

# The Character Map in Equivariant Twistorial Cohomotopy

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## Abstract

The fundamental notion of *non-abelian* generalized cohomology gained recognition in algebraic topology as the non-abelian Poincaré-dual to “factorization homology”, and in theoretical physics as providing flux-quantization for non-linear Gauß laws. However, already the archetypical example — unstable Cohomotopy, first studied almost a century ago by Pontrjagin — may remain underappreciated as a cohomology theory.

In illustration and amplification of its cohomological nature, we construct the non-abelian generalization of the Chern character map on  $\mathbb{Z}_2$ -equivariantized 7-Cohomotopy — in fact on its “twistorial” version classified by complex projective 3-space — essentially by computing its equivariant Sullivan model, and we highlight some interesting integral cohomology classes which are extracted this way.

We end with an outlook on the application of this result to the rigorous deduction of anyonic quantum states on M5-branes wrapped over Seifert 3-orbifolds.

## Contents

|          |   |           |
|----------|---|-----------|
| <b>1</b> | <b>Introduction &amp; Overview</b>                                      | <b>2</b>  |
| <b>2</b> | <b>Equivariant non-abelian cohomology</b>                               | <b>8</b>  |
| 2.1      | Homotopy theory of $\infty$ -group actions . . . . .                    | 9         |
| 2.2      | Proper equivariant homotopy theory . . . . .                            | 12        |
| 2.3      | Equivariant non-abelian cohomology theories . . . . .                   | 18        |
| 2.4      | Equivariant twisted non-abelian cohomology theories . . . . .           | 20        |
| <b>3</b> | <b>Equivariant non-abelian de Rham cohomology</b>                       | <b>23</b> |
| 3.1      | Equivariant dgc-algebras and equivariant $L_\infty$ -algebras . . . . . | 23        |
| 3.2      | Equivariant rational homotopy theory . . . . .                          | 37        |
| 3.3      | Equivariant non-abelian de Rham theorem . . . . .                       | 42        |
| 3.4      | Equivariant non-abelian character map . . . . .                         | 49        |
| <b>4</b> | <b>Application to flux quantization</b>                                 | <b>51</b> |

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# 1 Introduction & Overview

Algebraic topology, of course, is the study of spaces via systems of (co)homological invariants. Ever more generalized versions of cohomology are routinely discussed these days, but an ancient and archetypical example — namely unstable Cohomotopy (6), which we will refer to as just *Cohomotopy* — has received little attention as a cohomology theory, since as such it falls outside the scope even of the generalized cohomology theories as commonly understood today: it is a *non-abelian* generalized cohomology theory, as we recall in a moment.

Motivated by recent application [SS25a] in theoretical physics (exposition in §4) of non-abelian generalized cohomology in general, and of Cohomotopy in particular, we present here a case study of constructions on and phenomena exhibited by Cohomotopy when regarding it as a cohomology theory. Concretely, our main result (Thm. 1.1) is the construction and analysis of the nonabelian character map [FSS23] on the “twistorial” variant of low-degree Cohomotopy from [FSS20b], now generalized to  $\mathbb{Z}_2$ -equivariant form.

At the heart of this computation is, for reasons explained in a moment, the computation of minimal Sullivan models of fibrations of some basic cell complexes (like  $S^7$  and  $\mathbb{C}P^3$ ) but in *equivariant* homotopy theory (recalled in §2). Since in this context even such basic examples have not been discussed in print before – to the best of our knowledge – the reader may take §3 as an exposition of the notoriously more intricate equivariant version of dg-algebraic rational homotopy theory (which has seen little application in the past) along some illustrative examples and under the perspective of the equivariant generalization (in §3.3) of the non-abelian de Rham theorem from [FSS23, §6].

After the proof of the main theorem is thereby completed, for the inclined reader we end in §4 with a brief outlook on the somewhat remarkable implications of our computations to recent questions in theoretical physics, specifically to the rigorous derivation of anyonic quantum states on M5-branes.

But first, to set the scene, it is worthwhile to briefly take a step back and reconsider the notion of cohomology as such:

**Cohomology via classifying spaces.** It is a classical and yet possibly undervalued fact that *reasonable cohomology theories have classifying spaces* (and more generally *classifying stacks*). To quickly recall (more details and pointers in [FSS23, §2]<sup>1</sup>):

– **Ordinary cohomology.** This begins with the observation that (reduced) ordinary singular cohomology, with coefficients in a discrete abelian group  $A$ , is classified in degree  $n$  by Eilenberg-MacLane spaces  $K(A, n)$  – in that on well-behaved topological spaces  $X$ , notably on smooth manifolds, there are natural isomorphisms between the ordinary cohomology groups and the connected components of the respective (pointed) mapping spaces:

$$H^n(X; A) \simeq \pi_0 \text{Maps}(X, K(A, n)), \quad \tilde{H}^n(X; A) \simeq \pi_0 \text{Maps}^*/(X, K(A, n)). \quad (1)$$

This equivalence makes manifest the characteristic properties of cohomology: homotopy invariance, exactness and wedge property, since these are now immediately implied by general abstract properties of mapping spaces.

Moreover, these EM-spaces are in fact loop spaces of each other, via weak homotopy equivalences

$$\sigma_n : K(A, n) \xrightarrow{\sim} \Omega K(A, n+1) \quad (2)$$

that thereby represent the *suspension isomorphisms* between ordinary cohomology groups, as follows:

$$\tilde{H}^n(X; A) \underset{(1)}{\simeq} \text{Maps}^*/(X, K(A, n)) \xrightarrow[\text{adjunction}]{(\sigma_n)_*} \text{Maps}^*/(X, \Omega K(A, n+1)) \underset{(1)}{\simeq} \text{Maps}^*/(\Sigma X, K(A, n+1)) \underset{(1)}{\simeq} \tilde{H}^{n+1}(\Sigma X; A).$$

– **Ordinary non-abelian cohomology.** Note here that it is the loop space property (2), and hence the corresponding suspension isomorphism, which reflect the fact that the coefficient  $A$  has been assumed to be an *abelian* group: For a non-abelian group  $G$ , an Eilenberg-MacLane space  $K(G, 1) \simeq BG$  still exists, but is *not a loop space*.

While the suspension isomorphism is thus lost for non-abelian coefficients, the assignment

$$X \mapsto H^1(X; G) := \pi_0 \text{Maps}(X, BG) \in \text{Set}^* \quad (3)$$

<sup>1</sup>The numbering in [FSS23] we refer to is that of the published version, which differs from the numbering in the preprint version; see [ncatlab.org/schreiber/show/The+Character+Map+in+Non-Abelian+Cohomology].

still satisfies homotopy invariance, exactness and wedge property, just by the general properties of mapping spaces, and hence has all the characteristic properties of ordinary cohomology – except for its abelian-ness. Accordingly, (3) is known as *non-abelian cohomology*, famous from early applications in Chern-Weil theory.

– **Whitehead-generalized cohomology theory.** But if or as long as we do insist on abelian cohomology groups related by suspension isomorphisms, we may still immediately generalize ordinary cohomology in the form (1), simply by using any other sequence of classifying spaces  $(E_n)_{n=0}^\infty$ , being successive loop spaces of each other as in (2),

$$\sigma_n : E_n \xrightarrow{\sim} \Omega E_{n+1},$$

as such called a *sequential  $\Omega$ -spectrum of spaces*, or just a *spectrum*, for short. The Brown representability theorem says that the resulting assignments

$$X \mapsto E^n(X) := \pi_0 \text{Maps}(X; E_n)$$

are equivalently the *generalized cohomology theories* as introduced by Whitehead, including examples such as K-theory, elliptic cohomology and cobordism cohomology.

– **Non-abelian generalized cohomology.** But as we just saw, suspension isomorphisms are to be regarded as *extra* structure on cohomology. Not necessarily requiring them leads to consider *any pointed space*  $\mathcal{A}$  (which we may as well assume to be connected) as the classifying space of a non-abelian generalized cohomology theory, defined in evident generalization of (3) simply by

$$H^1(X; \Omega\mathcal{A}) := \pi_0 \text{Maps}(X, \mathcal{A}). \quad (4)$$

Here the notation on the left is suggestive of the fact that any loop space  $\Omega\mathcal{A}$  canonically carries the structure of a higher homotopy-coherent group – a groupal  $A_\infty$ -space or  $\infty$ -group, for short – whose de-looping is equivalent to the connected component of the original space:

$$\mathcal{A} \simeq B\Omega\mathcal{A}. \quad (5)$$

For instance, in the archetypical case where  $\mathcal{A} \equiv S^n$  is the  $n$ -sphere, then the non-abelian generalized cohomology theory that it classifies is known as (unstable) Cohomotopy  $\pi^n$

$$\tilde{H}^1(X; \Omega S^n) \equiv \pi_0 \text{Maps}^{*/}(X, S^n) \equiv \pi^n(X), \quad (6)$$

in dual reference to the familiar *homotopy* groups

$$\pi_n(X) \simeq \pi_0 \text{Maps}^{*/}(S^n, X).$$

Another example of non-abelian generalized cohomology is unstable topological K-theory [HK04], whose classifying spaces are taken to be finite stages  $U(n)$  of the sequential colimits which construct the classifying spaces of topological K-theory.

**Developing non-abelian cohomology.** Fundamental, elementary, and compelling as the notion of non-abelian generalized cohomology in (4) is, it has long remained underappreciated. For example, none of the original authors [Bo36][Po38][Sp49] on Cohomotopy (6) address their subject as a cohomology theory, instead the early development revolves around partial fixes for the perceived defect of co-homotopy sets to not in general carry group structure. The situation does not improve with the early development of “non-abelian gerbes”, whose original description [Gi71] appears unwieldy.

Explicit acknowledgment of (stacky) non-abelian generalized cohomology in the transparent guise (4) appears only in a lecture [To02] (possibly following [Si02]). Two independent developments in 2009 finally put non-abelian generalized cohomology into fruitful context:

- The discovery of non-abelian Poincaré duality [Lu09b, §3.8], relating non-abelian cohomology (later made explicit in [Lu14, Def. 6]) of manifolds to “non-abelian homology” in the guise of “factorization homology” (which, in contrast to non-abelian cohomology, takes work to define);
- The observation in theoretical physics [SS08][Sc09][SSS12] that charge/flux-quantization laws [SS25a] for higher gauge fields are generally in non-abelian cohomology.

With non-abelian generalized cohomology thus recognized as a worthwhile subject, we are led to generalize familiar constructions in abelian cohomology, as far as possible, and to explore the consequences.

First, we may straightforwardly equip non-abelian cohomology with further attributes: Considering the right-hand side of (4) not just for plain spaces but for sheaves of spaces (higher stacks) leads to non-abelian generalized sheaf cohomology, including, in particular, non-abelian generalized versions of twisted cohomology and of equivariant cohomology (also of differential cohomology, but this shall not concern us here):

– **Equivariant non-abelian cohomology.** Via the above identification of cohomology sets with homotopy classes of maps to a classifying space, every flavor of homotopy theory comes with its corresponding flavor of cohomology theories. In *equivariant homotopy theory* one considers (cf. [SS25b]) topological spaces  $\mathcal{A}$  equipped with the action  $G \curvearrowright \mathcal{A}$  of a (finite, for our purposes) group  $G$  and with  $G$ -equivariant maps between them – and the corresponding flavor of cohomology is *equivariant cohomology* (which we also call *proper equivariant cohomology* in order to distinguish it from the coarser form of Borel-equivariance):

$$H_G^1(X; \Omega \mathcal{A}) = \pi_0 \text{Maps}(G \curvearrowright X, G \curvearrowright \mathcal{A})^G. \quad (7)$$

Here the notion of  $G$ -homotopy equivalence of maps is straightforward but, at face value, technically cumbersome to reason about. However, Elmendorf’s theorem (recalled as Prop. 2.26 below) reveals that  $G$ -homotopy equivalences (between  $G$ -cell complexes) are nothing but systems of ordinary weak homotopy equivalences between the  $H$ -fixed spaces  $\mathcal{A}^H$  for all subgroups  $H \subset G$ . These systems of fixed spaces are conveniently re-packaged as presheaves on a small category called the *orbit category*  $\text{Orb}(G)$  of  $G$ , whence  $G$ -equivariant homotopy theory is equivalently the homotopy theory of presheaves of spaces on  $\text{Orb}(G)$ .

– **Twisted non-abelian cohomology.** Somewhat similarly, given any space  $\mathcal{B}$  in any homotopy theory, the  $\mathcal{B}$ -slice is the homotopy theory whose objects are spaces fibered over  $\mathcal{B}$  with maps between them respecting the fibration up to specified homotopy. If we assume, without essential restriction, that the base space is connected, then we may identify it as  $\mathcal{B} \simeq B\mathcal{G}$ , as in (5), which exhibits any fibration over it as the Borel construction  $\mathcal{A} // \mathcal{G}$  of the homotopy-quotient of a homotopy-coherent action  $\mathcal{G} \curvearrowright \mathcal{A}$ .

If we now think of a domain object  $X \xrightarrow{\tau} B\mathcal{G}$  in this  $B\mathcal{G}$ -slice as a *twist* and of a codomain object  $\mathcal{A} // \mathcal{G} \xrightarrow{p} B\mathcal{G}$  as a *local coefficient bundle*, then the corresponding non-abelian cohomology is just the homotopy classes of sections of the  $\tau$ -associated  $\mathcal{A}$ -fiber bundle, and as such is  $\tau$ -twisted  $\mathcal{A}$ -cohomology [FSS23, §3]:

$$H^{1+\tau}(X, \Omega \mathcal{A}) := \pi_0 \text{Maps}(X, \mathcal{A} // \mathcal{G})_{/B\mathcal{G}} = \left\{ \begin{array}{ccc} & & \mathcal{A} // \mathcal{G} \\ & \nearrow \text{dashed} & \downarrow p \\ X & \xrightarrow{\tau} & B\mathcal{G} \end{array} \right\} / \text{relative homotopy} \quad (8)$$

This works generally: If all spaces here are *in addition* equipped with  $G$ -actions as in (7), hence if we are looking at a slice of equivariant homotopy, then the above is automatically *twisted & equivariant* non-abelian cohomology. This is what we shall be concerned with here, concretely with the character map in this generality:

– **The non-abelian character.** A famous construction on abelian cohomology is the *Chern-Dold character map* to de Rham cohomology, which in the case of  $\mathbb{K}$ -cohomology becomes the familiar Chern character (and which on ordinary cohomology is essentially just the de Rham theorem). One may think of the Chern-Dold character as universally extracting the non-torsion data in the cohomology groups. Its generalization to non-abelian cohomology was developed in [FSS23]:

Observe that the Chern-Dold character is essentially just the cohomology operation induced by *rationalization* of the classifying space,

$$\mathcal{A} \xrightarrow[\text{rationalization}]{\eta^{\mathbb{Q}}} L^{\mathbb{Q}} \mathcal{A}.$$

As such, it makes sense in the generality of non-abelian classifying spaces (immediately so under mild technical assumptions, such as nilpotency, but with more work also more generally). In view of this, the fundamental theorem of dg-algebraic rational homotopy theory may be re-cast as a *non-abelian de Rham theorem* which identifies, over smooth manifolds  $X$ , the resulting non-abelian rational cohomology with the concordance classes of flat differential

forms having coefficients in the real Whitehead-bracket  $L_\infty$ -algebra  $\mathcal{L}\mathcal{A}$  of the classifying space:

$$\begin{array}{ccccc}
 & & \text{character map} & & \\
 & & \xrightarrow{\hspace{10em}} & & \\
 H^1(X; \mathcal{A}) \equiv \pi_0 \text{Maps}(X, \mathcal{A}) & \xrightarrow{(\eta^\mathbb{Q})_*} & \pi_0 \text{Maps}(X, L^\mathbb{Q}\mathcal{A}) & \longrightarrow & H_{\text{dR}}^1(X; \mathcal{L}\mathcal{A}) \\
 \text{non-abelian generalized cohomology} & \text{rationalization of} & \text{non-abelian} & \text{non-abelian} & \text{non-abelian} \\
 & \text{classifying space} & \text{rational cohomology} & \text{de Rham theorem} & \text{de Rham cohomology}
 \end{array} \tag{9}$$

Since generalized cohomology theories are typically hard to analyze, in particular non-abelian ones, this character map may be regarded as extracting the first non-trivial stage of more tractable invariants. For instance, the character of a non-abelian class is the first obstruction to a trivialization of that class. <sup>2</sup>

It is fairly straightforward to generalize the non-abelian character (9) to twisted non-abelian cohomology (8), now using *relative* minimal Sullivan models.

The following table shows some examples of the resulting form of twisted non-abelian character maps that we have computed elsewhere before – the first few examples are for general illustration and orientation, the last one is the one of concern here: Our goal here is to equivariantize it.

|   |                   |  |   |  |  |
|---|-------------------|--|---|--|--|
| $A // G$<br>$\downarrow$<br>$BG$  | [FSS23] Def. 5.4  | $H^\tau(X; A)$<br>twisted non-abelian cohomology<br>$[c]_\tau$<br>cocycle in twisted $A$ -cohomology | $\xrightarrow{\text{ch}_A}$<br>twisted non-abelian character map    | $H_{\text{dR}}^{\tau} (X; \mathcal{L}A)$<br>twisted non-abelian de Rham cohomology<br>$\text{ch}_A([c])$<br>flux densities satisfying Bianchi identities   | $\xrightarrow{\hspace{10em}}$<br>twist in $H(X, BG)$ spacetime manifold classifying space<br>twist in $H_{\text{dR}}(X; \mathcal{L}BG)$ spacetime manifold Whitehead $L_\infty$ -algebra |
| $B^n \mathbb{Z}$<br>$\downarrow$<br>$*$                                   | [FSS23] Ex. 4.9   | $H^n(X; \mathbb{Z})$<br>ordinary cohomology  | $\xrightarrow{\text{dR}}$<br>de Rham homomorphism                   | $\{ F_n \in \Omega_{\text{dR}}^n(X) \mid dF_n = 0 \} / \sim$   |  |
| $BU(n)$<br>$\downarrow$<br>$*$  | [FSS23] Thm. 4.26 | $H^1(X; U(n))$<br>ordinary non-abelian cohomology  | $\xrightarrow{\text{cw}}$<br>Chern-Weil homomorphism                | $\left\{ \begin{array}{l} \vdots \\ c_2(A), \in \Omega_{\text{dR}}^{2\bullet}(X) \\ c_1(A) \end{array} \middle  \begin{array}{l} \vdots \\ d c_2(A) = 0 \\ d c_1(A) = 0 \end{array} \right\} / \sim$   |  |
| $(\mathbb{Z} \times BU) // BU(1)$<br>$\downarrow$<br>$B^2U(1)$            | [FSS23] Prop. 5.5 | $KU^\tau(X)$<br>twisted complex K-theory   | $\xrightarrow{\text{ch}^\tau}$<br>twisted Chern character           | $\left\{ \begin{array}{l} F_{2\bullet}, \\ H_3 \end{array} \in \Omega_{\text{dR}}^\bullet(X) \middle  \begin{array}{l} dF_{2\bullet+2} = H_3 \wedge F_{2\bullet} \\ dH_3 = 0 \end{array} \right\} / \sim$  |  |
| $S^4 // \widehat{BSp}(2)$<br>$\downarrow$<br>$\widehat{BSp}(2)$           | [FSS23] Ex. 5.23a | $\pi^\tau(X)$<br>J-twisted 4-Cohomotopy  | $\xrightarrow{\text{ch}_\pi^\tau}$<br>twisted FSS-character         | $\left\{ \begin{array}{l} 2G_7, \\ G_4 \end{array} \in \Omega_{\text{dR}}^\bullet(X) \middle  \begin{array}{l} d2G_7 = -G_4 \wedge G_4 + (\frac{1}{4}p_1(\omega))^2 \\ dG_4 = 0 \end{array} \right\} / \sim$   |  |
| $\mathbb{C}P^3 // \widehat{BSp}(2)$<br>$\downarrow$<br>$\widehat{BSp}(2)$ | [FSS23] Ex. 5.23b | $\mathcal{T}^\tau(X)$<br>twistorial Cohomotopy   | $\xrightarrow{\text{ch}_\mathcal{T}^\tau}$<br>twisted FSS-character | $\left\{ \begin{array}{l} H_3 \\ F_2 \\ 2G_7, \\ G_4 \end{array} \in \Omega_{\text{dR}}^\bullet(X) \middle  \begin{array}{l} dH_3 = G_4 - \frac{1}{4}p_1(\omega) - F_2 \wedge F_2 \\ dF_2 = 0 \\ d2G_7 = -G_4 \wedge G_4 + (\frac{1}{4}p_1(\omega))^2 \\ dG_4 = 0 \end{array} \right\} / \sim$ |  |

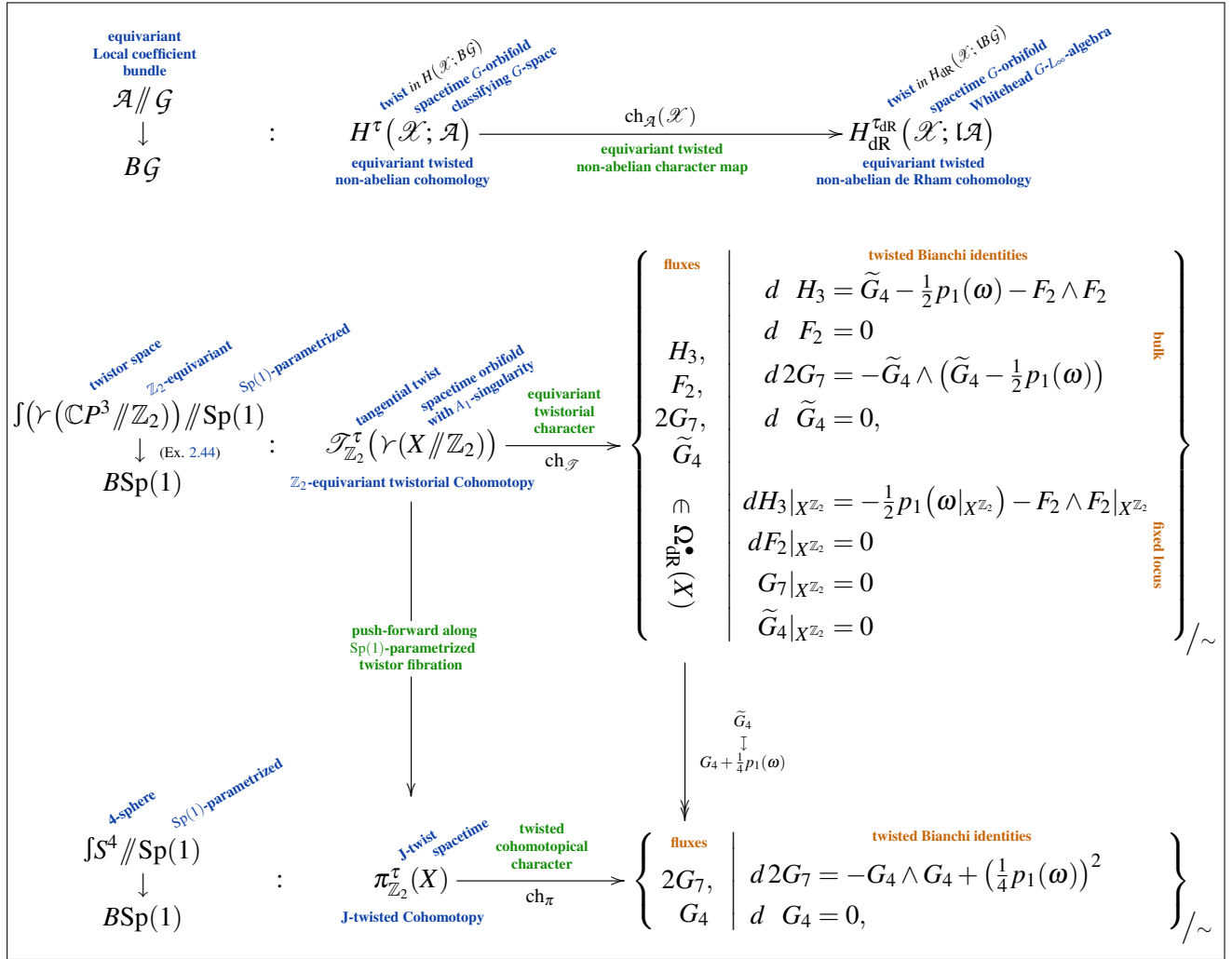
<sup>2</sup>In the mentioned application to physics, the flux densities of a higher gauge field are sourced by charges that appear as classes in non-abelian de Rham cohomology on the right, and the completion of the higher gauge theory by a flux-quantization law means to lift these charges through the character map to classes in a chosen non-abelian cohomology theory on the left.

This identification of the character map on non-abelian cohomology with the passage of classifying spaces to their minimal dgc-algebraic models in rational homotopy theory yields a new perspective on both subjects:

- On the one hand, it becomes clear at once how to make sense of the twisted equivariant non-abelian character, namely by construction of equivariant relative Sullivan models using the theory of [Tr82][Scu02, §11][Scu08];
- and conversely it provides a sudden wealth of motivation and applications of the latter (which arguably has led a niche existence in the literature).

**Main result.** Here we lay out this construction of the twisted equivariant non-abelian character map <sup>3</sup> and apply it to the example of twistorial Cohomotopy., with the following main result (whose proof culminates in Rem. 3.79 below):

**Theorem 1.1. (i)** *The character map (Def. 3.78) in  $\mathbb{Z}_2$ -equivariant twistorial Cohomotopy (Def. 2.48), on  $\mathbb{Z}_2$ -orbifolds (Def. 2.36) with  $\mathrm{Sp}(1)$ -structure  $\tau$  and -connection  $\omega$  (Example 3.70), is of the following form (3.79):*



(ii) Moreover, a necessary condition for differential forms to be in the image of this character map is their (shifted) integrality, as follows:

$$[\tilde{G}_4] := [G_4 + \frac{1}{4}p_1(\omega)] \in H^4(X; \mathbb{Z}) \rightarrow H^4(X; \mathbb{R}), \quad [F_2] \in H^2(X; \mathbb{Z}) \rightarrow H^2(X; \mathbb{R}). \quad (10)$$

<sup>3</sup>More specifically, here we develop the equivariant non-abelian character for the case of classifying spaces that are equivariantly (namely, fixed locus-wise) simply-connected. If one drops this assumption, then the discussion becomes much more involved, as one needs to rationalize the fixed locus-wise covering spaces while retaining the respective actions of the fundamental groups by Deck transformations over each fixed locus — all this on top of the action of the equivariance group  $G$  and of the twisting group  $\mathcal{G}$ .

**The computation at the heart of our proof.** At the heart of the proof of Theorem 1.1 is the computation (Prop. 3.56 below) of the equivariant relative minimal model ([Tr82, §5][Scu02, §11][Scu08, §4], recalled as Def. 3.40 below) of the  $\mathbb{Z}_2$ -equivariant  $\mathrm{Sp}(1)$ -parametrized twistor fibration in equivariant rational homotopy theory.

**The equivariant twistor fibration.** The *twistor fibration*  $t_{\mathbb{H}}$  ([At79, §III.1][Bry82], see [FSS20b, §2]) is the map from  $\mathbb{C}P^3$  (“twistor space”) to  $\mathbb{H}P^1 \simeq S^4$  which sends complex lines to the right quaternionic lines that they span:

$$\begin{array}{ccc}
 S^2 \simeq \mathbb{H}^\times / \mathbb{C}^\times & \xrightarrow{\mathrm{fib}(t_{\mathbb{H}})} & \mathbb{C}P^3 \simeq (\mathbb{C}^4 \setminus \{0\}) / \mathbb{C}^\times \ni \{v \cdot z \mid z \in \mathbb{C}^\times\} \\
 & \searrow & \downarrow t_{\mathbb{H}} \\
 & & \mathbb{H}P^1 \simeq (\mathbb{H}^2 \setminus \{0\}) / \mathbb{H}^\times \ni \{v \cdot q \mid q \in \mathbb{H}^\times\}
 \end{array} \quad (11)$$

The fiber of the twistor fibration is hence  $\mathbb{H}^\times / \mathbb{C}^\times \simeq \mathbb{C}P^1 \simeq S^2$ .

(i) There is the evident action of  $\mathrm{Sp}(2)$ , on both  $\mathbb{C}P^3$  and  $\mathbb{H}P^1$ , by left multiplication of homogeneous representatives with unitary quaternion  $2 \times 2$  matrices (58):

$$\begin{array}{ccc}
 \mathrm{Sp}(2) \times \mathbb{C}P^3 & \longrightarrow & \mathbb{C}P^3, & \mathrm{Sp}(2) \times \mathbb{H}P^1 & \longrightarrow & \mathbb{H}P^1, \\
 (A \cdot [v]) & \longmapsto & [A \cdot v] & (A \cdot [v]) & \longmapsto & [A \cdot v]
 \end{array} \quad (12)$$

and the twistor fibration (being given by quotienting on the right) is manifestly equivariant under this left action.

(ii) Consider the following subgroups:

$$\mathbb{Z}_2 := \left\{ 1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \subset \mathrm{Sp}(2), \quad \sigma : [z_1 : z_2 : z_3 : z_4] \mapsto [z_3 : z_4 : z_1 : z_2], \quad (13)$$

$$\mathrm{Sp}(1) := \left\{ q \cdot := \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \mid q \in S(\mathbb{H}) \right\} \subset \mathrm{Sp}(2). \quad (14)$$

Since these manifestly commute with each other, the homotopy quotient  $\mathbb{C}P^3 // \mathrm{Sp}(1)$  of twistor space (11) by  $\mathrm{Sp}(1)$  still admits the structure of a  $G$ -space (as in [tD79, §8][Ma96][Blu17]) for  $G = \mathbb{Z}_2$ , fibered over  $B\mathrm{Sp}(1)$  (see Ex. 2.44 below for details).

**The equivariant minimal relative dgc-algebra model of twistor space.** Our Prop. 3.56 gives its equivariant minimal model:

$$\begin{array}{ccc}
 \begin{array}{c} \mathbb{Z}_2 \\ \curvearrowright \\ \mathbb{C}P^3 // \mathrm{Sp}(1) \\ \text{twistor space} \\ \text{homotopy-quotiented by } \mathrm{Sp}(1) \\ \text{with residual } \mathbb{Z}_2\text{-action} \end{array} & : & \begin{array}{c} \mathbb{Z}_2 \\ \curvearrowright \\ \mathbb{Z}_2/1 \\ \downarrow \mathbb{Z}_2\text{-orbit category} \\ \mathbb{Z}_2/\mathbb{Z}_2 \end{array} \\
 & & \begin{array}{ccc} \xrightarrow{\text{bulk}} & \mathbb{R}[\frac{1}{4}p_1] \left[ \begin{array}{c} h_3, \\ f_2 \\ \omega_7, \\ \tilde{\omega}_4 \end{array} \right] / \left( \begin{array}{l} dh_3 = \tilde{\omega}_4 - \frac{1}{2}p_1 - f_2 \wedge f_2 \\ df_2 = 0 \\ d\omega_7 = -\tilde{\omega}_4 \wedge (\tilde{\omega}_4 - \frac{1}{2}p_1) \\ d\tilde{\omega}_4 = 0 \end{array} \right) & \xrightarrow{\text{singularity}} \\ \downarrow \text{minimal } \mathbb{Z}_2\text{-equivariant model} \\ \text{relative to } B\mathrm{Sp}(1) & \mathbb{R}[\frac{1}{4}p_1] \left[ \begin{array}{c} h_3, \\ f_2 \end{array} \right] / \left( \begin{array}{l} dh_3 = -\frac{1}{2}p_1 - f_2 \wedge f_2 \\ df_2 = 0 \end{array} \right) \end{array} \end{array} \quad (15)$$

normalized (as in [FSS19a][FSS19b][FSS20b]) such that:

- (a) all closed generators shown are rational images of *integral* and *integrally in-divisible* cohomology classes;
- (b)  $\omega := \tilde{\omega} - \frac{1}{4}p_1$  is fiberwise the volume form on  $\mathbb{H}P^1 \simeq S^4$ , and  $f_2$  is fiberwise the volume form on  $\mathbb{C}P^1 \simeq S^2$ .

As a non-trivial example of a (relative) minimal model in rational equivariant homotopy theory, this may be of interest in its own right. Such examples computed in the literature are rare (we have not come across any). Here we are concerned with a most curious aspect of this novel example: Under substituting the algebra generators in (15) with differential forms on a  $\mathbb{Z}_2$ -orbifold (essentially the non-abelian character map, Def. 3.78), the relations in (15) are those expected for flux densities in “M-theory”, as briefly explained in §4:

$$\begin{array}{ccc}
 \left( \frac{1}{4}p_1, \tilde{\omega}_4, \omega_7, f_2, h_3 \right) & \longleftrightarrow & \left( \frac{1}{4}p_1(\omega), \overset{\text{shifted}}{\underset{\text{C-field flux density}}{G_4 + \frac{1}{4}p_1(\omega)}}, \overset{\text{gauge flux}}{2G_7}, F_2, H_3 \right). \\
 \text{dgc-algebra generators of} & & \text{Pontrjagin form} \quad \text{dual} \quad \text{C-field flux density} \quad \text{B-field flux} \\
 \text{equivariant relative minimal model} & & \text{(gravitational flux density)}
 \end{array}$$

**Outline.**

In §2 we introduce equivariant non-abelian cohomology theory (in equivariant generalization of [FSS23, §2]) and the example of equivariant twistorial Cohomotopy theory  $\mathcal{T}_{\mathbb{Z}_2}^r(-)$  (Def. 2.48).

In §3 we introduce equivariant non-abelian de Rham cohomology theory and the equivariant non-abelian character map (in equivariant generalization of [FSS23, §3-5]) and compute the  $\mathbb{Z}_2$ -equivariant relative minimal model of  $\mathrm{Sp}(1)$ -parametrized twistor space (Prop. 3.56).

In §4 we briefly indicate the application and impact of our result on the problem of flux-quantization of higher gauge fields arising in super-gravity.

**Notation.** For various types of symmetry groups and their quotients, we use the following notation:

|               |                                   |           |                              |  |           |
|---------------|-----------------------------------|-----------|------------------------------|--|-----------|
| $T$           | Compact Borel equivariance group  | Def. 2.11 | $\int (X//T)$                | Borel equivariant homotopy type                                | Ex. 2.8   |
| $G$           | Finite proper equivariance group  |           | $\gamma(X//G)$               | Orbifold   | Ex. 2.20  |
| $T \times G$  | Borel & proper equivariance group |           | $\int \gamma(X//G)$          | Proper equivariant homotopy type                               | Def. 2.23 |
|               |                                   |           | $\int (\gamma(X//G))//T$     | Proper $G$ -equivariant & Borel $T$ -equivariant homotopy type | Ex. 2.43  |
| $\mathcal{G}$ | Simplicial group/ $\infty$ -group | Not.2.2   | $\mathcal{A} // \mathcal{G}$ | Homotopy quotient  | Prop. 2.7 |
| $\mathcal{G}$ | $G$ -equivariant $\infty$ -group  | Rem. 2.42 | $\mathcal{A} // \mathcal{G}$ | $G$ -equivariant homotopy quotient                             | (72)      |

Our notation for equivariant homotopy theory follows [SS20]. The symbol “ $\gamma$ ” refers to proper equivariant objects (“orbi-singular objects”), parametrized over the orbit category (Def. 2.13) of the equivariance group (41):

| Symbol   |                                     | Meaning   | Details   |
|--|-------------------------------------|---|-----------|
| $G\mathrm{Actions}(\mathrm{TopologicalSpaces})$                    | $G$ -actions on topological spaces  | Category of topological spaces equipped with continuous action of the equivariance group $G$                  | Def. 2.11 |
| $G\mathrm{Orbits}$   | $G$ -orbits                         | Category of canonical orbits $G/H$ of the equivariance group, with equivariant maps between them              | Def. 2.13 |
| $\mathcal{G}\mathrm{SimplicialSets}$                               | $G$ -equivariant simplicial sets    | Category of contra-variant functors from $G$ -orbits to simplicial sets                                       | Def. 2.19 |
| $\mathcal{G}\mathrm{VectorSpaces}_{\mathbb{R}}^{\vee}$             | $G$ -equivariant dual vector spaces | Category of co-variant functors from $G$ -orbits to vector spaces   | Def. 3.5  |
| $\mathcal{G}\mathrm{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$ | $G$ -equivariant dgc-algebras       | Category of co-variant functors from $G$ -orbits to connective differential graded-commutative algebras       | Def. 3.30 |
| $\mathcal{G}\mathrm{HomotopyTypes}$                                | $G$ -equivariant homotopy types     | Homotopy category of projective model category of contra-variant functors from $G$ -orbits to simplicial sets | Def. 2.22 |

## 2 Equivariant non-abelian cohomology

In §2.1 we recall basics of  $\infty$ -groups and their  $\infty$ -actions and establish some technical Lemmas.

In §2.2 we recall basics of proper equivariant homotopy theory and introduce our running Example 2.44.

in §2.3 we introduce equivariant non-abelian cohomology theory.

in §2.4 we introduce twisted equivariant non-abelian cohomology theory.

Throughout, we illustrate all concepts in the running example of the  $\mathbb{Z}_2$ -equivariant and  $\mathrm{Sp}(1)$ -parametrized twistor fibration (Example 2.44), the induced equivariant twistorial Cohomotopy theory (Def. 2.48) and its character image in equivariant de Rham cohomology (Example 3.74). We highlight that here both flavors of equivariance are involved.

|                                | Borel equivariance                                    | Proper equivariance     |
|--------------------------------|---|-------------------------|
| Equivariance group (§2)        | $T = \mathrm{Sp}(1)$                                  | $G = \mathbb{Z}_2$      |
| Equivariant dR-cohomology (§3) | Borel-Weil-Cartan model                               | Bredon-type theory      |
| Physical effect (§4)           | Flux quantization: shift of $G_4$ by $\frac{1}{4}p_1$ | orbifolding of M5-brane |



We make free use of basic concepts from category theory and homotopy theory (for joint introduction see [Rie14][Ri20]), in particular of model category theory ([Qu67], review in [Ho99][Hi02][Lu09a, A.2]). Relevant concepts and facts are recalled in [FSS23, §A].

For  $\mathcal{C}$  a category, and  $X, A \in \mathcal{C}$  a pair of objects, we write

$$\mathcal{C}(X, A) \in \mathbf{Sets} \quad (16)$$

for its set of morphisms from  $X$  to  $A$ . This assignment is, of course, a contravariant functor in its first argument, to be denoted:

$$\mathcal{C}(-; A) : \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Sets} . \quad (17)$$

Elementary as it is, this is of profound interest whenever  $\mathcal{C}$  is the *homotopy category* of a homotopy topos [TV05][Lu09a][Re10], in which case the contravariant hom-functors (17) are *non-abelian cohomology theories* [To02][Sc13][SS20][FSS23]. These subsume generalized and ordinary cohomology theories ([FSS23, §2]), as well as their equivariant enhancements, which we consider below.

## 2.1 Homotopy theory of $\infty$ -group actions

### Plain homotopy theory.

**Notation 2.1** (Classical homotopy category). (i) We write

$$\mathbf{TopologicalSpaces}_{\text{Qu}}, \mathbf{SimplicialSets}_{\text{Qu}} \in \mathbf{ModelCategories} \quad (18)$$

for the classical model category structures on topological spaces and on simplicial sets, respectively ([Qu67, §II.3], review in [Hi15][GJ99]).

(ii) The classical Quillen equivalence

$$\mathbf{TopologicalSpaces}_{\text{Qu}} \begin{array}{c} \xleftarrow{|-|} \\ \xrightarrow[\text{Sing}]{\simeq_{\text{Qu}}} \end{array} \mathbf{SimplicialSets}_{\text{Qu}} \quad (19)$$

induces an equivalence between the corresponding homotopy categories, which we denote:

$$\mathbf{SimplicialSets} \xrightarrow[\text{localization}]{\gamma} \mathbf{HomotopyTypes} := \mathbf{Ho}(\mathbf{SimplicialSets}_{\text{Qu}}). \quad (20)$$

(iii) We denote the localization functor from topological spaces to this classical homotopy category by “ $\mathfrak{f}$ ”:<sup>4</sup>

$$\begin{array}{ccc} \mathbf{TopologicalSpaces} & \xrightarrow[\text{localization at weak homotopy equivalences}]{\text{shape } \mathfrak{f}} & \mathbf{HomotopyTypes} . \\ & \searrow \text{form singular simplicial set (19)} & \nearrow \gamma \text{ localization (20)} \\ & \mathbf{SimplicialSets} & \end{array} \quad (21)$$

**Borel-equivariant homotopy theory.** We recall basics of Borel-equivariant homotopy theory, but in the generality of equivariance for  $\infty$ -group actions (for the broader picture see [NSS12a][SS20, §2.2]).

**Notation 2.2** (Model category of simplicial groups). (i) We write

$$\mathbf{SimplicialGroups} := \mathbf{Groups}(\mathbf{SimplicialSets}) \quad (22)$$

for the category of simplicial groups.

(ii) This becomes ([Qu67, §II.3.7]) a model category

$$\mathbf{SimplicialGroups}_{\text{proj}} \in \mathbf{ModelCategories}$$

by taking the weak equivalences and fibrations to be those of  $\mathbf{SimplicialSets}_{\text{Qu}}$  (Notation 2.1).

(iii) We denote the homotopy category of this model structure by

$$\mathbf{SimplicialGroups}_{\text{proj}} \xrightarrow[\text{localization at weak homotopy equivalences}]{\gamma} \mathbf{Groups}_{\infty} := \mathbf{Ho}(\mathbf{SimplicialGroups}_{\text{proj}}) \quad (23)$$

and denote the generic object here by

$$\mathcal{G} \in \mathbf{SimplicialGroups} \xrightarrow{\gamma} \mathbf{Groups}_{\infty} .$$

<sup>4</sup>The “esh”-symbol “ $\mathfrak{f}$ ” stands for *shape* [Sc13, 3.4.5][Shu15, 9.7][SS20, §3.1.1], following [Bo75], which for the well-behaved topological spaces of interest here is another term for their *homotopy type* [Lu09a, 7.1.6][Wa17, 4.6].

**Example 2.3** (Shapes of topological groups are  $\infty$ -groups). For  $T \in \text{TopologicalGroups}$ , its singular simplicial set (19) is canonically a simplicial group (22)

$$\text{Sing}(T) \in \text{SimplicialGroups}, \quad (24)$$

and, since the weak equivalence of simplicial groups are those of the underlying simplicial sets, its image in the homotopy category is the shape  $\int T$  (21), now equipped with induced  $\infty$ -group structure (Notation 2.2):

$$\begin{array}{ccc} \text{TopologicalGroups} & \xrightarrow[\text{localization at weak homotopy equivalences}]{\infty\text{-group shape } \int} & \text{Groups}_\infty \\ & \searrow \text{form singular simplicial group (19), (24)} & \nearrow \gamma \text{ localization (23)} \\ & \text{SimplicialGroups} & \end{array} \quad (25)$$

**Notation 2.4** (Model category of reduced simplicial sets). (i) We write

$$\text{ReducedSimplicialSets} \hookrightarrow \text{SimplicialSets}$$

for the full subcategory on those  $S \in \text{SimplicialSets}$  that have a single 0-cell,  $S_0 = *$ .

(ii) This becomes ([GJ99, §V, Prop. 6.2]) a model category with weak equivalences and cofibrations those of  $\text{SimplicialSets}_{\text{Qu}}$  (Notation 2.1):

$$\text{ReducedSimplicialSets}_{\text{GJ}} \in \text{ModelCategories}.$$

(iii) Since reduced simplicial sets model those homotopy types (20) which are *pointed and connected* (e.g. [NSS12b, Prop. 3.16]), we denote the corresponding homotopy category by

$$\text{ReducedSimplicialSets}_{\text{GJ}} \xrightarrow{\gamma} \text{HomotopyTypes}_{\geq 1}^{*/} := \text{Ho}(\text{ReducedSimplicialSets}_{\text{GJ}}). \quad (26)$$

**Proposition 2.5** (Classifying space/loop space construction [GJ99, §V, Prop. 6.3][Ste12][NSS12b, §3.5]). *There exists a Quillen equivalence between the model categories of reduced simplicial sets (Notation 2.4) and that of simplicial groups (Notation 2.2)*

$$\text{SimplicialGroups}_{\text{proj}} \begin{array}{c} \xleftarrow{\simeq_{\text{Qu}}} \\ \xrightarrow{W} \end{array} \text{ReducedSimplicialSets} \quad (27)$$

whose derived adjunction is given by forming homotopy types of based loop spaces and of classifying spaces:

$$\begin{array}{ccc} \infty\text{-groups} & \text{Groups}_\infty & \xleftarrow[\text{classifying space } B(-) := \mathbb{R}\overline{W}(-)]{\text{based loop } \infty\text{-group } \Omega(-)} \text{HomotopyTypes}_{\geq 1}^{*/} \\ & & \text{pointed \& connected homotopy types} \end{array} \quad (28)$$

**Notation 2.6** (Homotopy theory of simplicial group actions). For  $\mathcal{G} \in \text{SimplicialGroups}$  (Notation 2.2)

(i) we write

$$\mathcal{G}\text{Actions} := \text{SimplicialFunctors}(B\mathcal{G}, \text{SimplicialSets})$$

for the category of simplicial functors from the simplicial groupoid with a single object and  $\mathcal{G}$  as its hom-object to the simplicial category of simplicial sets.

(ii) This becomes a model category by taking the weak equivalences and fibrations to be those of underlying simplicial sets (evaluating at the single vertex of  $B\mathcal{G}$ ):

$$\mathcal{G}\text{Actions}_{\text{proj}} \in \text{ModelCategories}$$

and we denote its homotopy category by:

$$\mathcal{G}\text{Actions}_{\text{proj}} \xrightarrow{\gamma} \text{Ho}(\mathcal{G}\text{Actions}_{\text{proj}}) =: \mathcal{G}\text{Actions}_\infty.$$

The following, Prop. 2.7, is pivotal for the discussion of twisted non-abelian cohomology, notably for the notion of equivariant local coefficient bundles below in Def. 2.45; for more background and context, see [NSS12a, §4][SS20, §2.2][FSS23, Prop. 2.28].

**Proposition 2.7** ( $\infty$ -Group actions equivalent to fibrations over classifying space [DDK80, Prop. 2.3][Sha15]).

(i) For  $\mathcal{G} \in \text{SimplicialGroups}$  (Notation 2.2), the simplicial Borel construction (e.g. [NSS12b, Prop. 3.37]) is the right adjoint of a Quillen equivalence

$$\mathcal{G}\text{Actions}_{\text{proj}} \begin{array}{c} \xleftarrow{\simeq_{\text{Qu}}} \\ \xrightarrow{\mathcal{G} \wr X \mapsto \frac{X \times W_{\mathcal{G}}}{\mathcal{G}}} \\ \text{simplicial Borel construction} \end{array} \text{SimplicialSets}_{\text{Qu}}^{\overline{W}_{\mathcal{G}}} \quad (29)$$

between the projective model structure on simplicial  $\mathcal{G}$ -actions (Notation 2.6) and the slice model structure ([Hi02, §7.6.4]) of the classical model structure on simplicial sets (18) over  $\overline{W}_{\mathcal{G}}$  (27).

(ii) Its derived equivalence of homotopy categories

$$\begin{array}{ccc} \infty\text{-actions of} & & \text{homotopy fiber} \\ \infty\text{-group } \mathcal{G} & \mathcal{G}\text{Actions}_{\infty} & \xleftarrow{\text{hofib}_*(p) \leftarrow (E \xrightarrow{p} BG)} \\ & \xrightarrow[\mathcal{G} \wr A \mapsto A // \mathcal{G}]{\simeq} & \text{Ho}\left(\text{SimplicialSets}_{\text{Qu}}^{\overline{W}_{\mathcal{G}}}\right) \\ & \text{homotopy quotient} & \text{homotopy types fibered} \\ & & \text{over classifying space } B\mathcal{G} \end{array} \quad (30)$$

is given in one direction by forming homotopy fibers of fibrations over  $B\mathcal{G}$  and in the other by forming homotopy quotients of  $\infty$ -actions ([NSS12b, Prop. 3.73]):

$$\begin{array}{ccc} \mathcal{G} \infty\text{-action on } A & & A \xrightarrow{\text{hofib}_*(\rho_A)} A // \mathcal{G} \\ \mathcal{G} \wr A & \longleftrightarrow & \begin{array}{c} \text{A-fibration over} \\ \mathcal{G}\text{-classifying space} \end{array} \downarrow \rho_A \\ & & B\mathcal{G} \end{array} \quad (31)$$

**Example 2.8** (Homotopy type of Borel construction).

For  $T \in \text{TopologicalGroups}$  and  $T \wr X \in T\text{Actions}(\text{TopologicalSpaces})$  (Def. 2.11), passage to singular simplicial sets (19) yields a simplicial action (Notation 2.6). The corresponding fibration (Prop. 2.7) is given by the topological shape (21) of the Borel construction:

$$\int X \xrightarrow{\text{hofib}(\rho_X)} \int \left( \frac{X \times ET}{T} \right) =: \int (X // T). \\ \downarrow \rho_X \\ \int BT$$

**Lemma 2.9** (Pasting law [Lu09a, Lem 4.4.2.1]). For  $\mathcal{C}$  a model category, and given a pasting composite of two commuting squares

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & \text{(hpb)} & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array}$$

such that the right square is homotopy Cartesian, then the left square is homotopy Cartesian if and only if the total rectangle is.

**Lemma 2.10** (Homotopy fibers of homotopy-quotiented morphisms).

Let  $\mathcal{G} \in \text{Groups}_{\infty}$  (Notation 2.2) and  $(A, \rho_A) \xrightarrow{(f, \rho_f)} (A', \rho_{A'}) \in \mathcal{G}\text{Actions}_{\infty}^*$  a morphism of  $\infty$ -actions (Notation 2.6) preserving an  $\mathcal{G}$ -fixed point  $\text{pt} : * \rightarrow A \xrightarrow{f} A'$  (see also [SS20, Def. 2.97]). Then:

(i) The homotopy fiber of the homotopy-quotiented morphism  $f // \mathcal{G}$  (30) coincides with the homotopy fiber of  $f$

$$\text{hofib}_*(f // \mathcal{G}) \simeq \text{hofib}_*(f). \quad (32)$$

(ii) The homotopy fiber of  $f$  is canonically equipped with an  $\infty$ -action by  $\mathcal{G}$ :

$$(\text{hofib}_*(f), \rho_h) \in \mathcal{G}\text{Actions}_{\infty}.$$

(iii) The corresponding homotopy quotient is equivalent to the homotopy fiber of the homotopy-quotiented morphism parametrized over  $B\mathcal{G}$ , namely the following homotopy pullback:

$$\begin{array}{ccc} \text{hofib}_*(f) // \mathcal{G} \simeq \text{hofib}_{B\mathcal{G}}(f // \mathcal{G}) & \longrightarrow & A // \mathcal{G} \\ \downarrow & \text{(hpb)} & \downarrow f // \mathcal{G} \\ B\mathcal{G} & \xrightarrow{\text{pt}' // \mathcal{G}} & A' // \mathcal{G} \end{array} \quad (33)$$

*Proof.* Consider the following pasting diagrams:

$$\begin{array}{ccc}
\text{hofib}_*(f // \mathcal{G}) & \longrightarrow & \text{hofib}_{B\mathcal{G}}(f // \mathcal{G}) & \longrightarrow & A // \mathcal{G} \\
\downarrow & & \downarrow & & \downarrow f // \mathcal{G} \\
* & \longrightarrow & B\mathcal{G} & \xrightarrow{\text{pt}' // \mathcal{G}} & A' // \mathcal{G}
\end{array}
\quad \simeq \quad
\begin{array}{ccccc}
\text{hofib}_*(f) & \longrightarrow & A & \longrightarrow & A // \mathcal{G} \\
\downarrow & & \downarrow f & & \downarrow f // \mathcal{G} \\
* & \xrightarrow{\text{pt}'} & A' & \longrightarrow & A' // \mathcal{G} \\
\downarrow & & \downarrow & & \downarrow \rho_{A'} \\
* & \longrightarrow & B\mathcal{G} & & \downarrow \rho_A \\
& & & & A // \mathcal{G}
\end{array}
\quad (34)$$

With the right Cartesian square (33) given, the pasting law (Lem. 2.9) identifies the top left objects on both sides as shown; in particular, the left square on the right gives (35). But, since the composite bottom morphism is the same basepoint inclusion on both sides, this implies:

$$\text{hofib}_*(f // \mathcal{G}) \simeq \text{hofib}_*(f). \quad (35)$$

Moreover, the left Cartesian square on the left of (34) exhibits, by Prop. 2.7, a  $\mathcal{G}$ -action on  $\text{hofib}_*(f // \mathcal{G})$  with homotopy quotient given by

$$\text{hofib}_*(f // \mathcal{G}) // \mathcal{G} \simeq \text{hofib}_{B\mathcal{G}}(f // \mathcal{G}). \quad (36)$$

The combination of the equivalences (32) and (36) yields the claimed equivalence in (33).  $\square$

## 2.2 Proper equivariant homotopy theory

We now recall relevant basics of proper<sup>5</sup> equivariant homotopy theory [tD79, §8][Ma96][Blu17] and introduce the examples of interest here.

### *G*-Actions.

**Definition 2.11** (Group actions on topological spaces). (i) For a given compact topological group, which serves as the symmetry group of *Borel equivariance* in the following, generically to be denoted

$$\text{Borel equivariance group } T \in \text{CompactTopologicalGroups}, \quad (37)$$

we write

$$T\text{Actions}(\text{TopologicalSpaces}) \in \text{Categories} \quad (38)$$

for the category whose objects are topological spaces  $X$  equipped with a continuous  $T$ -action

$$\begin{array}{c}
T \curvearrowright X : T \times X \xrightarrow{\text{continuous}} X \\
(t, x) \longmapsto t \cdot x
\end{array}
\quad \text{such that: } \forall_{x \in X} e \cdot x = x \quad \text{and} \quad \forall_{x \in X} \forall_{t_1, t_2 \in G} (t_1 \cdot (t_2 \cdot x)) = (t_1 \cdot t_2) \cdot x \quad (39)$$

and whose morphisms are  $T$ -equivariant continuous functions, which we denote as follows:

$$\begin{array}{ccc}
\begin{array}{c} \langle T \rangle \\ X_1 \end{array} & \xrightarrow{f} & \begin{array}{c} \langle T \rangle \\ X_2 \end{array} \\
\Leftrightarrow & & \forall_{x \in X} \forall_{t \in T} f(t \cdot x) = t \cdot f(x).
\end{array} \quad (40)$$

(ii) Throughout, our *proper* equivariance group is a finite group, to be denoted:

$$\text{proper equivariance group } G \in \text{FiniteGroups}. \quad (41)$$

This finite group can be viewed as a topologically discrete topological group and we have the corresponding category (38) of continuous actions:

$$G\text{Actions}(\text{TopologicalSpaces}) \in \text{Categories}. \quad (42)$$

(iii) The full subcategory of the latter category on those objects, where also the topological space being acted on is discrete, is that of  $G$ -actions on sets:

$$G\text{Actions}(\text{Sets}) \hookrightarrow G\text{Actions}(\text{TopologicalSpaces}). \quad (43)$$

<sup>5</sup>Here by “proper equivariance” we refer to the fine notion of equivariant homotopy/cohomology in the sense of Bredon, as opposed to the coarse notion in the sense of Borel. For in-depth conceptual discussion of this distinction see [SS20]. Besides the colloquial meaning of “proper”, the action of our finite equivariance groups is necessarily *proper* in the technical sense of general topology (see Lemma 2.34 below), whence this terminology nicely matches that recently advocated in [DHLPS19].

(iv) Regarding the direct product group of the Borel equivariance group (37) with the proper equivariance group (41) as a compact topological group

$$\text{Borel \& proper equivariance group } T \times G \in \text{CompactTopologicalGroups},$$

we have the category of topological actions of this product group. This contains the previous categories, (38) and (42), as full subcategories (via equipping a space with trivial action)

$$T\text{Actions}(\text{TopologicalSpaces}) \hookrightarrow (T \times G)\text{Actions}(\text{TopologicalSpaces}) \longleftarrow G\text{Actions}(\text{TopologicalSpaces}). \quad (44)$$

**Example 2.12** (Representation spheres). Let  $V \in T\text{Representations}_{\mathbb{R}}^{\text{fin}}$  be a finite-dimensional linear representation of a compact topological group (37). Then the one-point compactification of  $V$  (the topological sphere of the same dimension, e.g. [Ke75, p. 150]) inherits a topological  $T$ -action (Def. 2.11) via stereographic projection, denoted

$$S^V \in T\text{Actions}(\text{TopologicalSpaces})$$

and called the *representation sphere* of  $V$  (e.g. [Blu17, §1.1.5][SS19a, §3]).

**Definition 2.13** (Orbit category). The *category of  $G$ -orbits* or *orbit category* of the equivariance group  $G$  (41)

$$G\text{Orbits} \hookrightarrow G\text{Actions}(\text{Sets}) \in \text{Categories}$$

is (up to equivalence of categories) the full subcategory of discrete  $G$ -actions (43) on the coset spaces  $G/H$  (which are discrete spaces, since  $G$  is assumed to be finite) for all subgroup inclusions  $H \xrightarrow{!} G$ .

**Example 2.14** (Explicit parameterization of morphisms of  $G\text{Orbits}$ ). The hom-sets (16) in the  $G$ -orbit category (Def. 2.13) from any  $G/H_1$  to any  $G/H_2$  are in bijection with sets of conjugations, inside  $G$ , of  $H_1$  into subgroups of  $H_2$ , modulo conjugations in  $H_2$ :

$$G\text{Orbits}(G/H_1, G/H_2) \simeq \frac{\{\phi : H_1 \hookrightarrow H_2, g \in G \mid \text{Ad}_{g^{-1}} \circ \iota_1 = \iota_2 \circ \phi\}}{((\phi, g) \sim (\text{Ad}_{h_2^{-1}} \circ \phi, gh_2) \mid h_2 \in H_2)}. \quad (45)$$

(Here “Ad” denotes the adjoint action of the group on itself, and  $H_i \xrightarrow{\iota_i} G$  are the two subgroup inclusions.)

**Example 2.15** (Orbit category of  $\mathbb{Z}_2$ ). The orbit category (Def. 2.13) of the cyclic group  $\mathbb{Z}_2 := \{e, \sigma \mid \sigma \circ \sigma = e\}$  is

$$\mathbb{Z}_2\text{Orbits} \simeq \left\{ \begin{array}{c} \langle \mathbb{Z}_2 \rangle \\ \mathbb{Z}_2/1 \xrightarrow{\exists!} \mathbb{Z}_2/\mathbb{Z}_2 \end{array} \right\}.$$

Hence its hom-sets (16) are:

$$\begin{aligned} \mathbb{Z}_2\text{Orbits}(\mathbb{Z}_2/1, \mathbb{Z}_2/1) &\simeq \mathbb{Z}_2, & \mathbb{Z}_2\text{Orbits}(\mathbb{Z}_2/\mathbb{Z}_2, \mathbb{Z}_2/\mathbb{Z}_2) &\simeq 1, \\ \mathbb{Z}_2\text{Orbits}(\mathbb{Z}_2/1, \mathbb{Z}_2/\mathbb{Z}_2) &\simeq *, & \mathbb{Z}_2\text{Orbits}(\mathbb{Z}_2/\mathbb{Z}_2, \mathbb{Z}_2/1) &\simeq \emptyset. \end{aligned} \quad (46)$$

**Example 2.16** (Automorphism groups in orbit category). For  $G$  a finite group and  $H \subset G$  a subgroup, the endomorphisms of  $G/H \in G\text{Orbits}$  (Def. 2.13) form the *Weyl group*  $W_G(H)$  (e.g. [Ma96, p. 13]) of  $H$  in  $G$ ,

$$\text{End}_{G\text{Orbits}}(G/H) \simeq \text{Aut}_{G\text{Orbits}}(G/H) = W_G(H) := N_G(H)/H, \quad (47)$$

namely the quotient group by  $H$  of the normalizer  $N_G(H)$  of  $H$  in  $G$ . For instance:

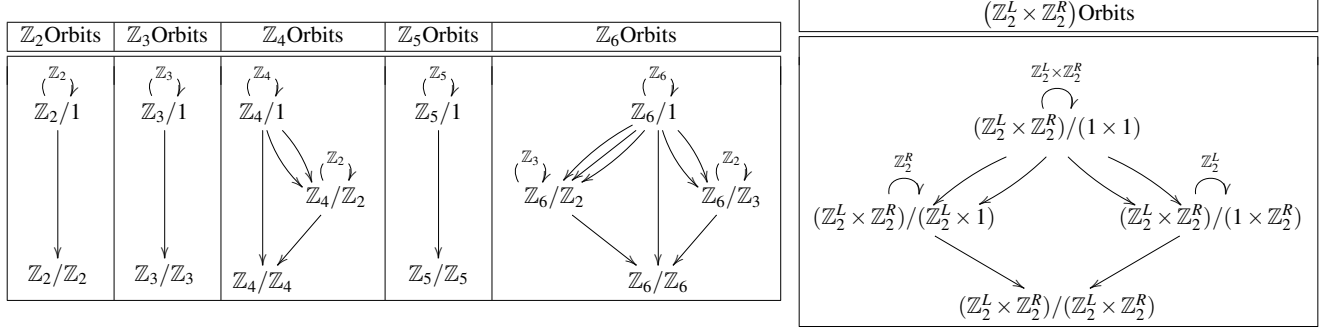
$$W_G(1) = G, \quad W_G(G) = 1; \quad \text{generally: } H \subset G \underset{\text{normal}}{\Rightarrow} W_G(H) = G/H.$$

Generally:

**Example 2.17** (Hom-sets in orbit category via Weyl groups). For any two subgroups  $K, H \subset G$ , the hom-set (16) in the  $G$ -orbit category (Def. 2.13) between their corresponding coset spaces is, as a right  $W_G(H)$ -set via Example 2.16, a disjoint union of copies of  $W_G(H)$ , one for each way of conjugating  $K$  into a subgroup of  $H$ :

$$G\text{Orbits}(G/K, G/H) \simeq \bigsqcup_{\substack{g \in G/N_G(K) \\ \text{s.t. } g^{-1}Kg \subset H}} gW_G(H) \in W_G(H)\text{Actions}(\text{Sets}). \quad (48)$$

**Example 2.18** (More examples of orbit categories).



### Equivariant homotopy types.

**Definition 2.19** (Equivariant simplicial sets). We write

$$\mathcal{G}\text{SimplicialSets} := \text{Functors}(\mathcal{G}\text{Orbits}^{\text{op}}, \text{SimplicialSets})$$

for the category of functors from the opposite of  $G$ -orbits (Def. 2.13) to simplicial sets.

**Example 2.20** (Systems of fixed loci of topological  $G$ -actions). Let  $G \curvearrowright X \in G\text{Actions}(\text{TopologicalSpaces})$  (Def. 2.11). For  $H \subset G$  any subgroup, a  $G$ -equivariant function (40)

$$\begin{array}{c} \langle G \rangle \\ G/H \end{array} \xrightarrow{f} \begin{array}{c} \langle G \rangle \\ X \end{array} \quad \Leftrightarrow \quad f([e]) \in \overset{H\text{-fixed locus}}{X^H} := \left\{ x \in X \mid \forall_{h \in H} (h \cdot x = x) \right\} \subset X \quad (49)$$

from the corresponding  $G$ -orbit (Def. 2.13) is determined by its image  $f([e]) \in X$  of the class of the neutral element, and that image has to be fixed by the action of  $H \subset G$  of  $X$ . Therefore, the corresponding  $G$ -equivariant mapping spaces

$$\text{Maps}(G/H, X)^G \simeq X^H$$

are the topological subspaces of  $H$ -fixed points inside  $X$ , the  $H$ -fixed loci in  $G \curvearrowright X$ . By functoriality of the mapping-space construction, these fixed point loci are exhibited as arranging into a contravariant functor on the  $G$ -orbit category (Def. 2.13):

$$\gamma(X//G) : \quad \mathcal{G}\text{Orbits}^{\text{op}} \xrightarrow{\text{Maps}(-, X)^G} \text{TopologicalSpaces} \quad (50)$$

$$\begin{array}{ccc} G/H_1 & \xrightarrow{\quad} & X^{H_1} & \text{\color{blue} } H_1\text{-fixed locus} \\ \downarrow \text{\color{black} } [(\text{id}, g)] & & \simeq \uparrow \text{\color{green} } g \cdot (-) & \text{\color{green} } \text{residual action on } H_2\text{-fixed locus} \\ G/H_1 & \xrightarrow{\quad} & X^{H_1} & \\ \downarrow \text{\color{black} } [(\phi, e)] & & \uparrow \text{\color{green} } \phi^* & \text{\color{green} } \text{inclusion of } H_2\text{-fixed locus} \\ G/H_2 & \xrightarrow{\quad} & X^{H_2} & \text{\color{blue} } H_2\text{-fixed locus} \end{array}$$

Here we used Example 2.14 to make explicit the nature of the continuous functions between fixed point spaces that this functor assigns to morphisms of  $\mathcal{G}\text{Orbits}$ . In particular, we see from Example 2.16 that the residual action on the  $H$ -fixed locus  $X^H$  is by the Weyl group  $W_G(H)$  (47). Postcomposing (50) with the singular simplicial set functor (19) yields an equivariant simplicial set (Def. 2.19), to be denoted (the notation follows [SS20, §3.2, 5.1]):

$$G \curvearrowright X \longmapsto \text{Sing}(\gamma(X//G)) := \text{Sing}(\text{Maps}(-, X)^G) \in \mathcal{G}\text{SimplicialSets}. \quad (51)$$

**Proposition 2.21** (Model category of equivariant simplicial sets [Hi02, Thm. 11.6.1][Gu06, Thm. 3.3][St16, §2.2]). *The category of equivariant simplicial sets (Def. 2.19) carries a model category structure whose*

- (a) **W** – weak equivalences are the weak equivalences of  $\text{SimplicialSets}_{\text{Qu}}$  over each  $G/H \in \mathcal{G}\text{Orbits}$ ;
- (b) **Fib** – fibrations are the weak equivalences of  $\text{SimplicialSets}_{\text{Qu}}$  over each  $G/H \in \mathcal{G}\text{Orbits}$ .

We denote this model category by  $\mathcal{G}\text{SimplicialSets}_{\text{proj}} \in \text{ModelCategories}$ .

**Definition 2.22** (Equivariant homotopy types). We denote the homotopy category of the projective model structure on equivariant simplicial sets (Prop. 2.21) by

$$\mathcal{G}\text{SimplicialSets}_{\text{proj}} \xrightarrow[\text{localization}]{\gamma} \mathcal{G}\text{HomotopyTypes} := \text{Ho}(\mathcal{G}\text{SimplicialSets}_{\text{proj}}). \quad (52)$$

The key source of equivariant homotopy types is the shapes of orbi-singularized homotopy quotients of topological spaces by continuous group actions (we follow [SS20, §3.2] in terminology and notation):

**Definition 2.23** (Equivariant shape). The composite of forming systems of fixed loci (Example 2.20) with localization to equivariant homotopy types (Def. 2.22) is the *equivariant shape* operation, generalizing the plain shape (21):

$$\begin{array}{ccc} \mathcal{G}\text{Actions}(\text{TopologicalSpaces}) & \xrightarrow[\text{localization at fixed locus-wise weak homotopy equivalences}]{\begin{array}{c} \text{equivariant shape} \\ G \zeta X \mapsto \int \mathcal{Y}(X//G) \end{array}} & \mathcal{G}\text{HomotopyTypes} \\ & \searrow[\text{form singular equivariant simplicial set (51)}] & \nearrow[\gamma \text{ localization (52)}] \\ & \mathcal{G}\text{SimplicialSets} & \end{array} \quad (53)$$

**Example 2.24** (Smooth equivariant homotopy types). A topological space  $X$  equipped with trivial  $G$ -action has equivariant shape (Def. 2.23) given by the functor on the orbit category which is constant on its ordinary shape (21)

$$\begin{array}{ccc} \text{TopologicalSpaces} & \xrightarrow[\text{shape}]{\int} & \text{HomotopyTypes} \\ \downarrow[\text{equip with trivial action}] & & \downarrow[\text{form constant functor on orbit category}] \\ \mathcal{G}\text{Actions}(\text{TopologicalSpaces}) & \xrightarrow[\text{equivariant shape}]{\int(-//G)} & \mathcal{G}\text{HomotopyTypes} \end{array} \quad (54)$$

For brevity, we will mostly leave this embedding notationally implicit and write

$$X := \text{Smth} \int X \in \mathcal{G}\text{HomotopyTypes}. \quad (55)$$

**Elmendorf's theorem.** In fact, every equivariant homotopy type (Def. 2.22) is the equivariant shape (Def. 2.23) of some topological space with  $G$ -action (Def. 2.11). This is the content of Elmendorf's theorem ([E183], see Prop. 2.26 below). Due to this fact, topological  $G$ -actions in equivariant homotopy theory are often conflated with their  $G$ -equivariant shape, and jointly referred to as  $G$ -spaces (e.g., [tD79, §8][Blu17, §1]).

**Proposition 2.25** (Model category of simplicial  $G$ -actions and fixed loci [Gu06, Thm. 3.12][St16, Prop. 2.6]). *The category  $\mathcal{G}\text{Actions}(\text{SimplicialSets})$  of  $G$ -actions  $G \zeta S$  on simplicial sets (analogous to Def. 2.11) carries a model category structure whose weak equivalences and fibrations are those that become so in the classical model structure on simplicial sets (18) under the functor (analogous to Example 2.20)*

$$\begin{array}{ccc} \mathcal{G}\text{Actions}(\text{SimplicialSets}) & \xrightarrow{\text{Maps}(-, -)^G} & \mathcal{G}\text{SimplicialSets} \\ G \zeta S & \mapsto & (G/H \mapsto S^H) \end{array} \quad (56)$$

which sends a  $G$ -action  $G \zeta S$  to its system of  $H$ -fixed loci parametrized over  $G/H \in G\text{Orbits}$ .

We denote this model category by

$$\mathcal{G}\text{Actions}(\text{SimplicialSets})_{\text{fine}} \in \text{ModelCategories}.$$

**Proposition 2.26** (Elmendorf's theorem via model categories [St16, Thm. 3.17][Gu06, Prop. 3.15]). *The functor assigning systems of simplicial fixed loci (56) is the right adjoint in a Quillen equivalence*

$$\begin{array}{ccc} \mathcal{G}\text{Actions}(\text{SimplicialSets})_{\text{fine}} & \xleftarrow[(-)(G/1)]{\simeq_{\text{Qu}}} & \mathcal{G}\text{SimplicialSets}_{\text{proj}} \\ & \xrightarrow{\text{Maps}(-, -)^G} & \end{array} \quad (57)$$

between the fine model structure on simplicial  $G$ -actions (Prop. 2.25) and the model category of equivariant simplicial sets (Prop. 2.21).

## Examples of equivariant homotopy types.

**Example 2.27** ( $G^{\text{ADE}}$ -equivariant 4-sphere). Let

$$G := G^{\text{ADE}} \subset \text{Spin}(3) \simeq \text{Sp}(1)$$

be a finite subgroup of the Spin group in dimension 3; these are famously classified along an ADE-pattern (reviewed in [HSS18, Rem. A.9]). Via the exceptional isomorphism with the quaternionic unitary group, this induces a canonical smooth action (Def. 2.35) on the Euclidean 4-space underlying the space of quaternions (reviewed as [HSS18, Prop. A.8]) and hence also on the corresponding representation 4-sphere (Example 2.12):

$$\begin{array}{c} \begin{array}{c} \text{G}^{\text{ADE}} \\ \curvearrowright \\ \mathbb{R}^4 \end{array}, \quad \begin{array}{c} \text{G}^{\text{ADE}} \\ \curvearrowright \\ S^4 \end{array} \in G^{\text{ADE}} \text{Actions}(\text{SmoothManifolds}). \end{array}$$

The corresponding  $G^{\text{ADE}}$ -equivariant homotopy types (Def. 2.22) (their equivariant shape, Def. 2.23)

$$\int \gamma(S^4 // G^{\text{ADE}}) \in \mathcal{G}^{\text{ADE}} \text{HomotopyTypes}$$

are the coefficients of ADE-equivariant Cohomotopy theory [HSS18, §5.2][SS19a, §3] (lifted to equivariant twistorial Cohomotopy theory below in Def. 2.48).

**Example 2.28** ( $\mathbb{Z}_2$ -equivariant twistor space). Consider the quaternion unitary group (e.g. [FSS20b, §A]) with its two commuting subgroups from (13) and (14):

$$\mathbb{Z}_2, \text{Sp}(1) \subset \text{Sp}(2) := \{g \in \text{Mat}_{2 \times 2}(\mathbb{H}) \mid g \cdot g^\dagger = 1\}. \quad (58)$$

Their canonical action on  $\mathbb{H}^2 \simeq_{\mathbb{R}} \mathbb{R}^8$  by left matrix multiplication induces an action (12) on  $\mathbb{C}P^3$  (“twistor space”). The fixed locus (49) of the subgroup  $\mathbb{Z}_2$  (13) under this action is evidently given by those  $[z_1 : z_2 : z_3 : z_4] \in \mathbb{C}P^3$  such that  $z_1 + j \cdot z_2 = z_3 + j \cdot z_4 \in \mathbb{H}$ . Since these are exactly the elements that are sent by the twistor fibration  $t_{\mathbb{H}}$  (11) to the base point  $[1 : 1] \in \mathbb{H}P^1$ , the  $\mathbb{Z}_2$ -fixed locus in twistor space  $\mathbb{C}P^3$  coincides with the  $S^2$ -fiber of the twistor fibration  $t_{\mathbb{H}}$  (11):

$$(\mathbb{C}P^3)^{\mathbb{Z}_2} \simeq S^2 \xrightarrow{\text{fib}(t_{\mathbb{H}})} \mathbb{C}P^3. \quad (59)$$

Hence the  $\mathbb{Z}_2$ -equivariant homotopy type (21) of twistor space with its  $\mathbb{Z}_2$  action (12) is given by the following functor on the  $\mathbb{Z}_2$ -orbit category (2.15):

$$\begin{array}{ccc} \begin{array}{c} \mathbb{Z}_2\text{-equivariant shape} \\ \text{of twistor space} \\ \int \gamma(\mathbb{C}P^3 // \mathbb{Z}_2) \end{array} : & \begin{array}{c} \begin{array}{c} \mathbb{Z}_2 \\ \curvearrowright \\ \mathbb{Z}_2/1 \end{array} \\ \downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 \end{array} & \begin{array}{c} \dashrightarrow \\ \dashrightarrow \end{array} & \begin{array}{c} \begin{array}{c} \mathbb{Z}_2 \\ \curvearrowright \\ \int \mathbb{C}P^3 \end{array} \\ \uparrow \text{fib}(t_{\mathbb{H}}) \\ \int S^2 \end{array} \end{array} \quad \begin{array}{c} \text{fiber inclusion of} \\ \text{twistor fibration} \end{array} \quad (60)$$

## Equivariant homotopy groups.

**Definition 2.29** (Equivariant groups). (i) We write

$$\mathcal{G}\text{Groups} := \text{Functors}(G\text{Orbits}^{\text{op}}, \text{Groups})$$

for the category of contravariant functors on the  $G$ -orbit category (Def. 2.13) with values in groups.

(ii) We write

$$\mathcal{G}\text{AbelianGroups} := \text{Functors}(G\text{Orbits}, \text{AbelianGroups})$$

for the sub-category of contravariant functors with values in abelian groups.

**Example 2.30** (Equivariant singular homology groups). For  $\mathcal{X} \in \mathcal{G}\text{HomotopyTypes}$  (Def. 2.22),  $A \in \text{AbelianGroups}$ , the ordinary  $A$ -homology groups in degree  $n \in \mathbb{N}$  of the stages of  $\mathcal{X}$  form an equivariant abelian group in the sense of Def. 2.29, to be denoted:

$$\underline{H}_n(\mathcal{X}; A) : G/H \mapsto H_n(\mathcal{X}(G/H); A).$$



**Definition 2.31** (Equivariant homotopy groups).

(i) For  $X \in \mathcal{G}\text{HomotopyTypes}$  (Def. 2.22),  $\gamma(*//G) \xrightarrow{x} X$  a base-point, and  $n \in \mathbb{N}$ , we say that the  $n$ th *equivariant homotopy group* of  $X$  at  $x$  is the equivariant group (Def. 2.29) which is stage-wise the ordinary  $n$ th homotopy group, to be denoted:

$$\underline{\pi}_n(X, x) := \left( G/H \mapsto \pi_n(X(G/H), x(G/H)) \right). \quad (61)$$

(ii) Similarly, for  $G \curvearrowright X \in G\text{Actions}(\text{TopologicalSpaces})$  (Def. 2.11),  $G \curvearrowright * \xrightarrow{x} G \curvearrowright X$  a fixed base point, and  $n \in \mathbb{N}$ , we say that the  $n$ th *equivariant homotopy group* of  $G \curvearrowright X$  is that (61) of its equivariant shape (21):

$$\underline{\pi}_n(X, x) := \underline{\pi}_n(\int \gamma(X//G), \int \gamma(x//G)) = \left( G/H \mapsto \pi_n(X^H, x) \right). \quad (62)$$

**Definition 2.32** (Equivariant connected homotopy types). We write

$$\mathcal{G}\text{HomotopyTypes}_{\geq 1} \hookrightarrow \mathcal{G}\text{HomotopyTypes} \quad (63)$$

for the full subcategory on those equivariant homotopy types  $X$  (Def. 2.22) which

- (a) are equivariantly connected, in that  $X(G/H) \in \text{HomotopyTypes}$  is connected for all  $H \subset G$ ;
- (b) admit an equivariant base point  $\gamma(*//G) \rightarrow X$ .

**Definition 2.33** (Equivariant 1-connected homotopy types).

(i) We write

$$\mathcal{G}\text{HomotopyTypes}_{\geq 2} \hookrightarrow \mathcal{G}\text{HomotopyTypes}_{\geq 1} \hookrightarrow \mathcal{G}\text{HomotopyTypes} \quad (64)$$

for the further full subcategory on those equivariant homotopy types  $X$  (Def. 2.22) which

- (a) are equivariantly connected and admit an equivariant base point (Def. 2.32);
- (b) have trivial first equivariant homotopy group (Def. 2.31) at that base point:

$$\underline{\pi}_1(X, x) = \underline{1}.$$

(ii) By the Hurewicz theorem, this implies that the equivariant real cohomology groups (Example 2.30) of these objects are trivial in degrees  $\leq 1$

$$X \in \mathcal{G}\text{HomotopyTypes}_{\geq 2} \quad \Rightarrow \quad (\underline{H}^0(X) \simeq \mathbb{R} \text{ and } \underline{H}^1(X) \simeq 0).$$

(iii) We write

$$\mathcal{G}\text{HomotopyTypes}_{\geq 2}^{\text{fin}_{\mathbb{R}}} \hookrightarrow \mathcal{G}\text{HomotopyTypes}_{\geq 2} \hookrightarrow \mathcal{G}\text{HomotopyTypes}$$

for the further full subcategory of those equivariant 1-connected homotopy types (64) which are of *finite type* over  $\mathbb{R}$ , in that all their equivariant real homology groups (Example 2.30) are finite-dimensional:

$$\forall_{\substack{H \subset G \\ n \in \mathbb{N}}} \dim_{\mathbb{R}} \left( H_n(X(G/H); \mathbb{R}) \right) < \infty.$$

**G-Orbifolds.** Given a smooth manifold  $X$  equipped with a smooth group action  $G \curvearrowright X$ , there are several somewhat different mathematical notions of what exactly counts as the corresponding *quotient orbifold* (review in [MoM03][Kap08, §6][IKZ10]).

- First, there is the singular quotient space  $X/G$  that dominates the early literature on orbifolds [Sa56][Sa57][Th80][Hae84] as well as the contemporary physics literature [BL99, §1.3].
- Second, there is the smooth stacky homotopy quotient  $X//G$  that has become the popular model for orbifolds among Lie theorists [MoP97][Mo02][Ler08][Am12].
- Third, there is the fine incarnation of orbifolds *orbisingular homotopy quotients*  $\gamma(X//G)$  in singular cohesive homotopy theory [SS20], which unifies the above two perspectives and lifts them to make orbifolds carry proper equivariant differential cohomology theories.

Here we extract from [SS20] the essence of this latter fine perspective that is necessary and convenient for the present purpose, as Def. 2.36 below.

**Lemma 2.34** (Fixed loci of finite smooth actions are smooth manifolds). *If  $G \curvearrowright X \in G\text{Actions}(\text{TopologicalSpaces})$  (Def. 2.11) is such that  $X$  admits the structure of a smooth manifold and such that the action (39) of  $G$  is smooth, then the fixed loci  $X^H \hookrightarrow X$  (49) are themselves smooth submanifolds.*

*Proof.* Since  $G$  is assumed to be finite (41), its smooth action is proper (e.g. [Lee12, Cor. 21.6]). But in smooth manifolds with proper smooth  $G$ -action, every closed submanifold inside a fixed locus has a  $G$ -equivariant tubular neighborhood [Br72, §VI, Thm. 2.2][Kan07, Thm. 4.4]. This applies, in particular, to individual fixed points, where it says that each such has a neighborhood in the fixed locus diffeomorphic to an open ball.  $\square$

**Definition 2.35** (Smooth group actions on smooth manifolds). (i) We write

$$G\text{Actions}(\text{SmoothManifolds}) \longrightarrow G\text{Actions}(\text{TopologicalSpaces})$$

for the category of smooth manifolds equipped with  $G$ -actions on the underlying topological spaces (Def. 2.11) which are smooth.

(ii) Similarly, if the compact Borel-equivariance group (37) is equipped with smooth structure making it a Lie group

$$T \in \text{CompactLieGroups} \longrightarrow \text{CompactTopologicalGroups},$$

we write

$$(T \times G)\text{Actions}(\text{SmoothManifolds}) \longrightarrow (T \times G)\text{Actions}(\text{TopologicalSpaces})$$

for the category of smooth manifolds equipped with  $T \times G$ -actions on the underlying topological spaces (Def. 2.11) which are smooth.

**Definition 2.36** ( $G$ -Orbifolds [SS20]). (i) We write

$$G\text{Orbifolds} := \text{Functors}(G\text{Orbits}^{\text{op}}, \text{SmoothManifolds}) \quad (65)$$

for the category of contravariant functors from  $G$ -orbits (Def. 2.13) to smooth manifolds.

(ii) By Lemma 2.34, the system of fixed loci (50) of a smooth action  $G \curvearrowright X$  (Def. 2.35) takes values in smooth manifolds

$$G \curvearrowright X \text{ smoothly} \Rightarrow \gamma(X//G) : G\text{Orbits}^{\text{op}} \dashrightarrow \text{SmoothManifolds} \longrightarrow \text{TopologicalSpaces}, \quad (66)$$

and hence witnesses an object  $\gamma(X//G) \in G\text{Orbifolds}$  (2.36) which is a smooth geometric refinement of the underlying equivariant homotopy type (Def. 2.23), in that we have the following commuting diagram of functors:

$$\begin{array}{ccc} G\text{Actions}(\text{SmoothManifolds}) & \xrightarrow{G \curvearrowright X \mapsto \gamma(X//G)} & G\text{Orbifolds} \\ \text{forget smooth structure} \downarrow (66) & & \downarrow \int \text{equivariant shape (Def. 2.23)} \\ G\text{Actions}(\text{TopologicalSpaces}) & \xrightarrow{G \curvearrowright X \mapsto \int \gamma(X//G) (51)} & \mathcal{G}\text{HomotopyTypes}. \end{array}$$

### 2.3 Equivariant non-abelian cohomology theories

We introduce the general concept of equivariant non-abelian cohomology theories, in direct generalization of [FSS23, §2.1], and consider some examples. This is in preparation for the twisted case in the next subsection.

In equivariant generalization of [FSS23, §2.1], we set:

**Definition 2.37** (Equivariant non-abelian cohomology). Let  $\mathcal{X}, \mathcal{A} \in \mathcal{G}\text{HomotopyTypes}$  (Def. 2.22).

(i) The *proper  $G$ -equivariant non-abelian cohomology* of  $\mathcal{X}$  with coefficients in  $\mathcal{A}$  is the hom-set (16)

$$H(\mathcal{X}; \mathcal{A}) := \mathcal{G}\text{HomotopyTypes}(\mathcal{X}, \mathcal{A}).$$

(ii) For  $X \in G\text{Actions}(\text{TopologicalSpaces})$  (Def. 2.11), with induced equivariant homotopy type  $\int \gamma(X//G)$  (21), we write

$$H_G(X; \mathcal{A}) := H(\int \gamma(X//G); \mathcal{A}) := \mathcal{G}\text{HomotopyTypes}(\int \gamma(X//G), \mathcal{A}).$$

(iii) We call the corresponding contravariant functor

$$G\text{Actions}(\text{TopologicalSpaces})^{\text{op}} \xrightarrow{\int \gamma(-//G)} \mathcal{G}\text{HomotopyTypes}^{\text{op}} \xrightarrow{H(-; \mathcal{A})} \text{Sets} \quad (67)$$

$\xrightarrow{H_G(-; \mathcal{A})}$

the *equivariant non-abelian cohomology theory* with coefficients in  $\mathcal{A}$ .

### Equivariant ordinary cohomology.

**Example 2.38** (Equivariant representation ring). For  $H$  a finite group and  $\mathbb{F}$  a field, write

$$\text{Rep}_{\mathbb{F}}(X) \in \text{Rings} \longrightarrow \text{AbelianGroups} \quad (68)$$

for the additive abelian group underlying the representation ring of  $H$  (i.e., the Grothendieck group of the semi-group of finite-dimensional  $\mathbb{F}$ -linear  $H$ -representations under tensor product of representations, review in [BSS19, §2.1]). Under the evident restriction of representations to subgroups and under conjugation action on representations, these groups arrange into a contravariant functor on the  $G$ -orbit category (Def. 2.13)

$$\begin{array}{ccc} \text{Rep}_{\mathbb{F}} : G\text{Orbits}^{\text{op}} & \longrightarrow & \text{AbelianGroups} \in \mathcal{G}\text{AbelianGroups} \\ G/H & \longmapsto & \text{Rep}_{\mathbb{F}}(H) \end{array} \quad (69)$$

and hence constitute an equivariant abelian group (Def. 2.29).

**Example 2.39** (Bredon cohomology [Br67a, p. 3][Br67b, Thm. 2.11 & (6.1)][GMa95, p. 10]).

Given  $\underline{A} \in \mathcal{G}\text{AbelianGroups}$  (Def. 2.29) and  $n \in \mathbb{N}$ :

(i) There is the *Eilenberg-MacLane  $G$ -space*

$$\mathcal{K}(\underline{A}, n) \in \mathcal{G}\text{HomotopyTypes} \quad (70)$$

in equivariant connected homotopy types (Def. 2.22), characterized by the fact that it admits a fixed point with equivariant homotopy groups (Def. 2.31) given by

$$\underline{\pi}_k(\mathcal{K}(\underline{A}, n)) \simeq \begin{cases} \underline{A} & | \quad k = n, \\ 0 & | \quad \text{otherwise.} \end{cases}$$

(ii) The *ordinary equivariant cohomology* or *Bredon cohomology* in degree  $n$  of  $X \in G\text{Actions}(\text{TopologicalSpaces})$  (Def. 2.11) with coefficients in  $\underline{A}$  is its equivariant non-abelian cohomology (Def. 2.37) with coefficients in  $\mathcal{K}(\underline{A}, n)$  (70):

$$\begin{array}{c} \text{Bredon cohomology} \\ \text{(equivariant ordinary cohomology)} \\ H_G^n(X; \underline{A}) \end{array} \simeq H_G(X; \mathcal{K}(\underline{A}, n)) = H(\gamma(X//G), \mathcal{K}(\underline{A}, n)).$$

### Equivariant Cohomotopy.

**Example 2.40** (Equivariant non-abelian Cohomotopy [tD79, §8.4][Pe94][Cr03] [SS19a]). For  $G \curvearrowright V$  a linear  $G$ -representation on a finite-dimensional real vector space  $V$ , the *representation sphere* (e.g. [Blu17, Ex. 1.1.5])

$$S^V := V^{\text{cpt}} \in G\text{Actions}(\text{TopologicalSpaces}) \xrightarrow{\int \gamma(-//G)} \mathcal{G}\text{HomotopyTypes}$$

defines an equivariant homotopy type (21). This is the coefficient space for the equivariant non-abelian cohomology theory (Def. 2.37) called (unstable) *equivariant Cohomotopy* in RO-degree  $V$ :

$$\begin{array}{c} \text{equivariant} \\ \text{Cohomotopy} \\ \pi_G^V(X) \end{array} := H_G(X; \gamma(S^V//G)) \simeq H(\gamma(X//G); \gamma(S^V//G)).$$

### Equivariant non-abelian cohomology operations.

**Definition 2.41** (Equivariant non-abelian cohomology operations). For  $\mathcal{A}, \mathcal{B} \in \mathcal{G}\text{HomotopyTypes}$  (Def. 2.22), a *cohomology operation* from equivariant non-abelian  $\mathcal{A}$ -cohomology to  $\mathcal{B}$ -cohomology (Def. 2.37) is a natural transformation

$$H(-; \mathcal{A}) \xrightarrow{\phi_*} H(-; \mathcal{B})$$

of the corresponding equivariant non-abelian cohomology theories (67). By the Yoneda lemma, such operations are induced by post-composition with morphisms between equivariant coefficient spaces:

$$\mathcal{A} \xrightarrow{\phi} \mathcal{B} \in \mathcal{G}\text{HomotopyTypes}. \quad (71)$$

## 2.4 Equivariant twisted non-abelian cohomology theories

We introduce equivariant twisted non-abelian cohomology, in direct generalization of [FSS23, §2.2], and introduce the main example of interest here (Def. 2.48 below).

### Equivariant $\infty$ -Actions.

**Remark 2.42** (Equivariant  $\infty$ -actions). (i) In equivariant generalization of Prop. 2.5 (and as a special case of [NSS12a, Thm. 2.19][NSS12b, Thm. 3.30, Cor. 3.34]), every equivariantly pointed and equivariantly connected equivariant homotopy type (Def. 2.32) is, equivalently, the equivariant classifying space  $B\mathcal{G}$  of an *equivariant  $\infty$ -group*

$$\mathcal{G} \in \mathcal{G}\text{EquivariantGroups}_\infty := \text{Ho}\left(\text{Functors}(G\text{Orbits}^{\text{op}}, \text{SimplicialGroups})_{\text{proj}}\right).$$

(ii) In equivariant generalization of Prop. 2.7 (and as a special case of [NSS12a, §4][SS20, §2.2]),  $\infty$ -actions of such equivariant  $\infty$ -groups on equivariant homotopy types  $\mathcal{A}$  are, equivalently, homotopy fibrations of equivariant homotopy types over  $B\mathcal{G}$  with homotopy fiber  $\mathcal{A}$ , hence a system of non-equivariant homotopy fibration (31) parametrized by the  $G/H \in G\text{Orbits}$  (Def. 2.13), denoted as follows <sup>6</sup>

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\text{hofib}(\rho_{\mathcal{A}})} & \mathcal{A} // \mathcal{G} \\ & \text{equivariant homotopy fibration} & \downarrow \rho_{\mathcal{A}} \\ & \text{associated to } \infty\text{-action of } \mathcal{G} \text{ on } \mathcal{A} & B\mathcal{G} \\ & & \in \mathcal{G}\text{HomotopyTypes} \\ \\ G/H \longmapsto & \mathcal{A}(G/H) \xrightarrow{\text{hofib}(\rho_{\mathcal{A}(G/H)})} & \mathcal{A}(G/H) // \mathcal{G}(G/H) \\ & \text{homotopy fibration} & \downarrow \rho_{\mathcal{A}(G/H)} \\ & \text{associated to } \infty\text{-action of } \mathcal{G}(G/H) \text{ on } \mathcal{A}(G/H) & B\mathcal{G}(G/H) \\ & & \in \text{HomotopyTypes} \end{array} \quad (72)$$

A key source of equivariant  $\infty$ -actions are equivariant parametrized homotopy types, in the following sense:

**Example 2.43** (Equivariant parametrized homotopy types). Consider  $T \in \text{CompactTopologicalGroups}$  (37),  $G \in \text{FiniteGroups}$  (41), and  $X \in (T \times G)\text{Actions}(\text{TopologicalSpaces})$  (44).

(i) Since the two group actions separately commute with each other, we may consider forming the combined

(a) proper equivariant shape (Def. 2.23) with respect to the  $G$ -action;

(b) ordinary shape (21) of the homotopy quotient (Borel construction, Example 2.8) with respect to the  $T$ -action:

$$\mathcal{G}\text{HomotopyTypes} \ni \left( (\gamma(X // G)) // T \right); \quad G/H \longmapsto \int (X^H // T). \quad (73)$$

This is the  $G$ -equivariant homotopy type (Def. 2.22) given on  $G/H \in G\text{Orbits}$  (Def. 2.13) by the Borel homotopy quotient construction (Example 2.8) of the  $T$ -action on the  $G \supset H$ -fixed locus (Example 2.20).

(ii) With the classifying space  $BT$  regarded as a smooth  $G$ -equivariant homotopy type (i.e., with trivial  $G$ -action, Example 2.24) the  $G$ -equivariant  $T$ -parametrized space (73) sits in an equivariant fibration (72) over  $BT$  with homotopy fiber the  $G$ -equivariant shape of  $X$  (Def. 2.23):

$$\begin{array}{ccc} \int \gamma(X // G) & \xrightarrow{\text{hofib}(\rho_{\gamma(X // G)})} & \int \left( (\gamma(X // G)) // T \right) \\ & & \downarrow \rho_{\gamma(X // G)} \\ & & BT \\ & & \in \mathcal{G}\text{HomotopyTypes} \\ \\ G/H \longmapsto & \int X^H \xrightarrow{\text{hofib}(\rho_{X^H})} & \int (X^H // T) \\ & & \downarrow \rho_{X^H} \\ & & BT \\ & & \in \text{HomotopyTypes} \end{array}$$

We may refer to these objects as *proper  $G$ -equivariant and Borel  $T$ -equivariant homotopy types*, but for brevity and due to their above fibration over  $BT$ , we will say  *$G$ -equivariant  $T$ -parametrized homotopy types*.

<sup>6</sup>Here and in the following we indicate the ambient category of a given diagram. The notation “Diagram  $\in$  Category” means that each vertex of the diagram is an object in that category, and each arrow is a morphism in that category.

**Example 2.44** ( $\mathbb{Z}_2$ -equivariant  $\mathrm{Sp}(1)$ -parametrized twistor fibration). Recall the  $\mathbb{Z}_2$ -equivariant twistor fibration (11) from Example 2.28. Since the  $\mathrm{Sp}(2)$ -subgroups  $\mathbb{Z}_2$  (13) and  $\mathrm{Sp}(1)$  (14) commute with each other, the quotient by the action of  $\mathrm{Sp}(1)$  of the Cartesian product of the twistor fibration (11) with (the identity map on) the total space  $E\mathrm{Sp}(2)$  of the universal principal  $\mathrm{Sp}(2)$ -bundle still has a residual equivariance under  $\mathbb{Z}_2$ :

$$\begin{array}{ccccc}
\frac{S^2 \times E\mathrm{Sp}(2)}{\mathrm{Sp}(1)} & \xrightarrow{\frac{\mathrm{fib}(t_{\mathbb{R}}) \times \mathrm{id}}{\mathrm{Sp}(1)}}} & \frac{\mathbb{C}P^3 \times E\mathrm{Sp}(2)}{\mathrm{Sp}(1)} & \xrightarrow{\frac{t_{\mathbb{R}} \times \mathrm{id}}{\mathrm{Sp}(1)}}} & \frac{S^4 \times E\mathrm{Sp}(2)}{\mathrm{Sp}(1)} \\
\downarrow & & \downarrow & & \downarrow \\
\frac{E\mathrm{Sp}(2)}{\mathrm{Sp}(1)} & \xlongequal{\quad} & \frac{E\mathrm{Sp}(2)}{\mathrm{Sp}(1)} & \xlongequal{\quad} & \frac{E\mathrm{Sp}(2)}{\mathrm{Sp}(1)}
\end{array} \in \mathbb{Z}_2 \text{Actions}(\text{TopologicalSpaces})^{\frac{E\mathrm{Sp}(2)}{\mathrm{Sp}(1)}} \quad (74)$$

Hence, using Example 2.28 and identifying the Borel construction of homotopy quotients (e.g. [NSS12b, Prop. 3.73], here for subgroups  $H \subset G$ )

$$\frac{X \times EG}{H} \simeq X // H \in \text{HomotopyTypes}, \quad (75)$$

Borel construction
homotopy quotient

the  $\mathbb{Z}_2$ -equivariant homotopy type (Def. 2.22) of the middle vertical morphism in (74) exhibits a  $\mathbb{Z}_2$ -equivariant  $\mathrm{Sp}(1)$ -parametrized homotopy type (in the sense of Example 2.43) of this form:

$$\begin{array}{ccc}
\int(\gamma(\mathbb{C}P^3 // \mathbb{Z}_2)) // \mathrm{Sp}(1) & : & \int(\gamma(* // \mathbb{Z}_2)) // \mathrm{Sp}(1) \\
\downarrow \rho_{\int \gamma(\mathbb{C}P^3 // \mathbb{Z}_2)} & & \downarrow \rho_{\int \gamma(* // \mathbb{Z}_2)} \\
\int(\gamma(* // \mathbb{Z}_2)) // \mathrm{Sp}(1) & : & \int(\gamma(* // \mathbb{Z}_2)) // \mathrm{Sp}(1)
\end{array}
\quad \begin{array}{ccc}
\mathbb{Z}_2 / 1 & \mapsto & \int \mathbb{C}P^3 // \mathrm{Sp}(1) \\
\downarrow & & \downarrow \rho_{\int \mathbb{C}P^3} \\
\mathbb{Z}_2 / \mathbb{Z}_2 & \mapsto & \int S^2 // \mathrm{Sp}(1) \\
& & \downarrow \rho_{\int S^2} \\
& & \int B\mathrm{Sp}(1)
\end{array} \quad (76)$$

The analogous statement holds for the vertical morphism on the right of (74), so that the full square on the right of (74) exhibits a morphism in  $\mathbb{Z}_2$ -equivariant  $\mathrm{Sp}(1)$ -parametrized homotopy types (Example 2.43) of this form:

$$\begin{array}{ccc}
\int(\gamma(\mathbb{C}P^3 // \mathbb{Z}_2)) // \mathrm{Sp}(1) & \xrightarrow{\int \gamma(t_{\mathbb{R}} // \mathbb{Z}_2) // \mathrm{Sp}(1)}} & \int(\gamma(S^4 // \mathbb{Z}_2)) // \mathrm{Sp}(1) \\
\downarrow & & \downarrow \\
& & B\mathrm{Sp}(1)
\end{array} \in \text{Ho}\left(\mathbb{Z}_2 \text{SimplicialSets}_{\text{proj}}^{\int B\mathrm{Sp}(1)}\right), \quad (77)$$

where  $B\mathrm{Sp}(1) := \text{Smth} \int B\mathrm{Sp}(1)$  (Example 2.24).

### Twisted equivariant non-abelian cohomology.

In twisted generalization of Def. 2.37 and in equivariant generalization of [FSS23, §2.2], we set:

**Definition 2.45** (Twisted equivariant non-abelian cohomology). Let

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\text{hofib}(\rho_{\mathcal{A}})} & \mathcal{A} // \mathcal{G} \\
\downarrow \rho_{\mathcal{A}} & & \downarrow \rho_{\mathcal{A}} \\
& & B\mathcal{G}
\end{array} \in \mathcal{GHomotopyTypes} \quad (78)$$

equivariant local coefficient bundle

be an homotopy fibration as in Remark 2.42, to be regarded now as an *equivariant local coefficient bundle*, and let  $X \in \mathcal{GHomotopyTypes}$  (Def. 2.22) equipped with an *equivariant twist*

$$[\tau] \in H(X; B\mathcal{G}) \quad (79)$$

in equivariant non-abelian cohomology (Def. 2.37) with coefficients in  $B\mathcal{G}$ . We say that the  $\tau$ -twisted equivariant non-abelian cohomology of  $X$  with coefficients in  $\mathcal{A}$  is the hom-set from  $\tau$  to  $\rho_{\mathcal{A}}$  in the homotopy category of the slice model structure (see [FSS23, Ex. A.10]) over  $B\mathcal{G}$  of the projective model structure on equivariant simplicial sets (Prop. 2.21):

$$H^{\tau}(X; \mathcal{A}) := \text{Ho}\left(\mathcal{G}\text{SimplicialSets}_{\text{proj}}^{/B\mathcal{G}}\right)(\tau, \rho_{\mathcal{A}}).$$

### Twisted equivariant ordinary cohomology.

**Example 2.46** (Twisted Bredon cohomology). Let  $G \curvearrowright X \in G\text{Actions}(\text{TopologicalSpaces})$  (Def. 2.11) with a base point  $G \curvearrowright * \xrightarrow{x} G \curvearrowright X$ , let  $\underline{A} \in \mathcal{G}\text{AbelianGroups}$  (Def. 2.29), and let

$$r : \underline{\pi}_1(X) \times \underline{A} \longrightarrow \underline{A}$$

be an action of the equivariant fundamental group (Def. 2.31) of  $X$  on  $\underline{A}$ . For  $n \in \mathbb{N}$ , there is an equivariant local coefficient bundle (78)

$$\begin{array}{ccc} \mathcal{K}(\underline{A}, n) & \longrightarrow & \mathcal{K}(\underline{A}, n) // \underline{\pi}_1(X) \\ \text{equivariant ordinary} & & \downarrow \rho \\ \text{local coefficients} & & B\underline{\pi}_1(X) \end{array}$$

with typical fiber the equivariant Eilenberg-MacLane space (70), such that the twisted equivariant non-abelian cohomology with local coefficients in  $\rho$  coincides (by [Go97a, Cor. 3.6][MuS10, Thm. 5.10]) with traditional  $r$ -twisted Bredon cohomology in degree  $n$  ([MoS93, Def. 2.1][MuM96, Def. 3.8][MuP02]):

$$H_G^{n+r}(X; \underline{A}) \simeq H^{\tau}(X; \mathcal{K}(\underline{A}, n)).$$

**Equivariant tangential structure.** In equivariant generalization of [FSS23, Example 2.33], we have:

**Definition 2.47** (Equivariant tangential structure). Let  $G \curvearrowright X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35) of dimension  $n := \dim(X)$ , and let  $\mathcal{G} \xrightarrow{\phi} BGL(n)$  be a topological group homomorphism. An *equivariant tangential*  $(\mathcal{G}, \phi)$ -structure (or just  $\mathcal{G}$ -structure, for short) on the orbifold  $\gamma(X // G)$  (Def. 2.36) is a class in the equivariant twisted non-abelian cohomology (Def. 2.45) of the equivariant shape (Def. 2.23) of the orbifold with equivariant local coefficients (78) in

$$\begin{array}{ccc} GL(n) // \mathcal{G} & \longrightarrow & B\mathcal{G} \\ & & \downarrow B\phi \\ & & BGL(n) \end{array}$$

and with twist given by the classifying map  $\tau_{\text{Fr}}$  of the frame bundle:

$$(\mathcal{G}, \phi)\text{Structures}(\gamma(X // G)) := H^{\tau_{\text{Fr}}}(\gamma(X // G); GL(n) // \mathcal{G}).$$

**Equivariant twistorial Cohomotopy.** In equivariant generalization of [FSS23, Ex. 2.44] we have:

**Definition 2.48** (Equivariant twistorial Cohomotopy theory). Let  $X^8 \in \mathbb{Z}_2\text{Actions}(\text{TopologicalSpaces})$  (Def. 2.11) be a smooth spin 8-manifold equipped with tangential structure (see [FSS19a, Ex. 2.33]) for the subgroup  $\text{Sp}(1) \subset \text{Sp}(2) \subset \text{Spin}(8)$  (where the first inclusion is (13) and the second is again given by left quaternion multiplication, e.g. [FSS19a, Ex. 2.12])

$$[\tau] \in H_{\mathbb{Z}_2}(X^8; B\text{Sp}(1)).$$

We say that:

- (a) its  $\mathbb{Z}_2$ -equivariant twistorial Cohomotopy  $\mathcal{T}_{\mathbb{Z}_2}^{\tau}(-)$  is the  $\tau$ -twisted equivariant non-abelian cohomology theory (Def. 2.45) with local coefficients in the  $\mathbb{Z}_2$ -equivariant  $\text{Sp}(1)$ -parametrized twistor space;
- (b) its  $\mathbb{Z}_2$ -equivariant  $J$ -twisted Cohomotopy  $\pi_{\mathbb{Z}_2}^{\tau}(-)$  is the  $\tau$ -twisted equivariant non-abelian cohomology theory (Def. 2.45) with local coefficients in the  $\mathbb{Z}_2$ -equivariant  $\text{Sp}(1)$ -parametrized 4-sphere;

(c) the twisted equivariant cohomology operation  $\mathcal{F}_{\mathbb{Z}_2}^{\tau}(-) \longrightarrow \pi_{\mathbb{Z}_2}^{\tau}(-)$  is that induced by the  $\mathbb{Z}_2$ -equivariant  $\mathrm{Sp}(1)$ -parametrized twistor fibration;

all as induced by the (morphism of) local coefficient bundles (77) in Example 2.44:

$$\begin{array}{ccc} \text{equivariant} & & \text{equivariant} \\ \text{twistorial Cohomology} & & \text{J-twisted Cohomology} \\ \mathcal{F}_{\mathbb{Z}_2}^{\tau}(X) & := H_{\mathbb{Z}_2}^{\tau}\left(X; \int \gamma(\mathbb{C}P^3 // \mathbb{Z}_2)\right) & \xrightarrow[\left(\int \gamma(t_{\mathbb{H}} // \mathbb{Z}_2) // \mathrm{Sp}(1)\right)_*]{\text{push-forward along}} H_{\mathbb{Z}_2}^{\tau}\left(X; \int \gamma(S^4 // \mathbb{Z}_2)\right) := \pi_{\mathbb{Z}_2}^{\tau}(X). \end{array} \quad (80)$$

### 3 Equivariant non-abelian de Rham cohomology

We had shown in [FSS23, §3] how the fundamental theorem of dgc-algebraic rational homotopy theory ([BG76, §9.4, §11.2]), augmented by differential-geometric observations [GMo13, §9], provides a non-abelian de Rham theorem for  $L_{\infty}$ -algebra valued differential forms, which serve as the recipient of non-abelian character maps.

The equivariant generalization of this fundamental theorem had been obtained in [Scu08] (following [Tr82]) without having found much attention yet. Here we review, in streamlined form and highlighting examples and applications, the underlying theory of injective equivariant dgc-algebras/ $L_{\infty}$ -algebras in §3.1 and how these serve to model equivariant rational homotopy theory in §3.2. Then we use this in §3.3 to prove the equivariant non-abelian de Rham theorem (Prop. 3.63) including its twisted version (Prop. 3.67); which, in turn, we use in §3.4 to construct the equivariant non-abelian character map (Def. 3.76) and its twisted version (Def. 3.78).

#### 3.1 Equivariant dgc-algebras and equivariant $L_{\infty}$ -algebras

We discuss here the generalization of the homotopy theory of connective dgc-algebras and of connective  $L_{\infty}$ -algebras (following [FSS23, §3.1]) to  $G$ -equivariant homotopy theory, for any finite equivariance group  $G$  (41). While the homotopy theory of equivariant connective dgc-algebras has been developed in [Tr82][Scu02][Scu08], previously little to no examples or applications have been worked out. Here we develop equivariantized twistor space as a running example (culminating in Prop. 3.56 below).

While the general form of the homotopy theory of plain dgc-algebras generalizes to equivariant dgc-algebras, the crucial new aspect is that equivariantly not every connective cochain complex, and hence not every connective dgc-algebra, is fibrant. The fibrant equivariant cochain complexes must be degreewise injective, which is now a non-trivial condition (Prop. 3.12 below).

The key effect on the theory is that equivariant minimal Sullivan models (Def. 3.40) – which still exist and still have the expected general properties – are no longer given just by iterative adjoining of (equivariant systems of) generators, but by adjoining of injective resolutions (Example 3.28) of systems of generators. This has interesting effects, as shown in Example 3.42, which is at the heart of the proof of Prop. 3.56 and thus of Theorem 1.1.

**Plain homological algebra.** For plain (i.e., non-equivariant) dgc-algebra, we follow the conventions of [FSS23, §3.1]. In particular, we make use of the following notation:

**Notation 3.1** (Generators/relations presentation of cochain complexes).

We may denote any  $V \in \mathrm{CochainComplexes}_{\mathbb{R}}^{\geq 0, \mathrm{fin}}$  by generators (a graded linear basis) and relations (the linear relations given by the differential). For instance:

$$\begin{aligned} \mathbb{R}\langle c_2 \rangle / (dc_2 = 0) &\simeq (0 \longrightarrow 0 \longrightarrow \mathbf{1} \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots), \\ \mathbb{R}\left\langle \begin{array}{c} c'_3 \\ c_3 \\ b_2 \end{array} \right\rangle / \left( \begin{array}{c} dc'_3 = 0 \\ dc_3 = 0 \\ db_2 = c_3 \end{array} \right) &\simeq (0 \longrightarrow 0 \longrightarrow \mathbf{1} \hookrightarrow \mathbf{2} \longrightarrow 0 \longrightarrow \dots). \end{aligned}$$

**Notation 3.2** (Generators/relations presentation of dgc-algebras). We may denote the Chevalley-Eilenberg algebra  $\text{CE}(\mathfrak{g}) \in \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0, \text{fin}}$  of any  $\mathfrak{g} \in L_{\infty}\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0}$  ([FSS23, Def. 3.25]) by generators (a graded linear basis) and relations (the polynomial relations given by the differential). For instance (see [FSS23, Ex. 3.67, 3.68]):

$$\mathbb{R}[c_2]/(dc_2 = 0) \simeq \text{CE}(\mathfrak{b}\mathbb{R}) \quad \text{and} \quad \mathbb{R}\left[\begin{array}{c} \omega_7, \\ \omega_4 \end{array}\right]/\left(\begin{array}{c} d\omega_7 = -\omega_4 \wedge \omega_4 \\ d\omega_4 = 0 \end{array}\right) \simeq \text{CE}(\mathfrak{IS}^4).$$

Similarly, for  $T$  a finite-dimensional compact and simply-connected Lie group with Lie algebra

$$\mathfrak{t} \simeq \left\{ \langle t_a \rangle_{a=1}^{\dim(T)}, [-, -] \right\} \in \text{LieAlgebras}_{\mathbb{R}, \text{fin}},$$

the abstract Chern-Weil isomorphism (e.g. [FSS23, §4.2]) reads:

$$\left( \mathbb{R}[\{r_2^a\}_{a=1}^{\dim(T)}]/(dr_2^a = 0) \right)^T \simeq \text{CE}(\mathfrak{IBT}), \quad (81)$$

where on the left  $(-)^T$  denotes the  $T$ -invariant elements with respect to the coadjoint action on the dual vector space of the Lie algebra.

### Equivariant vector spaces.

**Example 3.3** (Linear representations as functors). For  $G$  any finite group, write  $BG$  for the category with a single object and with  $G$  as its endomorphisms (hence its automorphisms). Then functors on  $BG$  with values in vector spaces are, equivalently, linear  $G$ -representations with  $G$  acting either from the left or from the right, depending on whether the functor is contravariant or covariant:

$$\begin{aligned} G\text{Representations}_{\mathbb{R}}^l &\simeq \text{Functors}(BG^{\text{op}}, \text{VectorSpaces}_{\mathbb{R}}), \\ G\text{Representations}_{\mathbb{R}}^r &\simeq \text{Functors}(BG, \text{VectorSpaces}_{\mathbb{R}}). \end{aligned} \quad (82)$$

**Example 3.4** (Irreducible  $\mathbb{Z}_2$ -representations). We write

$$\mathbf{1}, \mathbf{1}_{\text{sgn}} \in \mathbb{Z}_2\text{Representations}_{\mathbb{R}}^r$$

for the two irreducible right representations (Example 3.3) of  $\mathbb{Z}_2$ , namely the trivial representation and the sign representation, respectively.

**Definition 3.5** (Equivariant vector spaces). We write

$$\begin{aligned} \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\text{fin}} &:= \text{Functors}(G\text{Orbits}^{\text{op}}, \text{VectorSpaces}_{\mathbb{R}}^{\text{fin}}), \\ \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee, \text{fin}} &:= \text{Functors}(G\text{Orbits}, \text{VectorSpaces}_{\mathbb{R}}^{\text{fin}}) \end{aligned} \quad (83)$$

for the categories of contravariant or covariant functors, respectively, from the  $G$ -orbit category (Def. 2.13) to the category of finite-dimensional vector spaces over the real numbers.

Notice that forming linear dual vector spaces constitutes an equivalence of categories

$$\text{VectorSpaces}_{\mathbb{R}}^{\text{fin}} \xrightarrow[\simeq]{(-)^{\vee}} (\text{VectorSpaces}_{\mathbb{R}}^{\text{fin}})^{\text{op}}$$

and hence induces an equivalence:

$$\begin{aligned} (\mathcal{G}\text{VectorSpaces}_{\mathbb{R}})^{\text{op}} &= \left( \text{Functors}(G\text{Orbits}^{\text{op}}, \text{VectorSpaces}_{\mathbb{R}}^{\text{fin}}) \right)^{\text{op}} \\ &\simeq \text{Functors}\left(G\text{Orbits}, (\text{VectorSpaces}_{\mathbb{R}}^{\text{fin}})^{\text{op}}\right) \\ &\simeq \text{Functors}(G\text{Orbits}, \text{VectorSpaces}_{\mathbb{R}}^{\text{fin}}) \\ &= \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee, \text{fin}}. \end{aligned}$$

This justifies extending the notation (83) to vector spaces which are not necessarily finite-dimensional

$$\begin{aligned} \mathcal{G}\text{VectorSpaces}_{\mathbb{R}} &:= \text{Functors}(G\text{Orbits}^{\text{op}}, \text{VectorSpaces}_{\mathbb{R}}) \\ \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee} &:= \text{Functors}(G\text{Orbits}, \text{VectorSpaces}_{\mathbb{R}}) \end{aligned}$$

and to speak of the latter as the category of *equivariant dual vector spaces* (denoted  $\text{Vec}_G^*$  in [Tr82]).



**Example 3.6** (Equivariant dual vector spaces of real cohomology groups). For  $\mathcal{X} \in \mathcal{GHomotopyTypes}$  (Def. 2.22) and  $n \in \mathbb{N}$ , the stage-wise real cohomology groups in degree  $n$  form an equivariant dual vector space (Def. 3.5)

$$\underline{H}^n(\mathcal{X}; \mathbb{R}) : G/H \longmapsto H^n(\mathcal{X}(G/H); \mathbb{R}).$$

If these are stage-wise finite-dimensional, then these are the linear dual equivariant vector spaces of the equivariant singular real homology groups  $\underline{H}_n(\mathcal{X}; \mathbb{R})$  from Example 2.30.

**Example 3.7** ( $\mathbb{Z}_2$ -equivariant dual vector spaces). A (finite-dimensional) dual  $\mathbb{Z}_2$ -equivariant vector space (Def. 3.5) is a diagram of (finite-dimensional) vector spaces indexed by the  $\mathbb{Z}_2$ -orbit category (Example 2.15)

$$\left( \begin{array}{ccc} \begin{array}{c} \mathbb{Z}_2 \\ \downarrow \\ \mathbb{Z}_2/1 \end{array} & \mapsto & \begin{array}{c} \mathbb{Z}_2 \\ \downarrow \\ \mathbf{N} \end{array} \\ \downarrow & & \downarrow \phi \\ \begin{array}{c} \mathbb{Z}_2/\mathbb{Z}_2 \end{array} & \mapsto & \begin{array}{c} V \end{array} \end{array} \right) \in \mathbb{Z}_2 \mathcal{G} \text{VectorSpaces}_{\mathbb{R}}^{\vee}$$

hence constitutes:

- a right  $\mathbb{Z}_2$ -representation  $\mathbf{N}$  (Example 3.3),
- a vector space  $V$  (finite-dimensional),
- a linear map  $\phi$  from the underlying vector space of  $\mathbf{N}$  to  $V$ .

**Example 3.8** (Restriction of equivariant vector spaces to Weyl group linear representation). For  $H \subset G$  a subgroup, with Weyl group  $W_G(H) = \text{Aut}_{G\text{Orbits}}(G/H)$  (Example 2.16), the canonical inclusion of categories

$$BW_G(H) \xleftarrow{i_H} G\text{Orbits} \quad (84)$$

induces restriction functors of equivariant vector spaces (Def. 3.5) to linear representations (Example 3.3):

$$\begin{aligned} W_G(H)\text{Representations}_{\mathbb{R}}^l &\xleftarrow{i_H^*} \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}, \\ W_G(H)\text{Representations}_{\mathbb{R}}^r &\xleftarrow{i_H^*} \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee}. \end{aligned} \quad (85)$$

**Example 3.9** (Regular equivariant vector space). For any subgroup  $K \subset G$  we have an equivariant dual vector space (Def. 3.5) given by the  $\mathbb{R}$ -linear spans of the hom-sets (16) out of  $G/K$  in the orbit category (Def. 2.13):

$$\mathbb{R}[G\text{Orbits}(G/K, -)] \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee}.$$

For any further subgroup  $H \subset G$ , its restriction (Example 3.8) to a linear representation from the right (Example 3.3) of the Weyl group of  $H$  (Def. 2.16) is

$$i_H^*(\mathbb{R}[G\text{Orbits}(G/K, -)]) = \mathbb{R}[G\text{Orbits}(G/K, G/H)] \in W_G(H)\text{Representations}_{\mathbb{R}}^r,$$

where  $W_G(H)$  acts in linear extension of its canonical right action on the hom-set of the orbit category (Example 2.16).

**Lemma 3.10** (Extension of linear representations to equivariant vector spaces). *For any  $H \subset G$ , the restriction of equivariant vector spaces to linear representations (Example 3.8) has a right adjoint*

$$W_G(H)\text{Representations}_{\mathbb{R}}^r \begin{array}{c} \xleftarrow{i_H} \\ \perp \\ \xrightarrow{\text{Inj}_H} \end{array} \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee},$$

where

$$\text{Inj}_H(V^*) \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee} = \text{Functors}(G\text{Orbits}, \text{VectorSpaces}_{\mathbb{R}})$$

is given by

$$\text{Inj}_H(V^*) : G/K \longmapsto W_G(H)\text{Representations}_{\mathbb{R}}^r(\mathbb{R}[G\text{Orbits}(G/K, G/H)], V^*) \quad (86)$$

$$= \bigoplus_{\substack{g \in G/N_G(K) \\ \text{s.t. } g^{-1}Kg \subset H}} V^*. \quad (87)$$

Here the regular  $W_G(H)$ -representation in the first argument on the right of (86) is from Example 3.9.

*Proof.* Formula (86) is a special case of the general formula for right Kan extension [Kel82, (4.24)], here applied to the inclusion (84) regarded in  $\text{VectorSpaces}_{\mathbb{R}}$ -enriched category theory. Its equivalence to (87) follows with Example 2.17. See also [Tr82, (4.1)][Scu08, Lemma 2.3].  $\square$

**Injective equivariant dual vector spaces.** Recall the general definition of injective objects (e.g. [HS71, p. 30]), applied to equivariant dual vector spaces:

**Definition 3.11** (Injective equivariant dual vector spaces). An object  $I \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee}$  (Def. 3.5) is called *injective* if morphisms into it extend along all injections, hence if every solid diagram of the form

$$\begin{array}{ccc} W & \xleftarrow{\quad \exists \quad} & I \\ & \searrow \text{injection} & \nearrow \\ & V & \end{array} \quad \begin{array}{l} \text{injective} \\ \text{object} \end{array} \quad (88)$$

admits a dashed morphism that makes it commute, as shown. We write

$$\mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee, \text{inj}} \hookrightarrow \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee}$$

for the full sub-category on the injective objects.

**Proposition 3.12** (Injective envelope of equivariant dual vector spaces [Tr82, p. 2][Scu02, Prop. 7.34][Scu08, Lem. 2.4, Prop. 2.5]). For  $V \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee}$  (Def. 3.5), the direct sum of extensions  $\text{Inj}_{(-)}$  (Def. 3.10)

$$\text{Inj}(V) := \bigoplus_{[H \subset G]} \text{Inj}_H(V_H) \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee}, \quad (89)$$

of those components at stage  $H$  which vanish on all deeper stages

$$V_H := \begin{cases} \bigcap_{[K \supseteq H]} \ker \left( V(G/H) \xrightarrow{V(G/(H \rightarrow K))} V(G/K) \right) & | \quad H \neq G \\ V(G/G) & | \quad H = G \end{cases} \quad (90)$$

receives an injection

$$V \hookrightarrow \text{Inj}(V) \quad (91)$$

that extends the canonical inclusion of the  $V_H$ , and which is an injective envelope (e.g. [HS71, §I.9]) of  $V$  in  $\mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee}$ . In particular:

- (i) the summands  $\text{Inj}_H(V)$  (Example 3.10) are injective objects (Def. 3.11);
- (ii)  $V$  is injective (Def. 3.11) precisely if (91) is an isomorphism.

**Example 3.13** (Ground field is injective as equivariant dual vector space). The equivariant dual vector space (Def. 3.5) which is constant on the ground field

$$\underline{\mathbb{R}} := \text{const}_{G\text{Orbits}}(\mathbb{R}) : G/H \mapsto \mathbb{R}$$

is isomorphic to the right extension (Lemma 3.10)  $\underline{\mathbb{R}} \simeq \text{Inj}_G(\mathbf{1})$  of  $\mathbb{R} \simeq \mathbf{1} \in 1\text{Representations}_{\mathbb{R}}$ , and hence is injective, by Prop. 3.12.

**Example 3.14** (Injective  $\mathbb{Z}_2$ -equivariant dual vector spaces, cf. [ST23, Prop. 4.1]). For  $G = \mathbb{Z}_2$  (Example 2.15) the irreducible representations

$$\mathbf{1}, \mathbf{1}_{\text{sgn}} \in \mathbb{Z}_2\text{Representations}_{\mathbb{R}}, \quad \mathbf{1} \in 1\text{Representations}_{\mathbb{R}} \simeq \text{VectorSpaces}_{\mathbb{R}}$$

of the respective Weyl groups (Example 2.16, Example 3.4) induce by right extension (Def. 3.10) the following three  $\mathbb{Z}_2$ -equivariant vector spaces (Example 3.7), which, by Prop. 3.12, are the direct summand building blocks of all injective  $\mathbb{Z}_2$ -equivariant dual vector spaces:

$$\text{Inj}_1(\mathbf{1}) : \begin{array}{ccc} \begin{array}{c} (\mathbb{Z}_2) \\ \downarrow \\ \mathbb{Z}_2/1 \end{array} & \mapsto & \mathbf{1} \\ \downarrow & & \downarrow 0 \\ \mathbb{Z}_2/\mathbb{Z}_2 & \mapsto & 0, \end{array} \quad \text{Inj}_1(\mathbf{1}_{\text{sgn}}) : \begin{array}{ccc} \begin{array}{c} (\mathbb{Z}_2) \\ \downarrow \\ \mathbb{Z}_2/1 \end{array} & \mapsto & \mathbf{1}_{\text{sgn}} \\ \downarrow & & \downarrow 0 \\ \mathbb{Z}_2/\mathbb{Z}_2 & \mapsto & 0, \end{array} \quad (92)$$

and

$$\text{Inj}_{\mathbb{Z}_2}(\mathbf{1}) : \begin{array}{ccc} \begin{array}{c} (\mathbb{Z}_2) \\ \downarrow \\ \mathbb{Z}_2/1 \end{array} & \mapsto & \mathbf{1} \\ \downarrow & & \downarrow \text{id} \\ \mathbb{Z}_2/\mathbb{Z}_2 & \mapsto & \mathbf{1}. \end{array} \quad (93)$$

To see this, use (46) in (86) to get, for two cases,

$$\text{Inj}_1(\mathbf{1}) : \begin{array}{ccc} \begin{array}{c} \xrightarrow{(\mathbb{Z}_2)} \\ \mathbb{Z}_2/1 \end{array} & \longmapsto & \mathbb{Z}_2\text{Representations}_{\mathbb{R}}\left(\underbrace{\mathbb{R}[\mathbb{Z}_2\text{Orbits}(\mathbb{Z}_2/1, \mathbb{Z}_2/1)]}_{\simeq \mathbf{1} \oplus \mathbf{1}_{\text{sgn}}}, \mathbf{1}\right) \simeq \mathbf{1} \\ \downarrow & & \downarrow 0 \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbb{Z}_2\text{Representations}_{\mathbb{R}}\left(\underbrace{\mathbb{R}[\mathbb{Z}_2\text{Orbits}(\mathbb{Z}_2/\mathbb{Z}_2, \mathbb{Z}_2/1)]}_{\simeq 0}, \mathbf{1}\right) \simeq 0 \end{array}$$

and

$$\text{Inj}_{\mathbb{Z}_2}(\mathbf{1}) : \begin{array}{ccc} \begin{array}{c} \xrightarrow{(\mathbb{Z}_2)} \\ \mathbb{Z}_2/1 \end{array} & \longmapsto & \mathbf{1}\text{Representations}_{\mathbb{R}}\left(\underbrace{\mathbb{R}[\mathbb{Z}_2\text{Orbits}(\mathbb{Z}_2/1, \mathbb{Z}_2/\mathbb{Z}_2)]}_{\simeq \mathbf{1}}, \mathbf{1}\right) \simeq \mathbf{1} \\ \downarrow & & \downarrow \text{id} \\ \mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \mathbf{1}\text{Representations}_{\mathbb{R}}\left(\underbrace{\mathbb{R}[\mathbb{Z}_2\text{Orbits}(\mathbb{Z}_2/\mathbb{Z}_2, \mathbb{Z}_2/\mathbb{Z}_2)]}_{\simeq \mathbf{1}}, \mathbf{1}\right) \simeq \mathbf{1}. \end{array}$$

**Lemma 3.15** (Tensor product preserves injectivity of finite-dim dual vector  $G$ -spaces [Go97b, Lem. 3.6, Rem 1.2] [Scu02, Prop. 7.36]). *Let  $V, W \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee, \text{fin}}$  (Def. 3.5). If  $V$  and  $W$  are both injective (Def. 3.11), then so is their tensor product  $V \otimes W : G/H \mapsto V(G/H) \otimes W(G/H)$ .*

**Equivariant smooth differential forms.** In preparation of discussing equivariant de Rham cohomology, consider:

**Example 3.16** (Equivariant smooth differential forms). Let  $G \curvearrowright X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35) and  $n \in \mathbb{N}$ . Then there is the equivariant dual vector space (Def. 3.30)

$$\Omega_{\text{dR}}^n(\gamma(X//G)) \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee}$$

given by the system of vector spaces of smooth differential  $n$ -forms (e.g. [BT82]) of the fixed submanifolds (66), with pullback of differential forms along residual actions and along inclusions of fixed loci:

$$\begin{array}{ccc} \begin{array}{c} \text{equivariant dual vector space} \\ \text{of equivariant smooth} \\ \text{differential } n\text{-forms} \\ \Omega_{\text{dR}}^n(\gamma(X//G)) \end{array} : & \begin{array}{ccc} \begin{array}{c} \xrightarrow{g_1 \in \mathbb{W}_G(H_1)} \\ G/H_1 \\ \downarrow p \\ G/H_2 \\ \xrightarrow{g_2 \in \mathbb{W}_G(H_2)} \end{array} & \longmapsto & \begin{array}{c} \xrightarrow{X^{s_1^*}} \\ \Omega_{\text{dR}}^n(X^{H_1}) \\ \downarrow X^{p^*} \\ \Omega_{\text{dR}}^n(X^{H_2}) \\ \xrightarrow{X^{s_2^*}} \end{array} \end{array} \end{array} \quad \begin{array}{l} \text{ordinary differential forms} \\ \text{on fixed submanifold} \\ \text{pullback along inclu-} \\ \text{sion of fixed loci} \end{array}$$

**Remark 3.17** (Equivariant smooth differential forms are injective). The following Lemmas 3.19, 3.20, 3.21 show that the equivariant dual vector spaces of smooth differential  $n$ -forms (Def. 3.16) are injective objects (Def. 3.11), at least if the equivariance group is of order 4 or cyclic of prime order (in which case cf. [ST23, Prop. 4.1]):

$$G \in \{\mathbb{Z}_p \mid p \text{ prime}\} \cup \{\mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2\}.$$

From the proofs of these lemmas, given below, it is fairly clear how to approach the proof of the general case. But since this is heavy on notation if done properly, and since we do not need further generality for our application here, we will not go into that.

**Notation 3.18** (Extension of smooth differential forms away from fixed loci).

For  $G \curvearrowright X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35) and  $H \subset G$ , choose a tubular neighborhood (e.g. [Ko96, §1.2])  $\mathcal{N}_X(X^H) \subset X$  of the fixed locus (which exists by Lemma 2.34). Then multiplication of smooth  $n$ -forms on  $X^H$  with a choice of bump function in the neighborhood coordinates induces a linear section, which we denote

$\text{ext}_H$ , of the operation of restricting differential forms to the fixed locus:

$$\Omega_{\text{dR}}^n(X^H) \begin{array}{c} \xrightarrow{\text{ext}_H} \\ \xrightarrow{\text{id}} \end{array} \Omega_{\text{dR}}^n(X) \xrightarrow{(-)|_{X^H}} \Omega_{\text{dR}}^n(X^H).$$

**Lemma 3.19** ( $\mathbb{Z}_p$ -Equivariant smooth differential forms are injective). *Let the equivariance group  $G = \mathbb{Z}_p$  be a cyclic group of prime order. Then, for  $\mathbb{Z}_p \curvearrowright X \in \mathbb{Z}_p\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35), the equivariant dual vector space of  $\mathbb{Z}_p$ -equivariant smooth differential  $n$ -forms (Def. 3.33) is injective (Def. 3.11):*

$$\Omega_{\text{dR}}^n(\gamma(X//\mathbb{Z}_p)) \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee, \text{inj}}. \quad (94)$$

*Proof.* By extension of differential forms away from the fixed locus (Notation 3.18), we obtain the following isomorphism of equivariant dual vector spaces to a direct sum of injective extensions (Lemma 3.10)

$$\begin{array}{c} \begin{array}{c} \mathbb{Z}_p \\ \downarrow \\ \mathbb{Z}_p/1 \\ \downarrow \\ \mathbb{Z}_p/\mathbb{Z}_p \end{array} \\ \Omega_{\text{dR}}^n(\gamma(X//\mathbb{Z}_p)) \xrightarrow{\cong} \text{Inj}_{\mathbb{Z}_p} \left( \Omega_{\text{dR}}^n(X^{\mathbb{Z}_p}) \right) \oplus \text{Inj}_1 \left( \left\{ \omega \in \Omega_{\text{dR}}^n(X) \mid \omega|_{X^{\mathbb{Z}_p}} = 0 \right\} \right) \\ \alpha \longmapsto \left( \alpha|_{X^{\mathbb{Z}_p}}, \quad \alpha - \text{ext}_{\mathbb{Z}_p}(\alpha|_{X^{\mathbb{Z}_p}}) \right) \\ \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \\ \alpha|_{X^{\mathbb{Z}_p}} \longmapsto \left( \alpha|_{X^{\mathbb{Z}_p}}, \quad 0 \right), \end{array}$$

where we used, since  $p$  is assumed to be prime, that the only subgroups of  $G$  are 1 and  $\mathbb{Z}_p$  itself (Example 2.18). By Prop. 3.12, this implies the claim (94).  $\square$

**Lemma 3.20** ( $\mathbb{Z}_4$ -Equivariant smooth differential forms are injective). *Let the equivariance group  $G = \mathbb{Z}_4$  be the cyclic group of order 4. Then, for  $\mathbb{Z}_4 \curvearrowright X \in \mathbb{Z}_4\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35), the equivariant dual vector space of  $\mathbb{Z}_4$ -equivariant smooth differential  $n$ -forms (Def. 3.33) is injective (Def. 3.11):*

$$\Omega_{\text{dR}}^n(\gamma(X//\mathbb{Z}_4)) \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee, \text{inj}}. \quad (95)$$

*Proof.* Since the subgroups of  $\mathbb{Z}_4$  are linearly ordered  $1 \subset \mathbb{Z}_2 \subset \mathbb{Z}_4$  (Example 2.18), the proof of Lemma 3.19 generalizes immediately. Using extensions of differential  $n$ -forms (Notation 3.18), both from  $X^{\mathbb{Z}_4}$  as well as from  $X^{\mathbb{Z}_2}$ , we obtain the following isomorphism of equivariant dual vector spaces to a direct sum of injective extensions (Lemma 3.10)

$$\begin{array}{c} \begin{array}{c} \mathbb{Z}_4 \\ \downarrow \\ \mathbb{Z}_4/1 \\ \downarrow \\ \mathbb{Z}_4/\mathbb{Z}_2 \\ \downarrow \\ \mathbb{Z}_4/\mathbb{Z}_4 \end{array} \\ \Omega_{\text{dR}}^n(\gamma(X//\mathbb{Z}_4)) \xrightarrow{\cong} \text{Inj}_{\mathbb{Z}_4} \left( \Omega_{\text{dR}}^n(X^{\mathbb{Z}_4}) \right) \oplus \text{Inj}_{\mathbb{Z}_2} \left( \left\{ \omega \in \Omega_{\text{dR}}^n(X^{\mathbb{Z}_2}) \mid \omega|_{X^{\mathbb{Z}_4}} = 0 \right\} \right) \oplus \text{Inj}_1 \left( \left\{ \omega \in \Omega_{\text{dR}}^n(X) \mid \omega|_{X^{\mathbb{Z}_2}} = 0 \right\} \right) \\ \alpha \longmapsto \left( \alpha|_{X^{\mathbb{Z}_4}}, \quad \left( \alpha - \text{ext}_{\mathbb{Z}_4}(\alpha|_{X^{\mathbb{Z}_4}}) \right)|_{X^{\mathbb{Z}_2}}, \quad \alpha - \text{ext}_{\mathbb{Z}_2}(\alpha|_{X^{\mathbb{Z}_2}}) \right) \\ \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \\ \alpha|_{X^{\mathbb{Z}_2}} \longmapsto \left( \alpha|_{X^{\mathbb{Z}_4}}, \quad \alpha|_{X^{\mathbb{Z}_2}} - \left( \text{ext}_{\mathbb{Z}_4}(\alpha|_{X^{\mathbb{Z}_4}}) \right)|_{X^{\mathbb{Z}_2}}, \quad 0 \right) \\ \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \\ \alpha|_{X^{\mathbb{Z}_4}} \longmapsto \left( \alpha|_{X^{\mathbb{Z}_4}}, \quad 0, \quad 0 \right) \end{array}$$

By Prop. 3.12, this implies the claim (95).  $\square$

**Lemma 3.21** ( $\mathbb{Z}_2 \times \mathbb{Z}_2$ -Equivariant smooth differential forms are injective). *Let the equivariance group  $G = \mathbb{Z}_2^L \times \mathbb{Z}_2^R$  be the Klein 4-group. Then, for  $\mathbb{Z}_2^L \times \mathbb{Z}_2^R \curvearrowright X \in \mathbb{Z}_2^L \times \mathbb{Z}_2^R\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35), the equivariant dual vector space of equivariant smooth differential  $n$ -forms (Def. 3.33) is injective (Def. 3.11):*

$$\Omega_{\text{dR}}^n(\gamma(X//\mathbb{Z}_2^L \times \mathbb{Z}_2^R)) \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee, \text{inj}}. \quad (96)$$

*Proof.* We obtain an isomorphism to a direct sum of injective extensions (Lemma 3.10), much as in the proofs of Lemmas 3.19 and 3.20,

$$\begin{aligned}
& \Omega_{\mathrm{dR}}^n(\gamma(X//\mathbb{Z}_4)) \xrightarrow{\cong} \text{Inj}_{\mathbb{Z}_4} \left( \Omega_{\mathrm{dR}}^n(X^{\mathbb{Z}_4}) \right) \oplus \text{Inj}_{\mathbb{Z}_2^L} \left( \left\{ \omega \in \Omega_{\mathrm{dR}}^n(X^{\mathbb{Z}_2^L}) \mid \omega|_{X^{\mathbb{Z}_4}} = 0 \right\} \right) \oplus \text{Inj}_1 \left( \left\{ \omega \in \Omega_{\mathrm{dR}}^n(X) \mid \begin{array}{l} \omega|_{X^{\mathbb{Z}_2^L}} = 0 \\ \omega|_{X^{\mathbb{Z}_2^K}} = 0 \end{array} \right\} \right) \\
& \quad \begin{array}{l} \text{equivariant smooth} \\ \text{differential } n\text{-forms} \end{array} \quad \begin{array}{l} \text{differential } n\text{-forms} \\ \text{on deep fixed locus} \end{array} \quad \begin{array}{l} \text{differential } n\text{-forms on shallow fixed loci whose} \\ \text{restriction to the deep fixed locus vanishes} \end{array} \quad \begin{array}{l} \text{differential } n\text{-forms whose} \\ \text{restriction to the shallow fixed loci vanishes} \end{array} \\
\end{aligned}$$

$$\begin{array}{c}
\begin{array}{ccc}
\begin{array}{c} \mathbb{Z}_4 \\ \downarrow \\ G/1 \end{array} & \xrightarrow{\alpha} & \left( \alpha|_{X^{\mathbb{Z}_4}} \quad , \quad \overbrace{\left( \alpha - \text{ext}_{\mathbb{Z}_2^L \times \mathbb{Z}_2^K}(\alpha|_{X^{\mathbb{Z}_2^L \times \mathbb{Z}_2^K}} \right)}{=: \beta} \Big|_{X^{\mathbb{Z}_2^L} + |X^{\mathbb{Z}_2^K}} \quad , \quad \begin{array}{c} \beta - \text{ext}_{\mathbb{Z}_2^K}(\beta|_{X^{\mathbb{Z}_2^K}}) \\ - \text{ext}_{\mathbb{Z}_2^L}(\beta|_{X^{\mathbb{Z}_2^L}}) \end{array} \right) \\
\downarrow & & \downarrow \\
\begin{array}{c} \mathbb{Z}_2 \\ \downarrow \\ G/\mathbb{Z}_2^L \end{array} & \xrightarrow{\alpha|_{X^{\mathbb{Z}_2}}} & \left( \alpha|_{X^{\mathbb{Z}_4}} \quad , \quad \left( \alpha - \text{ext}_{\mathbb{Z}_2^L \times \mathbb{Z}_2^K}(\alpha|_{X^{\mathbb{Z}_2^L \times \mathbb{Z}_2^K}} \right) \Big|_{X^{\mathbb{Z}_2^L} + |X^{\mathbb{Z}_2^K}} \quad , \quad 0 \right) \\
\downarrow & & \downarrow \\
\begin{array}{c} \mathbb{Z}_2^L \times \mathbb{Z}_2^K \\ \downarrow \\ G/\mathbb{Z}_2^L \times \mathbb{Z}_2^K \end{array} & \xrightarrow{\alpha|_{X^{\mathbb{Z}_4}}} & \left( \alpha|_{X^{\mathbb{Z}_4}} \quad , \quad 0 \quad , \quad 0 \right)
\end{array}
\end{array}$$

and hence conclude the result, again by Prop. 3.12. The only further subtlety to take care of here is that the two extensions  $\text{ext}_{\mathbb{Z}_2^L}$  and  $\text{ext}_{\mathbb{Z}_2^K}$  (Notation 3.18) need to be chosen compatibly, such as to ensure that each preserves the property of a form to vanish on the corresponding other fixed locus:

$$\left( \text{ext}_{\mathbb{Z}_2^L}(\beta|_{X^{\mathbb{Z}_2^K}}) \Big|_{\mathbb{Z}_2^K} \right) = 0, \quad \left( \text{ext}_{\mathbb{Z}_2^K}(\beta|_{X^{\mathbb{Z}_2^L}}) \Big|_{\mathbb{Z}_2^L} \right) = 0.$$

This is achieved by choosing an *equivariant* tubular neighborhood (by [Br72, §VI, Thm. 2.2][Kan07, Thm. 4.4]) around the intersection  $X^{\mathbb{Z}_2^K} \cap X^{\mathbb{Z}_2^L}$  and using this to choose the extension away from  $X^{\mathbb{Z}_2^L}$  to be orthogonal to that away from  $X^{\mathbb{Z}_2^K}$ .  $\square$

### Equivariant graded vector spaces.

**Definition 3.22** (Equivariant graded vector spaces). We write

$$\mathcal{G}\text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0} := \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\mathbb{N}} \simeq \text{Functors}(G\text{Orbits}^{\text{op}}, \text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0})$$

for the category of  $\mathbb{N}$ -graded objects in equivariant vector spaces (Def. 3.5).

**Definition 3.23** (Equivariant rational homotopy groups). For  $X \in \mathcal{G}\text{HomotopyTypes}_{\geq 1}$  (Def. 2.32) and  $n \in \mathbb{N}$ , the rationalized  $n$ th equivariant homotopy group (Def. 2.31) hence the stage-wise rationalized simplicial homotopy group (Def. 2.31)

$$\pi_n(X) \otimes_{\mathbb{Z}} \mathbb{R} : G/H \longmapsto \pi_n(X(G/H)) \otimes_{\mathbb{Z}} \mathbb{R},$$

form an equivariant graded vector space (Def. 3.22):

$$\pi_{\bullet+1}(X) \otimes_{\mathbb{Z}} \mathbb{R} \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}.$$

**Example 3.24** ( $\mathbb{Z}_2$ -Equivariant rational homotopy groups of twistor space). The  $\mathbb{Z}_2$ -equivariant rational homotopy groups (Def. 3.23) of  $\mathbb{Z}_2$ -equivariant twistor space (Example 2.28) are, by (60), given by the rational homotopy groups of  $\mathbb{C}P^3$  and, on the fixed locus, of  $S^2$ . Hence these look as follows (using, e.g., [FSS20b, Lemma 2.13] with [FSS23, Prop. 3.65]):

$$\pi_{\bullet}^{\mathbb{Z}_2/2}(\mathbb{C}P^3) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \begin{array}{c|cccccccccc}
\mathbb{Z}_2/H & (\mathbb{C}P^3)^H & \pi_2 \otimes \mathbb{R} & \pi_3 \otimes \mathbb{R} & \pi_4 \otimes \mathbb{R} & \pi_5 \otimes \mathbb{R} & \pi_6 \otimes \mathbb{R} & \pi_7 \otimes \mathbb{R} & \pi_8 \otimes \mathbb{R} & \pi_9 \otimes \mathbb{R} & \dots \\
\hline
\mathbb{Z}_2/1 & \mathbb{C}P^3 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \dots \\
\hline
\mathbb{Z}_2/\mathbb{Z}_2 & S^2 & \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \dots
\end{array} \quad (97)$$

### Equivariant cochain complexes.

**Definition 3.25** (Equivariant cochain complexes). We write

$$\mathcal{G}\text{CochainComplexes}_{\mathbb{R}}^{\geq 0} := \text{Functors}(G\text{Orbits}, \text{CochainComplexes}_{\mathbb{R}}^{\geq 0})$$

for the category of functors from the  $G$ -orbit category (Def. 2.13) to the category of connective cochain complexes (i.e., in non-negative degrees with differential of degree +1) over the real numbers.

**Definition 3.26** (Delooping of equivariant cochain complexes). For  $V \in \mathcal{G}\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}$  (Def. 3.25), we denote its delooping as

$$\mathfrak{b}V : G/H \mapsto \left( 0 \longrightarrow V^0(G/H) \xrightarrow{d_V^0} V^1(G/H) \xrightarrow{d_V^1} V^2(G/H) \longrightarrow \dots \right).$$

As an instance of the general notion of mapping cones (e.g. [Scha11, Def. 3.2.2]), we get:

**Example 3.27** (Cone on an equivariant cochain complex). For  $V \in \mathcal{G}\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}$  (Def. 3.25), we say that the *cone* on its delooping  $\mathfrak{b}V$  (Def. 3.26) is the equivariant cochain complex  $\mathfrak{c}V \in \mathcal{G}\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}$  given by

$$\mathfrak{c}V := \text{Cone}(\mathfrak{b}V) : G/H \mapsto \left( \begin{array}{ccccccc} V^0(G/H) & \xrightarrow{-d_V^0} & V^1(G/H) & \xrightarrow{-d_V^1} & V^2(G/H) & \xrightarrow{-d_V^2} & V^3(G/H) & \xrightarrow{-d_V^3} & \dots \\ \oplus & \searrow \text{id} & \oplus & \searrow \text{id} & \oplus & \searrow \text{id} & \oplus & \searrow \text{id} & \\ 0 & \xrightarrow{0} & V^0(G/H) & \xrightarrow{d_V^0} & V^1(G/H) & \xrightarrow{d_V^1} & V^2(G/H) & \xrightarrow{d_V^2} & \dots \end{array} \right).$$

This sits in the evident cofiber sequence:

$$\begin{array}{ccc} V & \xleftarrow{\text{cofib}(i_{\mathfrak{b}V})} & \mathfrak{c}V \\ & & \uparrow i_{\mathfrak{b}V} \\ & & \mathfrak{b}V \end{array} \in \mathcal{G}\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}. \quad (98)$$

As an instance of the general notion of injective resolutions (e.g. [Scha11, §4.5]), we have:

**Example 3.28** (Injective resolution of equivariant dual vector spaces). Let  $V \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee}$  (Def. 3.5).

Then, by Prop. 3.12, we obtain an *injective resolution* (e.g. [HS71, p. 129]) of  $V$  given by the equivariant cochain complex (Def. 3.25) which in degree 0 is the injective envelope (89) of  $V$ , and whose differentials are, recursively, the injective envelope inclusions (91) of the quotients by the image of the previous degree.

$$\begin{array}{ccc} \vdots & & \vdots \\ \uparrow & & \uparrow \\ 0 & \longrightarrow & \text{Inj}(\text{coker}(d^1)) \\ \uparrow & & \uparrow d^2 \\ 0 & \longrightarrow & \text{Inj}(\text{coker}(d^0)) \\ \uparrow & & \uparrow d^1 \\ 0 & \longrightarrow & \text{Inj}(\text{Inj}(V)/V) \\ \uparrow & & \uparrow d^0 \\ V & \hookrightarrow & \text{Inj}(V) \end{array} \quad =: \text{Inj}^\bullet(V) \in \mathcal{G}\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}$$

This is such that for any  $A^\bullet \in \mathcal{G}\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}$  which is degreewise injective (Def. 3.11) and any morphism of equivariant dual vector spaces

$$\{ V \xrightarrow{\phi} A_{\text{clsd}}^n \} \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^{\vee}$$

from  $V$  to the subspace of closed elements (cocycles) in  $A^n$ , there exists an extension to a morphism

$$\{ \mathfrak{b}^n \text{Inj}^\bullet(V) \xrightarrow{\phi^\bullet} A^\bullet \} \in \mathcal{G}\text{CochainComplexes}_{\mathbb{R}}^{\geq 0} \quad (99)$$

of equivariant cochain complexes (as shown on the right) given recursively by using injectivity of  $A^{n+i+1}$  to obtain dashed extensions (88)

$$\begin{array}{ccc} \text{Inj}^{i+1}(V) & \dashrightarrow & A^{n+i+1} \\ \uparrow & \nearrow d_A \circ \phi^i & \\ \text{Inj}^i(V)/\text{im}(d^{i-1}) & & \end{array}$$

$$\begin{array}{ccc} \vdots & & \vdots \\ \uparrow & & \uparrow d_A^{n+3} \\ \text{Inj}(\text{coker}(d^1)) & \dashrightarrow \phi^{n+3} & A^{n+3} \\ \uparrow d^2 & & \uparrow d_A^{n+2} \\ \text{Inj}(\text{coker}(d^0)) & \dashrightarrow \phi^{n+2} & A^{n+2} \\ \uparrow d^1 & & \uparrow d_A^{n+1} \\ \text{Inj}(\text{Inv}(V)/V) & \dashrightarrow \phi^{n+1} & A^{n+1} \\ \uparrow d^0 & & \uparrow d_A^n \\ \text{Inj}(V) & \dashrightarrow \phi^n & A^n \\ \uparrow & \nearrow \phi =: \phi_V^n & \uparrow \\ V & \longrightarrow & A_{\text{clsd}}^n \end{array}$$

**Example 3.29** (Injective resolution of  $\mathbb{Z}_2$ -equivariant dual vector spaces).



with all structure maps induced by the functoriality of the non-equivariant Sym-construction.

(iii) This extends to a functor

$$\mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \begin{array}{c} \xleftarrow{\text{Sym}} \\ \perp \\ \xrightarrow{\text{CchnCmplx}} \end{array} \mathcal{G}\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}, \quad (102)$$

which is left adjoint to the evident assignment of underlying equivariant cochain complexes.

In terms of generators and relations (Notation 3.1, 3.2), passing to free dgc-algebras means to replace angular brackets by square brackets:

**Example 3.35** (Free  $\mathbb{Z}_2$ -equivariant dgc-algebra on injective resolution). In the case  $G = \mathbb{Z}_2$  (Example 2.15), the free  $\mathbb{Z}_2$ -equivariant dgc-algebra (Example 3.34) on the  $n$ -fold delooping (Def. 3.26) of the injective resolution (101) from Example 3.29 is:

$$\text{Sym} \circ \mathfrak{b}^n \circ \text{Inj}^\bullet \left( \begin{array}{ccc} \left( \begin{array}{c} \mathbb{Z}_2 \\ \mathbb{Z}_2/1 \end{array} \right) & \mapsto & 0 \\ \downarrow & & \downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 & \mapsto & \mathbb{R}\langle c_0 \rangle / (dc_0 = 0) \end{array} \right) = \left( \begin{array}{ccc} \left( \begin{array}{c} \mathbb{Z}_2 \\ \mathbb{Z}_2/1 \end{array} \right) & \mapsto & \mathbb{R} \left[ \begin{array}{c} c'_n, c_{n+1} \\ c_n \end{array} \right] / \left( \begin{array}{c} dc'_n = c_{n+1} \\ dc_{n+1} = 0 \\ dc_n = 0 \end{array} \right) \\ \downarrow & & \downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 & \mapsto & \mathbb{R}[c_n] / (dc_n = 0) \end{array} \right). \quad (103)$$

In equivariant generalization of [FSS23, Def. 3.25], we have:

**Definition 3.36** (Equivariant  $L_\infty$ -algebras). We write

$$\begin{array}{ccc} \mathcal{G}L_\infty\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0} & \xhookrightarrow{\text{CE}} & (\mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})^{\text{op}} \\ \mathfrak{g} & \mapsto & \text{CE}(\mathfrak{g}) \end{array} \quad (104)$$

for the opposite of the full subcategory of equivariant dgc-algebras (Def. 3.30) on those that are stage-wise Chevalley-Eilenberg algebras of  $L_\infty$ -algebras (connective and finite-type over the real numbers, as in [FSS23, Def. 3.25]).

In generalization of Example 3.33, we have:

**Example 3.37** (Proper  $G$ -equivariant and Borel-Weil-Cartan  $T$ -equivariant smooth de Rham complex).

Let  $(T \times G) \curvearrowright X \in (T \times G)\text{Actions}(\text{SmoothManifolds})$  (Def 2.35), where  $T \in \text{CompactLieGroups}$  is finite-dimensional with Lie algebra denoted (as in Notation 3.1)

$$\mathfrak{t} \simeq \left\{ \langle t_a \rangle_{a=1}^{\dim(T)}, [-, -] \right\} \in \text{LieAlgebras}_{\mathbb{R}, \text{fin}}. \quad (105)$$

Consider the equivariant dgc-algebra (Def. 3.30)

$$\Omega_{\text{dR}}^\bullet \left( (\mathcal{Y}(X//G))//T \right) \in \mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$$

of  $T$ -invariants in the tensor product of proper  $G$ -equivariant smooth differential forms (Example 3.16) with the free symmetric graded algebra on

$$\mathfrak{b}^2 \mathfrak{t}^\vee \simeq \langle r_2^a \rangle_{a=1}^{\dim(T)},$$

(the linear dual space of (105) in degree 2) and equipped with the sum of the de Rham differential

$$d_{\text{dR}} : \omega \wedge r_2^{a_1} \wedge \cdots \wedge r_2^{a_p} \mapsto (d_{\text{dR}} \omega) \wedge r_2^{a_1} \wedge \cdots \wedge r_2^{a_p}$$

and the operator

$$r_2^a \wedge \iota_{t_a} : \omega \wedge r_2^{a_1} \wedge \cdots \wedge r_2^{a_p} \mapsto (\iota_{t_a} \omega) \wedge r_2^a \wedge r_2^{a_1} \wedge \cdots \wedge r_2^{a_p},$$

where

- $\omega \in \Omega_{\text{dR}}^\bullet(-)$ ,
- $\iota_{t_a}$  denotes the contraction of differential forms with the vector field that is the derivative of the action  $T \times X \rightarrow X$  along  $t_a$ ,
- summation over the index  $a \in \{1, \dots, \dim(T)\}$  is understood, and



- the  $T$ -action on  $\mathfrak{t}^\vee$  is the coadjoint action and on that differential forms is by pullback along the given action on  $X$ :

$$\begin{array}{c} \text{proper } G\text{-equivariant \& Borel } T\text{-equivariant} \\ \text{smooth de Rham complex} \end{array} \quad \begin{array}{c} \text{Cartan model for } T\text{-equivariant Borel cohomology of } H\text{-fixed locus } X^H \end{array} \\ \Omega_{\text{dR}}^\bullet \left( (\gamma(X//G))//T \right) : G/H \longmapsto \left( \Omega_{\text{dR}}^\bullet(X^H) \left[ \{r_2^a\}_{a=1}^{\dim(T)} \right], d_{\text{dR}} + r_2^a \wedge \iota_{t_a} \right)^T. \quad (106)$$

This is, stage-wise over  $G/H \in G\text{Orbits}$  (Def. 2.13), the *Cartan model* dgc-algebra for Borel  $T$ -equivariant de Rham cohomology ([AB84][MQ86, §5][Kal93][GST99], review in [Me06][KT15][Pes17]), here formed for the fixed submanifolds (Lemma 2.34) of all the subgroups of the  $G$ -action.

### Homotopy theory of equivariant dgc-algebras.

**Proposition 3.38** (Projective model structure on connective equivariant dgc-algebras [Scu02, Theorem 3.2]). *There is the structure of a model category on  $\mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (Def. 3.30) whose*

*W – weak equivalences are the quasi-isomorphisms over each  $G/H \in G\text{Orbits}$ ;*

*Fib – fibrations are the degreewise surjections whose degreewise kernels are injective (Def. 3.11).*

We denote this model category by

$$\left( \mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \right)_{\text{proj}} \in \text{ModelCategories}.$$

A key technical subtlety of the model structure on equivariant dgc-algebras (Prop. 3.38), compared to its non-equivariant version ([BG76, §4.3][GM96, §V.3.4][FSS23, Prop. 3.36]), is that not all objects are fibrant anymore, since equivariantly the injectivity condition (Def. 3.11) is non-trivial (Prop. 3.12). However, we have the following class of examples of fibrant objects:

**Proposition 3.39** (Equivariant smooth de Rham complex is projectively fibrant).

*For  $G \curvearrowright X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35), the equivariant smooth de Rham complex (Example 3.33) is a fibrant object in the projective model structure (Prop. 3.38)*

$$\Omega_{\text{dR}}^\bullet(\gamma(X//G)) \xrightarrow{\in \text{Fib}} 0 \in \left( \mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \right)_{\text{proj}},$$

*at least if  $G$  is of order 4 or cyclic of prime order.*

*Proof.* By Prop. 3.38, the statement is equivalent to the claim that the equivariant dual vector spaces of equivariant smooth differential  $n$ -forms are injective. This is indeed the case, by Lemmas 3.19, 3.20, 3.21 (Remark 3.17).  $\square$

Next we turn to discussion of fibrant and cofibrant equivariant dgc-algebras.

### Minimal equivariant dgc-algebras.

**Definition 3.40** (Minimal equivariant dgc-algebras [Tr82, Construction 5.10][Scu02, §11][Scu08, §4]).

Let  $A \in \mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (Def. 3.30) be such that, for all  $k \in \mathbb{N}$ , the underlying  $\text{ChnCmplx}(A)^k \in \mathcal{G}\text{CochainComplexes}_{\mathbb{R}}^{\geq 0}$  is injective (Def. 3.11).

(i) For  $n \in \mathbb{N}$ , an *elementary extension*  $A \hookrightarrow A[\mathfrak{b}^n V_n]_\phi$  of  $A$  in degree  $n$  is a pushout of the image under  $\text{Sym}$  (Example 3.34) of the cone inclusion (Example 3.27) of the  $(n+1)$ -fold delooping (Def. 3.26) of the injective resolution  $\text{Inj}^\bullet(V)$  (Example 3.28)

$$\begin{array}{ccc} A[\mathfrak{b}^n V_n]_{\phi_n} & \longleftarrow & \text{Sym}(\mathfrak{e}\mathfrak{b}^n \text{Inj}^\bullet(V_n)) \\ \uparrow & \text{(po)} & \uparrow \text{Sym}(i_{\mathfrak{b}^{n+1} \text{Inj}^\bullet(V_n)}) \\ A & \xleftarrow{\tilde{\phi}_n} & \text{Sym}(\mathfrak{b}^{n+1} \text{Inj}^\bullet(V_n)) \end{array} \in \mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \quad (107)$$

along the adjunct  $\tilde{\phi}^\bullet$  (102) of an injective extension (99)

$$A^\bullet \xleftarrow{\phi_n^\bullet} \mathfrak{b}^{n+1} \text{Inj}^\bullet(V_n) \in \mathcal{G}\text{CoChainComplexes}_{\mathbb{R}}^{\geq 0} \quad (108)$$

of a given *attaching map* out of a given equivariant dual vector space  $V_n$  (Def. 3.5):

$$(ii) \text{ An inclusion } A_{\text{clsd}}^{n+1} \xleftarrow{\phi_n} V_n \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^\vee. \quad (109)$$

$$B^\bullet \xrightarrow{\text{min}} A^\bullet \in \mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \quad (110)$$

of degreewise injective (Def. 3.11) equivariant dgc-algebras (Def. 3.30) which are equivariantly 1-connected

$$B^0 \simeq \mathbb{R}, \quad B^1 \simeq \mathbb{R}$$

is called *relative minimal* if it is isomorphic under  $B^\bullet$  to the result of a sequence of elementary extensions (107) in strictly increasing degrees (noticing with Lemma 3.15, that the result of an elementary extension (107) is again degreewise injective).

(iii) An equivariant dgc-algebra  $A^\bullet$ , such that the unique inclusion of the equivariant ground field  $\mathbb{R}$  (which is clearly 1-connected and injective, by Example 3.13) is a relative minimal dgc-algebra (110)

$$\mathbb{R} \xrightarrow{\text{min}} A^\bullet \in \mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}, \quad (111)$$

is called a *minimal equivariant dgc-algebra*.

**Definition 3.41** (Minimal equivariant  $L_\infty$ -algebra). Any minimal equivariant dgc-algebra  $A$  (Def. 3.40) is the equivariant Chevalley-Eilenberg algebra (104)

$$A \simeq \text{CE}(\mathfrak{g}^A)$$

of an equivariant  $L_\infty$ -algebra  $\mathfrak{g}^A \in \mathcal{G}L_\infty\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0}$  (Def. 3.36), defined uniquely up to isomorphism. We say that the *underlying* graded equivariant vector space (Def. 3.22)

$$\mathfrak{g}_\bullet^A \in \mathcal{G}\text{GradedVectorSpaces}_{\mathbb{R}}^{\geq 0}$$

of this equivariant  $L_\infty$ -algebra is the linear dual of the spaces of generators  $V_n^A \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}^\vee$  (109) of the elementary extensions (107) that exhibit the minimality of  $A$ :

$$\mathfrak{g}_n^A := (V_n^A)^\vee \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}.$$

**Example 3.42** (A minimal  $\mathbb{Z}_2$ -equivariant dgc-algebra). We spell out the construction of an equivariant minimal dgc-algebra (Def. 3.40), for  $G = \mathbb{Z}_2$  (Example 2.15), which involves three basic cases of the elementary extensions (107):

(i) In the first stage, begin with the equivariant base algebra  $\mathbb{R}$  (Example 3.32) and consider the attaching map (109) in degree 2 given by

$$\phi_2 : \begin{array}{ccc} \begin{array}{c} \mathbb{Z}_2/1 \\ \downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 \end{array} & \longmapsto & \begin{array}{ccc} \mathbb{R} & \xleftarrow{0 \leftarrow c_3} & \mathbb{R}\langle c_3 \rangle \\ \downarrow \text{id} & & \downarrow \text{id} \\ \mathbb{R} & \xleftarrow{0 \leftarrow c_3} & \mathbb{R}\langle c_3 \rangle \end{array} \end{array} \quad (112)$$

By Example 3.14, the equivariant dual vector space on the right is already injective (93), so that we may extend this attaching map immediately to an equivariant cochain map (108)

$$\phi_2^\bullet : \begin{array}{ccc} \begin{array}{c} \mathbb{Z}_2/1 \\ \downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 \end{array} & \longmapsto & \begin{array}{ccc} \mathbb{R} & \xleftarrow{0 \leftarrow c_3} & \mathbb{R}\langle c_3 \rangle / (dc_3 = 0) \\ \downarrow \text{id} & & \downarrow \text{id} \\ \mathbb{R} & \xleftarrow{0 \leftarrow c_3} & \mathbb{R}\langle c_3 \rangle / (dc_3 = 0), \end{array} \end{array}$$

where on the right we are using the generators-and-relations Notation 3.1. By Example 3.35, its adjunct morphism

of equivariant dgc-algebras is

$$\tilde{\phi}_2 : \begin{array}{ccc} \begin{array}{c} \langle \mathbb{Z}_2 \rangle \\ \mathbb{Z}_2/1 \\ \downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 \end{array} & \longmapsto & \begin{array}{ccc} \mathbb{R} & \xleftarrow{0 \leftarrow c_3} & \mathbb{R}[c_3]/(dc_3 = 0) \\ \text{id} \downarrow & & \downarrow \text{id} \\ \mathbb{R} & \xleftarrow{0 \leftarrow c_3} & \mathbb{R}[c_3]/(dc_3 = 0) \end{array} \end{array}$$

Since all these diagrams so far are constant on the orbit category, the resulting pushout (107) is computed over both objects  $\mathbb{Z}_2/H \in \mathbb{Z}_2\text{Orbits}$  as in non-equivariant dgc-theory, and thus yields this minimal equivariant dgc-algebra:

$$\begin{array}{ccc} \begin{array}{c} \langle \mathbb{Z}_2 \rangle \\ \mathbb{Z}_2/1 \\ \downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 \end{array} & \longmapsto & \begin{array}{c} \mathbb{R}[f_2]/(df_2 = 0) \\ \text{id} \downarrow \\ \mathbb{R}[f_2]/(df_2 = 0) \end{array} \end{array} \quad (113)$$

(ii) Consider next the following attaching map (109) in degree 3 to the equivariant dgc-algebra (113):

$$\phi_3 : \begin{array}{ccc} \begin{array}{c} \langle \mathbb{Z}_2 \rangle \\ \mathbb{Z}_2/1 \\ \downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 \end{array} & \longmapsto & \begin{array}{ccc} \mathbb{R}[f_2]/(df_2 = 0) & \xleftarrow{\quad} & 0 \\ \text{id} \downarrow & & \downarrow \\ \mathbb{R}[f_2]/(df_2 = 0) & \xleftarrow{f_2 \wedge f_2 \leftarrow c_4} & \mathbb{R}\langle c_4 \rangle \end{array} \end{array} \quad (114)$$

Here the equivariant dual vector space on the right is *not* injective: Its injective envelope is given in Example 3.29, and the free dgc-algebra on this is given in Example 3.35, which says that the required extension (108) of the attaching map  $\phi$  is hence of this form:

$$\tilde{\phi}_3 : \begin{array}{ccc} \begin{array}{c} \langle \mathbb{Z}_2 \rangle \\ \mathbb{Z}_2/1 \\ \downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 \end{array} & \longmapsto & \begin{array}{ccc} \mathbb{R}[f_2]/(df_2 = 0) & \xleftarrow{f_2 \wedge f_2 \leftarrow c_4} & \mathbb{R} \begin{bmatrix} c_5 \\ c_4 \end{bmatrix} / \begin{pmatrix} dc_5 = 0 \\ dc_4 = c_5 \end{pmatrix} \\ \text{id} \downarrow & & \downarrow \\ \mathbb{R}[f_2]/(df_2 = 0) & \xleftarrow{f_2 \wedge f_2 \leftarrow c_4} & \mathbb{R}[c_4]/(dc_4 = 0) \end{array} \end{array}$$

The pushout (107) along this map is the following, yielding the next stage of the minimal equivariant dgc-algebra on the rear left:

$$\begin{array}{ccc} \mathbb{R} \begin{bmatrix} h_3, \\ \omega_4, \\ f_2 \end{bmatrix} / \begin{pmatrix} dh_3 = \omega_4 - f_2 \wedge f_2 \\ d\omega_4 = 0 \\ df_2 = 0 \end{pmatrix} & \xleftarrow{\begin{array}{c} 0 \leftarrow c_5 \\ f_2 \wedge f_2 \leftarrow c_4 \\ \omega_4 \leftarrow b_4 \\ h_3 \leftarrow b_3 \end{array}} & \mathbb{R} \begin{bmatrix} c_5, & b_4, \\ c_4, & b_3 \end{bmatrix} / \begin{pmatrix} dc_5 = 0, & db_4 = c_5 \\ dc_4 = c_5, & db_3 = b_4 - c_4 \end{pmatrix} \\ \downarrow & & \downarrow \\ \mathbb{R} \begin{bmatrix} h_3, \\ f_2 \end{bmatrix} / \begin{pmatrix} dh_3 = -f_2 \wedge f_2 \\ df_2 = 0 \end{pmatrix} & \xleftarrow{\begin{array}{c} f_2 \wedge f_2 \leftarrow c_4 \\ h_3 \leftarrow b_3 \end{array}} & \mathbb{R}[c_4, b_3] / (dc_4 = c_5, db_3 = -c_4) \\ \downarrow & & \downarrow \\ \mathbb{R} \begin{bmatrix} h_3, \\ \omega_4, \\ f_2 \end{bmatrix} / \begin{pmatrix} dh_3 = \omega_4 - f_2 \wedge f_2 \\ d\omega_4 = 0 \\ df_2 = 0 \end{pmatrix} & \xleftarrow{\begin{array}{c} 0 \leftarrow c_5 \\ f_2 \wedge f_2 \leftarrow c_4 \end{array}} & \mathbb{R} \begin{bmatrix} c_5 \\ c_4 \end{bmatrix} / \begin{pmatrix} dc_5 = 0 \\ dc_4 = c_5 \end{pmatrix} \\ \downarrow & & \downarrow \\ \mathbb{R} \begin{bmatrix} h_3, \\ \omega_4, \\ f_2 \end{bmatrix} / \begin{pmatrix} dh_3 = \omega_4 - f_2 \wedge f_2 \\ d\omega_4 = 0 \\ df_2 = 0 \end{pmatrix} & \xleftarrow{\begin{array}{c} f_2 \wedge f_2 \leftarrow c_4 \\ h_3 \leftarrow b_3 \end{array}} & \mathbb{R}[c_4, b_3] / (dc_4 = c_5, db_3 = -c_4) \\ \downarrow & & \downarrow \\ \mathbb{R} \begin{bmatrix} h_3, \\ \omega_4, \\ f_2 \end{bmatrix} / \begin{pmatrix} dh_3 = \omega_4 - f_2 \wedge f_2 \\ d\omega_4 = 0 \\ df_2 = 0 \end{pmatrix} & \xleftarrow{\begin{array}{c} 0 \leftarrow c_5 \\ f_2 \wedge f_2 \leftarrow c_4 \end{array}} & \mathbb{R}[c_4] / (dc_4 = 0) \end{array}$$



(ii) This is unique up to isomorphism, in that for  $A'_{\min} \xrightarrow{\in W} A$  any other such, there is a commuting diagram of the form

$$\begin{array}{ccc} A_{\min} & \xrightarrow{\cong} & A'_{\min} \\ & \searrow \in W & \swarrow \in W \\ & A & \end{array}$$

with the top morphism an isomorphism of equivariant dgc-algebras.

**Remark 3.45** (Existence of equivariant relative minimal models). By analogy with the theory of (relative) minimal models in non-equivariant dgc-algebraic rational homotopy theory (e.g., [BG76, §7][Hal83][FHT00, Thm. 14.12][FSS23, Prop. 3.50]), it is to be expected that Prop. 3.44 holds in greater generality:

- (a) The existence of equivariant minimal models should hold more generally for fixed locus-wise nilpotent  $G$ -spaces (not necessarily fixed-locus wise simply-connected).
- (b) There should exist also equivariant *relative* minimal models, unique up to relative isomorphism, of any morphism between fixed locus-wise nilpotent spaces of  $\mathbb{R}$ -finite homotopy type.

While a proof of these more general statements should be a fairly straightforward generalization of the proofs of the existing results, it does not seem to be available in the literature. Nonetheless, for our main example of interest (Example 2.44) we explicitly find the equivariant relative minimal model (in Prop. 3.56 below).

## 3.2 Equivariant rational homotopy theory

We review the fundamentals of equivariant rational homotopy theory [Tr82][Tr96][Go97b][Scu02][Scu08] and prove our main technical result (Prop. 3.56 below). Throughout we make free use of plain (non-equivariant) dgc-algebraic rational homotopy theory [BG76] (review in [FHT00][He07][GMo13][FSS23, §3.2]).

**Equivariant rationalization.** Equivariant rational homotopy theory is concerned with the following concept:

**Definition 3.46** (Equivariant rationalization [Ma96, §II.3][Tr82, §2.6]).

Let  $X \in \mathcal{G}\text{HomotopyTypes}_{\geq 2}$  (Def. 2.33).

(i)  $X$  is called *rational* (here: over the real numbers, see [FSS23, Rem. 3.51]) if all its equivariant homotopy groups (Def. 2.31) carry the structure of equivariant vector spaces (here: over the real numbers, Def. 3.5):

$$X \text{ is rational over the reals} \quad \Leftrightarrow \quad \pi_{\bullet+1}(X) \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}} \longrightarrow \mathcal{G}\text{Groups}. \quad (120)$$

(ii) A *rationalization* of  $X$  (here: over the real numbers) is a morphism

$$X \xrightarrow{\eta_X^{\mathbb{R}}} L_{\mathbb{R}}X \in \mathcal{G}\text{HomotopyTypes} \quad (121)$$

to a rational equivariant homotopy type (120) which induces isomorphisms on all equivariant rational cohomology groups (Example 3.6):

$$\underline{H}^{\bullet}(L_{\mathbb{R}}X; \mathbb{R}) \xrightarrow[\cong]{(\eta_X^{\mathbb{R}})^*} \underline{H}^{\bullet}(X; \mathbb{R}).$$

In other words: equivariant rationalization is plain rationalization (e.g. [FSS23, Def. 3.55]) at each stage  $G/H \in G\text{Orbits}$ .

**Proposition 3.47** (Uniqueness of equivariant rationalization [Ma96, §II, Thm. 3.2]). *Equivariant rationalization (Def. 3.46) of equivariantly simply-connected equivariant homotopy types exists essentially uniquely.*

### Equivariant PL de Rham theory.

**Definition 3.48** (Equivariant PL de Rham complex). Write

$$\begin{array}{ccc} \mathcal{G}\text{SimplicialSets} & \xrightarrow{\Omega_{\text{PLdR}}^{\bullet}} & (\mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})^{\text{op}} \\ X & \mapsto & (G/H \mapsto \Omega_{\text{PLdR}}^{\bullet}(X(G/H))) \end{array}$$

for the functor from equivariant simplicial sets (Def. 2.19) to the opposite of equivariant dgc-algebras (Def. 3.30). This applies the plain PL de Rham functor [Su77][BG76, p. 1.-7][FSS23, Def. 3.56] (assigning dgc-algebras of piecewise polynomial differential forms) to diagrams of simplicial sets parametrized over the orbit category.

**Proposition 3.49** (Equivariant PL de Rham theorem [Tr82, Thm. 4.9]). *For any  $X \in \mathcal{G}\text{SimplicialSets}$  (Def. 2.19) and  $\mathcal{A}_{\mathbb{R}} \in \mathcal{G}\text{VectorSpaces}_{\mathbb{R}}$  (Def. 3.5), we have a natural isomorphism*

$$H^{\bullet}(X; \mathcal{A}_{\mathbb{R}}) \simeq H^{\bullet}(\Omega_{\text{PLdR}}^{\bullet}(X; \mathcal{A}_{\mathbb{R}}))$$

*between the Bredon cohomology of  $X$  (Example 2.39) with coefficients in  $\mathcal{A}_{\mathbb{R}}$ , and the cochain cohomology of the equivariant PL de Rham complex of  $X$  (Def. 3.48) with coefficients in  $\mathcal{A}_{\mathbb{R}}$ .*

**Proposition 3.50** (Quillen adjunction between equivariant simplicial sets and equivariant dgc-algebras [Scu08, Prop. 5.1]). *The equivariant PL de Rham complex construction (Def. 3.48) is the left adjoint in a Quillen adjunction*

$$\left( \mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \right)_{\text{proj}}^{\text{op}} \begin{array}{c} \xleftarrow{\Omega_{\text{PLdR}}^{\bullet}} \\ \xrightarrow[\text{exp}]{\perp_{\text{Qu}}} \end{array} \mathcal{G}\text{SimplicialSets}_{\text{proj}}$$

*between the projective model structure on equivariant simplicial sets (Prop. 2.21) and the opposite of the projective model structure on connective equivariant dgc-algebras (Prop. 3.38).*

### The fundamental theorem of dgc-algebraic equivariant rational homotopy theory.

**Proposition 3.51** (Fundamental theorem of dgc-algebraic equivariant rational homotopy theory [Scu08, Thm. 5.6]). *On equivariant 1-connected  $\mathbb{R}$ -finite homotopy types (Def. 2.33):*

(i) *The derived PL de Rham adjunction (Prop. 3.50) restricts to an equivalence of homotopy categories*

$$\left( \mathcal{G}\text{HomotopyTypes}_{\geq 2}^{\text{fin}_{\mathbb{R}}} \right)^{\mathbb{R}} \begin{array}{c} \xleftarrow{\mathbb{L}\Omega_{\text{PLdR}}^{\bullet}} \\ \xrightarrow[\mathbb{R}\text{exp}]{\simeq} \end{array} \text{Ho} \left( \left( \mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \right)_{\text{proj}}^{\text{op}} \right)_{\text{fin}}^{\geq 2}$$

*between those simply-connected  $\mathbb{R}$ -finite equivariant homotopy types (Def. 2.33) which are rational (Def. 3.46) over the real numbers and formal duals of cohomologically connected 1-connected (118) equivariant dgc-algebras.*

(ii) *The derived adjunction unit is equivariant rationalization (Def. 3.46):*

$$X \in \mathcal{G}\text{HomotopyTypes}_{\geq 2}^{\text{fin}_{\mathbb{R}}} \quad \Rightarrow \quad \begin{array}{ccc} X & \xrightarrow{\mathbb{D}\eta_X^{\text{PLdR}}} & \mathbb{R}\text{exp} \circ \mathbb{L}\Omega_{\text{PLdR}}^{\bullet}(X) \\ \parallel & & \downarrow \simeq \\ X & \xrightarrow{\eta_X^{\mathbb{R}}} & L_{\mathbb{R}}X \end{array} \quad (122)$$

**Remark 3.52.** That the equivariant derived PLdR-unit (122) models equivariant rationalization is not made explicit in [Scu08], but it follows immediately from the fact that:

- (a) by definition, the equivariant PLdR adjunction is stage-wise over  $G/H \in G\text{Orbits}$  the plain PLdR adjunction;
- (b) the derived unit of the plain PLdR-adjunction models plain rationalization by the non-equivariant fundamental theorem (e.g. [FSS23, Prop. 3.60]); and
- (c) that equivariant rationalization (Def. 3.46) is stage-wise plain rationalization.

### Equivariant rational Whitehead $L_{\infty}$ -algebras

**Definition 3.53** (Equivariant Whitehead  $L_{\infty}$ -algebra). For  $\int \gamma(X//G) \in \mathcal{G}\text{HomotopyTypes}_{\geq 2}^{\text{fin}_{\mathbb{R}}}$  (Def. 2.33), we say that its *equivariant Whitehead  $L_{\infty}$ -algebra*

$$\mathfrak{l}\gamma(X//G) \in \mathcal{G}L_{\infty}\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0}$$

*is the equivariant  $L_{\infty}$ -algebra (Def. 3.36) whose equivariant Chevalley-Eilenberg algebra (104) is the minimal model (well-defined by Prop. 3.44) of the equivariant PL de Rham complex (Def. 3.48) of  $\int \gamma(X//G)$ :*

$$\text{CE}(\mathfrak{l}\gamma(X//G)) := \Omega_{\text{PLdR}}^{\bullet}(X)_{\text{min}} \xrightarrow[\in \mathbb{W}]{p^{\text{min}}} \Omega_{\text{PLdR}}^{\bullet}(X) \in \mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}. \quad (123)$$

**Proposition 3.54** (Equivariant rational homotopy groups in the equivariant Whitehead  $L_\infty$ -algebra [Tr82, Thm. 6.2 (2)]). For  $\int \gamma(X//G) \in \mathcal{GHomomorphismTypes}_{\geq 2}^{\text{fin}_\mathbb{R}}$  (Def. 2.33), the equivariant rational homotopy groups of  $\Omega X$  (Example 3.23) are equivalent to the underlying equivariant graded vector space (Def. 3.41) of the equivariant Whitehead  $L_\infty$ -algebra (Def. 3.53) of  $\gamma(X//G)$ :

$$\begin{array}{c} \text{equivariant} \\ \text{Whitehead } L_\infty\text{-algebra} \\ (\int \gamma(X//G))_\bullet \end{array} \simeq \begin{array}{c} \text{equivariant rational} \\ \text{homotopy groups of} \\ \text{equivariant loop space} \\ \pi_\bullet(\Omega X) \otimes_{\mathbb{Z}} \mathbb{R}. \end{array} \quad (124)$$

### Examples of equivariant Whitehead $L_\infty$ -algebras.

**Proposition 3.55** ( $\mathbb{Z}_2$ -Equivariant minimal model of twistor space). The equivariant minimal model (Def. 3.40) of the  $\mathbb{Z}_2$ -equivariant twistor space (Example 2.28) is the following  $\mathbb{Z}_2$ -equivariant dgc-algebra (Def. 3.30):

$$\begin{array}{ccc} \mathbb{Z}_2/\mathbb{Z}_2 & \mapsto & \mathbb{R} \left[ \begin{array}{c} h_3, \\ f_2 \end{array} \right] / \left( \begin{array}{l} dh_3 = -f_2 \wedge f_2 \\ df_2 = 0 \end{array} \right) \\ \uparrow & & \uparrow \\ \text{CE}(\int \gamma(\mathbb{C}P^3 // \mathbb{Z}_2)) : & & \in \mathbb{Z}_2 \text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \quad (125) \\ \mathbb{Z}_2/1 & \mapsto & \mathbb{R} \left[ \begin{array}{c} h_3, \\ f_2 \\ \omega_7, \\ \omega_4 \end{array} \right] / \left( \begin{array}{l} dh_3 = \omega_4 - f_2 \wedge f_2 \\ df_2 = 0 \\ d\omega_7 = -\omega_4 \wedge \omega_4 \\ d\omega_4 = 0 \end{array} \right) \\ \downarrow \text{(\mathbb{Z}_2)} & & \downarrow \end{array}$$

*Proof.* (i) Checking that (125) is indeed a minimal equivariant dgc-algebra is the content of Example 3.42, where this minimal algebra is obtained in (116).

(ii) It remains to see that (125) has indeed the algebraic homotopy type of the rationalized equivariant twistor space, under the fundamental theorem (Prop. 3.51). By (60), this amounts to showing that the right vertical morphism of ordinary dgc-algebras in (125) is a dgc-algebraic model (under the non-equivariant fundamental theorem of rational homotopy theory, [BG76, §8] reviewed as [FSS23, Prop. 3.59]) of the inclusion of the fiber of the twistor fibration (11). But, by [FSS23, Lem. 3.71]), the dgc-algebra model for this fiber is the cofiber of the minimal relative model of the twistor fibration. The latter is given in [FSS20b, Lem. 2.13], and its cofiber manifestly coincides with (125).

(iii) As a consistency check, notice that the equivariant rational homotopy groups of twistor space (97) do match the generators (117) of this minimal model; as it must be, by Prop. 3.54.  $\square$

**Proposition 3.56** ( $\mathbb{Z}_2$ -Equivariant relative minimal model of  $\text{Sp}(1)$ -parametrized twistor space). The equivariant relative minimal model (Def. 3.40) of the  $\mathbb{Z}_2$ -equivariant  $\text{Sp}(1)$ -parametrized twistor space (Example 2.44) is the following  $\mathbb{Z}_2$ -equivariant dgc-algebra (Def. 3.30) under  $\text{CE}(\int \text{BSp}(1)) = \mathbb{R}[\frac{1}{4}p_1] / (d\frac{1}{4}p_1 = 0)$ :

$$\begin{array}{ccc} \begin{array}{c} \text{twistor} \\ \text{space} \\ \text{orbitfolded} \\ \text{wrt } \mathbb{Z}_2 \\ \text{parametrized} \\ \text{wrt } \text{Sp}(1) \end{array} & & \\ \text{CE} \left( \left( \int_{\text{BSp}(1)} (\gamma(\mathbb{C}P^3 // \mathbb{Z}_2)) // \text{Sp}(1) \right) \right) : & & \\ \mathbb{Z}_2/1 & \mapsto & \text{CE}(\int \text{BSp}(1)) \left[ \begin{array}{c} h_3, \\ f_2 \\ \omega_7, \\ \tilde{\omega}_4 \end{array} \right] / \left( \begin{array}{l} dh_3 = \tilde{\omega}_4 - \frac{1}{2}p_1 - f_2 \wedge f_2 \\ df_2 = 0 \\ d\omega_7 = -\tilde{\omega}_4 \wedge (\tilde{\omega}_4 - \frac{1}{2}p_1) \\ d\tilde{\omega}_4 = 0 \end{array} \right) \\ \downarrow & & \downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 & \mapsto & \text{CE}(\int \text{BSp}(1)) \left[ \begin{array}{c} h_3, \\ f_2 \end{array} \right] / \left( \begin{array}{l} dh_3 = -\frac{1}{2}p_1 - f_2 \wedge f_2 \\ df_2 = 0 \end{array} \right), \end{array} \quad (126)$$

where

- (a) all closed generators are normalized such as to be rational images of integral and integrally in-divisible classes;
- (b)  $\omega := \tilde{\omega} - \frac{1}{4}p_1$  is fiberwise the pullback along  $\mathbb{C}P^3 \xrightarrow{t_{\mathbb{H}}} S^4$  (11) of the volume element on  $S^4$ ;
- (c)  $f_2$  is fiberwise the volume element on  $S^2 \xrightarrow{\text{fib}(t_{\mathbb{H}})} \mathbb{C}P^3$ .

*Proof.* (i) To see that (126) is relative minimal, observe that it is obtained from the equivariant base dgc-algebra

$$\begin{array}{ccc}
\begin{array}{c} \mathbb{Z}_2 \\ \downarrow \\ \mathbb{Z}_2/1 \end{array} & \longmapsto & \text{CE}(\mathfrak{t}B\text{Sp}(1)) \equiv \mathbb{R}[\frac{1}{4}p_1]/(d\frac{1}{4}p_1 = 0) \\
\downarrow & & \text{id} \downarrow \\
\mathbb{Z}_2/\mathbb{Z}_2 & \longmapsto & \text{CE}(\mathfrak{t}B\text{Sp}(1))
\end{array}$$

by the same three cell attachments as in the construction of the absolute minimal model of Example, 3.42 for the plain equivariant twistor space (Prop. 3.55), subject only to these replacements:

$$\begin{aligned}
f_2 \wedge f_2 &\longmapsto f_2 \wedge f_2 + \frac{1}{2}p_1 \\
\omega_4 \wedge \omega_4 &\longmapsto \tilde{\omega}_4 \wedge (\tilde{\omega}_4 - \frac{1}{2}p_1)
\end{aligned}$$

in the attaching maps  $\phi_3$  (114) and  $\phi_7$  (115), respectively.

(ii) By the fundamental theorem (Prop. 3.51), it remains to see that (126) is weakly equivalent to the relative equivariant PL de Rham complex of equivariant parametrized twistor space:

(ii.1) First observe that the relative minimal model  $\text{CE}(l(t_{\mathbb{H}} // \text{Sp}(1)))$  for the *non*-equivariant  $\text{Sp}(1)$ -parametrized twistor fibration  $t_{\mathbb{H}}$ , relative to the minimal model of  $S^4 // \text{Sp}(1)$  relative to  $B\text{Sp}(1)$ , is as follows, with generators normalized as stated in the claim above:

$$\begin{array}{ccc}
\begin{array}{c} S^2 // \text{Sp}(1) \\ \downarrow \text{hofib}_{B\text{Sp}(1)}(t_{\mathbb{H}} // \text{Sp}(1)) \\ \simeq \text{hofib}(t_{\mathbb{H}} // \text{Sp}(1)) \\ \text{(by Lemma 2.10)} \\ \mathbb{C}P^3 // \text{Sp}(1) \\ \downarrow t_{\mathbb{H}} // \text{Sp}(1) \\ S^4 // \text{Sp}(1) \end{array} & & \begin{array}{c} \mathbb{R}[\frac{1}{4}p_1] \left[ \begin{array}{c} h_3, \\ f_2, \end{array} \right] / \left( \begin{array}{l} dh_3 = -\frac{1}{2}p_1 - f_2 \wedge f_2 \\ df_2 = 0 \end{array} \right) \\ \uparrow \text{cof}_{B\text{Sp}(1)}(\text{CE}(l(t_{\mathbb{H}} // \text{Sp}(1)))) \\ \mathbb{R}[\frac{1}{4}p_1] \left[ \begin{array}{c} h_3, \\ f_2, \\ \omega_7, \\ \tilde{\omega}_4 \end{array} \right] / \left( \begin{array}{l} dh_3 = \tilde{\omega}_4 - \frac{1}{2}p_1 - f_2 \wedge f_2 \\ df_2 = 0 \\ d\omega_7 = -\tilde{\omega}_4 \wedge (\tilde{\omega}_4 - \frac{1}{2}p_1) \\ d\tilde{\omega}_4 = 0 \end{array} \right) \quad (127) \\ \uparrow \text{CE}(l(t_{\mathbb{H}} // \text{Sp}(1))) \quad \text{relative minimal model} \\ \text{for } t_{\mathbb{H}} // \text{Sp}(1) \text{ (by [FSS20b, Thm. 2.14])} \\ \mathbb{R}[\frac{1}{4}p_1] \left[ \begin{array}{c} \omega_7, \\ \tilde{\omega}_4 \end{array} \right] / \left( \begin{array}{l} d\omega_7 = -\tilde{\omega}_4 \wedge (\tilde{\omega}_4 - \frac{1}{2}p_1) \\ d\tilde{\omega}_4 = 0 \end{array} \right) \end{array} \\
\begin{array}{c} \rho_{S^2} \\ \swarrow \\ B\text{Sp}(1) \\ \nwarrow \rho_{S^4} \end{array} & & \begin{array}{c} \rho_{\mathbb{C}P^3} \\ \swarrow \\ B\text{Sp}(1) \\ \nwarrow \rho_{S^4} \end{array}
\end{array}$$

This is the statement of [FSS20b, Thm. 2.14], using the following notational simplifications in the present case:

(a) the Euler 8-class  $\mathcal{X}_8$  appearing in [FSS20b, (39)] vanishes here under restriction along  $B\text{Sp}(1) \rightarrow B\text{Sp}(2)$ ;

(b) we have applied to [FSS20b, (49)] the dgc-algebra isomorphism given by

$$h_3 \leftrightarrow h_3, \quad f_2 \leftrightarrow f_2, \quad \omega_7 \leftrightarrow \omega_7, \quad \omega_4 \leftrightarrow \tilde{\omega}_4 - \frac{1}{4}p_1. \quad (128)$$

(ii.2) This being a non-equivariant relative minimal model, it comes with horizontal weak equivalences of non-equivariant dgc-algebras as shown in the bottom square of the following commuting diagram (by, e.g., [FHT00, Thm. 14.12]), which induces (by the *fiber lemma* [BK72, §II] in the form [FHT00, Prop. 15.5][FHT15, Thm. 5.1]) a weak equivalence on plain cofibers (which is forms on  $S^2$ , by Lemma 2.10), as shown in the following top square:



$$\begin{array}{ccc}
\Omega_{\text{PLdR}}^\bullet(S^2) & \leftarrow \text{---} \in W \text{---} & \mathbb{R} \left[ \begin{array}{c} h_3, \\ f_2, \end{array} \right] / \left( \begin{array}{l} dh_3 = -f_2 \wedge f_2 \\ df_2 = 0 \end{array} \right) \\
\uparrow \Omega_{\text{PLdR}}^\bullet(\text{fib}(t_{\mathbb{H}} // \text{Sp}(1))) & & \uparrow \text{cof}(\text{CE}(t(t_{\mathbb{H}} // \text{Sp}(1)))) \\
\Omega_{\text{PLdR}}^\bullet(\mathbb{C}P^3 // \text{Sp}(1)) & \leftarrow \in W & \mathbb{R} \left[ \begin{array}{c} h_3, \\ f_2, \\ \omega_7, \\ \tilde{\omega}_4 \end{array} \right] / \left( \begin{array}{l} dh_3 = \tilde{\omega}_4 - \frac{1}{2}p_1 - f_2 \wedge f_2 \\ df_2 = 0 \\ d\omega_7 = -\tilde{\omega}_4 \wedge (\tilde{\omega}_4 - \frac{1}{2}p_1) \\ d\tilde{\omega}_4 = 0 \end{array} \right) \\
\uparrow \Omega_{\text{PLdR}}^\bullet(t_{\mathbb{H}} // \text{Sp}(1)) & & \uparrow \text{CE}(t(t_{\mathbb{H}} // \text{Sp}(1))) \\
\Omega_{\text{PLdR}}^\bullet(S^4 // \text{Sp}(1)) & \leftarrow \in W & \mathbb{R} \left[ \begin{array}{c} \omega_7, \\ \tilde{\omega}_4 \end{array} \right] / \left( \begin{array}{l} d\omega_7 = -\tilde{\omega}_4 \wedge (\tilde{\omega}_4 - \frac{1}{2}p_1) \\ d\tilde{\omega}_4 = 0 \end{array} \right)
\end{array} \quad (129)$$

(Here we are using that with  $t_{\mathbb{H}}$  also  $t_{\mathbb{H}} // \text{Sp}(1) := \frac{t_{\mathbb{H}} \times W \text{Sp}(1)}{\text{Sp}(1)}$  is a fibration, by the right Quillen functor (29) in Prop. 2.7, and that all spaces involved are simply-connected, so that all the technical assumptions in [FHT15, (5.1)] are indeed met.)

(ii.3) Then observe that

$$H^\bullet(S^2 // \text{Sp}(1); \mathbb{R}) \simeq \mathbb{R}[\omega_2, \frac{1}{4}p_1] / ((\omega_2)^2) \simeq H^\bullet(B\text{Sp}(1); \mathbb{R}) \otimes_{\mathbb{R}} H^\bullet(S^2; \mathbb{R}). \quad (130)$$

This follows readily from the Gysin exact sequence (e.g. [Sw75, §15.30])

$$\dots \rightarrow H^\bullet(B\text{Sp}(1); \mathbb{R}) \xrightarrow{\rho_{S^2}^*} H^\bullet(S^2 // \text{Sp}(1); \mathbb{R}) \xrightarrow{f_{S^2}} H^{\bullet-2}(B\text{Sp}(1); \mathbb{R}) \xrightarrow[=0]{c \cup (-)} H^{\bullet+1}(B\text{Sp}(1); \mathbb{R}) \rightarrow \dots \quad (131)$$

for the  $S^2$ -fiber sequence  $S^2 \xrightarrow{\text{hofib}(\rho_{S^2})} S^2 // \text{Sp}(1) \xrightarrow{\rho_{S^2}} B\text{Sp}(1)$  that corresponds to the  $\text{Sp}(1)$ -action on  $S^2$ , by Prop. 2.7; and using that  $H^\bullet(B\text{Sp}(1); \mathbb{R}) \simeq \mathbb{R}[\frac{1}{4}p_1]$  (e.g. [FSS23, Lemma 4.24]) is concentrated in degrees divisible by 4 (so that, in particular, the Euler class  $c \in H^3(B\text{Sp}(1); \mathbb{R}) \simeq 0$  in (131) vanishes).

But using (130) in (129) implies that also the induced map on *relative fibers* (33) over  $B\text{Sp}(1)$  is a weak equivalence:

$$\begin{array}{ccc}
\mathbb{Z}_2 / \mathbb{Z}_2 & & \Omega_{\text{PLdR}}^\bullet(S^2 // \text{Sp}(1)) \leftarrow \text{---} \in W \text{---} \mathbb{R} \left[ \begin{array}{c} h_3, \\ f_2, \end{array} \right] / \left( \begin{array}{l} dh_3 = -\frac{1}{2}p_1 - f_2 \wedge f_2 \\ df_2 = 0 \end{array} \right) \\
\uparrow & \Omega_{\text{PLdR}}^\bullet(\text{fib}_{B\text{Sp}(1)}(t_{\mathbb{H}} // \text{Sp}(1))) \simeq \Omega_{\text{PLdR}}^\bullet(\text{fib}(t_{\mathbb{H}}) // \text{Sp}(1)) & \uparrow \text{cof}_{B\text{Sp}(1)}(\text{CE}(t(t_{\mathbb{H}} // \text{Sp}(1)))) \\
\mathbb{Z}_2 / 1 & \Omega_{\text{PLdR}}^\bullet(\mathbb{C}P^3 // \text{Sp}(1)) \leftarrow \in W & \mathbb{R} \left[ \begin{array}{c} h_3, \\ f_2, \\ \omega_7, \\ \tilde{\omega}_4 \end{array} \right] / \left( \begin{array}{l} dh_3 = \tilde{\omega}_4 - \frac{1}{2}p_1 - f_2 \wedge f_2 \\ df_2 = 0 \\ d\omega_7 = -\tilde{\omega}_4 \wedge (\tilde{\omega}_4 - \frac{1}{2}p_1) \\ d\tilde{\omega}_4 = 0 \end{array} \right) \\
\uparrow \mathbb{Z}_2 & &
\end{array} \quad (132)$$

(ii.4) By Lemma 2.10 applied to (76), we see that the left morphism in (132) is equivalently the inclusion of the fixed-locus in the  $\mathbb{Z}_2$ -equivariant  $\text{Sp}(1)$ -parametrized twistor space (Example 2.44). Thus, by the stage-wise definition of the equivariant PL de Rham complex (Def. 3.48), it follows that the left morphism in (132) is the PL de Rham complex of  $\mathbb{Z}_2$ -equivariant  $\text{Sp}(1)$ -parametrized twistor space (as indicated by alignment with the  $\mathbb{Z}_2$ -orbit category on the far left of (129)). Finally this means, by the fundamental theorem (Prop. 3.51), that the commuting square in (129) exhibits the claimed equivariant dgc-algebra (15) as indeed modeling the equivariant rational homotopy type of the  $\mathbb{Z}_2$ -equivariant  $\text{Sp}(1)$ -parametrized twistor space. (The images on the left of the generators on the right of (129) are indeed all invariant under the  $\mathbb{Z}_2 \subset \text{Sp}(2)$ -action, by [BMSS19, Lemma 5.5]).  $\square$

### 3.3 Equivariant non-abelian de Rham theorem

We introduce properly equivariant non-abelian de Rham cohomology with coefficients in equivariant  $L_\infty$ -algebras, in direct generalization of the non-equivariant discussion in [FSS23, §3.3]. Our key example here is the non-abelian cohomology of equivariant twistorial differential forms (Example 3.74 below). The main result is the proper equivariant non-abelian de Rham theorem (Prop. 3.63) and its twisted version (Prop. 3.67). The specialization to traditional Borel-equivariant abelian de Rham cohomology is the content of Prop. 3.72 below.

#### Flat equivariant $L_\infty$ -algebra valued differential forms.

In equivariant generalization of [FSS23, Def. 3.77], we set:

**Definition 3.57** (Flat equivariant  $L_\infty$ -algebra valued differential forms). Let  $\underline{\mathfrak{g}} \in \mathcal{G}L_\infty\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0}$  (Def. 3.36) and  $G \curvearrowright X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35). Then the set of *flat equivariant  $\underline{\mathfrak{g}}$ -valued differential forms* on  $X$  is the hom-set (16)

$$\Omega_{\text{dR}}(\gamma(X//G); \underline{\mathfrak{g}})_{\text{flat}} := \mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}(\text{CE}(\underline{\mathfrak{g}}), \Omega_{\text{dR}}^\bullet(\gamma(X//G)))$$

of equivariant dgc-algebras (Def. 3.30) from the equivariant Chevalley-Eilenberg algebra (104) of  $\underline{\mathfrak{g}}$  to the equivariant smooth de Rham complex (Def. 3.33) of  $X$ .

In equivariant generalization of [FSS23, Def. 3.92], we set:

**Definition 3.58** (Flat twisted equivariant  $L_\infty$ -algebra valued differential forms on  $G$ -orbifold). Consider an *equivariant  $L_\infty$ -algebraic local coefficient bundle* in the form of a fibration of equivariant  $L_\infty$ -algebras (Def. 3.36) whose equivariant Chevalley-Eilenberg algebras (104), are relative minimal (Def. 3.40)

$$\begin{array}{ccc} \underline{\mathfrak{g}} & \xrightarrow{\text{fib}(\underline{\mathfrak{p}})} & \widehat{\underline{\mathfrak{b}}} \\ & \text{equivariant } L_\infty\text{-algebraic} & \downarrow \underline{\mathfrak{p}} \\ & \text{local coefficient bundle} & \underline{\mathfrak{b}} \end{array} \in \mathcal{G}L_\infty\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0}. \quad (133)$$

Then, for  $G \curvearrowright X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35) equipped with an *equivariant non-abelian de Rham twist*

$$\tau_{\text{dR}} \in \Omega_{\text{dR}}(\gamma(X//G); \underline{\mathfrak{b}})_{\text{flat}} \quad (134)$$

given by a flat equivariant  $\underline{\mathfrak{b}}$ -valued differential form (Def. 3.57) on  $X$ , the set of *flat  $\tau_{\text{dR}}$ -twisted equivariant  $\underline{\mathfrak{g}}$ -valued differential forms* on  $X$  is the hom-set (16) in the co-slice category of  $\mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}$  (Def. 3.30) under  $\text{CE}(\underline{\mathfrak{g}})$  from  $\text{CE}(\underline{\mathfrak{p}})$  to  $\tau_{\text{dR}}$ :

$$\begin{aligned} \Omega_{\text{dR}}^{\tau_{\text{dR}}}(\gamma(X//G), \underline{\mathfrak{g}})_{\text{flat}} &:= \left( \mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \right)^{\text{CE}(\underline{\mathfrak{b}})} / (\text{CE}(\underline{\mathfrak{p}}), \tau_{\text{dR}}) \\ &= \left\{ \begin{array}{ccc} \Omega_{\text{dR}}^\bullet(\gamma(X//G)) & \xleftarrow{\text{flat twisted equivariant } \underline{\mathfrak{g}}\text{-valued differential form}} & \text{CE}(\widehat{\underline{\mathfrak{b}}}) \\ & \xleftarrow{\text{twist } \tau_{\text{dR}}} & \text{CE}(\underline{\mathfrak{b}}) \xrightarrow{\text{CE}(\underline{\mathfrak{p}}) \text{ local coefficients}} & \end{array} \right\}. \quad (135) \end{aligned}$$

#### Equivariant non-abelian de Rham cohomology.

**Notation 3.59** (Cylinder orbifold). For  $G \curvearrowright X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35), let the product manifold  $X \times \mathbb{R}$  be equipped with the  $G$ -action given by

$$\begin{aligned} G \times (X \times \mathbb{R}) &\longrightarrow X \times \mathbb{R} \\ (g, (x, t)) &\longmapsto (g \cdot x, t). \end{aligned}$$

We say that the resulting  $G$ -orbifold (Def. 2.36)  $\gamma((X \times \mathbb{R})//G) \in G\text{Orbifolds}$  is the *cylinder orbifold* of  $\gamma(X//G)$ , and we write

$$\gamma(X//G) \simeq \gamma((X \times \{0\})//G) \xleftarrow{i_0} \gamma((X \times \mathbb{R})//G) \xleftarrow{i_1} \gamma((X \times \{1\})//G) \simeq \gamma(X//G) \quad (136)$$

for the canonical inclusion maps and

$$\gamma((X \times \mathbb{R})//G) \xrightarrow{p_X} \gamma(X//G) \quad (137)$$

for the canonical projection map.

In equivariant generalization of [FSS23, Def. 3.83], we set:

**Definition 3.60** (Coboundaries between flat equivariant  $L_\infty$ -algebra valued differential forms).

Let  $\mathfrak{g} \in \mathcal{G}L_\infty\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0}$  (Def. 104) and  $G \curvearrowright X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35).

(i) Then, given flat differential forms  $A_0, A_1 \in \Omega_{\text{dR}}(\gamma(X//G); \mathfrak{g})_{\text{flat}}$  (Def. 3.57), a *coboundary* between them

$$A_0 \xrightarrow{\tilde{A}} A_1$$

is a flat equivariant  $\mathfrak{g}$ -valued differential form (Def. 3.57) on the cylinder orbifold (Notation 3.59)

$$\tilde{A} \in \Omega_{\text{dR}}^{\text{cylinder orbifold}}(\gamma((X \times \mathbb{R})//G); \mathfrak{g})_{\text{flat}} \quad (138)$$

such that this restricts to the given pair of forms

$$i_0^*(\tilde{A}) = A_0 \quad \text{and} \quad i_1^*(\tilde{A}) = A_1 \quad (139)$$

along the canonical inclusions (136).

(ii) We denote the relation given by existence of such a coboundary by  $A_1 \sim A_2$ .

**Lemma 3.61** (Equivalence of equivariant smooth and PL de Rham complex of smooth orbifold). *Let  $G \curvearrowright X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35). Then the corresponding equivariant PL de Rham complex (Def. 3.48) is isomorphic to the equivariant smooth de Rham complex (Example 3.33) in the homotopy category of equivariant dgc-algebras (Prop. 3.38):*

$$\Omega_{\text{dR}}^\bullet(\gamma(X//G)) \simeq \Omega_{\text{PLdR}}^\bullet(\gamma(X//G)) \in \text{Ho}\left(\left(\mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}\right). \quad (140)$$

*Proof.* Observe that the analogous non-equivariant statement holds by [FSS23, Lem. 3.90], using [GMo13, Cor. 9.9], and that its proof proceeds by analyzing natural constructions applied to a choice of smooth triangulation of the given smooth manifold  $X$ .

Now, for a smooth manifold equipped with a smooth  $G$ -action  $G \curvearrowright X$ , we may choose a  $G$ -equivariant smooth triangulation, by the equivariant triangulation theorem [II78][II83]. Given this, the remainder of the non-equivariant proof applies stage-wise over the orbit category. Since the weak equivalences of equivariant dgc-algebras are the stage-wise weak equivalences of non-equivariant dgc-algebras (Prop. 3.38), the claim follows.  $\square$

In an equivariant generalization of [FSS23, Def. 3.84], we set:

**Definition 3.62** (Equivariant non-abelian de Rham cohomology). Let  $G \curvearrowright X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35) and  $\mathfrak{g} \in \mathcal{G}L_\infty\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0}$  (Def. 3.36). The *equivariant non-abelian de Rham cohomology* of  $G \curvearrowright X$  with coefficients in  $\mathfrak{g}$  is the quotient of the set of flat equivariant differential forms (Def. 3.57) by the coboundary relation (Def. 3.60):

$$H_{\text{dR}}(\gamma(X//G); \mathfrak{g}) := \left( \Omega_{\text{dR}}(\gamma(X//G); \mathfrak{g})_{\text{flat}} \right) / \sim$$

In equivariant generalization of [FSS23, Thm. 3.87], we have:

**Proposition 3.63** (Equivariant non-abelian de Rham theorem). *Let  $\mathcal{A} \in \mathcal{G}\text{HomotopyTypes}_{\geq 2}^{\text{fin}\mathbb{R}}$  (Def. 2.33) and  $G \curvearrowright X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35), such that its equivariant shape (Def. 2.23) is also equivariantly simply-connected and of  $\mathbb{R}$ -finite type:  $\int \gamma(X//G) \in \mathcal{G}\text{HomotopyTypes}_{\geq 2}^{\text{fin}\mathbb{R}}$ . Then, at least if  $G$  has order 4 or is cyclic of prime order (Remark 3.17), there is an equivalence between:*

(a) *real equivariant non-abelian cohomology (Def. 2.37) with coefficients in the equivariant rationalization  $L_{\mathbb{R}}\mathcal{A}$  (Def. 3.46) and*

(b) *equivariant non-abelian de Rham cohomology (Def. 3.62) of the  $G$ -orbifold  $\gamma(X//G)$  (Def. 2.36) with coefficients in the equivariant Whitehead  $L_\infty$ -algebra  $\mathcal{A}$  (Def. 3.53):*

$$H\left(\int \gamma(X//G); L_{\mathbb{R}}\mathcal{A}\right) \simeq H_{\text{dR}}\left(\gamma(X//G); \mathcal{A}\right). \quad (141)$$

*Proof.* Consider the following sequence of bijections:

$$\begin{aligned}
H(\gamma(X//G); L_{\mathbb{R}}\mathcal{A}) &:= \mathcal{G}\text{HomotyTypes}(\gamma(X//G), L_{\mathbb{R}}\mathcal{A}) \\
&\simeq \text{Ho}\left(\left(\mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}\left(\Omega_{\text{PLdR}}^{\bullet}(\mathcal{A}), \Omega_{\text{PLdR}}^{\bullet}(\gamma(X//G))\right)\right) \\
&\simeq \text{Ho}\left(\left(\mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}\left(\text{CE}(\mathcal{L}\mathcal{A}), \Omega_{\text{dR}}^{\bullet}(\gamma(X//G))\right)\right) \\
&\simeq \left(\mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0}\right)_{\text{proj}}\left(\text{CE}(\mathcal{L}\mathcal{A}), \Omega_{\text{dR}}^{\bullet}(\gamma(X//G))\right) \Big/_{\sim_{\text{right homotopy}}} \\
&\simeq \left(\Omega_{\text{dR}}(\gamma(X//G); \mathcal{L}\mathcal{A})_{\text{flat}}\right) \Big/_{\sim} \\
&=: H_{\text{dR}}(\gamma(X//G); \mathcal{L}\mathcal{A}).
\end{aligned}$$

The first step is Def. 2.37, while the second step is the fundamental theorem (Prop. 3.51). In the third step we are:

- (a) post-composing in the homotopy category with the isomorphism  $\Omega_{\text{PLdR}}^{\bullet}(-) \simeq \Omega_{\text{dR}}^{\bullet}(-)$  (140);
- (b) pre-composing with the isomorphism  $\text{CE}(\mathcal{L}\mathcal{A}) \simeq \Omega_{\text{PLdR}}^{\bullet}(\mathcal{A})$  exhibiting the minimal model (123).

Now the domain object  $\text{CE}(\mathcal{L}\mathcal{A})$  is cofibrant (by Lemma 3.43) and the codomain object  $\Omega_{\text{dR}}^{\bullet}(\gamma(X//G))$  is fibrant (by Prop. 3.39). Consequently, the hom-set in the homotopy category is equivalently given ([Qu67, §I.1 Cor. 7], see [FSS23, Prop. A.16]) by right-homotopy classes of equivariant dgc-algebra homomorphisms between these objects, shown in the fourth step.

To exhibit these right homotopies, we may choose as path-space object ([Qu67, Def. I.4], see [FSS23, A.11]) the equivariant de Rham complex on the cylinder orbifold (Notation 3.59): this qualifies as a path space object by stage-wise application of [FSS23, Lem. 3.88] and using again the argument of Lemmas 3.19, 3.20, 3.21 for equivariant fibrancy. But with this choice of path space object, the right homotopy relation manifestly coincides (by stage-wise application of [FSS23, Lem. 3.89]) with the coboundary relation on equivariant non-abelian forms (Def. 3.60). which is the fifth step above. With this, the last step is Def. 3.62.

In conclusion, the composite of this chain of bijections gives the claimed bijection (141).  $\square$

### Twisted equivariant non-abelian de Rham cohomology.

In equivariant generalization of [FSS23, Def. 3.97], we set:

**Definition 3.64** (Coboundaries between flat twisted equivariant  $L_{\infty}$ -algebra valued differential forms). Given an equivariant  $L_{\infty}$ -algebraic local coefficient bundle (133)

$$\begin{array}{ccc}
\underline{\mathfrak{g}} & \xrightarrow{\text{fib}(p)} & \widehat{\underline{\mathfrak{b}}} \\
& \text{equivariant } L_{\infty}\text{-algebraic} & \downarrow p \\
& \text{local coefficient bundle} & \underline{\mathfrak{b}}
\end{array} \in \mathcal{G}L_{\infty}\text{Algebras}_{\mathbb{R}, \text{fin}}^{\geq 0}, \quad (142)$$

and given  $G \curvearrowright X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35) equipped with an equivariant non-abelian de Rham twist (134)

$$\tau_{\text{dR}} \in \Omega_{\text{dR}}(\gamma(X//G); \underline{\mathfrak{b}}),$$

(i) we say that a *coboundary* between a pair

$$A_0, A_1 \in \Omega_{\text{dR}}^{\tau_{\text{dR}}}(\gamma(X//G); \underline{\mathfrak{g}})$$

of flat equivariant  $\tau_{\text{dR}}$ -twisted  $\underline{\mathfrak{g}}$ -valued differential forms (Def. 3.57) is such a form on the cylinder orbifold (Notation 3.59)

$$\tilde{A} \in \Omega_{\text{dR}}^{p_X^*}(\overset{\text{cylinder orbifold}}{\tau_{\text{dR}}}(\gamma((X \times \mathbb{R})//G); \underline{\mathfrak{g}})$$

twisted by the pullback of the given twist to the cylinder orbifold (along the canonical projection (137)), such that this restricts to the given pair of forms

$$i_0^*(\tilde{A}) = A_0 \quad \text{and} \quad i_1^*(\tilde{A}) = A_1 \quad (143)$$

along the canonical inclusions (136).

(ii) We denote the relation that there exists such a coboundary by  $A_0 \sim A_1$ .

In equivariant generalization of [FSS23, Def. 3.98], we set:

**Definition 3.65** (Twisted equivariant non-abelian de Rham cohomology). Let  $G \curvearrowright X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35) and let  $\underline{\mathfrak{g}} \rightarrow \widehat{\underline{\mathfrak{b}}} \rightarrow \underline{\mathfrak{b}}$  be an equivariant  $L_\infty$ -algebraic local coefficient bundle (133), and let

$$[\tau_{\text{dR}}] \in H_{\text{dR}}(\gamma(X//G); \underline{\mathfrak{b}})_{\text{flat}} \quad (144)$$

be the equivariant non-abelian de Rham cohomology class (Def. 3.62) of an equivariant twist (134). Then we say that the *equivariant  $\tau_{\text{dR}}$ -twisted de Rham cohomology* of the  $G$ -orbifold  $\gamma(X//G)$  (Def. 2.36) with coefficients in  $\underline{\mathfrak{g}}$  is the quotient of the set of *equivariant  $\tau_{\text{dR}}$ -twisted  $\underline{\mathfrak{g}}$ -valued differential forms* (Def. 3.58) by the coboundary relation from Def. 3.64:

$$H_{\text{dR}}^{\tau_{\text{dR}}}(\gamma(X//G); \underline{\mathfrak{g}}) := \Omega_{\text{dR}}^{\tau_{\text{dR}}}(\gamma(X//G); \underline{\mathfrak{g}}) / \sim.$$

**Notation 3.66** (Equivariant local coefficient bundle with relative minimal model). Given an equivariant local coefficient bundle (78)

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\text{hofib}(\rho_{\mathcal{A}})} & \mathcal{A} // G \\ \text{equivariant} & & \downarrow \rho_{\mathcal{A}} \\ \text{local coefficient} & & \in \mathcal{GHomomorphismTypes}_{\geq 2}^{\text{fin}_{\mathbb{R}}} \\ \text{bundle} & & BG \end{array} \quad (145)$$

all of whose objects are equivariantly 1-connected and of  $\mathbb{R}$ -finite type (Def. 2.33), assume (Remark 3.45) that  $\rho_{\mathcal{A}}$  admits an equivariant relative minimal model (Def. 3.40). This is to be denoted as follows:

$$\begin{array}{ccccc} & & \text{cofib}(\text{CE}(\iota_{\rho_{\mathcal{A}}})) & & \\ & & \longleftarrow & & \longrightarrow \\ \text{CE}(\iota_{\mathcal{A}}) & & & & \text{CE}(\iota_{BG}(\mathcal{A} // G)) \\ \longleftarrow & \Omega_{\text{PLdR}}^{\bullet}(\text{hofib}(\rho_{\mathcal{A}})) & \longrightarrow & \Omega_{\text{PLdR}}^{\bullet}(\mathcal{A} // G) & \longleftarrow & \begin{array}{l} p_{\mathcal{A} // G}^{\text{min}_{\mathbb{R}}} \in W \\ \text{equivariant relative} \\ \text{minimal model} \end{array} \\ \text{equivariant dgc-algebra model} & & \Omega_{\text{PLdR}}^{\bullet}(\rho_{\mathcal{A}}) & & \text{CE}(\iota_{\rho_{\mathcal{A}}}) \\ \text{of local coefficient bundle} & & \uparrow & & \text{equivariant relative} \\ & & \text{equivariant dgc-algebra model} & & \text{minimal model} \\ & & \text{of local coefficient bundle} & & \text{equivariant} \\ & & \Omega_{\text{dR}}^{\bullet}(BG) & & \text{minimal model} \\ & & \longleftarrow & & \text{CE}(\iota_{BG}) \\ & & p_{BG}^{\text{min}} \in W & & \end{array} \quad (146)$$

Notice that the corresponding fibration of equivariant  $L_\infty$ -algebras (Def. 3.36) serves as an equivariant  $L_\infty$ -algebraic local coefficient bundle (133).

In equivariant generalization of [FSS23, Thm. 3.104], we have:

**Proposition 3.67** (Twisted equivariant non-abelian de Rham theorem). *Consider the following*

- Let  $\rho_{\mathcal{A}}$  be an equivariant local coefficient bundle of equivariantly 1-connected  $G$ -spaces of finite  $\mathbb{R}$ -homotopy type, which admits an equivariant relative minimal model; all as in Notation 3.66.
- Moreover, let  $G \curvearrowright X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35) be such that also its equivariant shape (Def. 2.23) is equivariantly 1-connected and of  $\mathbb{R}$ -finite type,  $\int \gamma(X//G) \in \mathcal{GHomomorphismTypes}_{\geq 2}^{\text{fin}_{\mathbb{R}}}$  and let this be equipped with an equivariant twist  $\tau$  (79) with coefficients in the equivariant rationalization (Def. 3.46) of  $BG$ .
- Write  $\tau_{\text{dR}}$  for a representative of the image under the equivariant non-abelian de Rham theorem (Prop. 3.63) of the class of this twist in equivariant  $\mathbb{B}\mathcal{A}$ -valued de Rham cohomology (Def. 3.62) that the equivariant local coefficient bundle (145) admits an equivariant relative minimal model (Def. 3.40)

$$\begin{array}{ccc} H\left(\int \gamma(X//G); L_{\mathbb{R}}BG\right) & \xrightarrow{\simeq} & H_{\text{dR}}\left(\gamma(X//G); \mathbb{B}G\right) \\ \text{rational twist} & \text{equivariant non-abelian} & \text{de Rham twist} \\ [\tau] & \longmapsto & [\tau_{\text{dR}}] \end{array} \quad (147)$$

Then there is an equivalence between:

- (a) the  $\tau$ -twisted equivariant real non-abelian cohomology (Def. 2.45) with local coefficients in  $\rho_{\mathcal{A}}$ , and
- (b) the  $\tau_{\text{dR}}$ -twisted equivariant de Rham cohomology (Def. 3.65) with local coefficients in  $\iota_{BG}\rho_{\mathcal{A}}$  (146):

$$\begin{array}{ccc} \text{twisted equivariant} & & \text{twisted equivariant} \\ \text{non-abelian real cohomology} & \simeq & \text{non-abelian de Rham cohomology} \\ H^{\tau}\left(\int \gamma(X//G); L_{\mathbb{R}}\mathcal{A}\right) & \simeq & H^{\tau_{\text{dR}}}\left(\gamma(X//G); \mathbb{B}\mathcal{A}\right) \end{array} \quad (148)$$

*Proof.* The proof proceeds in direct joint generalization of the proofs of Prop. 3.63 (equivariant case) and [FSS23, Thm. 3.104] (twisted case).

First, by the fundamental theorem (Prop. 3.51), the twisted real cohomology is given by morphisms in the homotopy category of the co-slice model category of this form:

$$\Omega_{\text{PLdR}}^\bullet \left( \int \gamma (X // G) \right) \leftarrow \text{-----} \Omega_{\text{PLdR}}^\bullet (\mathcal{A} // \mathcal{G}) \in \text{Ho} \left( \left( \mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \right)_{\text{proj}} \right). \quad (149)$$

$$\begin{array}{ccc} & \swarrow \Omega_{\text{PLdR}}^\bullet(\tau) & \searrow \Omega_{\text{PLdR}}^\bullet(\rho_{\mathcal{A}}) \\ & \Omega_{\text{PLdR}}^\bullet(B\mathcal{G}) & \end{array}$$

Second, by

(a) post-composition with the isomorphism  $\Omega_{\text{PLdR}}^\bullet(-) \simeq \Omega_{\text{dR}}^\bullet(-)$  (140),

(b) pre-composition with the equivalence from the equivariant relative minimal model (146),

this becomes equivalent to morphisms of this form:

$$\Omega_{\text{dR}}^\bullet \left( \int \gamma (X // G) \right) \leftarrow \text{-----} \Omega_{\text{PLdR}}^\bullet (\mathcal{A} // \mathcal{G}) \in \text{Ho} \left( \left( \mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0} \right)_{\text{proj}} \right). \quad (150)$$

$$\begin{array}{ccc} & \swarrow \tau_{\text{dR}} & \searrow \text{CE}(\iota_{\rho_{\mathcal{A}}}) \\ & \text{CE}(\iota_{B\mathcal{G}}) & \end{array}$$

But, in this form,

- (a) the codomain  $\tau_{\text{dR}}$  is a fibrant object in the coslice model category, since  $\Omega_{\text{dR}}^\bullet(-)$  is fibrant in the un-sliced model structure (Prop. 3.39);
- (b) the relative minimal model domain  $\text{CE}(\iota_{\rho_{\mathcal{A}}})$  is cofibrant, by Lemma 3.43.

It follows ([Qu67, §I.1 Cor. 7], see [FSS23, Prop. A.16]) that a morphism of the form (150) in the homotopy category is equivalently the right homotopy class of an actual homomorphism of equivariant dgc-algebras in the coslice, hence is equivalently the right homotopy class of a flat equivariant twisted  $\mathcal{A}$ -valued differential form, by Def. 3.58.

Finally, in joint generalization of the proof of Prop. 3.63 (equivariant case) and [FSS23, Lem. 3.105] (twisted case), we see that a path space object ([Qu67, Def. 1.4], see [FSS23, A.11]) exhibiting these right homotopies in the coslice is given by pullback to the equivariant smooth de Rham complex of the cylinder orbifold (138). But with that choice, right homotopies are manifestly the same as coboundaries of flat equivariant twisted  $\mathcal{A}$ -valued differential forms (Def. 3.64), and hence the claim follows.  $\square$

**Twisted non-abelian Borel-Weil-Cartan equivariant de Rham cohomology.** Finally, we combine traditional Borel(-Weil-Cartan)  $T$ -equivariant de Rham cohomology ([AB84][MQ86, §5][Kal93][GSt99], review in [Me06][KT15][Pes17]), with proper  $G$ -equivariance and generalize it to non-abelian  $L_\infty$ -algebra coefficients.

By Prop. 2.7 and Remark 2.42, any Borel  $T$ -equivariantized  $G$ -orbifold carries a canonical twist in equivariant non-abelian cohomology  $H^1(-, T) \simeq H(-, BT)$ . The following is the de Rham image of that twist:

**Definition 3.68** (Canonical de Rham twist on Borel  $T$ -equivariant  $G$ -orbifolds).

Let  $(T \times G) \curvearrowright X \in (T \times G)\text{Actions}(\text{SmoothManifolds})$  (Def 2.35) for  $T \in \text{CompactLieGroups}$  finite-dimensional and simply-connected, with Lie algebra  $\mathfrak{t}$  (105), regarded as a smooth  $G$ -equivariant  $L_\infty$ -algebra (Def. 3.36). We say that the *canonical de Rham twist* on the corresponding  $T$ -parametrized  $G$ -orbifold is the canonical inclusion of equivariant dgc-algebras (Def. 3.30) from the minimal model for the classifying space of  $T$  (regarded as a smooth  $G$ -equivariant homotopy type, Example 2.24) into the proper  $G$ -equivariant & Borel  $T$ -equivariant smooth de Rham complex (Example 3.37):

$$\begin{array}{ccc} \Omega_{\text{dR}}^\bullet \left( (\gamma(X // G)) // T \right) & & \begin{array}{c} \text{Cartan model for } T\text{-equivariant Borel cohomology of } H\text{-fixed locus } X^H \\ \left( \Omega_{\text{dR}}^\bullet(X^H) \otimes \mathbb{R} \left[ \{r_2^a\}_{a=1}^{\dim(T)} \right], d_{\text{dR}} + r_2^a \wedge \iota_{t_a} \right)^T \\ \uparrow \\ \left( \mathbb{R} \left[ \{r_2^a\}_{a=1}^{\dim(T)} \right] \right)^T \end{array} \\ \uparrow \tau_{\text{dR}}^{\text{can}} & : & G/H \longmapsto \\ \text{CE}(\iota_{BT}) & & \end{array}$$

where on the bottom we used the abstract Chern-Weil isomorphism (81) in the form discussed in [FSS23, §4.2].

**Example 3.69** (Equivariant Cartan map). In the situation of Def. 3.68, consider the case when the  $T$ -action is free, hence that  $X := P$  is the total space of a  $G$ -equivariant  $T$ -principal bundle  $P \rightarrow B := P/T$  (e.g. [KT15, p. 2]). Then, for any choice of  $G$ -invariant  $N$ -principal connection  $\nabla \in N\text{Connections}(P)^G$ , we have the following weak equivalence (in the sense of Prop. 3.38) of  $G$ -equivariant dgc-algebras (Def. 3.30) in the co-slice under the minimal model dgc-algebra of the classifying space (81):

$$\begin{array}{ccc}
\Omega_{\text{dR}}^{\bullet} \left( (\gamma(X//G)) // T \right) & \xrightarrow{\in W} & \Omega_{\text{dR}}^{\bullet} (\gamma(B//G)) \\
\tau_{\text{dR}}^{\text{can}} \swarrow & & \nearrow \text{cw}_T \\
\text{CE}(\mathfrak{l}BT) & & \\
& & : G/H \mapsto \\
& & \left( \Omega_{\text{dR}}^{\bullet}(X^H) \left[ \{r_2^a\}_{a=1}^{\dim(T)} \right]^T \right) \xrightarrow{\begin{array}{l} \omega \mapsto \omega_{\text{hor}} \\ r_2^a \mapsto F_{\nabla}^a \end{array}} \Omega_{\text{dR}}^{\bullet}(B^H) \\
& & \searrow \quad \nearrow \text{Chern-Weil hom.} \\
& & \left( \mathbb{R} \left[ \{r_2^a\}_{a=1}^{\dim(T)} \right]^T \right) \xrightarrow{c \mapsto c(F_{\nabla})}
\end{array}$$

This is from the proper  $G$ -equivariant Borel  $T$ -equivariant smooth de Rham complex of  $X$  (Example 3.37) to the proper  $G$ -equivariant smooth de Rham complex over  $X/T$  (Example 3.33), which is stage-wise over  $G/H$  the *Cartan map* quasi-isomorphism [GS99, §5] (review in [Me06, (20), (30)]) from the Cartan model of  $X^H$  (106) to the ordinary smooth de Rham complex of  $B^H = (X/N)^H$ . This sends the Cartan model generators  $r_2^a$  to the curvature form component  $F_{\nabla}^a$  of the given connection, and hence restricts on universal real characteristic classes, represented by invariant polynomials  $c$ , to the Chern-Weil homomorphism assigning characteristic forms:  $c \mapsto c(F_{\nabla})$ .

**Example 3.70** (Tangential de Rham twists on  $G$ -orbifolds with  $T$ -structure). In further specialization of Example 3.69, let  $X \overset{\zeta}{\leftarrow} B \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35) be equipped with  $G$ -equivariant  $T \subset \text{GL}(\dim(X))$ -structure (see [SS20, p. 9] for pointers), namely with a  $G$ -equivariant reduction of its  $\text{GL}(\dim(X))$ -frame bundle to a  $T$ -principal  $T$ -frame bundle  $T\text{Fr}(X)$ :

$$\begin{array}{ccc}
T \times G & & T \times G \\
\downarrow & & \downarrow \\
T\text{Fr}(X) & \xrightarrow{\text{G-equivariant } T\text{-structure}} & \text{Fr}(X) \\
\downarrow & & \downarrow \\
X & & X \\
\downarrow & & \downarrow \\
G & & G
\end{array}$$

Then Example 3.69 induces on the  $G$ -orbifold  $\gamma(X//G)$  (Def. 2.36) an equivariant non-abelian de Rham twist (144) encoding all the real characteristic forms of the given  $G$ -equivariant  $T$ -structure on  $X$  (the *tangential twist*):

$$\begin{array}{ccc}
\Omega_{\text{dR}}^{\bullet} \left( (\gamma(T\text{Fr}(X)//G)) // T \right) & \xrightarrow{\text{Cartan map equivalence}} & \Omega_{\text{dR}}^{\bullet} (\gamma(X//G)) \\
\tau_{\text{dR}}^{\text{can}} \swarrow & \in W & \nearrow \text{cw}_T \\
\text{CE}(\mathfrak{l}BT) & & \\
\text{canonical de Rham twist on orbifold's } T\text{-frame bundle} & & \text{tangential de Rham twist on } G\text{-orbifold}
\end{array}$$

In further generalization of Def. 3.65, we set:

**Definition 3.71** (Proper  $G$ -equivariant & Borel  $T$ -equivariant twisted non-abelian de Rham cohomology). Let  $(T \times G) \overset{\zeta}{\leftarrow} X \in (T \times G)\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35) for  $T$  finite-dimensional, compact and simply-connected, and let

$$\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\text{hofib}(\mathfrak{p})} & \widehat{\mathfrak{b}} \\
& & \downarrow \mathfrak{p} \\
& & \mathfrak{l}BT
\end{array} \tag{151}$$

be an equivariant  $L_{\infty}$ -algebraic local coefficient bundle (133) over the Whitehead  $L_{\infty}$ -algebra of  $BT$  (i.e., whose Chevalley-Eilenberg algebra is (81)).

(i) We say that the set of *flat, canonically twisted, proper  $G$ -equivariant & Borel  $T$ -equivariant,  $\mathfrak{g}$ -valued differential forms* on  $X$  is the hom-set (16) in the co-slice of  $G$ -equivariant dgc-algebras (Def. 3.30) from  $\text{CE}(\mathfrak{p})$  (104) to the canonical de Rham twist (Def. 3.68) on the corresponding  $T$ -parametrized  $G$ -orbifold:

$$\begin{aligned}
\Omega_{\text{dR}}^{\tau_{\text{dR}}^{\text{can}}} \left( (\gamma(X//G))//T; \mathfrak{g} \right) &:= \left( (\mathcal{G}\text{DiffGradedCommAlgebras}_{\mathbb{R}}^{\geq 0})_{\text{proj}} \right)^{\text{CE}(\text{IBT})/} \left( \text{CE}(\mathfrak{p}), \tau_{\text{dR}}^{\text{can}} \right) \\
&= \left\{ \Omega_{\text{dR}}^{\bullet} \left( (\gamma(X//G))//T \right) \begin{array}{c} \xleftarrow{\tau_{\text{dR}}^{\text{can}}} \\ \xrightarrow{\text{CE}(\mathfrak{p})} \end{array} \text{CE}(\widehat{\mathfrak{b}}) \right\}. \quad (152)
\end{aligned}$$

flat canonically-twisted  
proper  $G$ -equivariant & Borel  $T$ -equivariant  
 $\mathfrak{g}$ -valued differential form

(ii) A *coboundary* between two such elements is defined, as in Def. 3.60, by a concordance form on the cylinder orbifold:

$$\tilde{A} \in \Omega_{\text{dR}}^{p_X^*(\tau_{\text{dR}}^{\text{can}})} \left( (\gamma((X \times \mathbb{R})//G))//T; \mathfrak{g} \right). \quad (153)$$

The corresponding twisted equivariant non-abelian de Rham cohomology is defined, as in Def. 3.65, to be the set of coboundary-classes of the elements in the set (152):

$$H_{\text{dR}}^{\tau_{\text{dR}}^{\text{can}}} \left( (\gamma(X//G))//T; \mathfrak{g} \right) := \Omega_{\text{dR}}^{\tau_{\text{dR}}^{\text{can}}} \left( (\gamma(X//G))//T; \mathfrak{g} \right) / \sim.$$

In Borel-equivariant generalization of [FSS23, Prop. 3.86], we have:

**Proposition 3.72** (Reproducing traditional Borel-Weil-Cartan equivariant de Rham cohomology). *For the case of trivial proper equivariance,  $G = 1$ , consider  $T \zeta X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35) and let the equivariant  $L_{\infty}$ -algebraic coefficient bundle (151) be the trivial bundle with fiber the line Lie  $n$ -algebra  $\mathfrak{b}^{n+1}\mathbb{R}$  ([FSS23, Ex. 3.27]). Then the canonically twisted proper  $G$ -equivariant & Borel  $T$ -equivariant non-abelian de Rham cohomology of  $X$  (Def. 3.71) reduces to the traditional Borel-Weil-Cartan equivariant de Rham cohomology (the cochain cohomology of the Cartan model complex (106)) in degree  $n$ :*

Borel-Weil-Cartan equivariant  
de Rham cohomology

$$H_{\text{dR},T}^n(X) \simeq H_{\text{dR}}(X//T; \mathfrak{b}^n\mathbb{R}).$$

*Proof.* From unravelling the definitions it is clear that, under the given assumptions, the defining set of cochains (152) reduces to the set of closed degree  $n$  elements in the Cartan model complex (106) on  $X = X^1$ . Hence, given any pair of such, it is sufficient to see that the coboundaries according to (153) exist precisely if a coboundary with respect to the Cartan model differential  $d_{\text{dR}} + r_2^a \wedge \iota_{t_a}$  exists.

In the case when the second summand  $r_2^a \wedge \iota_{t_a}$  vanishes, this is shown by the proof in [FSS23, Prop. 3.86], using the fiberwise Stokes theorem for fiber integration over  $[0, 1] \subset \mathbb{R}$ . Inspection shows that this proof generalizes verbatim in the presence of the second summand in the Cartan differential, using that this second summand evidently anti-commutes with the fiber integration operation:

$$r^a \wedge \iota_{t_a} \int_{[0,1]} \tilde{C} = - \int_{[0,1]} r^a \wedge \iota_{t_a} \tilde{C}. \quad \square$$

**Remark 3.73** (Localization in gauge theory). Prop. 3.72 means that the equivariant de Rham cohomology considered here subsumes the traditional Borel-equivariant de Rham cohomology that is used, for instance, in localization of gauge theories (see [Pes12][PZ+17]), and generalizes it to finite proper equivariance groups and to non-abelian coefficients.

In equivariant generalization of [FSS23, Ex. 3.96], we have:

**Example 3.74** (Flat equivariant twistorial differential forms). Consider the equivariant relative Whitehead  $L_{\infty}$ -algebra (126) of  $\mathbb{Z}_2$ -equivariant &  $\text{Sp}(1)$ -parametrized twistor space (76) (from Thm. 3.56) as an equivariant  $L_{\infty}$ -algebraic local coefficient bundle (133)

$$\begin{aligned}
\mathfrak{l}\gamma(\mathbb{C}P^3//\mathbb{Z}_2) &\longrightarrow \mathfrak{l}_{\text{BSp}(1)}(\gamma(\mathbb{C}P^3//\mathbb{Z}_2)//\text{Sp}(1)) \\
&\quad \downarrow \rho_{\gamma(\mathbb{C}P^3//\mathbb{Z}_2)} \\
&\quad \mathfrak{l}\text{BSp}(1) \quad (154)
\end{aligned}$$



Let  $X \in \mathbb{Z}_2\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35) be a spin 8-manifold with fixed locus (49) denoted

$$\gamma(X // \mathbb{Z}_2) : \begin{array}{ccc} \begin{array}{c} (\mathbb{Z}_2) \downarrow \\ \mathbb{Z}_2/1 \\ \downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 \end{array} & \longmapsto & \begin{array}{c} (\mathbb{Z}_2) \downarrow \\ X^{11} \\ \uparrow \\ X^{\mathbb{Z}_2} \end{array} \end{array} \quad (155)$$

and equipped with  $\mathbb{Z}_2$ -invariant  $\text{Sp}(1)$ -structure  $\tau$ , compatible  $\mathbb{Z}_2$ -invariant  $\text{Sp}(1)$ -connection  $\nabla \in \text{Sp}(1)\text{Connections}(X)$ , and corresponding tangential de Rham twist (Example 3.70)

$$\Omega_{\text{dR}}^\bullet(\gamma(X // \mathbb{Z}_2)) \xleftarrow{\tau_{\text{dR}}} \text{CE}(\text{IBSp}(1)).$$

$$\frac{1}{4}p_1(\nabla) \longleftarrow \frac{1}{4}p_1$$

Then the set of flat  $\tau_{\text{dR}}$ -twisted equivariant differential forms (Def. 135) with local coefficients in (154) is of the following form:

flat equivariant twistorial differential forms on  $\mathbb{Z}_2$ -orbifold  $X$

$$\Omega_{\text{dR}}^{\tau_{\text{dR}}}(\gamma(X // \mathbb{Z}_2); \iota\gamma(\mathbb{C}P^3 // \mathbb{Z}_2))_{\text{flat}} = \left\{ \begin{array}{l} \left. \begin{array}{l} H_3, \\ F_2, \\ 2G_7, \\ \tilde{G}_4 \end{array} \in \Omega_{\text{dR}}^\bullet(X^{11}) \right| \begin{array}{l} \text{twisted Bianchi identities in bulk } \mathbb{Z}_2\text{-orientifold} \\ d H_3 = \tilde{G}_4 - \frac{1}{2}p_1(\nabla) - F_2 \wedge F_2, \\ d F_2 = 0, \\ d 2G_7 = -\tilde{G}_4 \wedge (\tilde{G}_4 - \frac{1}{2}p_1(\nabla)) \\ d \tilde{G}_4 = 0, \end{array} \right. \left. \begin{array}{l} \text{restriction to } \mathbb{Z}_2\text{-fixed locus} \\ dH_3|_{X^{\mathbb{Z}_2}} = -\frac{1}{2}p_1(\nabla|_{X^{\mathbb{Z}_2}}) - F_2 \wedge F_2|_{X^{\mathbb{Z}_2}} \\ G_7|_{X^{\mathbb{Z}_2}} = 0, \\ \tilde{G}_4|_{X^{\mathbb{Z}_2}} = 0 \end{array} \right\}. \quad (156)$$

This follows as an immediate consequence of Prop. 3.56, according to which an element  $\mathcal{F}$  of this set of forms is a morphism of equivariant dgc-algebras of the following form

$$\mathcal{F} : \begin{array}{ccc} \begin{array}{c} (\mathbb{Z}_2) \downarrow \\ \mathbb{Z}_2/1 \\ \downarrow \\ \mathbb{Z}_2/\mathbb{Z}_2 \end{array} & \longmapsto & \begin{array}{c} \Omega_{\text{dR}}^\bullet(X) \\ \downarrow \alpha \\ \Omega_{\text{dR}}^\bullet(X^{\mathbb{Z}_2}) \end{array} \end{array} \xleftarrow{\begin{array}{l} H_3 \leftarrow h_3 \\ F_2 \leftarrow f_2 \\ 2G_7 \leftarrow \omega_7 \\ \tilde{G}_4 \leftarrow \tilde{\omega}_4 \end{array}} \text{CE}(\text{IBSp}(1)) \left[ \begin{array}{l} h_3, \\ f_2 \\ \omega_7, \\ \tilde{\omega}_4 \end{array} \right] / \left( \begin{array}{l} dh_3 = \tilde{\omega}_4 - \frac{1}{2}p_1 - f_2 \wedge f_2 \\ df_2 = 0 \\ d\omega_7 = -\tilde{\omega}_4 \wedge (\tilde{\omega}_4 - \frac{1}{2}p_1) \\ d\tilde{\omega}_4 = 0 \end{array} \right)$$

$$\xleftarrow{\quad} \text{CE}(\text{IBSp}(1)) \left[ \begin{array}{l} h_3, \\ f_2 \end{array} \right] / \left( \begin{array}{l} dh_3 = -\frac{1}{2}p_1 - f_2 \wedge f_2 \\ df_2 = 0 \end{array} \right). \quad (157)$$

### 3.4 Equivariant non-abelian character map

The Chern character in K-theory is just one special case of a plethora of character maps in a variety of flavors of generalized cohomology theories. As highlighted in [FSS23][SS25a], from the point of view of homotopy-theoretic non-abelian cohomology theory – where all cohomology classes are represented by (relative, parametrized) homotopy classes of maps into a classifying space (fibered, parametrized  $\infty$ -stack) – character maps are naturally realized as the non-abelian cohomology operations induced by *rationalization* of the classifying space (followed by a de Rham-Dold-type equivalence bringing the resulting rational cohomology theory into canonical shape).

Seen through the lens of Elmendorf’s theorem (Prop. 2.26), rationalization in proper equivariant homotopy theory (Def. 3.46) is stage-wise, on fixed loci, given by rationalization in non-equivariant homotopy theory. Consequently, the equivariant character maps are fixed loci-wise given by non-equivariant characters, hence are fixed loci-wise given by rationalization (followed by a de Rham equivalence).

For this reason, we will be brief here and refer to [FSS23] for background and further detail. We just make explicit now the concrete model of the equivariant non-abelian character map by means of the equivariant PL de Rham Quillen adjunction from Prop. 3.50. and then we discuss one example (in ??): the character map in equivariant twistorial Cohomotopy theory.

### The character map in equivariant non-abelian cohomology.

In equivariant generalization of [FSS23, Def. 4.1], we set:

**Definition 3.75** (Rationalization in equivariant non-abelian cohomology). Let  $\mathcal{A} \in \mathcal{G}\text{HomotopyTypes}_{\geq 2}^{\text{fin}\mathbb{R}}$  (Def. 2.33). Then we say that *rationalization in  $\mathcal{A}$ -cohomology* is the equivariant non-abelian cohomology operation (Def. 2.41) from  $\mathcal{A}$ -cohomology to real  $L_{\mathbb{R}}\mathcal{A}$ -cohomology which is induced (71) by the rationalization unit (121) on  $\mathcal{A}$ :

$$H(-; \mathcal{A}) \xrightarrow{(\eta_{\mathcal{A}}^{\mathbb{R}})_*} H(-; L_{\mathbb{R}}\mathcal{A}).$$

In an equivariant generalization of [FSS23, Def. 4.2], we set:

**Definition 3.76** (Equivariant non-abelian character map). Let  $G \curvearrowright X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35) and  $\underline{\mathfrak{g}}$  (Def. 3.36). Then the *equivariant non-abelian character map* on equivariant non-abelian  $\mathcal{A}$ -cohomology (Def. 2.37) over the orbifold  $\gamma(X // G)$  (Def. 2.36) is the composite of the rationalization cohomology operation (Def. 3.75) with the equivariant non-abelian de Rham theorem (Prop. 3.63) over the orbifold  $\gamma(X // G)$  (Def. 2.36)

$$\text{ch}_{\mathcal{A}}(X) : \underbrace{H(\gamma(X // G); \mathcal{A})}_{\substack{\text{equivariant non-abelian} \\ \mathcal{A}\text{-cohomology}}} \xrightarrow[\substack{\text{rationalization} \\ (\eta_{\mathcal{A}}^{\mathbb{R}})_*}]{} \underbrace{H(\gamma(X // G); L_{\mathbb{R}}\mathcal{A})}_{\substack{\text{equivariant non-abelian de Rham cohomology} \\ \text{with coefficient in equivariant Whitehead } L_{\infty}\text{-algebra}}} \xrightarrow[\simeq]{\substack{\text{equivariant non-abelian} \\ \text{de Rham theorem}}} H_{\text{dR}}(\gamma(X // G); L_{\mathbb{R}}\mathcal{A}). \quad (158)$$

### The character map in twisted equivariant non-abelian cohomology.

In equivariant generalization of [FSS23, Def. 5.2], we set:

**Definition 3.77** (Rationalization in twisted equivariant non-abelian cohomology). Let  $\rho_{\mathcal{A}}$  be an equivariant local coefficient bundle of equivariantly 1-connected  $G$ -spaces of finite  $\mathbb{R}$ -homotopy type, which admits an equivariant relative minimal model; all as in Notation 3.66. Then *rationalization in twisted equivariant non-abelian cohomology with local coefficients in  $\rho_{\mathcal{A}}$*  (Def. 2.45) is the equivariant non-abelian cohomology operation

$$(\eta_{\rho_{\mathcal{A}}}^{\mathbb{R}})_* : H^{\tau}(X; \mathcal{A}) \xrightarrow{(\mathbb{D}\eta_{\rho_{\mathcal{A}}}^{\text{PLdR}} \circ (-)) \circ \mathbb{L}(\eta_{B\mathcal{G}}^{\mathbb{R}})_!} H^{L_{\mathbb{R}}\tau}(X; L_{\mathbb{R}}\mathcal{A})$$

which is induced (as shown in [FSS23, (264)]) by the pasting composite with the naturality square on  $\rho_{\mathcal{A}}$  of the rationalization unit (Def. 3.46). By the fundamental theorem (Prop. 3.51), this means explicitly: the left derived base change (e.g. [FSS23, Ex. A.18]) along the PLdR-adjunction unit (Prop. 3.50) on  $B\mathcal{G}$  followed by composition with the following commuting square, regarded as a morphism in the slice over its bottom right object:

$$\mathbb{D}\eta_{\rho_{\mathcal{A}}}^{\mathbb{R}} := \left( \begin{array}{ccccc} & & \mathbb{D}\eta_{\mathcal{A} // \mathcal{G}}^{\text{PLdR}} \simeq \eta_{\mathcal{A} // \mathcal{G}}^{\mathbb{R}} & & \\ & \swarrow & & \searrow & \\ \mathcal{A} // \mathcal{G} & \xrightarrow{\eta_{\mathcal{A} // \mathcal{G}}^{\text{PLdR}}} & \exp \circ \Omega_{\text{PLdR}}^{\bullet}(\mathcal{A} // \mathcal{G}) & \xrightarrow{p_{\mathcal{A} // \mathcal{G}}^{\text{min}_{B\mathcal{G}}}} & \exp \circ \text{CE}(l_{B\mathcal{G}}(\mathcal{A} // \mathcal{G})) \\ \rho_{\mathcal{A}} \downarrow & & \exp \circ \Omega_{\text{PLdR}}^{\bullet}(\rho_{\mathcal{A}}) \downarrow & & \exp \circ \text{CE}(l_{\rho_{\mathcal{A}}}) \downarrow \\ B\mathcal{G} & \xrightarrow{\eta_{B\mathcal{G}}^{\text{PLdR}}} & \exp \circ \Omega_{\text{PLdR}}^{\bullet}(B\mathcal{G}) & \xrightarrow{p_{B\mathcal{G}}^{\text{min}}} & \exp \circ \text{CE}(l(B\mathcal{G})) \\ & \swarrow & & \searrow & \\ & & \mathbb{D}\eta_{B\mathcal{G}}^{\text{PLdR}} \simeq \eta_{B\mathcal{G}}^{\mathbb{R}} & & \end{array} \right).$$

Here the left-hand side is the naturality square of the equivariant PL de Rham adjunction (Prop. 3.50), while the right-hand side is the image under  $\exp$  of the relative minimal model (146). (Hence the composite represents the naturality square of the derived PL de Rham adjunction unit, see e.g. [FSS23, Ex. A.21]).

In equivariant generalization of [FSS23, Def. 5.4], we set:

**Definition 3.78** (Twisted equivariant non-abelian character map). Let  $G \curvearrowright X \in G\text{Actions}(\text{SmoothManifolds})$  (Def. 2.35), and let  $\rho_{\mathcal{A}}$  be an equivariant local coefficient bundle of equivariantly 1-connected  $G$ -spaces of finite  $\mathbb{R}$ -homotopy type, which admits an equivariant relative minimal model; all as in Notation 3.66. Then the *twisted*



**Table CG.** While cohomology has of course many and diverse applications, in physics no less than in other fields, the role of cohomology specifically in the *global* description of (higher) gauge fields (“force fields”) is profound: In a generalization of the seminal historical observation (“Dirac charge quantization”) that electromagnetic field configurations are globally to be identified with 2-cocycles in ordinary differential cohomology of spacetime, higher gauge field species are similarly to be identified with generalized cohomology theories whose further properties and attributes closely reflect the field’s physical nature, as indicated on the right.

| Cohomology    | Gauge fields          |
|---------------|-----------------------|
| -Theory       | Flux quantization law |
| Cocycle       | Field configuration   |
| Coboundary    | Gauge transformation  |
| Character     | Flux densities        |
| Ordinary-     | Electromagnetic       |
| Differential- | Gauge potentials      |
| Twisted-      | Background fields     |
| Equivariant-  | on orbifolds          |
| Real-         | on orientifolds       |
| Nonabelian-   | Nonlinear Gauß law    |

Conversely, this means that a fair amount of algebro-topological sophistication may be needed to propose or construct a cohomology theory suitable for flux quantization of a given higher gauge theory, and then to deduce its implications to be compared with physical expectations and, ultimately, with experiment. Much room is left here for applied algebraic topologists to get involved.

We briely indicate how the equivariant twistorial Cohomotopy from the main text is motivated as a flux-quantization law, and what some of its implications are:

**Characters arising in supergravity.** With the character map (9) describing which generalized cohomology theories  $\mathcal{A}$  may serve as flux-quantization laws for given generalized Gauß laws  $\mathcal{L}\mathcal{A}$  on flux densities, we have to ask: *What are natural generalized such Gauß laws  $\mathcal{L}\mathcal{A}$ ?* Remarkably, a profound source is *super-gravity*, in the following way (pointers in [GSS24a]):

It is a century-old observation due to É. Cartan that a field configuration of gravity is most usefully understood as a torsion-free *coframe field*  $E$  (Cartan’s “moving frame”) on spacetime with coefficients in the typical tangent space  $\mathbb{R}^{1,d}$  (Minkowski spacetime), subject to a corresponding “1st order”-formulation of Einstein’s equations. A miracle happens as this situation is generalized from ordinary tangent spaces to tangent super-spaces  $\mathbb{R}^{1,d|\mathbf{N}}$ , meaning to super-vector spaces (namely:  $\mathbb{Z}_2$ -graded vector spaces regarded with the unique non-trivial symmetric braided monoidal category structure) whose odd component carries the structure of a real spinor representation  $\mathbf{N} \in \text{Rep}_{\mathbb{R}}(\text{Spin}(1, d))$ :

– **11D Super-gravity.** Namely a field configuration of 11D super-gravity is a *supertorsion-free super-coframe field*  $(E, \Psi)$  on super-spacetime with coefficients in  $\mathbb{R}^{1,10|32}$ , where – remarkably – the corresponding Einstein-Rarita-Schwinger equations of motion are now *equivalent* [GSS24a, Thm. 3.1] simply to the statement that flux super-densities of the following form (meaning: super-differential forms whose local expansion in the co-frame field is of a prescribed form):<sup>7</sup>

$$\begin{aligned} G_4^s &\equiv (G_4)_{a_1 \dots a_4} E^{a_1} \dots E^{a_4} + \frac{1}{2} (\overline{\Psi} \Gamma_{a_1 a_2} \Psi) E^{a_1} E^{a_2}, \\ G_7^s &\equiv -(G_7)_{a_1 \dots a_7} E^{a_1} \dots E^{a_7} - \frac{1}{5!} (\overline{\Psi} \Gamma_{a_1 \dots a_5} \Psi) E^{a_1} \dots E^{a_5} \end{aligned} \quad (160)$$

satisfy the non-linear Bianchi/Gauß law encoded by the Whitehead  $L_\infty$ -algebra of the 4-sphere:

$$X \xrightarrow{(G_4^s, G_7^s)} \Omega^1(-; \mathbb{S}^4)_{\text{flat}} \Leftrightarrow \left\{ \begin{array}{l} dG_4^s = 0 \\ dG_7^s = -\frac{1}{2} G_4^s G_4^s \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \text{Equations of Motion} \\ \text{of 11D Supergravity} \\ \text{on supertorsion-free} \\ \text{super-coframe } (E, \Psi) \end{array} \right. \quad (161)$$

(In particular, the equations of motion include the Hodge duality relation  $G_7^s = -\star G_4^s$  over the underlying ordinary spacetime.)

Hence the non-linear Gauß law  $\mathbb{S}^4$  not only arises in but effectively *constitutes* 11D supergravity. But the miracle does not end here:

<sup>7</sup>In (160) we include a convetional sign in the definition of  $G_7^s$  to comply with the sign convention used in the main text.

– **M5-brane probes.** Given the above Cartan-geometric formulation of 11D super-gravity, all based on consideration of the Kleinian local model space  $\mathbb{R}^{1,d|\mathbb{N}}$ , it is natural to consider Kleinian sub-spaces and ask for their globalization to sub-supermanifolds of 11D spacetime. These are the “worldvolumes” of “probe super-branes”. Concretely, any Clifford algebra basis element  $\Gamma_{p+1\dots 10} \in \text{Pin}^+(1,d)$  which squares to  $+1$  (a “ $p$ -brane involution” [HSS18, §4.1]) corresponds to a projection operator on the Kleinian model space

$$P := \frac{1}{2}(\text{id} + \Gamma_{p+1\dots 10}) : \mathbb{R}^{1,d|\mathbb{N}} \longrightarrow \mathbb{R}^{1,p|\mathbb{N}/2} \hookrightarrow \mathbb{R}^{1,d|\mathbb{N}} \quad (162)$$

which projects out a sub-space  $\mathbb{R}^{1,p|\mathbb{N}/2}$  of half the odd dimensionl (jargon: “ $1/2$ BPS”). Thus we may ask for supermanifolds  $\Sigma^{1,p|\mathbb{N}/2}$  carrying such a  $1/2$ BPS-valued coframe field  $(e, \psi)$  and immersed into an ambient  $X^{1,d|\mathbb{N}}$  with coframe field  $(E, \Psi)$  such the inclusion relation (162) is suitably exhibited tangentspace-wise.

Such  $1/2$ BPS *super-immersions* ([GSS24b, Def. 2.19] essentially known in the literature as “super-embeddings”) exist in 11D supergravity in particular for  $p = 5$ , known as immersions of *probe M5-branes* into spacetime. Remarkably, the  $1/2$ BPS-immersion condition entails and is essentially implied by the existence of a 3-flux density super-form

$$H_3^s \equiv (H_3)_{a_1 a_2 a_3} e^{a_1} e^{a_2} e^{a_3}$$

on the M5’s worldvolume  $\Sigma^{1,5|2\mathbf{8}_+}$ , such that it is a coboundary for the pullback of the 4-flux to the worldvolume, and hence constitutes a lift to the Gauß law encoded by the quaternionic Hopf fibration  $I_{S^4} S^7$ :

$$\begin{array}{ccc} \Sigma^{1,5|2\mathbf{8}_+} & \xrightarrow{H_3^s} & \Omega_{\text{dR}}^1(-; I_{S^4} S^7)_{\text{flat}} \\ \downarrow \phi & & \downarrow I(\mathbb{H}\text{-Hopf fib.}) \\ X^{1,10|3\mathbf{2}} & \xrightarrow{(G_4^s, G_7^s)} & \Omega_{\text{dR}}^1(-; I_{S^4})_{\text{flat}} \end{array} \Leftrightarrow \left\{ \begin{array}{l} dH_3 = \phi^* G_4^s \\ dG_4 = 0 \\ dG_7 = -G_4^s G_4^s \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} 1/2\text{BPS immersion} \\ \text{of M5-worldvolume} \\ \text{in 11D SuGra solution} \end{array} \right. \quad (163)$$

This means that at this point, a valid flux quantization law for these fields is given by Cohmotopy: 4-Cohomotopy for the bulk C-field (as such proposed in [Sat13, §2.5] and developed in [FSS19a][GS21][GSS24a]), twisting 3-Cohomotopy (classified by the  $S^3$ -fiber of the quaternionic Hopf fibration) on the brane’s worldvolume (discussed in [FSS19b][FSS21][GSS24b]).

But here we take into account one more field:

– **Chern-Simons gauge field.** In view of this effective re-definition – of on-shell 11D supergravity with probe branes – in terms of (non-linear) Gauß laws for super-flux densities on super-space, we may go ahead and consider a further super-flux density

$$F_2^s \equiv (F_2)_{a_1 a_2} e^{a_1} e^{a_2}$$

on the M5-worldvolume, subjected to the Gauß law for an ordinary gauge field, but again imposed on super-space:

$$dF_2^s = 0.$$

Analysis of the super-components immediately shows that this is equivalent to the further equation of motion

$$F_2 = 0 \quad (164)$$

as befits a(n abelian) Chern-Simons gauge field.

While this equation of motion (164) means that such super-flux  $F_2^s$  is actually “rationally invisible”, under flux-quantization it may still contribute pure torsion-effects to the other higher gauge fields: Namely if we add — without changing the above equations of motion!, due to (164) — a summand of  $F_2^s F_2^s$  to the Gauß law for  $H_3^s$ , then (according to Thm. 1.1) it is no longer controlled by the quaternionic Hopf fibration but by the twistor fibration:

$$\begin{array}{ccc} \Sigma^{1,5|2\mathbf{8}_+} & \xrightarrow{(F_2^s, H_3^s)} & \Omega_{\text{dR}}^1(-; I_{S^4} \mathbb{C}P^3)_{\text{flat}} \\ \downarrow \phi & & \downarrow I(\text{twistor fib.})_* \\ X^{1,10|3\mathbf{2}} & \xrightarrow{(G_4^s, G_7^s)} & \Omega_{\text{dR}}^1(-; I_{S^4})_{\text{flat}} \end{array} \Leftrightarrow \left\{ \begin{array}{l} dF_2^s = 0 \\ dH_3 = \phi^* G_4^s - F_2^s F_2^s \\ dG_4 = 0 \\ dG_7 = -G_4^s G_4^s \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} 1/2\text{BPS immersion} \\ \text{of M5-worldvolume} \\ \text{with CS gauge field} \\ \text{in 11D SuGra solution} \end{array} \right. \quad (165)$$

As the notation already suggests, an admissible flux-quantization law admissible for this system of non-linear Gauß laws is the non-abelian cohomology theory whose classifying space is  $\mathbb{C}P^3$  (over  $S^4$ ), hence the “twistorial Cohomotopy” of [FSS20b][SS23a]. The character map on this cohomology theory is just what we develop in the main text, in twisted equivariant generalization, and we close by commenting on the consequences:

– **Tangential twisting and shifted integrality.** The higher gauge fields (flux densities) considered above are all defined on given (immersions of) super-spacetimes, and as such with respect to the *background field* of (super-gravity). According to the dictionary of *Table CG*, background fields manifest as *twisting* of the flux-quantizing cohomology theory. Since the topological charges of gravity are encoded in the frame bundle (or its associated tangent bundle) of spacetime, classified by a map  $X \xrightarrow{\text{Fr}_X} B\text{Spin}(1, 10)$ , we are looking for a corresponding “tangential” twisting of twistorial Cohomotopy. The subgroup of  $\text{Spin}(1, 10)$  that preserves the quaternionic Hopf fibration is  $\text{Sp}(2) \cdot \text{Sp}(1)$  [FSS19a, Prop. 2.20], and the subgroup that preserves the twistor fibration is still  $\text{Sp}(2)$  [FSS20b, Prop. 2.2], of which finally in the main text we consider the further subgroup  $\text{Sp}(1)$ , for definiteness.

The shifted flux quantization (10) of  $G_4$ , which is implied [FSS19a, Prop. 3.13] by this tangential twisting is thought to be [Wi97] a key aspect of the completion of 11D SuGra to “M-theory”.

– **Equivariance and anyonic solitons.** As indicated in *Table CG*, passage to equivariant cohomology on  $G$ -spaces corresponds to considering higher gauge fields on (super)  $G$ -orbifolds (cf. [SS19a][BSS19]). In the present context, an interesting situation are M5-branes wrapped on  $S^1$ -bundles over 2-dimensional orbifolds locally of the following form (cf. [GP17, p. 7]):

$$\Sigma^{1,5} \equiv \mathbb{R}^{1,0} \times \mathbb{R}_{\cup\{\infty\}}^2 \times S^1 \times \mathbb{R}^2, \quad \begin{matrix} \mathbb{Z}_2 \\ \downarrow \end{matrix}$$

where  $\mathbb{Z}_2$  acts on  $\mathbb{R}^2$  by point reflection, and where  $(-)\cup\{\infty\}$  denotes one-point compactification by adding a “point at infinity”, as suitable for measuring solitonic charges ([SS25a, §2.2]).

On worldvolume domains of this form, flux-quantization in equivariant twistorial Cohomotopy restricts on the orbi-singularity to flux-quantization in 2-Cohomotopy (76).

$$\mathcal{T}_{\mathbb{Z}_2} \left( \gamma(\Sigma^{1,5} // \mathbb{Z}_2) \right) = \left\{ \begin{array}{ccc} \begin{matrix} \mathbb{Z}_2 \\ \downarrow \\ \mathbb{Z}_2/1 \end{matrix} : & \mathbb{R}^{1,0} \times \mathbb{R}_{\cup\{\infty\}}^2 \times S^1 \times \mathbb{R}^2 & \dashrightarrow \mathbb{C}P^3 \\ & \uparrow & \uparrow \\ \mathbb{Z}_2/\mathbb{Z}_2 : & \mathbb{R}^{1,0} \times \mathbb{R}_{\cup\{\infty\}}^2 \times S^1 & \dashrightarrow S^2 \end{array} \right\} / \sim$$

**equivariant twistorial Cohomotopy  
of M5-brane worldvolume  
wrapped on Seifert-orbisingularity**

By a recent result [SS24b], this has the interesting consequence of implying that the corresponding solitonic field configurations have *anyonic quantum states* described by abelian Chern-Simons quantum observables. Such a derivation is of considerable interest in application to quantum materials and to quantum computation (cf. [SS23b]); several authors have argued for a similar conclusion on more informal grounds, following [CGK20]. We will further discuss this in a companion article [SS24c].

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