Twisted Cohomotopy implies M-Theory anomaly cancellation

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To Mike Duff on the occasion of his 70th birthday

Abstract

We show that all the expected anomaly cancellations in M-theory follow from charge-quantizing the C-field in the non-abelian cohomology theory *twisted Cohomotopy*. Specifically, we show that such cocycles exhibit all of the following:

(1) the half-integral shifted flux quantization condition,
(2) the cancellation of the total M5-brane anomaly,
(3) the M2-brane tadpole cancellation,
(4) the cancellation of the W_7 spacetime anomaly,
(5) the C-field integral equation of motion, and
(6) the C-field background charge.

Along the way, we find that the calibrated $N = 1$ exceptional geometries (Spin(7), G_2, SU(3), SU(2)) are all induced from the classification of twists in Cohomotopy. Finally we show that the notable factor of $1/24$ in the anomaly polynomial reflects the order of the 3rd stable homotopy group of spheres.

Contents

1 Introduction

The general open problem. One of the key open problems in theoretical high energy physics remains the actual formulation of the non-perturbative completion of string theory, with working title "M-theory" (see [Moo14, Sec. 12][HSS18, Sec. 2][BSS18, Sec. 1]). A plethora of indications and plausibility arguments about the elusive M-theory exist ([Wi95]; see [Du99][BBS06] for overviews), constituting a tantalizing but informal folklore.

The open problem of C-field charge quantization. A core sub-problem is the identification of the cohomological nature of the higher gauge field in M-theory: the "C-field". In generalization of Dirac's seminal identification of the electromagnetic field with a cocycle in differential cohomology, known as *Dirac charge quantization* (see [Fr00]) one expects that an analogous charge quantization of the M-theory C-field reveals it as a cocycle in cohomology with some extra structure [DFM03][HS05][FSS14a][FSS14b].

Accounting for all anomaly cancellation. However, it has long been argued [Sa05a][Sa05b][Sa06][Sa10] that the C-field should not just be in ordinary cohomology, albeit shifted, but in some generalized cohomology theory. Indeed, the M-theory folklore knows not just one, but a whole list of subtle *anomaly cancellation* constraints on the C-field. Any candidate cohomology theory charge-quantizing the C-field, should *imply all* of these conditions (reviewed below in §2):

Anomaly cancellation condition		folklore	Cohomotopy
Half-integral flux quantization	$\big[\underbrace{G_4 + \tfrac{1}{4}p_1}_{=:\widetilde{G}_4 \text{ integral flux}}\big] \in H^4(X,\mathbb{Z})$	§2.2	§4.2
Background charge	$\underbrace{q(\widetilde{G}_4)}_{\text{quadratic form}} = \widetilde{G}_4\big(\underbrace{\widetilde{G}_4 - \tfrac{1}{2}p_1}_{=(\widetilde{G}_4)_0}\big)$	§2.4	§4.4
DMW-anomaly cancellation	$W_7\big(TX\big) = 0$	§2.1	§4.1
Integral equation of motion	$\underbrace{\mathrm{Sq}^3}_{=\beta \mathrm{Sq}^2}\big(\widetilde{G}_4\big) = 0$	§2.3	§4.3
M5-brane anomaly cancellation	$\underbrace{I^{\mathrm{M5}}_{\mathrm{ferm}}}_{\substack{\text{chiral} \\ \text{fermion}}} + \underbrace{I^{\mathrm{M5}}_{\mathrm{sd}}}_{\substack{\text{self-dual} \\ \text{3-flux}}} + \underbrace{I^{\mathrm{bulk}}_{\mathrm{infl}}}_{\substack{\text{bulk} \\ \text{inflow}}} = 0$	§2.5	§4.5
M2-brane tadpole cancellation	$\underbrace{N_{\mathrm{M2}}}_{\substack{\text{number of} \\ \text{M2-branes}}} + q(\widetilde{G}_4) = I_8$	§2.6	§4.6

All previous proposals [DFM03][HS05][FSS14a][FSS14b] deal with the first of these conditions – enforcing it essentially "by hand". But one would hope the identification of the fundamental cohomological nature of the C-field to inform us about the unknown fundamental nature of M-theory, instead of just partially encoding existing folklore into complex mathematics.

First principles. To provide a solid ground, we initiated a program to exhibit M-theoretic structure *emerging* from first principles – and carried it out successfully in the *rational* approximation [FSS13][FSS15][FSS16a][FSS16b] [HS17][BSS18][HSS18]; see [FSS19] for review. One obtains a rigorous derivation in rational homotopy theory showing that, in the same way that the NS/RR fields of string theory are quantized in twisted K-theory, the C-field in M-theory is quantized in *Cohomotopy* cohomology theory [Bo36][Sp49], as originally proposed in [Sa13, 2.5]. See Figure R. This now leads us to study:

Hypothesis H. *The C-field is charge-quantized in Cohomotopy theory*, even non-rationally. (Def. 4.3)

Results. We lay out twisted Cohomotopy theory in §3; see Figure T. Then we prove in §4 that C-field charge quantization in twisted Cohomotopy implies all of the above anomaly cancellation conditions.

The emergence of rational Cohomotopy. We recall in more detail how *Hypothesis H* is motivated: in the approximation of rational homotopy theory (e.g. [FHT00]), i.e., ignoring torsion subgroups in cohomology and working super-tangent space wise, the charge quantization of M-brane charge in Cohomotopy *follows* by systematic analysis (see [FSS19] for review):

First of all, as observed in [Sa13, Sec. 2.5], the equations of motion for the C-field flux forms G_4 and G_7 in plain 11-dimensional supergravity [CJS78], which are (see also [D'AF82, Table 3] and [CDF91, III.8 and V.4-V.11])

$$\begin{aligned} dG_4 &= 0 \\ dG_7 &= -\tfrac{1}{2} G_4 \wedge G_4 \,, \end{aligned} \tag{1}$$

have the same form as the differential relations that define the Sullivan model for the 4-sphere in rational homotopy theory and suggested Cohomotopy as the proper setting (see [FSS16a, Appendix A] for review, and see Remark **??** below for the issue of the normalization factor). This indeed means that the pair (G_4, G_7) constitutes a cocycle in *rational Cohomotopy* in degree 4, namely a map from spacetime to the rationalized (equivalently "real-ified") 4-sphere [FSS15] (see [FSS16a, Sec. 2])

$$X \xrightarrow{(G_4, G_7)} S^4_{\mathbb{R}} \,. \tag{2}$$

This becomes yet more pronounced in the superspace formulation of 11d supergravity, which is fully controlled [D'AF82] by an iterated pair of invariant super-cocycles $\mu_{\rm M2}$ and $\mu_{\rm M5}$ on $D = 11, N = 1$ super Minkowski spacetime. In the super homotopy-theoretic formulation [FSS13, p. 12] [FSS15, (2.1)] this appears as maps

$$\begin{tikzcd} & K(\mathbb{R},7) \\ \widehat{\mathbb{T}^{10,1|\mathbf{32}}} \arrow[ur, "\mu_{\rm M5}"] \arrow[d, "\mathrm{fib}(\mu_{\rm M2})"'] & \\ \mathbb{T}^{10,1|\mathbf{32}} \arrow[r, "\mu_{\rm M2}"] & K(\mathbb{R},4) \end{tikzcd} \qquad \begin{aligned} \mu_{\rm M5} &= \tfrac{1}{5!}\left(\overline{\psi}\Gamma_{a_1\cdots a_5}\psi\right) e^{a_1}\wedge\cdots\wedge e^{a_5} \\ &\quad + h_3 \wedge \mu_{\rm M2} \\ \\ \mu_{\rm M2} &= \tfrac{i}{2}\left(\overline{\psi}\Gamma_{a_1 a_2}\psi\right) e^{a_1}\wedge e^{a_2} \end{aligned} \tag{3}$$

which are the super-flux forms to which the M2-brane and M5-brane couple, in their incarnation as Green-Schwarz-type sigma models [FSS13][FSS16a][FSS16b]. Here $\widehat{\mathbb{T}^{10,1|\mathbf{32}}} = \mathfrak{m2brane}$ arises as the homotopy fiber of $\mu_{\rm M2}$ [FSS13, p. 12] and is the extended super Minkowski spacetime that can be traced back to [CdAIB99] or the M2-brane super Lie 3-algebra [SSS09, p. 54]. This is crucial for the following discussion, as it means that:

- $\mu_{\rm M2}$ is the super-form component of the *magnetic flux* sourced by charged M5-branes, while
- $\mu_{\rm M5}$ is the super-form component of the *electric flux* source by charged M2-branes.

Hence these cocycles are avatars of M-brane charge/flux at the level of super rational homotopy theory.

This is amplified by the result of [BSS18], that the double dimensional reduction of rational M-brane supercocycles $(\mu_{\rm M2}, \mu_{\rm M5})$ is indeed the tuple of F1/Dp-brane supercocycles $(\mu_{\rm F1}\,\mu_{\rm D0}, \mu_{\rm D2}, \mu_{\rm D4}, \mu_{\rm D6}, \mu_{\rm D8})$ in rational twisted K-theory, which folklore demands to be the rational image of a cocycle in actual twisted K-theory (see [GS19] for general treatment of twisted differential cocycles for Ramond-Ramond fields)

Objects	Cohomology theory
M-branes	twisted Cohomotopy
D-branes	twisted K-theory

double dimensional reduction/oxidation

Hence our goal must be to lift this gauge quantization of M-brane charge in Cohomotopy beyond the rational approximation.

The rational quaternionic Hopf fibration. To see how this should work, we showed in [FSS15] that unification of the rational super-cocycles in (3) to a single cocycle (2) in rational Cohomotopy is induced via the *quaternionic Hopf fibration* $h_{\mathbb{H}}$, as shown in Figure R.

super rational C-field   in terms of   rational Cohomotopy

$$
\begin{tikzcd}
 & & S^7_{\mathbb{R}} \arrow[d, "h_{\mathbb{H}}"] \\
\widehat{\mathbb{T}^{10,1|\mathbf{32}}} \arrow[urr, dashed, "\mu_{\mathrm{M5}}"] \arrow[d, "\mathrm{fib}"'] & & S^4_{\mathbb{R}} \arrow[dl, "c_2"] \\
\mathbb{T}^{10,1|\mathbf{32}} \arrow[urr, dashed, "\mu_{\mathrm{M2/M5}}"] \arrow[r, "\mu_{\mathrm{M2}}"] & K(\mathbb{R},4) &
\end{tikzcd}
$$

Figure R – The C-field in rational Cohomotopy. *The incarnation of the C-field in rational super homotopy theory – hence its bifermionic differential form component on super Minkowski spacetime – may systematically be derived from first principles [FSS13][FSS15][FSS16a][FSS16b][HS17][BSS18][HSS18], as reviewed in [FSS19]: it is given by the curvatures* μ_{M2} *and* μ_{M5} *of the WZW-terms of the GS-sigma model for the M2- and the M5-brane* (3)*, but unified to form one single cocycle* $\mu_{\mathrm{M2/M5}}$ *in super rational Cohomotopy theory. See also Remark 4.25 below.*

The diagram in Figure R teaches us that, in the rational approximation:

(i) The M2/M5-brane charge is jointly quantized in Cohomotopy theory in degree 4;

(ii) the electric charge sourced by M2-branes factors through the quaternionic Hopf fibration.

Beyond the rational approximation. One lift of rational Cohomotopy stands out as being *minimal* in number of cells: this is *actual* Cohomotopy. In general this will be twisted, but by **(ii)** the twists need to respect the quaternionic Hopf fibration $h_{\mathbb{H}}$. We prove in §3 that this implies *Figure T*:

topological C-field   in terms of: twisted Cohomotopy,   calibrated submanifolds,   topological G-structure.

$$
\begin{tikzcd}[column sep=small]
 & S^7 /\!\!/ \big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[r, equal] \arrow[dd, "h_{\mathbb{H}} /\!\!/ (\mathrm{Sp}(2)\cdot\mathrm{Sp}(1))"'] & \mathrm{CAY}_{\mathrm{sL}} /\!\!/ \mathrm{Spin}(6) \arrow[r, equal] \arrow[dd] & B\big(\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\big) \arrow[dd] \arrow[dr] & \\
 & & & & B\mathrm{Spin}(6) \arrow[dd] \\
 & S^4 /\!\!/ \big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[r, equal] \arrow[dd] & \mathrm{CAY} /\!\!/ \mathrm{Spin}(7) \arrow[r, equal] & B\big(\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\big) \arrow[dd] \arrow[dr] & \\
\widehat{\mathbb{R}^{2,1}\times X^8} \arrow[uuur, dashed, "G_7+H_3\wedge G_4"] \arrow[dd] & & & & B\mathrm{Spin}(7) \arrow[dd] \\
 & \ast /\!\!/ \big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[rr, equal] & & B\big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[dr] & \\
\mathbb{R}^{2,1}\times X^8 \arrow[uuur, dashed, "(G_4,G_7)"] \arrow[ur, dashed, "N_{Q_{\mathrm{M2}}}Q_{\mathrm{M5}}\cdot N_X Q_{\mathrm{M5}}"'] \arrow[rrrr, "N_X\, Q_{\mathrm{M2}}"] & & & & B\mathrm{Spin}(8)
\end{tikzcd}
$$

Figure T – The C-field in twisted Cohomotopy. *The established rational situation shown in Figure R suggests that the C-field is also topologically a cocycle in Cohomotopy, in degrees 4 and 7 related by the quaternionic Hopf fibration* $h_{\mathbb{H}}$ *– this is* Hypothesis H. *We prove in §3.3 (Prop. 3.20) that this implies the twist structure as displayed here, and then show in §4 that this implies anomaly cancellation. Characterization in terms of topological G-structures is given in §3.2 and in terms of calibrated submanifolds in §3.6.*

Compactifications to $D = 4$, $N = 1$ from twisted Cohomotopy in degree 7. Conversely, this says that the *fluxless* sector, where G_4 vanishes but M2-branes may be present, is controlled by twisted Cohomotopy in degree 7 alone (we discuss in §3.6 and §2.6 the precise formulation of fluxlessness in Cohomotopy). We show in §3.4 that, under the relation of Cohomotopy to topological G-structures discussed in §3.2, there is a hierarchy of exceptional twists of degree 7 Cohomotopy given by iterated homotopy pullback, which reproduces precisely the special holonomy structures well-known to correspond to the $D = 4$, $N = 1$ compactifications of M/F-theory (see [AG04] for a review).

$$
\begin{tikzcd}
 & S^7 /\!\!/ \mathrm{Spin}(8) \arrow[d] \arrow[r, equal] & B\mathrm{Spin}(7) \arrow[d] \\
X \arrow[ur, dashed] \arrow[r] & B\mathrm{Spin}(8) \arrow[r, equal] & B\mathrm{Spin}(8)
\end{tikzcd}
\quad
\begin{tikzcd}
 & S^7 /\!\!/ \mathrm{Spin}(7) \arrow[d] \arrow[r, equal] & BG_2 \arrow[d] \\
X \arrow[ur, dashed] \arrow[r] & B\mathrm{Spin}(7) \arrow[r, equal] & B\mathrm{Spin}(7)
\end{tikzcd}
\quad
\begin{tikzcd}
 & S^7 /\!\!/ \mathrm{Spin}(6) \arrow[d] \arrow[r, equal] & B\mathrm{SU}(3) \arrow[d] \\
X \arrow[ur, dashed] \arrow[r] & B\mathrm{Spin}(6) \arrow[r, equal] & B\mathrm{Spin}(6)
\end{tikzcd}
\quad
\begin{tikzcd}
 & S^7 /\!\!/ \mathrm{Spin}(5) \arrow[d] \arrow[r, equal] & B\mathrm{SU}(2) \arrow[d] \\
X \arrow[ur, dashed] \arrow[r] & B\mathrm{Spin}(5) \arrow[r, equal] & B\mathrm{Spin}(5)
\end{tikzcd}
$$

Figure D. Twisted Cohomotopy in degree 7 *systematically induces the special holonomy structures that correspond to $D = 4$, $N = 1$ compactifications of M/F-theory. We discuss this in §3.4, based on §3.2.*

This corresponds, in particular, to the sequence of exceptional coset space realizations of the 7-sphere:

$$
S^7 \simeq \underset{\xrightarrow[\text{reduction of holonomy group}]{}}{\overset{\xrightarrow{\text{reduction of structure group}}}{\overbrace{\frac{\mathrm{Spin}(8)}{\mathrm{Spin}(7)}}^{8-\text{manifolds}} = \overbrace{\frac{\mathrm{Spin}(7)}{G_2}}^{7-\text{manifolds}} = \overbrace{\frac{\mathrm{Spin}(6)}{\mathrm{SU}(3)}}^{6-\text{manifolds}} = \overbrace{\frac{\mathrm{Spin}(5)}{\mathrm{SU}(2)}}^{5-\text{manifolds}}}}
\tag{4}
$$

We find this remarkable in several ways. First, the fact that the topological 7-sphere admits these various descriptions as a coset space G/K. Second, the fact that the 'numerator groups' G form the sequence of reductions $\mathrm{Spin}(8) \supset \mathrm{Spin}(7) \supset \mathrm{Spin}(6) \supset \mathrm{Spin}(5)$ of the structure groups of manifolds of dimensions eight, seven, six, and five, respectively, as internal spaces. Third, the fact that the 'denominator groups' K form the sequence of reductions $\mathrm{Spin}(7) \supset G_2 \supset \mathrm{SU}(3) \supset \mathrm{SU}(2)$ of holonomy groups in the corresponding dimensions, with the latter two associated with Calabi-Yau structures for complex threefolds and twofolds. Finally, homotopy theory shows that this sequence of reductions arises as an iterated homotopy pullback of 7-sphere fibrations classifying the corresponding twisted Cohomotopy theories (Prop. 3.21 below).

Four-spheres and Seven-spheres in 11d supergravity spacetimes. While *Figure R*, *Figure T*, and *Figure D* discover the 4-sphere and 7-sphere in various coset space realizations as *coefficients* for M-brane charge, of course these same spheres have long been known to prominently appear in *spacetime* solutions of 11-dimensional supergravity – we have discussed in [HSS18, Sec. 2] that this confluence between shapes of near horizon spacetime geometries and Cohomotopy coefficients is not a coincidence.

The importance of the 7-sphere in supergravity goes back to it being the first example of a consistent Kaluza-Klein reduction from eleven to four dimensions on a curved manifold, giving rise to maximal $N = 8$ gauged $\mathrm{SO}(8)$ supergravity [DP83]. After the round sphere, a family of squashed seven-spheres appeared which can be described in several ways, including the distance sphere in the quaternionic projective space $\mathbb{H}P^2$ or a single-instanton $\mathrm{SU}(2)$ bundle over S^4. The squashed S^7 is famously a coset space

$$
S^7 \simeq \mathrm{Sp}(2)/\mathrm{Sp}(1) \simeq \mathrm{SO}(5)/\mathrm{SO}(3)
\tag{5}
$$

with isometry group $\mathrm{SO}(5) \times \mathrm{SO}(3)$ [ADP83][DP83][DNP83][Du83]. The breaking of the symmetry from $\mathrm{SO}(8)$ to the latter has been used to study the standard model [DKN84]. See [DNP86] for a survey.

While these are *spacetime* phenomena, once again we find below in §3.3 that the coset space realization (5) controls also the *coefficient* of M-brane charge, and it does so in twisted Cohomotopy.

Incarnations of Cohomotopy. We observe that twisted Cohomotopy unifies several classical theorems in differential topology.

Table 1. Twisted Cohomotopy, introduced in §3.1, has interesting mathematical properties, independent of its role in *Hypothesis H*. We may think of it as a grand unified theory of classical results in differential topology.

Incarnations of Cohomotopy	
via the Hopf degree theorem	(28)
via topological G-structure	§3.2
via the Poincaré-Thom theorem	§3.5
via the Pontrjagin-Thom theorem	§3.6

Specifically, the cohomotopical formulation of the Poincaré-Hopf theorem relates to the presence of M2-branes and their cancellation of the C-field tadpole, by identifying the Cohomotopy charge around codimension 8-singularities with a fraction of the Euler characteristic $\chi[X]$; this is discussed in §4 below.

The I_8 anomaly polynomials. An $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$-structure implies that this multiple of the Euler class equals the M-theory one-loop anomaly polynomial I_8 (16) introduced in [DLM95][VW95]: $\frac{1}{24}\chi = I_8$. This is Prop. 4.4 below. The topological structures associated with I_8 have been studied from the point of view of generalized cohomology in [Sa08], used for mathematical consistency for the NS5-brane partition function in [Sa11b] where analogous anomaly conditions arise as for the M5-brane, and used for anomalies on String manifolds in [Sa11a], as well as for the study of the non-abelian higher gauge theory for multiple M5-branes [FSS14b]. In hindsight, one can revisit early results [IP88][IPW88] on spinors and triality automorphisms, based on the constructions of [GrGr70], and use them to interpret I_8 as an obstruction related to $\mathrm{Spin}(8)$. Furthermore, it was already pointed out in [Sa13] that I_8 can detect G_2 holonomy, by defining a new $\mathbb{Z}_{24}$-valued invariant, the "I_8-defect" for 8-manifolds whose boundary admits a G_2-structure, and which is related to the $\mathbb{Z}_{48}$-valued ν-invariant introduced in [CN15].

The factor of 24. Further consequences of our proposed approach and setting is that they provide a natural and fundamental interpretation of the factor of 24 appearing in the anomaly formulas. We show that this can be traced, via the Pontrjagin-Thom construction, to the stable homotopy group of degree 3, i.e., $\pi_3^s \cong \mathbb{Z}_{24}$. Note that a similar factor associated with the M2-brane viewed through the lens of Chern-Simons theory allows for an interpretation of the factor of 24, the framing anomaly, via String cobordism in dimension 3, which is equivalent to the above stable homotopy group [Sa10]. Thus our current discussion can be viewed as an analogue for the M5-brane.

Organization of the paper. The paper is organized in the following very simple form.

– In §2 we review the existing informal literature on the various anomaly cancellations in M-theory.
– In §3 we introduce our topological setting, which is twisted Cohomotopy theory, and prove some fundamental facts about it.
– In §4 we use the results of §3 to prove that *Hypothesis H* implies the anomaly cancellation conditions from §2.
– We conclude in Remark 4.29.

Outlook. The ideas, constructions, and results in this paper lead naturally to several topics which deserve discussion in the future, including the following:

◦ **Generalized Riemannian geometry.** One may also consider *generalized Cohomotopy* with coefficients products of spheres $S^n \times S^n$; see Remark 3.22. Twists for such generalized Cohomotopy arise from topological G-structure for Spin groups in split signature. We will discuss this elsewhere.

◦ **Equivariant Cohomotopy.** While here we explicitly consider only the plain topological sector of the C-field, hence its charge quantization in plain homotopy theory, the natural form of the charge quantization formulation in *Figure T* immediately generalizes to global equivariant and differential Cohomotopy (in the sense of [HSS18] and [FSS15][FSS16a]). Specifically, the enhancement to global equivariant Cohomotopy yields a definition of C-field charge quantization on orbifold spacetimes. Elsewhere in [RSS] we show that this generalization correctly captures further statements from the folklore, such as the tadpole cancellation for M5-branes at MO5-planes, according to [Wi96a, Sec. 2.3][Ho99].

2 M-theory anomaly cancellation in the folklore

For precise reference and complete discussion, in this section we review the state of the art on anomaly cancellation conditions in M-theory. These conditions all revolve around the C-field.

Folklore. A note on string theory folklore is in order. For a reasonable discussion of the open problem of formulating M-theory, it is necessary to distinguish established facts from plausibility arguments. The latter in the string theory literature, remarkably, do form a tantalizingly tight web, which is quite undoubtedly pointing to the existence of an actual underlying theory. Consequently, various conjectured phenomena of M-theory have become folklore statements that much of the string theory literature treats as established facts. But there remain problems with this (see §2.5). Ultimately, progress on foundations of M-theory will only be possible if one disentangles plausible assumptions from established facts.

For instance, the all-important shifted flux quantization condition of the C-field (reviewed in §2.2) is introduced in [Wi96a] by stating that it is "motivated" by the expected M/heterotic duality (in [Wi96a, Sec. 2.1]) and that there is "belief" in a more conclusive argument from M2-brane anomalies (in [Wi96a, Sec. 2.2]). When other authors consider other plausibility arguments, the common base is not readily established, e.g. [Ts04, p. 3].

Hence when we refer to such arguments as "folklore", this is not to doubt them, but to clarify what it means when we rigorously derive these conditions in a systematic fashion from first principles, below in §4.

2.1 DMW anomaly cancellation

In the comparison to type IIA string theory, Diaconescu, Moore and Witten [DMW03a] consider M-theory spacetime Y^{11} to be a product $X^{10} \times S^1$ and the C-field lifted from the type IIA base (see [MS04] for generalizing these conditions). The phase of the partition function of the C-field is given by $\Phi_a = (-1)^{f(a)}$, where a is the integral class characterizing the E_8 bundle over 11-dimensional Spin manifolds $Y^{11} = X^{10} \times S^1$ and $f(a)$ is $\mathbb{Z}_2$-invariant which satisfies:

- $f(a) = 0$ for $a = 0$,
- for $a, b \in H^4(X^{10};\mathbb{Z})$, there is a bilinear relation $f(a+b) = f(a) + f(b) + \int_{X^{10}} a \cup \mathrm{Sq}^2(\rho_2(b))$, where ρ_2 is mod 2 reduction.

For torsion classes, the torsion pairing $T : H^4_{\mathrm{tor}}(X^{10};\mathbb{Z}) \times H^7(X^{10};\mathbb{Z}) \to U(1)$ is given by $T(a,b) = \int_X a \cup c$ with $\beta(c) = b$ where β is the Bockstein corresponding to the exponential sequence $\mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z}$. In terms of this, $f(2b) = T(b, \mathrm{Sq}^3(\rho_2(\tfrac{1}{2}p_1)))$. Nondegeneracy requires that the third Steenrod operation Sq^3 annihilates the (mod 2 reduction of) the first fractional Pontrjagin class of the Spin tangent bundle [DMW03a]: $\mathrm{Sq}^3(\rho_2(\tfrac{1}{2}p_1(TX))) = 0$. But the expression on the left is equal to the integral Stiefel-Whitney class W_7 of the tangent bundle, hence the DMW anomaly cancellation condition is the following condition (see [DMW03b, p. 14])

$$W_7(TX) \;=\; 0\,. \tag{6}$$

The DMW anomaly cancellation is interpreted in [KS04] as an orientation condition for a connected closed Spin 10-dimensional spacetime with respect to second integral Morava K-theory $\widetilde{K}(2)$ and E-theory $E(2)$ at the prime 2. Twisted and differential versions were developed in [SW15] and [GS17], respectively. These results also mean that the corresponding expressions should take values in the above generalized cohomology theories (see [Sa10]), but we will not take that route here.

We show in §4.1 that *Hypothesis H*, implies the DMW anomaly cancellation condition. This will require a cohomological characterization of $\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)$-structures, which we establish in Prop. 4.4 below.

2.2 Half-integral flux quantization

In the approximation of M-theory by 11-dimensional supergravity, the C-field flux/field strength G_4 is simply a higher-degree analogue of the Faraday tensor of electromagnetism, hence a closed differential 4-form on 11-dimensional spacetime. Regarded as a representative in de Rham cohomology this is equivalently, via the classical de Rham theorem, a cocycle in the singular real cohomology of spacetime:

$$G_4 \in Z^4(X;\mathbb{R}) , \quad [G_4] \in H^4(X;\mathbb{R}) . \tag{7}$$

In [Wi96a], three arguments are given that this 4-form must have integral or half-integral periods, depending on whether the rational class $\frac{1}{4}p_1$ of the tangent bundle has integral or half-integral periods, respectively:

$$\underbrace{[G_4] + \tfrac{1}{4}p_1(TX)}_{=:[\widetilde{G}_4]} \in H^4(X;\mathbb{Z}) . \tag{8}$$

Notice that these arguments rely on some assumptions: the argument in [Wi96a, 2.2] considers an M2-brane spacetime of the form $\mathbb{R}^{2,1} \times X^8$ such that the space X^8 transverse to an M2-brane locus has a circle factor, and hence in particular that the structure group is reduced along

$$\mathrm{Spin}(7) \hookrightarrow \mathrm{Spin}(10,1) . \tag{9}$$

The argument in [Wi96a, 2.3] considers an M5-brane spacetime of the form $\mathbb{R}^{5,1} \times X^5$, and hence in particular that the structure group is reduced along

$$\mathrm{Spin}(5) \hookrightarrow \mathrm{Spin}(10,1) . \tag{10}$$

Beware that these folklore arguments, while plausible and interesting, they are circumstantial. The condition is "motivated" by the expected M-theory/heterotic duality in [Wi96a, Sec. 2.1] and there is "belief" in a more conclusive argument from M2-brane anomalies in [Wi96a, Sec. 2.2], and in [Wi96b, (3.5)] this is "suggested". A more rigorous derivation has been missing; however, see [Sa10].

We show below in §4.2 how *Hypothesis H* implies the half-integral flux quantization (8).

2.3 Integral equation of motion

According to [DMW03a], the class of the shifted C-field flux $[\widetilde{G}_4] := [G_4] + \frac{1}{4}p_1$ (which is an integral cohomology class according to §2.2) must satisfy the "integral equation of motion"

$$\mathrm{Sq}^3(\widetilde{G}_4) = 0 , \tag{11}$$

where Sq^3 (mod 2 reduction followed by) the degree three Steenrod square operation acting on integral cohomology. From the point of view of M-theory this is argued to come about from the M-theoretic path integral over the torsion component of $\widetilde{G}_4$ acting like a projection operator on those elements satisfying this condition; see [DMW03b, Sec. 5].

But condition (11) also implies, or is implied by (depending on perspective) the argument that after Kaluza-Klein (KK)-compactification to type IIA string theory, the 4-class of $\widetilde{G}_4$, which then is interpreted as the Ramond-Ramond (RR) 4-flux, has to lift to complex K-theory KU [MMS01, p. 11 (12 of 45)], [ES06]. This is of course itself another famous piece of string theory folklore [Wi98] (see [Ev06] for a survey and [GS19] for a mathematically solid treatment), whose relation to *Hypothesis H* has been discussed in [BSS18].

We show below in §4.3 how *Hypothesis H* implies the integral equation of motion (11).

2.4 Background charge

A priori, the C-field flux G_4 seems to appear via its plain cup square $\big(G_4\big)^2$ in the Bianchi identity from the supergravity equations of motion (1), $dG_7 = -\frac{1}{2}\big(G_4\big)^2$, as well as in the Chern-Simons term for the 7-dimensional Chern-Simons theory. In [Wi96b, 3.4] it was argued that this needs to be refined by a quadratic form

$$\begin{aligned} q(G_4) &:= \tfrac{1}{2}\big((G_4)^2 - \big(\tfrac{1}{4}p_1\big)^2\big) \\ &= \tfrac{1}{2}\big((\tilde{G}_4)^2 - \tilde{G}_4 \tfrac{1}{2}p_1\big) \end{aligned} \tag{12}$$

in order to imply divisibility by two of the prequantum line bundle of the 7d Chern-Simons theory. Here in the second line we re-expressed this in terms of the integral shifted expression $\widetilde{G}_4 = G_4 + \frac{1}{4}p_1$ from (8). In the course of formalizing and proving this divisibility statement, [HS05, 1.1] amplified that this quadratic form is a quadratic refinement of the intersection pairing, in that

$$q\big(\widetilde{G}_4 + \widetilde{G}'_4\big) - q\big(\widetilde{G}_4\big) - q\big(\widetilde{G}'_4\big) + q(0) \;=\; \widetilde{G}_4\widetilde{G}'_4 \,.$$

The *center* of such a quadratic refinement is the value $\big(\widetilde{G}_4\big)_0$ such that reflecting the field value *around this center* leaves the quadratic form invariant

$$q\big((\widetilde{G}_4)_0 - \widetilde{G}_4\big) \;=\; q\big(\widetilde{G}_4\big)\,.$$

The physics interpretation is that $\big(\widetilde{G}_4\big)_0$ is the *background charge* of the field, in the sense explained in [Fr00] [Fr09, p. 11]. In the present case of (12) the center/background charge of the C-field is given by the first fractional Pontrjagin class

$$(\widetilde{G}_4)_0 \;=\; \tfrac{1}{2}p_1\,.$$

For more review see [FSS14a, 3.2], where this setting is generalized and refined to the context of smooth stacks. The quadratic refinement was also studied via Spin bundles and their K-theory in [Sa08].

We show how *Hypothesis H* leads to the C-field background charge in §4.4.

2.5 M5-brane anomaly cancellation

We review here the approaches and results leading to the tradition on M5-brane anomaly cancellation.

The worldvolume QFT on the M5-brane. A fundamental aspect of string theory folklore is that the worldvolumes of D-branes are supposed to carry quantum gauge field theories of Yang-Mills type (in this context, see [BSS18, p. 3] for review and pointers to the literature). This and other arguments lead to the expectation that the worldvolume of the M5-brane in M-theory should carry a 6-dimensional quantum field theory of a self-dual higher gauge field (see [Moo12] for an extensive review).

M5-brane worldvolume anomaly. However, by itself such a quantum field theory would exhibit a quantum anomaly, making it inconsistent: the presence of the self-dual higher gauge field implies an anomaly term $I^{\mathrm{M5}}_{\mathrm{SD}}$ and the presence of chiral fermions implies an anomaly term $I^{\mathrm{M5}}_{\mathrm{ferm}}$, and the resulting total worldvolume anomaly of the M5-brane

$$I^{\mathrm{M5}} := I^{\mathrm{M5}}_{\mathrm{SD}} + I^{\mathrm{M5}}_{\mathrm{ferm}} \tag{13}$$

is in general non-vanishing.

Meaning of anomaly cancellation. Here these "anomaly terms" are meant to be classes of degree 8 differential cohomology on a universal moduli stack of field configurations. This implies that their transgression to the worldvolume 6-manifold Σ_6 of the 5-brane is a class in degree 2 differential cohomology on the moduli stack of fields on Σ_6, hence the class of a complex line bundle with connection. The action functional which defines the worldvolume quantum field theory is generally a covariantly constant section of such an anomaly line bundle. But, traditionally, in order to potentially make sense of the path integral over the action functional (which famously has not actually been made sense of) the action functional must be a genuine complex-valued function, hence the anomaly line bundle must trivialize, as a line bundle with connection, hence as a class in *differential cohomology*. A choice of such trivialization is then called *anomaly cancellation* (see [Fr00, p. 4]). Now, full-blown differential cohomology on universal moduli stacks of fields is a demanding subject, and doing full justice to this requires considerable technology, as exemplified in [HS05][FSS14b][FSS14a] already in a small sub-sector of the expected 5-brane theory. Collecting all available ideas on the full "global" M5-brane anomaly is taken up in [Mo14][Mo15].

Local anomaly in rational/de Rham cohomology. However, by the definition of differential cohomology and by the nature of rational homotopy theory, the image of these anomaly terms in rational/real cohomology, equivalently in de Rham cohomology, is the primary obstruction to anomaly cancellation: vanishing of the anomaly terms in rational/de Rham cohomology is in general not sufficient to deduce anomaly cancellation, but is always *necessary*. This image is essentially what is called the "local anomaly", namely the anomaly curvature form.[1] This is what the literature has mostly concentrated on and is most sure about. For instance, the M5-brane worldvolume anomaly terms recorded in [Wi96b, Sec. 5] are really (only) such local anomaly terms in rational/de Rham cohomology. This is what we focus on for the remainder of this section.

Anomaly inflow from the bulk. Since the M5-brane is not meant to exist abstractly by itself, but to propagate, within M-theory, in an ambient ("bulk") 11-dimensional spacetime, there is supposed to be a "bulk anomaly inflow" which contributes a further term $I^{\text{bulk}}|_{\text{M5}}$ to the 5-brane anomaly. The folklore argues roughly that

(i) if M-theory exists it should be consistent and hence anomaly-free,

(ii) both the 5-brane worldvolume QFT as well as the ambient supergravity should be limiting cases of M-theory, hence,

(iii) the sum of the worldvolume anomaly and the "bulk anomaly inflow" should vanish:

$$I^{\text{M5}} + I^{\text{bulk}}|_{\text{M5}} \overset{!?}{=} 0\,. \tag{14}$$

Now, at least at the level of rational or de Rham cohomology, the relevant part of the bulk action functional is supposed to be the integral of

$$I^{\text{bulk}} := \underbrace{-\tfrac{1}{6} G_4 \wedge G_4 \wedge G_4}_{I^{\text{bulk}}_{\text{Sugra}}} + G_4 \wedge I_8 \tag{15}$$

over a 12-dimensional manifold cobounding the given 11-dimensional spacetime. Here the first summand $I^{\text{bulk}}_{\text{Sugra}}$ is the contribution visible in plain classical supergravity. The second contribution is called the "one-loop anomaly term" ([VW95, Sec. 3][DLM95, (3.10) with (3.14)]) proportional to a combination of Pontrjagin classes:

$$I_8 \;=\; \tfrac{1}{48}\big(p_2 - \tfrac{1}{4}p_1^2\big). \tag{16}$$

First attempt to argue M5-brane anomaly cancellation. The traditional idea was that the I_8 term alone is to be regarded as the bulk anomaly [Wi96b, p. 32], hence that M5-brane anomaly cancellation should mean the vanishing of $I^{\text{M5}} + I_8$. However, straightforward computation showed that this sum is instead proportional to the second Pontrjagin class p_2 of the normal bundle N of the M5-brane [Wi96b, (5.7)] (which we will denote $N_X Q_{M5}$ below in §4):

$$I^{\text{M5}} + I_8 \;=\; \tfrac{1}{24} p_2(N)\,. \tag{17}$$

In view of the expected cancellation (14), this result was felt to be "somewhat puzzling" [Wi96b, p. 35], since this term does not generally vanish; and arguments for conditions to impose under which it would vanish were "not clear" [Wi96b, p. 37].

[1] Given an anomaly line bundle with connection, its curvature form is called the "local anomaly", while its holonomies are called the "global anomaly" [Fr86, p.1]. But the holonomies completely characterize a line bundle with connection [Ba91] [CP94], and hence the corresponding cocycle in differential cohomology, so that the "global anomaly" is equivalently the full differential cocycle, including its class in integral cohomology, as opposed to just in rational/de Rham cohomology.

Second attempt. In reaction to this state of affairs, it was argued in [FHMM98] that the computation in [Wi96b] overlooked the fact that the 4-flux G_4 must be required to have a singularity at the locus of the M5-brane, and that taking this singular behavior into account reveals an extra contribution to the anomaly term of exactly $-\frac{1}{24}p_2$, thus cancelling the anomaly after all.

Third attempt. In further reaction to this, [Mo15, Sec. 2.3] argued that,

1. First, the mathematical setup should be revisited by

(i) removing the M5-brane locus from spacetime (just as in Dirac's original argument on magnetic monopoles), thereby doing away with any singularities in G_4 and instead regarding a non-singular field configuration on an S^4-fibered spacetime X^{11} (the $4 = (11-6-1)$-sphere being the unit sphere around a 5-dimensional submanifolds inside an 11-dimensional manifold

$$S^4 \hookrightarrow X \xrightarrow{\pi} X_{\text{base}} \tag{18}$$

where typically

$$X_{\text{base}} \simeq Q_{\text{M5}} \times \mathbb{R}_{>0} \times U$$

is the product of the abstract M5-brane worldvolume Q_{M5}, the positive distances $\mathbb{R}_{>0}$ away from it, and an auxiliary finite-dimensional manifold U over which the situation is parametrized in families (see also Remark 4.19 below),

(ii) and declaring that "restriction to the 5-brane $(-)|_{\text{M5}}$" (which does not literally make sense, as the actual singular M5-brane locus is not part of the space X) should really be *fiber integration $\int_{S^4}$ over the 4-sphere fibration*:

$$I^{\text{bulk}}|_{\text{M5}} := \int_{S^4} I^{\text{bulk}} := \pi_*\big(I^{\text{bulk}}\big). \tag{19}$$

2. Second, [Mo15, Sec. 3.3] asserted that the "bulk anomaly inflow" should be induced not just by the I_8 term, as traditionally assumed, but by the *whole* supergravity term (15).

Accepting these two proposals, a straightforward and rigorous computation of the bulk inflow contribution, using [BC97, Lemma 2.1] (which already played the central role in the argument of [FHMM98]), yields

$$I^{\text{bulk}}|_{\text{M5}} := \underbrace{-\int_{S^4} \tfrac{1}{6} G_4 \wedge G_4 \wedge G_4}_{\frac{1}{24}p_2(N) + \frac{1}{2}(G_4^{\text{basic}})^2} \quad + \quad \underbrace{\int_{S^4} G_4 \wedge I_8}_{I_8}. \tag{20}$$

Here G_4 has been assumed (see Def. 4.17 below for details) to be the sum of the unit half-Euler class $\frac{1}{2}\chi$ on the 4-sphere fiber (reflecting the unit flux/charge associated with a single 5-brane, just as in Dirac's old argument) plus the pullback $\pi^*\big(G_4^{\text{basic}}\big)$ of a form on the base of the fibration, not contributing to the flux through the 4-sphere. As shown under the braces, this proposal implies a contribution of $-\frac{1}{24}p_2(N)$ appearing as "anomaly inflow" from the previously neglected supergravity term, which thus cancels the "puzzling" remainder of [Wi96b] in (17), along the lines of [FHMM98].

State of the folklore. However, there is a second term appearing under the first brace in (20). Therefore, in summary at this point of the development, the conclusion of the folklore argument is that the total anomaly of the M5-brane is

$$\begin{aligned} I^{\text{M5}}_{\text{tot}} &= \underbrace{I^{\text{M5}} + I_8}_{\frac{1}{24}p_2(N)} \quad - \quad \underbrace{I^{\text{bulk}}_{\text{Sugra}}|_{\text{M5}}}_{\frac{1}{24}p_2(N) + \frac{1}{2}(G_4^{\text{basic}})^2} \\ &= -\tfrac{1}{2}(G_4^{\text{basic}})^2. \end{aligned} \tag{21}$$

This still does not vanish – the refined proposal (19), (20) for the bulk anomaly inflow has, at this point, served to replace the residual 5-brane anomaly $\frac{1}{24}p_2$ of [Wi96b, (5.7)] not with zero, as argued in [FHMM98], but with $-\frac{1}{2}\big(G_4^{\text{basic}}\big)^2$. In order to make this residual term disappear, in accord with the expected result (14), [Mo15, (3.7)] gives an alternative formula for the worldvolume anomaly contribution $I_{\mathrm{SD}}^{\mathrm{M5}}$ (13) of the self-dual higher gauge field, by adding to it a summand proportional to $\left(G_4^{\mathrm{basic}}\right)^2$, with opposite sign.

But this does not seem to be completely justifiable: while, by the discussion above, it is true that there is ambiguity in the *torsion* components of this term not visible to rational/de Rham cohomology, which are explored in [Mo14, Sec. 4], the term $\frac{1}{2}\left(G_4^{\mathrm{basic}}\right)^2$ is not in general a torsion class, hence adding it to $I_{\mathrm{SD}}^{\mathrm{M5}}$ would in general violate the known form [Wi96b, (5.4)] of this local anomaly in real/de Rham cohomology.

Conclusion. In summary, the M5-brane anomaly is, a priori, given by (21). Hence, in view of (14), a coherent formulation of M-theory as a consistent theory should systematically imply the remaining cancellation condition

$$\left[G_4^{\mathrm{basic}}\right]^2 \;=\; 0 \tag{22}$$

at least in real/de Rham cohomology.

We discuss in §4.5 how this follows from *Hypothesis H* and derive (22) in Prop. 4.18 below.

2.6 M2-brane tadpole cancellation

Compactifications of M-theory on 8-manifolds X^8 and with vanishing C-field flux were argued in [SVW96] [Wi96a, Sec. 3] to require that the number of M2-branes N_{M2} equals the integral of the I_8-class (16)

$$N_{\mathrm{M2}} \;=\; I_8([X^8]) \tag{23}$$

in order to cancel a tadpole anomaly. If X^8 is assumed to have Spin(7)-structure then I_8 is related to the Euler 8-class χ_8 via

$$I_8 = \tfrac{1}{24}\chi_8\,. \tag{24}$$

For Calabi-Yau 4-folds CY4, hence for $\mathrm{SU}(4) \simeq \mathrm{Spin}(6)$-structures, this is [BB96, (2.22)][SVW96, p. 2], while more generally for Spin(7)-structure this is discussed in [GST02] following [IP88]. We notice below in expression (73) of Prop. 4.4 that relation (24) is also implied by G-structure for $G = \mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$, defined in (42).

In any case, in applications, relation (24) typically holds and hence implies that the tadpole cancellation condition becomes equivalently the condition that the number of M2-branes is the Euler characteristic of the compactification manifold:

$$N_{\mathrm{M2}} \;=\; \tfrac{1}{24}\chi_8([X^8])\,. \tag{25}$$

At the same time, [BB96, (2.58)] gave a complementary argument that in the absence of any M2-branes but in the presence of possibly non-vanishing squared C-field flux G_4, the tadpole cancellation condition is

$$-\tfrac{1}{2}\int_{X^8} G_4 \wedge G_4 \;=\; \tfrac{1}{24}\chi_8([X^8])\,. \tag{26}$$

In reaction to this situation, [DM96, (1)] assumed that there is a sign error in [BB96, (2.58)] and that, in the general situation, when neither the number of M2-branes nor the squared C-field-flux is taken to vanish, equations (25) and (26) should be jointly generalized to the equation

$$N_{\mathrm{M2}} + \tfrac{1}{2}\int_{X^8} G_4 \wedge G_4 \;=\; \tfrac{1}{24}\chi_8[X^8]\,. \tag{27}$$

In support of this assumption, [DM96, p. 3] offered a consistency check in the special case where $X^8 = K3 \times K3$, arguing that under the expected duality between M-theory and both the heterotic as well as the type IIA-string, equation (27) is compatible with similar formulas expected in these theories. From here on, starting with [GVW99, (2.1)] and [DRS99, (2.1)], the string theory literature takes (27) for granted. These days condition (27) plays a prominent role also in string model building; see for instance [CHLLT19, (9)].

We discuss how these M2-brane tadpole cancellation conditions appear from *Hypothesis H*, below in §4.6.

3 Cohomotopy theory

We introduce *J-twisted Cohomotopy* theory, the twisted generalization of *Cohomotopy theory*, in §3.1. In §3.2 we discuss how twisted Cohomotopy is equivalently a special sector in the theory of G-structures hence a special sector of Cartan geometry: every coset space realization of the n-sphere reflects an exceptional sector of twisted Cohomotopy in degree n. Using this, we turn to analyze particularly the twists of Cohomotopy in degree 7 and 4: in §3.3 we discuss how twisted Cohomotopy in degrees 4 and 7 combined, respecting the quaternionic Hopf fibration, singles out $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$-structure and $\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)$-structure, related by triality. In §3.4 we show how exceptional twists of Cohomotopy in degree 7 alone isolates the exceptional G-structures which happen to control $N=1$ compactifications of F-theory, M-theory, and type IIA string theory. In §3.5 we observe that the classical Poincaré-Hopf theorem expresses twisted Cohomotopy in terms of the Euler characteristic, while in §3.6 we comment on how the classical Pontrjagin-Thom theorem relates Cohomotopy to cobordism classes of submanifolds (branes) with normal structure.

In summary, we may say that twisted Cohomotopy theory by itself is a *grand unified theory of differential topology*. Below in §4 we discuss how with its physics interpretation under *Hypothesis H*, twisted Cohomotopy theory implies anomaly cancellation in M-theory.

3.1 Twisted Cohomotopy

The non-abelian cohomology theory (see [NSS12], following [SSS12]) represented by the n-spheres is called *Cohomotopy*, going back to [Bo36][Sp49]. Hence for X a topological space, its *Cohomotopy set* in degree n is

$$\pi^n(X) \;=\; \pi_0 \mathrm{Maps}\big(X, S^n\big) \;=\; \Big\{ X \xrightarrow{\text{cocycle in Cohomotopy}} S^n \Big\}_{/\sim}. \tag{28}$$

A basic class of examples is Cohomotopy of a manifold X in the same degree as the dimension $\dim(X)$ of that manifold. The classical *Hopf degree theorem* (see, e.g., [Ko93, IX (5.8)], [Kob16, 7.5]) says that for X connected, orientable and closed, this is canonically identified with the integral cohomology of X, and hence with the integers

$$\pi^n(X) \xrightarrow[S^n \to K(\mathbb{Z},n)]{\simeq} H^n(X;\mathbb{Z}) \simeq \mathbb{Z}\,, \qquad \text{for } n = \dim(X). \tag{29}$$

In its generalization to the *equivariant Hopf degree theorem* this becomes a powerful statement about equivariant Cohomotopy theory and thus, via *Hypothesis H*, about brane charges at orbifold singularities [HSS18]. We discuss this in detail elsewhere [RSS].

Here we generalize ordinary Cohomotopy (28) to *twisted Cohomotopy* (Def. 3.1 below), following the general theory of non-abelian (unstable) twisted cohomology theory [NSS12, Sec. 4]. [2] Generally, Cohomotopy in degree n may by twisted by $\mathrm{Aut}(S^n)$-principal ∞-bundles, for $\mathrm{Aut}(S^n) \subset \mathrm{Maps}(S^n,S^n)$ the automorphism ∞-group of S^n inside the mapping space from S^n to itself.

A well-behaved subspace of twists comes from $\mathrm{O}(n+1)$-principal bundles, or their associated real vector bundles of rank $n+1$, under the inclusion

$$\widehat{J}_n \;:\; \mathrm{O}(n+1) \hookrightarrow \mathrm{Aut}(S^n) \hookrightarrow \mathrm{Maps}(S^n,S^n)\;, \tag{30}$$

which witnesses the canonical action of orthogonal transformations in Euclidean space $\mathbb{R}^{n+1}$ on the unit sphere $S^n = S(\mathbb{R}^{n+1})$. The restriction of these to $O(n)$-actions

$$J_n \;:\; \mathrm{O}(n) \hookrightarrow \mathrm{O}(n+1) \xhookrightarrow{\widehat{J}_n} \mathrm{Maps}(S^n,S^n)$$

[2] Since all the involved constructions will be homotopical, all group actions, principal bundles, etc. will be "up to coherent homotopy", in a sense that can be made completely rigorous via the notion of an ∞-group, an ∞-group action and an ∞-principal bundle [NSS12]. The reader unfamiliar with this higher homotopical version of the theory of principal bundles can safely translate each occurrence of the prefix "∞-" with the colloquial "up to homotopy, in a systematic way" and get a pretty accurate intuition of what is going on.

are known as the unstable *J-homomorphisms* [Wh42] (see [Ko93][Mat12] for expositions). By general principles [NSS12], the homotopy quotient $S^n /\!\!/ O(n+1)$ of S^n by the action via $\widehat{J}_n$ is canonically equipped with a map $\widetilde{J}_n$ to the classifying space $BO(n+1)$, such that the homotopy fiber is S^n:

$$
\begin{tikzcd}
S^n \arrow[r] & S^n /\!\!/ \mathrm{O}(n+1) \arrow[d] \\
& B\mathrm{O}(n+1)\,.
\end{tikzcd}
$$

One may think of this as the universal spherical fibration which is the S^n-fiber ∞-bundle associated to the universal $O(n+1)$-principal bundle via the homotopy action $\hat{J}_n$.

Definition 3.1 (Twisted Cohomotopy)**.** Given a map $\tau : X \to B\mathrm{O}(n+1)$, we define the *$\tau$-twisted cohomotopy set of X* to be

$$
\begin{aligned}
\pi^\tau(X) \;:=\; \mathbf{H}_{/B\mathrm{O}(n+1)}\big(\tau, \widetilde{J}_{\mathrm{O}(n+1)}\big) &= \left\{
\begin{tikzcd}
& S^n /\!\!/ \mathrm{O}(n+1) \arrow[d, "\widetilde{J}_{\mathrm{O}(n+1)}"] \\
X \arrow[ur, dashed, "{\text{cocycle in twisted Cohomotopy}}"] \arrow[r, "\tau", "\text{twist}"'] \arrow[ur, phantom, "\Swarrow" near end] & B\mathrm{O}(n+1)
\end{tikzcd}
\right\} \Big/ \sim \\
&= \left\{
\begin{tikzcd}
& E \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & S^n /\!\!/ \mathrm{O}(n+1) \arrow[d, "\widetilde{J}_{\mathrm{O}(n+1)}"] \arrow[dl, phantom, "\Swarrow"] \\
X \arrow[ur, dashed, "{\text{cocycle in twisted Cohomotopy}}"] \arrow[r, equal] & X \arrow[r, "\tau", "\text{twist}"'] & B\mathrm{O}(n+1)
\end{tikzcd}
\right\} \Big/ \sim
\end{aligned}
\tag{31}
$$

Here in the second line, $E \to X$ denotes the n-spherical fibration classified by τ and the universal property of the homotopy pullback shows that cocycles in τ-twisted equivariant Cohomotopy are equivalently sections of this n-spherical fibration.

Remark 3.2 (Notation)**.** Here the notation $\pi^\tau(X)$ is motivated, as usual in twisted cohomology, from thinking of the map τ as encoding, in particular, also the degree $n \in \mathbb{N}$.

Remark 3.3 (Cohomotopy twist by Spin structure)**.** In applications, the twisting map τ is often equipped with a lift through some stage of the Whitehead tower of $B\mathrm{O}(n+1)$, notably with a lift through $B\mathrm{SO}(n+1)$ or further to $B\mathrm{Spin}(n+1)$

$$
\begin{tikzcd}
X \arrow[r, "\widehat{\tau}"] \arrow[rr, bend right, "\tau"'] & B\mathrm{Spin}(n+1) \arrow[r] & B\mathrm{O}(n+1)\,.
\end{tikzcd}
$$

In this case, due to the homotopy pullback diagram

$$
\begin{tikzcd}
S^n /\!\!/ \mathrm{Spin}(n+1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & S^n /\!\!/ \mathrm{O}(\mathrm{n}+1) \arrow[d, "\widetilde{J}_{\mathrm{O}(n+1)}"] \\
B\mathrm{Spin}(n+1) \arrow[r] & B\mathrm{O}(n+1)
\end{tikzcd}
$$

the twisted cohomotopy set from Def. 3.1 is equivalently given by

$$
\pi^\tau(X) \;\simeq\; \mathbf{H}_{/B\mathrm{Spin}(n+1)}\big(\widehat{\tau}, \widetilde{J}_{\mathrm{Spin}(n+1)}\big) \;:=\; \left\{
\begin{tikzcd}
& S^n /\!\!/ \mathrm{Spin}(n+1) \arrow[d, "\widetilde{J}_{\mathrm{Spin}(n+1)}"] \\
X \arrow[ur, dashed, "{\text{cocycle in twisted Cohomotopy}}"] \arrow[r, "\widehat{\tau}", "\text{twist}"'] \arrow[ur, phantom, "\Swarrow" near end] & B\mathrm{Spin}(n+1)
\end{tikzcd}
\right\} \Big/ \sim
\tag{32}
$$

Most of the examples in §3.3 and §4 arise in this form.

In order to extract differential form data ("flux densities") from cocycles in twisted Cohomotopy, in Prop. 3.5 below, we consider rational twisted Cohomotopy (Def. 3.4) below. A standard reference on the rational homotopy theory involved in [FHT00]. Reviews streamlined to our context can be found in [FSS16a, Appendix A][BSS18].

Definition 3.4 (Rationalizing twisted Cohomotopy)**.** We write $\pi^\tau(X) \xrightarrow{(-)_{\mathbb{Q}}} \pi^\tau_{\mathbb{Q}}(X)$ for the rationalization of twisted Cohomotopy to *rational twisted Cohomotopy*, given by applying rationalization to all spaces and maps involved in a twisted Cohomotopy cocycle.

We now characterize cocycles in rational twisted Cohomotopy in terms of differential form data (which will be the corresponding "flux density" in §4).

Proposition 3.5 (Differential form data underlying twisted Cohomotopy)**.** *Let X be a simply connected smooth manifold and $\tau : X \to B\mathrm{O}(n+1)$ a twisting for Cohomotopy in degree n, according to Def. 3.1. Let ∇_τ be any connection on the real vector bundle V classified by τ with Euler form $\chi_{2k+2}(\nabla_\tau)$ (see [MQ86, below (7.3)][Wu06, 2.2]).*

(i) If $n = 2k+1$ **is odd**, $n \geq 3$, *a cocycle defining a class in the rational τ-twisted Cohomotopy of X (Def. 3.4) is equivalently given by a differential $2k+1$-form $G_{2k+1} \in \Omega^{2k+1}(X)$ on X which trivializes the Euler form*

$$\pi^\tau_{\mathbb{Q}}(X) \;\simeq\; \big\{ G_{2k+1} \,|\, d\,G_{2k+1} \;=\; \chi_{2k+2}(\nabla_\tau) \big\}\big/_{\sim}\,.$$

(ii) If $n = 2k$ **is even**, $n \geq 2$, *a cocycle defining a class in the rational τ-twisted Cohomotopy of X (Def. 3.4) is given by a pair of differential forms $G_{2k} \in \Omega^{2k}(X)$ and $G_{4k-1} \in \Omega^{4k-1}(X)$ such that*

$$\begin{aligned} dG_{2k} = 0; \qquad \pi^* G_{2k} = \tfrac{1}{2}\chi_{2k}(\nabla_{\hat{\tau}}) \\ dG_{4k-1} = -G_{2k} \wedge G_{2k} + \tfrac{1}{4} p_k(\nabla_\tau), \end{aligned}$$

where $p_k(\nabla_\tau)$ is the k-th Pontrjagin form of ∇_τ, $\pi\colon E \to X$ is the unit sphere bundle over X associated with τ, $\hat{\tau}\colon E \to B\mathrm{O}(n)$ classifies the vector bundle $\widehat{V}$ on E defined by the splitting $\pi^* V = \mathbb{R}_E \oplus \widehat{V}$ associated with the tautological section of $\pi^* V$ over E, and $\nabla_{\hat{\tau}}$ is the induced connection on $\widehat{V}$. That is,

$$\pi^\tau_{\mathbb{Q}}(X) \;\simeq\; \left\{ \big(G_{2k}, G_{4k-1}\big) \,\middle|\, \begin{array}{r} d\,G_{2k} = 0\,, \quad \pi^* G_{2k} = \tfrac{1}{2}\chi_{2k}(\nabla_{\hat{\tau}}) \\ d\,G_{4k-1} = -G_{2k} \wedge G_{2k} + \tfrac{1}{4} p_k(\nabla_\tau) \end{array} \right\} \Big/_{\sim}\,.$$

Proof. By the assumption that the smooth manifold X is simply connected, it has a Sullivan model dgc-algebra $\mathrm{CE}\big(\mathfrak{l}X\big)$ and we may compute the rational twisted Cohomotopy by choosing a Sullivan model $\mathfrak{l}E$ for the spherical fibration classified by τ. By definition of rational twisted Cohomotopy, we are interested in the set of homotopy equivalence classes of dgca morphisms $\mathrm{CE}(\mathfrak{l}E) \to \mathrm{CE}(\mathfrak{l}X)$ that are sections of the morphism $\mathrm{CE}(\mathfrak{l}X) \to \mathrm{CE}(\mathfrak{l}E)$ corresponding to the projection $E \to X$. The Sullivan model model for E is well known. We recall from [FHT00, Sec. 15, Example 4]:

(I). The Sullivan model for the total space of a $2k+1$-spherical fibration $E \to X$ is of the form

$$\mathrm{CE}(\mathfrak{l}E) \;=\; \mathrm{CE}(\mathfrak{l}X) \otimes \mathbb{R}[\omega_{2k+1}] / \big(d\,\omega_{2k+1} = c_{2k+2}\big)\,, \tag{33}$$

where

(a) $c_{2k+2} \in \mathrm{CE}\big(\mathfrak{l}X\big)$ is some element in the base algebra, which by (33) is closed and so it represents a rational cohomology class

$$[c_{2k+2}] \;=\; H^{2k+2}(X;\mathbb{Q}).$$

This class classifies the spherical fibration, rationally. Moreover, if the spherical fibration $E \to X$ happens to be the unit sphere bundle $E = S(V)$ of a real vector bundle $V \to X$, then the class of c_{2k+2} is the rationalized Euler class $\chi_{2k+2}(V)$ of V:

$$[c_{2k+2}] \;=\; \chi_{2k+2}(V) \;\in\; H^{2k+2}(X;\mathbb{Q})\,. \tag{34}$$

(b) and in this case, under the quasi-isomorphism $\mathrm{CE}(\mathfrak{l}E) \to \Omega^\bullet_{\mathrm{dR}}(E)$ the new generator ω_{2k+1} corresponds to a differential form that evaluates to minus the unit volume on each $(2k+1)$-sphere fiber (e.g., [Wa04, Ch. 6.6, Thm. 6.1]):

$$\langle \omega_{2k+1}, [S^{2k+1}] \rangle \;=\; -1\;. \tag{35}$$

The morphism $\mathrm{CE}(\mathfrak{l}X) \to \mathrm{CE}(\mathfrak{l}E)$ is the obvious inclusion, so a section is completely defined by the image of ω_{2k+1} in $\mathrm{CE}(\mathfrak{l}X)$. This image will be an element $g_{2k+1} \in \mathrm{CE}(\mathfrak{l}X)$ such that $dg_{2k+1} = c_{2k+2}$, and every such element defines a section $\mathrm{CE}(\mathfrak{l}E) \to \mathrm{CE}(\mathfrak{l}X)$ and so a cocycle in rational twisted cohomotopy. Under the quasi-isomorphism $\mathrm{CE}(\mathfrak{l}X) \to \Omega^\bullet_{\mathrm{dR}}(X)$ defining $\mathrm{CE}(\mathfrak{l}X)$ as a Sullivan model of X, the element c_{2k+2} is mapped to a closed differential form $\chi_{2k+2}(\nabla_\tau)$ representing the Euler class $\chi_{2k+2}(V)$ of V, and so g_{2k+1} corresponds to a differential form G_{2k+1} on X with $dG_{2k+1} = \chi_{2k+2}(\nabla_\tau)$.

(II). The Sullivan model for the total space of $2k$-spherical fibration $E \to X$ is of the form

$$\mathrm{CE}(\mathfrak{l}E) \;=\; \mathrm{CE}(\mathfrak{l}X) \otimes \mathbb{R}\big[\omega_{2k}, \omega_{4k-1}\big] \big/ \left(\begin{array}{ll} d\,\omega_{2k} & = 0 \\ d\omega_{4k-1} & = -\omega_{2k} \wedge \omega_{2k} + c_{4k} \end{array} \right), \tag{36}$$

where

(a) $c_{4k} \in \mathrm{CE}(\mathfrak{l}X)$ is some element in the base algebra, which by (36) is closed and represents the rational cohomology class of the cup square of the class of ω_{4k}:

$$[c_{4k}] \;=\; [\omega_{2k}]^2 \;\in\; H^{4k}(X;\mathbb{Q}).$$

This class classifies the spherical fibration, rationally.

(b) under the quasi-isomorphism $\mathrm{CE}(\mathfrak{l}E) \to \Omega^\bullet_{\mathrm{dR}}(E)$ the new generator ω_{2k} corresponds to a closed differential form that restricts to the volume form on the $2k$-sphere fibers $S^{2k} \simeq E_x \hookrightarrow E$ over each point $x \in X$:

$$\langle \omega_{2k}, [S^{2k}] \rangle \;=\; 1\;. \tag{37}$$

Note that the element $[\omega_{2k}]^2$ is a priori an element in $H^{4k}(E,\mathbb{Q})$. By writing $[c_{4k}] = [\omega_{2k}]^2 \;\in\; H^{4k}(X;\mathbb{Q})$ we mean that $[\omega_{2k}]^2$ is actually the pullback of the element $[c_{4k}]$ via the projection $\pi\colon E \to X$.

Moreover, if the spherical fibration $\pi\colon E \to X$ happens to be the unit sphere bundle $E = S(V)$ of a real vector bundle $V \to X$, then the tautological section of π^*V defines a splitting $\pi^*V = \mathbb{R}_E \oplus \widehat{V}$ and

(a) the class of ω_{2k} is half the rationalized Euler class $\chi_{2k}(\widehat{V})$ of $\widehat{V}$:

$$[\omega_{2k}] \;=\; \tfrac{1}{2}\chi(\widehat{V}) \;\in\; H^{2k}(E;\mathbb{Q})\;. \tag{38}$$

(b) the class of c_{4k} is one fourth the rationalized k-th Pontrjagin class $p_k(V)$ of V:

$$[c_{4k}] \;=\; \tfrac{1}{4}p_k(V) \;\in\; H^{4k}(X;\mathbb{Q})\;. \tag{39}$$

The second equation is actually a consequence of the first one and of the naturality and multiplicativity of the total rational Pontrjagin class:

$$\pi^* p_k(V) = p_k(\mathbb{R}_E \oplus \widehat{V}) = p_k(\widehat{V}) = \chi_{2k}(\widehat{V})^2.$$

Reasoning as in the odd sphere bundles case, a section of $\mathrm{CE}(\mathfrak{l}X) \to \mathrm{CE}(\mathfrak{l}E)$, and so a cocycle in rational twisted cohomotopy, is the datum of elements $g_{2k}, g_{4k-1} \in \mathrm{CE}(\mathfrak{l}X)$ such that $dg_{2k} = 0$ and $dg_{4k-1} = -g_{2k} \wedge g_{2k} + c_{4k}$. Under the quasi-isomorphism $\mathrm{CE}(\mathfrak{l}E) \to \Omega^\bullet_{\mathrm{dR}}(E)$, the element g_{2k}, seen as an element in $\mathrm{CE}(\mathfrak{l}E)$, is mapped to a closed differential form $\frac{1}{2}\chi_{2k}(\nabla_{\hat{\tau}})$ representing 1/2 the Euler class $\chi_{2k}(\widehat{V})$ of $\widehat{V}$, while under the quasi-isomorphism $\mathrm{CE}(\mathfrak{l}X) \to \Omega^\bullet_{\mathrm{dR}}(X)$ the element c_{4k} is mapped to a closed differential form $\frac{1}{4}p_k(\nabla_{\hat{\tau}})$ representing 1/4 the k-th Pontrjagin class $\frac{1}{4}p_k(V)$ of V. Therefore, the quasi-isomorphism $\mathrm{CE}(\mathfrak{l}X) \to \Omega^\bullet_{\mathrm{dR}}(X)$ turns the elements g_{2k} and g_{4k-1} into differential forms G_{2k} and G_{4k-1} on X, subject to the identities $dG_{2k} = 0$, $\pi^*G_{2k} = \frac{1}{2}\chi_{2k}(\nabla_{\hat{\tau}})$, and $dG_{4k-1} = -G_{2k} \wedge G_{2k} + \frac{1}{4}p_k(\nabla_{\hat{\tau}})$. □

Remark 3.6 (Simply-connectedness assumption)**.** The assumption in Prop. 3.5 that X be simply connected is just to ensure the existence of a Sullivan model for X, as used in the proof. (It would be sufficient to assume, for that purpose, that the fundamental group is nilpotent). If X is not simply connected and not even nilpotent, then a similar statement about differential form data underlying twisted Cohomotopy cocycles on X will still hold, but statement and proof will be much more involved. Hence we assume simply connected X here only for convenience, not for fundamental reasons. A direct consequence of this assumption, which will play a role in §4, is that, by the Hurewicz theorem and the universal coefficient theorem, the degree 2 cohomology of X with coefficients in $\mathbb{Z}_2$ is given by:

$$H^2(X;\mathbb{Z}_2) \simeq \mathrm{Hom}_{\mathrm{Ab}}\big(H_2(X,\mathbb{Z}),\, \mathbb{Z}_2\big)\,. \tag{40}$$

3.2 Twisted Cohomotopy via topological G-structure

We discuss how cocycles in J-twisted Cohomotopy are equivalent to choice of certain topological G-structures (Prop. 3.8 below).

The following fact will play a crucial role throughout:

Lemma 3.7 (Homotopy actions and reduction of structure group)**.** *Let G be a topological group and V any topological space.*
(i) *Then for every homotopy-coherent action of G on V, the corresponding homotopy quotient $V /\!\!/ G$ forms a homotopy fiber sequence of the form*

$$V \longrightarrow V /\!\!/ G \longrightarrow BG$$

and, in fact, this association establishes an equivalence between homotopy V-fibrations over BG and homotopy coherent actions of V on G.
(ii) *In particular, if $\iota : H \hookrightarrow G$ is an inclusion of topological groups, then the homotopy fiber of the induced map $B\iota$ of classifying spaces is the coset space G/H:*

$$G/H \xrightarrow{\ \mathrm{fib}\ } BH \xrightarrow{\ B\iota\ } BG$$

thus exhibiting the weak homotopy equivalence $\big(G/H\big) /\!\!/ G \simeq BH$.

Proof. This equivalence goes back to [DDK80]. A modern account which generalizes to geometric situations (relevant for refinement of all constructions here to differential cohomology) is in [NSS12, Sec. 4]. When the given homotopy-coherent action of the topological group G on V happens to be given by an actual topological action we may use the Borel construction to represent the homotopy quotient. For the case of $H \hookrightarrow G$ a topological subgroup inclusion, we may compute as follows:

$$\begin{aligned} BH &\simeq * \times_H EH \\ &\simeq * \times_H EG \\ &\simeq * \times_H (G \times_G EG) \\ &\simeq (* \times_H G) \times_G EG \\ &\simeq (G/H) \times_G EG \\ &\simeq (G/H) /\!\!/ G\,. \end{aligned}$$

Here the first weak equivalence is the usual definition of the classifying space, while the second uses that one may take a universal H-bundle EH, up to weak homotopy equivalence, any contractible space with free H-action, hence in particular EG. The third line uses that G is the identity under Cartesian product followed by the quotient by the diagonal G-action. □

Proposition 3.8 (Twisted cohomotopy cocycle is reduction of structure group)**.** *Cocycles in twisted Cohomotopy (Def. 3.1) are equivalent to choices of topological G-structure for $G = O(n) \hookrightarrow O(n+1)$:*

$$
\pi^\tau(X) \;=\; \left\{
\begin{tikzcd}
 & BO(n) \arrow[d] \\
X \arrow[ur, dashed, "\text{cocycle in twisted Cohomotopy}"] \arrow[r, "\tau"', "\text{twist}"] & BO(n+1)
\end{tikzcd}
\right\} \Big/ \sim
$$

Moreover, if the twist is itself is factored through $B\mathrm{Spin}(n+1)$ as in Remark 3.3, then τ-twisted Cohomotopy is equivalent to reduction along $\mathrm{Spin}(n) \hookrightarrow \mathrm{Spin}(n+1)$:

$$
\pi^\tau(X) \;=\; \left\{
\begin{tikzcd}
 & B\mathrm{Spin}(n) \arrow[d] \\
X \arrow[ur, dashed, "\text{cocycle in twisted Cohomotopy}"] \arrow[r, "\widehat{\tau}"', "\text{twist}"] & B\mathrm{Spin}(n+1)
\end{tikzcd}
\right\} \Big/ \sim
$$

Generally, if there is a coset realization of an n-sphere $S^n \simeq G/H$ and the twist is factored through G-structure, then τ-twisted Cohomotopy is further reduction to topological H-structure:

$$
\pi^\tau(X) \;=\; \left\{
\begin{tikzcd}
 & BH \arrow[d] \\
X \arrow[ur, dashed, "\text{cocycle in twisted Cohomotopy}"] \arrow[r, "\widehat{\tau}"', "\text{twist}"] & BG
\end{tikzcd}
\right\} \Big/ \sim
$$

Proof. This follows by applying Lemma 3.7 and using the fact that $S^n \simeq O(n+1)/O(n)$. □

Remark 3.9 (Interpretation)**.** Prop. 3.8 means that the structure of twisted Cohomotopy sets (Def. 3.1) in degree n is controlled by the set of ways in which the n-sphere arises as a coset space $S^n \simeq G/H$.

3.3 Twisted Cohomotopy in degrees 4 and 7 combined

We discuss here twisted Cohomotopy in degree 4 and 7 jointly, related by the quaternionic Hopf fibration $h_{\mathbb{H}}$. This requires first determining the space of twists that are compatible with $h_{\mathbb{H}}$, which is the content of Prop. 3.19 and Prop. 3.20 below. This yields the scenario of incremental G-structures shown in *Figure T*. The twists that appear are subgroups of $\mathrm{Spin}(8)$ related by triality (Prop. 3.16 below), and in fact the classifying space for the C-field implied by *Hypothesis H* comes out to be the homotopy-fixed locus of triality.

It will be useful to have the following notation for a basic but crucial operation on Spin groups:

Definition 3.10 (Central product of groups)**.** Given a tuple of groups $G_1, G_2, \cdots, G_n$, each equipped with a central $\mathbb{Z}_2$-subgroup inclusion $\mathbb{Z}_2 \simeq \{1,-1\} \subset Z(G_i) \subset G_i$, we write

$$
G_1 \cdot G_2 \cdot \cdots \cdot G_{n-1} \cdot G_n \;:=\; \big(G_1 \times G_2 \times \cdots \times G_n\big)/_{\mathrm{diag}}\mathbb{Z}_2 \tag{41}
$$

for the quotient group of their direct product group by the corresponding diagonal $\mathbb{Z}_2$-subgroup:

$$
\{(1,1,\cdots,1),\,(-1,-1,\cdots,-1)\} \hookrightarrow G_1 \times G_2 \times \cdots \times G_n \,.
$$

Just to save space we will sometimes suppress the dots and write $G_1 G_2 := G_1 \cdot G_2$, etc.

Example 3.11 (Central product of symplectic groups)**.** The notation in Def. 3.10 originates in [Ale68, Gra69] for the examples

$$\mathrm{Sp}(n)\cdot\mathrm{Sp}(1) \;:=\; \big(\mathrm{Sp}(n)\times\mathrm{Sp}(1)\big)/\{(1,1),(-1,-1)\}. \tag{42}$$

For $n\geq 2$ this is such that a $\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$-structure on a $4n$-dimensional manifold is equivalently a quaternion-Kähler structure. Specifically, for $n=2$ there is a canonical subgroup inclusion

$$\begin{tikzcd} & \mathrm{Spin}(8) \arrow[d] \\ \mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \arrow[r, hook] \arrow[ur, hook] & \mathrm{SO}(8)\simeq\mathrm{SO}(\mathbb{H}^2) \\ (A,q) \arrow[r, mapsto] & (x\mapsto A\cdot x\cdot\overline{q}) \end{tikzcd} \tag{43}$$

given by identifying elements of $\mathrm{Sp}(2)$ as quaternion-unitary 2×2-matrices A, elements of $\mathrm{Sp}(1)$ as multiples of the 2×2 identity matrix by unit quaternions q, and acting with such pairs by quaternionic matrix conjugation on elements $x\in\mathbb{H}^2\simeq_{\mathbb{R}}\mathbb{R}^8$ as indicated. This lifts to an inclusion into $\mathrm{Spin}(8)$ through the defining double-covering map (see [CV97, 2.]). Notice that reversing the Sp-factors gives an isomorphic group, but a different subgroup inclusion

$$\begin{tikzcd} & \mathrm{Spin}(8) \arrow[d] \\ \mathrm{Sp}(1)\cdot\mathrm{Sp}(2) \arrow[r, hook] \arrow[ur, hook] & \mathrm{SO}(8)\simeq\mathrm{SO}(\mathbb{H}^2) \\ (q,A) \arrow[r, mapsto] & (x\mapsto q\cdot x\cdot\overline{A}) \end{tikzcd} \tag{44}$$

For more on this see Prop. 3.16 below.

Example 3.12 (Central product of Spin groups)**.** For $n_1,n_2\in\mathbb{N}$, we have the central product (Def. 3.10) of the corresponding Spin groups

$$\mathrm{Spin}(n_1)\cdot\mathrm{Spin}(n_2) \;:=\; \big(\mathrm{Spin}(n_1)\times\mathrm{Spin}(n_2)\big)/\{(1,1),(-1,-1)\}\,. \tag{45}$$

(The notation is used for instance in [HN12, Prop. 17.13.1].) Here the canonical subgroup inclusions of Spin groups $\mathrm{Spin}(n)\xhookrightarrow{\iota_n}\mathrm{Spin}(n+k)$ induce a canonical subgroup inclusion of (45) into $\mathrm{Spin}(n_1+n_2)$:

$$\begin{tikzcd} & (\alpha,\beta) \arrow[r, mapsto] & \iota_{n_1}(\alpha)\cdot\iota_{n_2}(\beta) \\ (\mathbb{Z}_2)_{\mathrm{diag}} \arrow[r, hook, "\ker"] & \mathrm{Spin}(n_1)\times\mathrm{Spin}(n_2) \arrow[r] \arrow[d, "\mathrm{quot}"'] & \mathrm{Spin}(n_1+n_2)\,. \\ & \mathrm{Spin}(n_1)\cdot\mathrm{Spin}(n_2) \arrow[ur, hook] & \end{tikzcd} \tag{46}$$

Notice that these groups sit in short exact sequences as follows:

$$1 \longrightarrow \mathrm{Spin}(n_1) \xhookrightarrow{\;\iota_{n_1}\;} \mathrm{Spin}(n_1)\cdot\mathrm{Spin}(n_2) \xrightarrow{\;\mathrm{pr}_{n_2}\;} \mathrm{SO}(n_2) \longrightarrow 1\,. \tag{47}$$

For low values of n_1,n_2 there are exceptional isomorphisms between the groups (42) and (45) as abstract groups, but as subgroups under the inclusions (43) and (46) these are different. This is the content of Prop. 3.16 below. First we record the following, for later use:

Definition 3.13 (Universal class of central products)**.** For $n_1,n_2\in\mathbb{N}$, write

$$\varepsilon\;\in\;H^2\big(B(\mathrm{Spin}(n_1)\cdot\mathrm{Spin}(n_2));\mathbb{Z}_2\big)$$

for the *universal characteristic class* on the classifying space of the central product Spin group (Def. 3.12) which is the pullback of the second Stiefel-Whitney class $w_2\in H^2\big(B\mathrm{SO}(n_2),\mathbb{Z}_2\big)$ from the classifying space of the underlying $\mathrm{SO}(n_2)$-bundles, via the projection (47):

$$\varepsilon \;:=\; (B\mathrm{pr}_{n_2})^*(w_2)\,. \tag{48}$$

See also [Sal82, Def. 2.1], following [MR76].

Lemma 3.14 (Characterization via the universal class)**.** *For $n_1, n_2 \in \mathbb{N}$, let $X \xrightarrow{\tau} B\big(\mathrm{Spin}(n_1)\cdot\mathrm{Spin}(n_2)\big)$ be a classifying map for a central product Spin structure (Def. 3.12). Then the following are equivalent:*

(i) *the class ε from Def. 3.13 vanishes:*

$$\varepsilon(\tau) \;=\; 0 \;\in\; H^2(X;\mathbb{Z}_2)\,,$$

(ii) *τ has a lift to the direct product Spin structure:*

$$\begin{tikzcd} & B\big(\mathrm{Spin}(n_1)\times\mathrm{Spin}(n_2)\big) \arrow[d] \\ X \arrow[ur, dashed, "\widehat{\tau}"] \arrow[r, "\tau"'] & B\big(\mathrm{Spin}(n_1)\cdot\mathrm{Spin}(n_2)\big)\,. \end{tikzcd}$$

(iii) *the underlying $\mathrm{SO}(n_2)$-bundle admits Spin structure:*

$$\begin{tikzcd} & B\mathrm{Spin}(n_2) \arrow[d] \\ X \arrow[ur, dashed, "\widehat{B\mathrm{pr}_{n_2}\circ\tau}"] \arrow[r, "B\mathrm{pr}_{n_2}\circ\tau"'] & B\mathrm{SO}(n_2)\,. \end{tikzcd}$$

Proof. By (45) and (47) we have the following short exact sequence of short exact sequences of groups:

$$\begin{tikzcd} 1 \arrow[r, hook] \arrow[d, hook] & \mathrm{Spin}(5) \arrow[r, equal] \arrow[d, hook] & \mathrm{Spin}(5) \arrow[d, hook] \\ \mathbb{Z}_2 \arrow[r, hook] \arrow[d, equal] & \mathrm{Spin}(5)\times\mathrm{Spin}(3) \arrow[r, two heads] \arrow[d, two heads] & \mathrm{Spin}(5)\cdot\mathrm{Spin}(3) \arrow[d, two heads, "\mathrm{pr}_3"] \\ \mathbb{Z}_2 \arrow[r, hook] & \mathrm{Spin}(3) \arrow[r, two heads] & \mathrm{SO}(3) \end{tikzcd}$$

Since the bottom left morphism is an identity, it follows that also after passing to classifying spaces and forming connecting homomorphisms, the corresponding morphism on the bottom right in the following diagram is a weak homotopy equivalence:

$$\begin{tikzcd} B\mathbb{Z}_2 \arrow[r] \arrow[d, equal] & B\big(\mathrm{Spin}(5)\times\mathrm{Spin}(3)\big) \arrow[r] \arrow[d] & B\big(\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)\big) \arrow[r, "\varepsilon"] \arrow[d, "B\mathrm{pr}_3"] & B^2\mathbb{Z} \arrow[d, equal] \\ B\mathbb{Z}_2 \arrow[r] & B\mathrm{Spin}(3) \arrow[r] & B\mathrm{SO}(3) \arrow[r, "w_2"] & B^2\mathbb{Z} \end{tikzcd}$$

By the top homotopy fiber sequence, this exhibits ε as the obstruction to the lift from central product Spin structure to direct product Spin structure. □

Example 3.15. Applying Def. 3.10 to three copies of $\mathrm{Sp}(1)$ yields the group

$$\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\cdot\mathrm{Sp}(1) \;:=\; \big(\mathrm{Sp}(1)\times\mathrm{Sp}(1)\times\mathrm{Sp}(1)\big)/\big\{(1,1,1),(-1,-1,-1)\big\}\,. \tag{49}$$

The notation appears for instance in [OP01][BM14].

- Observe that, due to the exceptional isomorphisms $\mathrm{Spin}(3) \simeq \mathrm{Sp}(1)$ and $\mathrm{Spin}(4) \simeq \mathrm{Spin}(3) \times \mathrm{Spin}(3)$ there are isomorphisms

$$\mathrm{Spin}(4) \cdot \mathrm{Spin}(3) \;\simeq\; \mathrm{Spin}(3) \cdot \mathrm{Spin}(3) \cdot \mathrm{Spin}(3) \;\simeq\; \mathrm{Sp}(1) \cdot \mathrm{Sp}(1) \cdot \mathrm{Sp}(1)\,. \tag{50}$$

- The group (49) is acted upon via automorphisms interchange the three dot-factors by the symmetric group on three elements:

$$S_3 \circlearrowright \; \big(\mathrm{Sp}(1) \cdot \mathrm{Sp}(1) \cdot \mathrm{Sp}(1)\big) \tag{51}$$

- Beware that the central product of groups with central $\mathbb{Z}_2$-subgroup (Def. 3.10) is not a binary associative operation: for instance, we have

$$\mathrm{Spin}(3) \cdot \mathrm{Spin}(3) \;\simeq\; \mathrm{SO}(4)\,,$$

  which does not even contain the $\mathbb{Z}_2$-subgroup anymore that one would diagonally quotient out in (50), hence the would-be iterated binary expression "$\big(\mathrm{Spin}(3) \cdot \mathrm{Spin}(3)\big) \cdot \mathrm{Spin}(3)$" does not even make sense.

Proposition 3.16 (Triality of quaternionic subgroups of $\mathrm{Spin}(8)$)**.** *The subgroup inclusions into* $\mathrm{Spin}(8)$ *of* $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$ *via* (43)*,* $\mathrm{Sp}(1)\cdot\mathrm{Sp}(2)$ *via* (44)*, and* $\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)$ *via* (46)*, represent three distinct conjugacy classes of subgroups, and under the defining projection to* $\mathrm{SO}(8)$ *they map to subgroups of* $\mathrm{SO}(8)$ *as follows:*

$$\begin{tikzcd}
\mathrm{Sp}(1)\cdot\mathrm{Sp}(2) \arrow[drr, hook, "\iota'"] \arrow[dddr, equal] & & & & & \\
& & \mathrm{Spin}(8) \arrow[dddr] & & \mathrm{Spin}(3)\cdot\mathrm{Spin}(5) \arrow[ll, hook', "\iota"'] \arrow[dddr] & \\
\mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \arrow[urr, hook, "\iota''"] \arrow[dddr, equal] & & & & & \\
& \mathrm{Sp}(1)\cdot\mathrm{Sp}(2) \arrow[drr, hook, "\iota'"] & & & & \\
& & & \mathrm{SO}(8) & & \mathrm{SO}(3)\times\mathrm{SO}(5). \arrow[ll, hook', "\iota"'] \\
& \mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \arrow[urr, hook, "\iota''"] & & & &
\end{tikzcd}$$

Moreover, the triality group $\mathrm{Out}(\mathrm{Spin}(8))$ *acts transitively by permutation on the set of these three conjugacy classes.*

$$\begin{tikzcd}
\mathrm{Spin}(8) \arrow[dddddd, leftrightarrow, bend right, "\simeq"'] & & & & & \\
& \mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \arrow[ul, hook] \arrow[dddd, leftrightarrow, bend right, "\simeq"'] & & & & \\
& & \mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\cdot\mathrm{Sp}(1) \arrow[ul, hook] \arrow[dd, leftrightarrow, "\simeq"] \arrow[dr, leftrightarrow, "\simeq"] & & & \\
& & & \mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\cdot\mathrm{Sp}(1) \arrow[r, hook] & \mathrm{Spin}(5)\cdot\mathrm{Spin}(3) \arrow[r, hook] \arrow[uulll, leftrightarrow, bend right, "\simeq"'] \arrow[ddlll, leftrightarrow, bend left, "\simeq"] & \mathrm{Spin}(8)\,. \arrow[uuulllll, leftrightarrow, bend right, "\simeq"'] \arrow[dddlllll, leftrightarrow, bend left, "\simeq"] \\
& & \mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\cdot\mathrm{Sp}(1) \arrow[dl, hook] \arrow[ur, leftrightarrow, "\simeq"] & & & \\
& \mathrm{Sp}(1)\cdot\mathrm{Sp}(2) \arrow[dl, hook] & & & & \\
\mathrm{Spin}(8) & & & & &
\end{tikzcd}$$

Proof. This follows by analysis of the action of triality on the corresponding Lie algebras; see [CV97, Sec. 2], [Ko02, Prop. 3.3 (3)]. $\square$

Remark 3.17 (Subgroups)**.** **(i)** For emphasis, notice that the subgroups appearing in Prop. 3.16 are all isomorphic as abstract groups

$$\mathrm{Sp}(1)\cdot\mathrm{Sp}(2) \;\simeq\; \mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \;\simeq\; \mathrm{Spin}(5)\cdot\mathrm{Spin}(3) \;\simeq\; \mathrm{Spin}(3)\cdot\mathrm{Spin}(5)$$

due to the classical exceptional isomorphisms

$$\mathrm{Sp}(1) \;\simeq\; \mathrm{Spin}(3)\,, \qquad \mathrm{Sp}(2) \;\simeq\; \mathrm{Spin}(5)$$

and via the evident automorphisms that permutes central product factors. However, when each is equipped with *its* canononical subgroup inclusion into Spin(8), via (43), (44) and (46), then these are distinct subgroups. Moreover, Prop. 3.16 says that the first three of these are even in distinct conjugacy classes of subgroups, while $\mathrm{Spin}(3)\cdot\mathrm{Spin}(5)$ and $\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)$ are in the same conjugacy class.

(ii) In the following, when considering these subgroup inclusions and their induced morphisms on classifying spaces, we will always mean that *canonical* inclusion of the subgroup of that name. When we need to refer to another, non-canonical embedding of any of these groups G, then we will always make this explicit as a triality automorphism $G \overset{\simeq}{\to} G'$ followed by the canonical inclusion of G'. See for instance (86) below for an example.

For the development in §4 we need to know in particular how universal characteristic classes behave under the triality automorphisms:

Lemma 3.18. *The integral cohomology ring of $B\mathrm{Spin}(8)$ is*

$$H^\bullet\big(B\mathrm{Spin}(8),\mathbb{Z}\big) \;\simeq\; \mathbb{Z}\Big[\tfrac{1}{2}p_1,\; \tfrac{1}{4}\big(p_2 - \big(\tfrac{1}{2}p_1\big)^2\big) - \tfrac{1}{2}\chi,\; \chi_8,\; \beta(w_6)\Big]\big/\big(2\beta(w_6)\big)\,,$$

where p_k are Pontrjagin classes, χ_8 is the Euler class, w_6 is a Stiefel-Whitney class, β is the Bockstein homomorphism, so that $W_7 := \beta(w_6)$ is an integral Stiefel-Whitney class.

Moreover, under the delooping of the triality automorphism from Prop. 3.16 to classifying spaces

$$\begin{tikzcd} B\big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[r, "\simeq"] \arrow[d] & B\big(\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)\big) \arrow[d] \\ B\mathrm{Spin}(8) \arrow[r, "\simeq", "B\mathrm{tri}"'] & B\mathrm{Spin}(8) \end{tikzcd} \tag{52}$$

these classes pull back as follows:

$$\big(B\mathrm{tri}\big)^* \;:\; \begin{aligned} \tfrac{1}{2}p_1 &\mapsto \tfrac{1}{2}p_1 \\ \chi_8 &\mapsto \tfrac{1}{4}\big(p_2 - \big(\tfrac{1}{2}p_1\big)^2\big) + \tfrac{1}{2}\chi_8 \\ &\quad = \Big(\tfrac{1}{4}\big(p_2 - \big(\tfrac{1}{2}p_1\big)^2\big) - \tfrac{1}{2}\chi_8\Big) + \chi_8 \\ \tfrac{1}{4}\big(p_2 - \big(\tfrac{1}{2}p_1\big)^2\big) - \tfrac{1}{2}\chi &\mapsto -\Big(\tfrac{1}{4}\big(p_2 - \big(\tfrac{1}{2}p_1\big)^2\big) - \tfrac{1}{2}\chi\Big) \end{aligned} \tag{53}$$

Notice that in particular

$$\Big(\big(B\mathrm{tri}\big)^*\Big)^{-1} \;=\; \big(B\mathrm{tri}\big)^*.$$

Proof. This follows by combining [CV97, Lemmas 2.5, 4.1, 4.2], following [GrGr70, Thm. 2.1], and using the property $\mathrm{tri}^{-1} = \mathrm{tri}$, recalled in [CV97, 2.]. □

Now we may have a closer look at the quaternionic Hopf fibration $S^7 \simeq S(\mathbb{H}^2) \xrightarrow{h_{\mathbb{H}}} \mathbb{H}P^1 \simeq S^4$:

Proposition 3.19 (Symmetries of the quaternionic Hopf fibration)**.**

(i) *The symmetry group of $h_{\mathbb{H}}$ and hence the group of twists for Cohomotopy jointly in degrees 4 and 7, is the group* (42), $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \hookrightarrow \mathrm{O}(8)$, *with its canonical action* (43), *in that this is the largest subgroup of* $\mathrm{O}(8) \simeq \mathrm{O}(\mathbb{H}^2)$ *under which $h_{\mathbb{H}}$ is equivariant.*

(ii) *Moreover, the corresponding action on the codomain 4-sphere $S^4 \simeq S(\mathbb{R}^5)$ is via the canonical projection* (47) *to* $\mathrm{SO}(5)$

$$\mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \xrightarrow{\ \simeq\ } \mathrm{Spin}(5)\cdot\mathrm{Spin}(3) \xrightarrow{\ \mathrm{pr}_5\ } \mathrm{SO}(5)\ .$$

Proof. This statement essentially appears as [GWZ86, Prop. 4.1] and also, somewhat more implicitly, in [Po95, p. 263]. To make this more explicit, we may observe that the quaternionic Hopf fibration has the following coset space description:

$$\begin{tikzcd}
S^3 \arrow[r, "\mathrm{fib}(h_{\mathbb{H}})"] \arrow[d, equal] & S^7 \arrow[r, "h_{\mathbb{H}}"] \arrow[d, equal] & S^4 \arrow[d, equal] \\
\frac{\mathrm{Spin}(4)}{\mathrm{Spin}(3)} \arrow[r, "\frac{\iota_4}{\mathrm{id}}"'] & \frac{\mathrm{Spin}(5)}{\mathrm{Spin}(3)} \arrow[r, "\frac{\mathrm{id}}{\iota_3}"'] & \frac{\mathrm{Spin}(5)}{\mathrm{Spin}(4)}
\end{tikzcd} \tag{54}$$

where $\iota_n : \mathrm{Spin}(n) \hookrightarrow \mathrm{Spin}(n+1)$ denotes the canonical inclusion. This can also be deduced from [HaTo09, Table 1]. (In the analogous octonionic case this is noticed in [OPPV12, p. 7].) ☐

The following homotopy-theoretic version of Prop. 3.19 is crucial for the discussion in §4 below.

Proposition 3.20. *The homotopy quotient $h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$ of the quaternionic Hopf fibration by its equivariance group (Prop. 3.19) is equivalently the map of classifying spaces induced by the canonical inclusion* $\mathrm{diag}\cdot\mathrm{id} : \mathrm{Sp}(1)\cdot\mathrm{Sp}(1) \hookrightarrow \mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)$*:*

$$\begin{tikzcd}
S^7 /\!\!/ \mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \arrow[r, equal] \arrow[d, "h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)\cdot\mathrm{Sp}(1)"'] & B\big(\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\big) \arrow[d, "B(\mathrm{diag}\cdot\mathrm{id})"] \\
S^4 /\!\!/ \mathrm{Sp}(2)\cdot\mathrm{Sp}(1) \arrow[r, equal] & B\big(\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\big)
\end{tikzcd}$$

Proof. This follows by using the right hand side of diagram (54), applying Lemma 3.7 to the two pairs $(G,H) = (\mathrm{Sp}(2),\mathrm{Sp}(1))$ and $(G,H) = (\mathrm{Sp}(2),\mathrm{Sp}(1)\times\mathrm{Sp}(1))$, and considering the following diagram:

$$\begin{tikzcd}
S^7 \arrow[r, equal] \arrow[d, "\mathrm{quot}"'] \arrow[rrrr, bend left, "h_{\mathbb{H}}"] & \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)} \arrow[rr, "\frac{\mathrm{id}}{\mathrm{diag}}"'] \arrow[d, "\mathrm{fib}"'] & & \frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)} \arrow[r, equal] \arrow[d, "\mathrm{fib}"] & S^4 \arrow[d, "\mathrm{quot}"] \\
S^7 /\!\!/ \big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[r, equal] \arrow[rrrr, bend right, "h_{\mathbb{H}} /\!\!/ (\mathrm{Sp}(2)\cdot\mathrm{Sp}(1))"'] & B\big(\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\big) \arrow[rr, "B(\mathrm{diag}\cdot\mathrm{id})"] \arrow[dr] & & B\big(\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\big) \arrow[r, equal] \arrow[dl] & S^4 /\!\!/ \big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \\
& & B\big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) & &
\end{tikzcd}$$

☐

3.4 Twisted Cohomotopy in degree 7 alone

If we do not require the twists of Cohomotopy in degree 7 to be compatible with the quaternionic Hopf fibration (as we did in the previous section, §3.3) then there are more exceptional twists. We give a homotopy-theoretic classification of these in Prop. 3.21 below. In Remark 3.23 below we highlight how this recovers precisely the special holonomy structures of $N = 1$ compactifications of M/F-theory.

Further below in §3.6, we explain how these $N = 1$ structures are *fluxless* in a precise Cohomotopical sense, which crucially enters the M2-tadpole cancellation in §4.6.

Proposition 3.21 (G-structures induced by Cohomotopy in degree 7)**.** *We have the following sequence of homotopy pullbacks of universal 7-spherical fibrations, hence of twists for Cohomotopy in degree 7 (see Figure D):*

$$
\begin{tikzcd}
S^7 \arrow[r, "\mathrm{fib}"] \arrow[d, equal] & B\mathrm{SU}(2) \arrow[r] \arrow[d] \arrow[dr, phantom, "(pb)"] & B\mathrm{Spin}(5) \arrow[d] \\
S^7 \arrow[r, "\mathrm{fib}"] \arrow[d, equal] & B\mathrm{SU}(3) \arrow[r] \arrow[d] \arrow[dr, phantom, "(pb)"] & B\mathrm{Spin}(6) \arrow[d] \\
S^7 \arrow[r, "\mathrm{fib}"] \arrow[d, equal] & B\mathrm{G}_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "(pb)"] & B\mathrm{Spin}(7) \arrow[d, "B\iota"] \\
S^7 \arrow[r, "\mathrm{fib}"] & B\mathrm{Spin}(7) \arrow[r, "B\iota'"'] & B\mathrm{Spin}(8)
\end{tikzcd}
$$

Proof. First, observe that there is the following analogous commuting diagram of Lie groups:

$$
\begin{tikzcd}
\mathrm{SU}(2) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "(pb)"] & \mathrm{SU}(3) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "(pb)"] & \mathrm{G}_2 \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "(pb)"] & \mathrm{Spin}(7) \arrow[d, hook, "\iota'"] \\
\mathrm{Spin}(5) \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "(pb)"] & \mathrm{Spin}(6) \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "(pb)"] & \mathrm{Spin}(7) \arrow[r, hook] \arrow[d, two heads] \arrow[dr, phantom, "(pb)"] & \mathrm{Spin}(8) \arrow[d, two heads] \\
\mathrm{SO}(5) \arrow[r, hook] & \mathrm{SO}(6) \arrow[r, hook] & \mathrm{SO}(7) \arrow[r, hook] & \mathrm{SO}(8)\,.
\end{tikzcd}
\tag{55}
$$

Here the bottom squares evidently commute and are pullback squares by the definition of Spin groups, while the three total vertical rectangles commute and are pullback squares by [On93, Table 2, p. 144]. By the pasting law, [3] this implies that also the top squares are pullbacks, hence exhibiting intersections of subgroup inclusions. Notice that the top right vertical inclusion ι' is *not* the canonical inclusion of $\mathrm{Spin}(7)$ in $\mathrm{Spin}(8)$, but is a subgroup inclusion in a distinct $\mathrm{Spin}(7)$-conjugacy class, of which there are three [Va01, Thm. 5 on p. 6]. The intersection in the top right square is also proven in [Va01, Thm. 5 on p. 13], and that of the middle square in [Va01, Lem. 9 on p. 10]. Again, by the pasting law, this implies that also the top squares are pullbacks, hence exhibiting intersections of subgroup inclusions.

[3] Recall that this says that if

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "(pb)"] & C \arrow[d] \\
D \arrow[r] & E \arrow[r] & F
\end{tikzcd}
$$

is a commuting diagram, where the right square is a pullback, then the left square is a pullback precisely if the full outer rectangle is a pullback. The same holds for homotopy-commutative diagrams and homotopy-pullback squares.

Applying delooping (passage to classifying spaces) to these top squares, this shows that we have a homotopy commuting diagram as follows:

$$
\begin{tikzcd}
 & & & S^7 \arrow[rr, equal] \arrow[dd, "\mathrm{fib}"] & & S^7 \arrow[rr, equal] \arrow[dd, "\mathrm{fib}"] & & S^7 \arrow[r, equal] \arrow[dd, "\mathrm{fib}"] & S^7 \arrow[dd, "\mathrm{fib}"] \\
S^3 \arrow[dr, "\mathrm{fib}"] & & S^5 \arrow[dr, "\mathrm{fib}"] & & S^6 \arrow[dr, "\mathrm{fib}"] & & S^7 \arrow[dr, "\mathrm{fib}"] & & \\
 & {*} \arrow[rr] \arrow[d] & & B\mathrm{SU}(2) \arrow[rr] \arrow[d] \arrow[rr, phantom, shift right=4ex, "\text{(pb)}"] & & B\mathrm{SU}(3) \arrow[rr] \arrow[d] \arrow[rr, phantom, shift right=4ex, "\text{(pb)}"] & & B\mathrm{G}_2 \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4ex, "\text{(pb)}"] & B\mathrm{Spin}(7) \arrow[d, "B\iota'"] \\
 & B\mathrm{Spin}(4) \arrow[rr] & & B\mathrm{Spin}(5) \arrow[rr] & & B\mathrm{Spin}(6) \arrow[rr] & & B\mathrm{Spin}(7) \arrow[r] & B\mathrm{Spin}(8) \\
S^4 \arrow[ur, "\mathrm{fib}"'] & & S^5 \arrow[ur, "\mathrm{fib}"'] & & S^6 \arrow[ur, "\mathrm{fib}"'] & & S^7 \arrow[ur, "\mathrm{fib}"'] & &
\end{tikzcd}
\tag{56}
$$

The spherical homotopy fibers shown in this diagram follow by using Lemma 3.7 with classical results about coset space structures of *topological* spheres, as summarized in the following table:

spherical coset spaces	[MoSa43], see [GrGr70, p.2]
$S^{n-1} \simeq \mathrm{Spin}(n)/\mathrm{Spin}(n-1)$	standard, e.g. [BS53, 17.1]
$S^{2n-1} \simeq \mathrm{SU}(n)/\mathrm{SU}(n-1)$	
$S^{4n-1} \simeq \mathrm{Sp}(n)/\mathrm{Sp}(n-1)$	
$S^7 \simeq \mathrm{Spin}(7)/\mathrm{G}_2$	[Va01, Thm. 3]
$S^7 \simeq \mathrm{Spin}(6)/\mathrm{SU}(3)$	by $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$
$S^7 \simeq \mathrm{Spin}(5)/\mathrm{SU}(2)$	by $\mathrm{Spin}(5) \simeq \mathrm{Sp}(2)$ and $\mathrm{SU}(2) \simeq \mathrm{Sp}(1)$ [ADP83] [DNP83]
$S^6 \simeq \mathrm{G}_2/\mathrm{SU}(3)$	[FI55]

In order to see that each square in the diagram of classifying spaces is a homotopy pullback, we now use the following basic fact from homotopy theory (see e.g. [CPS05, 5.2]): Assume that Y_1, Y_2 are connected spaces, and we are given a homotopy-commutative square as on the right in the following diagram

$$
\begin{tikzcd}
\mathrm{fib}(f_1) \arrow[r] \arrow[d, dotted, "\simeq"'] & X_1 \arrow[r, "f_1"] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & Y_1 \arrow[d] \\
\mathrm{fib}(f_2) \arrow[r] & X_2 \arrow[r, "f_2"] & Y_2\,.
\end{tikzcd}
$$

Then the square is a homotopy pullback square if and only if the induced left vertical morphism between horizontal homotopy fibers is a weak homotopy equivalence; as indicated. To see that in our case these induced left vertical morphisms are indeed weak homotopy equivalences, we first observe that for each of the squares above the horizontal homotopy fibers are n-spheres of the same dimension n:

$$
\begin{tikzcd}
S^7 \simeq \frac{\mathrm{Spin}(7)}{\mathrm{G}_2} \arrow[r] \arrow[d, dashed, "\simeq"'] & B\mathrm{G}_2 \arrow[r] \arrow[d] & B\mathrm{Spin}(7) \arrow[d] \\
S^7 \simeq \frac{\mathrm{Spin}(8)}{\mathrm{Spin}(7)} \arrow[r] & B\mathrm{Spin}(7) \arrow[r] & B\mathrm{Spin}(8)
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
S^6 \simeq \frac{\mathrm{G}_2}{\mathrm{SU}(3)} \arrow[r] \arrow[d, dashed, "\simeq"'] & B\mathrm{SU}(3) \arrow[r] \arrow[d] & B\mathrm{G}_2 \arrow[d] \\
S^6 \simeq \frac{\mathrm{Spin}(7)}{\mathrm{Spin}(6)} \arrow[r] & B\mathrm{Spin}(6) \arrow[r] & B\mathrm{Spin}(7)
\end{tikzcd}
$$

(for the coset realization of S^6 on the top left see [FI55]) and

$$
\begin{tikzcd}
S^5 \simeq \frac{\mathrm{SU}(3)}{\mathrm{SU}(2)} \arrow[r] \arrow[d, dashed, "\simeq"'] & B\mathrm{SU}(2) \arrow[r] \arrow[d] & B\mathrm{SU}(3) \arrow[d] \\
S^5 \simeq \frac{\mathrm{Spin}(6)}{\mathrm{Spin}(5)} \arrow[r] & B\mathrm{Spin}(5) \arrow[r] & B\mathrm{Spin}(6)\,.
\end{tikzcd}
$$

To see in detail that the homotopy fibers on the left are not only pairwise weakly homotopy equivalent, but that the universally induced dashed morphism exhibits such a weak homotopy equivalence, we proceed as follows. For $G := \mathrm{Spin}(n)$ one of the Spin groups appearing above, pick any one topological space EG modelling the total space of the universal G bundle (hence any weakly contractible topological space equipped with a free continuous G-action). Then for $G' \overset{\iota}{\hookrightarrow} G$ any subgroup, we have that the projection $(EG)/G' \to (EG)/G$ is a Serre fibration modelling $BG' \xrightarrow{B\iota} BG$ (e.g. [Mi11, 11.4]). Since ordinary pullbacks of Serre fibrations are already homotopy puyllbacks, this means that the above homotopy pullback squares are represented by actual pullback squares of topological spaces in the following diagram:

$$
\begin{tikzcd}
S^n \simeq \frac{G'}{G' \cap G''} \arrow[r] \arrow[d, dashed, "\simeq"'] \arrow[dr, phantom, "\text{(pb)}"] & (EG)/(G' \cap G'') \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & (EG)/G' \arrow[d] \\
S^n \simeq \frac{G}{G''} \arrow[r] & (EG)/G'' \arrow[r] & (EG)/G\,.
\end{tikzcd}
$$

Here the dashed morphism is the canonical continuous function induced by the given group inclusions, so that it is now sufficient to observe that this is a homeomorphism.

While this does not follow for general subgroup intersections, but it does follow as soon as the given coset spaces are homeomorphic, as is the case here. Namely, pick any point $x \in S^n$ and observe that we have a commuting square of continuous functions as follows.

$$
\begin{tikzcd}
S^n \arrow[d, equal] & \frac{G'}{G' \cap G''} \arrow[l, "{[g'] \mapsto g'(x)}"', "\simeq_{\mathrm{homeo}}"] \arrow[d] \\
S^n & \frac{G}{G''}\,. \arrow[l, "\simeq_{\mathrm{homeo}}"', "{[g] \mapsto g(x)}"]
\end{tikzcd}
$$

Since in this diagram the top, bottom and left maps are homeomorphisms, it follows that the right map is also a homeomorphism. □

Remark 3.22 (Twisted generalized cohomotopy)**.** We may also consider *generalized Cohomotopy* with coefficients $S^n \times S^n$. Twists for such generalized Cohomotopy arise from topological G-structure for Spin groups in split signature. The cohomology of the indefinite Lie groups (and their classifying spaces) can be determined using the homotopy equivalence with the maximal compact subgroups (and their classifying spaces) via the polar decomposition (see, e.g., [HN12, Sec. 17.2]). For instance, $\mathrm{Spin}(p,q)$ is homeomorphic to $\mathrm{Spin}(p) \times \mathrm{Spin}(q) \times \mathbb{R}^n$ for an appropriate n, and similarly for connective covers (see [SS19]). Indeed, we also have

$$
\begin{tikzcd}
S^7 \times S^7 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & BG_2 \arrow[d, "i_{G_2}"] \\
S^7 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & B\mathrm{Spin}(7) \arrow[d, "i_{\mathrm{Spin}(7)}"] \\
* \arrow[r] & B\mathrm{Spin}(8)
\end{tikzcd}
\quad \text{equivalently} \quad
\begin{tikzcd}
S^7 \times S^7 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & S^7 /\!\!/ \mathrm{Spin}(7) \arrow[d, "i_{G_2}"] \\
S^7 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & S^7 /\!\!/ \mathrm{Spin}(8) \arrow[d, "i_{\mathrm{Spin}(7)}"] \\
* \arrow[r] & B\mathrm{Spin}(8)
\end{tikzcd}
$$

This follows analogously as in Prop. 3.21, with [On93, p. 146]. We will discuss the details elsewhere.

Remark 3.23 ($N = 1$ structures via exceptional twists of Cohomotopy)**.**

(i) The types of G-structures that appear in the vertical columns in the diagram in Prop. 3.21 happen to be precisely those that, famously, correspond to $D = 4$, $N = 1$ compactifications of F-theory, M-theory, and string theory, respectively. See e.g. [AG04].

(ii) The horizontal relations in the rows of that diagram encode the well-known relation between these compactifications, where, for instance, an elliptically fibered Spin(7)-compactification of F-theory first reduces to a G_2-compactification of M-theory on a circle and then to a CY3-reduction of type IIA string theory. See, e.g., [GSZ14].

(iii) In view of this, it may be worth re-emphasizing that in Prop. 3.21 all these structures, and their relation to each other, are entirely induced by Cohomotopy in degree 7.

(iv) Supergravity in 11 dimensions admits SU(4)-invariant compactifications [PW85]. Since the one-loop term takes a special form on CY4s (see §2.6), this will allow Cohomotopy to reduce an SU(4)-structure to SU(3)-structure. This should be relevant, for instance, for elliptically fibered CY4s [KLRY98].

3.5 Twisted Cohomotopy via Poincaré-Hopf

We characterize here the TX-twisted Cohomotopy of compact orientable smooth manifolds X in terms of the "Cohomotopy charge" carried by a finite number of point singularities in X. This is the content of Prop. 3.24 below. The proof is a Cohomotopical restatement of the classical Poincaré-Hopf (PH) theorem (see e.g. [DNF85, Sec. 15.2]), but the perspective of twisted Cohomotopical is noteworthy in itself and is crucial for the discussion of M2-brane tadpole cancellation in §4.6 below.

Proposition 3.24 (Twisted cohomotopy and the Euler characteristic)**.** *Let X be an orientable compact smooth manifold. Then:*

(i) *A cocycle in the TX-twisted Cohomotopy of X (Def. 3.1) exists if and only if the Euler characteristic of X vanishes:*

$$
\pi^{TX}(X) \neq \varnothing \quad \Longleftrightarrow \quad \chi[X] \neq 0\,.
$$

(ii) *Generally, there exists a finite set of points $\{x_i \in X\}$ such that the operation of restriction to open neighbourhoods of these points exhibits an injection of the TX-twisted Cohomotopy of their complement $\pi^{TX}\big(X \setminus \coprod_i \{x_i\}\big)$ (Def. 3.1) into the product of untwisted Cohomotopy sets* (28) *$\pi^{\dim(X)}\big(U_{x_i} \setminus \{x_i\}\big)$ of these pointed neighborhoods. Moreover, the latter are integers which sum to the Euler characteristic $\chi[X]$ of X:*

$$
\begin{tikzcd}
\pi^{TX}\big(X \setminus \coprod_i \{x_i\}\big) \arrow[r, hook, "\text{restr.}"] \arrow[d] & \prod_i \pi^{\dim(X)-1}\big(U_{x_i} \setminus \{x_i\}\big) \arrow[r, "\simeq"] & \prod_i \mathbb{Z} \arrow[d, "\sum_i"] \\
* \arrow[rr, "\chi[X]"] & & \mathbb{Z}
\end{tikzcd}
\tag{57}
$$

Proof. This follows with the classical Poincaré-Hopf theorem, (60) below. We recall the relevant terminology:

(i) For v a vector field on X, a point $x \in X$ is called an *isolated zero* of v if there exists an open contractible neighborhood $U_x \subset X$ such that the restriction $v|_{U_x}$ of v to this neighborhood vanishes at x and only at x.

(ii) This means that on $U_x \setminus \{x\}$ the vector field v induces a map to the $(\dim(X)-1)$-sphere

$$v/|v| \;:\; U_x \setminus \{x\} \xrightarrow{\;v/|v|\;} S(T_x X) \simeq S^{\dim(X)-1}\;. \tag{58}$$

Here the equivalence on the right is to highlight that the sphere arises as the fiber of the unit sphere bundle of the tangent bundle TU_x, which may be identified with the unit sphere in $T_x X$, by the assumed contractibility of U_x.

(iii) Given an isolated zero x, the *Poincaré-Hopf index* of v at that point is the degree of the associated map (58) to the sphere, for any choice of local chart:

$$\mathrm{index}_x(v) \;:=\; \deg\big(U_x \setminus \{x\} \xrightarrow{\;v/|v|\;} S(T_x X) \simeq S^{\dim(X)-1}\big). \tag{59}$$

Now for X orientable and compact, the *Poincaré-Hopf theorem* (e.g. [DNF85, Sec. 15.2]) says that for any vector field $v \in \Gamma(TX)$ with a finite set $\{x_i \in X\}$ of isolated zeros, the sum of the indices (59) of v equals the Euler characteristic $\chi[X]$ of X:

$$\sum_{\substack{\text{isolated zero} \\ x_i \in X}} \mathrm{index}_{x_i}(v) \;=\; \chi[X]\,. \tag{60}$$

To conclude, observe that the maps to spheres in (58) are but the restriction of the corresponding cocycle in the TX-Cohomotopy of $X \setminus \coprod_i \{x_i\}$:

$$\begin{tikzcd}
& S^{\dim(X)} /\!\!/ \mathrm{SO}(\dim(X)) \arrow[d] \\
X \setminus \coprod_i \{x_i\} \arrow[ur, dashed, "v/|v|"] \arrow[r, "TX"'] & B\mathrm{SO}(\dim(X))
\end{tikzcd}$$

Finally, the identification of the PH-index with an integer is via the Hopf degree theorem (29), now understood as the characterization of untwisted Cohomotopy in (29). $\square$

We may equivalently use the differential form data that underlies a cocycle in twisted Cohomotopy, by Prop. 3.5, to re-express the cohomotopical PH-theorem, Prop. 3.24, via Stokes' theorem. Let X be an orientable compact smooth manifold of even dimension $\dim(X) = 2n+2$, for $n \in \mathbb{N}$ and let $v \in TX$ be a vector field with isolated zeros $\{x_i \in X\}$. For any fixed choice of Riemannian metric on X and any small enough positive real number ε, write

$$D^{\varepsilon}_{x_i} \;:=\; \big\{x \in X \mid d(x, x_i) < \varepsilon\big\} \subset X$$

for the open ball of radius ε around x_i. The complement of these open balls is hence a manifold with boundary a disjoint union of $(2n+1)$-spheres:

$$\partial\big(X \setminus \coprod_i \{x_i\}\big) \;\simeq\; \coprod_i S^{2n+1}.$$

Then, by Prop. 3.5, the cocycle in twisted Cohomotopy on $X \setminus \coprod_i \{x_i\}$ which corresponds to the vector field v has underlying it a differential $(2n+1)$-form G_{2n+1} which satisfies

$$dG_{2n+1} \;=\; \chi_{2n+2}(\nabla)\,.$$

By Stokes' theorem we thus have

$$\begin{aligned}\chi[X] &= \lim_{\varepsilon\to 0} \int_{X\setminus \coprod_i D^{\varepsilon}_{x_i}} \chi \\ &= \lim_{\varepsilon\to 0} \sum_i \int_{\partial D^{\varepsilon}_{x_i}} G_{2n+1}\end{aligned}$$

Hence, we summarize the above by the following.

Lemma 3.25 (Cohomological PH-theorem)**.** *In the above setting, the Euler characteristic is given by the integral of G_{2n+1} over the boundary components around the zeros of v:*

$$\chi[X] = \sum_i \int_{S_i^{2n+1}} G_{2n+1}\,. \tag{61}$$

3.6 Twisted Cohomotopy via Pontrjagin-Thom

We recall the unstable Pontrjagin-Thom theorem relating untwisted Cohomotopy to normally framed submanifolds, (62) below. Then we show that twisted Cohomotopy jointly in degrees 4 and 7 (as per §3.3) knows about *calibrated submanifolds* in 8-manifolds, Prop. 3.26 below. Finally we observe that in this case vanishing submanifolds under a twisted Pontrjagin-Thom construction means, equivalently, a factorization through the quaternionic Hopf fibration, (68) below.

Framed submanifolds from untwisted Cohomotopy. One striking aspect of *Hypothesis H*, is that unstable Cohomotopy of a manifold X is exactly the cohomology theory which classifies (cobordism classes of) *submanifolds* $\Sigma \subset X$, subject to constraints on the normal bundle $N_X\Sigma$ of the embedding.

In the case of vanishing twist, this is the statement of the classical *unstable Pontrjagin-Thom isomorphism* (e.g. [Ko93, IX.5])

$$\pi^n(X) \quad \underset{\mathrm{fib}_0\circ\mathrm{reg}}{\overset{\mathrm{PT}^n}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \quad \mathrm{FrSubMfd}^{\mathrm{codim}=n}(X)\big/_{\sim_{\mathrm{bord}}}. \tag{62}$$

For a closed smooth manifold X and any degree $n\in\mathbb{N}$, this identifies degree n cocycles

$$\big[X \xrightarrow{c} S^n\big] \;\in\; \pi^n(X)$$

in the untwisted unstable Cohomotopy (28) of X with the cobordism classes of normally framed submanifolds Σ of codimension n

$$\Big(\Sigma \hookrightarrow X\,,\quad N_X\Sigma \underset{\simeq}{\xrightarrow{\mathrm{fr}}} \Sigma\times\mathbb{R}^n\;,\;\; \dim(\Sigma)=\dim(X)-n\,\Big)$$

given as the preimage of a chosen base point

$$\mathrm{pt}\;\in\; S^n \tag{63}$$

under a smooth function representative c of $[c]$ for which pt is a regular value $c^{-1}\big(\{\mathrm{pt}\}\big) \;=:\; \Sigma\subset X$.

As advocated in [Sa13], we may naturally think of the submanifolds $\Sigma\subset X$ appearing in the unstable Pontrjagin-Thom isomorphism (62) as *branes* whose charge is given by the Cohomotopy class $[c]$. This reveals Cohomotopy as the canonical cohomology theory for measuring charges of branes given as (cobordism classes of) submanifolds. To see this in full detail one needs to consider the refinement of (62) to twisted and *equivariant* Cohomotopy. In the rational approximation this is discussed in [HSS18], the full non-rational theory of M-branes at singularities classified by equivariant Cohomotopy will be discussed elsewhere [RSS].

Here we content ourselves with highlighting two related facts, which are needed for the discuss in §4.

Calibrated submanifolds from twisted Cohomotopy. The manifold $\mathbb{R}^8$ carries an exceptional *calibration* by the *Cayley 4-form* $\Phi\in\Omega^4(\mathbb{R}^8)$ [HL82], which singles out 4-dimensional submanifold embeddings $\Sigma_4\hookrightarrow\mathbb{R}^8$ as the corresponding *calibrated submanifolds*. The space of all such *Cayley 4-planes*, canonically a subspace of the Grassmannian space $\mathrm{Gr}(4,8)$ of *all* 4-planes in 8 dimensions, is denoted

$$\mathrm{CAY} \subset \mathrm{Gr}(4,8) \tag{64}$$

in [BH89, (2.19)][GMM95, (5.20)]. We will write

$$\mathrm{CAY}_{\mathrm{sL}} \subset \mathrm{CAY} \subset \mathrm{Gr}(4,8) \tag{65}$$

for the further subspace of those Cayley 4-planes which are also special Lagrangian submanifolds. There are canonical symmetry actions of $\mathrm{Spin}(7)$ and of $\mathrm{Spin}(6)$, respectively, on these spaces [HL82, Prop. 1.36]:

$$\underset{\mathrm{CAY}}{\overset{\mathrm{Spin}(7)}{\curvearrowright}} \quad \text{and} \quad \underset{\mathrm{CAY}_{\mathrm{sL}}}{\overset{\mathrm{Spin}(6)}{\curvearrowright}} . \tag{66}$$

Hence the corresponding homotopy quotients

$$\mathrm{CAY} /\!\!/ \mathrm{Spin}(7) \quad \text{and} \quad \mathrm{CAY}_{\mathrm{sL}} /\!\!/ \mathrm{Spin}(6) \tag{67}$$

are the *moduli spaces* for Cayley 4-planes and for special Lagrangian Cayley 4-planes, respectively: for X a $\mathrm{Spin}(7)$-manifold, a dashed lift in

$$\begin{tikzcd} & \mathrm{CAY} /\!\!/ \mathrm{Spin}(7) \arrow[d] \\ X \arrow[ur, dashed] \arrow[r] & B\mathrm{Spin}(7) \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{CAY}_{\mathrm{sL}} /\!\!/ \mathrm{Spin}(6) \arrow[d] \\ X \arrow[ur, dashed] \arrow[r] & B\mathrm{Spin}(6) \end{tikzcd}$$

is a distribution on X by tangent spaces to (special Lagrangian) calibrated submanifolds.

Proposition 3.26 (Calibrations from twisted cohomotopy)**.** *The moduli spaces of (special Lagrangian) Cayley 4-planes* (67) *are compatibly weakly homotopy equivalent to the coefficient spaces for twisted Cohomotopy jointly in degrees 4 and 7, according to Prop. 3.19, as shown in Figure T:*

$$\begin{tikzcd} \mathrm{CAY}_{\mathrm{sL}} /\!\!/ \mathrm{Spin}(6) \arrow[d] & \simeq & S^7 /\!\!/ \big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[d] \\ \mathrm{CAY} /\!\!/ \mathrm{Spin}(7) & \simeq & S^4 /\!\!/ \big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \end{tikzcd}$$

Proof. By [HL82, Theorem 1.38] (see also [BH89, (3.19)], [GMM95, (5.20)]) we have a coset space realization

$$\mathrm{CAY} \simeq \mathrm{Spin}(7)/\big(\mathrm{Spin}(4)\cdot\mathrm{Spin}(3)\big) .$$

and by [BBMOOY96, p. 7] we have a coset space realization

$$\mathrm{CAY}_{\mathrm{sl}} \simeq \mathrm{Spin}(6)/\big(\mathrm{Spin}(3)\cdot\mathrm{Spin}(3)\big) \simeq \mathrm{SU}(6)/\mathrm{SO}(4) .$$

By Lemma 3.7 this means equivalently that there are weak homotopy equivalences

$$\mathrm{CAY} /\!\!/ \mathrm{Spin}(7) \simeq B\big(\mathrm{Spin}(4)\cdot\mathrm{Spin}(3)\big) \simeq B\big(\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\cdot\mathrm{Sp}(2)\big)$$

and

$$\mathrm{CAY}_{\mathrm{sL}} /\!\!/ \mathrm{Spin}(6) \simeq B\big(\mathrm{Spin}(3)\cdot\mathrm{Spin}(3)\big) \simeq B\big(\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\big) .$$

This then implies the claim by Prop. 3.20. □

Vanishing PT-charge in twisted Cohomotopy. Even without discussing a full generalization of the untwisted Pontrjagin-Thom theorem (62) to the case of twisted Cohomotopy (Def. 3.1), we may say what it means for a cocycle in twisted Cohomotopy to correspond to the empty submanifold, hence to correspond to vanishing brane charge in the sense discussed above. This is all that we will need to refer to below in §4.6:

(i) In the case of untwisted cohomotopy it is immediate that the zero-charge cocycle is simply the one represented by any function that does not meet the given base point $\mathrm{pt} \in S^n$ (63).

(ii) In the case of twisted Cohomotopy according to Def. 3.1, this chosen point must be a chosen *section* of the given spherical fibration corresponding to the given twist τ:

$$\begin{tikzcd}
 & S^n /\!\!/ \mathrm{O}(n+1) \arrow[d] \\
X \arrow[ur, dashed, "\mathrm{pt}"] \arrow[r, "\tau"'] & B\mathrm{O}(n+1)
\end{tikzcd}$$

which serves over each $x \in X$ as the point $\mathrm{pt}_x \in E_x \simeq S^4$ at which we declare to form the inverse image of another given section, under a parametrized inverse Pontrjagin-Thom construction.

(iii) With that section pt chosen, any other twisted Cohomotopy cocycle $[c_0] \in \pi^\tau(X)$ which will yield the empty submanifold under parametrized Pontrjagin-Thom must be represented by a section c_0 which is everywhere distinct from pt,

$$c_0(x) \neq \mathrm{pt}_x$$

so that $c_0^{-1}(\mathrm{pt}(x)) = \varnothing$ for all $x \in X$.

(iv) But such a choice of a pair of pointwise distinct sections is equivalently a reduction of the structure group not just along $\mathrm{O}(4) \hookrightarrow \mathrm{O}(5)$ as in Remark 3.8, but is rather a reduction all the way along $\mathrm{O}(3) \hookrightarrow \mathrm{O}(5)$.

Specified to the $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$-twisted Cohomotopy jointly in degrees 4 and 7, from §3.3 this says that vanishing of the brane charge seen by degree 4 Cohomotopy cocycle via a putative parameterized PT theorem is witnessed by a lift from $B\big(\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)\big)$ all the way to $B\big(\mathrm{Spin}(3)\cdot\mathrm{Spin}(3)\big)$. But comparison with Prop. 3.20 (see also *Figure T*) shows the following.

Lemma 3.27 (Joint Cohomotopy cocyles via parametrized PT)**.** *The vanishing of brane charge is, equivalently, a factorization of the degree 4 cocycle through degree 7 Cohomotopy, via the quaternionic Hopf fibration:*

$$\begin{tikzcd}
 & S^7 /\!\!/ \big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[r, "\simeq"] \arrow[d, "h_{\mathbb{H}} /\!\!/ (\mathrm{Sp}(2)\cdot\mathrm{Sp}(1))"] & B\big(\mathrm{Spin}(3)\cdot\mathrm{Spin}(3)\big) \arrow[d] \\
 & S^4 /\!\!/ \big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[r, "\simeq"] \arrow[d] & B\big(\mathrm{Spin}(4)\cdot\mathrm{Spin}(3)\big) \arrow[d] \\
X \arrow[uur, dashed, "\substack{\text{PT-vanishing of}\\ \text{cocycle in}\\ \text{twisted Cohomotopy}\\ \text{in degree 4}}"] \arrow[ur, dashed, "\substack{\text{cocycle in}\\ \text{twisted}\\ \text{Cohomotopy}\\ \text{in degree 4}}"] \arrow[r, "\tau"'] & B\big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[r, "\simeq", "B\mathrm{tri}"'] & B\big(\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)\big)\,.
\end{tikzcd} \tag{68}$$

We come back to this in Prop. 4.15 below.

This concludes our discussion of general properties of twisted Cohomotopy theory. Now we turn, in §4, to discussing how, under *Hypothesis H*, these serve to yield anomaly cancellation in M-theory.

4 M-theory anomaly cancellation via twisted Cohomotopy

In this section we show how *Hypothesis H* implies all the M-theory anomaly cancellation conditions reviewed in §2. Concretely, we have shown in §3 that Cohomotopy jointly in degrees 4 and 7, related by the quaternionic Hopf fibration, is $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$-twisted Cohomotopy, hence, under Triality (Prop. 3.16) is $\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)$-twisted Cohomotopy (by Prop. 3.19 and Prop. 3.20). Hence we have the following.

Corollary 4.1 (C-Field Cocycles in twisted Cohomotopy)**.** *Assume with Hypothesis H, that the C-field is a cocycle in twisted Cohomotopy (Def. 3.1), twisted by the tangent bundle of spacetime via the J_n-homomorphism* (30)*, compatibly in joint degree 4 and (for the dual C-field) degree 7, related via the quaternionic Hopf fibration $h_{\mathbb{H}}$. Then the equivariance property of the latter (Prop. 3.19 and Prop. 3.20) require that C-field configurations be dashed morphisms as in the following homotopy-commutative diagram (showing part of the diagram in Figure T):*

topological C-field   in terms of: twisted Cohomotopy,

$$
\begin{tikzcd}
 & S^7 /\!\!/ \big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[d, "h_{\mathbb{H}} /\!\!/ (\mathrm{Sp}(2)\cdot\mathrm{Sp}(1))"] \\
 & S^4 /\!\!/ \big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[dd] \\
\widehat{X^{11}} \arrow[uur, dashed, bend left, "G_7 + H_3 \wedge G_4"] \arrow[dd] & \\
 & B\big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[d] \\
\underbrace{X^{11}}_{\mathbb{R}^{2,1}\times X^8} \arrow[uuur, dashed, "(G_4, G_7)"] \arrow[ur, dashed, "N_{Q_{\mathrm{M2}}} Q_{\mathrm{M5}} \cdot N_{X^{11}} Q_{\mathrm{M5}}"] \arrow[r, "N_{X^{11}} Q_{\mathrm{M2}}"] \arrow[dr, "TX^{11}"'] & B\mathrm{Spin}(8) \arrow[d] \\
 & B\mathrm{Spin}(10,1)
\end{tikzcd}
\tag{69}
$$

where $N_{Q_{M2}} Q_{M5} \cdot N_{X^{11}} Q_{M2}$ denotes topological G-structure for $G = \mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$ (43) *(see Remark 4.2 below) and where (G_4, G_7) denotes the Cocycle in $N_{Q_{M2}} Q_{M5} \cdot N_{X^{11}} Q_{M2}$-twisted Cohomotopy (see Def. 4.3 below):*

Remark 4.2 (M-brane configurations)**.** The emergence of $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$-twisted Cohomotopy in Cororllary 4.1 *implies* that we are to consider 11d spacetimes whose frame bundle is incrementally equipped with $\mathrm{Spin}(2,1)\cdot\mathrm{Spin}(3)\cdot\mathrm{Spin}(5)$-structure as shown in diagram (69). Locally, this corresponds to configurations of M2-brane inside M5-branes inside spacetime:

		$\mathrm{Spin}(2,1)$ ↻		$\mathrm{Spin}(3)$ ↻		$\mathrm{Spin}(5)$ ↻
$\mathbb{R}^{10,1}$	$\simeq$	$\mathbb{R}^{2,1}$	$\oplus$	$\mathbb{R}^3$	$\oplus$	$\mathbb{R}^5$
M5		$\times$		$\times$		–
M2		$\times$		–		–

Such M2-M5 mixed/bound state configurations have been discussed in [ILPT96, p. 22][GLPT96, p. 13] [CR02, Sec. 1] [PT03, p. 19]. It is in this sense that we are labelling the classifying maps of the bundles in diagram (69), by suggestive abuse of notation:

- $N_X Q_{\mathrm{M2}}$ and $N_X Q_{\mathrm{M5}}$ refer to the normal bundle of an M2-brane or M5-branes, respectively, relative to all of the ambient 11d spacetime X;
- $N_{Q_{\mathrm{M5}}} Q_{\mathrm{M2}}$ refers to the normal bundle of an M2-brane (only) relative to the ambient M5-brane worldvolume.

Beware that we are abusing notation here, in that actual normal bundles are supported only on the corresponding submanifold locus $Q_p \hookrightarrow X$, while in diagram (69) we are showing bundles that extend over all of spacetime, as the dashed map here:

$$\begin{tikzcd}
Q_p \arrow[r, "TQ_p \cdot N_{X^{11}} Q_p"] \arrow[d, hook] & B\big(\mathrm{Spin}(p,1)\cdot \mathrm{Spin}(d-p)\big)\,. \\
X^{11} \arrow[ur, dashed] &
\end{tikzcd}$$

However, in relevant examples it is indeed the case that normal bundles to brane inclusions extend to all of spacetime, notably in M5-brane anomaly cancellation, see (96) in Prop. 4.18 below. Moreover, further below in §4.6 we discover actual M2-branes appear as point singularities in X^8, and then this setup ensures that *wherever these points appear*, the restriction of $N_X Q_{\mathrm{M2}}$ to these points will be the actual normal bundle to the M2-brane at that point.

By Prop. 3.5 there are differential form data G_4 and G_7 associated with such a cocycle in twisted Cohomotopy, as in diagram (69). For reference, and since this is the key that connects twisted Cohomotopy to form flux data, we make this explicit:

Definition 4.3 (Differential forms underlying cocycles in degree 4 twisted Cohomotopy)**.** Let $X^{11} := \mathbb{R}^{2,1} \times X^8$ be a spacetime equipped with topological $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$-structure

$$\begin{tikzcd}
 & B(\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)) \arrow[d] \\
\mathbb{R}^{2,1}\times X^8 \arrow[ur, dashed, "N_{X^{11}} Q_{\mathrm{M5}} \cdot N_{Q_{\mathrm{M5}}} Q_{\mathrm{M2}}"] \arrow[r, "N_{X^{11}} Q_{\mathrm{M2}}"] \arrow[dr, "TX^{11}"'] & B\mathrm{Spin}(8) \arrow[d] \\
 & B\mathrm{Spin}(10,1)
\end{tikzcd} \tag{70}$$

Prop. 3.19 implies that this structure serves as a twist for Cohomotopy in degree 4 (Def. 3.1)

$$\tau \;:=\; N_{Q_{\mathrm{M5}}} Q_{\mathrm{M2}} \cdot N_{X^{11}} Q_{\mathrm{M5}}\,,$$

and thus Prop. 3.5 provides a function

$$\pi^\tau\big(X^{11}\big) \simeq \pi^\tau(X^8) \longrightarrow \big\{(G_4, G_7) \;\in\; \Omega^4_{\mathrm{cl}}(X^8)\times\Omega^7(X^8)\big\}_{\big/\sim} \tag{71}$$

which extracts out of a full cocycle in τ-twisted Cohomotopy a pair of differential forms in degree 4 and 7, satisfying

$$\begin{aligned}
dG_4 &= 0\,,\\
dG_7 &= \tfrac{1}{4}p_2(\nabla_\tau) - G_4\wedge G_4\,.
\end{aligned} \tag{72}$$

Hence *Hypothesis H* now says more concretely that: These differential forms G_4 and G_7 are identified with the C-field flux form and its dual as in the 11d supergravity/M-theory literature, but their refinement through the map (71) to a cocycle (G_4, G_7) in $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$-twisted Cohomotopy is the actual nature of the C-field, in particular incorporating/implying the M-theory anomaly cancellation conditions.

We will now turn to a detailed elaboration and unpacking of this.

4.1 DMW anomaly cancellation

We show here that *Hypothesis H*, as in (69), implies the DMW anomaly cancellation condition (§2.1). The key argument is a cohomological characterization of $\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)$-structures, Prop. 4.4 below. We provide a conclusion in Remark 4.5 below.

Proposition 4.4 (Consequences of central product structure)**.** *Let X^8 be a closed connected smooth Spin manifold of dimension 8. Then a reduction of its structure group to* $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$ *(Def. 3.10) along the canonical inclusion* (43) *as in Def. 4.3*

$$\begin{tikzcd} & & B\big(\mathrm{Spin}(3)\cdot\mathrm{Spin}(5)\big) \arrow[d] \\ X^8 \arrow[urr, dashed, "N_{Q_{\mathrm{M5}}}Q_{\mathrm{M2}}\cdot N_{X^{11}}Q_{\mathrm{M5}}"] \arrow[rr, "N_{X^{11}}Q_{\mathrm{M2}}"'] & & B\mathrm{Spin}(8) \end{tikzcd}$$

implies the following conditions:

- *The I_8-polynomial* (16) *is proportional to the Euler class:*

$$I_8(N_{X^{11}}Q_{\mathrm{M2}}) = \tfrac{1}{24}\chi_8(N_{X^{11}}Q_{\mathrm{M2}})\,. \tag{73}$$

If in addition $H^2(X^8;\mathbb{Z}_2)=0$, then also the following conditions hold:

- *The sixth Stiefel-Whitney class vanishes:*

$$w_6(N_{X^{11}}Q_{\mathrm{M2}}) = 0\,. \tag{74}$$

Proof. This follows by combining [CV98b, Thm. 8.1 with Rem. 8.2] with the definition of I_8 (16). □

Remark 4.5 (Deriving the DMW-anomaly cancellation from Hypothesis H)**.** In the situation (69), where $TX^{11}\simeq N_{X^{11}}Q_{\mathrm{M2}}$, the condition (74) $w_6(N_{X^{11}}Q_{\mathrm{M2}})=0$ found in Prop. 4.4 becomes $w_6(TX^{11})=0$, and directly implies the weaker condition

$$W_7(TX^{11}) = \ \beta\big(w_6(TX^{11})\big) = 0\,,$$

where β is the Bockstein homomorphism, and $W_7 := \beta\circ w_6$ by definition of integral Stiefel-Whitney classes. This is manifestly the DMW-anomaly cancellation (6) from §2.1.

4.2 Half-integral flux quantization

We show here that the topological charge quantization of the C-field in twisted Cohomotopy, as in diagram (69), implies the half-integral flux quantization of the C-field (8). The key argument is Prop. 4.10 below. We conclude in Remark 4.11 below. The basic observation here is Remark 4.7 below, but to put this to full use we need to go into some technicalities in Lemma 4.8 and Prop. 4.9 below.

First we need to recall some classical facts about the integral cohomology of $B\mathrm{Spin}(n)$ for low n:

Lemma 4.6. **(i)** *The integral cohomology ring of $B\mathrm{SO}(3)$ is*

$$H^\bullet(B\mathrm{SO}(3);\mathbb{Z}) \ \simeq\ \mathbb{Z}\big[p_1,W_3\big]/(2W_3)\,, \tag{75}$$

and the integral cohomology of $B\mathrm{Spin}(3)$ is free on one generator

$$H^\bullet\big(B\mathrm{Spin}(3);\mathbb{Z}\big)\ \cong\ \mathbb{Z}\big[\tfrac{1}{4}p_1\big], \tag{76}$$

while the integral cohomology ring of $B\mathrm{Spin}(4)$ is free on two generators

$$H^\bullet\big(B\mathrm{Spin}(4);\mathbb{Z}\big)\ \simeq\ \mathbb{Z}\big[\tfrac{1}{2}p_1, \underbrace{\underbrace{\tfrac{1}{2}\chi_4}_{=:\Gamma_4} + \tfrac{1}{4}p_1}_{=:\widetilde{\Gamma}_4}\big]\,, \tag{77}$$

where p_1 is the first Pontrjagin class and χ_4 the Euler class.

(ii) *Under the exceptional isomorphism* $\vartheta : \mathrm{Spin}(3) \times \mathrm{Spin}(3) \xrightarrow{\simeq} \mathrm{Spin}(4)$ *these classes are related by*

$$
\begin{aligned}
\vartheta^*\big(\tfrac{1}{2}p_1\big) &= \tfrac{1}{4}p_1 \otimes 1 + 1 \otimes \tfrac{1}{4}p_1 \,, \\
\vartheta^*\big(\tfrac{1}{2}\chi + \tfrac{1}{4}p_1\big) &= 1 \otimes \tfrac{1}{4}p_1 \,, \\
\text{hence} \qquad \vartheta^*(\chi) &= -\tfrac{1}{4}p_1 \otimes 1 + 1 \otimes \tfrac{1}{4}p_1 \,.
\end{aligned}
\tag{78}
$$

Proof. This follows from classical results [Pi91]. More explicitly, (75) is a special case of [Br82, Thm. 1.5], recalled for instance as [RS17, Thm. 4.2.23 with Remark 4.2.25]. The other statements are recalled for instance in [CV98a, Lemma 2.1]. □

Remark 4.7 (Universal avatar of the integral C-field)**.** We highlight from (77), under the braces, the universal integral class

$$
\widetilde{\Gamma}_4 \;:=\; \underbrace{\tfrac{1}{2}\chi_4}_{=:\Gamma_4} + \tfrac{1}{4}p_1 \;\in\; H^4(B\mathrm{Spin}(4);\mathbb{Z})
\tag{79}
$$

for use below. Prop. 4.10 below says that, under *Hypothesis H*, these universal characteristic classes are the avatars of the half-integral shifted C-field flux $\widetilde{G}_4$. Since $\widetilde{\Gamma}_4$ is an integral cohomology class, its concrete realization on any given spacetime is an integral class. This is what implements the half-integral flux quantization condition in M-theory; see Remark 4.11 below.

We now trace the integral generator $\widetilde{\Gamma}_4$ in (79) to the larger group $\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)$.

Lemma 4.8 (Cohomology of the central group)**.** *The integral cohomology in degree 4 of the classifying space of the group* (50)

$$
\mathrm{Spin}(4)\cdot\mathrm{Spin}(3) \;\simeq\; \mathrm{Spin}(3)\cdot\mathrm{Spin}(3)\cdot\mathrm{Spin}(3)
$$

is the free lattice

$$
H^4\big(B(\mathrm{Spin}(4)\cdot\mathrm{Spin}(3));\mathbb{Z}\big) \;\simeq\; \mathbb{Z}\left\langle
\begin{array}{ccccc}
\tfrac{1}{4}p_1^{(1)} & + & \tfrac{1}{4}p_1^{(2)} & + & \tfrac{2}{4}p_1^{(3)}, \\
\tfrac{1}{4}p_1^{(1)} & + & \tfrac{2}{4}p_1^{(2)} & + & \tfrac{1}{4}p_1^{(3)}, \\
\tfrac{2}{4}p_1^{(1)} & + & \tfrac{1}{4}p_1^{(2)} & + & \tfrac{1}{4}p_1^{(3)}
\end{array}
\right\rangle
\tag{80}
$$

where $p_1^{(k)} \;:=\; (B\mathrm{pr}_k)^*(p_1)$ *is the pullback of the first Pontrjagin class along the projection* (47)

$$
B\big(\mathrm{Spin}(4)\cdot\mathrm{Spin}(3)\big) \simeq B\big(\mathrm{Spin}(3)\cdot\mathrm{Spin}(3)\cdot\mathrm{Spin}(3)\big) \xrightarrow{B\mathrm{pr}_k} B\mathrm{SO}(3)\,.
$$

Proof. The defining short exact sequence of groups (Def. 3.10)

$$
1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(3)\cdot\mathrm{Spin}(3)\cdot\mathrm{Spin}(3) \longrightarrow \mathrm{Spin}(3)\times\mathrm{Spin}(3)\times\mathrm{Spin}(3) \longrightarrow 1
$$

induces a homotopy fiber sequence of classifying spaces (e.g. [Mi11, 11.4])

$$
B\mathbb{Z}_2 \longrightarrow B\big(\mathrm{Spin}(3)\times\mathrm{Spin}(3)\times\mathrm{Spin}(3)\big) \longrightarrow B\big(\mathrm{Spin}(3)\cdot\mathrm{Spin}(3)\cdot\mathrm{Spin}(3)\big).
$$

The corresponding Serre spectral sequence shows that

$$
\begin{aligned}
H^4\big(B(\mathrm{Spin}(3)\cdot\mathrm{Spin}(3)\cdot\mathrm{Spin}(3));\mathbb{Z}\big) \;\hookrightarrow\; & H^4\big(B(\mathrm{Spin}(3)\times\mathrm{Spin}(3)\times\mathrm{Spin}(3)),\mathbb{Z}\big) \\
& \simeq \mathbb{Z}\big\langle \tfrac{1}{4}p_1^{(1)}, \tfrac{1}{4}p_1^{(2)}, \tfrac{1}{4}p_1^{(3)} \big\rangle
\end{aligned}
$$

is a sublattice of index 4. This sublattice must include the integral class $\frac{1}{2}p_1$ pulled back along the inclusion into $\mathrm{Spin}(7)$, which by Lemma 4.6 is

$$
\begin{array}{ccc}
B\big(\mathrm{Spin}(4)\cdot\mathrm{Spin}(3)\big) & \longrightarrow & B\mathrm{Spin}(7)\,. \\
\tfrac{1}{4}p_1 + \tfrac{1}{4}p_1 + \tfrac{2}{4}p_1 & \longleftarrow\!\!\shortmid & \tfrac{1}{2}p_1
\end{array}
\tag{81}
$$

But then it must also contain the images of this element under the delooping of the S_3-automorphisms (51). This yields the other two elements shown in (80). Finally, it is clear that the sublattice spanned by these three elements already has full rank and index 4:

$$\mathbb{Z}\left\langle \begin{array}{ccccc} \tfrac{1}{4}p_1^{(1)} & + & \tfrac{1}{4}p_1^{(2)} & + & \tfrac{2}{4}p_1^{(3)}, \\ \tfrac{1}{4}p_1^{(1)} & + & \tfrac{2}{4}p_1^{(2)} & + & \tfrac{1}{4}p_1^{(3)}, \\ \tfrac{2}{4}p_1^{(1)} & + & \tfrac{1}{4}p_1^{(2)} & + & \tfrac{1}{4}p_1^{(3)} \end{array} \right\rangle \simeq \left\{ \tfrac{a}{4}p_1^{(1)} + \tfrac{b}{4}p_1^{(2)} + \tfrac{c}{4}p_1^{(3)} \,\middle|\, a,b,c \in \mathbb{Z},\; a+b+c = 0 \bmod 4 \right\} \tag{82}$$

which means that there are no further generators. □

As a direct consequence we obtain the following identification.

Proposition 4.9 (Integral classes)**.** *The following cohomology class on the classifying space of the group* $\mathrm{Spin}(4)\cdot\mathrm{Spin}(3)$ (50)*, which a priori is in rational cohomology, is in fact integral:*

$$\underbrace{\tfrac{1}{2}\chi_4 + \tfrac{1}{4}p_1}_{=:\widetilde{\Gamma}_4} + \tfrac{1}{2}p_1^{(3)} \in H^4\big(\mathrm{Spin}(4)\cdot\mathrm{Spin}(3);\mathbb{Z}\big)$$

and hence so is its image on the classifying space of $\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)$ (49) *under the delooping of the triality isomorphism from Prop. 3.16, which we will denote by the same symbols:*

$$\underbrace{\tfrac{1}{2}\chi_4 + \tfrac{1}{4}p_1}_{=:\widetilde{\Gamma}_4} + \tfrac{1}{2}p_1^{(3)} \in H^4\big(\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\cdot\mathrm{Sp}(1);\mathbb{Z}\big) \simeq H^4\big(\mathrm{Spin}(4)\cdot\mathrm{Spin}(3),\mathbb{Z}\big). \tag{83}$$

Here $\tfrac{1}{2}\chi_4$ *is the Euler class pulled back back from the left* $B\mathrm{SO}(4)$ *factor and* $p_1^{(3)}$ *is the first Pontrjagin class pulled back from the right* $B\mathrm{SO}(3)$ *factor, both along the respective projections* (47)*, while* p_1 *is the first Pontrjagin class pulled back from the ambient* $B\mathrm{Spin}(8)$ *along the canonical inclusion* (46)*:*

$$\begin{tikzcd} & B\big(\mathrm{Spin}(4)\cdot\mathrm{Spin}(3)\big) \arrow[dl, "B\mathrm{pr}_4"'] \arrow[d, "B\iota_8"] \arrow[dr, "B\mathrm{pr}_3"] & \\ B\mathrm{SO}(4) & B\mathrm{Spin}(8) & B\mathrm{SO}(3) \\ \chi_4 & p_1 & p_1^{(3)} \end{tikzcd}$$

Proof. In terms of the contributions from the three factors under the identification $\mathrm{Spin}(4)\cdot\mathrm{Spin}(3) \simeq \mathrm{Spin}(3)\cdot\mathrm{Spin}(3)\cdot\mathrm{Spin}(3)$ the class in question is

$$\underbrace{-\tfrac{1}{8}p_1^{(1)} + \tfrac{1}{8}p_1^{(2)}}_{=\frac{1}{2}\chi_4} + \underbrace{\tfrac{1}{8}p_1^{(1)} + \tfrac{1}{8}p_1^{(2)} + \tfrac{1}{4}p_1^{(3)}}_{=\frac{1}{4}p_1} + \tfrac{2}{4}p_1^{(3)} = \tfrac{1}{4}p_1^{(2)} + \tfrac{3}{4}p_1^{(3)},$$

where under the braces we used Lemma 4.6 as in (81). The equivalent expression on the right makes manifest that this is in the sublattice (82). Therefore, Lemma 4.8 implies the claim. □

Now we may finally state and prove the main result of this section.

Proposition 4.10 (Integrality of the shifted class)**.** *Let* X^8 *be a 8-manifold which is simply connected (Remark 3.6) and equipped with topological* $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$*-structure (Def. 42)*

$$\begin{tikzcd} & B\big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[d] \\ X^8 \arrow[ur, "\tau"] \arrow[r, "TX^8"'] & B\mathrm{Spin}(8) \end{tikzcd}$$

such that its characteristic class ε, from Def. 3.13, vanishes:

$$\varepsilon(\tau) \in H^2\big(X^8;\mathbb{Z}_2\big) = 0\,. \tag{84}$$

Then the closed differential 4-form $G_4 \in \Omega^4_{\mathrm{cl}}(X^8)$ which comes, via Def. 4.3, with a cocycle in τ-twisted Cohomotopy $\pi^\tau\big(X^8\big)$ (Def. 3.1), is such that the cohomology class $[G_4] + \frac{1}{4}p_1\big(TX^8\big)$ — which a priori is an element in the cohomology of X with real coefficients — is actually an integral class:

$$[G_4] + \tfrac{1}{4}p_1\big(TX^8\big) \;\in\; H^4\big(X^8;\mathbb{Z}\big)\,. \tag{85}$$

Proof. The proof proceeds by considering the following diagram, which we will discuss below in stages:

$$\begin{tikzcd}
 & & & \overbrace{\tfrac{1}{2}\chi_4 + \tfrac{1}{4}p_1}^{\widetilde{\Gamma}_4} + \;\tfrac{1}{2}p_1^{(3)} & \tfrac{1}{2}p_1^{(3)} \arrow[l, mapsto] \\
 & & B\big(\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\big) \arrow[r, "\simeq"] \arrow[d] & B\big(\mathrm{Spin}(4)\cdot\mathrm{Spin}(3)\big) \arrow[r, "B\mathrm{pr}_4"] \arrow[d] & B\mathrm{SO}(4) \arrow[d] \\
 {[G_4] + \tfrac{1}{4}p_1(TX^8)} & & B\big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[r, "\simeq"] \arrow[d] & B\big(\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)\big) \arrow[r, "B\mathrm{pr}_5"] \arrow[d] & B\mathrm{SO}(5) \arrow[d] \\
 & X \arrow[r, "TX"] \arrow[uur, dashed, "c", "\text{cocycle in } \tau\text{-twisted Cohomotopy}"'] \arrow[ur, dashed, "\tau"'] & B\mathrm{Spin}(8) \arrow[r, "B\mathrm{tri}", "\simeq"'] & B\mathrm{Spin}(8) \arrow[r] & B\mathrm{SO}(8) \\
 & p_1\big(TX^8\big) & p_1 \arrow[l, mapsto] & p_1 \arrow[l, mapsto] &
 \arrow[from=1-4, to=3-1, mapsto, bend right]
\end{tikzcd} \tag{86}$$

Here the vertical maps are the deloopings of the canonical group inclusions (Remark 3.17) and the horizontal equivalences $B\mathrm{tri}$ are the deloopings (52) of the respective triality automorphism from Prop. 3.16, while the horizontal maps $B\mathrm{pr}_n$ are the deloopings of the canonical projections (47). On the left we used that, by Def. 3.1, an element

$$[c] \in \pi^\tau\big(X^8\big)$$

in the τ-twisted Cohomotopy of X^8 is the homotopy class of a section c of the S^4-bundle classified by $B\mathrm{pr}_5 \circ B\mathrm{tri} \circ \tau$:

$$\begin{tikzcd}
S^4 \arrow[r] & E \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\text{(pb)}"] & B\mathrm{SO}(4) \simeq S^4 /\!\!/ \mathrm{SO}(5) \arrow[d] \\
 & X^8 \arrow[r, "B\mathrm{pr}_5 \circ B\mathrm{tri} \circ \tau"'] \arrow[u, dashed, bend left, "c"] & B\mathrm{SO}(5)
\end{tikzcd}$$

and we used Prop. 3.20 to identify various homotopy quotients of S^4 with classifying spaces, as shown. This shows that E is the unit sphere bundle of a rank 5 real vector bundle V classified by $B\mathrm{pr}_5 \circ B\mathrm{tri} \circ c$. Therefore, by Prop. 3.5 we have

$$\pi^*[G_4] = \tfrac{1}{2}\chi_4\big(\widehat{V}\big)\,,$$

where $\widehat{V}$ is defined by the splitting $\pi^* V = \mathbb{R}_E \oplus \widehat{V}$ determined by the tautological section of $\pi^* V$ over E, i.e., it is the rank 4 real vector bundle on E classified by $E \to B\mathrm{SO}(4)$. Hence, by (83) in Prop. 4.9, we have that

$$\pi^*\Big(\underbrace{[G_4] + \tfrac{1}{4}p_1\big(B\mathrm{tri}\circ TX^8\big) + \tfrac{1}{2}p_1^{(3)}\big(B\mathrm{tri}\circ\tau\big)}_{=:K}\Big) \;\in\; H^4(E;\mathbb{Z})$$

is an integral class.

We now claim that the class K is integral already before the pullback, as a class on X. For this, consider the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^4\big(X^8;\mathbb{Z}\big) \arrow[r] \arrow[d, "\pi^*"] & H^4\big(X^8;\mathbb{Q}\big) \arrow[r, "q"] \arrow[d, "\pi^*"] & H^4\big(X^8;\mathbb{Q}/\mathbb{Z}\big) \arrow[r] \arrow[d, "\pi^*"] & \cdots \\
\cdots \arrow[r] & H^4\big(E;\mathbb{Z}\big) \arrow[r] & H^4\big(E;\mathbb{Q}\big) \arrow[r, "q"] & H^4\big(E;\mathbb{Q}/\mathbb{Z}\big) \arrow[r] & \cdots
\end{tikzcd}$$

induced by the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. From the Serre spectral sequence for the fibration $\pi\colon E \to X$ one sees that the vertical maps in the above diagram are injective. Consequently, from

$$\begin{aligned}
\pi^* q(K) &= q\pi^*(K) \\
&= 0
\end{aligned}$$

it follows that already $q(K) = 0$, which means that K itself is integral:

$$[G_4] + \tfrac{1}{4}p_1\big(B\mathrm{tri}\circ TX^8\big) + \tfrac{1}{2}p_1^{(3)}(B\mathrm{tri}\circ\tau) \;\in\; H^4(X;\mathbb{Z})\,. \tag{87}$$

Now observe that the third summand in (87) is the first fractional Pontrjagin class of the underlying SO(3)-bundle. By the assumption (84) this admits Spin structure, by Lemma 3.13. This in turn implies that its first Pontrjagin class is divisible by two, hence that the last summand in (87) is integral by itself

$$\tfrac{1}{2}p_1^{(3)}(B\mathrm{tri}\circ\tau) \;\in\; H^4\big(X^8;\mathbb{Z}\big)\,,$$

and hence that also the remaining summand

$$[G_4] + \tfrac{1}{4}p_1\big(B\mathrm{tri}\circ TX^8\big) \;\in\; H^4\big(X^8;\mathbb{Z}\big) \tag{88}$$

is integral by itself. Finally, pullback along the triality automorphism preserves the first Pontrjagin class, by Lemma 3.18

$$p_1(B\mathrm{tri}\circ\tau) \;=\; p_1\big(TX^8\big) \tag{89}$$

and hence (88) indeed becomes $[G_4] + \tfrac{1}{4}p_1\big(TX^8\big) \in H^4\big(X^8;\mathbb{Z}\big)$. □

Remark 4.11 (Deriving the shifted flux quantization from Hypothesis H)**.** Under *Hypothesis H* the statement (85) of Prop. 4.10 is manifestly the shifted flux quantization condition (8). Notice that the assumption of Spin(5)-structure (10) made in [Wi96a, 2.3], implies the assumption $\varepsilon = 0$ (84) in Prop. 4.10.

4.3 Integral equation of motion

We now show how *Hypothesis H* implies the "integral equation of motion" for the C-field (§2.3). The key argument is Prop. 4.12 below, and the conclusion is stated in Remark 4.13.

Proposition 4.12 (Sq^2-closedness of twisted cohomotopy 4-cocycles)**.** *Let X be a closed Spin 8-manifold which is simply connected (Remark 3.6) and equipped with topological* Sp(2)*-structure*

$$\begin{tikzcd}
 & B\mathrm{Sp}(2) \arrow[d] \\
X^8 \arrow[ur, dashed, "\tau"] \arrow[r, "TX^8"'] & B\mathrm{Spin}(8)\,.
\end{tikzcd}$$

Then the closed differential 4-form $G_4 \in \Omega^4_{\mathrm{cl}}(X^8)$ which comes, via Def. 4.3, with a cocycle in τ-twisted Cohomotopy $\pi^\tau(X)$ (Def. 3.1), is such that the cohomology class

$$[G_4] + \tfrac{1}{4}p_1(TX^8) \;\in\; H^4(X^8;\mathbb{Z})$$

which is integral by Prop. 4.10, is annihilated by (mod 2 reduction followed by) the second Steenrod operation:

$$\mathrm{Sq}^2\big([\widetilde{G}_4]\big) \;=\; 0\,. \tag{90}$$

Proof. By Prop. 3.20 and under triality (Prop. 3.16) the τ-twisted Cohomotopy cocycle exhibits reduction to $\mathrm{Spin}(4)$-structure:

$$\begin{tikzcd}
 & S^4/\!\!/\mathrm{Sp}(2) \arrow[r, "\simeq"] \arrow[d] & B\mathrm{Spin}(4) \arrow[d] \\
 & B\mathrm{Sp}(2) \arrow[r, "\simeq"] \arrow[d] & B\mathrm{Spin}(5) \arrow[d] \\
X^8 \arrow[uur, dashed, "c", "{\text{cocycle in } \tau\text{-twisted Cohomotopy}}"'] \arrow[ur, "\tau"] \arrow[r, "TX^8"'] & B\mathrm{Spin}(8) \arrow[r, "\simeq", "B\mathrm{tri}"'] & B\mathrm{Spin}(8)
\end{tikzcd}$$

But, by Prop. 4.10, the class of $\widetilde{G}_4$ is the pullback of the class $\widetilde{\Gamma}_4 \in H^4(B\mathrm{Spin}(4);\mathbb{Z})$ (77) along this reduction:

$$[\widetilde{G}_4] \;=\; (B\mathrm{tri}\circ c)^*\big(\widetilde{\Gamma}_4\big) \;\in\; H^4\big(X^8;\mathbb{Z}\big)\,.$$

Under these identifications, the statement follows upon using [CV98a, Cor. 4.2 (1)], where the element corresponding to $\widetilde{\Gamma}_4$ is denoted s, while the class $[\widetilde{G}_4]$ is denoted S. □

Remark 4.13 (Deriving the integral equation of motion from Hypothesis H)**.**

(i) The condition (90) in Prop. 4.12 directly implies the weaker condition $\mathrm{Sq}^3\big([\widetilde{G}_4]\big) = 0$, since, by the Adem relations, we have $\mathrm{Sq}^3 = \beta \circ \mathrm{Sq}^2$, with β being the Bockstein homomorphism. Under *Hypothesis H* this is manifestly the integral equation of motion (11) for the C-field.

(ii) Notice that the stronger condition (90) *also* has the interpretation as the vanishing of an obstruction to lifting to K-theory, but this stronger condition arises for lifting not to complex K-theory KU as in §2.3, but to orthogonal K-theory KO (see [GS18] for an extensive treatment). This stronger lift is necessary for string/M-theory on *orientifold* spacetimes [Wi98, Sec. 5][Gu00], and these in turn are thought to be crucially necessary for further cancellation of tadpole anomalies. We discuss elsewhere that this also follows from *Hypothesis H*.

4.4 Background charge

We discuss how *Hypothesis H* leads to the C-field background charge, according to §2.4. The key derivation is Prop. 4.14 and the conclusion is stated in Remark 4.16 below.

Proposition 4.14 (Background charge via twisted Cohomotopy)**.** *Let X^8 be a closed smooth Spin 8-manifold which is simply connected (Remark 3.6) and equipped with topological G-structure for $G = \mathrm{Sp}(2)$ (Def. 3.10) along the canonical inclusion* (43) *as in Def. 4.3*

$$\begin{tikzcd}
 & B\mathrm{Sp}(2) \arrow[r, "\simeq"] \arrow[d] & B\mathrm{Spin}(5) \arrow[d] \\
X^8 \arrow[ur, dashed, "\tau"] \arrow[r, "TX^8"'] & B\mathrm{Spin}(8) \arrow[r, "\simeq", "B\mathrm{tri}"'] & B\mathrm{Spin}(8)\,.
\end{tikzcd}$$

Then the differential forms G_4, G_7 which are associated via Def. 4.3 to a cocycle in τ-twisted Cohomotopy (Def. 3.1) satisfy

$$dG_7 \;=\; \tfrac{1}{2}\chi_8\big(TX^8\big) - \underbrace{\big(\widetilde{G}_4 \wedge \widetilde{G}_4 - \widetilde{G}_4 \wedge \tfrac{1}{2}p_1(\nabla_{TX^8})\big)}_{=:2q(G_4)}\,, \tag{91}$$

where ∇_{TX^8} is the connection chosen on the Sp(2)*-principal bundle in Def. 4.3 (via Prop. 3.5), $\chi_8(\nabla_{TX^8})$ is its Euler form, and*

$$\widetilde{G}_4 \;:=\; G_4 + \tfrac{1}{4}p_1(\nabla_{TX^8}) \tag{92}$$

is the corresponding differential form representative, of the class $[\widetilde{G}_4]$ from Prop. 4.10.

Proof. By equation (73) in Prop. 4.4, the $\mathrm{Sp}(2) \hookrightarrow \mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$ structure given by τ implies that the Pontrjagin forms are related to the Euler form by the relation

$$p_2 \;=\; 2\chi_8 + \big(\tfrac{1}{2}p_1\big)^2 . \tag{93}$$

Using this and the definition of $\widetilde{G}_4$ (92) in the general equation (72) satisfied by G_4 according to Prop. 3.5 we directly compute as follows:

$$\begin{aligned} dG_7 &= \tfrac{1}{4}p_2 \qquad\qquad\quad - G_4 \wedge G_4 \\ &= \tfrac{1}{2}\chi_8 + \big(\tfrac{1}{4}p_1\big)^2 - G_4 \wedge G_4 \\ &= \tfrac{1}{2}\chi_8 - \big(G_4 + \tfrac{1}{4}p_1\big) \wedge \big(G_4 - \tfrac{1}{4}p_1\big) \\ &= \tfrac{1}{2}\chi_8 - \widetilde{G}_4 \wedge \big(\widetilde{G}_4 - \tfrac{1}{2}p_1\big) \\ &= \tfrac{1}{2}\chi_8 - \widetilde{G}_4 \wedge \widetilde{G}_4 + \widetilde{G}_4 \wedge \tfrac{1}{2}p_1 \,. \end{aligned} \tag{94}$$

Here in the first line we used Prop. 3.5 with the assumed Sp(2)-structure, which implies that $p_2(\tau) = p_2\big(TX^8\big)$ in (72). □

Proposition 4.15 (PT-vanishing 4-flux)**.** *Let X^8 be a smooth 8-manifold which is simply connected (Remark 3.6) and equipped with* $\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)$*-structure τ. Then, if a cocycle in τ-twisted Cohomotopy (Def. 3.1) has a factorization through the quaternionic Hopf fibration, exhibiting its vanishing PT-charge according to* (68) *in* §*3.6, it follows that the differential 4-form G_4* (71) *which is associated to it by Prop. 3.5 vanishes. Consequently, the corresponding integral 4-form $\widetilde{G}_4$* (85) *from Prop. 4.10 has class $\tfrac{1}{4}p_1\big(TX^8\big)$:*

$$\begin{tikzcd} & S^7 /\!\!/ \big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[d, "h_{\mathbb{H}} /\!\!/ (\mathrm{Sp}(2)\cdot\mathrm{Sp}(1))"] \\ & S^4 /\!\!/ \big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[d] \\ X^8 \arrow[uur, dashed, bend left, "\exists"] \arrow[ur, dashed, "c"] \arrow[r, "\tau"] \arrow[dr, "TX^8"'] & B\big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \arrow[d] \\ & B\mathrm{Spin}(8) \end{tikzcd} \qquad \Longrightarrow \qquad [\widetilde{G}_4] \;=\; \tfrac{1}{4}p_1\big(TX^8\big) \;\in\; H^4\big(X^8;\mathbb{R}\big) .$$

Proof. By Prop. 3.20, the cocycle c itself is equivalently reduction of τ to topological $\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)$-structure; and the further assumed lift is equivalently reduction to topological $\mathrm{Sp}(1)\cdot\mathrm{Sp}(1)$-structure, through the diagonal inclusion on the first two central product factors. By (78) this further pullback annihilates the Euler 4-class χ_4 but preserves the Pontrjagin form. With this the statement follows from (86) in the proof of Prop. 4.10. □

Remark 4.16 (Deriving the background charge from Hypothesis H)**.** Equation (91) expresses the quadratic refinement (12) with background charge

$$\big(\widetilde{G}_4\big)_0 \;=\; \tfrac{1}{2}p_1\,,$$

as expected by the folklore as presented in §2.4.

4.5 M5-brane anomaly cancellation

We now discuss how *Hypothesis H* relates to the folklore of M5-brane anomaly cancellation, reviewed in §2.5. The relevant computation is Prop. 4.18 below. We conclude in Remark 4.19.

In order to formalize the situation with a single unit of M5-brane charge, we consider the following definition (see also [Mo15, (3.12)]):

Definition 4.17 (Form with unit flux through 4-sphere fibrations)**.** Consider a smooth manifold X which is exhibited as an S^4-fibration $S^4 \longrightarrow X \xrightarrow{\pi} Y_{\mathrm{base}}$ over a base manifold Y_{base}. Then a *flux density with unit 4-flux through the 4-sphere* is a differential 4-form which, up to an exact term, is the sum

$$G_4 \;=\; \tfrac{1}{2}\chi(\nabla_\pi) + \pi^*\big(G_4^{\mathrm{basic}}\big) + d\gamma \tag{95}$$

of half the Euler form of the connection $\nabla_{\hat{\tau}}$, as in Prop. 3.5, with any closed differential 4-form pulled back from the base of the fibration.

Now we may state the main result of this section:

Proposition 4.18 (Triviality of square of basic flux)**.** *Let X be a manifold which is simply connected (Remark 3.6) and which is a 4-spherical fibration associated to a* $\mathrm{Spin}(5)\cdot\mathrm{Spin}(n)$*-principal bundle $N_X Q_{\mathrm{M5}}\cdot\mathscr{T}$. Write τ for its canonically associated Cohomotopy twist, as in the following diagram (shown for the special case that $n=3$ and $\mathscr{T} = N_{Q_{\mathrm{M5}}}Q_{\mathrm{M2}}$):*

$$\begin{tikzcd}
 & & & S^4/\!\!/\mathrm{SO}(4) \arrow[r] \arrow[d] \arrow[dr, phantom, "(pb)"] & S^4/\!\!/\mathrm{SO}(5) \arrow[d] \\
X \arrow[r] \arrow[d, "\pi"', "\substack{\text{M5 near-horizon}\\ \text{spacetime}\\ \text{being}\\ \text{4-sphere fibration}}"] \arrow[urrrr, dashed, bend left, "\substack{\text{cocycle in}\\ \text{twisted}\\ \text{Cohomotopy}}"] \arrow[rrrr, bend left, "\tau"] \arrow[rrr, bend left, "\widehat{NQ_{\mathrm{M5}}}"] \arrow[dr, phantom, "(pb)"] & S^4/\!\!/\big(\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)\big) \arrow[r] \arrow[d] \arrow[dr, phantom, "(pb)"] & S^4/\!\!/\mathrm{SO}(5) \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "(pb)"] & B\mathrm{SO}(4) \arrow[r, "B\iota"] \arrow[d, "B\iota"] & B\mathrm{SO}(5) \\
Y_{\mathrm{base}} \arrow[r, "NQ_{\mathrm{M5}}\cdot N_{Q_{\mathrm{M2}}}Q_{\mathrm{M5}}"] \arrow[rrr, bend right, "NQ_{\mathrm{M5}}"'] & B\big(\mathrm{Spin}(5)\cdot\mathrm{Spin}(3)\big) \arrow[r, "B\mathrm{pr}_5"] & B\mathrm{SO}(5) \arrow[r, equal] & B\mathrm{SO}(5) &
\end{tikzcd} \tag{96}$$

If a differential 4-form G_4 which is associated to a cocycle in τ-twisted Cohomotopy $\pi^\tau(X)$, via Def. 4.3, is a unit flux form according to Def. 4.17 then the wedge square of its basic component (95) *has trivial class in cohomology:*

$$\big[G_4^{\mathrm{basic}} \wedge G_4^{\mathrm{basic}}\big] \;=\; 0 \;\in\; H^8(Y_{\mathrm{base}};\mathbb{R})\,. \tag{97}$$

Proof. By Prop. 3.5 the class of the wedge square of the full 4-form G_4 associated with the cocycle in τ-twisted Cohomotopy satisfies equation (72)

$$\big[G_4 \wedge G_4\big] \;=\; \big[\tfrac{1}{4}p_2(\tau)\big] \;\in\; H^8(X;\mathbb{R})\,. \tag{98}$$

Consequently, under cup product with $\frac{1}{2}[G_4]$, it in particular satisfies also the following equation:

$$\tfrac{1}{2}[G_4 \wedge G_4 \wedge G_4] \;=\; \tfrac{1}{8}[G_4 \wedge p_2(\tau)] \;\in\; H^{12}(X;\mathbb{R})\,. \tag{99}$$

As $\tau = B\iota \circ \widehat{NQ_{\mathrm{M5}}} = NQ_{\mathrm{M5}} \circ \pi$, we have $p_2(\tau) = \pi^* p_2(NQ_{\mathrm{M5}})$ and so

$$\tfrac{1}{2}[G_4 \wedge G_4 \wedge G_4] \;=\; \tfrac{1}{8}[G_4 \wedge \pi^* p_2(NQ_{\mathrm{M5}})] \;\in\; H^{12}(X;\mathbb{R})\,. \tag{100}$$

We now consider the image of this equation under fiber integration

$$\pi_* \;:\; H^{\bullet}(X;\mathbb{R}) \longrightarrow H^{\bullet-4}(Y_{\mathrm{base}};\mathbb{R}) \tag{101}$$

along the fibers of the given 4-spherical fibration $S^4 \longrightarrow X \xrightarrow{\pi} Y_{\mathrm{base}}$. By [BC97, Lemma 2.1], the fiber integration of the odd cup powers χ^{2k+1} of the Euler class $\chi \in H^4(X;\mathbb{R})$ of the fibration π are proportional to cup powers of the second Pontrjagin class of the SO(5)-principal bundle to which it is associated:

$$\pi_*(\chi^{2k+1}) = 2\big(p_2(NQ_{\mathrm{M5}})\big)^k \;\in\; H^{4k}(Y_{\mathrm{base}};\mathbb{R})\,, \tag{102}$$

while the fiber integration of the even cup powers of the Euler class vanishes for all $k \in \mathbb{N}$:

$$\pi_*(\chi^{2k}) = 0 \;\in\; H^{8k-1}(Y_{\mathrm{base}};\mathbb{R})\,. \tag{103}$$

Using these relations (102) and (103) together with the unit flux assumption (95)

$$[G_4] \;=\; \tfrac{1}{2}[\chi] + \pi^*\big([G_4^{\mathrm{basic}}]\big)$$

in the image of equation (99) under fiber integration (101), a direct computation, making use of the projection formula [4] yields the following:

$$\begin{aligned}
0 =\;& \pi_*\big[-\tfrac{1}{2}G_4 \wedge G_4 \wedge G_4 + \tfrac{1}{8}G_4 \wedge \pi^* p_2(NQ_{\mathrm{M5}})\big]\\
=\;& \pi_*\Big[-\tfrac{1}{16}\chi^3 - \tfrac{3}{4}\chi \wedge \pi^*(G_4^{\mathrm{basic}} \wedge G_4^{\mathrm{basic}}) + \tfrac{1}{16}\chi \wedge \pi^* p_2(NQ_{\mathrm{M5}})\Big]\\
=\;& \quad\Big[-\tfrac{1}{8}p_2\big(N_X Q_{\mathrm{M5}}\big) - \tfrac{3}{2}G_4^{\mathrm{basic}} \wedge G_4^{\mathrm{basic}} + \tfrac{1}{8}p_2\big(N_X Q_{\mathrm{M5}}\big)\Big]\\
=\;& -\tfrac{3}{2}\Big[G_4^{\mathrm{basic}} \wedge G_4^{\mathrm{basic}}\Big]\,.
\end{aligned}$$

Here in the second line the identification under the brace is manifest from the diagram in the statement of the proposition. ☐

Remark 4.19 (Deriving M5-brane anomaly cancellation from Hypothesis H)**.**

(ii) Under *Hypothesis H*, condition (97) following by Prop. 4.18 is manifestly the remaining M5-brane anomaly cancellation condition (22) as discussed in §2.5.

(ii) Notice that in these considerations, as in (18), the base manifold in Prop. 4.18 is to be taken as a product manifold

$$Y_{\mathrm{base}} \;=\; Q_{\mathrm{M5}} \times \mathbb{R}_{>0} \times U\,,$$

where Q_{M5} is the given 5-brane worldvolume (a 6-manifold), $\mathbb{R}_{>0}$ is the positive real line, representing the radial direction away from the 5-brane locus (see [HSS18, Sec. 2.2] for review), and U is *any* finite-dimensional manifold, which serves to parameterize a family of 4-sphere fibered spacetimes equipped with cocycle data. In a more high-brow discussion than we need here, the above forms would be understood on the moduli stack of cocycle data on $Q_{\mathrm{M5}} \times \mathbb{R}_{>0}$ and U would be any given object in the site of manifolds on which these may be evaluated (see [FSS14c] for cocycles on moduli stacks etc.).

[4] See [FSS18a] for extensive illustrations.

4.6 M2-brane tadpole cancellation

We discuss here how *Hypothesis H* implies the M2-brane tadpole cancellation condition from §2.6. We first explain and then formally define the concepts of "number of M2-branes in a fluxless background" (Def. 4.20 below) and of "fluxless C-field configurations" (Def. 4.21 below) under *Hypothesis H*. Then we prove the cancellation of C-field tadpoles in the fluxless case, Prop. 4.22 below. We conclude the fluxless situation in Remark 4.23. Finally we generalize this result to the general fluxed case, by considering the extended spacetimes (Def. 4.24 below) on which the flux is universally trivialized by the higher gauge field on the M5-brane worldvolume; Remark 4.25 below. This introduces a flux correction term to the number of M2-branes (Prop. 4.27) below which, via the cohomological PH-theorem (Lemma 3.25), yields the general fluxed C-field tadpole cancellation formula. We provide our conclusion in Remark 4.28.

So to start with, consider the scenario found in (69), specified to the *fluxless* case. By the discussion in §3.6 and Prop. 4.15, in this fluxless case *Hypothesis H* implies that the (dual) C-field is exhibited by a cocycle in twisted Cohomotopy of degree 7:

$$\begin{tikzcd} & S^7 /\!\!/ \mathrm{Spin}(8) \arrow[d] \\ \mathbb{R}^{2,1} \times X^8 \arrow[ur, dashed, bend left, "c"] \arrow[r, "\tau = TX^8"'] & B\mathrm{Spin}(8)\,. \end{tikzcd} \tag{104}$$

But for this situation, the first clause of Prop. 3.24 asserts that for such a cocycle

$$[c] \;\in\; \pi^{TX}(\mathbb{R}^{2,1} \times X^8) \;\simeq\; \pi^{TX^8}(X^8)$$

to even exist, it is necessary that the Euler characteristic of X^8 vanishes, $\chi[X^8] = 0$.

On the other hand, the second clause of Prop. 3.24 says that in general a cocycle will exist if a finite set $\{x_i \in X^8\}_i$ of singular points is removed from the "compactification" space X^8. This corresponds to removing from spacetime X a finite set of submanifolds of the form

$$\mathbb{R}^{2,1} \times \{x_i\} \hookrightarrow \mathbb{R}^{2,1} \times X^8\,.$$

These are naturally interpreted as the worldvolumes of M2-branes, which are removed from spacetime in just the same way as the worldlines of magnetic monopoles are removed from spacetime in the classical argument for Dirac charge quantization. If we do adopt this interpretation, then *Hypothesis H* implies that the number of M2-branes at each locus x_i separately is proportional to the restriction of the cocycle c to the vicinity of x_i, as a cocycle in untwisted Cohomotopy (28). We record this conclusion formally as follows:

Definition 4.20 (Number of M2-branes)**.** Let X^8 be a closed smooth Spin manifold of dimension 8, equipped with a finite set $\{x_i \in X^8\}_i$ of points, to be called the *loci of M2-branes*. For

$$[c] \;\in\; \pi^{TX^8}\big(X^8 \setminus \coprod_i \{x_i\}\big)$$

a cocycle in degree 7 Cohomotopy twisted by the tangent bundle (Def. 3.1) and for

$$k \in \mathbb{R} \tag{105}$$

a number, we say that the total *Cohomotopical M2-brane charge in units of* k is the integer N_{M2} which is the image of $[c]$ under restriction to the vicinity U_{x_i} of the points x_i, followed by forming Hopf degrees (29):

$$\begin{array}{ccccccc} \pi^{TX^8}\big(X^8 \setminus \coprod_i \{x_i\}\big) & \xhookrightarrow{\text{restr.}} & \prod_i \pi^7\big(U_{x_i} \setminus \{x_i\}\big) & \xrightarrow{\simeq} & \prod_i \mathbb{Z} & \xrightarrow{\Sigma_i} & \mathbb{Z} \\ [c] & \longmapsto & & & & & k \cdot N_{\mathrm{M2}} \end{array} \tag{106}$$

To determine the minimal proportionality constant k in (105), hence to determine which Cohomotopy charge in degree 7 is to count as *unit charge* of an M2-brane, we have a closer look at the meaning of *fluxlessness* under *Hypothesis H*. So far we used that under the coarse approximation to Cohomotopy given by ordinary cohomology, fluxlessness means factorization through Cohomotopy in degree 7, by Prop. 4.15. But Cohomotopy is finer than ordinary cohomology. In between full non-abelian Cohomotopy and abelian ordinary cohomology is *stable Cohomotopy*, represented not by actual spheres, but by their stabilization to the sphere spectrum (see [BSS18]):

Cohomology theory	Rational cohomology	Integral cohomology	Stable Cohomotopy	Non-abelian Cohomotopy
Cocycle	G_4	$\widetilde{G}_4$	$\Sigma^\infty c$	c

Table 2 – *Incremental approximations to full non-abelian Cohomotopy.*

Observe that it makes no sense to interpret fluxlessness in full non-abelian Cohomotopy. Since the quaternionic Hopf fibration represents the non-torsion generator of

$$\pi_7(S^4) \;=\; \pi^4(S^7) \;\simeq\; \mathbb{Z} \oplus \mathbb{Z}_{12}\,,$$

there is *no* non-trivial way in which a cocycle in full non-abelian degree 7 Cohomotopy could induce a trivial, hence fluxless, cocycle in degree 4 Cohomotopy. This means that the stronger consistent formalization of *fluxlessness* for C-field charge in Cohomotopy is via stable Cohomotopy.

Definition 4.21 (Fluxless Cohomotopy cocycles)**.** Let X^8 be an 8-manifold equipped with topological G-structure τ for

$$G \;:=\; \big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \,\cap\, \mathrm{Spin}(7) \;\subset\; \mathrm{Spin}(8)$$

the intersection of the subgroups of (43) and (55). Then we say that a *fluxless cocyle* in τ-twisted Cohomotopy on $X := \mathbb{R}^{2,1} \times X^8$ (69) is a cocycle $[c] \,\in\, \pi^\tau(X^8)$ in τ-twisted Cohomotopy in degree 7 (Def. 3.1) such that its image in τ-twisted Cohomotopy in degree 4, under the equivariant quaternionic Hopf fibration (Prop. 3.19) is trivial after fiberwise stabilization [BSS18, 2.1], hence trivial in τ-twisted stable Cohomotopy:

$$\Sigma^\infty_X\big(h_{\mathbb{H}}(c)\big) \;\simeq\; 0 \;\in\; \pi^\tau_{\mathrm{st}}(X^8)\,.$$

With the concepts of *number of M2-branes* and of *fluxless cocycles* formalized in terms of Cohomotopy this way, we may now state the main result of this section:

Proposition 4.22 (M2-brane tadpole cancellation from Poincaré-Hopf)**.** *Let X^8 be an 8-manifold equipped with topological G-structure τ for*

$$G \;:=\; \big(\mathrm{Sp}(2)\cdot\mathrm{Sp}(1)\big) \,\cap\, \mathrm{Spin}(7) \;\subset\; \mathrm{Spin}(8)$$

the intersection of the subgroups of (43)*, and* (55)*, and equipped with a set $\{x_i \in X^8\}$ of M2-brane loci (Def. 4.20). Then:*

(i) *The smallest k* (105) *such that for all pairs $[c],[c'] \in \pi^\tau\big(X\big)$ of fluxless cocycles (Def. 4.21) on $X^8 \setminus \coprod_i \{x_i\}$, the difference of number of M2-branes* (106) *is integer $N'_{\mathrm{M2}} - N_{\mathrm{M2}} \,\in\, \mathbb{Z}$ is:*

$$k = 24\,. \tag{107}$$

(ii) *With this minimal unit of Cohomotopical M2-brane charge we have for $[c] \in \pi^\tau\big(X\big)$ any fluxless cocycle (Def. 4.21), that the number of M2-branes (Def. 4.20) equals $\frac{1}{24}$ times the Euler characteristic of X^8:*

$$N_{\mathrm{M2}} \;=\; \tfrac{1}{24}\chi[X^8]\,. \tag{108}$$

Proof. Since the third stable homotopy group of spheres is $\mathbb{Z}_{24}$, generated from the stabilization of the quaternionic Hopf fibration $h_{\mathbb{H}}$

$$
\begin{array}{ccccc}
 & & \pi_7\big(S^4\big) & \xrightarrow{\ \text{stabilization}\ } & \pi_3^{\mathrm{stab}}\big(\mathbb{S}\big) \\
 & & \| & & \| \\
24\mathbb{Z} \oplus \mathbb{Z}_{12} & \xrightarrow{\ \ker\ } & \mathbb{Z} \oplus \mathbb{Z}_{12} & \xrightarrow[\ \mathrm{mod}\, 24 \,\oplus\, 0\ ]{} & \mathbb{Z}_{24} \\
 & & \langle h_{\mathbb{H}} \rangle & \longmapsto & \langle \Sigma^\infty h_{\mathbb{H}} \rangle
\end{array}
$$

this means that $k = 24$. We now spell out this derivation more explicitly, specializing to the untwisted case for ease of presentation. We start with a cocycle c in Cohomotopy of degree 7, whose image under the quaternionic Hopf fibration in *stable* Cohomotopy of degree 4 is some value $(G_4)_0$ interpreted as vanishing flux, up to, possibly, a background charge:

$$
\begin{tikzcd}
X^8 \setminus \coprod_i \{x_i\} \arrow[r, "cs"'] \arrow[rrr, bend left, "(G_4)_0"] & S^7 \arrow[r, "h_{\mathbb{H}}"'] & S^4 \arrow[r, "\omega_4"'] & \Omega^\infty \Sigma^\infty S^4 \,.
\end{tikzcd}
$$

Since X^8 is 8-dimensional, the stable homotopy class of c is fully determined by its Hopf degree in $\mathbb{Z}$.

We would like to know if we may increase this degree by some $n \in \mathbb{Z}$ while keeping the total G_4-flux fixed at the given value $(G_4)_0$:

$$
\begin{tikzcd}
X^8 \setminus \coprod_i \{x_i\} \arrow[r, "(G_4, G_7) + n"'] \arrow[rrr, bend left, "(G_4)_0\ (?)"] & S^7 \arrow[r, "h_{\mathbb{H}}"'] & S^4 \arrow[r, "\omega_4"'] & \Omega^\infty \Sigma^\infty S^4 \,.
\end{tikzcd}
$$

But since the final coefficient is now stable, we may equivalently stabilize all the way through

$$
\begin{tikzcd}
\Sigma^\infty_+\big(X \setminus \coprod_i \{x_i\}\big) \arrow[r, "\Sigma^\infty\left((G_4, G_7) + n\right)"'] \arrow[rrr, bend left, "(G_4)_0\ (?)"] & \Sigma^\infty S^7 \arrow[r, "\Sigma^\infty h_{\mathbb{H}}"'] & \Sigma^\infty S^4 \arrow[r, "\omega_4"'] & \Sigma^\infty S^4 \,.
\end{tikzcd}
$$

This shows that $n \in \mathbb{Z} \hookrightarrow \pi_7(S^4)$ contributes to the total G_4-flux seen in stable Cohomotopy only via its stabilization in $\pi_7(\Sigma^\infty S^4) = \pi_3^S = \mathbb{Z}_{24}$. Hence all n of the form $n = 24k$ lead to the the same G_4-flux:

$$
\begin{tikzcd}
\Sigma^\infty_+\big(X \setminus \coprod_i \{x_i\}\big) \arrow[r, "\Sigma^\infty\left((G_4, G_7) + 24k\right)"'] \arrow[rrr, bend left, "(G_4)_0"] & \Sigma^\infty S^7 \arrow[r, "\Sigma^\infty h_{\mathbb{H}}"'] & \Sigma^\infty S^4 \arrow[r, "\omega_4"'] & \Sigma^\infty S^4 \,.
\end{tikzcd}
$$

This says that the Cohomotopy charge of the M2-branes must change by multiples of 24 if no M5-brane charge is to be generated, and hence that 24 units of Cohomotopy charge should be thought of as one unit of M2-brane charge. With this, the statement of (108) follows from the Poincaré-Hopf theorem, in its Cohomotopical formulation of Prop. 3.24. □

Remark 4.23 (Deriving M2-brane tadpole cancellation in fluxless background from Hypothesis H)**.** Under *Hypothesis H*, relation (108) is manifestly the M2-brane tadpole cancellation condition (25) in fluxless backgrounds, discussed in §2.6.

Now we generalize this discussion to non-vanishing flux.

Definition 4.24 (Extended spacetime)**.** Let X be a smooth manifold which is simply connected (Remark 3.6), equipped with topological $\mathrm{Sp}(2)$-structure τ (43) and equipped with a cocycle c in τ-twisted Cohomotopy (Def. 3.1). Then we say that the corresponding *extended spacetime* is the fibration $\widehat{X} \to X$ arising as the homotopy pullback of the $\mathrm{Sp}(2)$-equivariant quaternionic Hopf fibration (Prop. 3.20) along c:

$$
\begin{tikzcd}
\widehat{X} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & S^7 /\!/ \mathrm{Sp}(2) \arrow[d, "h_{\mathbb{H}} /\!/ \mathrm{Sp}(2)"] \\
X \arrow[rr, dashed, "c"] \arrow[dr, "\tau"'] & & S^4 /\!/ \mathrm{Sp}(2) \arrow[dl] \\
& B\mathrm{Sp}(2) &
\end{tikzcd}
\tag{109}
$$

Remark 4.25 (Nature of extended spacetime in parametrized super homotopy theory)**.**

(i) The extended spacetime $\widehat{X}$ in Def. 4.24 is an S^3-fibration over X, since the homotopy fiber of $h_{\mathbb{H}} /\!/ \mathrm{Sp}(2)$ over any point is S^3:

$$
\begin{tikzcd}
S^3 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & S^7 \arrow[r] \arrow[d, "h_{\mathbb{H}}"] \arrow[dr, phantom, "\text{(pb)}"] & S^7 /\!/ \mathrm{Sp}(2) \arrow[d, "h_{\mathbb{H}} /\!/ \mathrm{Sp}(2)"] \\
\ast \arrow[r] & S^4 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{(pb)}"] & S^4 /\!/ B\mathrm{Sp}(2) \arrow[d] \\
& \ast \arrow[r] & B\mathrm{Sp}(2)
\end{tikzcd}
$$

As such, this is the incarnation in non-rational parameterized homotopy theory of the super rational S^3-fibration (3) over 11-dimensional superspacetime from Figure R, discussed in detail in [FSS18b][SS18], which is classified by the bifermionic component μ_{M2} of the C-field super flux form [FSS13, p. 12] [FSS15, (2.1)]:

$$
\begin{tikzcd}
\mathfrak{m2brane} \simeq \widehat{\mathbb{T}^{10,1\vert \mathbf{32}}} \arrow[rr, "\mu_{\mathrm{M5}} + h_3 \wedge \mu_{\mathrm{M2}}"] \arrow[d] \arrow[drr, phantom, "\text{(pb)}"] & & S^7_{\mathbb{R}} \arrow[d, "(h_{\mathbb{H}})_{\mathbb{R}}"] \\
\mathbb{T}^{10,1\vert \mathbf{32}} \arrow[rr, "(\mu_{\mathrm{M2}}, \mu_{\mathrm{M5}})"] \arrow[dr, "\tau"'] & & S^4_{\mathbb{R}} \arrow[dl] \\
& K(\mathbb{R}, 4) &
\end{tikzcd}
$$

(ii) By the universal property of homotopy pullbacks, the extended spacetime $\widehat{X}$ in Def. 4.24 is the classifying space for maps ϕ to X equipped with a cocycle $\widehat{c}$ in degree 7 twisted Cohomotopy that exhibits the degree 4 twisted Cohomotopy cocycle $\phi^*(c)$ as fluxless, via a homotopy H_3:

$$
\begin{tikzcd}
Q \arrow[dr, dashed, "{(\widehat{\phi},\widehat{c},H_3)}"] \arrow[ddr, bend right, "\phi"'] \arrow[drrr, bend left, "\widehat{c}"] & & & \\
& \widehat{X} \arrow[d] \arrow[rr] & & S^7 /\!\!/ \mathrm{Sp}(2) \arrow[d, "h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)"] \\
& X \arrow[rr, "c"] \arrow[dr, "\tau"'] \arrow[urr, phantom, "\Uparrow H_3"] & & S^4 /\!\!/ \mathrm{Sp}(2) \arrow[dl] \\
& & B\mathrm{Sp}(2) &
\end{tikzcd}
\tag{110}
$$

As such, this is, under *Hypothesis H*, the classifyng space for fundamental M5-brane sigma-model configurations in X with worldvolume Q carrying twisted 3-form field strength H_3, as explained in [FSS13, Rem. 3.11][FSS15, p. 4].

Next we characterize the differential form data encoded in (110), in Prop. 4.27 below. For that we need the following Lemma (see e.g. [FHT00, Section 12]):

Lemma 4.26 (Sullivan model of $\mathrm{Sp}(2)$-equivariant Hopf fibration)**.** *The Sullivan model for the* $\mathrm{Sp}(2)$*-equivariant quaternionic Hopf fibration (Prop. 3.20) is as shown on the right here:*

$$
\begin{tikzcd}
S^7 /\!\!/ \mathrm{Sp}(2) \arrow[r, leftrightarrow, "\simeq"] \arrow[d, "h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)"] & S^7 /\!\!/ \mathrm{Spin}(5) \arrow[d, "h_{\mathbb{H}} /\!\!/ \mathrm{Spin}(5)"] \\
S^4 /\!\!/ \mathrm{Sp}(2) \arrow[r, leftrightarrow, "\simeq"] & S^4 /\!\!/ \mathrm{Spin}(5)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big) \otimes \mathbb{R}[\omega_7] \big/ \big(d\omega_7 = \tfrac{1}{2}\chi_8\big) \\
\mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big) \otimes \mathbb{R}[\omega_4, \omega_7] \Big/ \left(\begin{aligned} d\omega_4 &= 0 \\ d\omega_7 &= -\omega_4 \wedge \omega_4 \\ &\quad + \tfrac{1}{4} p_2 \end{aligned} \right) \arrow[u, "\big(h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)\big)^*", "{\begin{smallmatrix} \omega_4 \mapsto \frac{1}{4} p_1 \\ \omega_7 \mapsto \omega_7 \end{smallmatrix}}"']
\end{tikzcd}
$$

where $\mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big)$ *denotes the Sullivan model of the classifying space of* $\mathrm{Sp}(2)$.

Proof. The domain and codomain Sullivan algebras are as shown, by [FHT00, Sec. 15, Example 4] as in the proof of Prop. 3.5. To see that the map is given on generators as claimed, use that over the (any) base point of $B\mathrm{Sp}(2)$ the parameterized Hopf fibration restricts to the ordinary quaternionic Hopf fibration, making the following diagram homotopy commutative

$$
\begin{tikzcd}
S^7 \arrow[rr] \arrow[dd, "h_{\mathbb{H}}"'] \arrow[dr] & & S^7 /\!\!/ \mathrm{Sp}(2) \arrow[dd, "h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)"] \arrow[dr] & \\
& \ast \arrow[rr] & & B\mathrm{Sp}(2)\,. \\
S^4 \arrow[rr] \arrow[ur] & & S^4 /\!\!/ \mathrm{Sp}(2) \arrow[ur] &
\end{tikzcd}
$$

This means that the Sullivan model of $h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)$ must be the dashed homomorphism that makes the following diagram of dg-algebras commute

$$
\begin{tikzcd}
\mathbb{R}[\omega_7]\big/\big(d\omega_7 = 0\big) & & \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big)\otimes\mathbb{R}[\omega_7]\big/\big(d\omega_7 = \chi_8\big) \arrow[ll] \\
\mathbb{R}[\omega_4,\omega_7]\Big/\begin{pmatrix} d\omega_4 = 0 \\ d(2\omega_7) = -\omega_4\wedge\omega_4 \end{pmatrix} \arrow[u, "\substack{\omega_4 \mapsto 0 \\ \omega_7 \mapsto \omega_7}"] & & \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big)\otimes\mathbb{R}[\omega_4,\omega_7]\Bigg/\begin{pmatrix} d\omega_4 = 0 \\ d(2\omega_7) = -\omega_4\wedge\omega_4 + \underbrace{\tfrac{1}{4}p_2}_{(\frac{1}{4}p_1)^2+\frac{1}{2}\chi_8} \end{pmatrix} \arrow[ll] \arrow[u, dashed, "\substack{\omega_4 \mapsto \frac{1}{4}p_1 \\ \omega_7 \mapsto \omega_7}"']
\end{tikzcd}
$$

where the horizontal morphisms project away the base algebra $\mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big)$. Here under the brace we used the $\mathrm{Sp}(2)$-structure relation

$$\tfrac{1}{4}p_2 = \big(\tfrac{1}{4}p_1\big)^2 + \tfrac{1}{2}\chi_2 \tag{111}$$

from Prop. 4.4. The commutativity of this diagram requires that the dashed morphism sends $\omega_7 \mapsto \frac{1}{2}\omega_7$. Moreover, by degree reasons, we have that $\omega_4 \mapsto c\cdot p_1$, for some $c\in\mathbb{R}$. The unique choice for c that makes the map respect the differentials is $c = \frac{1}{4}$, by (111). $\square$

Proposition 4.27 (Differential form data on extended spacetime)**.** *Let X be a smooth manifold which is simply connected (Remark 3.6), equipped with topological* $\mathrm{Sp}(2)$*-structure* τ (43) *and equipped with a cocycle c in τ-twisted Cohomotopy (Def. 3.1) with underlying differential forms* (G_4, G_7) *according to Def. 4.3*

$$
\begin{tikzcd}
X \arrow[rr, "(G_4,G_7)"] \arrow[dr, "\tau"'] & & S^4/\!\!/\mathrm{Sp}(2) \,. \arrow[dl] \\
& B\mathrm{Sp}(2) &
\end{tikzcd}
$$

Then the pullback of these differential forms to the corresponding extended spacetime $\widehat{X}$ (Def. 4.24) satisfies

$$dH_3^{\mathrm{univ}} \;=\; \widetilde{G}_4 - \tfrac{1}{2}p_1(\nabla) \tag{112}$$

$$d\big(G_7 + H_3^{\mathrm{univ}}\wedge\widetilde{G}_4\big) \;=\; \chi_8(\nabla) \tag{113}$$

where H_3^{univ} is the universal 3-form H_3^{univ} (110) *on $\widehat{X}$.*

Proof. To extract the differential form data following Def. 4.3 we may compute the defining homotopy pullback (109) in rational homotopy theory and read off the resulting assignment of generators in the Sullivan model.

By general facts of rational homotopy theory (recalled e.g. in [FSS16a]) the Sullivan model for $\widehat{X}$ is given as the pushout along the map corresponding to (G_4, G_7) of a minimal cofibration resolution of the Sullivan model for the equivariant quaternionic Hopf fibration $h_{\mathbb{H}}/\!\!/\mathrm{Sp}(2)$. The latter was obtained in Lemma 4.26. By direction inspection one sees that the minimal cofibration resolution is given as shown on the right of the following diagram:

$$
\begin{tikzcd}
 & & \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big) \otimes \mathbb{R}[\omega_7]\big/(d\omega_7 = \chi_8) \\
\mathrm{CE}\big(\mathfrak{l}\widehat{X}\big) & & \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big) \otimes \mathbb{R}[h_3, \omega_4, \omega_7]\Big/\left(\begin{aligned} dh_3 &= \omega_4 - \tfrac{1}{4}p_1 \\ d\omega_4 &= 0 \\ d(2\omega_7) &= -dh_3 \wedge (\omega_4 + \tfrac{1}{4}p_1) \\ & \quad + \tfrac{1}{2}\chi_8 \end{aligned} \right) \arrow[ll, "{\begin{array}{c} \omega_4 \mapsto G_4 \\ \omega_7 \mapsto G_7 \\ h_3 \mapsto H_3^{\mathrm{univ}} \end{array}}"'] \arrow[u, "\simeq", "{\begin{array}{l} h_3 \mapsto 0 \\ \omega_4 \mapsto \frac{1}{4}p_1 \\ \omega_7 \mapsto \omega_7 \end{array}}"'] \\
\mathrm{CE}\big(\mathfrak{l}X\big) \arrow[u] \arrow[urr, phantom, "\text{(po)}"] & & \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big) \otimes \mathbb{R}[\omega_4, \omega_7]\Big/\left(\begin{aligned} d\omega_4 &= 0 \\ d(2\omega_7) &= -\omega_4 \wedge \omega_4 + \tfrac{1}{4}p_2 \\ &= -\omega_4 \wedge \omega_4 + \tfrac{1}{4}p_1 \wedge \tfrac{1}{4}p_1 + \tfrac{1}{2}\chi_8 \end{aligned} \right) \arrow[ll, "{\begin{array}{c} \omega_4 \mapsto G_4 \\ \omega_7 \mapsto G_7 \end{array}}"'] \arrow[u, hook, "{\begin{array}{l} \omega_4 \mapsto \omega_4 \\ \omega_7 \mapsto \omega_7 \end{array}}"'] \arrow[uu, bend left, "\left(h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)\right)^*"] \\
 & \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big) \arrow[ul, "\tau^*"] \arrow[ur]
\end{tikzcd}
$$

Therefore, the differential relations appearing on the right imply the claim. □

Remark 4.28 (Deriving fluxed M2-brane tadpole cancellation from Hypothesis H)**.** To conclude, we just need to observe now that, due to the self-interaction of the C-field according to the supergravity equation of motion (1), a contribution of $-\frac{1}{2}G_4 \wedge G_4$ to dG_7 is not due to M2-brane charge, so that the flux-corrected M2-brane number density (i.e. PH-index density, via the cohomological PH-theorem, Remark 3.25) in the presence of G_4-flux is, in the topologically trivial case, $G_7 + \frac{1}{2}G_4 \wedge G_4$. With this and the cohomological PH-theorem (Remark 3.25), equation (113) from Prop. 4.27 generalizes the fluxless tadpole cancellation condition (108) from Prop. 4.22 to read

$$N_{\mathrm{M2}} \;+\; q(G_4) \;=\; \tfrac{1}{24}\chi[X]\,,$$

where $q(G_4)$ is the quadratic form (12). This is the general (fluxed) M2-brane tadpole cancellation condition (27), discussed in §2.6.

Remark 4.29 (The global picture.)**.** In conclusion, we have considered, with *Hypothesis H*, a single mathematically clean unifying picture of the C-field in M-theory with flux and M2-brane sources, and have proven that with this hypothesis a plethora of situations and effects considered informally in the string theory literature follow by rigorous mathematical analysis. Besides informing us about the plausibility, interrelation and fine print of the notoriously subtle anomaly cancellation conditions in M-theory, we suggest that the main impact of this result is that it indicates that *Hypothesis H* is correct, and indeed an ingredient of the solution to the open problem of fundamentally formulating M-theory.

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