Twistorial Cohomotopy implies Green-Schwarz anomaly cancellation

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Abstract

We characterize the integral cohomology and the rational homotopy type of the maximal Borel-equivariantization of the combined Hopf/twistor fibration, and find that subtle relations satisfied by the cohomology generators are just those that govern Hořava-Witten's proposal for the extension of the Green-Schwarz mechanism from heterotic string theory to heterotic M-theory. We discuss how this squares with the *Hypothesis H* that the elusive mathematical foundation of M-theory is based on charge quantization in tangentially twisted unstable Cohomotopy theory.

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1   Introduction and overview

The Green-Schwarz mechanism in heterotic M-theory. At the heart of M-theory (the conjectural non-perturbative completion of type IIA string theory, see [Du96][Du98][Du99]) is the proposal [Wi97a, (1.2)][Wi97b, (1.2)] that the *difference* of the classes of:

  (i) the flux density G_4 of the higher gauge field of M-theory (the *C-field*, or *3-index A-field* [CJS78]),

  (ii) $^1\!/4$th of the Pontrjagin form of the spin connection ω on spacetime Y^{11} (e.g. [KN63, §XII.4][GSa18, p. 10]), lifts to an integral class:

$$\underset{\substack{\text{C-field}\\\text{4-flux density}}}{\big[G_4\big]} \;-\; \underset{\substack{^1/2\text{ gravitational}\\\text{instanton density}}}{\big[\tfrac{1}{4}p_1(\omega)\big]} \;\in\; \underset{\substack{\text{integral cohomology of}\\\text{11-d spacetime}}}{H^4\big(Y^{11};\mathbb{Z}\big)} \xrightarrow{\;\text{rationalization}\;} \underset{\substack{\text{real cohomology of}\\\text{11-d spacetime}}}{H^4\big(Y^{11};\mathbb{R}\big).} \tag{1}$$

One motivation for this proposal [Wi97a, §2.1] comes from heterotic M-theory (Hořava-Witten theory [HW95] [Wi96][HW96][DOPW99][DOPW00][Ov02], the conjectural non-perturbative completion of heterotic string theory [GHMR85][GHMR86][AGLP12]). Here the celebrated ("first superstring revolution", see [Schw07]) *Green-Schwarz anomaly cancellation mechanism* [GSc84][CHSW85] (review in [Wi99, §2.2][Fr00]) in heterotic string theory, which in itself is understood clearly, is argued to imply, upon lift to heterotic M-theory, that the restriction of the 4-flux G_4 to an MO9-plane X^{10} inside 11-dimensional spacetime Y^{11} satisfies this relation [HW96, (1.13)]:[1]

$$\underset{\substack{\text{C-field 4-flux density}\\\text{restricted to }X^{10}}}{\big[G_4\big]\big|_{X^{10}}} \;=\; \underset{\substack{^1/2\text{ gravitational}\\\text{instanton density}}}{\big[\tfrac{1}{4}p_1(\omega)\big]\big|_{X^{10}}} \;-\; \underset{\substack{\text{gauge instanton}\\\text{density on }X^{10}}}{\big[c_2(A)\big]} \;\in\; \underset{\substack{\text{integral cohomology}\\\text{of 10d MO9-plane}}}{H^4\big(X^{10};\mathbb{Z}\big)} \xrightarrow{\;\text{rationalization}\;} \underset{\substack{\text{real cohomology}\\\text{of 10d MO9-plane}}}{H^4\big(X^{10};\mathbb{R}\big),} \tag{2}$$

where A (the *gauge field*) is a connection on an E_8-principal bundle over X^{10}, and $c_2(A)$ is its second Chern-form. But the summand $[c_2(A)]$ *is* integral by itself: it is the real image of the second Chern class of the E_8-bundle. Therefore, (2) implies that (1) holds at least upon restriction to MO9-planes; and it suggests [DMW00][ES03][DFM03] [Sa06b][FSS14a] that the integral 4-class in (1) is to be thought of as the second Chern class of an extension $\widetilde{A}$ of the E_8-gauge field from X^{10} to all of Y^{11}:

$$\underset{\substack{\text{C-field}\\\text{4-flux density}}}{\big[G_4\big]} \;-\; \underset{\substack{^1/2\text{ gravitational}\\\text{instanton density}}}{\big[\tfrac{1}{4}p_1(\omega)\big]} \;=\; \underset{=:a}{\underbrace{-\underset{\substack{\text{gauge instanton density of}\\\text{auxiliary gauge field on }X^{11}}}{\big[c_2(\widetilde{A})\big]}}} \;\in\; \underset{\substack{\text{integral cohomology of}\\\text{11-d spacetime}}}{H^4\big(Y^{11};\mathbb{Z}\big)} \xrightarrow{\;\text{rationalization}\;} \underset{\substack{\text{real cohomology of}\\\text{11-d spacetime}}}{H^4\big(Y^{11};\mathbb{R}\big).} \tag{3}$$

Open problem. Despite the tight web of hints and consistency checks like these, actually formulating M-theory remains an open problem (e.g., [Du96, 6][HLW98, p. 2][Du98, p. 6][NH98, p. 2][Du99, p. 330][Mo14, 12][CP18, p. 2][Wi19, @21:15][Du19, @17:14]). In particular:

  (i) The conditions (1) and (2) had not actually been derived from any theory (see the comments around [HW96, (1.13)] and [Wi97a, (2.1)]).

  (ii) The ontology of the gauge field on Y^{11} in (3) has remained mysterious, as no such gauge field is seen in 11-dimensional supergravity [CJS78][D'AF82][CDF91], which, however, is famously argued to be the low-energy limit of M-theory [Wi95].

By Charge quantization in generalized cohomology? On the other hand, the Green-Schwarz mechanism in perturbative string theory is well understood as an index-theoretic phenomenon, resulting from charge quantization in a generalized cohomology theory, namely in K-theory [Fr00][Cl05][Bu11]. This mathematical understanding has been most fruitful, spawning understanding of elliptic genera (e.g. [HLZ07][CHZ11]), twisted higher bundles/gerbes (e.g. [SSS09b][Wa13]), Hermitian and generalized geometry (e.g. [GF16]), and more.

There have been various proposals for lifting this situation to heterotic M-theory, understanding also the shifted integrality condition (3) as an effect of charge quantization in *some* generalized cohomology theory [DFM03][HS05] [Sa05a][Sa05b][Sa06a][Sa10][FSS14a][FSS14b], which then would control M-theory in generalization of how K-theory controls string theory (for the latter, see [GSa19] and references therein). However, while advancements in understanding have certainly been made, the situation had remained inconclusive.

---

[1]Our normalization convention for G_4 absorbs a factor of $^{-1}\!/2\pi$ compared to [Wi96].

Non-abelian characters. Our strategy is to invoke the further generalization of generalized cohomology to *non-abelian cohomology* [To02][RS12][SSS12][NSS12a][NSS12b][FSS19b][SS20b]. In §3.1 below we discuss how the *Chern-Dold character* in generalized cohomology ([Bu70], see [LSW16, §2.1]) extends to non-abelian cohmology theories, where it is given by passage to *rational homotopy theory*, here over the real numbers [FSS20c, §3.2][2]:

$$
\begin{array}{ccc}
\overset{E\text{-cohomology}}{E^\bullet(X)} & \xrightarrow{\text{Non-abelian character map}} & \overset{\text{Non-abelian real cohomology}}{H^\bullet(X; L_{\mathbb{R}}E)} \\
\wr\! & & \wr\! \\
\pi_{-\bullet}\mathrm{Maps}\big(X, \underbrace{E}_{\text{classifying space}}\big) & \xrightarrow[\mathbb{R}\text{-rationalization}]{} & \pi_{-\bullet}\mathrm{Maps}\big(X, \underbrace{L_{\mathbb{R}}E}_{\mathbb{R}\text{-rationalized classifying space}}\big)
\end{array}
\tag{4}
$$

For X a smooth manifold, the right hand side of (4) is a subquotient of the set of differential forms on X (Prop. 3.3); thus the image of the nonabelian character identifies equivalence classes of differential forms satisfying certain conditions. If these forms are interpreted as flux densities, then these are **non-abelian charge-quantization** conditions. For example, in the abelian case $E = \mathrm{KU}, \mathrm{KO}$, the Chern-Dold character (4) reduces to the traditional Chern character in K-theory [FSS20c, Ex. 4.13, 4.14], and the corresponding charge quantization condition is that thought to hold for RR-fields/D-brane charges in type II/I string theory (for more discussion see [GSa18][GSa19]).

Charge quantization in J-twisted Cohomotopy. The most fundamental non-abelian cohomology theory is *Cohomotopy theory* [Bo36][Sp49][Pe56][Ta09][KMT12] whose classifying spaces are the *n*-spheres $S^n \simeq B(\Omega S^n)$. Accordingly, twisted Cohomotopy is classified by spherical fibrations, and we say *J-twisted Cohomotopy* [FSS19b] [FSS19c] for twisting by the unit sphere fibration in the tangent bundle. The main theorem of [FSS19b, 3.9] says that the Chern-Dold character (4) in J-twisted 4-Cohomotopy encodes Witten's shifted C-field flux quantization condition (1):

$$
\underset{\substack{\text{manifold} \\ \text{with} \\ \text{tangential Sp(2)-structure } \tau}}{\overset{\substack{\text{J-twisted} \\ \text{4-Cohomotopy}}}{\pi^\tau(X)}} \xrightarrow[\mathrm{ch}]{\substack{\text{Non-abelian} \\ \text{character map}}}
\left\{
\begin{array}{l}
G_4, \\
G_7
\end{array}
\in \Omega^\bullet(X)
\;\middle|\;
\begin{array}{l}
\overset{\substack{\text{C-field} \\ \text{4-flux}}}{d}\; \overset{\substack{\text{shifted flux quantization (1)}}}{G_4 = 0, \; [G_4] - [\tfrac{1}{4}p_1(\omega)] \in H^4(X; \mathbb{Z})} \\[4pt]
d\, 2G_7 = -\big(G_4 - \tfrac{1}{4}p_1(\omega)\big) \wedge \big(G_4 + \tfrac{1}{4}p_1(\omega)\big) \\[4pt]
\overset{\substack{\text{dual} \\ \text{7-flux}}}{} \quad -\tfrac{1}{2}\big(p_2(\omega) - \tfrac{1}{4}\big(p_1(\omega)\big)^2\big)
\end{array}
\right\}
\tag{5}
$$

In fact, further constraints are implied, matching a whole list of further topological conditions expected in M-theory (see [FSS19b, Table 1]). This motivates, following [Sa13, §2.5], **Hypothesis H**: *Charge quantization in M-theory happens in J-twisted Cohomotopy theory* [FSS19b][FSS19c][SS19a][BSS19][SS19b][SS20b]. While J-twisted Cohomotopy in degree 4 alone (5) does not reflect the appearance of a heterotic gauge field as in (2), its lifting to 7-Cohomotopy, through the quaternionic Hopf fibration $h_{\mathbb{H}}$, does encode structure expected on heterotic M5-branes [FSS19c][FSS19d][FSS20a][FSS20b]. Therefore we turn attention to an intermediate lift:

Charge quantization in Twistorial Cohomotopy. In between J-twisted Cohomotopy in degrees 4 and 7, when lifting along the quaternionic Hopf fibration, we find *Twistorial Cohomotopy* (§3.2): The twisted non-abelian cohomology theory classified by the Borel-equivariantized *twistor space* $\mathbb{C}P^3$ (§2.1). Our main Theorem 2.14, illustrated in *Figure I*, implies (Corollary 3.11) that charge quantization in Twistorial Cohomotopy
(i) makes the class of an $S\big(U(1)^2\big)$-gauge field $\widetilde{A}$ appear, hence a "heterotic line bundle" [AGLP12][BBL17], and
(ii) enforces on this gauge field Hořava-Witten's Green-Schwarz mechanism (3) in heterotic M-theory:

$$
\underset{\substack{\text{manifold} \\ \text{with} \\ \text{tangential Sp(2)-structure } \tau}}{\overset{\substack{\text{Twistorial} \\ \text{Cohomotopy}}}{\mathscr{T}^\tau(X)}} \xrightarrow[\mathrm{ch}]{\substack{\text{Twisted Non-abelian} \\ \text{character map}}}
\left\{
\begin{array}{l}
F_2, \\
G_4, \\
G_7
\end{array}
\in \Omega^\bullet(X)
\;\middle|\;
\begin{array}{l}
\overset{\substack{\text{1st Chern form of} \\ \text{heterotic line bundle}}}{d}\; \overset{\substack{\text{2nd Chern class of corresponding} \\ S(U(1)^2)\text{-gauge field } \widetilde{A}}}{F_2 = 0, \quad -[F_2 \wedge F_2] \in H^4(X; \mathbb{Z})} \\[4pt]
\overset{\text{C-field 4-flux}}{d}\; \overset{\substack{\text{Hořava-Witten's Green-Schwarz mechanism (3)}}}{G_4 = 0, \; [G_4] - [\tfrac{1}{4}p_1(\omega)] = [F_2 \wedge F_2] \in H^4(X; \mathbb{Z})} \\[4pt]
d\, 2G_7 = -\big(G_4 - \tfrac{1}{4}p_1(\omega)\big) \wedge \big(G_4 + \tfrac{1}{4}p_1(\omega)\big) \\[4pt]
\overset{\substack{\text{dual} \\ \text{7-flux}}}{} \quad -\tfrac{1}{2}\big(p_2(\omega) - \tfrac{1}{4}\big(p_1(\omega)\big)^2\big)
\end{array}
\right\}
\tag{6}
$$

---

[2]For general introduction and review of rational homotopy theory [Qu69] and its Sullivan models [Su77] see for instance [FHT00][He07], for brief discussion in our context see [FSS16a, §A][FSS17, §2.2] and for extensive details see the companion article [FSS20c]. As in all applications to differential geometry and physics, we consider rational homotopy theory over the real numbers [FSS20c, Rem. 3.64], as in [DGMS75][GM13]. Notice that in the supergravity literature these real Sullivan models are known as "FDA's" (following [vN82][D'AF82]); for details and translation see [FSS13][FSS16a][FSS16b][HSS18][BMSS19] with review in [FSS19a].

Here X (6) denotes a spacetime manifold with $\mathrm{Sp}(2) \hookrightarrow \mathrm{Spin}(8)$-structure (65), as befits backgrounds expected in M-theory compactified on 8-manifolds (see [FSS19b, Rem. 3.1] for pointers). This reduction is a requirement/prediction of *Hypothesis H* by Prop. 2.2 below. Besides the GS-anomaly cancellation presented here and in [SS20c], this turns out to imply several M5-brane consistency conditions [FSS19c][FSS20a][SS20a].

The crux of the proof of (6) is this cohomological analysis of the $\mathrm{Sp}(2)$-equivariantized Hopf/twistor fibration (§2):

$$
\begin{array}{ccccccc}
S^7 /\!\!/ \mathrm{Sp}(2) & \xleftrightarrow{\ \simeq\ }_{\text{coset space realization}} & B\mathrm{Sp}(1) & \overset{\text{exc. iso}}{\simeq} & B\mathrm{SU}(2)_L & \times & * \\
\downarrow & & \downarrow & & & & \downarrow {\scriptstyle Be} \\
\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2) & \xleftrightarrow{\ \simeq\ } & B(\mathrm{Sp}(1)\times\mathrm{U}(1)) & \overset{\text{exc. iso}}{\simeq} & B\mathrm{SU}(2)_L & \times & B\mathrm{U}(1)_R \\
\downarrow & & \downarrow & & & & \downarrow \\
& & & & & & BS(\mathrm{U}(1)^2) \quad {\scriptstyle B(c\mapsto\mathrm{diag}(c,\bar c))} \\
S^4 /\!\!/ \mathrm{Spin}(5) & \xleftrightarrow{\ \simeq\ } & B\mathrm{Spin}(4) & \overset{\text{exc. iso}}{\simeq} & B\mathrm{SU}(2)_L & \times & B\mathrm{SU}(2)_R
\end{array}
$$

Labels (green): coset space realization; exceptional isomorphism; Borel-equivariant complex Hopf fibration $h_{\mathbb{C}} /\!\!/ \mathrm{Sp}(2)$; Borel-equivariant quaternionic Hopf fibration $h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)$; Borel-equivariant twistor fibration $t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)$.

$$
\begin{array}{ccccc}
H^4(\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2);\mathbb{Z}) & \xleftrightarrow{\ \simeq\ } & H^4(B\mathrm{Sp}(1)\times\mathrm{U}(1);\mathbb{Z}) & \xleftrightarrow{\ \simeq\ } & H^4(B\mathrm{SU}(2);\mathbb{Z})\oplus H^4(B\mathrm{U}(1);\mathbb{Z}) \\
\uparrow & & & & \uparrow \\
-a & = & c_1 \cup c_1 & = & c_1^R \cup c_1^R \\
\uparrow & & & & \\
\widetilde{\Gamma}_4 - \widetilde{\Gamma}_4^{\mathrm{vac}} & = & \tfrac{1}{2}\mathcal{X}_4 - \tfrac{1}{4}p_1 & = & -c_2^R \\
\widetilde{\Gamma}_4 & = & \tfrac{1}{2}\mathcal{X}_4 + \tfrac{1}{4}p_1 & = & c_2^L \\
\widetilde{\Gamma}_4^{\mathrm{vac}} & = & \tfrac{1}{2}p_1 & = & c_2^L + c_2^R \\
H^4(S^4 /\!\!/ \mathrm{Spin}(5);\mathbb{Z}) & \xleftrightarrow{\ \simeq\ } & H^4(B\mathrm{Spin}(4);\mathbb{Z}) & \xleftrightarrow{\ \simeq\ } & H^4(B\mathrm{Sp}(1);\mathbb{Z})\oplus H^4(B\mathrm{Sp}(1);\mathbb{Z})
\end{array}
$$

Orange labels: universal gauge instanton density in heterotic M-theory; Hořava-Witten's Green-Schwarz mechanism in heterotic M-theory; 2nd Chern class of $S(\mathrm{U}(1)^2)$-field; $\big(B(c\mapsto\mathrm{diag}(c,\bar c))\big)^*$.

Blue labels: pullback in cohomology along Borel-equivariant twistor fibration $(t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2))^*$; shifted integral C-field flux relative to background flux; universal shifted integral C-field flux; C-field background flux; right 2nd Chern class; left 2nd Chern class [FSS19b, 3.9]; total 2nd Chern class.

(7)

Figure I. Integral cohomology of Borel-equivariant Hopf/twistor fibration and its interpretation under Hypothesis H.

  (i) The top part shows the equivalent incarnations of the Borel-equivariant Hopf/twistor fibration (Def. 2.5) according to Prop. 2.7.

  (ii) The bottom part shows the corresponding identifications of the integral cohomology generators (32) according to [FSS19b, 3.9].

  (iii) This makes manifest, shown in the middle of the diagram, how these generators pull back along the fibration (Theorem 2.9), which

  (iv) allows to normalize the generators in the Sullivan model for the rational homotopy type of the fibrations, below in Theorem 2.14.

  (v) Blue labels indicate the interpretation of the bottom generators as universal fluxes in M-theory, according to [FSS19b][FSS20a];

  (vi) while orange label indicate the interpretation of the new top generators as universal fluxes in heterotic M-theory, discussed in §3.

Conclusion and Outlook.

The heterotic gauge field. It is remarkable that the class of a gauge field $\widetilde{A}$, which had remained mysterious in (3), does appear from charge-quantization in Twistorial Cohomotopy, according to (6).

(i) Missing generality? Of course, the gauge field in (6) has the abelian structure group $G = S\big(\mathrm{U}(1)^2\big)$ instead of the non-abelian structure group $G = E_8$ that could be expected to apply to (3). In terms of characteristic classes, this means that charge quantization in Twistorial Cohomotopy constrains the class $a = [c_2(\widetilde{A})] \in H^4\big(X^{11};\mathbb{Z}\big)$, which for $G = E_8$ may be *any* element in degree four integral cohomology (since $\tau_{11} BE_8 \simeq_{\mathrm{whe}} \tau_{11} K(\mathbb{Z};4)$, e.g. [DFM03, 3.2]), to factor as minus a cup square of an element in degree two integral cohomology. This might indicate that Twistorial Cohomotopy as presented does not capture full heterotic M-theory; or that one should look for other factorizations of the quaternionic Hopf fibration, or for variants of the construction presented here.

(ii) Or predictive constraint? On the other hand, it is curious to notice that in heterotic string *phenomenology* the reduction of the heterotic gauge bundle along the inclusions

$$\big(\mathrm{U}(1)\big)^{n-1} \simeq S\big(\mathrm{U}(1)^n\big) \subset \mathrm{SU}(n) \subset E_8, \qquad \text{for } 2 \le n \le 5, \tag{8}$$

has led to a little revolution in string phenomenology [AGLP11][AGLP12]. These *heterotic line bundle models* turn out to be an abundant source of low energy theories with the *exact* field content of the (minimally supersymmetric) standard model of particle physics (up to decoupled and ultra-heavy fields), amenable to effective computational classification [ACGLP14][HLLS13][BBL17][GW19] (used for $n = 4, 5$ in the observable sector, while our $n = 2$ is used in the hidden sector [ADO20a, §4.2][ADO20b, §2.2][DM21][DM22]). Before considering the reduction (8), only a small handful of hand-crafted semi-realistic models were known.

Notice that this works because the structure group of the heterotic gauge bundle is part of what *breaks* E_8 down to the low-energy gauge group: the latter is within the commutant of the former in E_8. Therefore, realistic phenomenology does not require $\widetilde{A}$ in (3) to be in a non-abelian GUT-group – in fact it must instead be complementary to (be in the commutant within E_8 of) the low-energy gauge group ($\widetilde{A}$ is a background field/vev, not the dynamical gauge field fluctuating about it); and analysis of heterotic line bundle models indicates that restricting $\widetilde{A}$ to be reduced along (8) narrows in heterotic M-theory onto its phenomenologically realistic sector.

This might indicate that *Hypothesis H* captures not only mathematical but also phenomenological constraints of M-theory.

The degree-8 polynomial. Beyond encoding the shifted heterotic flux quantization in the first two lines of (6), the third line there shows that charge-quantization in Twistorial Cohomotopy also enforces (Corollary 3.10) the trivialization of this 8-class:

$$
\begin{aligned}
I_8^H &:= \big([G_4] - \tfrac{1}{4}p_1\big) \cup \big([G_4] + \tfrac{1}{4}p_1\big) + \tfrac{1}{2}\big(p_2 - \tfrac{1}{4}p_1 \cup p_1\big) \\
&= \big([F_2 \wedge F_2] + \tfrac{1}{2}p_1\big) \cup [F_2 \wedge F_2] + \tfrac{1}{2}\big(p_2 - \tfrac{1}{4}p_1 \cup p_1\big) \quad \in H^8(X, \mathbb{R}).
\end{aligned}
\tag{9}
$$

In the form of the first line of (9), the condition $I_8^H = 0$ is inherited from charge-quantization in J-twisted Cohomotopy (5). We had shown previously that this condition guarantees the vanishing *under general conditions* [3] of
(a) the anomaly in the Hopf-WZ term on the M5-brane [FSS19c],
(b) the total remaining anomaly of the M5-brane [SS20a].

In the form of the second line in (9) – now equivalently re-expressed in terms of the emergent heterotic gauge flux instead of the G_4-flux – this class is seen to be closely related to the 8-class denoted $\widehat{I_8}$ in [HW96, (1.10)]: Up to global and relative rescaling of $\widehat{I_8}$ (as on the bottom of [HW96, p. 15]) both are related by

$$I_8^H = \widehat{I_8} - \tfrac{1}{4}p_1 \cup p_1. \tag{10}$$

It might be interesting to understand the potential significance of this relation. Notice that
(i) in [HW96] there is no condition that $\widehat{I_8}$ should vanish;
(ii) the shift in (10) is what makes I_8^H an integral class (using (41) and (76));
(iii) it is expected [Mos08] that $\widehat{I_8}$ is just approximate: it receives an infinite but unknown tower of corrections;
(iv) while *Hypothesis H* suggests that $I_8^H = 0$ is a statement about fully-fledged M-theory.

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[3]Both cancellations had previously been discussed only subject to tacit assumptions on the C-field; see [FSS19b, p. 2] and [SS20a, (6)].

2 Borel-equivariant Hopf/twistor fibration

The *twistor fibration*

$$S^2 \longrightarrow \mathbb{C}P^3 \atop \downarrow \atop S^4$$

(due to [At79, §III.1], see also, e.g., [Br82, §1][AS13][ABS19, §6]) or *Penrose fibration* (as in [ES85]), is the canonical map $\mathbb{C}P^3 \to \mathbb{H}P^1$ (under the identification $\mathbb{H}P^1 \simeq S^4$, recalled below as Prop. 2.1) that sends complex lines to the quaternionic lines which they span [At79, §III (1.1)]. While, as the name suggests, this is traditionally motivated from the role of $\mathbb{C}P^3$ as a *twistor space* (a division-algebraic account is in [BC88]), our interest in the twistor fibration here comes from its appearance as an intermediate stage of the *quaternionic Hopf fibration* [GWY83, §6]. We observe that it is given by the following iterative quotienting by multiplicative groups in the four real normed division algebras (reals $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$, octonions $\mathbb{O}$):

$$
\begin{array}{llll}
S^1 & \simeq & \mathbb{C}^\times/\mathbb{R}_+^\times & & & & & & \text{for} \\
& & & S^7 & \simeq & (\mathbb{R}^8\backslash\{0\})/\mathbb{R}_+^\times & \ni & \{v\cdot t \,|\, t \in \mathbb{R}_+^\times\} & & v \neq 0 \\
& & & & & \text{7d complex} & & & & \cap \\
S^2 & \simeq & \mathbb{H}^\times/\mathbb{C}^\times & h_\mathbb{C} & \text{Hopf fibration} & & & & & \mathbb{R}^8 \\
& & & \mathbb{C}P^3 & \simeq & (\mathbb{C}^4\backslash\{0\})/\mathbb{C}^\times & \ni & \{v\cdot z \,|\, z \in \mathbb{C}^\times\} & & \mathbb{C}^4 \quad (11)\\
S^4 & \simeq & \mathbb{O}^\times/\mathbb{H}^\times & t_\mathbb{H} & \text{twistor fibration} & & & & & \\
& & & \mathbb{H}P^1 & \simeq & (\mathbb{H}^2\backslash\{0\})/\mathbb{H}^\times & \ni & \{v\cdot q \,|\, q \in \mathbb{H}^\times\} & & \mathbb{H}^2 \\
& & & * & \simeq & (\mathbb{O}^1\backslash\{0\})/\mathbb{O}^\times & \ni & \{v\cdot o \,|\, o \in \mathbb{O}^\times\} & & \mathbb{O}
\end{array}
$$

Alternatively, in its coset-space realization

$$\mathrm{SU}(4)/\mathrm{U}(2) \longrightarrow \mathrm{SO}(5)/\mathrm{U}(2) \atop \downarrow \atop \mathrm{SO}(5)/\mathrm{SO}(4)$$

the twistor fibration is also called *Calabi-Penrose fibration* (following [La85, §3], see also [Lo89] and see [No08, 2.31] for a review of Calabi's construction [Ca67][Ca68]). We observe that the $\mathrm{Sp}(2)$-coset realization ([On60, Table 1], see [GO93, Table 3]) of the Hopf/twistor fibrations is given as follows (see also [FSS19b, (73)]):

$$
\begin{array}{ccc}
S^7 & \simeq & \mathrm{Sp}(2)/\,\mathrm{Sp}(1)_L \\
& h_\mathbb{C} & \text{7d complex Hopf fibration} \\
\mathbb{C}P^3 & \simeq & \mathrm{Sp}(2)/(\mathrm{Sp}(1)_L \times \mathrm{U}(1)_R) \qquad (12)\\
& t_\mathbb{H} & \text{Calabi-Penrose twistor fibration} \\
S^4 & \simeq & \mathrm{Sp}(2)/(\mathrm{Sp}(1)_L \times \mathrm{Sp}(1)_R)
\end{array}
$$

where the maps are induced by the canonical subgroup inclusions (recalled in Example A.4).

We discuss in this paper the enhancement (Prop. 2.2 below) of these classical fibrations (11) (12) to Borel-equivariant *parametrized fibrations* (Def. 2.5 below) over the classifying space of the group $\mathrm{Sp}(2)$ (recalled as Def. A.3 below), generalizing the analogous discussion for just the quaternionic Hopf fibration in [FSS19b][FSS19c]. The main results are Theorem 2.9 and Theorem 2.14 below, which characterize the integral cohomology and the rational homotopy type of the Borel-equivariant Hopf/twistor fibrations (29) (generalizing the result for just the quaternionic Hopf fibration from [FSS19b, 3.19]).

2.1 Construction

Here we determine the maximal symmetry group of the joint Hopf/twistor fibrations (Prop. 2.2, Remark 2.3), construct the corresponding Borel-equivariantization (Def. 2.5) and characterize its integral cohomology (Theorem 2.9).

The following isomorphisms (Prop. 2.1) are classical, but since the third of these is rarely made explicit in the literature, we spell out a proof:

Proposition 2.1 (Equivariant identification of 4-sphere with quaternionic projective space). *There are canonical isomorphisms*

(i) *of topological spaces:*
$$S^4 \xrightarrow[\simeq]{\alpha} \mathbb{H}P^1 \; ; \tag{13}$$

(ii) *of topological groups:*
$$\mathrm{Spin}(5) \xrightarrow[\simeq]{\gamma} \mathrm{Sp}(2) \; ; \tag{14}$$

(iii) *of canonical topological group actions:*
$$\mathrm{Spin}(5) \circlearrowright S^4 \xrightarrow[\simeq]{(\gamma,\alpha)} \mathrm{Sp}(2) \circlearrowright \mathbb{H}P^1 \; . \tag{15}$$

Proof. Quaternionic 2-component spinor formalism provides an isomorphism of real quadratic vector spaces ([KT82], streamlined review in [BH09][VB20][FSS20b, §3.2])

$$
\left(\mathbb{R}^{5,1}, \eta\right) \xrightarrow{\quad\simeq\quad} \left(\mathrm{Mat}^{\mathrm{herm}}\left(2 \times 2, \mathbb{H}\right), -\det\right) \tag{16}
$$
$$
\begin{bmatrix} x^0, \\ x^1, \\ x^2, \\ \vdots \\ x^5 \end{bmatrix}
\longmapsto
\begin{pmatrix} x^0 - x^1 & x^2 + \mathrm{i}x^3 + \mathrm{j}x^4 + \mathrm{k}x^5 \\ x^2 - \mathrm{i}x^3 - \mathrm{j}x^4 - \mathrm{k}x^5 & x^0 + x^1 \end{pmatrix}
$$

from **(a)** 6d Minkowski spacetime $\mathbb{R}^{5,1}$ with metric $\eta := \mathrm{diag}(-1,+1,\cdots,+1)$ to **(b)** the vector space of 2-by-2 quaternionic matrices which are hermitian, $A^\dagger = A$, with quadratic form the negative of the determinant operation. Under this identification, the canonical action of $\mathrm{Spin}(5,1)$ on $\mathbb{R}^{5,1}$ (through that of $\mathrm{SO}(5,1)$) translates to the conjugation action of $\mathrm{SL}(2,\mathbb{H})$ (79):

$$\mathrm{Spin}(5,1) \circlearrowright \mathbb{R}^{5,1} \xrightarrow{\quad\simeq\quad} \mathrm{SL}(2,\mathbb{H}) \circlearrowright \mathrm{Mat}^{\mathrm{herm}}\left(2 \times 2, \mathbb{H}\right) \tag{17}$$
$$A \mapsto G \cdot A \cdot G^\dagger$$

Now consider the restriction of this situation to the Euclidean spatial slice $\mathbb{R}^5 \hookrightarrow \mathbb{R}^{5,1}$ determined by $x^0 = 0$. Under the isomorphism (16), this clearly corresponds to restriction to the *traceless* hermitian matrices:

$$\left(\mathbb{R}^5, g\right) \xrightarrow{\quad\simeq\quad} \left(\mathrm{Mat}^{\mathrm{herm}}_{\mathrm{trless}}\left(2 \times 2, \mathbb{H}\right), -\det\right) \tag{18}$$

Notice here, from direct inspection (see also [BH09, Prop. 5]), that

$$A \in \mathrm{Mat}^{\mathrm{herm}}_{\mathrm{trless}}\left(2 \times 2, \mathbb{H}\right) \qquad \Rightarrow \qquad A \cdot A = -\det(A) \cdot \mathrm{I}. \tag{19}$$

Morever, the subgroup $\mathrm{Spin}(5) \subset \mathrm{Spin}(5,1)$ which fixes this subspace corresponds under (17) to that subgroup of $\mathrm{SL}(2,\mathbb{H})$ whose conjugation operation preserves traceless matrices. Since this means, equivalently, to act trivially on their orthogonal complement, given by the pure trace matrices, i.e. the real multiples of the 2-by-2 identity matrix $\mathrm{I} := \mathrm{Id}_{\mathbb{H}^2}$:

$$G \cdot \mathrm{I} \cdot G^\dagger = \mathrm{I} \qquad \Leftrightarrow \qquad G \cdot G^\dagger = \mathrm{I},$$

we see, using (85), that this is the quaternionic unitary group $\mathrm{Sp}(2) := \mathrm{U}(2,\mathbb{H})$ (81), hence that (17) restricts as follows:

$$\mathrm{Spin}(5) \circlearrowright \mathbb{R}^5 \xrightarrow{\quad\simeq\quad} \mathrm{Sp}(2) \circlearrowright \mathrm{Mat}^{\mathrm{herm}}_{\mathrm{trless}}\left(2 \times 2, \mathbb{H}\right). \tag{20}$$

Analogously, the further restriction to the unit sphere in $\mathbb{R}^5$ corresponds, under (16) and in view of (19), to those matrices which are all of:

(a) hermitian: $A^\dagger = A$, **(b)** traceless: $\mathrm{tr}(A) = 0$, **(c)** unitary: $A \cdot A = 1$.

From (c) it follows that the matrix

$$P_A := \tfrac{1}{2}\left(\mathrm{I} - A\right)$$

is a projector, $P_A \cdot P_A = P_A$; and from (b) it follows that this projector has unit rank:

$$\mathrm{tr}(P_A) = \tfrac{1}{2}\left(\underbrace{\mathrm{tr}(\mathrm{I})}_{=2} - \underbrace{\mathrm{tr}(A)}_{=0}\right) = 1.$$

Here a unit-rank projector is one for which there exists $v_A \in \mathbb{H}^2 \setminus \{0\}$ such that

$$P_A = \tfrac{1}{\|v_A\|^2}\, v_A \cdot v_A^\dagger.$$

Noticing that P_A, and hence $A = \mathrm{I} - 2P_A$, thus depends on v_A exactly only through the quaternionic line that it spans, we have thus found the following identification of the 4-sphere with quaternionic projective space:

$$S^4 \simeq S(\mathbb{R}^5) \xrightarrow{\ \simeq\ } \mathrm{Mat}_{\mathrm{trless}}^{\mathrm{herm}}\!\left(2 \times 2, \mathbb{H}\right) \cap \mathrm{U}(2,\mathbb{H}) \xleftarrow{\ \simeq\ } \mathbb{H}P^1 \tag{21}$$
$$\mathrm{I} - \tfrac{2}{\|v\|}\, v \cdot v^\dagger \qquad \longleftarrow\!\shortmid \qquad [v]$$

Finally, under the isomorphism on the right of (21) the canonical $\mathrm{Sp}(2)$-action on $\mathbb{H}P^1$

$$\mathrm{Sp}(2) \times \mathbb{H}P^1 \longrightarrow \mathbb{H}P^1$$
$$\left(A, [v]\right) \longmapsto \left[A \cdot v\right]$$

is manifestly identified with the conjugation action (17). This implies the claim (iii), by (20). $\qquad\square$

Summarizing (16), (20) & (21):

$\mathrm{Spin}(5,1) \underset{\mathrm{can}}{\circlearrowright} \mathbb{R}^{5,1} \quad \simeq \quad \mathrm{SL}(2,\mathbb{H}) \underset{\mathrm{adj}}{\circlearrowright} \left\{A \in \mathrm{Mat}_{2\times2}(\mathbb{H}) \,\middle|\, A^\dagger = A\right\}$

$\mathrm{Spin}(5) \underset{\mathrm{can}}{\circlearrowright} \mathbb{R}^5 \quad \simeq \quad \underbrace{\mathrm{U}(2,\mathbb{H})}_{} \underset{\mathrm{adj}}{\circlearrowright} \left\{A \in \mathrm{Mat}_{2\times2}(\mathbb{H}) \,\middle|\, A^\dagger = A,\, \mathrm{tr}(A) = 0\right\}$

$= \mathrm{Sp}(2)$

$\mathrm{Spin}(5) \underset{\mathrm{can}}{\circlearrowright} S^4 \quad \simeq \quad \overbrace{\mathrm{U}(2,\mathbb{H})}^{} \underset{\mathrm{adj}}{\circlearrowright} \underbrace{\left\{A \in \mathrm{Mat}_{2\times2}(\mathbb{H}) \,\middle|\, A^\dagger = A,\, \mathrm{tr}(A) = 0,\, A^\dagger \cdot A = \mathrm{I}\right\}}_{\simeq \mathbb{H}P^1}$

Of course, there is the analogous situation over the complex numbers:

$\mathrm{Spin}(3,1) \underset{\mathrm{can}}{\circlearrowright} \mathbb{R}^{3,1} \quad \simeq \quad \mathrm{SL}(2,\mathbb{C}) \underset{\mathrm{adj}}{\circlearrowright} \left\{A \in \mathrm{Mat}_{2\times2}(\mathbb{C}) \,\middle|\, A^\dagger = A\right\}$

$\mathrm{Spin}(3) \underset{\mathrm{can}}{\circlearrowright} \mathbb{R}^3 \quad \simeq \quad \mathrm{SU}(2,\mathbb{C}) \underset{\mathrm{adj}}{\circlearrowright} \left\{A \in \mathrm{Mat}_{2\times2}(\mathbb{C}) \,\middle|\, A^\dagger = A,\, \mathrm{tr}(A) = 0\right\}$

$\mathrm{Spin}(3) \underset{\mathrm{can}}{\circlearrowright} S^3 \quad \simeq \quad \mathrm{SU}(2,\mathbb{C}) \underset{\mathrm{adj}}{\circlearrowright} \underbrace{\left\{A \in \mathrm{Mat}_{2\times2}(\mathbb{C}) \,\middle|\, A^\dagger = A,\, \mathrm{tr}(A) = 0,\, A^\dagger \cdot A = \mathrm{I}\right\}}_{\simeq \mathbb{C}P^1}$

Proposition 2.2 (Equivariance of combined Hopf/twistor fibrations).

(i) *The quaternionic Hopf fibration* $S^7 \overset{h_{\mathbb{H}}}{\twoheadrightarrow} S^4$ *(Diagram (11)) is equivariant with respect to the action of* $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ *(Def. A.3)*

(a) *on* S^7, *by*

$$
\begin{array}{ccc}
\mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \times S^7 & \longrightarrow & S^7 \\
\big([A, q'], \{v \cdot t \,\big|\, t \in \mathbb{R}^{\times}_+ \}\big) & \longmapsto & \{A \cdot v \cdot t \cdot q' \,\big|\, t \in \mathbb{R}^{\times}_+ \}
\end{array}
\tag{22}
$$

(b) *on* S^4, *by*

$$
\begin{array}{ccc}
\mathrm{Sp}(2) \cdot \mathrm{Sp}(1) \times S^4 & \longrightarrow & S^4 \\
\big([A, q'], \{v \cdot q \,|\, q \in \mathbb{H}^{\times} \}\big) & \longmapsto & \{A \cdot v \cdot q \,|\, q \in \mathbb{H}^{\times} \}
\end{array}
\tag{23}
$$

(ii) *Its factorization* $h_{\mathbb{H}} = t_{\mathbb{H}} \circ h_{\mathbb{C}}$ *through the combined Hopf/twistor fibrations retains equivariance under the subgroup* $\mathrm{Sp}(2) \hookrightarrow \mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ *(86) with action on* $\mathbb{C}P^3$ *given by*

$$
\begin{array}{ccc}
\mathrm{Sp}(2) \times \mathbb{C}P^3 & \longrightarrow & \mathbb{C}P^3 \\
\big(A, \{x \cdot z \,|\, z \in \mathbb{C}^{\times} \}\big) & \longmapsto & \{A \cdot x \cdot z \,|\, z \in \mathbb{C} \}
\end{array}
\tag{24}
$$

In summary:

Proof. This is essentially immediate from the presentation of the fibrations in (11) (12):
(ii) Diagram (11) makes manifest that all maps here are equivariant with respect to left action by $\mathrm{GL}(8, \mathbb{R})$, hence in particular under left action by $\mathrm{Sp}(2)$, which is also manifest from (12):

$$
\{A \cdot v \cdot t \,\big|\, t \in \mathbb{R}^{\times}_+ \} \overset{h_{\mathbb{C}}}{\longmapsto} \{A \cdot v \cdot z \,|\, z \in \mathbb{C}^{\times} \} \overset{t_{\mathbb{H}}}{\longmapsto} \{A \cdot v \cdot q \,|\, q \in \mathbb{H}^{\times} \}.
\tag{25}
$$

(i) We see that the total quaternionic Hopf fibration is also equivariant under the right $\mathrm{Sp}(1)$-action, due to the fact that the reals commute with the quaternions:

$$
\{v \cdot t \cdot q' \,\big|\, t \in \mathbb{R}^{\times}_+ \} = \{v \cdot q' \cdot t \,\big|\, t \in \mathbb{R}^{\times}_+ \} \overset{h_{\mathbb{H}}}{\longmapsto} \{A \cdot q' \cdot q \,|\, q \in \mathbb{H}^{\times} \} = \{A \cdot q \,|\, q \in \mathbb{H}^{\times} \}.
\tag{26}
$$

Moreover, since the left multiplication action by $\mathrm{Sp}(2)$ evidently commutes with the right multiplication action by $\mathrm{Sp}(1)$ and since $-1 \in \mathrm{Sp}(n)$ is central, this generates the claimed $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$-action. (In fact, this is the maximal symmetry group of $h_{\mathbb{H}}$ [GWZ86, 4.1][FSS19b, 2.20].) $\qquad\square$

Remark 2.3 (Twistor space breaks equivariance to $\mathrm{Sp}(2)$)**.** Notice that the factorization of the quaternionic Hopf fibration through $\mathbb{C}P^3$ is *not* equivariant under the further right $\mathrm{Sp}(1)$-action from (22) and (23), Indeed, the computation analogous to (26) now gives

$$
\{v \cdot t \cdot q' \,\big|\, t \in \mathbb{R}^{\times}_+ \} = \{v \cdot q' \cdot t \,\big|\, t \in \mathbb{R}^{\times}_+ \} \overset{h_{\mathbb{C}}}{\longmapsto} \{A \cdot q' \cdot z \,|\, z \in \mathbb{C}^{\times} \} \underset{\text{in general}}{\neq} \{A \cdot z \cdot q' \,|\, z \in \mathbb{C}^{\times} \}
$$

since the complex numbers do not commute with the quaternions. Therefore, factoring the quaternionic Hopf fibration through the twistor fibration (11) breaks its symmetry from $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ to $\mathrm{Sp}(2)$ (86).

Remark 2.4 (Homotopy quotients and Borel construction)**.** For X a topological space equipped with a continuous action by a topological group G, the *homotopy quotient* $X /\!\!/ G$ is the homotopy type which is represented by the *Borel space* $(X \times EG)/_{\mathrm{diag}} G$, where EG denotes the universal G-principal bundle:

$$
\overset{\text{homotopy quotient}}{X /\!\!/ G} \quad \simeq_{\mathrm{whe}} \quad \overset{\text{Borel construction}}{(X \times EG)/_{\mathrm{diag}} G} .
\tag{27}
$$

9

(i) This construction is clearly functorial: On the right this is a 1-functor on the category of topological spaces equipped with group actions, while on the left this is an ∞-functor on the ∞-category Groupoids$_\infty$ equipped with ∞-actions, see [NSS12a, §4][SS20b, §2.2].

(ii) In the special case when $X = *$ is the point, the Borel space is the classifying space BG. With (i), this means that topological group homomorphisms $G_1 \xrightarrow{\phi} G_2$ induce maps of classifying spaces

$$BG_1 \xrightarrow{B\phi} BG_2 \ . \tag{28}$$

Definition 2.5 (Borel-equivariant Hopf/twistor fibrations). We say that the $\mathrm{Sp}(2)$-*Borel-equivariant Hopf-twistor fibrations* are the image (in homotopy types of topological spaces) of the Hopf/twistor fibrations (11) under taking the homotopy quotient (27) by their compatible $\mathrm{Sp}(2)$-action of Prop. 2.2:

$$\tag{29}$$

Lemma 2.6 (Coset spaces as homotopy fibers [FSS19b, 2.7][SS20b, 2.79]). *Let* $H \xhookrightarrow{i} G$ *be an inclusion of topological groups.*
(i) *The homotopy type of the corresponding coset space* G/H *is, equivalently, the homotopy fiber of the induced morphism* (28) *on classifying spaces.*
(ii) *The homotopy quotient of the coset space by* G *is homotopy equivalent to the classifying space of* H:

$$
\begin{array}{ccc}
G/H \xrightarrow{\ \mathrm{hofib}(B_1)\ } & BH & \simeq \quad \bigl(G/H\bigr)\!/\!/ G \\
& \Big\downarrow{\scriptstyle Bi} & \\
& BG &
\end{array}
$$

The following Prop. 2.7 is the twistorial version of [FSS19b, Prop. 2.22].

Proposition 2.7 (Borel-equivariant twistor fibration as sequence of classifying spaces). *The Borel-equivariant Hopf/twistor fibration (Def. 2.5) is homotopy equivalent to the following sequence of classifying spaces:*

$$
\begin{array}{ccccc}
S^7 /\!\!/ \mathrm{Sp}(2) & \xrightarrow{\ h_{\mathbb{C}} /\!\!/ \mathrm{Sp}(2)\ } & \mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2) & \xrightarrow{\ t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)\ } & S^4 /\!\!/ \mathrm{Sp}(2) \\
\big\| \wr & & \big\| \wr & & \big\| \wr \\
B\bigl(\mathrm{Sp}(1)_L\bigr) & \longrightarrow & B\bigl(\mathrm{Sp}(1)_L \times \mathrm{U}(1)_R\bigr) & \longrightarrow & B\bigl(\mathrm{Sp}(1)_L \times \mathrm{Sp}(1)_R\bigr)
\end{array}
$$

where the maps on the bottom are the deloopings (28) *of the canonical group inclusions (Example A.4).*

Proof. With Lemma 2.6 this follows from the $\mathrm{Sp}(2)$-coset space realization of the Hopf/twistor fibration in (12). $\square$

Lemma 2.8 (Borel-equivariant Hopf/twistor fibrations are spherical). *The Borel-equivariant Hopf/twistor fibrations* (29) *are still spherical fibrations:*

Proof. This follows on general grounds, as in [FSS19b, Remark 3.17]. More concretely, by Prop. 2.7 and using again Lemma 2.6 we have:

$$\mathrm{fib}\big(h_{\mathbb{C}} /\!\!/ \mathrm{Sp}(2)\big) \longrightarrow S^7 /\!\!/ \mathrm{Sp}(2) \qquad\qquad \mathrm{U}(1) \longrightarrow B\big(\mathrm{Sp}(1)_L\big)$$
$$\downarrow{\scriptstyle h_{\mathbb{C}} /\!\!/ \mathrm{Sp}(2)} \qquad \simeq \qquad\qquad \downarrow$$
$$\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2) \qquad\qquad B\big(\mathrm{Sp}(1)_L \times \mathrm{U}(1)_R\big)$$

and

$$\mathrm{fib}\big(t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)\big) \longrightarrow \mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2) \qquad\qquad \mathrm{SU}(2)/\mathrm{U}(1) \longrightarrow B\big(\mathrm{Sp}(1)_L \times \mathrm{U}(1)_R\big)$$
$$\downarrow{\scriptstyle t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)} \qquad \simeq \qquad\qquad \downarrow$$
$$S^4 /\!\!/ \mathrm{Sp}(2) \qquad\qquad B\big(\mathrm{Sp}(1)_L \times \mathrm{Sp}(1)_R\big)$$

$\square$

Theorem 2.9 (Integral cohomology of Borel-equivariant Hopf/twistor-fibration).
(i) *The integral cohomology of the space* $S^4 /\!\!/ \mathrm{Sp}(2)$ *in* (29) *is free on two generators in degree 4*

$$H^\bullet\big(S^4 /\!\!/ \mathrm{Sp}(2); \mathbb{Z}\big) \simeq \mathbb{Z}\big[\widetilde{\Gamma}_4, \widetilde{\Gamma}_4^{\mathrm{vac}}\big] \tag{30}$$

with the property that their evaluation on the fundamental class of the 4-sphere fiber $S^4 \twoheadrightarrow S^4 /\!\!/ \mathrm{Sp}(2)$ *is unity and zero, respectively:*

$$\big\langle \widetilde{\Gamma}_4, S^4 \big\rangle = 1\,, \qquad \big\langle \widetilde{\Gamma}_4^{\mathrm{vac}}, S^4 \big\rangle = 0\,. \tag{31}$$

(ii) *The integral cohomology of the space* $\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2)$ *in* (29) *is free on two generators in degrees 4 and 2, respectively:*

$$H^\bullet\big(\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2); \mathbb{Z}\big) = \mathbb{Z}\big[c_2^L, c_1^R\big]\,. \tag{32}$$

(iii) *The two are related in that pullback in integral cohomology along the Borel-equivariant twistor fibration (Def. 2.5) takes the difference of the former generators to the cup-square of the latter:*

$$H^\bullet\big(S^4 /\!\!/ \mathrm{Sp}(2); \mathbb{Z}\big) \xrightarrow{\ (t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2))^* \ } H^\bullet\big(\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2); \mathbb{Z}\big) \tag{33}$$
$$\widetilde{\Gamma}_4 - \widetilde{\Gamma}_4^{\mathrm{vac}} \qquad\longmapsto\qquad -a := c_1^R \cup c_1^R$$
$$\widetilde{\Gamma}_4 \qquad\longmapsto\qquad c_2^L$$

Proof. Consider Diagram (7) in Figure I. The top part shows the equivalence of the Borel-equivariant Hopf/twistor fibration to a sequence of classifying spaces, according to Prop. 2.7. On the top right we are making fully explicit the factor-wise nature of the corresponding maps, using Example A.4 and Remark 2.4.

The bottom part of the diagram shows the corresponding identification of the cohomology generators according to [FSS19b, 3.9]. This involves the observation that:
(a) Half the universal Euler 4-class on $B\mathrm{Spin}(4)$ is (e.g., [BC98, §2]) the class of the fiberwise unit volume form on the universal S^4-fibration, under the identification from Prop. 2.7:

$$1 \cdot [\mathrm{dvol}] \quad\longleftarrow\quad \tfrac{1}{2}\mathcal{X}_4 \in H^4\big(B\mathrm{Spin}(4); \mathbb{R}\big) \tag{34}$$
$$S^4 \longrightarrow S^4 /\!\!/ \mathrm{Spin}(5) \quad\simeq\quad B\mathrm{Spin}(4)$$

(b) The fractional Euler class by itself is not integral, but its shift by $\tfrac{1}{4}p_1$ (which is also not integral by itself) is (the rational image of) an integral generator $\widetilde{\Gamma}_4 = \tfrac{1}{2}\mathcal{X}_4 + \tfrac{1}{4}p_1$ (e.g. [CV98a, Lemma 2.1]).

Together, (a) and (b) yield the claim (31), and make manifest that the pullback in question is equivalently that of the negative of the left Chern class $-c_2^L$ along the map on classifying space $B\mathrm{U}(1) \xrightarrow{B(c\mapsto \mathrm{diag}(c,\bar{c}))} B\mathrm{SU}(2)$, induced from the inclusion $\mathrm{U}(1) \xrightarrow{\cong} \mathrm{S}\big(\mathrm{U}(1)^2\big) \hookrightarrow \mathrm{SU}(2)$, hence is $c_1 \cup (-c_1)$, which yields the last claim (33). $\square$

2.2 Rational homotopy type

We compute (in Theorem 2.14) the relative Sullivan model for the Borel-equivariant Hopf/twistor fibration from Def. 2.5, with generators normalized such as to match their integral pre-images from Theorem 2.9.

Notation. We use the following notation for dg-algebraic rational homotopy theory (following [FSS16a], exposition in [FSS19a, §3] full details in [FSS20c, §3]): For X a (nilpotent, e.g. simply connected) topological space, we write $\mathrm{CE}(\mathfrak{l}X)$ for its *Sullivan model*, namely for the *minimal* real differential graded-commutative (dgc) algebra ("FDA") which is quasi-isomorphic to the piecewise polynomial de Rham complex of X (which for X a smooth manifold is itself quasi-isomorphic to the ordinary de Rham complex).

Our notation is meant to be suggestive of the fact that this is the Chevalley-Eilenberg algebra $\mathrm{CE}(-)$ ([FSS19a, Def. 3.25], in generalization of classical CE-algebras computing Lie algebra cohomology [FSS19a, Ex. 3.24]) of an L_∞-algebra ([FSS19a, Rem. 3.45]), namely of the real *Whitehead L_∞-algebra* $\mathfrak{l}X$ of X ([FSS19a, Prop. 3.67]):[4]

$$
\underset{\substack{\text{rational} \\ \text{topological} \\ \text{space}}}{X} \quad \longleftrightarrow \quad \underset{\substack{\text{higher} \\ \text{Whitehead} \\ L_\infty\text{-algebra}}}{\mathfrak{l}X} \quad \longleftrightarrow \quad \underset{\substack{\text{Sullivan} \\ \text{dgc-algebra}}}{\mathrm{CE}(\mathfrak{l}X)} \quad \underset{\mathrm{qi}}{\simeq} \quad \Omega^\bullet_{\mathrm{PL}}(X) \tag{35}
$$

Moreover, we give these minimal dgc-algebras by their polynomial generators ω_n in some degree n, quotiented out by their differential relations $d\omega_n = P(...)$ for P some polynomial in generators of lower degree.

For example (e.g. [Me13][FSS19a, Ex. 3.71-2]), the Sullivan models of Eilenberg-MacLane spaces and of spheres are:

$$
\mathrm{CE}\big(B^n\mathbb{Z}\big) \simeq \mathbb{R}[n]/(d=0) \underset{n=2k+1}{\simeq} \mathrm{CE}(\mathfrak{l}S^{2k+1}), \quad \mathrm{CE}\big(S^{2k}\big) \simeq \mathbb{R}[\omega_{2k}, \omega_{4k-1}] \Big/ \begin{pmatrix} d\,\omega_{4k-1} & = -\omega_{2k} \wedge \omega_{2k} \\ d\,\omega_{2k} & = 0 \end{pmatrix}.
$$

Lemma 2.10 (Normalized Sullivan model of spherical fibrations [FHT00, p. 202]; see [FSS19b, 2.5]). *Let X be a topological space with Sullivan model $\mathrm{CE}(\mathfrak{l}X) \in \mathrm{dgcAlgebras}_\mathbb{R}$ (35). Then the relative minimal Sullivan model for a S^n-fibration $Y \twoheadrightarrow X$ is of the following form:*
(i) *for $n = 2k+1$ odd:*

$$
\begin{array}{ccc}
S^{2k+1} \longrightarrow Y & & \mathrm{CE}\big(\mathfrak{l}X\big)\big[\omega_{2k+1}\big]\Big/\big(d\,\omega_{2k+1} = \alpha_{2k+2}\big) \\
\downarrow & & \uparrow \\
X & & \mathrm{CE}\big(\mathfrak{l}X\big)
\end{array} \tag{36}
$$

(ii) *for $n = 2k$ even:*

$$
\begin{array}{ccc}
S^{2k} \longrightarrow Y & & \mathrm{CE}\big(\mathfrak{l}X\big)\begin{bmatrix}\omega_{2k}, \\ \omega_{4k-1}\end{bmatrix}\Big/\begin{pmatrix} d\,\omega_{2k} = 0 \\ d\,\omega_{4k-1} = -\omega_{2k} \wedge \omega_{2k} + \alpha_{4k} \end{pmatrix} \\
\downarrow & & \uparrow \\
X & & \mathrm{CE}\big(\mathfrak{l}X\big)
\end{array} \tag{37}
$$

for some closed $\alpha \in \mathrm{CE}\big(\mathfrak{l}X\big)$ (which can be characterized further, see [FHT00, p. 202][FSS19b, 2.5]).
(iii) *The differential in (37) is normalized so that the generators ω_d restrict to the unit volume forms on the respective sphere fibers (see [FSS19b, Lemma 3.19]):*

$$
\langle \omega_{2k}, S^{2k} \rangle = 1, \qquad \langle \omega_{4k-1}, S^{4k-1} \rangle = 1. \tag{38}
$$

The action of triality group automorphisms on $\mathrm{Spin}(8)$ famously relates three distinct conjugacy classes of subgroup inclusions of $\mathrm{Spin}(7)$. Less widely appreciated is another triple of subgroups of $\mathrm{Spin}(8)$ that is permuted under triality:

---

[4]This passage (35) through Whitehead L_∞-algebras makes transparent how it is that dgc-algebras know about homotopy types and how dgc-algebra homomorphisms between these encode L_∞-algebra valued higher gauge fields [FSS19a, §3.3], but for the purpose of the present article the reader may ignore L_∞-algebra theory and regard the notation $\mathrm{CE}(\mathfrak{l}(-))$ as a primitive for Sullivan models.

Lemma 2.11 (Triality on central product groups in $\mathrm{Spin}(8)$ [FSS19b, 2.17]). *Under the triality automorphisms of* $\mathrm{Spin}(8)$ *the canonical subgroup inclusions of the central product groups* $\mathrm{Spin}(5) \cdot \mathrm{Spin}(3)$ *and* $\mathrm{Sp}(2) \cdot \mathrm{Sp}(1)$ *(Def. A.1) turn into each other:*

$$
\begin{array}{ccc}
\mathrm{Sp}(2)\cdot\mathrm{Sp}(1) & \xhookrightarrow{\;i_{\mathrm{Sp}}\;} & \mathrm{Spin}(8) \\
\simeq \downarrow & & \simeq \downarrow \mathrm{tri} \\
\mathrm{Spin}(5)\cdot\mathrm{Spin}(3) & \xhookrightarrow{\;i_{\mathrm{Spin}}\;} & \mathrm{Spin}(8)
\end{array}
\tag{39}
$$

Lemma 2.12 (Sullivan model for $B\mathrm{Spin}(5)$ and $B\mathrm{Sp}(2)$ [FSS19b, 2.19]). *Minimal sullivan models for* $B\mathrm{Spin}(5)$ *and* $B\mathrm{Sp}(2)$, *and their relation under triality* (40) *are given, up to isomorphism, as follows:*

$$
\begin{array}{ccc}
B\mathrm{Spin}(8) & \xrightarrow{Bi_{\mathrm{Sp}}} & B\mathrm{Sp}(2) \\
B\mathrm{tri} \downarrow \simeq & & \downarrow \simeq \\
B\mathrm{Spin}(8) & \xrightarrow{Bi_{\mathrm{Spin}}} & B\mathrm{Spin}(5)
\end{array}
\qquad
\begin{array}{c}
\mathbb{R}\big[\tfrac{1}{2}p_1, \mathcal{X}_8\big] = \mathrm{CE}\big(lB\mathrm{Sp}(2)\big) \\
{\Large\uparrow}\; \substack{\tfrac{1}{2}p_1 \\ \uparrow \\ \tfrac{1}{2}p_1} \;\; \substack{-\mathcal{X}_8 + (\tfrac{1}{4}p_1)^2 \\ \uparrow \\ \tfrac{1}{4}p_2} \\
\mathbb{R}\big[\tfrac{1}{2}p_1, p_2\big] =: \mathrm{CE}\big(lB\mathrm{Spin}(5)\big)
\end{array}
\tag{40}
$$

where (by [CV98b, 8.1, 8.2][FSS19b, 3.7]):

$$
\tfrac{1}{2}\mathcal{X}_8 = \tfrac{1}{4}\Big(p_2 - \big(\tfrac{1}{2}p_1\big)^2\Big) \;\in\; H^4(B\mathrm{Sp}(2); \mathbb{R}).
\tag{41}
$$

Lemma 2.13 (Normalized Sullivan model for plain Hopf/twistor fibrations). *The minimal relative Sullivan model for the plain Hopf/twistor fibrations* (11) *is as follows:*

where the generators $\omega_4, \omega_7, f_2, h_3$ *are all normalized according to* (38), *in particular:*

$$
\big\langle \omega_4, S^4 \big\rangle = 1 \qquad \big\langle f_2, S^2 \big\rangle = 1.
\tag{43}
$$

Note that on the right in (42) we are showing the minimal Sullivan models of S^7 and of $\mathbb{C}P^3$ by themselves (which is classical, e.g. [FHT00, p. 142, 203][Me13, 1.2, 5.3]), while on the left we are showing their Sullivan models as fiber spaces, i.e., the relative minimal Sullivan models.

Proof. **(i)** It is classical that the Sullivan model for S^4 is as shown (it is also a special case of Lemma 2.10).
(ii) Since $\mathbb{C}P^3 \to S^4$ is an S^2-fibration (11), Lemma 2.10 implies from (i) that $\mathbb{C}P^3$ fibered over S^4 is modeled by

$$\mathrm{CE}\big(\mathfrak{l}\mathbb{C}P^3\big)_{S^4} \;=\; \mathbb{R}\big[\omega_4, \omega_7, f_2, h_3\big] \Big/ \begin{pmatrix} d\,\omega_4 = 0 \\ d\,\omega_7 = -\omega_4 \wedge \omega_4 \\ d\,f_2 = 0 \\ d\,h_3 = f_2 \wedge f_2 + \alpha_4 \end{pmatrix}$$

for *some* closed element $\alpha_4 \in \mathrm{CE}\big(\mathfrak{l}S^4\big)$. But in the present case, due to (i), there is a unique such element, up to a real factor, namely ω_4. Below in (46) we find this factor to be unity. This yields the middle part of (42).
(iii) Since $S^7 \to \mathbb{C}P^3$ is an S^1-fibration (11), Lemma 2.10 implies, via (ii), that S^7 fibered over $\mathbb{C}P^3$ is modeled by

$$\mathrm{CE}\big(\mathfrak{l}S^7\big)_{\mathbb{C}P^3} \;=\; \mathbb{R}\big[\omega_4, \omega_7, f_2, h_3, h_1\big] \Big/ \begin{pmatrix} d\,\omega_4 = 0 \\ d\,\omega_7 = -\omega_4 \wedge \omega_4 \\ d\,f_2 = 0 \\ d\,h_3 = f_2 \wedge f_2 + \omega_4 \\ d\,f_1 = \alpha_2 \end{pmatrix}$$

for *some* closed degree-2 element $\alpha_2 \in \mathrm{CE}\big(\mathfrak{l}\mathbb{C}P^3\big)_{S^2}$. But in the present case, due to (ii), there is a unique such element, up to a real factor, namely f_2. Thus, by suitably rescaling f_1, we obtain $\alpha_2 = f_2$ and the claim follows. $\square$

Theorem 2.14 (Normalized Sullivan model of Borel-equivariant Hopf/twistor fibrations). *The iterative relative Sullivan models for the parametrized Hopf/twistor fibrations* (29) *are as follows (here $\frac{1}{2}p_1, \mathcal{X}_8 \in \mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big)$, via Lemma 2.12):*

$S^7 /\!\!/ \mathrm{Sp}(2)$

$\mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big)\begin{bmatrix} h_1, \\ f_2, \\ h_3, \\ \omega_4, \\ \omega_7 \end{bmatrix} \Big/ \begin{pmatrix} d\,h_1 = f_2 \\ d\,f_2 = 0 \\ d\,h_3 = \omega_4 - \frac{1}{4}p_1 - f_2 \wedge f_2 \\ d\,\omega_4 = 0 \\ d\,\omega_7 = -\omega_4 \wedge \omega_4 + \big(\frac{1}{4}p_1\big)^2 \\ \qquad\qquad - \mathcal{X}_8 \end{pmatrix}$

$\begin{matrix} \omega_4 & \omega_7 & f_2 & h_3 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \omega_4 & \omega_7 & f_2 & h_3 \end{matrix}$

$h_{\mathbb{C}} /\!\!/ \mathrm{Sp}(2)$

$\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2)$

$\mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big)\begin{bmatrix} f_2, \\ h_3, \\ \omega_4, \\ \omega_7 \end{bmatrix} \Big/ \begin{pmatrix} d\,f_2 = 0 \\ d\,h_3 = \omega_4 - \frac{1}{4}p_1 - f_2 \wedge f_2 \\ d\,\omega_4 = 0 \\ d\,\omega_7 = -\omega_4 \wedge \omega_4 + \big(\frac{1}{4}p_1\big)^2 \\ \qquad\qquad - \mathcal{X}_8 \end{pmatrix}$ (44)

$\begin{matrix} \omega_4 & \omega_7 \\ \updownarrow & \updownarrow \\ \omega_4 & \omega_7 \end{matrix}$

$t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)$

$B\mathrm{Sp}(2)$

$\mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big)$

$S^4 /\!\!/ \mathrm{Sp}(2)$

$\mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big)\begin{bmatrix} \omega_4, \\ \omega_7 \end{bmatrix} \Big/ \begin{pmatrix} d\,\omega_4 = 0 \\ d\,\omega_7 = -\omega_4 \wedge \omega_4 + \big(\frac{1}{4}p_1\big)^2 \\ \qquad\qquad - \mathcal{X}_8 \end{pmatrix} \,,$

where the generators f_2 and ω_4 represent the classes c_1^R and $\frac{1}{2}\mathcal{X}_4$ in (7), *respectively:*

14

$$[\omega_4] \;=\; \tfrac{1}{2}\mathcal{X}_4 \;\in\; H^4\big(\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2);\mathbb{R}\big)\,, \qquad\quad [f_2] \;=\; c_1^R \;\in\; H^2\big(\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2);\mathbb{R}\big). \qquad (45)$$

Proof. That the composite vertical morphism, upon discarding the generators h_1, f_2 and h_3, is the minimal relative Sullivan model for $h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)$ with the identification $[\omega_4] = \tfrac{1}{2}\mathcal{X}_4$ (45) is the result of [FSS19b, 3.19].

Its factorization through $\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2)$ must have minimal Sullivan model given by adjoining generators f_2 and f_3, by Lemma 2.8 with Lemma 2.10. The fiberwise normalization (43) implies the identification $[f_2] = c_1^R$ in (45), using that c_1^R pulled back along $S^2 = \mathrm{SU}(2)/U(1) \to B\mathrm{U}(1)_R$ is the unit volume generator.

For the factorization of $h_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)$ through $\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2)$ to reproduce on fibers over $B\mathrm{Sp}(2)$ (hence upon discarding the generators $\tfrac{1}{2}p_1$ and $\mathcal{X}_8$) the minimal Sullivan model for the plain Hopf/twistor fibrations from Lemma 2.13 *at least* those monomials in f_2 shown in (44) have to appear. We just have to observe that the relative coefficients in the differential relations for h_3 are as shown. But under the identification (45) we have the second logical equivalence shown here:

$$dh_3 = \omega_4 - \tfrac{1}{4}p_1 - f_2 \wedge f_2 \;\;\Leftrightarrow\;\; \big[\omega_4 - \tfrac{1}{4}p_1\big] = f_2 \wedge f_2 \;\;\Leftrightarrow\;\; \widetilde{\Gamma}_4 - \widetilde{\Gamma}_4^{\mathrm{vac}} = c_1^R \cup c_1^R \;\; \in H^4\big(\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2);\mathbb{R}\big)\,. \quad (46)$$

That the relation on the right of (46) does hold follows immediately from (33) in Theorem 2.9.

Hence to conclude it suffices now to show that no further monomials in f_2 appear on the right of (44):

First, any further monomial in f_2 that does appear must contain as a factor a *basic* generator, namely a generator from $\mathrm{CE}(\mathfrak{l}B\mathrm{Sp}(2))$, to guarantee that it vanishes on fibers (where we already have the right terms). Since, by Lemma 2.12, the generators of $\mathrm{CE}\big(\mathfrak{l}B\mathrm{Sp}(2)\big)$ are in degrees 4 and 8, the only further term that could possibly appear, by degree reasons, is the blue term in the following expression:

$$d\,\omega_7 \;=\; -\omega_4 \wedge \omega_4 + \big(\tfrac{1}{2}p_1\big)^2 - \mathcal{X}_8 + a \cdot f_2 \wedge f_2 \wedge p_1 \qquad (47)$$

for some coefficient $a \in \mathbb{R}$. But we also know that $t_{\mathbb{H}} /\!\!/ \mathrm{Sp}(2)$ is an S^2-fibration (by Lemma 2.8), so that Lemma 2.10 rules out the appearance of the blue term in (47) (i.e., implies $a = 0$). $\qquad\square$

3 Charge quantization in Twistorial Cohomotopy

After recalling (in §3.1) general non-abelian cohomology and highlighting the non-abelian Chern-Dold character map, we introduce (in §3.2) the twisted non-abelian cohomology to be called *Twistorial Cohomotopy* and use the results from §2 to show (Corollary 3.10) that charge quantization in Twistorial Cohomotopy implies the heteoric shifted flux quantization condition (6).

3.1 Non-abelian character map

We recall the Chern-Dold character (55) in generalized cohomology and then introduce its generalization, to a non-abelian character map (63) on non-abelian cohomology. The full technical detail is laid out in [FSS20c].

From generalized to non-abelian cohomology. It is well-known, though perhaps under-appreciated, that cohomology theory is all about homotopy groups of mapping spaces into a given "coefficient space" or "classifying space". We recall this briefly for "bare" cohomology theories, with the domain spaces X assumed to a sufficiently nice topological space; but the statement remains true for structured cohomology theories such as differential and/or equivariant cohomology, when interpreted internal to suitable higher toposes, see [SS20b, p. 6].

For ordinary (e.g., singular) cohomology with coefficients in an *abelian* discrete group A, these classifying spaces are the Eilenberg-MacLane spaces $K(A,n)$ (e.g. [AGP02, §7.1, Cor. 12.1.20]):

$$\underset{\substack{\text{ordinary}\\\text{cohomology}}}{H^n\big(X;A\big)} \;\simeq\; \pi_0\,\underset{\substack{\text{homotopy classes of maps to}\\\text{Eilenberg-MacLane space}}}{\mathrm{Maps}\big(X, K(A,n)\big)}\,. \qquad (48)$$

These happen to be based loop spaces of each other, $K(A,n) \simeq_{\mathrm{whe}} \Omega K(A,n+1)$ (e.g. [AGP02, 7.1.1]), so that each of them is an *infinite loop space* (e.g. [Ad78]).

More generally, consider a *generalized cohomology theory*[5] $E^\bullet$ in the sense of [Wh62] (see [Ad75][Ad78]), such as K-theory, elliptic cohomology, tmf, stable Cobordism, stable Cohomotopy, etc. These are classified by such sequences of (pointed) spaces which are successively equipped with weak homotopy equivalences exhibiting them as based loop spaces of each other, called a *spectrum* of spaces:

$$\{E_n\}_{n\in\mathbb{N}}, \quad \text{s.t.} \quad E_n \simeq \Omega E_{n+1}, \tag{49}$$

in that

$$\underset{\substack{\text{generalized} \\ \text{cohomology}}}{E^n(X)} \simeq \underset{\substack{\text{homotopy classes of maps to} \\ \text{infinite loop space}}}{\pi_0 \mathrm{Maps}(X, E_n)}. \tag{50}$$

This is the *Brown representability theorem*, see e.g. [Ad75, §III.6][Ko96, §3.4]. But the right hand side of (50) makes sense for E_n *any* space, not necessarily part of a spectrum (49), and not necessarily even being a loop space. It is not the notion of cohomology itself, but rather only some extra *properties* enjoyed by these abelian cohomology groups (such as existence of connecting homomorphisms) which is what is reflected in the infinite loop space structure (49).

Indeed, for G a well-behaved topological group, *not* necessarily abelian (such as $G = \mathrm{U}(1), \mathrm{SU}(n), \mathrm{Sp}(n), \cdots$) the fundamental theorem of G-principal bundles ([St51, §19.3], review in [Add07, §5]) says that degree-1 *non-abelian cohomology* with coefficients in G is represented by the classifying space BG of G:

$$\underset{\substack{\text{non-abelian cohomology with} \\ \text{coefficients in topological group}}}{H^1(X; G)} \simeq \underset{\substack{\text{homotopy classes of maps} \\ \text{to classifying space of group}}}{\pi_0 \mathrm{Maps}(X, BG)}. \tag{51}$$

If $G = A$ is abelian and discrete, then $BA \simeq K(A, 1)$ and (51) reduces to (48), but not otherwise. Moreover, the May recognition theorem implies that *any* connected space A is weakly homotopy equivalent to a classifying space BG, namely for $G = \Omega A$ the based loop group of A (which may be rectified, up to weak homotopy equivalence, to an actual topological group). Thereby, the traditional equivalence (51) is re-interpreted as an elegant general notion of *non-abelian cohomology*:

$$\underset{\substack{\text{non-abelian cohomology} \\ \text{with coefficients in } A}}{H(X; A)} := \underset{\substack{\text{homotopy-classes of} \\ \text{maps to } A}}{\pi_0 \mathrm{Maps}(X, A)} = \left\{ X \underset{c'}{\overset{c}{\rightrightarrows}} A \right\}_{\Big/ \text{homotopy}} \tag{52}$$

Non-abelian cohomology in this generality is discussed in [To02][Ja09][RS12][NSS12a][NSS12b][SS20b]. For example, for $X = S^n$ an n-sphere, we have $S^n \simeq B(\Omega S^n)$ and the corresponding non-abelian cohomology theory (51) is *Cohomotopy theory*

$$\pi^n(X) := \pi_0 \mathrm{Maps}(X, S^n) \simeq H^1(X; \Omega S^n).$$

This perspective on generalized/non-abelian cohomology via classifying spaces makes many related concepts nicely transparent, for example the notions of *twisting in cohohomology* and of *generalized Chern characters*.

Twisted non-abelian cohomology. A *twist* of A-cohomology (51) is what is classified by a twisted parametrization of A over some base space B [NSS12a, §4][SS20b, §2.2][FSS20c, §2.2]: Instead of mapping into a fixed classifying spaces, a *twisted cocycle* maps into a varying classifying space that may twist and turn as one moves in the domain space. In other words, a *twisting* τ of A-cohomology theory on some X is a bundle over X with typical fiber A, and a τ-twisted cocycle is a *section* of that bundle [NSS12a, §4][ABGHR14][SS20a, §2.2]:

---

[5] The term is widely used but somewhat unfortunate, since various *further* generalizations of Whitehead's generalization of ordinary cohomology theories are relevant, such as twisted-, sheaf-, differential-, equivariant- and nonabelian-cohomology theories, as well as all their joint combinations.

$$
\begin{array}{c}
\substack{\tau\text{-twisted}\\ \text{non-abelian generalized}\\ A\text{-cohomology theory}} \quad A^\tau(X) \; := \; \left\{\begin{array}{c}
\xymatrix{ & P \ar[r] \ar[d]^{p} & A/\!\!/\operatorname{Aut}(A) \ar[d] \\ X \ar@{=}[r] & X \ar[r]^{\tau} & B\operatorname{Aut}(A) }
\end{array}\right\}\Big/\underset{\substack{\text{homotopy}\\ B\operatorname{Aut}(A)}}{\sim}
\end{array}
$$
$$
\simeq \left\{\begin{array}{c}
\xymatrix{ X \ar@{-->}[rr] \ar[dr]_{\text{twist } \tau} & & A/\!\!/\operatorname{Aut}(A) \ar[dl] \\ & B\operatorname{Aut}(A) & }
\end{array}\right\}\Big/\underset{\substack{\text{homotopy}\\ B\operatorname{Aut}(A)}}{\sim}
\qquad (53)
$$

Here the equivalent formulation shown in the second line follows because A-fiber bundles are themselves classified by nonabelian $\operatorname{Aut}(A)$-cohomology (see [NSS12a, 4.11][SS20b, 2.92]), as shown on the right of the first line.

With a general concept of twisted non-abelian cohomology theories in hand, we turn to discussion of their character maps. At their core, these come from the rationalization approximation on coefficient/classifying spaces:

Rationalization. For X a connected nilpotent space, we write

$$
X \xrightarrow[\substack{\text{rationalization}\\ \text{(over the real numbers)}}]{\eta_X^{\mathbb{R}}} L_{\mathbb{R}} X
\qquad (54)
$$

for its rationalization (e.g. [He07, 1.4]) over the real numbers. And we write $\mathfrak{l}X$ for the *Whitehead L_∞-algebra* that is the formal dual of the minimal Sullivan model for X [BFM06][BMSS19, §2.1][FSS20c, Prop. 3.61] (or in their rectified incarnation [FRS13, §1.0.2]: dg-Lie algebras as in the original [Qu69]).

Chern-Dold character in abelian cohomology. Given an abelian generalized cohomology theory $E^\bullet$ (50), rationalization (54) of its classifying spaces (49) induces a cohomology operation from E-cohomology theory to ordinary cohomology with coefficients in the rationalized stable homotopy groups of E:

$$
\underset{\substack{\text{Chern-Dold}\\ \text{character}}}{\operatorname{ch}_E} \; : \; \underset{\substack{\text{generalized}\\ \text{cohomology}}}{E^n(X)} \overset{\substack{\text{Brown}\\ \text{represen-}\\ \text{tability}}}{\simeq} \pi_0 \operatorname{Maps}(X, E_n) \xrightarrow[\pi_0 \operatorname{Maps}(X, L_{\mathbb{R}})]{\text{rationalization}} \pi_0 \operatorname{Maps}(X, L_{\mathbb{R}} E_n) \; =: \; \underset{\substack{\text{rational } E\text{-cohomology}}}{E_{\mathbb{R}}^n(X)} \overset{\substack{\text{Dold's}\\ \text{equiv-}\\ \text{alence}}}{\simeq} \bigoplus_k \underset{\substack{\text{ordinary cohomology}\\ \text{with coefficients in}\\ \text{rationalized homotopy groups of } E}}{H^{n+k}\big(X; \pi_k(E)\otimes_{\mathbb{Z}} \mathbb{R}\big)}, \quad (55)
$$

where Dold's equivalence on the far right is due to [Do65, Cor. 4], reviewed in [Rud98, §II.3.17]. This map (55) is called the *Chern-Dold character map*, due to [Bu70]. The modern formulation above is made fully explicit in [LSW16, §2.1]; see also [HS05, §4.8][BN14, p. 17][GS17]. For example:

   (i) When $E^\bullet = H^\bullet(-; \mathbb{Z})$ is ordinary integral cohomology, its rationalization is $E_{\mathbb{R}}^\bullet = H^\bullet(-; \mathbb{R})$ and the Chern-Dold character (55) reduces to extension of scalars from integral to real cohomology, as in (1), (2).

   (ii) When $E^\bullet = KU^\bullet$ is complex topological K-theory, its rationalization is $KU_{\mathbb{R}}^{0,1} \simeq H^{\text{even,odd}}(-; \mathbb{R})$ and the Chern-Dold character (55) reduces to the ordinary Chern character (see [GSa18][GSa19] for extensive discussions).

Character map in twisted non-abelian cohomology.

Definition 3.1 (Non-abelian de Rham cohomology [SSS09a, §6.5][FSSt10, §4.1][FSS20c, Def. 3.82]). The non-abelian de Rham cohomology of a smooth manifold X with coefficients in an L_∞-algebra $\mathfrak{g}$ of finite type is the quotient of the set of dg-algebra homomorphism from the Chevalley-Eilenberg algebra $CE(\mathfrak{g})$ of $\mathfrak{g}$ (which is the Sullivan model of a rational space) to the de Rham dg-algebra $\Omega_{\text{dR}}^\bullet$ of differential forms on X, quotiented by dg-algebra homotopies:

$$H_{\mathrm{dR}}\big(X;\mathfrak{l}A\big) \;:=\; \mathrm{Hom}\big(\mathrm{CE}(\mathfrak{l}A)\;,\;\Omega^{\bullet}_{\mathrm{dR}}(X)\big)_{\big/\sim} \;=\; \left\{ \Omega^{\bullet}_{\mathrm{dR}}(X) \overset{A}{\underset{A'}{\rightleftarrows}} \mathrm{CE}(\mathfrak{l}A) \right\}_{\Big/\mathrm{homotopy}} \tag{56}$$

Example 3.2 (Recovering ordinary de Rham cohomology [FSS20c, Prop. 3.84]). In the case that $\mathfrak{g} = \mathbb{R}[n]$ is the *line* L_∞-*algebra* concentrated in degree n, its Chevalley-Eilenberg algebra is the free graded-commutative algebra on a single generator in degree $n+1$ with vanishing differential; which is also the Sullivan model of the Eilenberg-MacLane space (48) in that degree:

$$\mathrm{CE}\big(\mathbb{R}[n]\big) \;=\; \mathbb{R}\big[c_{n_1}\big]\big/\big(d\,c_{n+1} = 0\ \big) \;\simeq\; \mathrm{CE}\big(\mathfrak{l}K(n+1,\mathbb{Z})\big). \tag{57}$$

Hence dg-algebra homomorphisms out of this into a de Rham algebra are equivalently closed differential $(n+1)$-forms:

$$\mathrm{Hom}\big(\mathrm{CE}(\mathbb{R}[1]),\Omega^{\bullet}_{\mathrm{dR}}(X)\big) \;\simeq\; \Omega^n(X)_{\mathrm{cl}}, \tag{58}$$

and dg-algebra homotopies between these are equivalently de Rham coboundaries. Therefore, the non-abelian de Rham cohomology (56) with these coefficients reduces to ordinary de Rham cohomology in that degree:

$$H_{\mathrm{dR}}\big(X;\mathbb{R}[n]\big) \;\simeq\; H^{n+1}_{\mathrm{dR}}(X). \tag{59}$$

Proposition 3.3 (Non-abelian de Rham theorem [FSS20c, Thm. 3.95]). *Let X be a smooth manifold and A a nilpotent topological space of finite rational homotopy type, hence with a minimal Sullivan model $\mathrm{CE}\big(\mathfrak{l}A\big)$ for its rationalization $\mathfrak{l}A$ (54). Then the non-abelian cohomology (52) of X with real coefficients $\mathfrak{l}A$ is equivalent to the non-abelian de Rham cohomology (56) with coefficient in $\mathfrak{l}A$:*

$$\overset{\substack{\text{non-abelian}\\\text{real cohomology}}}{H(X;L_{\mathbb{R}}A)} \;\simeq\; \overset{\substack{\text{non-abelian}\\\text{de Rham cohomology}}}{H_{\mathrm{dR}}(X;\mathfrak{l}A)}. \tag{60}$$

Proof. Unwinding the definitions, the equivalence (60) reduces to the fundamental theorem of rational homotopy theory [BG76, §9.4] (reviewed as [BMSS19, Prop. 2.11]; see also [He07, Cor. 1.26]) which identifies the hom-sets in the homotopy categories of **a)** nilpotent and finite-type rational topological spaces, and **b)** the opposite of dgc-algebras. $\square$

Proposition 3.4 (Non-abelian de Rham theorem for stable coefficients [FSS19a, Ex. 3.75]). *Let X be a smooth manifold, and E an infinite-loop space (49). Then non-abelian de Rham cohomology (56) of X with coefficients in $\mathfrak{l}E$ is equivalent to the real cohomology of X with coefficients in the rationalized homotopy groups of E:*

$$H_{\mathrm{dR}}\big(X;\mathfrak{l}E\big) \;\simeq\; \bigoplus_k H^k\big(X;\pi_k(E)\otimes_{\mathbb{Z}}\mathbb{R}\big). \tag{61}$$

Proof. The minimal Sullivan model of an infinite loop space is the free graded algebra generated by its rationalized homotopy groups, with vanishing differential (see [FHT00, p. 143], or, from a broader perspective of rational spectra, [BMSS19, Lemma 2.25, Prop. 2.30]). This implies the claim by Example 3.2, via the ordinary de Rham theorem (e.g. [FHT00, 10.15]). $\square$

In conclusion:

Proposition 3.5 (Non-abelian de Rham theorem recovers Dold's equivalence [FSS20c, Prop. 4.6]). *Let X be a smooth manifold, and E (the connective spectrum of) an infinite-loop space (49). Then Dold's equivalence is equivalent to the restriction of the non-abelian de Rham theorem (Prop. 3.3) to stable coefficients (Prop. 3.4):*

$$E^n_{\mathbb{R}}(X) \xrightarrow[\simeq]{\text{Dold's equivalence}} \bigoplus_k H^{n+k}\big(X; \pi_{n+k}(E) \otimes_{\mathbb{Z}} \mathbb{R}\big) \tag{62}$$

$$H(X; L_{\mathbb{R}} E_n) \xrightarrow[\substack{\text{non-abelian} \\ \text{de Rham theorem} \\ \text{Prop. 3.3}}]{\simeq} H_{\mathrm{dR}}\big(X; \mathfrak{l} E_n\big)$$

with the vertical map on the right labeled $\mathbb{R}$ (61).

Therefore, we obtain the following generalization of the Chern-Dold character (55):

Definition 3.6 (Character map in non-abelian cohomology [FSS20c, Def. 4.3]). Let X be a smooth manifold and A a nilpotent space of finite rational type. Then the *non-abelian Chern-Dold character* on non-abelian cohomology theory (52) represented by A is the composite of
(a) the rationalization map (54) on coefficients
(b) the non-abelian de Rham theorem 3.3:

$$
\underset{\substack{\text{non-abelian} \\ \text{Chern-Dold character}}}{\mathrm{ch}_A} \quad : \quad \underset{\substack{\text{non-abelian} \\ \text{cohomology} \\ \text{with coefficients in } A}}{H(X;A) := \pi_0 \mathrm{Maps}(X, A)} \xrightarrow[\pi_0 \mathrm{Maps}(X, \eta_A^{\mathbb{R}})]{\text{rationalization}} \underset{\substack{\text{non-abelian cohomology} \\ \text{with coefficients in } \mathfrak{l}A}}{\pi_0 \mathrm{Maps}(X, L_{\mathbb{R}} A) =: H(X; L_{\mathbb{R}} A)} \underset{\substack{\text{non-abelian} \\ \text{de Rham} \\ \text{theorem}}}{\simeq} \underset{\substack{\text{non-abelian} \\ \text{de Rham cohomology} \\ \text{with coefficients in } \mathfrak{l}A}}{H_{\mathrm{dR}}(X; \mathfrak{l}A)} .
\tag{63}
$$

Character map in twisted non-abelian cohomology. The above constructions immediately generalize to twisted nonabelian cohomology (53) to yield the twisted non-abelian Chern character cohomology operation:

Definition 3.7 (Character in twisted non-abelian cohomology [FSS20c, Def. 5.4]). The twisted non-abelian character map is the non-abelian character (Def. 3.6) in the slice over $B\mathrm{Aut}(A)$:

$$
\underset{\substack{\text{twisted non-abelian} \\ \text{Chern-Dold character}}}{\mathrm{ch}_A^\tau} \quad : \quad \underset{\substack{\tau\text{-twisted non-abelian} \\ \text{cohomology} \\ \text{with coefficients in } A}}{H^\tau(X;A) := \pi_0 \mathrm{Maps}_{B\mathrm{Aut}(A)}(X, A)} \longrightarrow \underset{\substack{L_{\mathbb{R}} \tau\text{-twisted non-abelian cohomology} \\ \text{with coefficients in } L_{\mathbb{R}} A}}{\pi_0 \mathrm{Maps}_{/L_{\mathbb{R}} B\mathrm{Aut}(A)}(X, L_{\mathbb{R}} A) =: H^{L_{\mathbb{R}}\tau}(X; L_{\mathbb{R}} A)} \underset{\substack{\text{twisted non-abelian} \\ \text{de Rham} \\ \text{theorem}}}{\simeq} \underset{\substack{\mathfrak{l}\tau\text{-twisted non-abelian} \\ \text{de Rham cohomology} \\ \text{with coefficients in } \mathfrak{l}A}}{H_{\mathrm{dR}}^{\tau_{\mathrm{dR}}}(X; \mathfrak{l}A)}
\tag{64}
$$

$$
\left\{ \begin{array}{c} X \xrightarrow{\ c\ } A /\!\!/ \mathrm{Aut}(A) \\ \tau \searrow \qquad \swarrow \\ B\mathrm{Aut}(A) \end{array} \right\}_{\Big/ \sim} \longmapsto \left\{ \begin{array}{c} \Omega^\bullet_{\mathrm{dR}}(X) \xleftarrow{\ A\ } \mathrm{CE}\Big(\mathfrak{l}_{B\mathrm{Aut}(A)}\big(A /\!\!/ \mathrm{Aut}(A)\big)\Big) \\ \tau^* \nwarrow \qquad \nearrow \\ \mathrm{CE}\Big(\mathfrak{l}\big(B\mathrm{Aut}(A)\big)\Big) \end{array} \right\}_{\Big/ \sim}
$$

This means that the twisted character on A-cohomology is the plain character on $A /\!\!/ \mathrm{Aut}(A)$-cohomology fibered over $B\mathrm{Aut}(A)$, hence is the fiberwise A-character on an A-fiber ∞-bundle. (The notation $\mathfrak{l}_B(-)$ in (64) denotes the *relative* Whitehead L_∞-algebra over a base B [FSS20c, Prop. 3.80], such that $\mathrm{CE}\big(\mathfrak{l}_B(-)\big)$ denotes the Sullivant minimal model *relative* to that of the base B ([FSS20c, Prop. 3.0.49]), thus ensuring that the domain on the right is still cofibrant in the co-sliced model structure, as in [FSS20c, proof of Prop. 3.115].)

Remark 3.8 (Charge quantization by lift through character map). Just as for the traditional Chern character on K-theory (see [GSa18] for a detailed account), the Chern-Dold character (55) is generally far from being surjective, and the same is true for its non-abelian (63) and its twisted non-abelian generalization (64).
(i) The obstruction to lifting de Rham form data through the Chern-Dold character maps are *integrality* conditions that disappear upon rationalization, hence are "quantization" conditions (in the original sense of Bohr-Sommerfeld quantization).
(ii) Therefore, if any given differential form data lifts through the Chern-Dold character of some twisted non-abelian A-cohomology theory, we say that that it is *quantized in A-theory*.
(iii) In typical examples the differential forms in question are flux densities, encoding charges of physical fields, and hence we speak of *charge-quantization in A-theory*. (For abelian cohomology this is discussed in [Fr00][GSa19].)

3.2 Twistorial Cohomotopy theory

We now identify and study the twisted non-abelian cohomology theory whose classifying space is the Borel-equivariant twistor fibration (Def. 2.5). The main result of this section is Theorem 3.11, which shows that charge-quantization (Remark 3.8) in this *Twistorial Cohomotopy* (Prop. 3.9) imposes a shifted integrality condition (78) on Chern-Dold character forms (Corollary 3.11) matching that of (6).

Tangential $Sp(2)$**-structure.** Consider smooth spin 8-manifolds X that are equipped with tangential $Sp(2)$-structure (e.g. [SS20b, 4.48]), hence with a homotopy-lift[6] of the classifying map of their tangent bundle to the classifying space $BSp(2)$ of the quaternionic unitary group (Def. A.3) along its canonical inclusion i_{Sp} (39):

$$
\begin{array}{ccc}
 & \overset{\text{tangential}}{\underset{\tau}{Sp(2)\text{-structure}}} & \\
X & \longrightarrow & BSp(2) \\
\end{array}
\tag{65}
$$

In the intended applications, this spin 8-manifold (65) is one factor in an 11-dimensional spacetime of the form $\mathbb{R}^{2,1} \times X$ (see [FSS19b, §3]). We write ω for any affine connection on TX ("spin connection") and write

$$
p_i(\omega) \in H^{2i}_{\mathrm{dR}}(X) \simeq H^{2i}(X;\mathbb{R}) .
\tag{66}
$$

for the induced Pontrjagin forms (e.g. [GSa18, p. 10]).

Associated twistor-space fibration. By Prop. 2.2, a tangential $Sp(2)$-structure (65) induces, via pullback of the parametrized Hopf/twistor fibration from Def. 2.5, an S^4-fibration E and a $\mathbb{C}P^3$-fibration $\widetilde{E}$ over X, connected by a morphism of fibrations over X which is fiberwise the plain twistor fibration $t_{\mathbb{H}}$ (11):

(67)

Twistorial Cohomotopy theory. A section (c,a) of the $\mathbb{C}P^3$-fibration $\widetilde{E}$ is a cocycle in a twisted non-abelian cohomology theory (53), which we call *Twistorial Cohomotopy theory* [7] of X. Notice that, as in (53), such a section is equivalently a lift of the classifying map τ to the parametrized twistor space:

---
[6]All diagrams in the following are filled with such homotopies, but for ease of presentation we mostly suppress them, notationally.

[7]Not to be confused with *twistor cohomology* (see, e.g., [EPW81]). The latter is abelian cohomology *of* twistor space, while Twistorial Cohomotopy is non-abelian cohomology with coefficients *in* (Borel-equivariantized) twistor space, hence with cocycles being maps *into* twistor space.

We write

$$
\underset{\substack{\text{Twistorial Cohomotopy} \\ \text{of tangentially } \mathrm{Sp}(2)\text{-structured} \\ \text{manifold}}}{\mathscr{T}^{\tau}(X)} \;:=\; \left\{ \begin{array}{c} \underset{\substack{\text{universally parametrized} \\ \text{twistor space}}}{\mathbb{C}P^3 /\!\!/ \mathrm{Sp}(2)} \\ \underset{\text{cocycle}}{(c,a)} \nearrow \qquad \downarrow \\ X \xrightarrow{ \underset{\mathrm{Sp}(2)\text{-structure}}{\tau} } B\mathrm{Sp}(2) \end{array} \right\}_{\big/\sim}
\tag{68}
$$

for the set of homotopy classes (relative X) of such sections, and call this the *cohomology set of Twistorial Coho-motopy*, when evaluated on spin-8 manifolds with tangential $\mathrm{Sp}(2)$-structure τ (65).

Twistor fibration as cohomology operation. Notice the direct analogy of Twistorial Cohomotopy theory (68) to J-twisted Cohomotopy theory [FSS19b, 2.1]:

$$
\underset{\substack{\text{J-twisted 4-Cohomotopy} \\ \text{of tangentially} \\ \mathrm{Sp}(2)\text{-structured} \\ \text{manifold}}}{\pi^{\tau}(X)} \;:=\; \left\{ \begin{array}{c} \underset{\substack{\text{universally parametrized} \\ \text{4-sphere}}}{S^4 /\!\!/ \mathrm{Sp}(2)} \\ \underset{\text{cocycle}}{c} \nearrow \qquad \downarrow \\ X \xrightarrow{ \underset{\mathrm{Sp}(2)\text{-structure}}{\tau} } B\mathrm{Sp}(2) \end{array} \right\}_{\big/\sim}
\tag{69}
$$

and the fact that postcomposition with the parametrized twistor fibration (Def. 2.5) constitutes a cohomology operation (a natural transformation of cohomology sets) between the two:

$$
\underset{\substack{\text{Twistorial} \\ \text{Cohomotopy}}}{\mathscr{T}^{\tau}} \xrightarrow[\underset{(t_{\mathbb{G}} /\!\!/ \mathrm{Sp}(2))_*}{}]{\underset{\substack{\text{cohomology operation by} \\ \text{parametrized twistor fibration}}}{}} \underset{\substack{\text{J-twisted} \\ \text{Cohomotopy}}}{\pi^{\tau}}
\tag{70}
$$

Chern-Dold character in Twistorial Cohomotopy. The Chern-Dold character (64) in J-twisted 4-Cohomotopy (69) is discussed in some detail [FSS19b]. The following Prop. 3.9 is its generalization to Twistorial Cohomotopy:

Proposition 3.9 (Character map in Twistorial Cohomotopy theory)**.** *The twisted non-abelian character* (64) *in Twistorial Cohomotopy* (68) *is of the following form:*

$$
\underset{\substack{\text{Twistorial} \\ \text{Cohomology} \\ \mathscr{T}^{\tau}(X)}}{} \xrightarrow[\substack{\text{twisted non-abelian} \\ \text{Chern-Dold character}}]{\mathrm{ch}_{\mathscr{T}}} \left\{ \begin{array}{l} F_2, \\ H_3, \\ G_4, \\ G_7 \end{array} \in \Omega^{\bullet}(X) \;\middle|\; \begin{array}{l} d\,F_2 = 0 \\ d\,H_3 = G_4 - \tfrac{1}{4}p_1(\omega) - F_2 \wedge F_2 \\ d\,G_4 = 0 \\ d\,G_7 = -\tfrac{1}{2}\big(G_4 - \tfrac{1}{4}p_1(\omega)\big) \wedge \big(G_4 + \tfrac{1}{4}p_1(\omega)\big) - \tfrac{1}{4}\big(p_2 - (\tfrac{1}{2}p_1(\omega))^2\big) \end{array} \right\}_{\big/\sim}
$$

$$
(c,a) \longmapsto (c,a)^* \begin{pmatrix} f_2, \\ h_3, \\ \omega_4, \\ \omega_7 \end{pmatrix}
\tag{71}
$$

Proof. By Theorem 2.14 the class of a section of the parametrized twistor fibration in rational homotopy theory is given equivalently by a dg-algebra homomorphism shown as the dashed arrow in the following diagram:

$$\tag{72}$$

Here the dg-algebras on the right are the Sullivan model for the Borel-equivariant twistor fibration (44) from Theorem 2.14. These being Sullivan models means that they are cofibrant as dg-algebras, which implies that all homotopy classes of rational sections are indeed represented this way. Therefore, a rational section is specified by the differential forms on X to which it pulls back the generators on the right. The condition for any such set of differential forms to arise this way is that it satisfies the same differential relations as the generators, now in the de Rham dg-algebra $\Omega^\bullet(X)$. This way the relation $d f_2 = 0$ in the Sullivan model pulls back to the relation $dF_2 = 0$ in $\Omega^\bullet(X)$ in (72), etc. $\qquad\square$

For use below, we record the de Rham-cohomological relations implied by the differential relations (71):

Corollary 3.10 (Cohomological relations in Twistorial Cohomotopy). *For X an 8-manifold with tangential* $\mathrm{Sp}(2)$-*structure (65), let* $F_2, H_3, G_4, G_7 \in \Omega^\bullet(X)$ *be differential form components in the image of the Chern-Dold character in Twistorial Cohomotopy on X (Def. 3.9). Then the real/de Rham cohomology classes these represent satisfy the following relations:*

$$[G_4] - \tfrac{1}{4}p_1 \;=\; [F_2 \wedge F_2] \;\in\; H^4(X, \mathbb{R})\,, \tag{73}$$

$$0 \;=\; \big([F_2 \wedge F_2] + \tfrac{1}{2}p_1\big) \cup [F_2 \wedge F_2] + \tfrac{1}{2}\big(p_2 - \tfrac{1}{4}p_1 \cup p_1\big) \;\in\; H^8(X, \mathbb{R})\,. \tag{74}$$

Proof. Equation (73) is the direct consequence of the second line in (71). From the fourth line of (71) we similarly get the relation

$$-[G_4 \wedge G_4] + \tfrac{1}{16}p_1 \cup p_1 - \chi_8 \;=\; 0 \tag{75}$$

Plugging (73) and (41) into (75) yields (74). $\qquad\square$

Charge quantization in Twistorial Cohomotopy. Finally we obtain the claimed result (6):

Corollary 3.11 (Shifted integrality of G_4, F_2 in Twistorial Cohomotopy). *Let X be a spin 8-manifold with tangential* $\mathrm{Sp}(2)$-*structure* τ *(65). Then differential form data* $(F_2, H_3, G_4, G_7) \in \Omega^\bullet(X)$ *which is in the image (71) of the Chern-Dold character from Prop. 3.9, hence which is charge-quantized (Remark 3.8) in Twistorial Cohomotopy (68), satisfies the following integrality conditions:*
(i) *The class of* G_4 *shifted by* $\tfrac{1}{4}p_1(\omega)$ *is integral, hence is the image in real cohomology of a class in integral cohomology:*

$$[G_4 + \tfrac{1}{4}p_1(\omega)] \;\in\; H^4(X, \mathbb{Z}) \longrightarrow H^4(X, \mathbb{R})\,. \tag{76}$$

(ii) *The class of F_2 is integral:*

$$[F_2] \in H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{R}). \tag{77}$$

(iii) *Hence the relation* (73) *is the image of such a relation in integral cohomology:*

$$[G_4 - \tfrac{1}{4}p_1(\omega)] = [F_2 \wedge F_2] \in H^4(X, \mathbb{Z}) \longrightarrow H^4(X, \mathbb{R}). \tag{78}$$

Proof. By Prop. 3.9 these de Rham classes are pullbacks of the generators in the Sullivan model from Theorem 2.14. By the normalization (45) there, the statement hence follows with Theorem 2.9. $\qquad\square$

In fact, we have a stronger statement:

Remark 3.12 (Cochain-level model of the C-field). While Corollary 3.11 produces Hořava-Witten's identity (6) between the cohomology classes related to the C-field in heterotic M-theory, the twistorial character map from Prop. 3.9 gives a little more information, namely an explicit differential form (cochain) model for these cohomology classes. Incidentally, this cochain expression for the C-field,

$$G_4 = \tfrac{1}{4}p_1(\omega) - c_2(A) + dH_3$$

as obtained from twistorial Cohomotopy in the second line of (71) (and from differential twistorial Cohomotopy in [FSS20c, (296)]), coincides with the proposed model for the C-field in [DFM03, (3.9)] (under identifying our H_3 with minus their c and our G_4 with minus their G).

A Quaternion-linear groups

For reference, we record some basics of quaternion-linear groups:

Definition A.1 (Special quaternion-linear group). The special quaternion-linear group

$$\mathrm{SL}(2, \mathbb{H}) := \big\{ G \in \mathrm{Mat}(2 \times 2, \mathbb{H}) \mid \det_{\mathrm{Di}}(G) = 1 \big\} \tag{79}$$

is the group of 2×2 quaternionic matrices with unit *Dieudonné determinant* [Di43] (review in [As96][VB20, §1]).

Remark A.2 (Size of $\mathrm{SL}(2, \mathbb{H})$). When restricted along the inclusion of complex matrices into quaternionic matrices

$$\mathrm{Mat}(2 \times 2, \mathbb{C}) \overset{i_{\mathbb{C}}}{\hookrightarrow} \mathrm{Mat}(2 \times 2, \mathbb{H})$$

the Dieudonné determinant does *not* reduce to the ordinary determinant, but to its absolute value:

$$\det_{\mathrm{Di}}\big(i_{\mathbb{C}}(A)\big) = \|\det(A)\|. \tag{80}$$

Accordingly, $\mathrm{SL}(2, \mathbb{H})$ (Def. A.1) is larger than the notation might suggest: For instance, it follows immediately from (80) that all complex unitary matrices have unit Dieudonné determinant. In fact, Example A.4 says that the full quaternion-unitary group (Def. A.3) is contained in $\mathrm{SL}(2, \mathbb{H})$ (85) (and hence coincides with what would otherwise be called $\mathrm{SU}(2, \mathbb{H})$).

Definition A.3 (Unitary quaternion-linear groups). Let $n \in \mathbb{N}$.
(i) The $n \times n$ *quaternionic unitary group* is

$$\mathrm{Sp}(n) := \mathrm{U}(n, \mathbb{H}) := \big\{ G \in \mathrm{GL}(n, \mathbb{H}) \mid G \cdot G^{\dagger} = 1 \big\}, \tag{81}$$

where $(-)^{\dagger}$ denotes matrix transpose combined with quaternionic conjugation.

(ii) The *central product group* of $\mathrm{Sp}(n_1)$ with $\mathrm{Sp}(n_2)$ is

$$\mathrm{Sp}(n_1) \cdot \mathrm{Sp}(n_2) := \big(\mathrm{Sp}(n_1) \times \mathrm{Sp}(n_2)\big) / \underbrace{\{(1,1), (-1,-1)\}}_{\simeq \mathbb{Z}_2} \tag{82}$$

Example A.4 (*Subgroups of quaternion-linear groups*). We have the following canonical subgroup inclusions into special quaternion-linear (Def. A.1) and unitary quaternion-linear groups (Def. A.3):

(i) The algebra inclusion of the complex numbers into the quaternions induces:

$$\mathbb{C} \hookrightarrow \mathbb{H} \tag{83}$$

$$\begin{array}{ccc} \mathrm{U}(n,\mathbb{C}) & \hookrightarrow & \mathrm{U}(n,\mathbb{H}) \\ \| & & \| \\ \mathrm{U}(n) & \hookrightarrow & \mathrm{Sp}(n) \end{array}$$

(ii) We write

$$\mathrm{Sp}(1)_L \times \mathrm{Sp}(1)_R \hookrightarrow \mathrm{Sp}(2) \tag{84}$$
$$(q_L, q_R) \longmapsto \mathrm{diag}(q_L, q_R)$$

for the subgroup of $\mathrm{Sp}(2)$ given by the diagonal matrices with coefficients in unit-norm quaternions q, hence the direct product group of two copies of $\mathrm{Sp}(1)$, equipped with their left and right factor embedding, as indicated.

(iii) The unitary quaternion-linear 2×2-matrices (Def. A.3) have Dieudonné-determinant (Def. A.1) equal to 1 [CDL00, 6.4] and hence include into the special quaternion-linear group:

$$\mathrm{Sp}(2) = \mathrm{U}(2,\mathbb{H}) \subset \mathrm{SL}(2,\mathbb{H}). \tag{85}$$

(iv) There is the canonical subgroup inclusion of symplectic-unitary groups into their central product groups (82)

$$\mathrm{Sp}(n_1) \hookrightarrow \mathrm{Sp}(n_1) \cdot \mathrm{Sp}(n_2) \tag{86}$$
$$A \longmapsto [A, 1]$$

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