotopy equivalent to the coefficient spaces for twisted Cohomotopy jointly in degrees 4 and 7, according to Prop. 2.20:*

$$\begin{array}{ccc} \text{CAY}_{\text{sL}} // \text{Spin}(6) & \simeq & S^7 // (\text{Sp}(2) \cdot \text{Sp}(1)) \\ \downarrow & & \downarrow \\ \text{CAY} // \text{Spin}(7) & \simeq & S^4 // (\text{Sp}(2) \cdot \text{Sp}(1)) \end{array}$$

Proof. By [HL82, Theorem 1.38] (see also [BH89, (3.19)], [GMM95, (5.20)]) we have a coset space realization

$$\text{CAY} \simeq \text{Spin}(7)/(\text{Spin}(4) \cdot \text{Spin}(3)) .$$

and by [BBMOOY96, p. 7] we have a coset space realization

$$\text{CAY}_{\text{sl}} \simeq \text{Spin}(6)/(\text{Spin}(3) \cdot \text{Spin}(3)) \simeq \text{SU}(6)/\text{SO}(4) .$$

By Lemma 2.7 this means equivalently that there are weak homotopy equivalences

$$\text{CAY} // \text{Spin}(7) \simeq B(\text{Spin}(4) \cdot \text{Spin}(3)) \simeq B(\text{Sp}(1) \cdot \text{Sp}(1) \cdot \text{Sp}(2))$$

and

$$\text{CAY}_{\text{sl}} // \text{Spin}(6) \simeq B(\text{Spin}(3) \cdot \text{Spin}(3)) \simeq B(\text{Sp}(1) \cdot \text{Sp}(1)) .$$

This then implies the claim by Prop. 2.22. □

Vanishing PT-charge in twisted Cohomotopy. Even without discussing a full generalization of the untwisted Pontrjagin-Thom theorem (85) to the case of twisted Cohomotopy (Def. 2.1), we may say what it means for a cocycle in twisted Cohomotopy to correspond to the empty submanifold, hence to correspond to vanishing brane charge in the sense discussed above. This is all that we will need to refer to below in §3.3 and §3.8:

- (i) In the case of untwisted cohomotopy it is immediate that the zero-charge cocycle is simply the one represented by any function that does not meet the given base point $\text{pt} \in S^n$ (86).
- (ii) In the case of twisted Cohomotopy according to Def. 2.1, this chosen point must be a chosen *section* of the given spherical fibration corresponding to the given twist τ :

$$\begin{array}{ccc} & & S^n // \text{O}(n+1) \\ & \nearrow \text{pt} & \downarrow \\ X & \xrightarrow{\tau} & \text{BO}(n+1) \end{array}$$

which serves over each $x \in X$ as the point $\text{pt}_x \in E_x \simeq S^4$ at which we declare to form the inverse image of another given section, under a parametrized inverse Pontrjagin-Thom construction.

- (iii) With that section pt chosen, any other twisted Cohomotopy cocycle $[c_0] \in \pi^\tau(X)$ which will yield the empty submanifold under parametrized Pontrjagin-Thom must be represented by a section c_0 which is everywhere distinct from pt ,

$$c_0(x) \neq \text{pt}_x$$

so that $c_0^{-1}(\text{pt}(x)) = \emptyset$ for all $x \in X$.

- (iv) But such a choice of a pair of pointwise distinct sections is equivalently a reduction of the structure group not just along $\text{O}(4) \hookrightarrow \text{O}(5)$ as in Remark 2.8, but is rather a reduction all the way along $\text{O}(3) \hookrightarrow \text{O}(5)$.

Specified to the $\text{Sp}(2) \cdot \text{Sp}(1)$ -twisted Cohomotopy jointly in degrees 4 and 7, from §2.3 this says that vanishing of the brane charge seen by degree 4 Cohomotopy cocycle via a putative parameterized PT theorem is witnessed by a lift from $B(\text{Spin}(5) \cdot \text{Spin}(3))$ all the way to $B(\text{Spin}(3) \cdot \text{Spin}(3))$. But comparison with Prop. 2.22 (see also *Figure T*) shows the following:

Lemma 2.28 (Vanishing of Cohomotopy charge means factorization through h_{III}). *The vanishing of cohomotopical brane charge of $\text{Sp}(2) \cdot \text{Sp}(1)$ -twisted Cohomotopy in degree 4 (§2.3), in the sense of the above parametrized*

Pontrjagin-Thom construction of corresponding branes, is exhibited by factorizations of the degree-4 cocycle through degree-7 Cohomotopy, via the equivariant quaternionic Hopf fibration $h_{\mathbb{H}}$ of Prop. 2.22:

$$\begin{array}{ccccc}
 & & S^7 // (\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)) & \xrightarrow{\cong} & B(\mathrm{Spin}(3) \cdot \mathrm{Spin}(3)) \\
 & \nearrow \text{PT-vanishing of} & \downarrow h_{\mathbb{H}} // (\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)) & & \downarrow \\
 & \text{cocycle in} & & & \\
 & \text{twisted Cohomotopy} & & & \\
 & \text{in degree 4} & & & \\
 & \nearrow & S^4 // (\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)) & \xrightarrow{\cong} & B(\mathrm{Spin}(4) \cdot \mathrm{Spin}(3)) \\
 & \text{cocycle in} & \downarrow & & \downarrow \\
 & \text{twisted} & & & \\
 & \text{Cohomotopy} & & & \\
 & \text{in degree 4} & & & \\
 X & \xrightarrow{\tau} & B(\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)) & \xrightarrow{\cong_{\text{Btri}}} & B(\mathrm{Spin}(5) \cdot \mathrm{Spin}(3)) .
 \end{array} \tag{91}$$

We come back to this in Prop. 3.14 and Prop. 3.20 below.

This concludes our discussion of general properties of twisted Cohomotopy theory. Now we turn, in §3, to discussing how, under *Hypothesis H*, these serve to yield anomaly cancellation in M-theory.

3 C-field charge-quantized in twisted Cohomotopy

We consider now the setup of M-theory on 8-manifolds:

Remark 3.1. For M-Theory on 8-manifolds [Wi95b][BB96][SVW96], spacetime is of the form $\mathbb{R}^{2,1} \times X^8$, corresponding to a background of parallel M2-branes which appear as singular points in the 8-dimensional space X^8 , or else as points that would be singular were they included in X^8 . See also [Ts06] [CMP13][PT13][BL14a][Sh15][BL14b][BL14c][BL14d].

M-theory on 8-manifolds with $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ -structure (as in Def. 3.5 below), specifically on the quaternionic projective plane $\mathbb{H}P^2$ [MV19, 4.3] (see also Example 3.2 below), has been argued in [AW03, pp. 75] to be dual to 4d M-theory on G_2 -manifolds in three different ways, such as to plausibly yield proof of confinement in 4d gauge theory.

If the 8-manifold X^8 is elliptically fibered then M-theory on X^8 has been argued to be T-dual to phenomenologically interesting F-theory compactifications on spacetimes of the form $\mathbb{R}^{3,1} \times \tilde{X}^8$ [CMP13][BGPP13]:

$$\begin{array}{ccc}
 \begin{array}{c} \text{M-theory} \\ \text{on 8-manifolds} \end{array} & \xleftarrow{\text{T-duality}} & \begin{array}{c} \text{F-theory on} \\ \text{8-manifolds} \end{array} \\
 \mathbb{R}^{2,1} \times X^8 & & \mathbb{R}^{3,1} \times \tilde{X}^8
 \end{array}$$

In particular, for X^8 of $\mathrm{Spin}(7)$ -structure, the resulting $\mathcal{N} = 1$ supersymmetry in 3d on the left is argued [Wi95b][Wi95c, 3][Va96, 4.3][Wi00, p. 7] to be dual to a peculiar “ $\mathcal{N} = 1/2$ ” supersymmetry in 4d on the right, which does enforce a vanishing cosmological constant, but does not constrain the finite energy particle spectrum to be supersymmetric. This is developed in [BGP13][BGPP13][HLLZ19][HLLSZ19].

For our purposes, the following is concretely the data in question:

Definition 3.2 (The 8-manifold X^8). We consider X^8 to be a smooth 8-dimensional spin-manifold, possibly with boundary, which is connected and simply connected. Let ∇ be any affine connection on the tangent bundle TX^8 . We assume that $H^2(X^8, \mathbb{Z}_2) = 0$.

Remark 3.3 (Role of technical assumptions on the 8-manifold). We highlight the following:

(i) The assumption in Def. 3.2 that X^8 be connected is convenient but immaterial and easily dropped.

(ii) The assumption that X^8 be simply connected should also be immaterial, but is not so easily dropped: All proofs invoking Sullivan models in the following should generalize at least to nilpotent fundamental groups, but will be much harder without this assumption.

(iii) The choice of affine connection ∇ in Def. 3.2 is just to bring in Chern-Weil theory and only affects the explicit representatives of characteristic forms in the following, not any of the gauge/homotopy invariant statements.

(iv) The assumption $H^2(X, \mathbb{Z}_2) = 0$ bluntly ensures that any specific obstruction classes that could appear in this group vanishes. This is used only in §3.2 and §3.4 below, and in §3.4 we only need that the specific obstruction class $\varpi \in H^2(X, \mathbb{Z}_2)$ to direct product $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ -structure vanishes (from Prop. 2.15). With this class thus assumed to vanish, there is no essential loss of generality in assuming $\mathrm{Sp}(2)$ -structure.

Example 3.4. The quaternionic projective plane $X^8 = \mathbb{H}P^2$ satisfies the assumptions of Def. 3.2. To see this, it is sufficient to observe that it is homotopy equivalent to the result of gluing an 8-cell to a 4-cell (with attaching map being the quaternionic Hopf fibration)

$$\begin{array}{ccc} S^7 & \xrightarrow{h_{\mathbb{H}}} & S^4 \\ \downarrow & \text{(po)} & \downarrow \\ \mathbb{D}^8 & \longrightarrow & \mathbb{H}P^2 \end{array}$$

This cell structure immediately implies vanishing of all cohomology in degree ≤ 3 .

Definition 3.5 (*Hypothesis H* for M-theory on 8-manifolds). Given an 8-manifold X^8 (Def. 3.2) we say that a pair of differential forms (G_4, G_7) on X^8 satisfies *Hypothesis H* if it is in the image of the non-abelian Chern character map (Def. 2.4) from J-twisted 4-Cohomotopy

$$\begin{array}{ccc} \pi^{i_7 \circ \tau}(X^8) & & \\ h_* \downarrow & & \\ \pi^{i_4 \circ \tau}(X^8) & \xrightarrow{L_{\mathbb{R}}} & \pi^{i_4 \circ \tau}(X^8)_{\mathbb{R}} \longleftarrow \{ (G_4, G_7) \mid \dots \} \hookrightarrow \Omega^4(X^8) \times \Omega^7(X^8) \\ [c] \longmapsto & \longrightarrow & [(G_4, G_7)] \end{array}$$

for twists compatible with the quaternionic Hopf fibration, which by Prop. 2.20 means that τ is a topological $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ -structure on X^8 , via (55).

We now discuss some consequences of Hypothesis H, as summarized in *Table 1*.

3.1 Special G -structures

We discuss how Hypothesis H implies $\mathcal{N} = 1$ G -structure as in (14).

Parallel spinors and G -structure. Conditions on a compactification manifold to admit suitably parallel spinor sections and hence preserve some amount of supersymmetry have commonly been phrased in terms of special holonomy metrics (see e.g. [Gu02]). But more generally, in the potential presence of flux, an alternative characterization is in terms of G -structure, i.e. reductions of the structure group of the tangent/frame bundle. This was used already in the classical [IPW88] but received more attention after it was re-amplified in the context of flux compactifications in [GMPW04, Sec. 2], see also [Koe11, Sec. 2][Ga11][DDG14]. Discussion of G -structure specifically in the context of M-theory on 8-manifolds (Remark 3.1) includes [Ts06][CMP13][PT13][BL14a][Sh15][BL14b][BL14c][BL14d].

Proposition 3.6. *Let X^d be a spin-manifold of dimension $d \in \{5, 6, 7, 8\}$. Then cocycles in J-twisted 7-Cohomotopy (Def. 2.1) are equivalent to topological G -structures on X^d as follows:*

$$\left[\begin{array}{ccc} X^d & \overset{\exists}{\dashrightarrow} & S^7 // \mathrm{Spin}(d) \\ \swarrow TX & \searrow & \downarrow \\ & & B\mathrm{Spin}(d) \end{array} \right] \in \pi^{i_7 \circ TX}(X^d) \Leftrightarrow \text{topological } G\text{-structure for } G = \begin{cases} \mathrm{Spin}(7) & | & d = 8 \\ \mathrm{G}_2 & | & d = 7 \\ \mathrm{SU}(3) & | & d = 6 \\ \mathrm{SU}(2) & | & d = 5 \end{cases} \quad (92)$$

hence are equivalent precisely to those G -structures that correspond to $\mathcal{N} = 1$ compactifications of F -theory, M -theory, and string theory, respectively. (e.g. [AG4][BBS10][GSZ14]).

Proof. This is Prop. 2.23 used in Prop. 2.8 □

3.2 DMW anomaly cancellation

We prove that Hypothesis H implies the DMW anomaly cancellation condition (13):

Proposition 3.7. *Let X^8 be an 8-manifold as in Def. 3.2. Then existence of topological $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ -structure on X^8 , as in Hypothesis H (Def. 3.5) implies the following:*

(i) *The Euler class of the tangent bundle is proportional to the one-loop anomaly polynomial (X_8 in [DLM95, (1.2)], cf. [Wi96b, (5.5)]):*

$$\frac{1}{24}\chi_8(TX^8) = I_8(TX^8) := \frac{1}{48}(p_2(TX^8) - \frac{1}{4}(p_1(TX^8))^2) \in H^8(X^8, \mathbb{R}). \quad (93)$$

(ii) *The degree-6 Stiefel-Whitney class vanishes:*

$$w_6(TX^8) = 0 \in H^6(X^8, \mathbb{Z}_2), \quad (94)$$

and hence so does the degree-7 integral Stiefel-Whitney class $W_7 := \beta(w_6)$:

$$W_7(TX^8) = 0 \in H^7(X^8, \mathbb{Z}). \quad (95)$$

Proof. This follows by applying [CV98b, Thm. 8.1 & Rem. 8.2]. □

3.3 Curvature-corrected Bianchi identity

We prove that Hypothesis H implies the higher curvature corrected Bianchi identities (19) (20).

Proposition 3.8 (Higher curvature corrections via Cohomotopy). *Let X^8 be an 8-manifold as in Def. 3.2. Then:*

(i) *The general form of the rationally twisted rational Cohomotopy sets in degrees 4 and 7 is as in (19) and (20).*

(ii) *If the differential forms (G_4, G_7) moreover satisfy Hypothesis H (Def. 3.5), then the Cohomotopy set is concretely given as follows:*

$$\pi^{i_4 \circ \tau}(X^8)_{\mathbb{R}} \simeq \left\{ \begin{array}{l} dG_4 = 0 \\ dG_7 = -\frac{1}{2}\tilde{G}_4 \wedge (\tilde{G}_4 - \frac{1}{2}p_1(\nabla)) - 12 \cdot I_8(\nabla) \end{array} \right\} / \sim \longleftarrow \{G_4, G_7 | \dots\} \hookrightarrow \Omega^4(X^8) \times \Omega^7(X^8),$$

where $\tilde{G}_4 := G_4 + \frac{1}{4}p_1(\nabla)$ from (108) and $I_8 = \frac{1}{48}(p_2 - \frac{1}{4}p_1^2)$ from (93).

Proof. The first statement is the specialization of Prop. 2.5 to degrees 4&7. For the second statement it then remains to re-express the class $\frac{1}{4}p_2$ of the effective $O(5)$ -twist (42) to the corresponding class of the given tangential $\mathrm{Sp}(2)$ -twist as we pass through triality (Prop. 2.17)

$$\begin{array}{ccccc} & & B\mathrm{Sp}(2) & \xrightarrow{\cong} & B\mathrm{Spin}(5) \\ & \nearrow \tau & \downarrow & & \downarrow \\ X^8 & \xrightarrow{TX^8} & B\mathrm{Spin}(8) & \xrightarrow[B_{\mathrm{Tri}}]{\cong} & B\mathrm{Spin}(8), \end{array} \quad (96)$$

with the delooped triality automorphism (68) shown on the right. We claim that this is the difference between the Euler class and the squared first fractional Pontrjagin class of X^8 :

$$\frac{1}{4}p_2(\text{Btri}_*(\tau)) = \left(\frac{1}{4}p_1(TX^8)\right)^2 - \chi_8(TX^8). \quad (97)$$

This follows by combining (70) from Prop. 2.19 with the $\text{Sp}(2)$ -structure relation (93)

$$\frac{1}{2}(p_2 - (\frac{1}{4}p_1)^2) = \chi_8 \quad \text{on } B\text{Sp}(2) \quad (98)$$

from Prop. 3.7. Inserting this in (42) yields the claim by completing the square on the right of the Bianchi identity:

$$\begin{aligned} dG_7 &= -\frac{1}{2}G_4 \wedge G_4 + \frac{1}{2}\frac{1}{4}p_2(\text{Btri}_*\nabla) \\ &= -\frac{1}{2}G_4 \wedge G_4 + \frac{1}{2}\frac{1}{4}p_1(\nabla) \wedge \frac{1}{4}p_1(\nabla) - \frac{1}{2}\chi_8(\nabla) \\ &= -\frac{1}{2}\underbrace{\left(G_4 + \frac{1}{4}p_1(\nabla)\right)}_{=\tilde{G}_4} \wedge \underbrace{\left(G_4 - \frac{1}{4}p_1(\nabla)\right)}_{=\tilde{G}_4 - \frac{1}{2}p_1(\nabla)} - \underbrace{\frac{1}{2}\chi_8(\nabla)}_{=12 \cdot I_8(\nabla)}. \end{aligned}$$

(Recalling that under the braces we use (108) and (93).) □

3.4 Shifted 4-flux quantization

We prove that Hypothesis H implies the shifted flux quantization condition (109). The result is Prop. 3.13 below. The basic observation that makes this work is highlighted in Remark 3.10 below. To put this to full use we need to go into some technicalities in Lemma 3.11 and Lemma 3.12 below.

First we recall some classical facts about the integral cohomology of $B\text{Spin}(n)$ for low n :

Lemma 3.9. (i) *The integral cohomology ring of $BSO(3)$ is*

$$H^*(BSO(3); \mathbb{Z}) \simeq \mathbb{Z}[p_1, W_3]/(2W_3), \quad (99)$$

and the integral cohomology of $B\text{Spin}(3)$ is free on one generator

$$H^*(B\text{Spin}(3); \mathbb{Z}) \cong \mathbb{Z}[\frac{1}{4}p_1], \quad (100)$$

while the integral cohomology ring of $B\text{Spin}(4)$ is free on two generators

$$H^*(B\text{Spin}(4); \mathbb{Z}) \simeq \mathbb{Z}[\underbrace{\frac{1}{2}p_1, \frac{1}{2}\chi_4 + \frac{1}{4}p_1}_{=\Gamma_4}], \quad (101)$$

$\underbrace{\hspace{10em}}_{=\tilde{\Gamma}_4}$

where p_1 is the first Pontrjagin class and χ_4 the Euler class.

(ii) *Under the exceptional isomorphism $\vartheta : \text{Spin}(3) \times \text{Spin}(3) \xrightarrow{\cong} \text{Spin}(4)$ these classes are related by*

$$\begin{aligned} \vartheta^*\left(\frac{1}{2}p_1\right) &= \frac{1}{4}p_1 \otimes 1 + 1 \otimes \frac{1}{4}p_1, \\ \vartheta^*\left(\frac{1}{2}\chi + \frac{1}{4}p_1\right) &= \frac{1}{4}p_1 \otimes 1, \\ \text{hence } \vartheta^*(\chi) &= \frac{1}{4}p_1 \otimes 1 - 1 \otimes \frac{1}{4}p_1. \end{aligned} \quad (102)$$

Proof. This follows from classical results [Pi91]. More explicitly, (99) is a special case of [Br82, Thm. 1.5], recalled for instance as [RS17, Thm. 4.2.23 with Remark 4.2.25]. The other statements are recalled for instance in [CV98a, Lemma 2.1]. □

Remark 3.10 (Universal avatar of the integral C-field). We highlight from (101), under the braces, the universal integral class

$$\tilde{\Gamma}_4 := \underbrace{\frac{1}{2}\chi_4 + \frac{1}{4}p_1}_{=:\Gamma_4} \in H^4(B\text{Spin}(4); \mathbb{Z}) \quad (103)$$

for use below. Prop. 3.13 below says that, under Hypothesis H, these universal characteristic classes are the avatars of the half-integral shifted C-field flux \tilde{G}_4 . Since $\tilde{\Gamma}_4$ is an integral cohomology class, its pullback to any given spacetime is an integral class, and such that its image in de Rham cohomology is $[G_4 + \frac{1}{4}p_1(\nabla)]$. This integral lift is what implements the shifted C-field flux quantization condition in M-theory §3.4.

We now trace the integral generator $\tilde{\Gamma}_4$ in (103) to the larger group $\text{Spin}(5) \cdot \text{Spin}(3)$.

Lemma 3.11 (Cohomology of the central group). *The integral cohomology in degree 4 of the classifying space of the central product group (62)*

$$\text{Spin}(4) \cdot \text{Spin}(3) \simeq \text{Spin}(3) \cdot \text{Spin}(3) \cdot \text{Spin}(3)$$

is the free lattice

$$H^4(B(\text{Spin}(4) \cdot \text{Spin}(3)); \mathbb{Z}) \simeq \mathbb{Z} \left\langle \begin{array}{ccc} \frac{1}{4}p_1^{(1)} & + & \frac{1}{4}p_1^{(2)} & + & \frac{2}{4}p_1^{(3)}, \\ \frac{1}{4}p_1^{(1)} & + & \frac{2}{4}p_1^{(2)} & + & \frac{1}{4}p_1^{(3)}, \\ \frac{2}{4}p_1^{(1)} & + & \frac{1}{4}p_1^{(2)} & + & \frac{1}{4}p_1^{(3)} \end{array} \right\rangle \quad (104)$$

where $p_1^{(k)} := (\text{Bpr}_k)^*(p_1)$ is the pullback of the first Pontrjagin class along the projection (59)

$$B(\text{Spin}(4) \cdot \text{Spin}(3)) \simeq B(\text{Spin}(3) \cdot \text{Spin}(3) \cdot \text{Spin}(3)) \xrightarrow{\text{Bpr}_k} B\text{SO}(3).$$

Proof. The defining short exact sequence of groups (Def. 2.11)

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Spin}(3) \cdot \text{Spin}(3) \cdot \text{Spin}(3) \longrightarrow \text{Spin}(3) \times \text{Spin}(3) \times \text{Spin}(3) \longrightarrow 1$$

induces a homotopy fiber sequence of classifying spaces (e.g. [Mi11, 11.4])

$$B\mathbb{Z}_2 \longrightarrow B(\text{Spin}(3) \times \text{Spin}(3) \times \text{Spin}(3)) \longrightarrow B(\text{Spin}(3) \cdot \text{Spin}(3) \cdot \text{Spin}(3)).$$

The corresponding Serre spectral sequence shows that

$$\begin{aligned} H^4(B(\text{Spin}(3) \cdot \text{Spin}(3) \cdot \text{Spin}(3)); \mathbb{Z}) &\hookrightarrow H^4(B(\text{Spin}(3) \times \text{Spin}(3) \times \text{Spin}(3)), \mathbb{Z}) \\ &\simeq \mathbb{Z} \langle \frac{1}{4}p_1^{(1)}, \frac{1}{4}p_1^{(2)}, \frac{1}{4}p_1^{(3)} \rangle \end{aligned}$$

is a sublattice of index 4. This sublattice must include the integral class $\frac{1}{2}p_1$ pulled back along the inclusion into $\text{Spin}(7)$, which by Lemma 3.9 is

$$\begin{aligned} B(\text{Spin}(4) \cdot \text{Spin}(3)) &\longrightarrow B\text{Spin}(7) . \\ \frac{1}{4}p_1^{(1)} + \frac{1}{4}p_1^{(2)} + \frac{2}{4}p_1^{(3)} &\longleftarrow \frac{1}{2}p_1 \end{aligned} \quad (105)$$

But then it must also contain the images of this element under the delooping of the S_3 -automorphisms (63). This yields the other two elements shown in (104). Finally, it is clear that the sublattice spanned by these three elements already has full rank and index 4:

$$\mathbb{Z} \left\langle \begin{array}{ccc} \frac{1}{4}p_1^{(1)} & + & \frac{1}{4}p_1^{(2)} & + & \frac{2}{4}p_1^{(3)}, \\ \frac{1}{4}p_1^{(1)} & + & \frac{2}{4}p_1^{(2)} & + & \frac{1}{4}p_1^{(3)}, \\ \frac{2}{4}p_1^{(1)} & + & \frac{1}{4}p_1^{(2)} & + & \frac{1}{4}p_1^{(3)} \end{array} \right\rangle \simeq \left\{ \frac{a}{4}p_1^{(1)} + \frac{b}{4}p_1^{(2)} + \frac{c}{4}p_1^{(3)} \mid a, b, c \in \mathbb{Z}, a + b + c = 0 \pmod{4} \right\} \quad (106)$$

which means that there are no further generators. □

As a direct consequence we obtain the following identification.

Lemma 3.12 (Integral classes). *The following cohomology class on the classifying space of the group $\text{Spin}(4) \cdot \text{Spin}(3)$ (62), which a priori is in rational cohomology, is in fact integral:*

$$\underbrace{\frac{1}{2}\mathcal{X}_4 + \frac{1}{4}p_1}_{=:\tilde{\Gamma}_4} + \frac{1}{2}p_1^{(3)} \in H^4(\text{Spin}(4) \cdot \text{Spin}(3); \mathbb{Z})$$

and hence so is its image on the classifying space of $\text{Sp}(1) \cdot \text{Sp}(1) \cdot \text{Sp}(1)$ (61) under the delooping of the triality isomorphism from Prop. 2.17, which we will denote by the same symbols:

$$\underbrace{\frac{1}{2}\mathcal{X}_4 + \frac{1}{4}p_1}_{=:\tilde{\Gamma}_4} + \frac{1}{2}p_1^{(3)} \in H^4(\text{Sp}(1) \cdot \text{Sp}(1) \cdot \text{Sp}(1); \mathbb{Z}) \simeq H^4(\text{Spin}(4) \cdot \text{Spin}(3), \mathbb{Z}). \quad (107)$$

Here $\frac{1}{2}\mathcal{X}_4$ is the Euler class pulled back back from the left $\text{BSO}(4)$ factor and $p_1^{(3)}$ is the first Pontrjagin class pulled back from the right $\text{BSO}(3)$ factor, both along the respective projections (59), while p_1 is the first Pontrjagin class pulled back from the ambient $\text{BSpin}(8)$ along the canonical inclusion (58):

$$\begin{array}{ccccc} & & B(\text{Spin}(4) \cdot \text{Spin}(3)) & & \\ & \swarrow^{B\text{pr}_4} & \downarrow^{B\iota_8} & \searrow^{B\text{pr}_3} & \\ \text{BSO}(4) & & \text{BSpin}(8) & & \text{BSO}(3) \\ \mathcal{X}_4 & & p_1 & & p_1^{(3)} \end{array}$$

Proof. In terms of the contributions from the three factors under the identification $\text{Spin}(4) \cdot \text{Spin}(3) \simeq \text{Spin}(3) \cdot \text{Spin}(3) \cdot \text{Spin}(3)$ the class in question is

$$\underbrace{\frac{1}{8}p_1^{(1)} - \frac{1}{8}p_1^{(2)}}_{=\frac{1}{2}\mathcal{X}_4} + \underbrace{\frac{1}{8}p_1^{(1)} + \frac{1}{8}p_1^{(2)} + \frac{1}{4}p_1^{(3)}}_{=\frac{1}{4}p_1} + \frac{2}{4}p_1^{(3)} = \frac{1}{4}p_1^{(1)} + \frac{3}{4}p_1^{(3)},$$

where under the braces we used Lemma 3.9 as in (105). The equivalent expression on the right makes manifest that this is in the sublattice (106). Therefore, Lemma 3.11 implies the claim. \square

Now we may finally state and prove the main result of this section.

Proposition 3.13 (Integrality of the shifted class). *Let X^8 be a 8-manifold as in Def. 3.2. If a differential 4-form G_4 on X^8 satisfies Hypothesis H (Def. 3.5), then its shift by 1/4th the first Pontrjagin form*

$$\tilde{G}_4 := G_4 + \frac{1}{4}p_1(\nabla) \quad (108)$$

is integral:

$$[G_4 + \frac{1}{4}p_1(\nabla)] = [G_4] + \frac{1}{4}p_1(TX^8) \in H^4(X^8; \mathbb{Z}) \rightarrow H^4(X^8; \mathbb{R}). \quad (109)$$

Proof. Notice that, by the assumption $H^2(X^8, \mathbb{Z}_2) = 0$ in Def. 3.2, it follows in particular that the characteristic class ϖ , from Def. 2.14, vanishes:

$$\varpi(\tau) \in H^2(X^8; \mathbb{Z}_2) = 0. \quad (110)$$

With this, the proof proceeds by considering the following diagram, which we will discuss below in stages:

$$\begin{array}{ccccccc}
& & & & \overbrace{\frac{1}{2}\mathcal{X}_4 + \frac{1}{4}p_1 + \frac{1}{2}p_1^{(3)}}^{\tilde{\Gamma}_4} & & \\
& & & & \downarrow & & \\
& & & & B(\mathrm{Sp}(1) \cdot \mathrm{Sp}(1) \cdot \mathrm{Sp}(1)) \xrightarrow{\simeq} B(\mathrm{Spin}(4) \cdot \mathrm{Spin}(3)) \xrightarrow{Bpr_4} BSO(4) & & \\
& & & & \downarrow & & \downarrow \\
& & & & B(\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)) \xrightarrow{\simeq} B(\mathrm{Spin}(5) \cdot \mathrm{Spin}(3)) \xrightarrow{Bpr_5} BSO(5) & & \\
& & & & \downarrow & & \downarrow \\
[G_4] + \frac{1}{4}p_1(TX^8) & \xrightarrow{c} & & & & & \\
& & & & & & \\
X & \xrightarrow{TX} & B\mathrm{Spin}(8) & \xrightarrow[\simeq]{Btri} & B\mathrm{Spin}(8) & \longrightarrow & BSO(8) \\
& & \downarrow & & \downarrow & & \downarrow \\
p_1(TX^8) & \xleftarrow{p_1} & & & & & p_1
\end{array} \tag{111}$$

cocycle in τ -twisted Cohomology

Here the vertical maps are the deloopings of the canonical group inclusions (Remark 2.18) and the horizontal equivalences $Btri$ are the deloopings (68) of the respective triality automorphism from Prop. 2.17, while the horizontal maps Bpr_n are the deloopings of the canonical projections (59). On the left we used that, by Def. 2.1, an element

$$[c] \in \pi^\tau(X^8)$$

in the τ -twisted Cohomology of X^8 is the homotopy class of a section c of the S^4 -bundle classified by $Bpr_5 \circ Btri \circ \tau$:

$$\begin{array}{ccc}
S^4 & \longrightarrow & E \\
& \nearrow c & \downarrow \pi \\
& & X^8 \\
& & \xrightarrow{Bpr_5 \circ Btri \circ \tau} BSO(5)
\end{array}
\quad
\begin{array}{ccc}
& & BSO(4) \simeq S^4 // SO(5) \\
& & \downarrow \\
& & BSO(5)
\end{array}$$

(pb)

and we used Prop. 2.22 to identify various homotopy quotients of S^4 with classifying spaces, as shown. This shows that E is the unit sphere bundle of a rank 5 real vector bundle V classified by $Bpr_5 \circ Btri \circ c$. Therefore, by Prop. 2.5 we have

$$\pi^*[G_4] = \frac{1}{2}\mathcal{X}_4(\widehat{V}),$$

where \widehat{V} is defined by the splitting $\pi^*V = \mathbb{R}_E \oplus \widehat{V}$ determined by the tautological section of π^*V over E , i.e., it is the rank 4 real vector bundle on E classified by $E \rightarrow BSO(4)$. Hence, by (107) in Lemma 3.12, we have that

$$\pi^* \left(\underbrace{[G_4] + \frac{1}{4}p_1(Btri \circ TX^8) + \frac{1}{2}p_1^{(3)}(Btri \circ \tau)}_{=:K} \right) \in H^4(E; \mathbb{Z})$$

is an integral class.

We now claim that the class K is integral already before the pullback, as a class on X . For this, consider the commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & H^4(X^8; \mathbb{Z}) & \longrightarrow & H^4(X^8; \mathbb{R}) & \xrightarrow{q} & H^4(X^8; \mathbb{R}/\mathbb{Z}) & \longrightarrow \cdots \\
& & \downarrow \pi^* & & \downarrow \pi^* & & \downarrow \pi^* & \\
\cdots & \longrightarrow & H^4(E; \mathbb{Z}) & \longrightarrow & H^4(E; \mathbb{R}) & \xrightarrow{q} & H^4(E; \mathbb{R}/\mathbb{Z}) & \longrightarrow \cdots
\end{array}$$

induced by the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow 0$. From the Serre spectral sequence for the fibration $\pi: E \rightarrow X$ one sees that the vertical maps in the above diagram are injective. Consequently, from

$$\begin{aligned}
\pi^*q(K) &= q\pi^*(K) \\
&= 0
\end{aligned}$$

it follows that already $q(K) = 0$, which means that K itself is integral:

$$[G_4] + \frac{1}{4}p_1(B\text{tri} \circ TX^8) + \frac{1}{2}p_1^{(3)}(B\text{tri} \circ \tau) \in H^4(X^8; \mathbb{Z}). \quad (112)$$

Now observe that the third summand in (112) is the first fractional Pontrjagin class of the underlying $\text{SO}(3)$ -bundle. By the assumption (110) this admits Spin structure, by Lemma 2.14. This in turn implies that its first Pontrjagin class is divisible by two, hence that the last summand in (112) is integral by itself

$$\frac{1}{2}p_1^{(3)}(B\text{tri} \circ \tau) \in H^4(X^8; \mathbb{Z}),$$

and hence that also the remaining summand

$$[G_4] + \frac{1}{4}p_1(B\text{tri} \circ TX^8) \in H^4(X^8; \mathbb{Z}) \quad (113)$$

is integral by itself. Finally, pullback along the triality automorphism preserves the first Pontrjagin class, by Lemma 2.19

$$p_1(B\text{tri} \circ \tau) = p_1(TX^8) \quad (114)$$

and hence (113) indeed becomes $[G_4] + \frac{1}{4}p_1(TX^8) \in H^4(X^8; \mathbb{Z})$. \square

3.5 Background charge

We prove that Hypothesis H implies the background charge (24) of the 4-flux.

Proposition 3.14 (Cohomotopically vanishing 4-flux form). *Let X^8 be a smooth 8-manifold which is simply connected (Remark 2.6) and equipped with topological $\text{Sp}(2)$ -structure τ (Example 2.12). Then, if a cocycle in τ -twisted Cohomotopy (Def. 2.1) has a factorization through the quaternionic Hopf fibration, exhibiting its vanishing PT-charge according to (91) in §2.6, it follows that the differential 4-form G_4 by Def. 3.5 has value*

$$G_4 = \frac{1}{4}p_1(\nabla\tau).$$

Consequently, the corresponding integral 4-form \tilde{G}_4 (109) from Prop. 3.13 has class $\frac{1}{2}p_1$:

$$\begin{array}{ccc} & \begin{array}{c} S^7 // \text{Sp}(2) \\ \downarrow h_{\mathbb{H}} // \text{Sp}(2) \\ S^4 // \text{Sp}(2) \\ \downarrow \\ B\text{Sp}(2) \\ \downarrow \\ B\text{Spin}(8) \end{array} & \\ \begin{array}{c} \exists \\ \dashrightarrow \\ \text{c} \end{array} & \begin{array}{c} \dashrightarrow \\ \dashrightarrow \\ \xrightarrow{\tau} \\ \searrow^{TX^8} \end{array} & \\ X^8 & \begin{array}{c} B\text{Sp}(2) \\ \downarrow \\ B\text{Spin}(8) \end{array} & \\ & \Rightarrow [\tilde{G}_4] = \frac{1}{2}p_1(TX^8) \in H^4(X^8; \mathbb{R}). & \end{array}$$

Proof. By Prop. 2.22, the cocycle c in degree 4 twisted Cohomotopy itself is equivalently further reduction of τ to topological $\text{Sp}(1) \cdot \text{Sp}(1) \cdot \mathbb{Z}_2$ -structure (Example 2.16). Similarly, the assumed factorization through degree-7 Cohomotopy is equivalently existence of yet further reduction to topological $\text{Sp}(1) \cdot \mathbb{Z}_2$ -structure, via inclusion of the first factor

$$\begin{array}{ccccc} S^7 // \text{Sp}(2) & \xrightarrow{\simeq} & B(\text{Sp}(1) \cdot \mathbb{Z}_2) & \simeq & B\text{Sp}(1) \\ \downarrow h_{\mathbb{H}} // \text{Sp}(2) & & \downarrow B([q,1] \rightarrow [q,1,1]) & & \downarrow B(q \rightarrow (q,1)) \\ S^4 // \text{Sp}(2) & \xrightarrow{\simeq} & B(\text{Sp}(1) \cdot \text{Sp}(1) \cdot \mathbb{Z}_2) & \simeq & B(\text{Sp}(1) \times \text{Sp}(1)) \end{array}$$

This means that the pullback along the equivariant quaternionic Hopf fibration is given by projection to the first component $p_1^{(1)}$ (in the notation of Lemma 3.11). But, by (102) in Lemma 3.9, the difference between the universal avatar class $\tilde{\Gamma}_4$ of the half integral shifted flux (Prop. 3.13 and Remark 3.10) and the class $\frac{1}{2}p_1$ has no such first component:

$$\tilde{\Gamma}_4 - \frac{1}{2}p_1 = -\frac{1}{4}p_1^{(2)} \xrightarrow{(h_{\mathbb{H}}//\mathrm{Sp}(2))^*} 0.$$

With this, the statement follows from (111) in the proof of Prop. 3.13. \square

3.6 Integral equation of motion

We prove that Hypothesis H implies the C-field's ‘‘integral equation of motion’’ (23).

Proposition 3.15 (*Sq²-closedness of shifted 4-flux*). *If G_4 is a differential 4-form on an 8-manifold X^8 as in Def. 3.2 and satisfying Hypothesis H (Def. 3.5), then the class of the shifted form $\tilde{G}_4 := G_4 + \frac{1}{4}p_1(\nabla)$, which is integral by Prop. 3.13, is annihilated by (mod 2 reduction followed by) the second Steenrod operation:*

$$\mathrm{Sq}^2([\tilde{G}_4]) = 0. \quad (115)$$

Proof. By Prop. 2.22 and under triality (Prop. 2.17) the τ -twisted Cohomotopy cocycle exhibits reduction to Spin(4)-structure:

$$\begin{array}{ccccc} & & S^4//\mathrm{Sp}(2) & \xrightarrow{\cong} & B\mathrm{Spin}(4) \\ & \nearrow c & \downarrow & & \downarrow \\ X^8 & \xrightarrow{\tau} & B\mathrm{Sp}(2) & \xrightarrow{\cong} & B\mathrm{Spin}(5) \\ & \nearrow \tau & \downarrow & & \downarrow \\ & & B\mathrm{Spin}(8) & \xrightarrow[B\mathrm{tri}]{\cong} & B\mathrm{Spin}(8) \end{array}$$

cocycle in τ -twisted Cohomotopy

But, by Prop. 3.13, the class of \tilde{G}_4 is the pullback of the class $\tilde{\Gamma}_4 \in H^4(B\mathrm{Spin}(4); \mathbb{Z})$ (101) along this reduction:

$$[\tilde{G}_4] = (B\mathrm{tri} \circ c)^*(\tilde{\Gamma}_4) \in H^4(X^8; \mathbb{Z}).$$

Under these identifications, the statement follows upon using [CV98a, Cor. 4.2 (1)], where the element corresponding to $\tilde{\Gamma}_4$ is denoted s , while the class $[\tilde{G}_4]$ is denoted S . \square

3.7 7-Flux quantization

We prove that Hypothesis H implies integrality (29) of the Page 7-flux (27).

The main result is Theorem 3.21 below. In order to formulate this, the key ingredient in the expression for the page charge is a 3-flux H_3 which locally trivializes the C-field 4-flux G_4 . The homotopy-theoretic manifestation of this local trivialization is the homotopy pullback of the quaternionic Hopf fibration. We first introduce this now in Def. 3.16, and then prove a few technical lemmas about it.

Definition 3.16 (Extended spacetime). Let X^8 be an 8-manifold as in Def. 3.2, equipped with topological Sp(2)-structure τ and with a cocycle in c in τ -twisted Cohomotopy as in Def. 3.5. Then we say that the corresponding

By Lemma 2.28 and Prop. 3.14, factorization through the quaternionic Hopf fibration is the intrinsic cohomotopical meaning of the concept of “vanishing 4-flux”, and here this is reflected by the trivializing homotopy H_3 . But this means that, under *Hypothesis H*, the extended spacetime of Def. 3.16 is really the classifying space for fundamental M5-brane sigma-model configurations in X with extended worldvolume \hat{Q}_{M5} . Hence the extended spacetime \hat{X}^8 is the classifying space for the fluxed M5-brane sigma model in the M2-brane background X^8 . This is discussed in detail in [FSS19c]. For the super-rational analog (117) this was discussed in [FSS13, Rem. 3.11][FSS15, p. 4].

Next we characterize, in Prop. 3.20 below, the differential form data encoded in (118). For that we need the following two lemmas. The statement of Lemma 3.18 is standard but rarely made fully explicit. We spell it out since it is crucial for our new result, Lemma 3.19. For background on Sullivan models see e.g. [FHT00, Sec. 12].

Lemma 3.18 (Sullivan model of the Hopf fibration). *The Sullivan model of the quaternionic Hopf fibration $h_{\mathbb{H}}$, with explicit normalization of its generators, is:*

$$\begin{array}{ccc}
 S^7 & \mathbb{R}[\omega_7]/(d\omega_7 = 0) & \langle \omega_7, [S^7] \rangle = 1 \\
 \downarrow h_{\mathbb{H}} & \uparrow (h_{\mathbb{H}})^* \begin{array}{l} \omega_4 \mapsto 0 \\ \omega_7 \mapsto \omega_7 \end{array} & \\
 S^4 & \mathbb{R}[\omega_4, \omega_7]/\left(\begin{array}{l} d\omega_4 = 0 \\ d\omega_7 = -\omega_4 \wedge \omega_4 \end{array}\right) & \langle \omega_4, [S^4] \rangle = 1
 \end{array}$$

Proof. One way to see this is with [AA78, Theorem 6.1], by which, under the identification of Sullivan generators with linear duals of homotopy groups, the co-binary component of the Sullivan differential equals the linear dual of the Whitehead product, $[\cdot, \cdot]_{\text{Wh}}$:

$$[d\omega]_{|\wedge^2} = -[\cdot, \cdot]_{\text{Wh}}^*(\omega).$$

Note that both the Whitehead product gives a factor of 2

$$[[\text{id}_{S^4}], [\text{id}_{S^4}]]_{\text{Wh}} = 2 \cdot [h_{\mathbb{H}}]$$

as does the evaluation $\langle \cdot, \cdot \rangle$ of the wedge square of ω_4 (by [AA78, top of p. 976]):

$$\langle \omega_4 \wedge \omega_4, S^4 \wedge S^4 \rangle = (-1)^{2 \cdot 2} \langle \omega_4, S^4 \rangle^2 + \langle \omega_4, S^4 \rangle^2 = 2,$$

which hence cancel out. See also [FHT00, Example 1 on p. 178].

Alternatively, this follows by considering the homotopy cofiber of $h_{\mathbb{H}}$, whose Sullivan model is the fiber product

$$\begin{array}{ccccc}
 & \begin{array}{l} \omega_4 \mapsto \omega_4 \\ \omega_7 \mapsto \omega_7 \\ \omega_8 \mapsto 0 \end{array} & \left(\begin{array}{l} d\omega_4 = 0 \\ d\omega_7 = h \cdot \omega_4 \wedge \omega_4 + \omega_8 \\ d\omega_8 = 0 \end{array} \right) & \begin{array}{l} \omega_4 \mapsto 0 \\ \omega_7 \mapsto \omega_7 \\ \omega_8 \mapsto \omega_8 \end{array} & \\
 \swarrow & & & & \searrow \\
 \left(\begin{array}{l} d\omega_4 = 0 \\ d\omega_7 = h \cdot \omega_4 \wedge \omega_4 \end{array} \right) & & & & \left(\begin{array}{l} d\omega_7 = \omega_8 \\ d\omega_8 = 0 \end{array} \right) \\
 \searrow & & & & \swarrow \\
 & \begin{array}{l} \omega_4 \mapsto 0 \\ \omega_7 \mapsto \omega_7 \end{array} & (d\omega_7 = 0) & \begin{array}{l} \omega_7 \mapsto \omega_7 \\ \omega_8 \mapsto 0 \end{array} &
 \end{array}$$

and then using the Hopf invariant one theorem [Ada60] which implies that $h = \pm 1$. \square

Lemma 3.19 (Sullivan model of $\mathrm{Sp}(2)$ -equivariant Hopf fibration). *The Sullivan model of the $\mathrm{Sp}(2)$ -parametrized quaternionic Hopf fibration $h_{\mathbb{H}} // \mathrm{Sp}(2)$ (Prop. 2.22) is as shown here:*

$$\begin{array}{ccc}
\begin{array}{ccc}
S^7 // \mathrm{Sp}(2) & & \\
\downarrow & \searrow & \\
h_{\mathbb{H}} // \mathrm{Sp}(2) & & B\mathrm{Sp}(2) \\
\downarrow & \nearrow & \\
S^4 // \mathrm{Sp}(2) & &
\end{array} & \begin{array}{c}
\mathrm{CE}(\mathfrak{B}\mathrm{Sp}(2)) \otimes \mathbb{R}[\omega_7] / (d\omega_7 = -\chi_8) \\
\uparrow \\
(h_{\mathbb{H}} // \mathrm{Sp}(2))^* \\
\omega_4 \mapsto \frac{1}{4}p_1 \\
\omega_7 \mapsto \omega_7
\end{array} & \langle \omega_7, [S^7] \rangle = 1 \\
& & (119) \\
& & \begin{array}{c}
\mathrm{CE}(\mathfrak{B}\mathrm{Sp}(2)) \otimes \mathbb{R}[\omega_4, \omega_7] / \left(\begin{array}{l} d\omega_4 = 0 \\ d\omega_7 = -\omega_4 \wedge \omega_4 \\ -\chi_8 + \left(\frac{1}{4}p_1\right)^2 \end{array} \right) \\
\langle \omega_4, [S^4] \rangle = 1
\end{array}
\end{array}$$

where $\mathrm{CE}(\mathfrak{B}\mathrm{Sp}(2))$ denotes the Sullivan model of the classifying space of $\mathrm{Sp}(2)$.

Proof. That the domain and codomain Sullivan algebras are as shown follows by [FHT00, Sec. 15, Example 4] as in the proof of Prop. 2.5, where the normalization of the generators is from Lemma 3.18. Here in the bottom right we translated, the summand $\frac{1}{4}p_2$ (49) from the $\mathrm{Spin}(5)$ -structure for which Prop. 2.5 applies, to the given $\mathrm{Sp}(2)$ -structure, by pullback along $B\mathrm{tri}$ (68), using (97)

$$(B\mathrm{tri})^* \left(\frac{1}{4}p_2 \right) = -\chi_8 + \left(\frac{1}{4}p_1 \right)^2. \quad (120)$$

Now to see that the map $(h_{\mathbb{H}} // \mathrm{Sp}(2))^*$ in (119) is given on generators as claimed, we use that over any base point of $B\mathrm{Sp}(2)$ the parameterized quaternionic Hopf fibration restricts to the ordinary quaternionic Hopf fibration, making the following diagram homotopy commutative:

$$\begin{array}{ccccc}
S^7 & \longrightarrow & S^7 // \mathrm{Sp}(2) & & \\
h_{\mathbb{H}} \downarrow & \searrow & \downarrow h_{\mathbb{H}} // \mathrm{Sp}(2) & \searrow & \\
& & * & \longrightarrow & B\mathrm{Sp}(2) \\
S^4 & \longrightarrow & S^4 // \mathrm{Sp}(2) & &
\end{array}$$

This means that the Sullivan model of $h_{\mathbb{H}} // \mathrm{Sp}(2)$ must be a dashed homomorphism that makes the following diagram of dg-algebras commute:

$$\begin{array}{ccc}
\mathbb{R}[\omega_7] / (d\omega_7 = 0) & \longleftarrow & \mathrm{CE}(\mathfrak{B}\mathrm{Sp}(2)) \otimes \mathbb{R}[\omega_7] / (d\omega_7 = \chi_8) \\
\uparrow \begin{array}{l} \omega_4 \mapsto 0 \\ \omega_7 \mapsto \omega_7 \end{array} & & \uparrow \begin{array}{l} \omega_4 \mapsto \frac{1}{4}p_1 \\ \omega_7 \mapsto \omega_7 \end{array} \\
\mathbb{R}[\omega_4, \omega_7] / \left(\begin{array}{l} d\omega_4 = 0 \\ d\omega_7 = -\omega_4 \wedge \omega_4 \end{array} \right) & \longleftarrow & \mathrm{CE}(\mathfrak{B}\mathrm{Sp}(2)) \otimes \mathbb{R}[\omega_4, \omega_7] / \left(\begin{array}{l} d\omega_4 = 0 \\ d\omega_7 = -\omega_4 \wedge \omega_4 \\ \underbrace{-\chi_8 + \left(\frac{1}{4}p_1\right)^2}_{=(B\mathrm{tri})^* \left(\frac{1}{4}p_2\right)} \end{array} \right)
\end{array}$$

where the horizontal morphisms project away the base algebra $\mathrm{CE}(\mathfrak{B}\mathrm{Sp}(2))$.

The commutativity of this diagram requires that the dashed morphism sends $\omega_7 \mapsto \omega_7$. and by degree reasons it must send $\omega_4 \mapsto c \cdot p_1$, for some $c \in \mathbb{R}$. The unique choice for c that makes the map respect the differentials, in that the second summand in (120) cancels out, is clearly $c = \frac{1}{4}$. Alternatively, this follows also by Prop. 3.14. \square

Proposition 3.20 (Differential form data on extended spacetime). *Let X^8 be an 8-manifold as in Def. 3.2 equipped with differential forms (G_4, G_7) that satisfy Hypothesis H (Def. 3.5), hence equipped with topological $\mathrm{Sp}(2)$ -structure τ (55) and equipped with a cocycle c in τ -twisted Cohomotopy (Def. 2.1) with underlying differential forms $(G_4, 2G_7)$ according to Def. 3.5*

$$\begin{array}{ccc} X & \xrightarrow{(G_4, 2G_7)} & S^4 // \mathrm{Sp}(2) . \\ & \searrow \tau & \swarrow \\ & & B\mathrm{Sp}(2) \end{array}$$

Then the pullback of these differential forms to the corresponding extended spacetime \widehat{X} from Def. 3.16 satisfies

$$dH_3^{\mathrm{univ}} = \widetilde{G}_4 - \frac{1}{2}p_1(\nabla), \quad (121)$$

$$d\widetilde{G}_7 = -\frac{1}{2}\chi_8(\nabla), \quad (122)$$

where H_3^{univ} is the universal 3-form H_3^{univ} (118) on \widehat{X} , and where \widetilde{G}_7 the shifted 7-flux form or Page flux

$$\widetilde{G}_7 := G_7 + \frac{1}{2}H_3^{\mathrm{univ}} \wedge \widetilde{G}_4. \quad (123)$$

Proof. To extract the differential form data we may compute the defining homotopy pullback (116) in rational homotopy theory and read off the resulting assignment of generators in the Sullivan model. By general facts of rational homotopy theory (recalled e.g. in [FSS16a, A]) the Sullivan model for \widehat{X}^8 is given as the pushout along the map corresponding to $(G_4, 2G_7)$ of a minimal cofibration resolution of the Sullivan model for the equivariant quaternionic Hopf fibration $h_{\mathbb{H}} // \mathrm{Sp}(2)$. The latter was obtained in Lemma 3.19. By direct inspection one sees that the minimal cofibration resolution is given as shown on the right of the following diagram:

$$\begin{array}{ccc} & & \mathrm{CE}(B\mathrm{Sp}(2)) \otimes \mathbb{R}[\omega_7] / (d\omega_7 = -\chi_8) \\ & & \uparrow (h_{\mathbb{H}} // \mathrm{Sp}(2))^* \simeq \begin{array}{l} h_3 \mapsto 0 \\ \omega_4 \mapsto \frac{1}{4}p_1 \\ \omega_7 \mapsto \omega_7 \end{array} \\ & & \left(\begin{array}{l} dh_3 = \omega_4 - \frac{1}{4}p_1 \\ d\omega_4 = 0 \\ d\omega_7 = -dh_3 \wedge (\omega_4 + \frac{1}{4}p_1) \\ -\chi_8 \end{array} \right) \\ \mathrm{CE}(\widehat{X}) \xleftarrow[\begin{array}{l} \omega_7 \mapsto 2G_7 \\ h_3 \mapsto H_3^{\mathrm{univ}} \end{array}]{\omega_4 \mapsto G_4} \mathrm{CE}(B\mathrm{Sp}(2)) \otimes \mathbb{R}[h_3, \omega_4, \omega_7] / \left(\begin{array}{l} dh_3 = \omega_4 - \frac{1}{4}p_1 \\ d\omega_4 = 0 \\ d\omega_7 = -dh_3 \wedge (\omega_4 + \frac{1}{4}p_1) \\ -\chi_8 \end{array} \right) & \xrightarrow{\text{(po)}} & \mathrm{CE}(B\mathrm{Sp}(2)) \otimes \mathbb{R}[\omega_4, \omega_7] / \left(\begin{array}{l} d\omega_4 = 0 \\ d\omega_7 = -\omega_4 \wedge \omega_4 + (\frac{1}{4}p_1)^2 - \chi_8 \end{array} \right) \\ & & \uparrow \begin{array}{l} \omega_4 \mapsto \omega_4 \\ \omega_7 \mapsto \omega_7 \end{array} \\ \mathrm{CE}(X) \xleftarrow[\omega_7 \mapsto 2G_7]{\omega_4 \mapsto G_4} \mathrm{CE}(B\mathrm{Sp}(2)) \otimes \mathbb{R}[\omega_4, \omega_7] / \left(\begin{array}{l} d\omega_4 = 0 \\ d\omega_7 = -\omega_4 \wedge \omega_4 + (\frac{1}{4}p_1)^2 - \chi_8 \end{array} \right) & & \\ \uparrow \tau^* & & \uparrow \tau^* \\ \mathrm{CE}(B\mathrm{Sp}(2)) & & \end{array}$$

The differential relations appearing on the right now immediately imply the claim. \square

Proposition 3.21 (Integrality of Page charge). *Let X^8 be an 8-manifold as in Def. 3.2, equipped with differential forms (G_4, G_7) that satisfy Hypothesis H, Def. 3.5 with respect to a topological $\mathrm{Sp}(2)$ -structure τ and a cocycle in c in τ -twisted Cohomotopy. Then for every map*

$$i : S^7 \longrightarrow \widehat{X}^8$$

from the 7-sphere to the corresponding extended spacetime (Def. 3.16), the integration of the pullback of the Page flux (123) over the 7-sphere is half-integral:

$$2 \int_{S^7} i^* \tilde{G}_7 \in \mathbb{Z}.$$

Proof. This is proven as [FSS19c, Theorem 4.6]. □

3.8 Tadpole cancellation

We discuss here how Hypothesis H implies the fluxless C-field tadpole cancellation condition (33).

The key point is to see what precisely “4-fluxless” is to mean. For this, recall that we discussed Cohomotopy cocycles at four levels of approximation, from the coarse approximation of rational/de Rham cohomology on the left to full non-abelian Cohomotopy on the right:

Cohomology theory	Rational cohomology	Integral cohomology	Stable Cohomotopy	Non-abelian Cohomotopy
Cocycle	G_4	\tilde{G}_4	$\Sigma^\infty c$	c

On the far left, for rational and integral cohomology we had found in Prop. 3.14 that cohomotopical fluxlessness is reflected by any factorization of the 4-cocycle through 7-Cohomotopy via the quaternionic Hopf fibration $h_{\mathbb{H}}$, and that this means that the differential flux 4-form takes its background charge value: $G_4 = \frac{1}{4}p_1(\nabla)$.

But stable Cohomotopy theory is finer than its approximation by de Rham cohomology. Indeed, not *every* factorization through $h_{\mathbb{H}}$ gives zero in 4-Cohomotopy, instead there are in general torsion side effects which disappear only in de Rham cohomology, as shown in (8): It is only those factorizations through $h_{\mathbb{H}}$ which occur in multiples of 24 that are strictly 4-fluxless as seen in stable Cohomotopy. Together with the 7-flux quantization of Prop. 3.21 this means that the cohomotopically normalized 7-flux, measuring the number of M2-branes, is as in (32). With this we have:

Proposition 3.22 (C-field tadpole cancellation via Cohomotopy). *Let X^8 be an 8-manifold as in Def. 3.2 which is (a) the complement of a tubular neighborhood around a finite number of points in a closed 8-manifold (the M2-brane loci) as in (30), and*

(b) equipped with a twisted 7-Cohomotopy cocycle as in (3.16), hence with a section i of the corresponding extended spacetime (Def. 3.16).

Then the fluxless C-field tadpole cancellation condition (33) holds:

$$N_{M_2} = I_8[X^8].$$

Proof. By definition of the cohomotopically normalized 7-flux (32), we compute as follows:

$$\begin{aligned} N_{M_2} &= \frac{-1}{12} \int_{\partial X^8} i^* \tilde{G}_7 \\ &= \frac{-1}{12} \int_{X^8} i^* d\tilde{G}_7 \\ &= \frac{-1}{12} \int_{X^8} \frac{-1}{2} \chi_8(\nabla) \\ &= \int_{X^8} I_8(\nabla) \\ &= I_8[X^8]. \end{aligned}$$

Here the third step is by (122) from Prop. 3.20 and the fourth step by (93) from Prop. 3.7. □

4 Conclusion

Perturbative string theory has a precise definition via 2d worldsheet SCFT. In contrast, the formulation of its non-perturbative completion to M-theory and of the brane physics this subsumes (see [Du99][BBS06]), remains an open problem (e.g. [Du96, 6][HLW97, p. 2][NH98, p. 2] [Du98, p. 6] [Du99, p. 330] [Mo14, 12][CP18, p. 2][Wi19]⁶). The lack of an actual set of fundamental laws of non-perturbative brane physics has recently surfaced in a debate on the extent of validity of the brane uplifts that have been widely discussed for 15 years [DvR18][Ba19, p. 14-22].

Besides the field of gravity, the only other field in M-theory at low-energy is the C-field [CJS78]. A list of cohomological conditions on the C-field, including those shown in Table 1, have been derived as plausible consistency conditions in various expected limiting cases of M-theory (effective field theory limits, decoupling limits etc.) assuming the conjectural string dualities to hold. One imagines that if M-theory exists then thereby it must be consistent, and hence ought to imply all these expected consistency conditions. In order to make this actually happen, the first step in formulating M-theory ought to be the identification of the generalized cohomology theory that charge-quantizes the C-field, just as the first step in formulating a quantum consistent theory of electromagnetism was Dirac's *charge quantization* of the electromagnetic field: as a cocycle in (differential) ordinary cohomology (see [Fr00]),

The string theory literature has mostly regarded the M-theory C-field as a cocycle in ordinary 4-cohomology, with extra constraints imposed on it by hand. A proposal to build at least one of these conditions, the shifted flux quantization condition (§3.4), into the definition of the cohomology theory (making it a “mildly generalized cohomology theory”) has been considered in [DFM03][HS05][SSS12][FSS14a]. Another condition, the “integral equation of motion” (§3.6) has been argued in [DMW03a][DMW03b] to be in correspondence with one differential of specific degree in the Atiyah-Hirzebruch spectral sequence for K-theory. In reaction to this state of affairs, it has been suggested [Sa05a][Sa05b][Sa06][Sa10] that the C-field should be regarded as a cocycle in some genuine generalized cohomology theory, such as Cohomotopy theory [Sa13, 2.5]. Indeed, if M-theory is as fundamental to physics as it should be, one may expect the generalized cohomology theory that charge quantizes the C-field to be more fundamental to mathematics than ordinary cohomology with some modifications.

In order to derive what this generalized cohomology theory actually is, we had initiated a systematic analysis of the bifermionic super p -brane charges from the point of view of super rational homotopy theory [FSS13]; see [FSS19a] for review. We proved in this fully super-geometric setting, albeit in rational approximation, that the expected charge quantization of the RR-field in twisted K-theory follows from systematic analysis of the D-brane super WZW terms [FSS16a][FSS16b][BSS18]. Then we showed that the exact same logic applies to the super WZW terms of the M-branes [FSS15]. The analysis in this case reveals their cohomology theory to be [FSS15, 3][FSS16a, 2]: Cohomotopy cohomology theory in compatible degrees $(4, 7)$, related by the quaternionic Hopf fibration; see [FSS19a, 7] for review of this super rational analysis. This proves that if full M-theory retains the super-space structure of its low-energy limit, then the cohomology theory that charge-quantizes the C-field must be such that its rationalization coincides with that of Cohomotopy cohomology theory in degrees $(4, 7)$. While there are many different cohomology theories with the same rationalization as Cohomotopy theory, one of these is *minimal* in number of CW-cells: This is Cohomotopy theory itself.

What we have shown in this article is that assuming, with *Hypothesis H*, that Cohomotopy cohomology theory in compatible degrees $(4, 7)$ indeed encodes the charge-quantization of the C-field even beyond the rational approximation, then the list in Table 1 of expected consistency conditions is implied. Further checks of *Hypothesis H* are presented in [SS19a][BSS19b] (for the case of M-theory orbifolds) and in [SS19c] (for intersecting branes).

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⁶[Wi19] at 21:15: “I actually believe that string/M-theory is on the right track toward a deeper explanation. But at a very fundamental level it's not well understood. And I'm not even confident that we have a good concept of what sort of thing is missing or where to find it.”

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