

Twistorial Cohomotopy implies Green-Schwarz anomaly cancellation

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Abstract

We characterize the integral cohomology and the rational homotopy type of the maximal Borel-equivariantization of the combined Hopf/twistor fibration, and find that subtle relations satisfied by the cohomology generators are just those that govern Hořava-Witten's proposal for the extension of the Green-Schwarz mechanism from heterotic string theory to heterotic M-theory. We discuss how this squares with the *Hypothesis H* that the elusive mathematical foundation of M-theory is based on charge quantization in J-twisted Cohomotopy theory.

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1 Introduction and overview

The Green-Schwarz mechanism in heterotic M-theory. At the heart of M-theory (the conjectural non-perturbative completion of type IIA string theory, see [Du96][Du98][Du99]) is the proposal [Wi97a, (1.2)][Wi97b, (1.2)] that the *difference* of the classes of:

- (i) the flux density G_4 of the higher gauge field of M-theory (the *C-field*, or *3-index A-field* [CJS78]),
- (ii) $1/4$ th of the Pontrjagin form of the spin connection ω on spacetime Y^{11} (e.g. [GSa18, p. 10]),

lifts to an integral class:

$$\begin{array}{c} \text{C-field} \\ \text{4-flux density} \end{array} [G_4] - \begin{array}{c} 1/2 \text{ gravitational} \\ \text{instanton density} \end{array} \left[\frac{1}{4} p_1(\omega) \right] \in \begin{array}{c} \text{integral cohomology of} \\ \text{11-d spacetime} \end{array} H^4(Y^{11}; \mathbb{Z}) \xrightarrow{\text{rationalization}} \begin{array}{c} \text{real cohomology of} \\ \text{11-d spacetime} \end{array} H^4(Y^{11}; \mathbb{R}). \quad (1)$$

One motivation for this proposal [Wi97a, §2.1] comes from heterotic M-theory (Hořava-Witten theory [HW95] [Wi96][HW96][DOPW99][DOPW00][Ov02], the conjectural non-perturbative completion of heterotic string theory [GHMR85][GHMR85][AGLP12]). Here the celebrated (“first superstring revolution”, see [Schw07]) *Green-Schwarz anomaly cancellation mechanism* [GSc84][CHSW85] (review in [Wi99, §2.2][Fr00]) in heterotic string theory, which in itself is understood clearly, is argued to imply, upon lift to heterotic M-theory, that the restriction of the 4-flux G_4 to an MO9-plane X^{10} inside 11-dimensional spacetime Y^{11} satisfies this relation [HW96, (1.13)]¹

$$\begin{array}{c} \text{C-field 4-flux density} \\ \text{restricted to } X^{10} \end{array} [G_4]|_{X^{10}} = \begin{array}{c} 1/2 \text{ gravitational} \\ \text{instanton density} \end{array} \left[\frac{1}{4} p_1(\omega) \right]|_{X^{10}} - \begin{array}{c} \text{gauge instanton} \\ \text{density on } X^{10} \end{array} [c_2(A)] \in \begin{array}{c} \text{integral cohomology} \\ \text{of 10d MO9-plane} \end{array} H^4(X^{10}; \mathbb{Z}) \xrightarrow{\text{rationalization}} \begin{array}{c} \text{real cohomology} \\ \text{of 10d MO9-plane} \end{array} H^4(X^{10}; \mathbb{R}), \quad (2)$$

where A (the *gauge field*) is a connection on an E_8 -principal bundle over X^{10} , and $c_2(A)$ is its second Chern-form. But the summand $[c_2(A)]$ is integral by itself: it is the real image of the second Chern class of the E_8 -bundle. Therefore, (2) implies that (1) holds at least upon restriction to MO9-planes; and it suggests [DMW00][ES03][DFM03] [Sa06b][FSS14a] that the integral 4-class in (1) is to be thought of as the second Chern class of an extension \tilde{A} of the E_8 -gauge field from X^{10} to all of Y^{11} :

$$\begin{array}{c} \text{C-field} \\ \text{4-flux density} \end{array} [G_4] - \begin{array}{c} 1/2 \text{ gravitational} \\ \text{instanton density} \end{array} \left[\frac{1}{4} p_1(\omega) \right] = \underbrace{- [c_2(\tilde{A})]}_{=:a} \in \begin{array}{c} \text{integral cohomology of} \\ \text{11-d spacetime} \end{array} H^4(Y^{11}; \mathbb{Z}) \xrightarrow{\text{rationalization}} \begin{array}{c} \text{real cohomology of} \\ \text{11-d spacetime} \end{array} H^4(Y^{11}; \mathbb{R}). \quad (3)$$

Open problem. Despite the tight web of hints and consistency checks like these, actually formulating M-theory remains an open problem (e.g., [Du96, 6][HLW98, p. 2][Du98, p. 6][NH98, p. 2][Du99, p. 330][Mo14, 12][CP18, p. 2][Wi19, @21:15][Du19, @17:14]). In particular:

- (i) The conditions (1) and (2) had not actually been derived from any theory (see the comments around [HW96, (1.13)] and [Wi97a, (2.1)]).
- (ii) The ontology of the gauge field on Y^{11} in (3) has remained mysterious, as no such gauge field is seen in 11-dimensional supergravity [CJS78][D’AF82][CDF91], which, however, is famously argued to be the low-energy limit of M-theory [Wi95].

By Charge quantization in generalized cohomology? On the other hand, the Green-Schwarz mechanism in perturbative string theory is well understood as an index-theoretic phenomenon, resulting from charge quantization in a generalized cohomology theory, namely in K-theory [Fr00][Cl05][Bu11]. This mathematical understanding has been most fruitful, spawning understanding of elliptic genera (e.g. [HLZ07][CHZ11]), twisted higher bundles/gerbes (e.g. [SSS09b][Wa13]), Hermitian and generalized geometry (e.g. [GF16]), and more.

There have been various proposals for lifting this situation to heterotic M-theory, understanding also the shifted integrality condition (3) as an effect of charge quantization in *some* generalized cohomology theory [DFM03][HS05] [Sa05a][Sa05b][Sa06a][Sa10][FSS14a][FSS14b], which then would control M-theory in generalization of how K-theory controls string theory (for the latter, see [GSa19] and references therein). However, while advancements in understanding have certainly been made, the situation had remained inconclusive.

¹ Our normalization convention for G_4 absorbs a factor of $-1/2\pi$ compared to [Wi96].

Non-abelian Chern-Dold characters. Our strategy is to invoke the further generalization of generalized cohomology to *non-abelian cohomology* [To02][RS12][SSS12][NSS12a][NSS12b][FSS19b][SS20b]. In §3.1 below we discuss how the *Chern-Dold character* in generalized cohomology ([Bu70], see [LSW16, §2.1]) extends to non-abelian cohomology theories, where it is given by passage to *rational homotopy theory*²:

$$\begin{array}{ccc}
\begin{array}{c} E\text{-cohomology} \\ E^\bullet(X) \end{array} & \xrightarrow{\begin{array}{c} \text{Non-abelian} \\ \text{Chern-Dold character} \\ \text{ch} \end{array}} & \begin{array}{c} \text{Non-abelian} \\ \text{Real cohomology} \\ H^\bullet(X; \mathbb{E}) \end{array} \\
\downarrow \wr & & \downarrow \wr \\
\begin{array}{c} \pi_{-\bullet}\text{-Maps}(X, \underbrace{E}_{\text{classifying space}}) \end{array} & \xrightarrow{\text{rationalization}} & \begin{array}{c} \pi_{-\bullet}\text{-Maps}(X, \underbrace{\mathbb{E}}_{\text{rationalized classifying space}}) \end{array}
\end{array} \quad (4)$$

For X a smooth manifold, the right hand side of (4) is a subquotient of the set of differential forms on X (Prop. 3.3); thus the image of the Chern-Dold character identifies equivalence classes of differential forms satisfying certain conditions. If these forms are interpreted as flux densities, then these are **non-abelian charge-quantization** conditions. For example, in the abelian case $E = \text{KU}, \text{KO}$, the Chern-Dold character (4) reduces to the traditional Chern character in K-theory, and the corresponding charge quantization condition is that thought to hold for RR-fields/D-brane charges in type II/I string theory (for detailed discussion and further pointers see [GSa18][GSa19]).

Charge quantization in J-twisted Cohomotopy. The most fundamental non-abelian cohomology theory is *Cohomotopy theory* [Bo36][Sp49][Pe56][Ta09][KMT12] whose classifying spaces are the n -spheres $S^n \simeq B(\Omega S^n)$. Accordingly, twisted Cohomotopy is classified by spherical fibrations, and we say *J-twisted Cohomotopy* [FSS19b][FSS19c] for twisting by the unit sphere fibration in the tangent bundle. The main theorem of [FSS19b, 3.9] says that the Chern-Dold character (4) in J-twisted 4-Cohomotopy encodes Witten’s shifted C-field flux quantization condition (1):

$$\begin{array}{ccc}
\begin{array}{c} \text{J-twisted} \\ \text{4-Cohomotopy} \\ \pi^\tau(X) \\ \text{manifold} \\ \text{with} \\ \text{tangential Sp}(2)\text{-structure } \tau \end{array} & \xrightarrow{\begin{array}{c} \text{Non-abelian} \\ \text{Chern-Dold character} \\ \text{ch} \end{array}} & \left\{ \begin{array}{l} G_4, \in \Omega^\bullet(X) \\ G_7 \end{array} \left| \begin{array}{l} \text{C-field} \\ \text{4-flux} \\ \text{shifted flux quantization (1)} \\ d \ G_4 = 0, \ [G_4] - [\frac{1}{4}p_1(\omega)] \in H^4(X; \mathbb{Z}) \\ d2G_7 = -(G_4 - \frac{1}{4}p_1(\omega)) \wedge (G_4 + \frac{1}{4}p_1(\omega)) \\ \text{dual} \\ \text{7-flux} \quad -\frac{1}{2}(p_2(\omega) - \frac{1}{4}(p_1(\omega))^2) \end{array} \right. \right\} \quad (5)
\end{array}$$

In fact, further constraints are implied, matching a whole list of further topological conditions expected in M-theory (see [FSS19b, Table 1]). This motivates, following [Sa13, §2.5], **Hypothesis H: Charge quantization in M-theory happens in J-twisted Cohomotopy theory** [FSS19b][FSS19c][SS19a][BSS19][SS19b][SS20b]. While J-twisted Cohomotopy in degree 4 alone (5) does not reflect the appearance of a heterotic gauge field as in (2), its lifting to 7-Cohomotopy, through the quaternionic Hopf fibration $h_{\mathbb{H}}$, does encode structure expected on heterotic M5-branes [FSS19c][FSS19d][FSS20a][FSS20b]. Therefore we turn attention to an intermediate lift:

Charge quantization in Twistorial Cohomotopy. In between J-twisted Cohomotopy in degrees 4 and 7, when lifting along the quaternionic Hopf fibration, we find *Twistorial Cohomotopy* (§3.2): The twisted non-abelian cohomology theory classified by the Borel-equivariantized *twistor space* $\mathbb{C}P^3$ (§2.1). Our main Theorem 2.14, illustrated in *Figure I*, implies (Corollary 3.11) that charge quantization in Twistorial Cohomotopy (i) makes the class of an $S(U(1)^2)$ -gauge field \tilde{A} appear, hence a “heterotic line bundle” [AGLP12][BBL17], and (ii) enforces on this gauge field Hořava-Witten’s Green-Schwarz mechanism (3) in heterotic M-theory:

$$\begin{array}{ccc}
\begin{array}{c} \text{Twistorial} \\ \text{Cohomotopy} \\ \mathcal{T}^\tau(X) \\ \text{manifold} \\ \text{with} \\ \text{tangential Sp}(2)\text{-structure } \tau \end{array} & \xrightarrow{\begin{array}{c} \text{Twisted Non-abelian} \\ \text{Chern-Dold character} \\ \text{ch} \end{array}} & \left\{ \begin{array}{l} F_2, \\ G_4, \in \Omega^\bullet(X) \\ G_7 \end{array} \left| \begin{array}{l} \text{1st Chern form of} \\ \text{heterotic line bundle} \\ d \ F_2 = 0, \ -[F_2 \wedge F_2] \in H^4(X; \mathbb{Z}) \\ \text{C-field 4-flux} \\ d \ G_4 = 0, \ [G_4] - [\frac{1}{4}p_1(\omega)] = [F_2 \wedge F_2] \in H^4(X; \mathbb{Z}) \\ \text{Hořava-Witten's Green-Schwarz mechanism (3)} \\ d2G_7 = -(G_4 - \frac{1}{4}p_1(\omega)) \wedge (G_4 + \frac{1}{4}p_1(\omega)) \\ \text{dual} \\ \text{7-flux} \quad -\frac{1}{2}(p_2(\omega) - \frac{1}{4}(p_1(\omega))^2) \end{array} \right. \right\} \quad (6)
\end{array}$$

² For general introduction and review of rational homotopy theory [Qu69] and its Sullivan models [Su77] see [FHT00][He07]; for detailed discussion in our context see [FSS16a, §A][FSS17, §2.2][FSS19a]. Notice that in the supergravity literature these (super-)Sullivan models are known as “FDA”s (following [vN82][D’AF82]); for details and translation see [FSS13][FSS16a][FSS16b][HSS18][BMSS19].

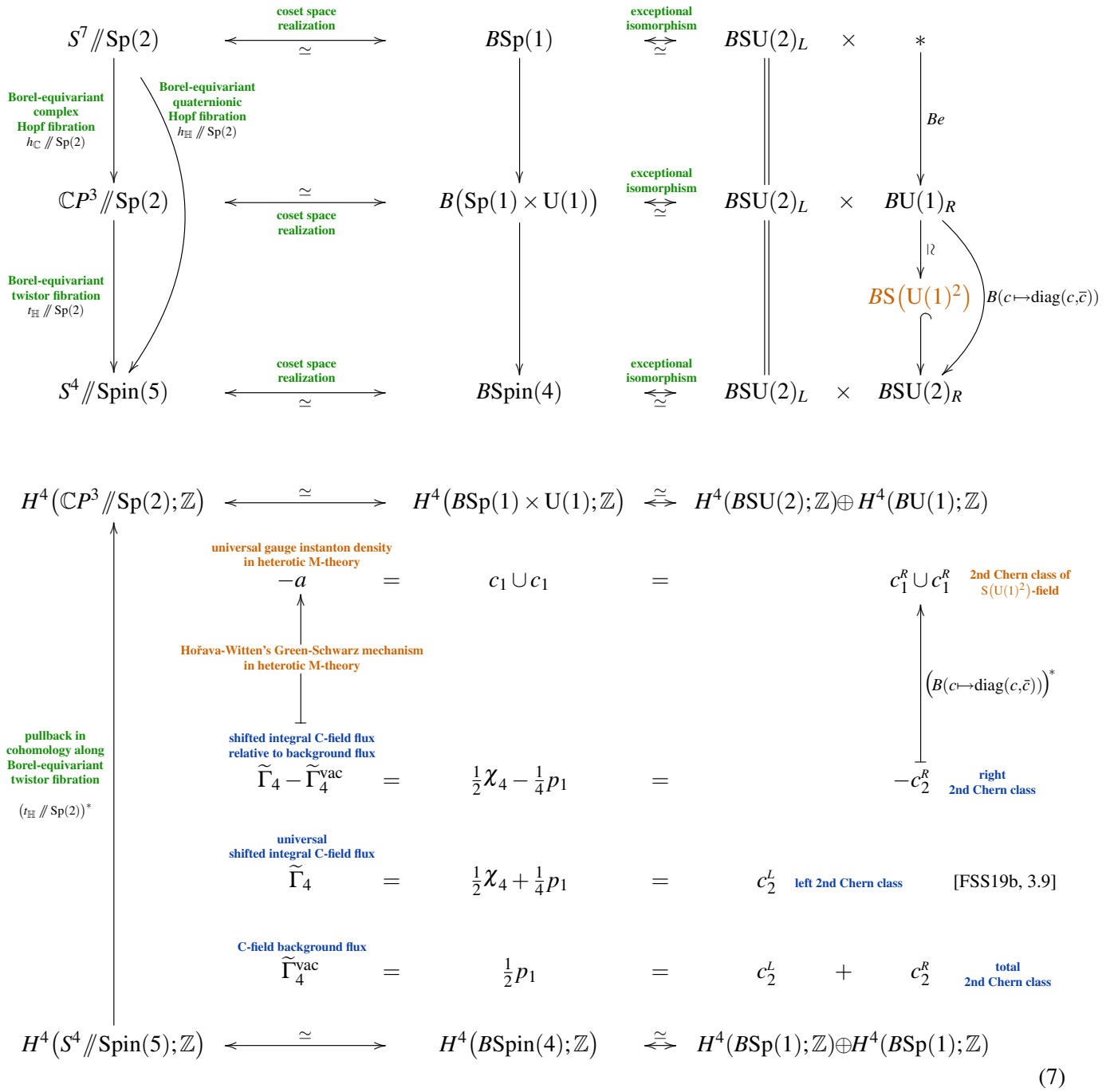


Figure I. Integral cohomology of Borel-equivariant Hopf/twistor fibration and its interpretation under Hypothesis H.

- (i) The top part shows the equivalent incarnations of the Borel-equivariant Hopf/twistor fibration (Def. 2.5) according to Prop. 2.7.
- (ii) The bottom part shows the corresponding identifications of the integral cohomology generators (32) according to [FSS19b, 3.9].
- (iii) This makes manifest, shown in the middle of the diagram, how these generators pull back along the fibration (Theorem 2.9), which
- (iv) allows to normalize the generators in the Sullivan model for the rational homotopy type of the fibrations, below in Theorem 2.14.
- (v) Blue labels indicate the interpretation of the bottom generators as universal fluxes in M-theory, according to [FSS19b][FSS20a];
- (vi) while orange labels indicate the interpretation of the new top generators as universal fluxes in heterotic M-theory, discussed in §3.

Conclusion and Outlook.

The heterotic gauge field. It is remarkable that the class of a gauge field \tilde{A} , which had remained mysterious in (3), does appear from charge-quantization in Twistorial Cohomotopy, according to (6).

(i) Missing generality? Of course, the gauge field in (6) has the abelian structure group $G = S(U(1)^2)$ instead of the non-abelian structure group $G = E_8$ that could be expected to apply to (3). In terms of characteristic classes, this means that charge quantization in Twistorial Cohomotopy constrains the class $a = [c_2(\tilde{A})] \in H^4(X^{11}; \mathbb{Z})$, which for $G = E_8$ may be *any* element in degree four integral cohomology (since $\tau_{11}BE_8 \simeq_{\text{whe}} \tau_{11}K(\mathbb{Z}; 4)$, e.g. [DFM03, 3.2]), to factor as minus a cup square of an element in degree two integral cohomology. This might indicate that Twistorial Cohomotopy as presented does not capture full heterotic M-theory; or that one should look for other factorizations of the quaternionic Hopf fibration, or for variants of the construction presented here.

(ii) Or predictive constraint? On the other hand, it is curious to notice that in heterotic string *phenomenology* the reduction of the heterotic gauge bundle along the inclusions

$$S(U(1)^n) \subset SU(n) \subset E_8, \quad \text{for } 2 \leq n \leq 5, \quad (8)$$

has led to a little revolution in string phenomenology [AGLP11][AGLP12]. These *heterotic line bundle models* turn out (at least for $n = 5$) to be an abundant source of low energy theories with the *exact* field content of the (minimally supersymmetric) standard model of particle physics (up to decoupled and ultra-heavy fields), amenable to effective computational classification [ACGLP14][HLLS13][BBL17][GW19]. Before considering the reduction (8), only a small handful of hand-crafted semi-realistic models were known.

Notice that this works because the structure group of the heterotic gauge bundle is part of what *breaks* E_8 down to the low-energy gauge group: the latter is within the commutant of the former in E_8 . Therefore, realistic phenomenology does not require \tilde{A} in (3) to be in a non-abelian GUT-group – in fact it must instead be complementary to (be in the commutant within E_8 of) the low-energy gauge group (\tilde{A} is a background field/vev, not the dynamical gauge field fluctuating about it); and analysis of heterotic line bundle models indicates that restricting \tilde{A} to be reduced along (8) narrows in heterotic M-theory onto its phenomenologically realistic sector.

This might indicate that *Hypothesis H* captures not only mathematical but also phenomenological constraints of M-theory.

The degree-8 polynomial. Beyond encoding the shifted heterotic flux quantization in the first two lines of (6), the third line there shows that charge-quantization in Twistorial Cohomotopy also enforces (Corollary 3.10) the trivialization of this 8-class:

$$\begin{aligned} I_8^H &:= ([G_4] - \frac{1}{4}p_1) \cup ([G_4] + \frac{1}{4}p_1) + \frac{1}{2}(p_2 - \frac{1}{4}p_1 \cup p_1) \\ &= ([F_2 \wedge F_2] + \frac{1}{2}p_1) \cup [F_2 \wedge F_2] + \frac{1}{2}(p_2 - \frac{1}{4}p_1 \cup p_1) \in H^8(X, \mathbb{R}). \end{aligned} \quad (9)$$

In the form of the first line of (9), the condition $I_8^H = 0$ is inherited from charge-quantization in J-twisted Cohomotopy (5). We had shown previously that this condition guarantees the vanishing *under general conditions*³ of

- (a) the anomaly in the Hopf-WZ term on the M5-brane [FSS19c],
- (b) the total remaining anomaly of the M5-brane [SS20a].

In the form of the second line in (9) – now equivalently re-expressed in terms of the emergent heterotic gauge flux instead of the G_4 -flux – this class is seen to be closely related to the 8-class denoted \widehat{I}_8 in [HW96, (1.10)]: Up to global and relative rescaling of \widehat{I}_8 (as on the bottom of [HW96, p. 15]) both are related by

$$I_8^H = \widehat{I}_8 - \frac{1}{4}p_1 \cup p_1. \quad (10)$$

It might be interesting to understand the potential significance of this relation. Notice that

- (i) in [HW96] there is no condition that \widehat{I}_8 should vanish;
- (ii) the shift in (10) is what makes I_8^H an integral class (using (40) and (75));
- (iii) it is expected [Mos08] that \widehat{I}_8 is just approximate: it receives an infinite but unknown tower of corrections;
- (iv) while *Hypothesis H* suggests that $I_8^H = 0$ is a statement about fully-fledged M-theory.

³ Both cancellations had previously been discussed only subject to tacit assumptions on the C-field; see [FSS19b, p. 2] and [SS20a, (6)].

2 Borel-equivariant Hopf/twistor fibration

The *twistor fibration*

$$S^2 \longrightarrow \mathbb{C}P^3 \\ \downarrow \\ S^4$$

(due to [At79, §III.1], see also, e.g., [Br82, §1][AS13][ABS19, §6]) or *Penrose fibration* (as in [ES85]), is the canonical map $\mathbb{C}P^3 \rightarrow \mathbb{H}P^1$ (under the identification $\mathbb{H}P^1 \simeq S^4$, recalled below as Prop. 2.1) that sends complex lines to the quaternionic lines which they span [At79, §III (1.1)]. While, as the name suggests, this is traditionally motivated from the role of $\mathbb{C}P^3$ as a *twistor space* (a division-algebraic account is in [BC88]), our interest in the twistor fibration here comes from its appearance as an intermediate stage of the *quaternionic Hopf fibration* [GWY83, §6]. We observe that it is given by the following iterative quotienting by multiplicative groups in the four real normed division algebras (reals \mathbb{R} , complex numbers \mathbb{C} , quaternions \mathbb{H} , octonions \mathbb{O}):

$$\begin{array}{c}
 S^1 \xrightarrow{\simeq} \mathbb{C}^\times / \mathbb{R}_+^\times \\
 \searrow \\
 S^7 \xrightarrow{\simeq} (\mathbb{R}^8 \setminus \{0\}) / \mathbb{R}_+^\times \ni \{v \cdot t \mid t \in \mathbb{R}_+^\times\} \\
 \downarrow h_{\mathbb{C}} \text{ 7d complex Hopf fibration} \\
 S^2 \xrightarrow{\simeq} \mathbb{H}^\times / \mathbb{C}^\times \\
 \searrow \\
 \mathbb{C}P^3 \xrightarrow{\simeq} (\mathbb{C}^4 \setminus \{0\}) / \mathbb{C}^\times \ni \{v \cdot z \mid z \in \mathbb{C}^\times\} \\
 \downarrow t_{\mathbb{H}} \text{ twistor fibration} \\
 S^4 \xrightarrow{\simeq} \mathbb{O}^\times / \mathbb{H}^\times \\
 \searrow \\
 \mathbb{H}P^1 \xrightarrow{\simeq} (\mathbb{H}^2 \setminus \{0\}) / \mathbb{H}^\times \ni \{v \cdot q \mid q \in \mathbb{H}^\times\} \\
 \downarrow \\
 * \xrightarrow{\simeq} (\mathbb{O}^1 \setminus \{0\}) / \mathbb{O}^\times \ni \{v \cdot o \mid o \in \mathbb{O}^\times\}
 \end{array}
 \left. \vphantom{\begin{array}{c} S^1 \\ S^2 \\ S^4 \end{array}} \right\} \begin{array}{l} \text{for } v \neq 0 \\ \cap \\ \mathbb{R}^8 \\ \downarrow \\ \mathbb{C}^4 \\ \downarrow \\ \mathbb{H}^2 \\ \downarrow \\ \mathbb{O} \end{array} \quad (11)$$

Alternatively, in its coset-space realization

$$\begin{array}{ccc}
 \mathrm{SU}(4)/\mathrm{U}(2) & \longrightarrow & \mathrm{SO}(5)/\mathrm{U}(2) \\
 & & \downarrow \\
 & & \mathrm{SO}(5)/\mathrm{SO}(4)
 \end{array}$$

the twistor fibration is also called *Calabi-Penrose fibration* (following [La85, §3], see also [Lo89] and see [No08, 2.31] for a review of Calabi's construction [Ca67][Ca68]). We observe that the $\mathrm{Sp}(2)$ -coset realization ([On60, Table 1], see [GO93, Table 3]) of the Hopf/twistor fibrations is given as follows (see also [FSS19b, (73)]):

$$\begin{array}{ccc}
 S^7 & \simeq & \mathrm{Sp}(2)/\mathrm{Sp}(1)_L \\
 \downarrow h_{\mathbb{C}} \text{ 7d complex Hopf fibration} & & \\
 \text{quaternionic Hopf fibration } h_{\mathbb{H}} \left(\mathbb{C}P^3 \right. & \simeq & \mathrm{Sp}(2)/(\mathrm{Sp}(1)_L \times \mathrm{U}(1)_R) \\
 \downarrow t_{\mathbb{H}} \text{ Calabi-Penrose twistor fibration} & & \\
 S^4 & \simeq & \mathrm{Sp}(2)/(\mathrm{Sp}(1)_L \times \mathrm{Sp}(1)_R)
 \end{array} \quad (12)$$

where the maps are induced by the canonical subgroup inclusions (recalled in Example A.4).

We discuss in this paper the enhancement (Prop. 2.2 below) of these classical fibrations (11) (12) to Borel-equivariant *parametrized fibrations* (Def. 2.5 below) over the classifying space of the group $\mathrm{Sp}(2)$ (recalled as Def. A.3 below), generalizing the analogous discussion for just the quaternionic Hopf fibration in [FSS19b][FSS19c]. The main results are Theorem 2.9 and Theorem 2.14 below, which characterize the integral cohomology and the rational homotopy type of the Borel-equivariant Hopf/twistor fibrations (29) (generalizing the result for just the quaternionic Hopf fibration from [FSS19b, 3.19]).

2.1 Construction

Here we determine the maximal symmetry group of the joint Hopf/twistor fibrations (Prop. 2.2, Remark 2.3), construct the corresponding Borel-equivariantization (Def. 2.5) and characterize its integral cohomology (Theorem 2.9).

The following isomorphisms (Prop. 2.1) are classical, but since the third of these is rarely made explicit in the literature, we spell out a proof:

Proposition 2.1 (Equivariant identification of 4-sphere with quaternionic projective space). *There are canonical isomorphisms*

$$(i) \text{ of topological spaces: } S^4 \xrightarrow[\simeq]{\alpha} \mathbb{H}P^1 ; \quad (13)$$

$$(ii) \text{ of topological groups: } \text{Spin}(5) \xrightarrow[\simeq]{\gamma} \text{Sp}(2) ; \quad (14)$$

(iii) of canonical topological group actions:

$$\text{Spin}(5) \wr S^4 \xrightarrow[\simeq]{(\gamma, \alpha)} \text{Sp}(2) \wr \mathbb{H}P^1 . \quad (15)$$

Proof. Quaternionic 2-component spinor formalism provides an isomorphism of real quadratic vector spaces ([KT82], streamlined review in [BH09][VB20][FSS20b, §3.2])

$$\begin{aligned} (\mathbb{R}^{5,1}, \eta) &\xrightarrow{\simeq} (\text{Mat}^{\text{herm}}(2 \times 2, \mathbb{H}), -\det) \\ \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ \vdots \\ x^5 \end{bmatrix} &\mapsto \begin{pmatrix} x^0 - x^1 & x^2 + ix^3 + jx^4 + kx^5 \\ x^2 - ix^3 - jx^4 - kx^5 & x^0 + x^1 \end{pmatrix} \end{aligned} \quad (16)$$

from (a) 6d Minkowski space $\mathbb{R}^{5,1}$ with metric $\eta := \text{diag}(-1, +1, \dots, +1)$ to (b) the vector space of 2-by-2 quaternionic matrices which are hermitian, $A^\dagger = A$, with quadratic form the negative of the determinant operation. Moreover, under this identification, the canonical action of $\text{Spin}(5, 1)$ on $\mathbb{R}^{5,1}$ (through that of $\text{SO}(5, 1)$) translates to the conjugation action of $\text{SL}(2, \mathbb{H})$ (78):

$$\begin{aligned} \text{Spin}(5, 1) \wr \mathbb{R}^{5,1} &\xrightarrow{\simeq} \text{SL}(2, \mathbb{H}) \wr \text{Mat}^{\text{herm}}(2 \times 2, \mathbb{H}) \\ &A \mapsto G \cdot A \cdot G^\dagger \end{aligned} \quad (17)$$

Now consider the restriction of this situation to the Euclidean spatial slice $\mathbb{R}^5 \hookrightarrow \mathbb{R}^{5,1}$ determined by $x^0 = 0$. Under the isomorphism (16), this clearly corresponds to restriction to the *traceless* hermitian matrices:

$$(\mathbb{R}^5, g) \xrightarrow{\simeq} (\text{Mat}_{\text{trless}}^{\text{herm}}(2 \times 2, \mathbb{H}), -\det) \quad (18)$$

Notice here, from direct inspection (see also [BH09, Prop. 5]), that

$$A \in \text{Mat}_{\text{trless}}^{\text{herm}}(2 \times 2, \mathbb{H}) \quad \Rightarrow \quad A \cdot A = -\det(A) \cdot \mathbf{I}. \quad (19)$$

Moreover, the subgroup $\text{Spin}(5) \subset \text{Spin}(5, 1)$ which fixes this subspace corresponds under (17) to that subgroup of $\text{SL}(2, \mathbb{H})$ whose conjugation operation preserves traceless matrices. Since this means equivalently to act trivially on their orthogonal complement, given by the pure trace matrices, i.e. the real multiples of the 2-by-2 identity matrix $\mathbf{I} := \text{Id}_{\mathbb{H}^2}$:

$$G \cdot \mathbf{I} \cdot G^\dagger = \mathbf{I} \quad \Leftrightarrow \quad G \cdot G^\dagger = \mathbf{I}$$

we see, using (84), that this is the quaternionic unitary group $\text{Sp}(2) := \text{U}(2, \mathbb{H})$ (80), hence that (17) restricts as follows:

$$\text{Spin}(5) \wr \mathbb{R}^5 \xrightarrow{\simeq} \text{Sp}(2) \wr \text{Mat}_{\text{trless}}^{\text{herm}}(2 \times 2, \mathbb{H}). \quad (20)$$

Analogously, the further restriction to the unit sphere in \mathbb{R}^5 corresponds, under (16) and in view of (19), to those matrices which are all of:

$$\text{(a) hermitian: } A^\dagger = A, \quad \text{(b) traceless: } \text{tr}(A) = 0, \quad \text{(c) unitary: } A \cdot A = 1.$$

6d Minkowski spacetime
5d Euclidean spacetime
4d sphere

From (c) it follows that the matrix

$$P_A := \frac{1}{2}(\mathbf{I} - A)$$

is a projector, $P_A \cdot P_A = P_A$; and from (b) it follows that this projector has unit rank:

$$\text{tr}(P_A) = \frac{1}{2}(\underbrace{\text{tr}(\mathbf{I})}_{=2} - \underbrace{\text{tr}(A)}_{=0}) = 1.$$

Here a unit-rank projector is one for which there exists $v_A \in \mathbb{H}^2 \setminus \{0\}$ such that

$$P_A = \frac{1}{\|v_A\|^2} v_A \cdot v_A^\dagger.$$

Noticing that P_A , and hence $A = \mathbf{I} - 2P_A$, thus depends on v_A exactly only through the quaternionic line that it spans, we have thus found the following identification of the 4-sphere with quaternionic projective space:

$$S^4 \simeq S(\mathbb{R}^5) \xrightarrow{\simeq} \text{Mat}_{\text{trless}}^{\text{herm}}(2 \times 2, \mathbb{H}) \cap \text{U}(2, \mathbb{H}) \xleftarrow{\simeq} \mathbb{H}P^1 \quad (21)$$

$$\mathbf{I} - \frac{2}{\|v\|^2} v \cdot v^\dagger \quad \longleftarrow \quad [v]$$

Finally, under the isomorphism on the right of (21) the canonical $\text{Sp}(2)$ -action on $\mathbb{H}P^1$

$$\text{Sp}(2) \times \mathbb{H}P^1 \longrightarrow \mathbb{H}P^1$$

$$(A, [v]) \quad \longmapsto \quad [A \cdot v]$$

is manifestly identified with the conjugation action (17). This implies the claim (iii), by (20). □

In summary:

$$\begin{array}{lcl}
 \text{6d Minkowski spacetime} & \text{Spin}(5, 1) \overset{\text{can}}{\hookrightarrow} \mathbb{R}^{5,1} & \simeq \text{SL}(2, \mathbb{H}) \overset{\text{adj}}{\hookrightarrow} \underbrace{\{A \in \text{Mat}_{2 \times 2}(\mathbb{H}) \mid A^\dagger = A\}}_{\text{relativistic quaternionic Pauli matrices}} \\
 & \uparrow & \\
 \text{5d Euclidean space} & \text{Spin}(5) \overset{\text{can}}{\hookrightarrow} \mathbb{R}^5 & \simeq \underbrace{\text{U}(2, \mathbb{H}) \overset{\text{adj}}{\hookrightarrow} \{A \in \text{Mat}_{2 \times 2}(\mathbb{H}) \mid A^\dagger = A, \text{tr}(A) = 0\}}_{\text{quaternionic Pauli matrices}} \\
 & & = \text{Sp}(2) \\
 & \uparrow & \\
 \text{4-sphere} & \text{Spin}(5) \overset{\text{can}}{\hookrightarrow} S^4 & \simeq \underbrace{\underbrace{\text{U}(2, \mathbb{H}) \overset{\text{adj}}{\hookrightarrow} \{A \in \text{Mat}_{2 \times 2}(\mathbb{H}) \mid A^\dagger = A, \text{tr}(A) = 0, A^\dagger \cdot A = \mathbf{I}\}}_{\text{quaternionic rank-1 projectors } P = \frac{1}{2}(\mathbf{I} - A)}}_{\simeq \mathbb{H}P^1} \\
 & & \text{quaternionic projective line}
 \end{array}$$

Of course, there is the analogous situation over the complex numbers:

$$\begin{array}{lcl}
 \text{4d Minkowski spacetime} & \text{Spin}(3, 1) \overset{\text{can}}{\hookrightarrow} \mathbb{R}^{3,1} & \simeq \text{SL}(2, \mathbb{C}) \overset{\text{adj}}{\hookrightarrow} \underbrace{\{A \in \text{Mat}_{2 \times 2}(\mathbb{C}) \mid A^\dagger = A\}}_{\text{relativistic complex Pauli matrices}} \\
 & \uparrow & \\
 \text{3d Euclidean space} & \text{Spin}(3) \overset{\text{can}}{\hookrightarrow} \mathbb{R}^3 & \simeq \underbrace{\text{SU}(2, \mathbb{C}) \overset{\text{adj}}{\hookrightarrow} \{A \in \text{Mat}_{2 \times 2}(\mathbb{C}) \mid A^\dagger = A, \text{tr}(A) = 0\}}_{\text{complex Pauli matrices}} \\
 & \uparrow & \\
 \text{3-sphere} & \text{Spin}(3) \overset{\text{can}}{\hookrightarrow} S^3 & \simeq \underbrace{\underbrace{\text{SU}(2, \mathbb{C}) \overset{\text{adj}}{\hookrightarrow} \{A \in \text{Mat}_{2 \times 2}(\mathbb{C}) \mid A^\dagger = A, \text{tr}(A) = 0, A^\dagger \cdot A = \mathbf{I}\}}_{\text{complex rank-1 projectors } P = \frac{1}{2}(\mathbf{I} - A)}}_{\simeq \mathbb{C}P^1} \\
 & & \text{complex projective line}
 \end{array}$$

Proposition 2.2 (Equivariance of combined Hopf/twistor fibrations).

(i) The quaternionic Hopf fibration $S^7 \xrightarrow{h_{\mathbb{H}}} S^4$ (Diagram (11)) is equivariant with respect to the action of $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ (Def. A.3)

(a) on S^7 , by

$$\begin{array}{ccc} \mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \times S^7 & \longrightarrow & S^7 \\ ([A, q'], \{v \cdot t \mid t \in \mathbb{R}_+^\times\}) & \mapsto & \{A \cdot v \cdot t \cdot q' \mid t \in \mathbb{R}_+^\times\} \end{array} \quad (22)$$

(b) on S^4 , by

$$\begin{array}{ccc} \mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \times S^4 & \longrightarrow & S^4 \\ ([A, q'], \{v \cdot q \mid q \in \mathbb{H}^\times\}) & \mapsto & \{A \cdot v \cdot q \mid q \in \mathbb{H}^\times\} \end{array} \quad (23)$$

(ii) Its factorization $h_{\mathbb{H}} = t_{\mathbb{H}} \circ h_{\mathbb{C}}$ through the combined Hopf/twistor fibrations retains equivariance under the subgroup $\mathrm{Sp}(2) \hookrightarrow \mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ (85) with action on $\mathbb{C}P^3$ given by

$$\begin{array}{ccc} \mathrm{Sp}(2) \times \mathbb{C}P^3 & \longrightarrow & \mathbb{C}P^3 \\ (A, \{x \cdot z \mid z \in \mathbb{C}^\times\}) & \mapsto & \{A \cdot x \cdot z \mid z \in \mathbb{C}^\times\} \end{array} \quad (24)$$

In summary:

$$\begin{array}{ccc} \begin{array}{ccc} \mathrm{Sp}(2) \cdot \mathrm{Sp}(1) & & \mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \\ \curvearrowright & & \curvearrowright \\ S^7 & \xrightarrow{h_{\mathbb{H}}} & S^4 \end{array} & \text{and} & \begin{array}{ccc} \mathrm{Sp}(2) & & \mathrm{Sp}(2) & & \mathrm{Sp}(2) \\ \curvearrowright & & \curvearrowright & & \curvearrowright \\ S^7 & \xrightarrow{h_{\mathbb{C}}} & \mathbb{C}P^3 & \xrightarrow{t_{\mathbb{H}}} & S^4 \\ & \searrow & & \nearrow & \\ & & h_{\mathbb{H}} & & \end{array} \end{array}$$

Proof. This is essentially immediate from the presentation of the fibrations in (11) (12):

(ii) Diagram (11) makes manifest that all maps here are equivariant with respect to left action by $\mathrm{GL}(8, \mathbb{R})$, hence in particular under left action by $\mathrm{Sp}(2)$, which is also manifest from (12):

$$\{A \cdot v \cdot t \mid t \in \mathbb{R}_+^\times\} \xrightarrow{h_{\mathbb{C}}} \{A \cdot v \cdot z \mid z \in \mathbb{C}^\times\} \xrightarrow{t_{\mathbb{H}}} \{A \cdot v \cdot q \mid q \in \mathbb{H}^\times\}. \quad (25)$$

(i) We see that the total quaternionic Hopf fibration is also equivariant under the right $\mathrm{Sp}(1)$ -action, due to the fact that the reals commute with the quaternions:

$$\{v \cdot t \cdot q' \mid t \in \mathbb{R}_+^\times\} = \{v \cdot q' \cdot t \mid t \in \mathbb{R}_+^\times\} \xrightarrow{h_{\mathbb{H}}} \{A \cdot q' \cdot q \mid q \in \mathbb{H}^\times\} = \{A \cdot q \mid q \in \mathbb{H}^\times\}. \quad (26)$$

Moreover, since the left multiplication action by $\mathrm{Sp}(2)$ evidently commutes with the right multiplication action by $\mathrm{Sp}(1)$ and since $-1 \in \mathrm{Sp}(n)$ is central, this generates the claimed $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ -action. (In fact, this is the maximal symmetry group of $h_{\mathbb{H}}$ [GWZ86, 4.1][FSS19b, 2.20].) \square

Remark 2.3 (Twistor space breaks equivariance to $\mathrm{Sp}(2)$). Notice that the factorization of the quaternionic Hopf fibration through $\mathbb{C}P^3$ is *not* equivariant under the further right $\mathrm{Sp}(1)$ -action from (22) and (23), Indeed, the computation analogous to (26) now gives

$$\{v \cdot t \cdot q' \mid t \in \mathbb{R}_+^\times\} = \{v \cdot q' \cdot t \mid t \in \mathbb{R}_+^\times\} \xrightarrow{h_{\mathbb{C}}} \{A \cdot q' \cdot z \mid z \in \mathbb{C}^\times\} \neq \{A \cdot z \cdot q' \mid z \in \mathbb{C}^\times\} \quad \text{in general}$$

since the complex numbers do not commute with the quaternions. Therefore, factoring the quaternionic Hopf fibration through the twistor fibration (11) breaks its symmetry from $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ to $\mathrm{Sp}(2)$ (85).

Remark 2.4 (Homotopy quotients and Borel construction). For X a topological space equipped with a continuous action by a topological group G , the *homotopy quotient* $X // G$ is the homotopy type of a topological space is represented by the *Borel space* $(X \times EG) / \mathrm{diag} G$, where EG denotes the universal G -principal bundle:

$$\begin{array}{ccc} \text{homotopy quotient} & & \text{Borel construction} \\ X // G & \simeq_{\text{whe}} & (X \times EG) / \mathrm{diag} G. \end{array} \quad (27)$$

(i) This construction is clearly functorial: On the right this is a 1-functor on the category of topological spaces equipped with group actions, while on the left this is an ∞ -functor on the ∞ -category Groupoids_∞ equipped with ∞ -actions, see [NSS12a, §4][SS20b, §2.2].

(ii) In the special case when $X = *$ is the point, the Borel space is the classifying space BG . With (i), this means that topological group homomorphisms $G_1 \xrightarrow{\phi} G_2$ induce maps of classifying spaces

$$BG_1 \xrightarrow{B\phi} BG_2 . \quad (28)$$

Definition 2.5 (Borel-equivariant Hopf/twistor fibrations). We say that the $\text{Sp}(2)$ -Borel-equivariant Hopf-twistor fibrations are the image (in homotopy types of topological spaces) of the Hopf/twistor fibrations (11) under taking the homotopy quotient (27) by their compatible $\text{Sp}(2)$ -action of Prop. 2.2:

$$\begin{array}{ccccc}
 & \text{parametrized quaternionic Hopf fibration} & & & \\
 & h_{\mathbb{H}} // \text{Sp}(2) & & & \\
 S^7 // \text{Sp}(2) & \xrightarrow{h_{\mathbb{C}} // \text{Sp}(2)} & \mathbb{C}P^3 // \text{Sp}(2) & \xrightarrow{t_{\mathbb{H}} // \text{Sp}(2)} & S^4 // \text{Sp}(2) \\
 & \text{parametrized complex Hopf fibration} & & \text{parametrized twistor fibration} & \\
 & & \downarrow & & \\
 & & B\text{Sp}(2) & &
 \end{array} \quad (29)$$

Lemma 2.6 (Coset spaces as homotopy fibers [FSS19b, 2.7][SS20b, 2.79]). Let $H \xrightarrow{i} G$ be an inclusion of topological groups.

(i) The homotopy type of the corresponding coset space G/H is, equivalently, the homotopy fiber of the induced morphism (28) on classifying spaces.

(ii) The homotopy quotient of the coset space by G is homotopy equivalent to the classifying space of H :

$$\begin{array}{ccc}
 G/H & \xrightarrow{\text{hofib}(B_i)} & BH \simeq (G/H) // G \\
 & & \downarrow B_i \\
 & & BG
 \end{array}$$

The following Prop. 2.7 is the twistorial version of [FSS19b, Prop. 2.22].

Proposition 2.7 (Borel-equivariant twistor fibration as sequence of classifying spaces). The Borel-equivariant Hopf/twistor fibration (Def. 2.5) is homotopy equivalent to the following sequence of classifying spaces:

$$\begin{array}{ccccc}
 S^7 // \text{Sp}(2) & \xrightarrow{h_{\mathbb{C}} // \text{Sp}(2)} & \mathbb{C}P^3 // \text{Sp}(2) & \xrightarrow{t_{\mathbb{H}} // \text{Sp}(2)} & S^4 // \text{Sp}(2) \\
 \wr & & \wr & & \wr \\
 B(\text{Sp}(1)_L) & \longrightarrow & B(\text{Sp}(1)_L \times \text{U}(1)_R) & \longrightarrow & B(\text{Sp}(1)_L \times \text{Sp}(1)_R)
 \end{array}$$

where the maps on the bottom are the deloopings (28) of the canonical group inclusions (Example A.4).

Proof. With Lemma 2.6 this follows from the $\text{Sp}(2)$ -coset space realization of the Hopf/twistor fibration in (12). \square

Lemma 2.8 (Borel-equivariant Hopf/twistor fibrations are spherical). The Borel-equivariant Hopf/twistor fibrations (29) are still spherical fibrations:

$$\begin{array}{ccccc}
 S^1 & \longrightarrow & S^3 & \longrightarrow & S^7 // \text{Sp}(2) \\
 \downarrow & \text{(pb)} & \downarrow & \text{(pb)} & \downarrow h_{\mathbb{C}} // \text{Sp}(2) \\
 * & \longrightarrow & S^2 & \longrightarrow & \mathbb{C}P^3 // \text{Sp}(2) \\
 & & \downarrow & \text{(pb)} & \downarrow t_{\mathbb{H}} // \text{Sp}(2) \\
 & & * & \longrightarrow & S^4 // \text{Sp}(2)
 \end{array}$$

Proof. This follows on general grounds, as in [FSS19b, Remark 3.17]. More concretely, by Prop. 2.7 and using again Lemma 2.6 we have:

$$\begin{array}{ccc} \text{fib}(h_{\mathbb{C}} // \text{Sp}(2)) & \longrightarrow & S^7 // \text{Sp}(2) \\ & & \downarrow h_{\mathbb{C}} // \text{Sp}(2) \\ & & \mathbb{C}P^3 // \text{Sp}(2) \end{array} \quad \simeq \quad \begin{array}{ccc} \text{U}(1) & \longrightarrow & B(\text{Sp}(1)_L) \\ & & \downarrow \\ & & B(\text{Sp}(1)_L \times \text{U}(1)_R) \end{array}$$

and

$$\begin{array}{ccc} \text{fib}(t_{\mathbb{H}} // \text{Sp}(2)) & \longrightarrow & \mathbb{C}P^3 // \text{Sp}(2) \\ & & \downarrow t_{\mathbb{H}} // \text{Sp}(2) \\ & & S^4 // \text{Sp}(2) \end{array} \quad \simeq \quad \begin{array}{ccc} \text{SU}(2)/\text{U}(1) & \longrightarrow & B(\text{Sp}(1)_L \times \text{U}(1)_R) \\ & & \downarrow \\ & & B(\text{Sp}(1)_L \times \text{Sp}(1)_R) \end{array}$$

□

Theorem 2.9 (Integral cohomology of Borel-equivariant Hopf/twistor-fibration).

(i) *The integral cohomology of the space $S^4 // \text{Sp}(2)$ in (29) is free on two generators in degree 4*

$$H^*(S^4 // \text{Sp}(2); \mathbb{Z}) \simeq \mathbb{Z}[\tilde{\Gamma}_4, \tilde{\Gamma}_4^{\text{vac}}] \quad (30)$$

with the property that their evaluation on the fundamental class of the 4-sphere fiber $S^4 \rightarrow S^4 // \text{Sp}(2)$ is unity and zero, respectively:

$$\langle \tilde{\Gamma}_4, S^4 \rangle = 1, \quad \langle \tilde{\Gamma}_4^{\text{vac}}, S^4 \rangle = 0. \quad (31)$$

(ii) *The integral cohomology of the space $\mathbb{C}P^3 // \text{Sp}(2)$ in (29) is free on two generators in degrees 4 and 2, respectively:*

$$H^*(\mathbb{C}P^3 // \text{Sp}(2); \mathbb{Z}) = \mathbb{Z}[c_2^L, c_1^R]. \quad (32)$$

(iii) *The two are related in that pullback in integral cohomology along the Borel-equivariant twistor fibration (Def. 2.5) takes the difference of the former generators to the cup-square of the latter:*

$$\begin{array}{ccc} H^*(S^4 // \text{Sp}(2); \mathbb{Z}) & \xrightarrow{(t_{\mathbb{H}} // \text{Sp}(2))^*} & H^*(\mathbb{C}P^3 // \text{Sp}(2); \mathbb{Z}) \\ \tilde{\Gamma}_4 - \tilde{\Gamma}_4^{\text{vac}} & \longmapsto & -a := c_1^R \cup c_1^R \\ \tilde{\Gamma}_4 & \longmapsto & c_2^L \end{array} \quad (33)$$

Proof. Consider Diagram (7) in Figure I. The top part shows the equivalence of the Borel-equivariant Hopf/twistor fibration to a sequence of classifying spaces, according to Prop. 2.7. On the top right we are making fully explicit the factor-wise nature of the corresponding maps, using Example A.4 and Remark 2.4.

The bottom part of the diagram shows the corresponding identification of the cohomology generators according to [FSS19b, 3.9]. This involves the observation that:

(a) Half the universal Euler 4-class on $B\text{Spin}(4)$ is (e.g., [BC98, §2]) the class of the fiberwise unit volume form on the universal S^4 -fibration, under the identification from Prop. 2.7:

$$\begin{array}{ccc} 1 \cdot [\text{dvol}] & \longleftarrow & \frac{1}{2} \chi_4 \in H^4(B\text{Spin}(4); \mathbb{R}) \\ S^4 & \longrightarrow & S^4 // \text{Spin}(5) \simeq B\text{Spin}(4) \end{array} \quad (34)$$

(b) The fractional Euler class by itself is not integral, but its shift by $\frac{1}{4}p_1$ (which is also not integral by itself) is (the rational image of) an integral generator $\tilde{\Gamma}_4 = \frac{1}{2}\chi_4 + \frac{1}{4}p_1$ (e.g. [CV98a, Lemma 2.1]).

Together, (a) and (b) yield the claim (31), and make manifest that the pullback in question is equivalently that of the negative of the left Chern class $-c_2^L$ along the map on classifying space $BU(1) \xrightarrow{B(c \rightarrow \text{diag}(c, \bar{c}))} BSU(2)$, induced from the inclusion $U(1) \xrightarrow{\cong} S(U(1)^2) \hookrightarrow SU(2)$, hence is $c_1 \cup (-c_1)$, which yields the last claim (33). □

2.2 Rational homotopy type

We compute (in Theorem 2.14) the relative Sullivan model for the Borel-equivariant Hopf/twistor fibration from Def. 2.5, with generators normalized such as to match their integral pre-images from Theorem 2.9.

First to establish some preparatory facts:

Lemma 2.10 (Normalized Sullivan model of spherical fibrations [FHT00, p. 202]; see [FSS19b, 2.5]). *Let X be a topological space with Sullivan model $\mathrm{CE}(IX) \in \mathrm{dgcAlgebras}_{\mathbb{R}}$. Then the relative Sullivan model for a S^n -fibration $Y \rightarrow X$ is of the following form:*

(i) for $n = 2k + 1$ odd:

$$\begin{array}{ccc} S^{2k+1} & \longrightarrow & Y \\ & & \downarrow \\ & & X \end{array} \quad \mathrm{CE}(IX)[\omega_{2k+1}] / (d\omega_{2k+1} = \alpha_{2k+2}) \quad (35)$$

$$\uparrow$$

$$\mathrm{CE}(IX)$$

(ii) for $n = 2k$ even:

$$\begin{array}{ccc} S^{2k} & \longrightarrow & Y \\ & & \downarrow \\ & & X \end{array} \quad \mathrm{CE}(IX) \left[\begin{array}{c} \omega_{2k}, \\ \omega_{4k-1} \end{array} \right] / \left(\begin{array}{c} d\omega_{2k} = 0 \\ d\omega_{4k-1} = -\omega_{2k} \wedge \omega_{2k} + \alpha_{4k} \end{array} \right) \quad (36)$$

$$\uparrow$$

$$\mathrm{CE}(IX)$$

for some closed $\alpha \in \mathrm{CE}(IX)$ (which can be characterized further, see [FHT00, p. 202][FSS19b, 2.5]).

(iii) The differential in (36) is normalized so that the generators ω_d restrict to the unit volume forms on the respective sphere fibers (see [FSS19b, Lemma 3.19]):

$$\langle \omega_{2k}, S^{2k} \rangle = 1, \quad \langle \omega_{4k-1}, S^{4k-1} \rangle = 1. \quad (37)$$

The action of triality group automorphisms on $\mathrm{Spin}(8)$ famously relates three distinct conjugacy classes of subgroup inclusions of $\mathrm{Spin}(7)$. Less widely appreciated is another triple of subgroups of $\mathrm{Spin}(8)$ that is permuted under triality:

Lemma 2.11 (Triality on central product groups in $\mathrm{Spin}(8)$ [FSS19b, 2.17]). *Under the triality automorphisms of $\mathrm{Spin}(8)$ the canonical subgroup inclusions of the central product groups $\mathrm{Spin}(5) \cdot \mathrm{Spin}(3)$ and $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ (Def. A.1) turn into each other:*

$$\begin{array}{ccc} \mathrm{Sp}(2) \cdot \mathrm{Sp}(1) & \xrightarrow{i_{\mathrm{Sp}}} & \mathrm{Spin}(8) \\ \simeq \downarrow & & \simeq \downarrow \mathrm{tri} \\ \mathrm{Spin}(5) \cdot \mathrm{Spin}(3) & \xrightarrow{i_{\mathrm{Spin}}} & \mathrm{Spin}(8) \end{array} \quad (38)$$

Lemma 2.12 (Sullivan model for $B\mathrm{Spin}(5)$ and $B\mathrm{Sp}(2)$ [FSS19b, 2.19]). *Sullivan models for $B\mathrm{Spin}(5)$ and $B\mathrm{Sp}(2)$, and their relation under triality (39) are given, up to isomorphism, as follows:*

$$\begin{array}{ccc} B\mathrm{Spin}(8) & \xrightarrow{Bi_{\mathrm{Sp}}} & B\mathrm{Sp}(2) \\ \downarrow \simeq & & \downarrow \simeq \\ B\mathrm{Spin}(8) & \xrightarrow{Bi_{\mathrm{Spin}}} & B\mathrm{Spin}(5) \end{array} \quad \begin{array}{c} \mathbb{R}[\frac{1}{2}p_1, \mathcal{X}_8] = \mathrm{CE}(lB\mathrm{Sp}(2)) \\ \uparrow \begin{array}{cc} \frac{1}{2}p_1 & -\mathcal{X}_8 + (\frac{1}{4}p_1)^2 \\ \uparrow & \uparrow \\ \frac{1}{2}p_1 & \frac{1}{2}p_2 \end{array} \\ \mathbb{R}[\frac{1}{2}p_1, p_2] =: \mathrm{CE}(lB\mathrm{Spin}(5)) \end{array} \quad (39)$$

where (by [CV98b, 8.1, 8.2][FSS19b, 3.7]):

$$\frac{1}{2}\mathcal{X}_8 = \frac{1}{4} \left(p_2 - \left(\frac{1}{2}p_1 \right)^2 \right) \in H^4(B\mathrm{Sp}(2); \mathbb{R}). \quad (40)$$

Lemma 2.13 (Normalized Sullivan model for plain Hopf/twistor fibrations). *The Sullivan model for the plain Hopf/twistor fibrations (11) is as follows:*

$$\begin{array}{ccc}
\begin{array}{c} S^7 \\ \downarrow h_{\mathbb{C}} \\ \mathbb{C}P^3 \\ \downarrow t_{\mathbb{H}} \\ S^4 \end{array} & \begin{array}{c} \mathbb{R} \left[\begin{array}{c} h_1, \\ f_2, \\ h_3, \\ \omega_4, \\ \omega_7 \end{array} \right] / \left(\begin{array}{c} dh_1 = f_2 \\ df_2 = 0 \\ dh_3 = \omega_4 - f_2 \wedge f_2 \\ d\omega_4 = 0 \\ d\omega_7 = -\omega_4 \wedge \omega_4 \end{array} \right) \\ \uparrow \begin{array}{c} f_2 \ h_3 \ \omega_4 \ \omega_7 \\ \uparrow \ \uparrow \ \uparrow \ \uparrow \\ f_2 \ h_3 \ \omega_4 \ \omega_7 \end{array} \\ \mathbb{R} \left[\begin{array}{c} f_2, \\ h_3, \\ \omega_4, \\ \omega_7 \end{array} \right] / \left(\begin{array}{c} df_2 = 0 \\ dh_3 = \omega_4 - f_2 \wedge f_2 \\ d\omega_4 = 0 \\ d\omega_7 = -\omega_4 \wedge \omega_4 \end{array} \right) \\ \uparrow \begin{array}{c} \omega_4 \ \omega_7 \\ \uparrow \ \uparrow \\ \omega_4 \ \omega_7 \end{array} \\ \mathbb{R} \left[\begin{array}{c} \omega_4, \\ \omega_7 \end{array} \right] / \left(\begin{array}{c} d\omega_4 = 0 \\ d\omega_7 = -\omega_4 \wedge \omega_4 \end{array} \right) \end{array} & \xrightarrow{\cong} & \begin{array}{c} \mathbb{R}[\omega_7] / (d\omega_7 = 0) \\ \mathbb{R} \left[\begin{array}{c} f_2, \\ \omega_7 \end{array} \right] / \left(\begin{array}{c} df_2 = 0 \\ d\omega_7 = -(f_2)^4 \end{array} \right) \end{array} \quad (41)
\end{array}$$

where the generators $\omega_4, \omega_7, f_2, h_3$ are all normalized according to (37), in particular:

$$\langle \omega_4, S^4 \rangle = 1 \quad \langle f_2, S^2 \rangle = 1. \quad (42)$$

Note that on the right in (41) we are showing the Sullivan models of S^7 and of $\mathbb{C}P^3$ by themselves (which is classical, e.g. [FHT00, p. 142, 203][Me13, 1.2, 5.3]), while on the left we are showing their Sullivan models as fiber spaces, i.e., the relative Sullivan models.

Proof. (i) It is classical that the Sullivan model for S^4 is as shown (it is also a special case of Lemma 2.10).

(ii) Since $\mathbb{C}P^3 \rightarrow S^4$ is an S^2 -fibration (11), Lemma 2.10 implies from (i) that $\mathbb{C}P^3$ fibered over S^4 is modeled by

$$\text{CE}(\mathbb{C}P^3)_{S^4} = \mathbb{R}[\omega_4, \omega_7, f_2, h_3] / \left(\begin{array}{c} d\omega_4 = 0 \\ d\omega_7 = -\omega_4 \wedge \omega_4 \\ df_2 = 0 \\ dh_3 = f_2 \wedge f_2 + \alpha_4 \end{array} \right)$$

for *some* closed element $\alpha_4 \in \text{CE}(\mathbb{C}P^3)_{S^4}$. But in the present case, due to (i), there is a unique such element, up to a real factor, namely ω_4 . Below in (45) we find this factor to be unity. This yields the middle part of (41).

(iii) Since $S^7 \rightarrow \mathbb{C}P^3$ is an S^1 -fibration (11), Lemma 2.10 implies, via (ii), that S^7 fibered over $\mathbb{C}P^3$ is modeled by

$$\text{CE}(S^7)_{\mathbb{C}P^3} = \mathbb{R}[\omega_4, \omega_7, f_2, h_3, h_1] / \left(\begin{array}{c} d\omega_4 = 0 \\ d\omega_7 = -\omega_4 \wedge \omega_4 \\ df_2 = 0 \\ dh_3 = f_2 \wedge f_2 + \omega_4 \\ dh_1 = \alpha_2 \end{array} \right)$$

for *some* closed degree-2 element $\alpha_2 \in \text{CE}(S^7)_{\mathbb{C}P^3}$. But in the present case, due to (ii), there is a unique such element, up to a real factor, namely f_2 . Thus, by suitably rescaling f_1 , we obtain $\alpha_2 = f_2$ and the claim follows. \square

With this, we may turn to the main theorem:

Theorem 2.14 (Normalized Sullivan model of Borel-equivariant Hopf/twistor fibrations). *The iterative relative Sullivan models for the parametrized Hopf/twistor fibrations (29) are as follows (here $\frac{1}{2}p_1, \chi_8 \in \text{CE}(\mathfrak{tBSp}(2))$), via Lemma 2.12):*

$$\begin{array}{ccc}
\begin{array}{c} S^7 // \text{Sp}(2) \\ \downarrow h_{\mathbb{C}} // \text{Sp}(2) \\ \mathbb{C}P^3 // \text{Sp}(2) \\ \downarrow t_{\mathbb{H}} // \text{Sp}(2) \\ B\text{Sp}(2) \\ \uparrow \\ S^4 // \text{Sp}(2) \end{array} & \begin{array}{c} \text{CE}(\mathfrak{tBSp}(2)) \left[\begin{array}{c} h_1, \\ f_2, \\ h_3, \\ \omega_4, \\ \omega_7 \end{array} \right] / \left(\begin{array}{l} dh_1 = f_2 \\ df_2 = 0 \\ dh_3 = \omega_4 - \frac{1}{4}p_1 - f_2 \wedge f_2 \\ d\omega_4 = 0 \\ d\omega_7 = -\omega_4 \wedge \omega_4 + (\frac{1}{4}p_1)^2 \\ -\chi_8 \end{array} \right) \\ \uparrow \\ \text{CE}(\mathfrak{tBSp}(2)) \left[\begin{array}{c} f_2, \\ h_3, \\ \omega_4, \\ \omega_7 \end{array} \right] / \left(\begin{array}{l} df_2 = 0 \\ dh_3 = \omega_4 - \frac{1}{4}p_1 - f_2 \wedge f_2 \\ d\omega_4 = 0 \\ d\omega_7 = -\omega_4 \wedge \omega_4 + (\frac{1}{4}p_1)^2 \\ -\chi_8 \end{array} \right) \\ \downarrow \\ \text{CE}(\mathfrak{tBSp}(2)) \left[\begin{array}{c} \omega_4, \\ \omega_7 \end{array} \right] / \left(\begin{array}{l} d\omega_4 = 0 \\ d\omega_7 = -\omega_4 \wedge \omega_4 + (\frac{1}{4}p_1)^2 \\ -\chi_8 \end{array} \right) \end{array} & \begin{array}{c} \begin{array}{c} \omega_4 \ \omega_7 \ f_2 \ h_3 \\ \uparrow \ \uparrow \ \uparrow \ \uparrow \\ \omega_4 \ \omega_7 \ f_2 \ h_3 \end{array} \\ \downarrow \\ \begin{array}{c} \omega_4 \ \omega_7 \\ \uparrow \ \uparrow \\ \omega_4 \ \omega_7 \end{array} \end{array} \end{array} \quad (43)
\end{array}$$

where the generators f_2 and ω_4 represent the classes c_1^R and $\frac{1}{2}\chi_4$ in (7), respectively:

$$[\omega_4] = \frac{1}{2}\chi_4 \in H^4(\mathbb{C}P^3 // \text{Sp}(2); \mathbb{R}), \quad [f_2] = c_1^R \in H^2(\mathbb{C}P^3 // \text{Sp}(2); \mathbb{R}). \quad (44)$$

Proof. That the composite vertical morphism, upon discarding the generators h_1, f_2 and h_3 , is the Sullivan model for $h_{\mathbb{H}} // \text{Sp}(2)$ with the identification $[\omega_4] = \frac{1}{2}\chi_4$ (44) is the result of [FSS19b, 3.19].

Its factorization through $\mathbb{C}P^3 // \text{Sp}(2)$ must have Sullivan model given by adjoining generators f_2 and f_3 , by Lemma 2.8 with Lemma 2.10. The fiberwise normalization (42) implies the identification $[f_2] = c_1^R$ in (44), using that c_1^R pulled back along $S^2 = \text{SU}(2)/U(1) \rightarrow \text{BU}(1)_R$ is the unit volume generator.

For the factorization of $h_{\mathbb{H}} // \text{Sp}(2)$ through $\mathbb{C}P^3 // \text{Sp}(2)$ to reproduce on fibers over $B\text{Sp}(2)$ (hence upon discarding the generators $\frac{1}{2}p_1$ and χ_8) the Sullivan model for the plain Hopf/twistor fibrations from Lemma 2.13 at least those monomials in f_2 shown in (43) have to appear. We just have to observe that the relative coefficients in the differential relations for h_3 are as shown. But under the identification (44) we have the second logical equivalence shown here:

$$dh_3 = \omega_4 - \frac{1}{4}p_1 - f_2 \wedge f_2 \Leftrightarrow [\omega_4 - \frac{1}{4}p_1] = f_2 \wedge f_2 \Leftrightarrow \tilde{\Gamma}_4 - \tilde{\Gamma}_4^{\text{vac}} = c_1^R \cup c_1^R \in H^4(\mathbb{C}P^3 // \text{Sp}(2); \mathbb{R}). \quad (45)$$

That the relation on the right of (45) does hold follows immediately from (33) in Theorem 2.9.

Hence to conclude it suffices now to show that no further monomials in f_2 appear on the right of (43):

First, any further monomial in f_2 that does appear must contain as a factor a *basic* generator, namely a generator from $\text{CE}(\text{IBSp}(2))$, to guarantee that it vanishes on fibers (where we already have the right terms). Since, by Lemma 2.12, the generators of $\text{CE}(\text{IBSp}(2))$ are in degrees 4 and 8, the only further term that could possibly appear, by degree reasons, is the blue term in the following expression:

$$d\omega_7 = -\omega_4 \wedge \omega_4 + \left(\frac{1}{2}p_1\right)^2 - \chi_8 + a \cdot f_2 \wedge f_2 \wedge p_1 \quad (46)$$

for some coefficient $a \in \mathbb{R}$. But we also know that $t_{\mathbb{H}} // \text{Sp}(2)$ is an S^2 -fibration (by Lemma 2.8), so that Lemma 2.10 rules out the appearance of the blue term in (46) (i.e., implies $a = 0$). \square

3 Charge quantization in Twistorial Cohomotopy

After recalling (in §3.1) general non-abelian cohomology and highlighting the non-abelian Chern-Dold character map, we introduce (in §3.2) the twisted non-abelian cohomology to be called *Twistorial Cohomotopy* and use the results from §2 to show (Corollary 3.10) that charge quantization in Twistorial Cohomotopy implies the heterotic shifted flux quantization condition (6).

3.1 Non-abelian Chern-Dold character

We recall the Chern-Dold character (54) in generalized cohomology and then introduce its generalization, to a non-abelian Chern-Dold character (62) on non-abelian cohomology. For full technical detail see [FSS20c].

From generalized to non-abelian cohomology. It is well-known, though perhaps under-appreciated, that cohomology theory is all about homotopy groups of mapping spaces into a given “coefficient space” or “classifying space”. We recall this briefly for “bare” cohomology theories, with the domain spaces X assumed to a sufficiently nice topological space; but the statement remains true for structured cohomology theories such as differential and/or equivariant cohomology, when interpreted internal to suitable higher toposes, see [SS20b, p. 6].

For ordinary (e.g., singular) cohomology with coefficients in an *abelian* discrete group A , these classifying spaces are the Eilenberg-MacLane spaces $K(A, n)$ (e.g. [AGP02, §7.1, Cor. 12.1.20]):

$$H^n(X; A) \simeq \pi_0 \text{Maps}(X, K(A, n)). \quad (47)$$

These happen to be based loop spaces of each other, $K(A, n) \simeq_{\text{wh}} \Omega K(A, n+1)$ (e.g. [AGP02, 7.1.1]), so that each of them is an *infinite loop space* (e.g. [Ad78]).

More generally, consider a *generalized cohomology theory*⁴ E^\bullet in the sense of [Wh62] (see [Ad75][Ad78]), such as K-theory, elliptic cohomology, tmf, stable Cobordism, stable Cohomotopy, etc. These are classified by such sequences of (pointed) spaces which are successively equipped with weak homotopy equivalences exhibiting them as based loop spaces of each other, called a *spectrum* of spaces:

$$\{E_n\}_{n \in \mathbb{N}}, \quad \text{s.t. } E_n \simeq \Omega E_{n+1}, \quad (48)$$

in that

$$E^n(X) \simeq \pi_0 \text{Maps}(X, E_n). \quad (49)$$

This is the *Brown representability theorem*, see e.g. [Ad75, §III.6][Ko96, §3.4]. But the right hand side of (49) makes sense for E_n *any* space, not necessarily part of a spectrum (48), and not necessarily even being a loop space. It is not the notion of cohomology itself, but rather only some extra *properties* enjoyed by these abelian cohomology groups (such as existence of connecting homomorphisms) which is what is reflected in the infinite loop space structure (48).

⁴ The term is widely used but somewhat unfortunate, since various *further* generalizations of Whitehead’s generalization of ordinary cohomology theories are relevant, such as twisted-, sheaf-, differential-, equivariant- and nonabelian-cohomology theories, as well as all their joint combinations.

Indeed, for G a well-behaved topological group, *not* necessarily abelian (such as $G = U(1), SU(n), Sp(n), \dots$) the fundamental theorem of G -principal bundles ([St51, §19.3], review in [Add07, §5]) says that degree-1 *non-abelian cohomology* with coefficients in G is represented by the classifying space BG of G :

$$\overset{\text{non-abelian cohomology with coefficients in topological group}}{H^1(X; G)} \simeq \overset{\text{homotopy classes of maps to classifying space of group}}{\pi_0 \text{Maps}(X, BG)}. \quad (50)$$

If $G = A$ is abelian and discrete, then $BA \simeq K(A, 1)$ and (50) reduces to (47), but not otherwise. Moreover, the May recognition theorem implies that *any* connected space A is weakly homotopy equivalent to a classifying space BG , namely for $G = \Omega A$ the based loop group of A (which may be rectified, up to weak homotopy equivalence, to an actual topological group). Thereby, the traditional equivalence (50) is re-interpreted as an elegant general notion of *non-abelian cohomology*:

$$\overset{\text{non-abelian cohomology with coefficients in } A}{H(X; A)} := \overset{\text{homotopy-classes of maps to } A}{\pi_0 \text{Maps}(X, A)} = \left\{ \begin{array}{ccc} & \begin{array}{c} \text{map/cocycle} \\ \xrightarrow{c} \\ X \end{array} & \\ & \begin{array}{c} \text{homotopy/coboundary} \\ \Downarrow \\ \text{map/cocycle} \\ \xrightarrow{c'} \end{array} & A \\ & \begin{array}{c} \text{map/cocycle} \\ \xrightarrow{c'} \\ X \end{array} & \end{array} \right\} / \text{homotopy} \quad (51)$$

Non-abelian cohomology in this generality is discussed in [To02][Ja09][RS12][NSS12a][NSS12b][SS20b]. For example, for $X = S^n$ an n -sphere, we have $S^n \simeq B(\Omega S^n)$ and the corresponding non-abelian cohomology theory (50) is *Cohomotopy theory*

$$\pi^n(X) := \pi_0 \text{Maps}(X, S^n) \simeq H^1(X; \Omega S^n).$$

This perspective on generalized/non-abelian cohomology via classifying spaces makes many related concepts nicely transparent, for example the notions of *twisting in cohomology* and of *generalized Chern characters*.

Twisted non-abelian cohomology. A *twist* of A -cohomology (50) is what is classified by a twisted parametrization of A over some base space B [NSS12a, §4][SS20b, §2.2]. Instead of mapping into a fixed classifying spaces, a *twisted cocycle* maps into a varying classifying space that may twist and turn as one moves in the domain space. In other words, a *twisting* τ of A -cohomology theory on some X is a bundle over X with typical fiber A , and a τ -twisted cocycle is a *section* of that bundle [NSS12a, §4][ABGHR14][SS20a, §2.2]:

$$\overset{\tau\text{-twisted non-abelian generalized } A\text{-cohomology theory}}{A^\tau(X)} := \left\{ \begin{array}{ccc} & \begin{array}{c} \text{continuous section} \\ = \text{twisted cocycle} \\ \nearrow \\ X \end{array} & \\ & \begin{array}{c} \text{A-fiber bundle} \\ \xrightarrow{P} \\ X \end{array} & \\ & \begin{array}{c} \text{universal A-fiber bundle} \\ \xrightarrow{\tau} \\ BAut(A) \end{array} & \\ & \begin{array}{c} \text{classifying map for } P \\ \xrightarrow{\tau} \\ BAut(A) \end{array} & \end{array} \right\} / \underset{BAut(A)}{\sim} \text{homotopy} \quad (52)$$

$$\simeq \left\{ \begin{array}{ccc} & \begin{array}{c} \text{continuous function} \\ \xrightarrow{\tau} \\ X \end{array} & \\ & \begin{array}{c} \text{twist } \tau \\ \searrow \\ BAut(A) \end{array} & \\ & \begin{array}{c} \text{homotopy} \\ \Downarrow \\ BAut(A) \end{array} & \end{array} \right\} / \underset{BAut(A)}{\sim} \text{homotopy}$$

Here the equivalent formulation shown in the second line follows because A -fiber bundles are themselves classified by nonabelian $\text{Aut}(A)$ -cohomology (see [NSS12a, 4.11][SS20b, 2.92]), as shown on the right of the first line.

With a general concept of twisted non-abelian cohomology theories in hand, we turn to discussion of their Chern-Dold characters. At their core, these come from the rationalization approximation on coefficient/classifying spaces:

Rationalization. For X a connected space, we write

$$X \xrightarrow[\substack{\text{rationalization} \\ \text{(over the real numbers)}}]{L_{\mathbb{R}}} \mathbb{L}X \quad (53)$$

for its rationalization (e.g. [He07, 1.4]) over the real numbers. Elsewhere we use the notation $\mathbb{L}X$ more specifically for the L_{∞} -algebra that is the formal dual of the Sullivan model for X [BFM06][BMSS19, §2.1] (or in their rectified incarnation [FRS13, §1.0.2]: dg-Lie algebras as in the original [Qu69]), but by the fundamental theorem of rational homotopy theory it is expedient to conflate the two.

Chern-Dold character in abelian cohomology. Given an abelian generalized cohomology theory E^{\bullet} (49), rationalization (53) of its classifying spaces (48) induces a cohomology operation from E -cohomology theory to ordinary cohomology with coefficients in the rationalized stable homotopy groups of E :

$$\begin{array}{c} \text{Chern-Dold} \\ \text{character} \\ \text{ch}_E : E^n(X) \end{array} \begin{array}{c} \text{Brown} \\ \text{representability} \\ \simeq \end{array} \begin{array}{c} \pi_0 \text{Maps}(X, E_n) \\ \text{generalized} \\ \text{cohomology} \end{array} \xrightarrow[\pi_0 \text{Maps}(X, L_{\mathbb{R}})]{\text{rationalization}} \begin{array}{c} \pi_0 \text{Maps}(X, \mathbb{L}E_n) \\ \text{rational } E\text{-cohomology} \end{array} =: E_{\mathbb{R}}^n(X) \begin{array}{c} \text{Dold's} \\ \text{equiv-} \\ \text{alence} \\ \simeq \bigoplus_k H^{n+k}(X; \pi_k(E) \otimes_{\mathbb{Z}} \mathbb{R}), \\ \text{ordinary cohomology} \\ \text{with coefficients in} \\ \text{rationalized homotopy groups of } E \end{array} \quad (54)$$

where Dold's equivalence on the far right is due to [Do65, Cor. 4], reviewed in [Rud98, §II.3.17]. This map (54) is called the *Chern-Dold character map*, due to [Bu70]. The modern formulation above is made fully explicit in [LSW16, §2.1]; see also [HS05, §4.8][BN14, p. 17][GS17]. For example:

- (i) When $E^{\bullet} = H^{\bullet}(-; \mathbb{Z})$ is ordinary integral cohomology, its rationalization is $E_{\mathbb{R}}^{\bullet} = H^{\bullet}(-; \mathbb{R})$ and the Chern-Dold character (54) reduces to extension of scalars from integral to real cohomology, as in (1), (2).
- (ii) When $E^{\bullet} = \text{KU}^{\bullet}$ is complex topological K-theory, its rationalization is $\text{KU}_{\mathbb{R}}^{0,1} \simeq H^{\text{even, odd}}(-; \mathbb{R})$ and the Chern-Dold character (54) reduces to the ordinary Chern character (see [GSa18][GSa19] for extensive discussions).

Chern-Dold character in non-abelian cohomology.

Definition 3.1 (Non-abelian de Rham cohomology [SSS09a, §6.5][FSSSt10, §4.1]). The non-abelian de Rham cohomology of a smooth manifold X with coefficients in an L_{∞} -algebra \mathfrak{g} of finite type is the quotient of the set of dg-algebra homomorphism from the Chevalley-Eilenberg algebra $\text{CE}(\mathfrak{g})$ of \mathfrak{g} (which is the Sullivan model of a rational space) to the de Rham dg-algebra $\Omega_{\text{dR}}^{\bullet}$ of differential forms on X , quotiented by dg-algebra homotopies:

$$H_{\text{dR}}(X; \mathbb{L}\mathfrak{A}) := \text{Hom}(\text{CE}(\mathbb{L}\mathfrak{A}), \Omega_{\text{dR}}^{\bullet}(X)) / \sim = \left\{ \begin{array}{c} \text{dg-algebra homomorphism/} \\ \text{flat } \mathbb{L}\mathfrak{A}\text{-valued differential form} \\ \text{A} \\ \text{dga-homotopy/} \\ \text{coboundary} \\ \text{A}' \\ \text{dg-algebra homomorphism/} \\ \text{flat } \mathbb{L}\mathfrak{A}\text{-valued differential form} \end{array} \right\} / \text{homotopy} \quad (55)$$

Example 3.2 (Recovering ordinary de Rham cohomology). In the case that $\mathfrak{g} = \mathbb{R}[n]$ is the *line* L_{∞} -algebra concentrated in degree n , its Chevalley-Eilenberg algebra is the free graded-commutative algebra on a single generator in degree $n+1$ with vanishing differential; which is also the Sullivan model of the Eilenberg-MacLane space (47) in that degree:

$$\text{CE}(\mathbb{R}[n]) = \mathbb{R}[c_{n+1}] / (dc_{n+1} = 0) \simeq \text{CE}(\mathbb{L}K(n+1, \mathbb{Z})). \quad (56)$$

Hence dg-algebra homomorphisms out of this into a de Rham algebra are equivalently closed differential $(n+1)$ -forms:

$$\text{Hom}(\text{CE}(\mathbb{R}[1]), \Omega_{\text{dR}}^{\bullet}(X)) \simeq \Omega^n(X)_{\text{cl}}, \quad (57)$$

and dg-algebra homotopies between these are equivalently de Rham coboundaries. Therefore, the non-abelian de Rham cohomology (55) with these coefficients reduces to ordinary de Rham cohomology in that degree:

$$H_{\text{dR}}(X; \mathbb{R}[n]) \simeq H_{\text{dR}}^{n+1}(X). \quad (58)$$

Proposition 3.3 (Non-abelian de Rham theorem). *Let X be a smooth manifold and A a nilpotent topological space of finite rational homotopy type, hence with a Sullivan model $\mathbb{C}E(\mathbb{1}A)$ for its rationalization $\mathbb{1}A$ (53). Then the non-abelian cohomology (51) of X with real coefficients $\mathbb{1}A$ is equivalent to the non-abelian de Rham cohomology (55) with coefficient in $\mathbb{1}A$:*

$$\overset{\text{non-abelian}}{\underset{\text{real cohomology}}{H(X; \mathbb{1}A)}} \simeq \overset{\text{non-abelian}}{\underset{\text{de Rham cohomology}}{H_{\text{dR}}(X; \mathbb{1}A)}}. \quad (59)$$

Proof. Unwinding the definitions, the equivalence (59) reduces to the fundamental theorem of rational homotopy theory [BG76, §9.4] (reviewed as [BMSS19, Prop. 2.11]; see also [He07, Cor. 1.26]) which identifies the hom-sets in the homotopy categories of **a**) nilpotent and finite-type rational topological spaces, and **b**) the opposite of dgc-algebras. \square

Proposition 3.4 (Non-abelian de Rham theorem for stable coefficients). *Let X be a smooth manifold, and E an infinite-loop space (48). Then non-abelian de Rham cohomology (55) of X with coefficients in $\mathbb{1}E$ is equivalent to the real cohomology of X with coefficients in the rationalized homotopy groups of E :*

$$H_{\text{dR}}(X; \mathbb{1}E) \simeq \bigoplus_k H^k(X; \pi_k(E) \otimes_{\mathbb{Z}} \mathbb{R}). \quad (60)$$

Proof. The Sullivan model of an infinite loop space is the free graded algebra generated by its rationalized homotopy groups, with vanishing differential (see [FHT00, p. 143], or, from a broader perspective of rational spectra, [BMSS19, Lemma 2.25, Prop. 2.30]). This implies the claim by Example 3.2, via the ordinary de Rham theorem (e.g. [FHT00, 10.15]). \square

In conclusion:

Proposition 3.5 (Non-abelian de Rham theorem recovers Dold's equivalence). *Let X be a smooth manifold, and E (the connective spectrum of) an infinite-loop space (48). Then Dold's equivalence is equivalent to the restriction of the non-abelian de Rham theorem (Prop. 3.3) to stable coefficients (Prop. 3.4):*

$$\begin{array}{ccc} E_{\mathbb{R}}^n(X) & \xrightarrow[\simeq]{\text{Dold's equivalence}} & \bigoplus_k H^{n+k}(X; \pi_{n+k}(E) \otimes_{\mathbb{Z}} \mathbb{R}) \\ \parallel & & \uparrow \text{(60)} \\ H(X; \mathbb{1}E_n) & \xrightarrow[\text{non-abelian de Rham theorem Prop. 3.3}]{\simeq} & H_{\text{dR}}(X; \mathbb{1}E_n) \end{array} \quad (61)$$

Therefore, we obtain the following generalization of the Chern-Dold character (54):

Definition 3.6 (Chern-Dold character in non-abelian cohomology). *Let X be a smooth manifold and A a nilpotent space of finite rational type. Then the non-abelian Chern-Dold character on non-abelian cohomology theory (51) represented by A is the composite of*

- (a) the rationalization map (53) on coefficients
- (b) the non-abelian de Rham theorem 3.3:

$$\overset{\text{non-abelian}}{\underset{\text{Chern-Dold character}}{\text{ch}_A}} : \overset{\text{non-abelian}}{\underset{\text{cohomology}}{\underset{\text{with coefficients in } A}{H(X; A)}}} := \pi_0 \text{Maps}(X, A) \xrightarrow[\pi_0 \text{Maps}(X, L_{\mathbb{R}})]{\text{rationalization}} \pi_0 \text{Maps}(X, \mathbb{1}A) =: \overset{\text{non-abelian}}{\underset{\text{cohomology}}{\underset{\text{with coefficients in } \mathbb{1}A}{H(X; \mathbb{1}A)}}} \simeq \overset{\text{non-abelian}}{\underset{\text{de Rham theorem}}{\underset{\text{de Rham cohomology}}{\underset{\text{with coefficients in } \mathbb{1}A}{H_{\text{dR}}(X; \mathbb{1}A)}}}}. \quad (62)$$

Chern-Dold character in twisted non-abelian cohomology. The above constructions immediately generalize to twisted nonabelian cohomology (52) to yield the twisted non-abelian Chern character cohomology operation:

Definition 3.7 (Chern-Dold character in twisted non-abelian cohomology). The twisted non-abelian Chern-Dold character is the non-abelian Chern-Dold character (Def. 3.6) in the slice over $B\text{Aut}(A)$:

$$\text{ch}_A^\tau : H^\tau(X; A) := \pi_0 \text{Maps}_{B\text{Aut}(A)}(X, A) \xrightarrow{\text{rationalization}} \pi_0 \text{Maps}_{/B\text{Aut}(A)}(X, \mathbb{A}) =: H^{\text{tr}}(X; \mathbb{A}) \xrightarrow{\text{twisted non-abelian de Rham theorem}} H_{\text{dR}}^{\text{tr}}(X; \mathbb{A}) \quad (63)$$

τ -twisted non-abelian cohomology with coefficients in A

 τ -twisted non-abelian cohomology with coefficients in \mathbb{A}

 τ -twisted non-abelian de Rham cohomology with coefficients in \mathbb{A}

$$\left\{ \begin{array}{c} X \xrightarrow{c} A // \text{Aut}(A) \\ \tau \searrow \swarrow \\ B\text{Aut}(A) \end{array} \right\} / \sim \quad \longmapsto \quad \left\{ \begin{array}{c} \Omega_{\text{dR}}^\bullet(X) \xleftarrow{A} \text{CE}(l(A // \text{Aut}(A))) \\ \tau^* \swarrow \searrow \\ \text{CE}(l(B\text{Aut}(A))) \end{array} \right\} / \sim$$

Remark 3.8 (Charge quantization by lift through Chern-Dold character). Just as for the traditional Chern character on K-theory (see [GSa18] for a detailed account), the Chern-Dold character (54) is generally far from being surjective, and the same is true for its non-abelian (62) and its twisted non-abelian generalization (63).

(i) The obstruction to lifting de Rham form data through the Chern-Dold character maps are *integrality* conditions that disappear upon rationalization, hence are “quantization” conditions (in the original sense of Bohr-Sommerfeld quantization).

(ii) Therefore, if any given differential form data lifts through the Chern-Dold character of some twisted non-abelian A -cohomology theory, we say that that it is *quantized in A -theory*.

(iii) In typical examples the differential forms in question are flux densities, encoding charges of physical fields, and hence we speak of *charge-quantization in A -theory*. (For abelian cohomology this is discussed in [FR00][GSa19].)

3.2 Twistorial Cohomotopy theory

We now identify and study the twisted non-abelian cohomology theory whose classifying space is the Borel-equivariant twistor fibration (Def. 2.5). The main result of this section is Theorem 3.11, which shows that charge-quantization (Remark 3.8) in this *Twistorial Cohomotopy* (Prop. 3.9) imposes a shifted integrality condition (77) on Chern-Dold character forms (Corollary 3.11) matching that of (6).

Tangential $\text{Sp}(2)$ -structure. Consider smooth spin 8-manifolds X that are equipped with tangential $\text{Sp}(2)$ -structure (e.g. [SS20b, 4.48]), hence with a homotopy-lift⁵ of the classifying map of their tangent bundle to the classifying space $B\text{Sp}(2)$ of the quaternionic unitary group (Def. A.3) along its canonical inclusion i_{Sp} (38):

$$\begin{array}{ccc} X & \xrightarrow{\tau} & B\text{Sp}(2) \\ & \searrow TX & \swarrow B i_{\text{Sp}} \\ & & B\text{Spin}(8) \end{array} \quad (64)$$

8-manifold tangential $\text{Sp}(2)$ -structure
 classifying map of tangent bundle

In the intended applications, this spin 8-manifold (64) is one factor in an 11-dimensional spacetime of the form $\mathbb{R}^{2,1} \times X$ (see [FSS19b, §3]). We write ω for any affine connection on TX (“spin connection”) and write

$$p_i(\omega) \in H_{\text{dR}}^{2i}(X) \simeq H^{2i}(X; \mathbb{R}). \quad (65)$$

for the induced Pontrjagin forms (e.g. [GSa18, p. 10]).

⁵ All diagrams in the following are filled with such homotopies, but for ease of presentation we mostly suppress them, notationally.

Associated twistor-space fibration. By Prop. 2.2, a tangential $\mathrm{Sp}(2)$ -structure (64) induces, via pullback of the parametrized Hopf/twistor fibration from Def. 2.5, an S^4 -fibration E and a $\mathbb{C}P^3$ -fibration \tilde{E} over X , connected by a morphism of fibrations over X which is fiberwise the plain twistor fibration $t_{\mathbb{H}}$ (11):

$$\begin{array}{ccccc}
 \mathbb{C}P^3 & \hookrightarrow & \tilde{E} & \hookrightarrow & \mathbb{C}P^3 // \mathrm{Sp}(2) \\
 \downarrow t_{\mathbb{H}} & & \downarrow & & \downarrow t_{\mathbb{H}} // \mathrm{Sp}(2) \\
 S^4 & \xrightarrow{\quad} & E & \xrightarrow{\quad} & S^4 // \mathrm{Sp}(2) \\
 \downarrow & \text{(pb)} & \downarrow & \text{(pb)} & \downarrow \\
 \{x\} & \hookrightarrow & X & \xrightarrow{\tau} & B\mathrm{Sp}(2) \\
 & & \text{spacetime} & \text{Sp}(2)\text{-structure} & \text{classifying space} \\
 & & \downarrow TX & & \downarrow \\
 & & B\mathrm{Spin}(8) & &
 \end{array}
 \tag{66}$$

Twistorial Cohomotopy theory. A section (c, a) of the $\mathbb{C}P^3$ -fibration \tilde{E} is a cocycle in a twisted non-abelian cohomology theory (52), which we call *Twistorial Cohomotopy theory*⁶ of X . Notice that, as in (52), such a section is equivalently a lift of the classifying map τ to the parametrized twistor space:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \tilde{E} & \xrightarrow{\quad} & \mathbb{C}P^3 // \mathrm{Sp}(2) \\
 \downarrow & \text{(pb)} & \downarrow \\
 X & \xrightarrow{\tau} & B\mathrm{Sp}(2)
 \end{array} & \Leftrightarrow & \begin{array}{ccc}
 \mathbb{C}P^3 // \mathrm{Sp}(2) & & \\
 \uparrow \text{lift of } \tau & \text{lift of } \tau & \\
 X & \xrightarrow{\tau} & B\mathrm{Sp}(2)
 \end{array}
 \end{array}$$

We write

$$\mathcal{T}^\tau(X) := \left\{ \begin{array}{c} \text{universally parametrized} \\ \text{twistor space} \\ \mathbb{C}P^3 // \mathrm{Sp}(2) \\ \uparrow \text{cocycle } (c, a) \\ X \xrightarrow{\text{Sp}(2)\text{-structure}} B\mathrm{Sp}(2) \end{array} \right\} / \sim
 \tag{67}$$

for the set of homotopy classes (relative X) of such sections, and call this the *cohomology set of Twistorial Cohomotopy*, when evaluated on spin-8 manifolds with tangential $\mathrm{Sp}(2)$ -structure τ (64).

Twistor fibration as cohomology operation. Notice the direct analogy of Twistorial Cohomotopy theory (67) to J-twisted Cohomotopy theory [FSS19b, 2.1]:

$$\pi^\tau(X) := \left\{ \begin{array}{c} \text{universally parametrized} \\ \text{4-sphere} \\ S^4 // \mathrm{Sp}(2) \\ \uparrow \text{cocycle } c \\ X \xrightarrow{\text{Sp}(2)\text{-structure}} B\mathrm{Sp}(2) \end{array} \right\} / \sim
 \tag{68}$$

⁶ Not to be confused with *twistor cohomology* (see, e.g., [EPW81]). The latter is abelian cohomology of twistor space, while Twistorial Cohomotopy is non-abelian cohomology with coefficients in (Borel-equivariantized) twistor space, hence with cocycles being maps into twistor space.

and the fact that postcomposition with the parametrized twistor fibration (Def. 2.5) constitutes a cohomology operation (a natural transformation of cohomology sets) between the two:

$$\begin{array}{ccc} \text{Twistorial Cohomology} & \xrightarrow{\text{cohomology operation by parametrized twistor fibration}} & \text{J-twisted Cohomology} \\ \mathcal{F}^\tau & \xrightarrow{(t_{\mathbb{G}} // \text{Sp}(2))_*} & \pi^\tau \end{array} \quad (69)$$

Chern-Dold character in Twistorial Cohomotopy. The Chern-Dold character (63) in J-twisted 4-Cohomotopy (68) is discussed in some detail [FSS19b]. The following Prop. 3.9 is its generalization to Twistorial Cohomotopy:

Proposition 3.9 (Chern-Dold character in Twistorial Cohomotopy theory). *The Chern-Dold character (63) in Twistorial Cohomotopy (67) is of the following form:*

$$\begin{array}{ccc} \text{Twistorial Cohomology} & \xrightarrow{\text{ch}_{\mathcal{F}}} & \left\{ \begin{array}{l} F_2, \\ H_3, \\ G_4, \\ G_7 \end{array} \in \Omega^\bullet(X) \left| \begin{array}{l} dF_2 = 0 \\ dH_3 = G_4 - \frac{1}{4}p_1(\omega) - F_2 \wedge F_2 \\ dG_4 = 0 \\ dG_7 = -\frac{1}{2}(G_4 - \frac{1}{4}p_1(\omega)) \wedge (G_4 + \frac{1}{4}p_1(\omega)) - \frac{1}{4}(p_2 - (\frac{1}{2}p_1(\omega))^2) \end{array} \right. \right\} / \sim \\ \mathcal{F}^\tau(X) & \xrightarrow{\text{twisted non-abelian Chern-Dold character}} & \end{array} \\ (c, a) & \longmapsto & (c, a)^* \begin{pmatrix} f_2, \\ h_3, \\ \omega_4, \\ \omega_7 \end{pmatrix} \end{array} \quad (70)$$

Proof. By Theorem 2.14 the class of a section of the parametrized twistor fibration in rational homotopy theory is given equivalently by a dg-algebra homomorphism shown in blue in the following diagram:

$$\begin{array}{ccc} & \begin{array}{c} \text{CP}^3 // \text{Sp}(2) \\ \uparrow \\ \text{cocycle in twisted T-theory} \\ (c, a) \\ \text{cocycle in twisted Cohomotopy} \\ c \end{array} & \\ & \begin{array}{c} \text{CE}(\text{BSp}(2)) \left[\begin{array}{l} f_2, \\ h_3, \\ \omega_4, \\ \omega_7 \end{array} \right] / \left(\begin{array}{l} d f_2 = 0 \\ d h_3 = \omega_4 - \frac{1}{4}p_1 - f_2 \wedge f_2 \\ d \omega_4 = 0 \\ d \omega_7 = -(\omega_4 - \frac{1}{4}p_1) \wedge (\omega_4 + \frac{1}{4}p_1) \\ -\frac{1}{2}(p_2 - (\frac{1}{2}p_1)^2) \end{array} \right) \\ \uparrow \\ \omega_4 \leftrightarrow \omega_7 \\ \uparrow \uparrow \\ \omega_4 \leftrightarrow \omega_7 \\ \uparrow \\ \left(\begin{array}{l} d \omega_4 = 0 \\ d \omega_7 = -(\omega_4 - \frac{1}{4}p_1) \wedge (\omega_4 + \frac{1}{4}p_1) \\ -\frac{1}{2}(p_2 - (\frac{1}{2}p_1)^2) \end{array} \right) \end{array} & \\ \text{rational dg-model for cocycle in twisted T-theory} & & \\ \begin{array}{l} F_2 \leftarrow f_2 \\ H_3 \leftarrow h_3 \\ G_4 \leftarrow \omega_4 \\ 2G_7 \leftarrow \omega_7 \end{array} & & \\ \begin{array}{c} \text{CE}(\text{BSp}(2)) \left[\begin{array}{l} \omega_4, \\ \omega_7 \end{array} \right] / \left(\begin{array}{l} d \omega_4 = 0 \\ d \omega_7 = -(\omega_4 - \frac{1}{4}p_1) \wedge (\omega_4 + \frac{1}{4}p_1) \\ -\frac{1}{2}(p_2 - (\frac{1}{2}p_1)^2) \end{array} \right) \\ \uparrow \\ \frac{1}{2}p_1(\omega) \quad \chi_8(\omega) \\ \uparrow \quad \uparrow \\ \frac{1}{2}p_1 \quad \chi_8 \\ \text{CE}(\text{BSp}(2)) \end{array} & \longleftarrow & \Omega^\bullet(X) \\ \begin{array}{c} \text{S}^4 // \text{Sp}(2) \\ \uparrow \\ \text{cocycle in twisted Cohomotopy} \\ c \end{array} & \xrightarrow{t_{\mathbb{H}} // \text{Sp}(2)} & \text{CP}^3 // \text{Sp}(2) \\ \begin{array}{c} X \\ \downarrow TX \\ \text{BSpin}(8) \leftarrow \text{BSp}(2) \end{array} & \xrightarrow{\tau} & \text{S}^4 // \text{Sp}(2) \end{array} \quad (71)$$

Here the dg-algebras on the right are the Sullivan model for the Borel-equivariant twistor fibration (43) from Theorem 2.14. These being Sullivan models means that they are cofibrant as dg-algebras, which implies that all homotopy classes of rational sections are indeed represented this way. Therefore, a rational section is specified by the differential forms on X to which it pulls back the generators on the right. The condition for any such set of differential forms to arise this way is that it satisfies the same differential relations as the generators, now in the de Rham dg-algebra $\Omega^\bullet(X)$. This way the relation $d f_2 = 0$ in the Sullivan model pulls back to the relation $d F_2 = 0$ in $\Omega^\bullet(X)$ in (71), etc. \square

For use below, we record the de Rham-cohomological relations implied by the differential relations (70):

Corollary 3.10 (Cohomological relations in Twistorial Cohomotopy). *For X an 8-manifold with tangential $\mathrm{Sp}(2)$ -structure (64), let $F_2, H_3, G_4, G_7 \in \Omega^\bullet(X)$ be differential form components in the image of the Chern-Dold character in Twistorial Cohomotopy on X (Def. 3.9). Then the real/de Rham cohomology classes these represent satisfy the following relations:*

$$[G_4] - \frac{1}{4}p_1 = [F_2 \wedge F_2] \in H^4(X, \mathbb{R}), \quad (72)$$

$$0 = ([F_2 \wedge F_2] + \frac{1}{2}p_1) \cup [F_2 \wedge F_2] + \frac{1}{2}(p_2 - \frac{1}{4}p_1 \cup p_1) \in H^8(X, \mathbb{R}). \quad (73)$$

Proof. Equation (72) is the direct consequence of the second line in (70). From the fourth line of (70) we similarly get the relation

$$-[G_4 \wedge G_4] + \frac{1}{16}p_1 \cup p_1 - \mathcal{X}_8 = 0 \quad (74)$$

Plugging (72) and (40) into (74) yields (73). \square

Charge quantization in Twistorial Cohomotopy. Finally we obtain the claimed result (6):

Corollary 3.11 (Shifted integrality of G_4, F_2 in Twistorial Cohomotopy). *Let X be a spin 8-manifold with tangential $\mathrm{Sp}(2)$ -structure τ (64). Then differential form data $(F_2, H_3, G_4, G_7) \in \Omega^\bullet(X)$ which is in the image (70) of the Chern-Dold character from Prop. 3.9, hence which is charge-quantized (Remark 3.8) in Twistorial Cohomotopy (67), satisfies the following integrality conditions:*

(i) *The class of G_4 shifted by $\frac{1}{4}p_1(\omega)$ is integral, hence is the image in real cohomology of a class in integral cohomology:*

$$[G_4 + \frac{1}{4}p_1(\omega)] \in H^4(X, \mathbb{Z}) \longrightarrow H^4(X, \mathbb{R}). \quad (75)$$

(ii) *The class of F_2 is integral:*

$$[F_2] \in H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{R}). \quad (76)$$

(iii) *Hence the relation (72) is the image of such a relation in integral cohomology:*

$$[G_4 - \frac{1}{4}p_1(\omega)] = [F_2 \wedge F_2] \in H^4(X, \mathbb{Z}) \longrightarrow H^4(X, \mathbb{R}). \quad (77)$$

Proof. By Prop. 3.9 these de Rham classes are pullbacks of the generators in the Sullivan model from Theorem 2.14. By the normalization (44) there, the statement hence follows with Theorem 2.9. \square

A Quaternion-linear groups

For reference, we record some basics of quaternion-linear groups:

Definition A.1 (Special quaternion-linear group). The special quaternion-linear group

$$\mathrm{SL}(2, \mathbb{H}) := \{G \in \mathrm{Mat}(2 \times 2, \mathbb{H}) \mid \det_{\mathrm{Di}}(G) = 1\} \quad (78)$$

is the group of 2×2 quaternionic matrices with unit *Dieudonné determinant* [Di43] (review in [As96][VB20, §1]).

Remark A.2 (Size of $\mathrm{SL}(2, \mathbb{H})$). When restricted along the inclusion of complex matrices into quaternionic matrices

$$\mathrm{Mat}(2 \times 2, \mathbb{C}) \xrightarrow{i_{\mathbb{C}}} \mathrm{Mat}(2 \times 2, \mathbb{H})$$

the Dieudonné determinant does *not* reduce to the ordinary determinant, but to its absolute value:

$$\det_{\mathrm{Di}}(i_{\mathbb{C}}(A)) = \|\det(A)\|. \quad (79)$$

Accordingly, $\mathrm{SL}(2, \mathbb{H})$ (Def. A.1) is larger than the notation might suggest: For instance, it follows immediately from (79) that all complex unitary matrices have unit Dieudonné determinant. In fact, Example A.4 says that the full quaternion-unitary group (Def. A.3) is contained in $\mathrm{SL}(2, \mathbb{H})$ (84) (and hence coincides with what would otherwise be called $\mathrm{SU}(2, \mathbb{H})$).

Definition A.3 (Unitary quaternion-linear groups). Let $n \in \mathbb{N}$.

(i) The $n \times n$ quaternionic unitary group is

$$\mathrm{Sp}(n) := \mathrm{U}(n, \mathbb{H}) := \{ G \in \mathrm{GL}(n, \mathbb{H}) \mid G \cdot G^\dagger = 1 \}, \quad (80)$$

where $(-)^{\dagger}$ denotes matrix transpose combined with quaternionic conjugation.

(ii) The central product group of $\mathrm{Sp}(n_1)$ with $\mathrm{Sp}(n_2)$ is

$$\mathrm{Sp}(n_1) \cdot \mathrm{Sp}(n_2) := (\mathrm{Sp}(n_1) \times \mathrm{Sp}(n_2)) / \underbrace{\{(1, 1), (-1, -1)\}}_{\simeq \mathbb{Z}_2} \quad (81)$$

Example A.4 (Subgroups of quaternion-linear groups). We have the following canonical subgroup inclusions into special quaternion-linear (Def. A.1) and unitary quaternion-linear groups (Def. A.3):

(i) The algebra inclusion of the complex numbers into the quaternions induces:

$$\begin{array}{ccc} \mathbb{C} & \hookrightarrow & \mathbb{H} \\ \mathrm{U}(n, \mathbb{C}) & \hookrightarrow & \mathrm{U}(n, \mathbb{H}) \\ \parallel & & \parallel \\ \mathrm{U}(n) & \hookrightarrow & \mathrm{Sp}(n) \end{array} \quad (82)$$

(ii) We write

$$\begin{array}{ccc} \mathrm{Sp}(1)_L \times \mathrm{Sp}(1)_R & \hookrightarrow & \mathrm{Sp}(2) \\ (q_L, q_R) & \longmapsto & \mathrm{diag}(q_L, q_R) \end{array} \quad (83)$$

for the subgroup of $\mathrm{Sp}(2)$ given by the diagonal matrices with coefficients in unit-norm quaternions q , hence the direct product group of two copies of $\mathrm{Sp}(1)$, equipped with their left and right factor embedding, as indicated.

(iii) The unitary quaternion-linear 2×2 -matrices (Def. A.3) have Dieudonné-determinant (Def. A.1) equal to 1 [CDL00, 6.4] and hence include into the special quaternion-linear group:

$$\mathrm{Sp}(2) = \mathrm{U}(2, \mathbb{H}) \subset \mathrm{SL}(2, \mathbb{H}). \quad (84)$$

(iv) There is the canonical subgroup inclusion of symplectic-unitary groups into their central product groups (81)

$$\begin{array}{ccc} \mathrm{Sp}(n_1) & \hookrightarrow & \mathrm{Sp}(n_1) \cdot \mathrm{Sp}(n_2) \\ A & \longmapsto & [A, 1] \end{array} \quad (85)$$

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