

Flux Quantization of Type IIA in Unstable K-Theory

PINAK BANERJEE¹, HISHAM SATI^{2,3}, AND URS SCHREIBER²
pinakb24@vt.edu hsati@nyu.edu us13@nyu.edu

¹Department of Physics, Virginia Tech, USA

²Mathematics Program and Center for Quantum and Topological Systems (CQTS),
New York University Abu Dhabi (NYUAD), UAE

³The Courant Institute of Mathematical Sciences, New York University, New York, USA

September 26, 2026

Abstract

The traditional conjecture that RR-fluxes are quantized in stable K-cohomology fails to account for the presence of NS-brane sources: These impose nonlinear relations — reductions of the famous quadratic relation on M-brane flux — that can only be captured by unstable nonabelian cohomology theories. Here we consider a deformation of unstable K-theory which properly quantizes the fluxes coupling to D0/D2/NS5-branes, find a twisted version that quantizes also the fluxes coupling to NS1/D4-branes, and show that this oxidizes to a proper electromagnetic quantization of M-brane fluxes.

Contents

1 The Flux Quantization	2
2 The Proofs	10
3 Conclusion	21
References	22

keywords: flux quantization, supergravity, type IIA, M-theory, unstable K-theory, nonabelian cohomology
MSC codes: Primary: 8E50, 81T30, 19L50; Secondary: 55N20, 55R35, 55P62
funding: by *Tamkeen UAE* under the *Abu Dhabi Research Institute Grant CG008*

1 The Flux Quantization

The Nonlinearity in the IIA Gauss Laws. The NS/RR-flux densities, H_3 , $H_7 := \star H_3$ and $F_{2\bullet} = \pm \star F_{10-2\bullet}$, in duality-symmetric massive type IIA 10D supergravity are subject to the following differential identities (cf. [Ort15, §22.1.3] after [GP84; HN85; DLT98; Cre+98, §3; Lav+99, (5.9); GH11, (2.13) & (s.14)]):

$$d F_{2\bullet} = H_3 \wedge F_{2\bullet-2} \quad (1a)$$

$$d H_3 = 0 \quad (1b)$$

$$d H_7 = \frac{1}{2} F_4 \wedge F_4 - F_2 \wedge F_6 + F_0 \wedge F_8. \quad (1c)$$

On spacetime these may be called the duality symmetric *Bianchi identities* (cf. [Ber+01, (2.11)]), but upon restriction to any Cauchy surface (“canonical formulation”) these differential equations become unambiguously the *Gauss law constraints* on flux data, and this is how we understand it here—we expand on this Rem. 1.1.

In discussing possible *flux/charge quantization laws*¹ [Fre02a; SS25a; MS25, §4.1; SS26c] for these relations, it has been tradition to ignore the last one (1c). Granting that, the first relation (1a) has the form characterizing the Chern character image in twisted complex topological K-theory (cf. [Bou+02, §6.3; MS03]), with the twist being in ordinary 3-cohomology quantizing the H_3 -flux in (1b).

This is essentially the origin of the famous conjecture that D/NS-brane charge is quantized in 3-twisted topological K-theory ([GHM97; MM97; MW00], cf. [Fre01, §3.7; SS25a, §4.1]). A lot of work has been done under this assumption (cf. [Wit98; DFM11; GS22; HTZ25]).

However, the last relation (1c) cannot actually be disregarded if one really means to speak about type IIA supergravity. Notice in particular that:

- (i) the standard K-theory conjecture means to quantize not only the magnetic ($F_{<5}$) but also the electric ($F_{>5}$) RR-flux densities, which highlights that there is no reason to disregard quantization of the electric NS flux H_7 , much less ignore the relation it imposes on the other fluxes already rationally, even if not itself quantized;
- (ii) the nonlinear H_7 Gauss law (1c) is (cf. [MS04, §4.2; FSS17, §3]) the direct image under dimensional reduction (cf. [Ort15, §22.1]) of the more widely appreciated nonlinearity in the Gauss law on the flux densities, G_4 and $G_7 := \star G_4$, in 11D supergravity (cf. [MS06, §3.1.3; GSS24a, Thm. 3.1] after [CJS78]):

$$d G_4 = 0, \quad (2a)$$

$$d G_7 = \frac{1}{2} G_4 \wedge G_4. \quad (2b)$$

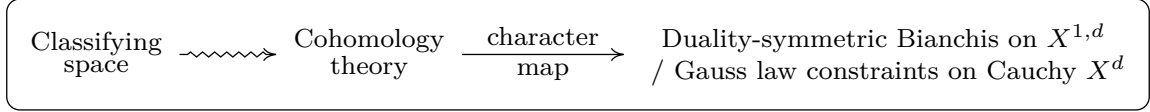
Hence if the global completion of 11D/10D supergravity by flux quantization is to be compatible with M/IIA duality, it must account for these nonlinearities (1c) and (2b).

¹ Beware that the term “quantization” in “flux/charge quantization” is in the sense of *discretization* (as in *flux quanta* and *charge units*) not in the sense of *quantum physics* (even if both are related: flux quantization of background fluxes tends to provide quantum anomaly cancellation for probe brane sigma-models coupled to these fluxes, cf. [WW85, §5; Alv85, §1] and [FSS21a]): All charges discussed here are classical observables; their possible quantum observable algebra is a separate discussion (for aspects of which see instead [FMS07a; FMS07b; SS25b]).

Proper Electromagnetic Flux Quantization. In general, flux/charge quantization means [SS25a; SS24b] to choose a generalized cohomology theory whose *character map* [FSS23] image reproduces the given duality-symmetric Bianchi identities (for which cf. Rem. 1.1), and then to ask for charge preimages of the flux densities under this map, cf. Fig. 1 and (58).

Specifically, this applies to the spatial components of the flux densities on any Cauchy surface X^d of spacetime $X^{1,d}$, where the Hodge duality partners become independent of each other and the Bianchi identities become the Gauss law constraints, cf. [SS24b, Thm. 2.2]:

Figure 1: The *flux/charge quantization* of higher gauge fields means to choose a generalized cohomology theory whose character map takes values in differential forms satisfying the given duality-symmetric Bianchi identities on spacetime $X^{1,d}$, or equivalently the Gauss law constraints on a Cauchy surface X^d . After this choice, the globally (topologically) completed field content consists, besides these flux densities, of a choice of cocycle in their essential character preimage.



Traditionally this has been considered (cf. [Fre02a]) for Whitehead-generalized abelian cohomology theories like ordinary cohomology, K-theory, elliptic cohomology, etc. In their twisted generalization these abelian cohomology theories capture *linear* Gauss laws like (1a) — but abelian cohomology theories can never (cf. [FSS23, Ex. 5.6]) capture nonlinear Gauss laws like (1c) and (2b).

Concretely in the case (1) at hand, the preimages of the RR-flux densities $F_{2\bullet>0}$ under the (Chern) character map are polynomials $\text{ch}_\bullet$ in the *Chern classes* $c_\bullet \in H^{2\bullet}(\text{KU}_0; \mathbb{Z})$ on the stable classifying space for complex K-theory (cf. [Kar78, §V.3; FHT08, §I.2; FSS23, Ex. 7.2]):

$$\text{KU}_0 \simeq \text{BU}(\infty) \times \mathbb{Z} := \bigcup_n \text{BU}(n) \times \mathbb{Z}. \quad (3)$$

Despite its superficial appearance, this space (3) is famously an infinite loop space (by homotopy-theoretic *Bott periodicity*, cf. [AGP02, §B.2; Hus+08, §15]), which makes complex K-theory be an abelian cohomology theory (cf. [FSS23, Ex. 2.10]).

Hence if we are to account for the nonlinearity in (1c), then we have to deform this space and break its infinite loop space property to obtain a classifying space of an unstable *nonabelian cohomology* theory (27) [Toë02, Def. 6.0.6; Lur14, Def. 6; FSS23, §2] (more exposition in [SS25c, §1; SS26d, §4]).

Remark 1.1 (Duality-symmetric formulation of flux dynamics). We are concerned here with the question of flux/charge quantization jointly in the magnetic and the electric sector (known in the literature as *duality-symmetric* [SS94; BBS98; Nur03] or *democratic* [MV23; Fer+24] or, historically, *pre-metric* [Dan34]).

The archetypical example is the already mentioned proposal of RR-flux/charge quantization in K-theory: This involves (cf. Tab. 1):

- (i) regarding electric and magnetic RR-fluxes $F_{2\bullet}$ on the same footing,
- (ii) observing that the equations of motion on the magnetic fluxes, rewritten via Hodge duality as differential equations on the electric fluxes, unify with the Bianchi identities

on the magnetic fluxes to a degree-uniform duality-symmetric Bianchi identity $d F_{2\bullet} = F_{2\bullet-2} \wedge H_3$ (cf. [Ber+01, (2.11)]),

- (iii) realizing that this duality-symmetric Bianchi identity is of the form seen in the image of the Chern character on twisted K-theory.

Table 1: The flux species in massless 10D IIA SuGra and the duality-symmetric Bianchi identities which they satisfy. On-shell the fluxes are further related by spacetime Hodge duality, $F_6 = \star F_4$, $F_8 = \star F_2$, and $H_7 = \star H_3$, which, together with the duality-symmetric Bianchi identities, are equivalent to the equations of motion (cf. Fig. 2).

Flux species	magnetic $F_2 \ H_3 \ F_4$	electric $F_6 \ H_7 \ F_8$
$ \begin{aligned} d F_2 &= 0 \wedge H_3 \\ d F_4 &= F_2 \wedge H_3 & d H_3 &= 0 \\ d F_6 &= F_4 \wedge H_3 & d H_7 &= \tfrac{1}{2} F_4 \wedge F_4 - F_2 \wedge F_6 \\ d F_8 &= F_6 \wedge H_3 \end{aligned} $		

This is just a special case of the general duality-symmetric situation (cf. Fig. 2), where we have a tuple of electromagnetic flux species (not *a priori* distinguished as being magnetic or electric),

$$\vec{F} := \{F^{(i)} \in \Omega_{\text{dR}}^{\deg_i}(X^{1,d})\}_{i=1}^N, \quad (4)$$

all subject to Bianchi identities with polynomial sources $P^{(i)}(\vec{F})$ and to a Hodge duality relation given by components of a linear isomorphism $\mu^{(i)}(\vec{F})$, as shown in Fig. 2.

It is only after a choice of flux basis $\{F^{(j)}\}_{j=1}^N$ for which (and this need not exist, such as when N is odd and generally when some fluxes are self-dual)

$$\mu = \begin{pmatrix} 0 & \pm I_{N/2} \\ I_{N/2} & 0 \end{pmatrix}, \quad (5)$$

that one may distinguish

- (i) *magnetic fluxes* $F^{(j \leq N/2)}$ from *electric fluxes* $F^{(j > N/2)}$,
- (ii) Bianchi identities $dF^{(j)} = P^{(j)}(\vec{F})$ from *equations of motion* $d \star F^{(j)} = \pm P^{(j+N/2)}(\vec{F})$ (where $1 \leq j \leq \frac{N}{2}$).

In traditional constructions of supergravity a magnetic/electric distinction is provided, but this is a convention, not a property of the theories (cf. Lagrangian constructions in [MV23; Fer+24] and the superspace construction of 11D SUGRA in [GSS24a, Thm. 3.1]).

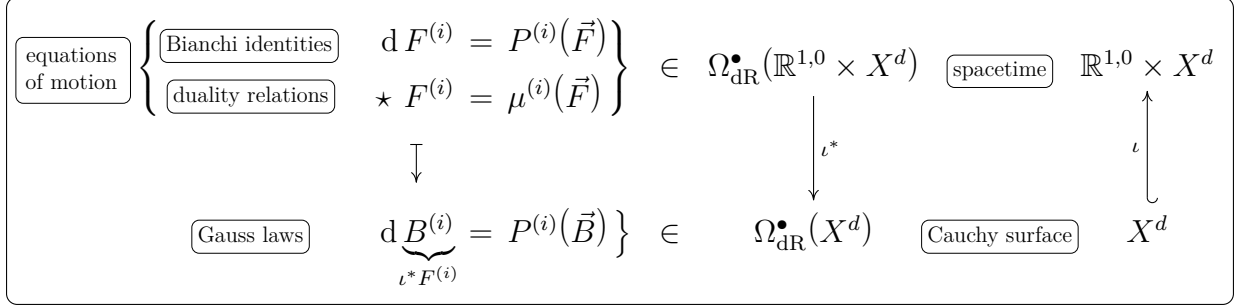
Note that here it is the *formulation* that is duality-symmetric, not necessarily the theory. In fact any Maxwell-type gauge sector appearing in descendants of 11D supergravity may be written in this form. Notably the ordinary vacuum Maxwell equations may be written in duality-symmetric form as

$$\begin{aligned}
d F_2 &= 0 & E_2 &= \star F_2 \\
d E_2 &= 0,
\end{aligned} \quad (6)$$

While, therefore, the duality-symmetric formulation is just a matter of perspective when working over spacetime, it acquires intrinsic physical meaning over phase space. Namely if spacetime is globally hyperbolic, $X^{1,d} \simeq \mathbb{R}^{1,0} \times X^d$ (which is what we assume throughout), with a chosen spatial *Cauchy surface*

$$X^d \xrightarrow[x \mapsto (0,x)]{\iota} \mathbb{R}^{1,0} \times X^d \simeq X^{1,d}, \quad (7)$$

Figure 2: **Duality-symmetric equations of motion, Bianchis and Gauss laws.** When flux equations of motion are written in the “*duality-symmetric form*”, where electric fluxes appear on the same footing as magnetic fluxes, then all differential equations are of the form of *Bianchi identities*, now imposed not just on the magnetic but also on the electric fluxes, cf. [DLM95, (3.9); BBS98, above (2.4) & (5.26); Lav+99, (5.9); MW00, (2.15); Ber+01, (2.11); HT03, (54) & (56); Ber+09, (1.1); GH11, (2.13) & (s.14); Fer+24, (4.30)]. On the latter, these become the actual equations of motion after enforcing the *duality relations*. Upon pullback to any Cauchy surface, these duality-symmetric Bianchi identities become the magnetic and electric *Gauss laws* [SS26c, §3.2.1; SS25a, §3.1; SS24b, Thm. 2.2]. For the special case of plain p -form electromagnetism, cf. also [Fre02b, Ex. 3.8; MS25, §31.3].



consider the restriction (pullback) of the fluxes (4) to the Cauchy surface, there to be denoted:

$$\vec{B} := \{B^{(i)} := \iota^* F^{(i)} \in \Omega_{\text{dR}}^{\deg_i}(X^d)\}_{i=1}^N. \quad (8)$$

For instance, in the example of 4D Maxwell theory (6), these pullbacks are the magnetic and electric field strengths as seen in the frame X^d .

Since pullback of differential forms is a dg-algebra homomorphism, it follows that the $\vec{B}$ satisfy differential equations of just the same form as the duality-symmetric Bianchi identities, namely $d\vec{B} = \vec{P}(\vec{B})$. Here on the Cauchy surface, these equations are the magnetic and electric *Gauss laws* (cf. bottom row of Fig. 2), exactly the constraints on *initial value data* that uniquely characterize (germs of) solutions to the equations of motion propagating away from the Cauchy surface [SS24b, Thm. 2.2; SS26c, §3.2.1].

The duality-symmetric formulation on spacetime is such that, under restriction to a Cauchy surface, the duality-symmetric Bianchi identities become exactly the electromagnetic Gauss law constraints.

And since on globally hyperbolic spacetimes—where all spacetime topology is in the spatial Cauchy surface X^d —electromagnetic charges are associated with the topology of the Cauchy surface alone, this shows that admissible flux/charge quantization in the joint magnetic and electric sectors must reflect the form $\vec{P}$ of the Gauss law constraints and hence equivalently the form of the duality-symmetric Bianchi identities on spacetime.

This way, it is, for instance, the above duality-symmetric formulation of Maxwell theory (6) which naturally relates to its duality-symmetric flux/charge quantization ([FMS07a, (1.26); Bec+17, Rem. 2.3; LS18; LS22a; LS22b]), where not only the magnetic but also the electric sector is required to be a cocycle in (differential) integral cohomology in degree 2.

The corresponding duality-symmetric formulation of the C-field in 11D SuGra is that shown in (2).

D0/D2/NS5-sector and Unstable K-Theory. To warm up to the idea of deforming the traditional situation (3) to account for the nonlinear Bianchi (1c), consider first the special case when only F_2 , F_4 and H_7 (the fluxes coupling to D0/D2/NS5-branes) are non-zero (understanding all flux densities from now on as their spatial components on a Cauchy surface X^d). In this case, (1) reduces to:

$$dF_2 = 0, \quad dF_4 = 0, \quad (9a)$$

$$dH_7 = \frac{1}{2}F_4 \wedge F_4. \quad (9b)$$

Taken at face value, the first relation (9a) is flux-quantized by the differential nonabelian cohomology theory whose classifying space is the finite stage $BU(2)$ of (3). This may be understood as a form of differential *unstable K-theory* (cf. [SS26b, Ex. 1.18; FSS23, Prop. 9.4]), following the terminology of “unstable K-theory” from [Kal80, §1.2], cf. [HK03; Ham03; HK04; CJ24, §1.3].

For comparison: On globally hyperbolic 10D spacetimes, the stable range for unstable K-theory is that classified by $BU(n > 4)$. Hence, while $BU(2)$ is “very unstable” in absolute terms, for 10D spacetimes it is halfway through the unstable range.

To obtain from this a classifying space $\widehat{BU(2)}$ whose induced flux quantization is compatible also with the nonlinearity (9b), the canonical step is to pass to the *homotopy fiber* of the map classifying $(c_2)^2 := c_2 \cup c_2$, making a homotopy fiber sequence (34):

$$\widehat{BU(2)} \xrightarrow{\text{hofib}((c_2)^2)} BU(2) \xrightarrow{(c_2)^2} K(\mathbb{Z}, 8). \quad (10)$$

(Here $\widehat{U(2)}$ itself may be understood as a Lie 6-group which as such is a cousin of the *MFivebrane 6-group* of [FSS21a, Ex. 3.2], following terminology of [SSS12; SSS09].)

Hence the nonabelian differential cohomology theory (cf. [FSS23, §§2, 9]) classified by this $\widehat{BU(2)}$ — being an admissible flux quantization law for (9), cf. Lem. 2.13 — is a deformation of unstable topological K-theory that properly takes into account the nonlinear Bianchi identity (1c) for the NS flux H_7 in the case that $H_3 = 0$ and $F_6 = 0$.

Proper Flux Quantization of Type IIA and Twisted Unstable K-theory. After this warm-up exercise for going beyond abelian cohomology theories, we next ask for a comprehensive flux quantization of (1) and a compatible lift to M-theory quantizing also (2). On this second point, we highlight (with [BSS19]) the open secret that the derivation of massive type IIA from 11D is elusive, in that what one really gets by double dimensional reduction of the 11D SuGra fluxes (2) is (cf. again [MS04, §4.2; FSS17, §3]) only the restriction of (1) to:

- (i) F_2 (the class of the 11D circle bundle),
- (ii) F_4 (the direct image of G_4),
- (iii) F_6 (the reduced image of G_7),
- (iv) H_3 (the reduced image of G_4)
- (v) H_7 (the direct image of G_7):

$$dF_2 = 0, \quad dF_4 = H_3 \wedge F_2, \quad dF_6 = H_3 \wedge F_4 \quad (11a)$$

$$dH_3 = 0 \quad (11b)$$

$$dH_7 = \frac{1}{2}F_4 \wedge F_4 - F_2 \wedge F_6. \quad (11c)$$

This being the exact reduction of the 11D Bianchis (2), we may appeal to the following theorem (cf. [BSS19, §2.2; SS24a], going back to [FSS18, §3]): For $\mathcal{B}$ any classifying space that quantizes the fluxes in 11D, its *cyclic loop space*

$$\mathrm{Cyc}(\mathcal{B}) := (L\mathcal{B})//S^1 \quad (12)$$

is an admissible flux quantization for the resulting Bianchis (11).

But we show in § 2 (Cor. 2.12) that an admissible choice for $\mathcal{B}$ in 11D is

$$\widehat{BSU(2)} := \mathrm{hofib}\left(BSU(2) \xrightarrow{(c_2)^2} K(\mathbb{Z}, 8)\right). \quad (13)$$

This implies (Cor. 2.16) that the space

$$\mathrm{Cyc}\left(\widehat{BSU(2)}\right) \quad (14)$$

classifies an admissible flux quantization for the type IIA fluxes (11).

The universal property of the Cyc-construction then implies (Cor. 2.2) that there is a canonical comparison map:

$$\widehat{BU(2)} \longrightarrow \mathrm{Cyc}\left(\widehat{BSU(2)}\right), \quad (15)$$

which reflects (Cor. 2.17) exactly the lift to flux-quantized fields of the inclusion of the restricted situation (9) into the full situation (11). In view of the previous discussion, we may hence think of (14) as classifying a deformed and *unstable* version of *3-twisted K-theory*.

This shows how the traditional conjecture, that D-brane charge is quantized in stable K-theory, may be corrected to account for traditionally ignored nonlinearities in the Bianchi identities, by passage to a deformed version of unstable K-theory, and how after this correction the flux-quantized situation is actually a reduction of a corresponding situation in 11D, as it should be.

Relation to Hypothesis H. Beware here that in saying so we are using a slightly different flux quantization in 11D than was considered in [FSS20; FSS21a] following [Sat18, §2.5], where the classifying space in 11D was instead taken to be S^4 , quantizing the C-field fluxes in 4-Cohomotopy (“Hypothesis H”).

But the two are closely related and in fact indistinguishable in compactifications to $\mathbb{R}^{1,3}$: There is a canonical comparison map

$$S^4 \xrightarrow{\sim \leq 7} \widehat{BSU(2)}, \quad (16)$$

which is an integral homotopy equivalence up to dimension 7 (this is our main Thm. 2.6), and hence in particular a rational homotopy equivalence (Cor. 2.11).

This highlights once more that the global completion of higher gauge theories by flux quantization is a choice which needs to be justified by checking that it implies desired or expected consequences.

The choice $\mathcal{B} \equiv S^4$ stands out in that it provably implies core topological effects expected in “M-theory” (cf. [Duf99]), such as the $\frac{1}{4}p_1$ -shifted flux quantization of G_4 [FSS20, Prop. 3.13] and anomaly cancellations of the M5-brane [FSS21a, Thm. 4.8; SS21]. These results are unlikely to carry over to $\mathcal{B} \equiv \widehat{BSU(2)}$, notably in that the relevant tangential twisting by $\mathrm{Spin}(5)$ -structure no longer applies.

However, we may turn this around: The 7-equivalence (16) shows that when the tangential twist is trivial (such as for compactification on tori or other group manifolds) and the compactified space is of dimension ≤ 7 , then the cohomotopical flux quantization with $\mathcal{B} \equiv S^4$ becomes equivalent to that with $\mathcal{B} \equiv \widehat{BSU(2)}$ and as such the deformed unstable K-theoretic nature of its dimensional reduction follows.

These equivalences in low dimension may hence serve to relate different hypotheses about flux quantization to each other. Compare also the fact that the stabilization (homotopy linearization) of S^4 is equivalent to the classifying space for *topological modular forms* (tmf) in degree 4, over 10-manifolds ([Hop02, §4.3; FSS23, Ex. 7.5]):

$$\Sigma^\infty S^4 \xrightarrow{\sim \leq 10} \Sigma^4 \text{tmf}. \quad (17)$$

Including Worldvolume Flux on D4/M5. More generally, in the presence of M5-brane probes ([HS97; HSW97; Aga+97b; Sor00, §5.2; GSS24b, §3]) of the 11D SuGra bulk along an embedding

$$\Sigma^{1,5} \hookrightarrow X^{1,10}, \quad (18)$$

there famously is a (nonlinearly) self-dual flux density $\mathcal{H}_3$ on the worldvolume $\Sigma^{1,5}$ (cf. [GSS24b, Rem. 3.19]) satisfying, on top of (2), the further worldvolume Bianchi identity (cf. [GSS24b, Prop. 3.18]):

$$d\mathcal{H}_3 = \phi^* G_4. \quad (19)$$

Equivalently, after double dimensional reduction [Duf+87] of the M5 to a D4-probe of type IIA (cf. [Tow96; Aga+97b, §6; Aga+97a, §6]),

$$\Sigma^{1,4} \hookrightarrow X^{1,9}, \quad (20)$$

this becomes (cf. [Ste01; Zho01, §3; Ban26, §2]):

$$d\mathcal{F}_2 = \varphi^* H_3 \quad (21a)$$

$$d\mathcal{H}_3 = \varphi^* F_4 - \varphi^* F_2 \wedge \mathcal{F}_2, \quad (21b)$$

where $\mathcal{F}_2$ is the twisted flux density of the D4-worldvolume Chan-Paton gauge field and $\mathcal{H}_3 = \star_{1,4} \mathcal{F}_2$ its dual electric flux density.

Flux quantization of such *twisted relative* Bianchi identities (namely twisted by the bulk fluxes relative to the worldvolume embedding) happens in *twisted relative nonabelian cohomology theories* ([BSS26, §3.2; SS26d, §2; FSS23, §3]), where classifying spaces $\mathcal{B}$ are enhanced to classifying fibrations $\mathcal{A} \twoheadrightarrow \mathcal{B}$:

$$H^0(\phi; \wp) := \pi_0 \text{Map} \left(\begin{array}{ccc} \Sigma^{1,5} & \dashrightarrow & \mathcal{A} \\ \downarrow \phi & & \downarrow \wp \\ X^{1,10} & \dashrightarrow & \mathcal{B} \end{array} \right). \quad (22)$$

The minimal (in number of CW-cells) admissible choice of flux quantization $\wp$ for higher gauge sector of M5@11Bulk turns out to be ([FSS20, §3.7; FSS21a; FSS21b]) the quaternionic Hopf fibration (51) $S^7 \xrightarrow{h_{\mathbb{H}}} S^4$, which gives the flux quantization of 11D SuGra with M5-probes according to *Hypothesis H*.

Hence, in view of the above discussion, we are now to ask whether the comparison map (16) to the alternative K-flavored flux quantization law extends to a 7-equivalence of classifying

fibrations. Indeed, we find a canonical $SU(2)$ -fiber bundle (52) over $\widehat{BSU(2)}$ equipped with compatible comparison maps (53) which are integral homotopy equivalences in dimensions ≤ 7 (Thm. 2.9) and hence are in particular rational homotopy equivalences (Cor. 2.15):

$$\begin{array}{ccc} S^7 & \xrightarrow{\sim \leq 7} & B^7\mathbb{Z} \\ \downarrow h_{\mathbb{H}} & & \downarrow \wp \\ S^4 & \xrightarrow{\sim \leq 7} & \widehat{BSU(2)}. \end{array} \quad (23)$$

Figure 3: **Overview for IIA.** On the bottom left, the bulk type IIA Bianchi identities (11) that appear via double dimensional reduction from 11D. On the top left, the further Bianchis (21) on a D4-probe embedding φ . On the right, the corresponding classifying maps of admissible flux quantization of this situation, first according to Hypothesis H and, on the far right, according to the more K-flavored flux quantization law established here. This dimensionally oxidizes to flux quantizations of 11D SuGra surveyed in Fig. 4.

$\begin{aligned} d\mathcal{F}_2 &= \varphi^* H_3 \\ d\mathcal{H}_3 &= \varphi^* F_4 - \varphi^* F_2 \wedge \mathcal{F}_2 \\ dF_2 &= 0 \\ dF_4 &= H_3 \wedge F_2, \quad dF_6 = H_3 \wedge F_4 \\ dH_3 &= 0 \\ dH_7 &= \tfrac{1}{2}F_4 \wedge F_4 - F_2 \wedge F_6. \end{aligned}$	$\Sigma^{1,4} \dashrightarrow \text{Cyc}(S^7) \xrightarrow{\text{Cyc}(\hat{h}_{\mathbb{H}})} \text{Cyc}(B^7\mathbb{Z})$ $\downarrow \varphi$ $X^{1,9} \dashrightarrow \text{Cyc}(S^4) \xrightarrow{\text{Cyc}(\hat{1})} \text{Cyc}(\widehat{BSU(2)})$	$\downarrow \text{Cyc}(h_{\mathbb{H}})$ $\downarrow \text{Cyc}(\wp)$
--	---	---

As before, by double dimensional reduction of flux quantization laws via cyclification (12) of classifying spaces [BSS19, §2.2], it follows at once that the cyclified fibration

$$\begin{array}{c} \text{Cyc}(B^7\mathbb{Z}) \\ \downarrow \text{Cyc}(\wp) \\ \text{Cyc}(\widehat{BSU(2)}) \end{array} \quad (24)$$

classifies a twisted relative flux quantization law for type IIA with D4-brane probes (cf. [Aga+97a, §6]). The situation is summarized in Fig. 3.

Figure 4: **Overview for 11D.** On the bottom left, the Bianchi identities in the bulk of 11D SuGra (2); on the top left, the further Bianchi on an M5 probe (19). On the right, the corresponding classifying maps of admissible flux quantization of this situation, first according to Hypothesis H and then, on the far right, according to the more K-flavored flux quantization law established here. Under double dimensional reduction, this yields exactly the situation in Fig. 3.

$\begin{aligned} d\mathcal{H}_3 &= \phi^* G_4 \\ dG_4 &= 0 \\ dG_7 &= \tfrac{1}{2}G_4 \wedge G_4 \end{aligned}$	$\Sigma^{1,5} \dashrightarrow S^7 \xrightarrow{\hat{h}_{\mathbb{H}}} B^7\mathbb{Z}$ $\downarrow \phi$ $X^{1,10} \dashrightarrow S^4 \xrightarrow{\hat{1}} \widehat{BSU(2)}$	$\downarrow h_{\mathbb{H}}$ $\downarrow \wp$
---	---	---

2 The Proofs

We freely use tools from algebraic topology and homotopy theory, as found in textbooks such as [Whi78; Jam84; Mas91; Koc96; Hat02; AGP02; Ark11; FF16]. In particular we use the homotopy long exact sequences (LES) induced by homotopy fibrations (cf. [FF16, §9.8]); for concise review in our context cf. [SS26a, §A.3].

Eilenberg-MacLane spaces.. For A a discrete abelian group and $n \in \mathbb{N}$, we write $B^n A := K(A, n)$ for the *Eilenberg-MacLane space* of A in degree n (cf. [AGP02, §6]), characterized uniquely (up to weak homotopy equivalence) by the property

$$\pi_k B^n A = \begin{cases} A & \text{if } k = n \\ \text{trivial} & \text{otherwise.} \end{cases} \quad (25)$$

This classifies ordinary cohomology (cf. [AGP02, §7.1; FSS23, Ex. 2.1]), in that cohomology classes correspond to homotopy classes of classifying maps:

$$H^n(X; A) \simeq \pi_0 \text{Map}(X, B^n A). \quad (26)$$

This classical fact is the archetype for the definition of generalized *nonabelian cohomology* [Lur14, Def. 6; FSS23, §2], defined for *any* (pointed) space $\mathcal{A}$ (its *classifying space*) as

$$H^0(X; \mathcal{A}) := \pi_0 \text{Map}(X, \mathcal{A}). \quad (27)$$

Quaternionic Projective Spaces. We recall that S^4 is homeomorphically the quaternionic projective line, while $BSU(2)$ is the infinite projective space (cf. [SS23, Rem. 3.88]), and that under this identification the canonical inclusion map represents the generator $1 \in \mathbb{Z} \simeq \pi_4(BSU(2))$:

$$\begin{array}{ccc} S^4 & \xrightarrow{1} & BSU(2) \\ \wr & & \wr \\ \mathbb{H}P^1 & \xrightarrow{\iota_1} \mathbb{H}P^2 \xrightarrow{\iota_2} \dots \hookrightarrow & \mathbb{H}P^\infty. \end{array} \quad (28)$$

Concretely, $\mathbb{H}P^2$ is obtained from $\mathbb{H}P^1$ by attaching an 8-cell along the quaternionic Hopf fibration $h_{\mathbb{H}}$ (51):

$$\begin{array}{ccc} S^7 & \xrightarrow{h_{\mathbb{H}}} & \mathbb{H}P^1 \\ \downarrow & \lrcorner & \downarrow \iota_1 \\ D^8 & \longrightarrow & \mathbb{H}P^2. \end{array} \quad (29)$$

Next, $\mathbb{H}P^3$ is obtained by adding a further 12-cell and so on.

In view of this cell decomposition, one finds that:

- (i) the map $\mathbb{H}P^2 \hookrightarrow \mathbb{H}P^\infty$ induces isomorphisms on homotopy up to dimension 10 (by the cellular approximation theorem, cf. [Hat02, Thm. 4.8])

$$\pi_{\leq 10} \mathbb{H}P^2 \xrightarrow{\sim} \pi_{\leq 10} \mathbb{H}P^\infty, \quad (30)$$

- (ii) the integral cohomology ring of quaternionic complex projective spaces is (as in [Hat02, Thm. 3.19]) freely generated by a degree=4 generator, which under the above equivalence we may address as the second Chern class c_2 (as in [MS74, §14]):

$$c_2 \in H^4(BSU(2); \mathbb{Z}) \subset H^\bullet(BSU(2); \mathbb{Z}) \simeq \mathbb{Z}[c_2]. \quad (31)$$

(alternatively, under the further homeomorphism $SU(2) \simeq \text{Spin}(3)$, as the first fractional Pontrjagin class $\frac{1}{2}p_1$).

Using the same symbol also for the pullback of this generator to any finite stage, we have the cohomology rings

$$H^\bullet(\mathbb{H}P^n; \mathbb{Z}) \simeq \mathbb{Z}[c_2] / ((c_2)^{n+1}). \quad (32)$$

In particular (and trivially so):

$$H^4(S^4; \mathbb{Z}) \simeq \mathbb{Z}[c_2] / ((c_2)^2). \quad (33)$$

The 6-Group Classifying Spaces. By (26) the squared universal second Chern class has a classifying map, which we denote by the same symbol:

$$\begin{array}{ccccc} BSU(2) & \longrightarrow & BU(2) & \xrightarrow{(c_2)^2} & B^7\mathbb{Z} \\ & \searrow & \downarrow & & \uparrow \\ & & & (c_2)^2 & \end{array} \quad (34)$$

The homotopy fibers of these maps are our hatted spaces $\widehat{BU(2)}$ (10) and $\widehat{BSU(2)}$ (13), which thereby come with a canonical comparison map:

Lemma 2.1. *This canonical map*

$$\widehat{BSU(2)} \longrightarrow \widehat{BU(2)} \quad (35)$$

is a homotopy $U(1)$ -fibration.

Proof. First, we note that the unhatted map is already a homotopy $U(1)$ -fibration (cf. [FSS20, Lem. 2.7(b)]):

$$\text{fib}(BSU(2) \rightarrow BU(2)) \simeq U(2)/SU(2) \simeq U(1). \quad (36)$$

Now, the fiber in question may be understood as the homotopy fiber X in the left vertical sequence of the following homotopy-commuting diagram

$$\begin{array}{ccccc} X & \longrightarrow & U(1) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{BSU(2)} & \longrightarrow & BSU(2) & \xrightarrow{(c_2)^2} & B^8\mathbb{Z} \\ \downarrow & & \downarrow & & \parallel \\ \widehat{BU(2)} & \longrightarrow & BU(2) & \xrightarrow{(c_2)^2} & B^8\mathbb{Z} \end{array} \quad (37)$$

Here the middle and bottom rows are homotopy fiber sequences by definition, and the middle and right columns are homotopy fiber sequences by (36). But since homotopy limits commute with each other, it follows that:

$$\begin{aligned} X &\equiv \text{fib}\left(\widehat{BSU(2)} \rightarrow \widehat{BU(2)}\right) \\ &\simeq \text{fib}(U(1) \rightarrow *) \\ &\simeq U(1). \end{aligned} \quad \square$$

Corollary 2.2. *Under the Ext/Cyc-adjunction ([BSS19, §2.2; SS24a, §2.2]), the adjunct of the identity on $\widehat{BSU(2)}$ is a map of the form (15):*

$$\begin{array}{ccc} \widehat{BU(2)} & \xrightarrow{\quad} & \text{Cyc}(\widehat{BSU(2)}) \\ & \searrow & \swarrow \\ & BU(1) & \end{array} \quad (38)$$

Blakers-Massey theorem. Recall that:

- (i) a topological space is called:
 - (-2) -connected: always,
 - (-1) -connected if it is not empty,
 - 0-connected if it is (-1) -connected and path connected,
 - $(k \geq 1)$ -connected if it is 0-connected and all its $\pi_{\leq k}$ are trivial;
- (ii) a map between spaces is called:
 - k -connected if all its homotopy fibers are $(k-1)$ -connected.

So by the homotopy LES, a k -connected map induces isomorphisms on $\pi_{\leq k-1}$.

Proposition 2.3 (Blakers-Massey theorem [Goo91, p. 309; Ane+20]). *Given a homotopy pushout square, where f_i is k_i -connected,*

$$\begin{array}{ccc} X & \xrightarrow{f_2} & B \\ \downarrow f_1 & \searrow & \downarrow \\ A & \xrightarrow{\quad} & A \sqcup_X B \end{array} \quad (39)$$

the dashed comparison map to the homotopy pullback is $(k_1 + k_2 - 1)$ -connected.

Example 2.4. Applied to the cell attachment (29), which is evidently the homotopy pushout of a 3-connected map along a 7-connected map, the BM theorem (Prop. 2.3) gives that the comparison map to the homotopy fiber $\text{fib}(\iota_1)$ of the first cell inclusion:

$$\begin{array}{ccc} S^7 & \xrightarrow{h_{\mathbb{H}}} & S^4 \\ \downarrow & \searrow & \downarrow \iota_1 \\ * & \xrightarrow{\quad} & \mathbb{H}P^2 \end{array} \quad (40)$$

is 9-connected, hence induces isomorphisms

$$\pi_{\leq 8} S^7 \xrightarrow{\sim} \pi_{\leq 8} \text{fib}(\iota_1). \quad (41)$$

Lemma 2.5. *The homotopy fibers of $S^4 \xrightarrow{\iota_1} \mathbb{H}P^2$ and of $S^4 \xrightarrow{1} \mathbb{H}P^\infty$ are identified in low degrees:*

$$\pi_{\leq 9} \text{fib}(\iota_1) \xrightarrow{\sim} \pi_{\leq 9} \text{fib}(1). \quad (42)$$

Proof. The map of homotopy fiber sequences

$$\begin{array}{ccccc} \text{fib}(\iota_1) & \longrightarrow & S^4 & \xrightarrow{\iota_1} & \mathbb{H}P^2 \\ \downarrow & & \parallel & & \downarrow \\ \text{fib}(1) & \longrightarrow & S^4 & \xrightarrow{1} & \mathbb{H}P^\infty \end{array} \quad (43)$$

induces the following map of corresponding homotopy LESs:

$$\begin{array}{ccccccc} \pi_{k+1} S^4 & \longrightarrow & \pi_{k+1} \mathbb{H}P^2 & \longrightarrow & \pi_k \text{fib}(\iota_1) & \longrightarrow & \pi_k S^4 \xrightarrow{(\iota_1)^*} \pi_k \mathbb{H}P^2 \\ \parallel & & \downarrow & & \downarrow & & \parallel \downarrow \\ \pi_{k+1} S^4 & \longrightarrow & \pi_{k+1} \mathbb{H}P^\infty & \longrightarrow & \pi_k \text{fib}(1) & \longrightarrow & \pi_k S^4 \xrightarrow{1_*} \pi_k \mathbb{H}P^\infty. \end{array} \quad (44)$$

Hence the five-lemma (cf. [Wei94, Ex. 1.3.3]) implies the claim, by (30). $\square$

The Comparison Map. By (26) and (33), the canonical map (28) has null-homotopic postcomposition with $(c_2)^2$ and hence factors, up to homotopy, through the latter's homotopy fiber (13) via an essentially unique map $\hat{1}$:

$$\begin{array}{ccc} & \widehat{BSU(2)} & \longrightarrow * \\ & \uparrow \hat{1} \quad \downarrow p \quad \dashv h & \downarrow \\ S^4 & \xrightarrow{1} BSU(2) & \xrightarrow{(c_2)^2} B^8\mathbb{Z}. \end{array} \quad (45)$$

Theorem 2.6. *The comparison map $\hat{1}$ (45) induces isomorphisms on $\pi_{\leq 7}$.*

Proof. In degrees $n \leq 6$ this is straightforward: The map 1 induces isomorphisms in these degrees (since the next cell of $BSU(2)$ is in dimension 8) as does the map fib (by the long exact sequence of homotopy groups and using that $\pi_{\leq 7}(B^8\mathbb{Z})$ is trivial).

More work is required to show isomorphism on π_7 . To this end, consider the following map of homotopy fiber sequences induced from the diagram (45):

$$\begin{array}{ccccc} \text{fib}(1) & \longrightarrow & S^4 & \xrightarrow{1} & BSU(2) \\ \downarrow \phi & & \downarrow \hat{1} & & \parallel \\ B^7\mathbb{Z} & \longrightarrow & \widehat{BSU(2)} & \xrightarrow{p} & BSU(2) \end{array} \quad (46)$$

The corresponding morphism of induced homotopy LESs is of this form:

$$\begin{array}{ccccccc} \pi_8 BSU(2) & \longrightarrow & \overbrace{\pi_7 \text{fib}(1)}^{\mathbb{Z}} & \longrightarrow & \pi_7 S^4 & \xrightarrow{1_*} & \pi_7 BSU(2) \longrightarrow \overbrace{\pi_6 \text{fib}(1)}^0 \\ \parallel & & \downarrow \phi_* & & \downarrow \hat{1}_* & & \parallel \downarrow \\ \pi_8 BSU(2) & \longrightarrow & \underbrace{\pi_7 B^7\mathbb{Z}}_{\mathbb{Z}} & \longrightarrow & \pi_7 \widehat{BSU(2)} & \xrightarrow{p_*} & \pi_7 BSU(2) \longrightarrow \underbrace{\pi_6 B^7\mathbb{Z}}_0, \end{array} \quad (47)$$

where over the braces we used (41) and (42).

Hence the five-lemma (cf. [Wei94, Ex. 1.3.3]) implies the claim as soon as we show that ϕ_* here is an isomorphism. We relegate this to Lem. 2.7. $\square$

Lemma 2.7. *The map ϕ_* in (47) is an isomorphism.*

Proof. Consider now the map of cohomology Serre LESs (cf. [McC01, Ex. 5.D with Thm. 5.2]) induced by (46):

$$\begin{array}{ccccc} \overbrace{H^8(S^4; \mathbb{Z})}^0 & \xleftarrow{1^*} & H^8(BSU(2); \mathbb{Z}) & \xleftarrow{\tau_8} & H^7(\text{fib}(1); \mathbb{Z}) & \xleftarrow{\quad} & \overbrace{H^7(S^4; \mathbb{Z})}^0 \\ & & \parallel & & \uparrow \phi^* & & \\ & & H^8(BSU(2); \mathbb{Z}) & \xleftarrow{\tau'_8} & H^7(B^7\mathbb{Z}; \mathbb{Z}) & & \end{array} \quad (48)$$

Here the top transgression map, τ_8 , is immediately seen to be an isomorphism by exactness, and the bottom map may be checked to be an isomorphism, too (Lem. 2.8). By commutativity, this implies that ϕ^* is an isomorphism. From this, the Hurewicz theorem (cf. [Hat02, Thm. 4.32]) implies that also ϕ_* is an isomorphism, since $B^7\mathbb{Z}$ and $\text{fib}(1)$ are 6-connected by (25) and (41). $\square$

Lemma 2.8. *The map τ'_8 in (48) is an isomorphism.*

Proof. By construction (45), the bottom homotopy fiber sequence in (46) is equivalently the pullback of the path fibration over $B^8\mathbb{Z}$:

$$\begin{array}{ccc} B^7\mathbb{Z} & \xlongequal{\quad} & B^7\mathbb{Z} \\ \downarrow & & \downarrow \\ \widehat{BSU(2)} & \longrightarrow & PB^8\mathbb{Z} \\ \downarrow p & \lrcorner & \downarrow \\ BSU(2) & \xrightarrow{(c_2)^2} & B^8\mathbb{Z}. \end{array} \quad (49)$$

Hence, again by the naturality of the cohomology Serre LES (cf. [McC01, Ex. 5.D with Thm. 5.2]), we have the commuting square

$$\begin{array}{ccc} H^8(BSU(2); \mathbb{Z}) & \xleftarrow{\tau'_8} & H^7(B^7\mathbb{Z}; \mathbb{Z}) \\ \wr \uparrow ((c_2)^2)^* & & \parallel \\ H^8(B^8\mathbb{Z}; \mathbb{Z}) & \xleftarrow[\sim]{\tau_8^{\text{univ}}} & H^7(B^7\mathbb{Z}; \mathbb{Z}). \end{array} \quad (50)$$

But as indicated, here the bottom map is an isomorphism (by [Mil21, Lem. 65.2]) as is the left map, by (31). This implies the claim, by commutativity. $\square$

This completes the proof of Thm. 2.6, establishing the integral 7-equivalence. Next we extend this to a 7-equivalence of classifying fibrations.

Relation to the Quaternionic Hopf fibration. We note that the homotopy fiber of our unit map (28) is the unit class $SU(2)$ -principal bundle over S^4 , which is the quaternionic Hopf fibration $h_{\mathbb{H}}$ (cf. [Nak18, (9.62); SS26d, §5.2.3]):

$$\begin{array}{ccc} S^7 & \xrightarrow{\quad} & * \\ h_{\mathbb{H}} \downarrow & \lrcorner h & \downarrow \\ S^4 & \xrightarrow{1} & BSU(2) \end{array} \quad (51)$$

Its hatted analogue in our context is hence the homotopy fiber $\wp$ of p (45):

$$\begin{array}{ccc} B^7\mathbb{Z} & \xrightarrow{\quad} & * \\ \wp \downarrow & \lrcorner h & \downarrow \\ \widehat{BSU(2)} & \xrightarrow{p} & BSU(2). \end{array} \quad (52)$$

By the homotopy commutativity of (45) and the universal property of homotopy fibers, we have an essentially unique canonical comparison map $\hat{h}_{\mathbb{H}}$ between these two fibrations:

$$\begin{array}{ccccc} S^7 & \xrightarrow{h_{\mathbb{H}}} & S^4 & \xrightarrow{1} & BSU(2) \\ \downarrow \hat{h}_{\mathbb{H}} & & \downarrow \hat{1} & & \parallel \\ B^7\mathbb{Z} & \xrightarrow{\wp} & \widehat{BSU(2)} & \xrightarrow{p} & BSU(2). \end{array} \quad (53)$$

Theorem 2.9. *The comparison map $\hat{h}_{\mathbb{H}}$ (53) induces isomorphisms on $\pi_{\leq 7}$.*

Proof. The map of homotopy LESs induced by (53) is of this form:

$$\begin{array}{ccccccc} \pi_{k+1} S^4 & \xrightarrow{1_*} & \pi_{k+1} BSU(2) & \xrightarrow{\partial_k} & \pi_k S^7 & \xrightarrow{(h_{\mathbb{H}})_*} & \pi_k S^4 \xrightarrow{1_*} \pi_k BSU(2) \\ \downarrow \hat{1}_* & & \parallel & & \downarrow (\hat{h}_{\mathbb{H}})_* & & \downarrow \hat{1}_* \parallel \\ \pi_{k+1} \widehat{BSU(2)} & \xrightarrow{p} & \pi_{k+1} BSU(2) & \xrightarrow{\partial'_k} & \pi_k B^7\mathbb{Z} & \xrightarrow{\wp_*} & \pi_k \widehat{BSU(2)} \xrightarrow{p_*} \pi_k BSU(2). \end{array} \quad (54)$$

Hence, by the five-lemma (cf. [Wei94, Ex. 1.3.3]), it is now sufficient to show that $\hat{1}_*$ is an isomorphism on $\pi_{\leq 7}$ and an epimorphism on π_8 . The first statement is Thm. 2.6, the second we relegate to Lem. 2.10. $\square$

Lemma 2.10. *The map $\hat{1}$ (45) induces an epimorphism on π_8 .*

Proof. The map of homotopy LESs induced by (53) looks in the relevant parts as follows:

$$\begin{array}{ccccccc} & & \pi_8 S^4 & \xrightarrow{1_*} & \overbrace{\pi_8 BSU(2)}^{\text{finite}} & \xrightarrow[\partial_7]{\quad} & \overbrace{\pi_7 S^7}^{\mathbb{Z}} \\ & & \downarrow \hat{1}_* & & \parallel & & \\ \underbrace{\pi_8 B^7\mathbb{Z}}_0 & \longrightarrow & \pi_8 \widehat{BSU(2)} & \xrightarrow[p_*]{\sim} & \underbrace{\pi_8 BSU(2)}_{\text{finite}} & \xrightarrow[\partial'_7]{\quad} & \underbrace{\pi_7 B^7\mathbb{Z}}_{\mathbb{Z}}. \end{array} \quad (55)$$

Using Serre finiteness (60) on the terms with braces implies that the shown connecting homomorphisms both vanish, whence exactness implies that 1_* is an epimorphism and p_* is an isomorphism. But then commutativity implies the claim. $\square$

This establishes that we have a 7-equivalence of classifying fibrations. Next we turn to the rationalization of this situation.

Rational Homotopy. For general background on rational homotopy theory, cf. [FHT00; Hes07; FSS23, §5] and for exposition in our context see [FSS19]. Here we denote the *minimal Sullivan model* (cf. [Men15]) for the $\mathbb{R}$ -rational homotopy type of a space $\mathcal{B}$ (simply connected and of finite rational type) by:

$$\mathrm{CE}(\mathfrak{t}\mathcal{B}) \in \mathrm{dgcAlg}_{\mathbb{R}}, \quad (56)$$

since it is the *Chevalley-Eilenberg algebra* of the *real Whitehead-bracket L_∞ -algebra* $\mathfrak{I}\mathcal{B}$ [FSS23, Prop. 5.11, Rem. 5.4]. Moreover, when we make these CE-algebras explicit we write (as in [GSS25]):

$$\mathrm{CE}(\mathfrak{I}\mathcal{B}) \equiv \mathbb{R}_d[\text{generators}] / (\text{differential relations}), \quad (57)$$

indicating how they are *free differential algebras* (FDA in supergravity terminology, cf. [CD25; FSS19; FSS15]) on the given graded generators, quotiented by the given differential ideal.

The subscript on a generator will always indicate its degree, directly analogous to the notation for the flux densities in § 1. Indeed, $\mathcal{B}$ is the classifying space of an admissible flux-quantization law if [SS25a, §3.1-2; SS24b] there is a presentation (57) such that these generators and relations correspond to the given flux species and their Bianchi/Gauss identities, respectively:

$$\begin{aligned} \mathrm{CE}(\mathfrak{I}\mathcal{B})\text{-generators} &\leftrightarrow \text{flux species} \\ \mathrm{CE}(\mathfrak{I}\mathcal{B})\text{-relations} &\leftrightarrow \begin{array}{l} \text{duality-symmetric Bianchi identities} \\ \text{/ Gauss laws on Cauchy surface.} \end{array} \end{aligned} \quad (58)$$

More generally, for $\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$ a fibration of spaces, we denote the *relative* minimal Sullivan model of $\mathcal{A}$, φ -relative to the minimal Sullivan model of $\mathcal{B}$, by (cf. [FSS23, Prop. 5.16]):

$$\mathrm{CE}(\mathfrak{I}_{\mathcal{B}}\mathcal{A}) \in \mathrm{dgcAlg}_{\mathbb{R}}, \quad (59)$$

obtained by adjoining a minimal set of *further* generators to the minimal model (57) of the base (without removing any of these generators).

Hence, while contractible pairs of generators, of the form $(de_p = f_{p+1}, df_{p+1} = 0)$, do not appear in minimal Sullivan models (being non-minimal), they may appear in relative minimal Sullivan models, namely whenever the cohomology class of f_{p+1} in the base space vanishes when pulled back along φ .

Now, first to recall that the classical *Serre finiteness theorem* (cf. [Rav04, §I, Thm. 1.1.8]) says that the homotopy groups of spheres are mostly all finite. Concretely, for $n \in \mathbb{N}$:

$$\pi_k S^{2n+1} \simeq \begin{cases} \mathbb{Z} & \text{if } k = 2n+1 \\ \text{finite} & \text{otherwise,} \end{cases} \quad (60a)$$

$$\pi_k S^{2(n+1)} \simeq \begin{cases} \mathbb{Z} & \text{if } k = 2(n+1) \\ \mathbb{Z} \times \text{finite} & \text{if } k = 4(n+1) - 1 \\ \text{finite} & \text{otherwise.} \end{cases} \quad (60b)$$

Accordingly, all finite group factors (*torsion subgroups*) disappear under rationalization, and the Sullivan minimal models of spheres are (cf. [Men15, §1.2]) concentrated on just the one or two remaining free homotopy groups:

$$\mathrm{CE}(\mathfrak{I}S^{2n+1}) \simeq \mathbb{R}_d[g_{2n+1}] / (dg_{2n+1} = 0) \quad (61a)$$

$$\mathrm{CE}(\mathfrak{I}S^{2(n+1)}) \simeq \mathbb{R}_d \left[\begin{array}{c} g_{2(n+1)} \\ g_{4(n+1)-1} \end{array} \right] / \left(\begin{array}{c} dg_{2(n+1)} = 0 \\ dg_{4(n+1)-1} = \frac{1}{2}g_{2(n+1)}^2 \end{array} \right). \quad (61b)$$

With this, Thm. 2.6 readily implies:

Corollary 2.11. *The comparison map $\hat{1}$ (45) is a rational homotopy equivalence.*

Proof. By Serre’s finiteness theorem (60), the only non-torsion homotopy groups of S^4 are in degrees 4 and 7. Therefore, with Thm. 2.6, it is sufficient to see that all homotopy groups of $\widehat{BSU(2)}$ in degree ≥ 8 are finite.

Now the homotopy LES in degrees $n \geq 9$,

$$\begin{array}{c} \overbrace{\pi_{n+1} B^8 \mathbb{Z}}^0 \\ \left. \begin{array}{c} \xrightarrow{\partial_n} \\ \searrow \end{array} \right\} \pi_n \widehat{BSU(2)} \xrightarrow{\sim} \pi_n BSU(2) \longrightarrow \underbrace{\pi_n B^8 \mathbb{Z}}_0 \end{array} \quad (62)$$

immediately implies that

$$\begin{aligned}\pi_n \widehat{BSU(2)} &\simeq \pi_n BSU(2) \\ &\simeq \pi_{n-1} SU(2) \\ &\simeq \pi_{n-1} S^3,\end{aligned}\tag{63}$$

which is indeed finite, again by Serre's theorem.

In the remaining degree $n = 8$, the homotopy LES instead gives

$$\begin{array}{c}
\overbrace{\pi_9 B^8 \mathbb{Z}}^0 \\
\downarrow \partial_8 \\
\pi_8 \widehat{BSU(2)} \longrightarrow \pi_8 BSU(2) \longrightarrow \underbrace{\pi_8 B^8 \mathbb{Z}}_{\mathbb{Z}} .
\end{array} \quad (64)$$

But since $\pi_8 BSU(2) \simeq \pi_7 S^3$ is again finite, the last map here is zero and hence exactness implies, as before, that also $\pi_8 \widehat{BSU}(2) \simeq \pi_7 S^3$ is finite. $\square$

Corollary 2.12. *Under rationalization of the comparison map $\hat{1}$ (45), the minimal Sullivan model of $\widehat{BSU(2)}$ (13) is identified with that of the 4-sphere:*

$$\mathfrak{I}S^4 \xrightarrow[\sim]{\mathfrak{I}\hat{1}} \mathfrak{I}\widehat{BSU(2)}, \quad (65)$$

hence:

$$\text{CE}(\widehat{\mathfrak{I}BSU(2)}) \simeq \mathbb{R}_d \left[\begin{matrix} g_4 \\ g_7 \end{matrix} \right] / \left(\begin{matrix} dg_4 = 0 \\ dg_7 = \frac{1}{2}g_4^2 \end{matrix} \right). \quad (66)$$

By comparison with (2), this shows that $\widehat{BSU(2)}$ is an admissible classifying space for electromagnetic flux quantization of the gauge sector of 11D supergravity, as discussed in § 1.

Of course, the minimal Sullivan model (66) by itself (disregarding the comparison map $\hat{1}$) also follows by more elementary rational homotopy theory, not requiring Cor. 2.11. And since rationally $U(2) \simeq_0 U(1) \times SU(2)$, we similarly have:

Lemma 2.13. *The minimal Sullivan model of $\widehat{BU(2)}$ (10) is:*

$$\text{CE}(\widehat{\text{IBU}(2)}) \simeq \mathbb{R}_d \left[\begin{array}{c} f_2 \\ f_4 \\ h_7 \end{array} \right] / \left(\begin{array}{l} \text{d } f_2 = 0 \\ \text{d } f_4 = 0 \\ \text{d } h_7 = \frac{1}{2} f_4^2 \end{array} \right). \quad (67)$$

Comparison with (9a) hence shows that $\widehat{BU(2)}$ is an admissible classifying space for that situation, as discussed there.

From combining (67) with (66) we also have:

Lemma 2.14. *The relative minimal Sullivan model (59) of $\widehat{BSU(2)}$ $U(1)$ -fibered (35) over $\widehat{BU(2)}$ is*

$$\mathrm{CE}\left(\mathfrak{l}_{\widehat{BU(2)}}\widehat{BSU(2)}\right) \simeq \mathbb{R}_d \left[\begin{array}{c} \theta_1 \\ f_2 \\ f_4 \\ h_7 \end{array} \right] / \left(\begin{array}{l} d\theta_1 = f_2 \\ df_2 = 0 \\ df_4 = 0 \\ dh_7 = \frac{1}{2}f_4^2 \end{array} \right), \quad (68)$$

whence the identity map on $\widehat{BSU(2)}$ is also modeled by the quasi-isomorphism:

$$\begin{array}{ccc} \mathfrak{l}_{\widehat{BU(2)}}\widehat{BSU(2)} & \xrightarrow{\sim} & \mathfrak{l}\widehat{BSU(2)} \\ f_4 & \longleftarrow & f_4 \\ h_7 & \longleftarrow & h_7. \end{array} \quad (69)$$

Similarly, Thm. 2.9 readily implies, furthermore:

Corollary 2.15. *The fiber comparison map $\hat{h}_{\mathbb{H}}$ (53) is a rational homotopy equivalence.*

Proof. By Thm. 2.9, it is sufficient to observe that all higher homotopy groups $\pi_{\geq 8}S^7$ are finite (60). $\square$

In particular, therefore, the *relative* minimal Sullivan model (59) of the fibration $\wp$ (52) coincides with that of the quaternionic Hopf fibration, which is [FSS20, Prop. 3.20; FSS22, Lem. 2.13]:

$$\mathrm{CE}\left(\mathfrak{l}_{\widehat{BSU(2)}}B^7\mathbb{Z}\right) \simeq \mathrm{CE}\left(\mathfrak{l}_{S^4}S^7\right) \simeq \mathbb{R}_d \left[\begin{array}{c} h_3 \\ g_4 \\ g_7 \end{array} \right] / \left(\begin{array}{l} dh_3 = g_4 \\ dg_4 = 0 \\ dg_7 = \frac{1}{2}g_4^2 \end{array} \right). \quad (70)$$

Comparison with Fig. 4 therefore shows that both $h_{\mathbb{H}}$ as well as our more K-flavored $\wp$ classify admissible twisted relative flux quantization laws for 11D SuGra with M5-probes.

Cyclification. Given the minimal Sullivan model of a space $\mathcal{B}$,

$$\mathrm{CE}(\mathfrak{l}\mathcal{B}) \simeq \mathbb{R}_d \left[(e^i)_{i \in I} \right] / \left(d_{\mathcal{B}} e^i = P^i(\vec{e}) \right), \quad (71)$$

we recall that the minimal Sullivan model of its cyclic loop space, $\mathrm{Cyc}(\mathcal{B})$ (12), is given ([BV85], cf. [FSS17, Prop. 3.2; SV23, Prop. 2.4(ii)]) by:

$$\mathrm{CE}(\mathfrak{l}\mathrm{Cyc}(\mathcal{B})) \simeq \mathbb{R}_d \left[\begin{array}{c} (e^i)_{i \in I} \\ (se^i)_{i \in I} \\ \omega_2 \end{array} \right] / \left(\begin{array}{l} de^i = d_{\mathcal{B}} e^i + \omega_2 se^i \\ dse^i = -s(d_{\mathcal{B}} e^i) \\ d\omega_2 = 0 \end{array} \right), \quad (72)$$

where se^i are degree-shifted generators, $\deg(se^i) = \deg(e^i) - 1$, and on the right we understand the *shift operator*

$$s : \begin{cases} e^i \mapsto se^i \\ se^i \mapsto 0 \end{cases} \quad (73)$$

as uniquely extended to a (-1) -graded derivation on $\mathbb{R}[(e^i)_{i \in I}, (se^i)_{i \in I}]$.

Using this with Cor. 2.12 it follows (as in [FSS17, §3], cf. [GSS25, Ex. 2.26]) that:

Corollary 2.16. *The minimal Sullivan model of the cyclified loop space of $\widehat{BSU(2)}$ is:*

$$\mathrm{CE}(\mathfrak{ICyc}(\widehat{BSU(2)})) \simeq \mathbb{R}_a \left[\begin{array}{c} f_2 \\ f_4 \\ f_6 \\ h_3 \\ h_7 \end{array} \right] / \left(\begin{array}{l} d f_2 = 0 \\ d f_4 = h_3 f_2 \\ d f_6 = h_3 f_4 \\ d h_3 = 0 \\ d h_7 = \frac{1}{2} f_4^2 - f_2 f_6 \end{array} \right). \quad (74)$$

Proof. In view of the general formula (72) applied to (66), this is a matter of naming the generators as follows

$$\begin{aligned} h_7 &:= +g_7 \\ f_6 &:= -sg_7 \\ f_4 &:= +g_4 \\ h_3 &:= +sf_4 \end{aligned} \quad (75)$$

(noting the conventional choice of relative signs). $\square$

Comparison with (11) thus shows that $\mathrm{Cyc}(\widehat{BSU(2)})$ is therefore indeed the classifying space of an admissible flux quantization law for the gauge sector of (non-massive) type IIA supergravity.

Corollary 2.17. *Pullback along the map $\mathfrak{I}BU(2) \longrightarrow \mathfrak{ICyc}(\widehat{BSU(2)})$ which is induced on minimal Sullivan models by the map (15) and (38), is the identity of f_2 , f_4 and h_7 and vanishes on f_6 and h_3 .*

Proof. By construction in Cor. 2.2, the map is the Ext/Cyc-adjunct of the identity on $\mathfrak{I}\widehat{BSU(2)}$. Applying the general formula [GSS25, Prop. 2.25; FSS18, Thm. 3.8] for these adjuncts to the above model (69) for that identity map yields the correspondence:

$$\begin{array}{ccc} \mathfrak{I}\widehat{BU(2)} & \longrightarrow & \mathfrak{ICyc}(\widehat{BSU(2)}) \\ \mathfrak{I}\widehat{BSU(2)} \xrightarrow{\sim} \mathfrak{I}\widehat{BSU(2)} & & \\ \begin{array}{ccc} f_4 & \longleftarrow & g_4 \\ h_7 & \longleftarrow & g_7 \end{array} & \rightsquigarrow & \begin{array}{ccc} f_2 & \longleftarrow & f_2 \\ f_4 & \longleftarrow & f_4 \\ 0 & \longleftarrow & f_6 \\ 0 & \longleftarrow & h_3 \\ h_7 & \longleftarrow & h_7 \end{array} \end{array} \quad (76)$$

On the right, this is the claimed pullback. $\square$

Further, with Cor. 2.15, the immediate generalization of this computation to the relative minimal Sullivan model of φ yields, as in [Ban26, §3]:

Corollary 2.18. *The relative minimal Sullivan model of the cyclification of $\wp$ is:*

$$\mathrm{CE}\left(\mathfrak{l}_{\mathrm{Cyc}(\widehat{BSU(2)})}\mathrm{Cyc}(B^7\mathbb{Z})\right) \simeq \mathbb{R}_d \left[\begin{array}{c} f_2 \\ h_3 \\ f_2 \\ f_4 \\ f_6 \\ h_3 \\ h_7 \end{array} \right] / \left(\begin{array}{l} d f_2 = h_3 \\ d h_3 = f_4 - f_2 f_2 \\ d f_2 = 0 \\ d f_4 = h_3 f_2 \\ d f_6 = h_3 f_4 \\ d h_3 = 0 \\ d h_7 = \frac{1}{2} f_4 f_4 - f_2 f_6 \end{array} \right). \quad (77)$$

Proof. In view of the general formula (72), now applied to (70), this is a matter of naming the further generators, beyond (75), as follows:

$$\begin{aligned} h_3 &:= +h_3 \\ f_2 &:= -s h_3 \end{aligned} \quad (78)$$

(noting again the conventional choice of signs). $\square$

Comparison with Fig. 3 shows that, therefore, $\wp$ is indeed an admissible flux quantization law for type IIA with D4-probes.

3 Conclusion

The global completion of a higher gauge theory by flux/charge quantization is a choice, but candidate quantization laws in nontrivial situations have hardly been explored.

The traditional proposal that RR-flux (D-brane charge) is quantized in twisted topological K-theory suffers (besides from the problematic ontology of the “massive” flux species F_0 and F_{10}) from its unjustified disregard of the nonlinear Bianchi identity for the dual NS-flux. At the same time, such nonlinear Bianchis do not admit flux quantization in the traditionally considered abelian (stable) cohomology theories like K-theory.

The solution is to pass to flux quantization in nonabelian cohomology theories. For 11D SUGRA, the minimal choice (“Hypothesis H”) of full electromagnetic flux quantization (minimal in number of cells in the corresponding classifying space) is in 4-Cohomotopy, classified by the 4-sphere. Under the process of double dimensional reduction of flux-quantization laws, namely by cyclification of their classifying spaces, this implies a consistent quantization of (actual, non-massive) type IIA SuGRA.

However, the resulting *cyclified 4-Cohomotopy* is in general not closely related to K-theory, at least not manifestly so. Hence, in order to see to what extent the K-theory conjecture may be salvaged while retaining dimensional oxidation to 11D (hence retaining M/IIA duality even after flux quantization), here we looked for a more “K-flavored” but still strictly admissible choice of electromagnetic flux quantization of the gauge sector of type IIA SuGRA.

One such choice is naturally suggested by understanding the classifying space $S^4 \simeq \mathbb{H}P^1$ as the quaternionic projective line, and as such as the 4-skeleton of the infinite quaternionic projective space $\mathbb{H}P^\infty \simeq BSU(2)$. The latter, being $U(1)$ -fibered over a finite stage in the classifying space for complex K-theory, is closer to K-theory, and as such might be understood as classifying a form of “unstable K-theory”, at least bringing in cocycle data in the form of unitary bundles.

Our main mathematical observation here (Thm. 2.6) is that, up to dimension 7, the homotopy theoretic identification of S^4 with $BSU(2)$ is obstructed exactly only by the square of the second Chern class, which vanishes on S^4 but not on $BSU(2)$: Forcing this squared class to vanish yields a deformed classifying space $\widehat{BSU(2)}$ which does classify an admissible electromagnetic flux quantization of 11D SuGRA but is more “K-theory flavored” than S^4 (with which however it agrees up to dimension 7), in that unitary bundles, at least of rank 2, serve as cocycle data.

Concretely, the double-dimensional reduction of this quantization law yields an admissible proper quantization of the IIA flux densities classified by the cyclic loop space $\text{Cyc}(\widehat{BSU(2)})$, which receives a canonical map from the “deformed unstable K-theory” classified by $\widehat{BU(2)}$.

This map thus exhibits $\text{Cyc}(\widehat{BSU(2)})$ as classifying a nonabelian cohomology theory that is an unstable form (correction) of the traditionally expected twisted K-theory, while being strictly admissible as a flux-quantization law for (non-massive) type IIA supergravity, including the nonlinear Bianchi identity of the dual NS-flux. A variant also extends to the twisted relative quantization of the further worldvolume fluxes on D4-branes.

Moreover, still by the main theorem (Thm. 2.6), the flux quantization of this “K-flavored” law strictly coincides with the cohomotopical one (Hypothesis H) on 10D spacetimes of the

form $\mathbb{R}^{1,3} \times X^6$. Hence, it immediately inherits the latter’s favorable properties regarding topological conditions expected in M-theory.

In this sense, $\text{Cyc}(\widehat{BSU(2)})$ classifies a nonabelian cohomology theory that fares fairly well in meeting both: (i) the strict technical demands on a proper flux quantization of type IIA SuGra, as well as (ii) further string/M-theoretic folklore expectations.

But it needs to be re-emphasized that global completion of any higher gauge theory, like type IIA SuGra, is a choice that requires attention, and that there are an infinitude of further admissible flux quantization laws waiting to be explored and to be compared to given desiderata and expectations.

Acknowledgments. We thank Alonso Perez-Lona for bringing further references to our attention.

References

- [Aga+97a] M. Aganagic et al. “Dual D-brane actions”. In: *Nuclear Physics B* 496.1-2 (1997), pp. 215–230. DOI: [10.1016/S0550-3213\(97\)00257-5](https://doi.org/10.1016/S0550-3213(97)00257-5). arXiv: [hep-th/9702133](https://arxiv.org/abs/hep-th/9702133) (cit. on pp. 8, 9).
- [Aga+97b] M. Aganagic et al. “World-volume action of the M theory five-brane”. In: *Nuclear Physics B* 496.1-2 (1997), pp. 191–214. DOI: [10.1016/S0550-3213\(97\)00227-7](https://doi.org/10.1016/S0550-3213(97)00227-7). arXiv: [hep-th/9701166](https://arxiv.org/abs/hep-th/9701166) (cit. on p. 8).
- [AGP02] M. Aguilar, S. Gitler, and C. Prieto. *Algebraic Topology from a Homotopical Viewpoint*. Universitext. Springer, 2002. ISBN: 9780387224893. DOI: [10.1007/b97586](https://doi.org/10.1007/b97586) (cit. on pp. 3, 10).
- [Alv85] O. Alvarez. “Topological Quantization and Cohomology”. In: *Commun. Math. Phys.* 100.2 (June 1985), pp. 279–309. DOI: [10.1007/bf01212452](https://doi.org/10.1007/bf01212452) (cit. on p. 2).
- [Ane+20] M. Anel et al. “A generalized Blakers-Massey theorem”. In: *Journal of Topology* 13.4 (2020), pp. 1521–1553. DOI: [10.1112/topo.12163](https://doi.org/10.1112/topo.12163). arXiv: [1703.09050](https://arxiv.org/abs/1703.09050) [[math.AT](https://arxiv.org/abs/math.AT)] (cit. on p. 12).
- [Ark11] M. Arkowitz. *Introduction to Homotopy Theory*. Universitext. New York, NY: Springer, 2011. ISBN: 978-1-4419-7329-0. DOI: [10.1007/978-1-4419-7329-0](https://doi.org/10.1007/978-1-4419-7329-0) (cit. on p. 10).
- [Ban26] P. Banerjee. “M5 Brane to D4 Brane via Cyclification of Rational Relative 3-Cohomotopy”. In: *JHEP* (in press) (2026). arXiv: [2601.02293](https://arxiv.org/abs/2601.02293) [[hep-th](https://arxiv.org/abs/hep-th)] (cit. on pp. 8, 19).
- [BBS98] I. Bandos, N. Berkovits, and D. Sorokin. “Duality-symmetric eleven-dimensional supergravity and its coupling to M-branes”. In: *Nucl. Phys. B* 522 (1998), pp. 214–233. DOI: [10.1016/S0550-3213\(98\)00102-3](https://doi.org/10.1016/S0550-3213(98)00102-3). arXiv: [hep-th/9711055](https://arxiv.org/abs/hep-th/9711055) (cit. on pp. 3, 5).
- [Bec+17] C. Becker et al. “Abelian Duality on Globally Hyperbolic Spacetimes”. In: *Communications in Mathematical Physics* 349.1 (May 2017), pp. 361–392. ISSN: 1432-0916. DOI: [10.1007/s00220-016-2669-9](https://doi.org/10.1007/s00220-016-2669-9). arXiv: [1511.00316](https://arxiv.org/abs/1511.00316) [[hep-th](https://arxiv.org/abs/hep-th)] (cit. on p. 5).

- [Ber+01] E. Bergshoeff et al. “New formulations of $D = 10$ supersymmetry and D8-O8 domain walls”. In: *Class. Quant. Grav.* 18 (2001), pp. 3359–3382. DOI: [10.1088/0264-9381/18/17/303](#). arXiv: [hep-th/0103233](#) (cit. on pp. 2, 4, 5).
- [Ber+09] E. A. Bergshoeff et al. “Gauge theories, duality relations and the tensor hierarchy”. In: *JHEP* 04 (2009), p. 123. DOI: [10.1088/1126-6708/2009/04/123](#). arXiv: [0901.2054 \[hep-th\]](#) (cit. on p. 5).
- [Bou+02] P. Bouwknegt et al. “Twisted K-theory and K-theory of bundle gerbes”. In: *Communications in Mathematical Physics* 228.1 (2002), pp. 17–49. DOI: [10.1007/s002200200646](#). arXiv: [hep-th/0106194](#) (cit. on p. 2).
- [BSS19] V. Braunack-Mayer, H. Sati, and U. Schreiber. “Gauge enhancement of Super M-Branes via rational parameterized stable homotopy theory”. In: *Communications in Mathematical Physics* 371 (2019), pp. 197–265. DOI: [10.1007/s00220-019-03441-4](#). arXiv: [1806.01115 \[hep-th\]](#) (cit. on pp. 6, 7, 9, 12).
- [BSS26] P. Banerjee, H. Sati, and U. Schreiber. “Flux Quantization on M-Strings”. In: *J. Phys. A* (in press) (2026). arXiv: [2603.14440 \[hep-th\]](#) (cit. on p. 8).
- [BV85] D. Burghelea and M. Vigué-Poirrier. “A model for cyclic homology and algebraic K-theory of 1-connected topological spaces”. In: *Journal of Differential Geometry* 22.2 (1985), pp. 243–253. DOI: [10.4310/jdg/1214439821](#) (cit. on p. 18).
- [CD25] L. Castellani and R. D’Auria. “The L_∞ -structure of Free Differential Algebras and Supergravity”. In: *International Journal of Modern Physics A* (2025). DOI: [10.1142/S0217751X25501787](#). arXiv: [2507.20344 \[hep-th\]](#) (cit. on p. 16).
- [CJ24] D. Clausen and M. Ø. Jansen. “The reductive Borel-Serre compactification as a model for unstable algebraic K-theory”. In: *Selecta Mathematica* 30, 10 (2024). DOI: [10.1007/s00029-023-00900-8](#). arXiv: [2108.01924 \[math.KT\]](#) (cit. on p. 6).
- [CJS78] E. Cremmer, B. Julia, and J. Scherk. “Supergravity theory in 11 dimensions”. In: *Physics Letters B* 76.4 (1978), pp. 409–412. DOI: [10.1016/0370-2693\(78\)90894-8](#) (cit. on p. 2).
- [Cre+98] E. Cremmer et al. “Dualisation of Dualities, II: Twisted Self-Duality of Doubled Fields and Superdualities”. In: *Nuclear Physics B* 535.1–2 (Nov. 1998), pp. 242–292. DOI: [10.1016/S0550-3213\(98\)00552-5](#). arXiv: [hep-th/9806106 \[hep-th\]](#) (cit. on p. 2).
- [Dan34] D. van Dantzig. “The fundamental equations of electromagnetism, independent of metrical geometry”. In: *Math. Proc. Cambridge Phil. Soc.* 30.4 (1934), pp. 421–427. DOI: [10.1017/S0305004100012664](#) (cit. on p. 3).
- [DFM11] J. Distler, D. S. Freed, and G. W. Moore. “Orientifold Précis”. In: *Mathematical Foundations of Quantum Field and Perturbative String Theory*. Ed. by H. Sati and U. Schreiber. Vol. 83. Proceedings of Symposia in Pure Mathematics. Providence, RI: American Mathematical Society, 2011. DOI: [10.1090/pspum/083](#). arXiv: [0906.0795 \[hep-th\]](#) (cit. on p. 2).

- [DLM95] M. J. Duff, J. T. Liu, and R. Minasian. “Eleven-dimensional origin of string / string duality: A one loop test”. In: *Nucl. Phys. B* 452 (1995), pp. 261–282. DOI: [10.1016/0550-3213\(95\)00368-3](#). arXiv: [hep-th/9506126](#) (cit. on p. 5).
- [DLT98] G. Dall’Agata, K. Lechner, and M. Tonin. “ $D = 10$, $N = IIB$ supergravity: Lorentz-invariant actions and duality”. In: *Journal of High Energy Physics* 1998.07 (1998), p. 017. DOI: [10.1088/1126-6708/1998/07/017](#). arXiv: [hep-th/9806140](#) (cit. on p. 2).
- [Duf+87] M. J. Duff et al. “Superstrings in $D = 10$ from supermembranes in $D = 11$ ”. In: *Physics Letters B* 191.1–2 (June 1987), pp. 70–74. DOI: [10.1016/0370-2693\(87\)91323-2](#) (cit. on p. 8).
- [Duf99] M. J. Duff. *The World in Eleven Dimensions: Supergravity, Supermembranes and M-Theory*. IOP Publishing, 1999. ISBN: 978-0-750-30672-0. DOI: [10.1201/9781482268737](#) (cit. on p. 7).
- [Fer+24] J. J. Fernandez-Melgarejo et al. “Democratic actions with scalar fields: symmetric sigma models, supergravity actions and the effective theory of the type IIB superstring”. In: *SciPost Phys. Core* 7.4 (2024), p. 068. DOI: [10.21468/SciPostPhysCore.7.4.068](#). arXiv: [2401.00549 \[hep-th\]](#) (cit. on pp. 3–5).
- [FF16] A. Fomenko and D. Fuchs. *Homotopical Topology*. Springer, 2016. ISBN: 9783319234885. DOI: [10.1007/978-3-319-23488-5](#) (cit. on p. 10).
- [FHT00] Y. Félix, S. Halperin, and J.-C. Thomas. *Rational Homotopy Theory*. Vol. 205. Graduate Texts in Mathematics. Springer, 2000. DOI: [10.1007/978-1-4613-0105-9](#) (cit. on p. 15).
- [FHT08] D. S. Freed, M. J. Hopkins, and C. Teleman. “Twisted equivariant K-theory with complex coefficients”. In: *Journal of Topology* 1.1 (2008), pp. 16–44. DOI: [10.1112/jtopol/jtm001](#). arXiv: [math/0206257 \[math.AT\]](#) (cit. on p. 3).
- [FMS07a] D. S. Freed, G. W. Moore, and G. Segal. “The Uncertainty of Fluxes”. In: *Communications in Mathematical Physics* 271.1 (Jan. 2007), pp. 247–274. ISSN: 1432-0916. DOI: [10.1007/s00220-006-0181-3](#). arXiv: [hep-th/0605198 \[hep-th\]](#) (cit. on pp. 2, 5).
- [FMS07b] D. S. Freed, G. W. Moore, and G. Segal. “Heisenberg Groups and Noncommutative Fluxes”. In: *Annals of Physics* 322.1 (Jan. 2007), pp. 236–285. DOI: [10.1016/j.aop.2006.07.014](#). arXiv: [hep-th/0605200](#) (cit. on p. 2).
- [Fre01] D. S. Freed. “K-Theory in Quantum Field Theory”. In: *Current Developments in Mathematics, 2001*. Vol. 2001. Current Developments in Mathematics 1. International Press of Boston, 2001, pp. 41–87. DOI: [10.4310/cdm.2001.v2001.n1.a2](#). arXiv: [math-ph/0206031](#) (cit. on p. 2).
- [Fre02a] D. S. Freed. “Dirac Charge Quantization and Generalized Differential Cohomology”. In: *Surveys in Differential Geometry* 7.1 (2002), pp. 129–194. DOI: [10.4310/SDG.2002.v7.n1.a6](#). arXiv: [hep-th/0011220 \[hep-th\]](#) (cit. on pp. 2, 3).

- [Fre02b] D. S. Freed. “Dirac Charge Quantization and Generalized Differential Cohomology”. In: *Surveys in Differential Geometry* 7.1 (2002), pp. 129–194. DOI: [10.4310/SDG.2002.v7.n1.a6](#). arXiv: [hep-th/0011220 \[hep-th\]](#) (cit. on p. 5).
- [FSS15] D. Fiorenza, H. Sati, and U. Schreiber. “Super Lie n -algebra extensions, higher WZW models, and super p -branes with tensor multiplet fields”. In: *International Journal of Geometric Methods in Modern Physics* 12.02 (2015), p. 1550018. DOI: [10.1142/S0219887815500188](#). arXiv: [1308.5264 \[hep-th\]](#) (cit. on p. 16).
- [FSS17] D. Fiorenza, H. Sati, and U. Schreiber. “Rational sphere valued supercocycles in M-theory and type IIA string theory”. In: *Journal of Geometry and Physics* 114 (Apr. 2017), pp. 91–108. DOI: [10.1016/j.geomphys.2016.11.024](#). arXiv: [1606.03206 \[hep-th\]](#) (cit. on pp. 2, 6, 18, 19).
- [FSS18] D. Fiorenza, H. Sati, and U. Schreiber. “T-Duality from super Lie n -algebra cocycles for super p -branes”. In: *Adv. Theor. Math. Phys.* 22.5 (2018), pp. 1209–1270. DOI: [10.4310/ATMP.2018.v22.n5.a3](#). arXiv: [1611.06536 \[hep-th\]](#) (cit. on pp. 7, 19).
- [FSS19] D. Fiorenza, H. Sati, and U. Schreiber. “The Rational Higher Structure of M-theory”. In: *Fortschritte der Physik* 67.8–9 (May 2019). DOI: [10.1002/prop.201910017](#). arXiv: [1903.02834 \[hep-th\]](#) (cit. on pp. 15, 16).
- [FSS20] D. Fiorenza, H. Sati, and U. Schreiber. “Twisted Cohomotopy Implies M-Theory Anomaly Cancellation on 8-Manifolds”. In: *Commun. Math. Phys.* 377.3 (Apr. 2020), pp. 1961–2025. DOI: [10.1007/s00220-020-03707-2](#). arXiv: [1904.10207 \[hep-th\]](#) (cit. on pp. 7, 8, 11, 18).
- [FSS21a] D. Fiorenza, H. Sati, and U. Schreiber. “Twisted Cohomotopy Implies Level Quantization of the Full 6d Wess-Zumino Term of the M5-Brane”. In: *Commun. Math. Phys.* 384.1 (Apr. 2021), pp. 403–432. DOI: [10.1007/s00220-021-03951-0](#). arXiv: [1906.07417 \[hep-th\]](#) (cit. on pp. 2, 6–8).
- [FSS21b] D. Fiorenza, H. Sati, and U. Schreiber. “Twisted Cohomotopy implies Twisted String Structure on M5-branes”. In: *Journal of Mathematical Physics* 62.4 (2021), p. 042301. DOI: [10.1063/5.0037786](#). arXiv: [2002.11093 \[hep-th\]](#) (cit. on p. 8).
- [FSS22] D. Fiorenza, H. Sati, and U. Schreiber. “Twistorial Cohomotopy Implies Green-Schwarz Anomaly Cancellation”. In: *Reviews in Mathematical Physics* 34.05 (Mar. 2022), p. 2250013. DOI: [10.1142/S0129055X22500131](#). arXiv: [2008.08544 \[hep-th\]](#) (cit. on p. 18).
- [FSS23] D. Fiorenza, H. Sati, and U. Schreiber. *The Character Map in Non-abelian Cohomology: Twisted, Differential, and Generalized*. <https://ncatlab.org/schreiber/show/The+Character+Map+in+Non-Abelian+Cohomology>. World Scientific, 2023. ISBN: 9789811276705. DOI: [10.1142/13422](#) (cit. on pp. 3, 6, 8, 10, 15, 16).
- [GH11] J. Greitz and P. S. Howe. “Maximal supergravity in $D = 10$: forms, Borcherds algebras and superspace cohomology”. In: *JHEP* 08 (2011), p. 146. DOI: [10.1007/JHEP08\(2011\)146](#). arXiv: [1103.5053 \[hep-th\]](#) (cit. on pp. 2, 5).

- [GHM97] M. B. Green, J. A. Harvey, and G. W. Moore. “I-brane inflow and anomalous couplings on D-branes”. In: *Class. Quant. Grav.* 14 (1997), pp. 47–52. DOI: [10.1088/0264-9381/14/1/008](https://doi.org/10.1088/0264-9381/14/1/008). arXiv: [hep-th/9605033](https://arxiv.org/abs/hep-th/9605033) [hep-th] (cit. on p. 2).
- [Goo91] T. G. Goodwillie. “Calculus II: Analytic functors”. In: *K-Theory* 5.4 (1991), pp. 295–332. DOI: [10.1007/BF00535644](https://doi.org/10.1007/BF00535644) (cit. on p. 12).
- [GP84] F. Giani and M. Pernici. “ $N = 2$ supergravity in ten dimensions”. In: *Phys. Rev. D* 30 (1984), pp. 325–333. DOI: [10.1103/PhysRevD.30.325](https://doi.org/10.1103/PhysRevD.30.325) (cit. on p. 2).
- [GS22] D. Grady and H. Sati. “Ramond–Ramond fields and twisted differential K-theory”. In: *Advances in Theoretical and Mathematical Physics* 26.5 (2022), pp. 1097–1155. ISSN: 1095-0753. DOI: [10.4310/atmp.2022.v26.n5.a2](https://doi.org/10.4310/atmp.2022.v26.n5.a2) (cit. on p. 2).
- [GSS24a] G. Giotopoulos, H. Sati, and U. Schreiber. “Flux quantization on 11-dimensional superspace”. In: *Journal of High Energ. Physics* 2024.7 (July 2024), p. 82. DOI: [10.1007/JHEP07\(2024\)082](https://doi.org/10.1007/JHEP07(2024)082). arXiv: [2403.16456](https://arxiv.org/abs/2403.16456) [hep-th] (cit. on pp. 2, 4).
- [GSS24b] G. Giotopoulos, H. Sati, and U. Schreiber. “Flux Quantization on M5-Branes”. In: *Journal of High Energy Physics* 2024.10 (Oct. 2024), p. 140. DOI: [10.1007/JHEP10\(2024\)140](https://doi.org/10.1007/JHEP10(2024)140). arXiv: [2406.11304](https://arxiv.org/abs/2406.11304) [hep-th] (cit. on p. 8).
- [GSS25] G. Giotopoulos, H. Sati, and U. Schreiber. “Super-Lie $_{\infty}$ T-Duality and M-Theory”. In: *Rev. Math. Phys.* (2025). DOI: [10.1142/S0129055X25500187](https://doi.org/10.1142/S0129055X25500187). arXiv: [2411.10260](https://arxiv.org/abs/2411.10260) [hep-th] (cit. on pp. 16, 19).
- [Ham03] H. Hamanaka. “Adams e -invariant, Toda bracket and $[X, U(n)]$ ”. In: *Journal of Mathematics of Kyoto University* 43.4 (2003), pp. 815–827. DOI: [10.1215/kjm/1250281737](https://doi.org/10.1215/kjm/1250281737) (cit. on p. 6).
- [Hat02] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002. ISBN: 9780521795401. URL: <https://pi.math.cornell.edu/~hatcher/AT/AT.pdf> (cit. on pp. 10, 14).
- [Hes07] K. Hess. “Rational homotopy theory: a brief introduction”. In: *Interactions between Homotopy Theory and Algebra*. Ed. by L. L. Avramov et al. Vol. 436. Contemporary Mathematics. Providence, RI: American Mathematical Society, 2007, pp. 175–202. DOI: [10.1090/conm/436/08409](https://doi.org/10.1090/conm/436/08409). arXiv: [math/0604626](https://arxiv.org/abs/math/0604626) [math.AT] (cit. on p. 15).
- [HK03] H. Hamanaka and A. Kono. “On $[X, U(n)]$ when $\dim X$ is $2n$ ”. In: *Journal of Mathematics of Kyoto University* 43.2 (2003), pp. 333–348. DOI: [10.1215/kjm/1250283730](https://doi.org/10.1215/kjm/1250283730) (cit. on p. 6).
- [HK04] H. Hamanaka and A. Kono. “An application of unstable K-theory”. In: *Journal of Mathematics of Kyoto University* 44.2 (2004), pp. 451–456. DOI: [10.1215/kjm/1250283560](https://doi.org/10.1215/kjm/1250283560) (cit. on p. 6).
- [HN85] M. Huq and M. A. Namazie. “Kaluza-Klein supergravity in ten dimensions”. In: *Classical and Quantum Gravity* 2.3 (1985), pp. 293–308. DOI: [10.1088/0264-9381/2/3/007](https://doi.org/10.1088/0264-9381/2/3/007) (cit. on p. 2).

- [Hop02] M. J. Hopkins. “Algebraic topology and modular forms”. In: *Proceedings of the International Congress of Mathematicians (ICM, Beijing 2002)*. Vol. 1. Higher Education Press, 2002, pp. 283–309. ISBN: 7-04-008690-5. arXiv: [math/0212397](#) (cit. on p. 8).
- [HS97] P. S. Howe and E. Sezgin. “ $D = 11, p = 5$ ”. In: *Phys. Lett. B* 394 (1997), pp. 62–66. DOI: [10.1016/S0370-2693\(96\)01672-3](#). arXiv: [hep-th/9611008](#) [[hep-th](#)] (cit. on p. 8).
- [HSW97] P. S. Howe, E. Sezgin, and P. C. West. “Covariant field equations of the M theory five-brane”. In: *Physics Letters B* 399.1-2 (1997), pp. 49–59. DOI: [10.1016/S0370-2693\(97\)00257-8](#). arXiv: [hep-th/9702008](#) (cit. on p. 8).
- [HT03] P. S. Howe and D. Tsimpis. “On higher-order corrections in M theory”. In: *JHEP* 09 (2003), p. 038. DOI: [10.1088/1126-6708/2003/09/038](#). arXiv: [hep-th/0305129](#) (cit. on p. 5).
- [HTZ25] S. S. Hosseini, Y. Tachikawa, and H. Y. Zhang. “Type I anomaly cancellation revisited”. In: (2025). DOI: [10.48550/arXiv.2505.07933](#). arXiv: [2505.07933](#) [[hep-th](#)] (cit. on p. 2).
- [Hus+08] D. Husemöller et al. *Basic Bundle Theory and K-Cohomology Invariants*. Vol. 726. Lecture Notes in Physics. Springer, 2008. DOI: [10.1007/978-3-540-74956-1](#) (cit. on p. 3).
- [Jam84] I. M. James. *General Topology and Homotopy Theory*. Springer, 1984. DOI: [10.1007/978-1-4613-8283-6](#) (cit. on p. 10).
- [Kal80] W. van der Kallen. “Homology stability for linear groups”. In: *Inventiones mathematicae* 60 (1980), pp. 269–295. DOI: [10.1007/BF01390018](#) (cit. on p. 6).
- [Kar78] M. Karoubi. *K-Theory: An Introduction*. Vol. 226. Grundlehren der mathematischen Wissenschaften. Springer, 1978. DOI: [10.1007/978-3-540-79890-3](#) (cit. on p. 3).
- [Koc96] S. O. Kochman. *Bordism, Stable Homotopy and Adams Spectral Sequences*. Vol. 7. Fields Institute Monographs. Providence, RI: American Mathematical Society, 1996. ISBN: 978-1-4704-3134-1. DOI: [10.1090/fim/007](#). URL: <https://bookstore.ams.org/fim-7> (cit. on p. 10).
- [Lav+99] I. V. Lavrinenko et al. “Superdualities, brane tensions and massive IIA/IIB duality”. In: *Nucl. Phys. B* 555 (1999), pp. 201–227. DOI: [10.1016/S0550-3213\(99\)00307-7](#). arXiv: [hep-th/9903057](#) (cit. on pp. 2, 5).
- [LS18] C. I. Lazaroiu and C. S. Shahbazi. “Section sigma models coupled to symplectic duality bundles on Lorentzian four-manifolds”. In: *Journal of Geometry and Physics* 128 (2018), pp. 58–86. DOI: [10.1016/j.geomphys.2018.02.003](#). arXiv: [1711.05651](#) [[hep-th](#)] (cit. on p. 5).
- [LS22a] C. Lazaroiu and C. S. Shahbazi. “The duality covariant geometry and DSZ quantization of abelian gauge theory”. In: *Advances in Theoretical and Mathematical Physics* 26.7 (2022), pp. 2213–2312. ISSN: 1095-0753. DOI: [10.4310/atmp.2022.v26.n7.a5](#). arXiv: [2101.07236](#) [[math.DG](#)] (cit. on p. 5).

- [LS22b] C. Lazaroiu and C. S. Shahbazi. “The geometry and DSZ quantization of four-dimensional supergravity”. In: *Letters in Mathematical Physics* 113.1 (Dec. 2022), p. 9. ISSN: 1573-0530. DOI: [10.1007/s11005-022-01626-y](#). arXiv: [2101.07778 \[math.DG\]](#) (cit. on p. 5).
- [Lur14] J. Lurie. *Nonabelian Poincaré Duality*. Lecture 8 in: Tamagawa Numbers via Nonabelian Poincare Duality (282y). 2014. URL: <http://www.math.harvard.edu/~lurie/282ynotes/LectureVIII-Poincare.pdf> (cit. on pp. 3, 10).
- [Mas91] W. S. Massey. *A Basic Course in Algebraic Topology*. Graduate Texts in Mathematics 127. Springer, 1991. ISBN: 9781493990634. DOI: [10.1007/978-1-4939-9063-4](#) (cit. on p. 10).
- [McC01] J. McCleary. *A User’s Guide to Spectral Sequences*. 2nd ed. Vol. 58. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2001. DOI: [10.1017/CBO9780511626289](#) (cit. on p. 14).
- [Men15] L. Menichi. “Rational homotopy – Sullivan models”. In: *Free Loop Spaces in Geometry and Topology*. Vol. 24. IRMA Lectures in Mathematics and Theoretical Physics. European Mathematical Society, 2015, pp. 111–136. DOI: [10.4171/153](#). arXiv: [1308.6685 \[math.AT\]](#) (cit. on pp. 15, 16).
- [Mil21] H. Miller. *Lectures on Algebraic Topology*. World Scientific, 2021. DOI: [10.1142/12132](#) (cit. on p. 14).
- [MM97] R. Minasian and G. Moore. “K-theory and Ramond-Ramond charge”. In: *Journal of High Energy Physics* 1997.11 (1997), p. 002. DOI: [10.1088/1126-6708/1997/11/002](#). arXiv: [hep-th/9710230 \[hep-th\]](#) (cit. on p. 2).
- [MS03] V. Mathai and D. Stevenson. “Chern character in twisted K-theory: equivariant and holomorphic cases”. In: *Communications in Mathematical Physics* 236.1 (2003), pp. 161–186. DOI: [10.1007/s00220-003-0807-7](#). arXiv: [hep-th/0201010](#) (cit. on p. 2).
- [MS04] V. Mathai and H. Sati. “Some relations between twisted K-theory and E_8 gauge theory”. In: *JHEP* 03 (2004), p. 016. DOI: [10.1088/1126-6708/2004/03/016](#). arXiv: [hep-th/0312033](#) (cit. on pp. 2, 6).
- [MS06] A. Miemiec and I. Schnakenburg. “Basics of M-Theory”. In: *Fortschritte der Physik* 54.1 (Jan. 2006), pp. 5–72. DOI: [10.1002/prop.200510256](#). arXiv: [hep-th/0509137 \[hep-th\]](#) (cit. on p. 2).
- [MS25] G. W. Moore and V. Saxena. *TASI Lectures On Topological Field Theories And Differential Cohomology*. Oct. 2025. arXiv: [2510.07408 \[hep-th\]](#) (cit. on pp. 2, 5).
- [MS74] J. W. Milnor and J. D. Stasheff. *Characteristic Classes*. Annals of Mathematics Studies 76. Princeton University Press, 1974. ISBN: 978-0-691-08122-9. DOI: [10.1515/9781400881826](#) (cit. on p. 10).
- [MV23] K. Mkrtchyan and F. Valach. “Democratic actions for type II supergravities”. In: *Phys. Rev. D* 107.6 (2023), p. 066027. DOI: [10.1103/PhysRevD.107.066027](#). arXiv: [2207.00626 \[hep-th\]](#) (cit. on pp. 3, 4).

- [MW00] G. W. Moore and E. Witten. “Self-duality, Ramond-Ramond fields, and K-theory”. In: *JHEP* 05 (2000), p. 032. DOI: [10.1088/1126-6708/2000/05/032](https://doi.org/10.1088/1126-6708/2000/05/032). arXiv: [hep-th/9912279](https://arxiv.org/abs/hep-th/9912279) (cit. on pp. 2, 5).
- [Nak18] M. Nakahara. *Geometry, Topology and Physics*. 2nd ed. Boca Raton, FL: CRC Press, Oct. 2018. ISBN: 9781315275826. DOI: [10.1201/9781315275826](https://doi.org/10.1201/9781315275826) (cit. on p. 14).
- [Nur03] A. J. Nurmagambetov. “The Sigma-Model Representation for the Duality-Symmetric D=11 Supergravity”. In: *eConf* C0306234 (2003), pp. 894–901. arXiv: [hep-th/0312157](https://arxiv.org/abs/hep-th/0312157) [[hep-th](#)] (cit. on p. 3).
- [Ort15] T. Ortín. *Gravity and Strings*. 2nd. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2015. DOI: [10.1017/CB09781139019750](https://doi.org/10.1017/CB09781139019750) (cit. on p. 2).
- [Rav04] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres*. Vol. 347. Originally published by Academic Press Orlando (1986). AMS Chelsea Publishing, 2004. ISBN: 978-0-8218-2967-7. URL: <https://www.sas.rochester.edu/mth/sites/doug-ravenel/mu.html> (cit. on p. 16).
- [Sat18] H. Sati. “Framed M-branes, corners, and topological invariants”. In: *Journal of Mathematical Physics* 59.6 (2018), p. 062304. DOI: [10.1063/1.5007185](https://doi.org/10.1063/1.5007185). arXiv: [1310.1060](https://arxiv.org/abs/1310.1060) [[hep-th](#)] (cit. on p. 7).
- [Sor00] D. Sorokin. “Superbranes and Superembeddings”. In: *Phys. Rept.* 329 (2000), pp. 1–101. DOI: [10.1016/S0370-1573\(99\)00104-0](https://doi.org/10.1016/S0370-1573(99)00104-0). arXiv: [hep-th/9906142](https://arxiv.org/abs/hep-th/9906142) [[hep-th](#)] (cit. on p. 8).
- [SS21] H. Sati and U. Schreiber. “Twisted cohomotopy implies M5-brane anomaly cancellation”. In: *Letters in Mathematical Physics* 111.5 (Sept. 2021). DOI: [10.1007/s11005-021-01452-8](https://doi.org/10.1007/s11005-021-01452-8). arXiv: [2002.07737](https://arxiv.org/abs/2002.07737) [[hep-th](#)] (cit. on p. 7).
- [SS23] H. Sati and U. Schreiber. “M/F-theory as Mf -theory”. In: *Reviews in Mathematical Physics* 35 (2023), p. 23500289. DOI: [10.1142/s0129055x23500289](https://doi.org/10.1142/s0129055x23500289). arXiv: [2103.01877](https://arxiv.org/abs/2103.01877) [[hep-th](#)] (cit. on p. 10).
- [SS24a] H. Sati and U. Schreiber. “Cyclification of Orbifolds”. In: *Communications in Mathematical Physics* 405.3, 67 (2024), p. 67. DOI: [10.1007/s00220-023-04929-w](https://doi.org/10.1007/s00220-023-04929-w). arXiv: [2212.13836](https://arxiv.org/abs/2212.13836) [[math.AT](#)] (cit. on pp. 7, 12).
- [SS24b] H. Sati and U. Schreiber. “Flux Quantization on Phase Space”. In: *Annales Henri Poincaré* 26 (May 2024), pp. 895–919. DOI: [10.1007/s00023-024-01438-x](https://doi.org/10.1007/s00023-024-01438-x). arXiv: [2312.12517](https://arxiv.org/abs/2312.12517) [[hep-th](#)] (cit. on pp. 3, 5, 16).
- [SS25a] H. Sati and U. Schreiber. “Flux Quantization”. In: *Encyclopedia of Mathematical Physics*. Ed. by R. Szabo and M. Bojowald. 2nd ed. Vol. 4. Academic Press, 2025, pp. 281–324. ISBN: 9780323957069. DOI: [10.1016/b978-0-323-95703-8.00078-1](https://doi.org/10.1016/b978-0-323-95703-8.00078-1). arXiv: [2402.18473](https://arxiv.org/abs/2402.18473) [[hep-th](#)] (cit. on pp. 2, 3, 5, 16).
- [SS25b] H. Sati and U. Schreiber. “Quantum Observables of Quantized Fluxes”. In: *Annales Henri Poincaré* 26 (2025), pp. 4241–4269. DOI: [10.1007/s00023-024-01517-z](https://doi.org/10.1007/s00023-024-01517-z). arXiv: [2306.01214](https://arxiv.org/abs/2306.01214) [[hep-th](#)] (cit. on p. 2).

- [SS25c] H. Sati and U. Schreiber. “The Character Map in Twisted Equivariant Nonabelian Cohomology”. In: *Beijing Journal of Pure and Applied Mathematics* 2.2 (2025), pp. 515–617. DOI: [10.4310/bpam.250908174706](https://doi.org/10.4310/bpam.250908174706). arXiv: [2011.06533](https://arxiv.org/abs/2011.06533) [hep-th] (cit. on p. 3).
- [SS26a] H. Sati and U. Schreiber. “Bulk-Edge Correspondence via Higher Gauge Theory”. In: (2026). arXiv: [2605.10232](https://arxiv.org/abs/2605.10232) [hep-th] (cit. on p. 10).
- [SS26b] H. Sati and U. Schreiber. “Complete Topological Quantization of Higher Gauge Fields”. In: *SciPost Physics Lecture Notes* 127 (2026). DOI: <https://scipost.org/10.21468/SciPostPhysLectNotes.127>. arXiv: [2512.12431](https://arxiv.org/abs/2512.12431) [hep-th] (cit. on p. 6).
- [SS26c] H. Sati and U. Schreiber. “Higher Gauge Theory via Differential Nonabelian Cohomology”. In: *Reviews in Mathematical Physics* (2026). arXiv: [2606.12534](https://arxiv.org/abs/2606.12534) [hep-th] (cit. on pp. 2, 5).
- [SS26d] H. Sati and U. Schreiber. *Geometric Orbifold Cohomology*. CRC Press, 2026. ISBN: 9781041147510. URL: <https://ncatlab.org/schreiber/show/OrbCoh> (cit. on pp. 3, 8, 14).
- [SS94] J. H. Schwarz and A. Sen. “Duality symmetric actions”. In: *Nucl. Phys. B* 411 (1994), pp. 35–63. DOI: [10.1016/0550-3213\(94\)90053-1](https://doi.org/10.1016/0550-3213(94)90053-1). arXiv: [hep-th/9304154](https://arxiv.org/abs/hep-th/9304154) [hep-th] (cit. on p. 3).
- [SSS09] H. Sati, U. Schreiber, and J. Stasheff. “Fivebrane structures”. In: *Rev. Math. Phys.* 21.10 (2009), pp. 1197–1240. DOI: [10.1142/S0129055X09003840](https://doi.org/10.1142/S0129055X09003840). arXiv: [0805.0564](https://arxiv.org/abs/0805.0564) [math.AT] (cit. on p. 6).
- [SSS12] H. Sati, U. Schreiber, and J. Stasheff. “Twisted Differential String and Fivebrane Structures”. In: *Commun. Math. Phys.* 315.1 (2012), pp. 169–213. DOI: [10.1007/s00220-012-1510-3](https://doi.org/10.1007/s00220-012-1510-3). arXiv: [0910.4001](https://arxiv.org/abs/0910.4001) [math.AT] (cit. on p. 6).
- [Ste01] B. Stefanski. “WZ Couplings of D-branes and O-planes”. In: *Progress in String Theory and M-Theory*. Vol. 564. NATO Science Series. Springer, 2001, pp. 401–404. DOI: [10.1007/978-94-010-0852-5_31](https://doi.org/10.1007/978-94-010-0852-5_31). arXiv: [hep-th/9909105](https://arxiv.org/abs/hep-th/9909105) [hep-th] (cit. on p. 8).
- [SV23] H. Sati and A. A. Voronov. “Mysterious Triality and Rational Homotopy Theory”. In: *Communications in Mathematical Physics* 400.3 (June 2023), pp. 1915–1960. DOI: [10.1007/s00220-023-04643-7](https://doi.org/10.1007/s00220-023-04643-7). arXiv: [2111.14810](https://arxiv.org/abs/2111.14810) [hep-th] (cit. on p. 18).
- [Toë02] B. Toën. *Stacks and Non-abelian cohomology*. Lecture at the Introductory Workshop on Algebraic Stacks, Intersection Theory, and Non-Abelian Hodge Theory, MSRI. 2002. URL: <https://perso.math.univ-toulouse.fr/btoen/files/2015/02/msri2002.pdf> (cit. on p. 3).
- [Tow96] P. K. Townsend. “D-branes from M-branes”. In: *Physics Letters B* 373.1-3 (1996), pp. 68–75. DOI: [10.1016/0370-2693\(96\)00104-9](https://doi.org/10.1016/0370-2693(96)00104-9). arXiv: [hep-th/9512062](https://arxiv.org/abs/hep-th/9512062) (cit. on p. 8).

- [Wei94] C. A. Weibel. *An Introduction to Homological Algebra*. Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1994. DOI: [10.1017/CB09781139644136](#) (cit. on pp. 13, 15).
- [Whi78] G. W. Whitehead. *Elements of Homotopy Theory*. Vol. 61. Graduate Texts in Mathematics. New York: Springer, 1978. ISBN: 978-1-4612-6318-0. DOI: [10.1007/978-1-4612-6318-0](#) (cit. on p. 10).
- [Wit98] E. Witten. “D-branes and K-theory”. In: *JHEP* 12 (1998), p. 019. DOI: [10.1088/1126-6708/1998/12/019](#). arXiv: [hep-th/9810188](#) [[hep-th](#)] (cit. on p. 2).
- [WW85] X.-G. Wen and E. Witten. “Electric and Magnetic Charges in Superstring Models”. In: *Nuclear Physics B* 261 (1985), pp. 651–677. DOI: [10.1016/0550-3213\(85\)90592-9](#) (cit. on p. 2).
- [Zho01] J.-G. Zhou. “D-branes in B fields”. In: *Nuclear Physics B* 607.1–2 (2001), pp. 237–246. DOI: [10.1016/S0550-3213\(01\)00240-1](#). arXiv: [hep-th/0102178](#) [[hep-th](#)] (cit. on p. 8).