Higher geometric prequantum theory and The Brane Bouquet

Urs Schreiber

May 27, 2013

Talk at
Quantization, topological models, and generalized geometries
Bayrischzell, May 2013
http://hep.itp.tuwien.ac.at/~miw/bzell2013/
("The rain forest meeting")

based on

1. Domenico Fiorenza, Chris L. Rogers, Urs Schreiber,
   Higher geometric prequantum theory
   arXiv:1304.0236

2. Domenico Fiorenza, Hisham Sati, Urs Schreiber,
   Higher brane GS-WZW actions and Noether super Lie n-algebras from higher super-orbispaces
   http://ncatlab.org/schreiber/show/The+brane+bouquet

Contents

1 Motivation: The localized WZW σ-model

The WZW σ-model field theory describing a bosonic string on a simple Lie group G is all controled by the canonical Lie algebra 3-cocycle, which we may write as

$$\langle \theta, [\theta, \theta] \rangle : \; \mathfrak{g} \longrightarrow \mathbf{B}^2\mathbb{R} \; .$$

This σ-model famously has an affine Lie current algebra of Noether currents. This is the symmetry algebra of the transgression of the theory to loop space $[S^1, G]$:

$$\boxed{\text{affine Lie algebra}} \simeq \boxed{\text{Heisenberg Lie algebra of prequantum geometry on } [S^1, G]} .$$

This in turn is the infinitesimal approximation to the smooth group of global symmetries:

$$\boxed{\text{Kac-Moody loop group}} \simeq \boxed{\text{Heisenberg Lie group of prequantum geometry on } [S^1, G]}$$

(the geometric loop representation theory of Pressley-Segal).

But the WZW is a ***local* field theory**. It is not defined just on loop space. Its transgression to loop space loses information. Therefore we want to

- "**de-transgress**" or "**localize**" from $[S^1, G]$ to $[*, G] \simeq G$.

In [9] the following is made precise and proven (we come back to this below):

$$\boxed{\begin{array}{c}\text{String}\\ \text{Lie 2-algebra}\end{array}} \simeq \boxed{\begin{array}{c}\text{Noether current Lie}\\ \text{2-algebra of } L_{\mathrm{WZW}}\end{array}} \simeq \boxed{\begin{array}{c}\text{Heisenberg Lie 2-algebra of}\\ \text{prequantum 2-geometry } (G, \langle \theta \wedge [\theta \wedge \theta] \rangle)\end{array}}$$

Here the String Lie 2-algebra is the homotopy fiber of L_∞-algebras of the curvature of the WZW term:

$$\begin{array}{ccc} \mathfrak{string}(\mathfrak{g}) & \longrightarrow & 0 \\ {\scriptstyle \mathrm{hfib}}\downarrow & \swArrow & \downarrow \\ \mathfrak{g} & \xrightarrow[\langle \theta \wedge [\theta \wedge \theta] \rangle]{} & \mathbf{B}^2\mathbb{R} \; . \end{array}$$

The situation is even better for the corresponding smooth groups:

$$\boxed{\begin{array}{c}\text{smooth String}\\ \text{Lie 2-group}\end{array}} \simeq \boxed{\begin{array}{c}\text{exponentiated Noether current}\\ \text{smooth 2-group of WZW term}\end{array}} \simeq \boxed{\begin{array}{c}\text{Heisenberg Lie 2-group of}\\ \text{2-prequantum geometry on } G\end{array}}$$

Hence an interesting **question** is:

- How does this **generalize** to higher WZW-type field theories?
- What are **examples**?

2 Example I: Some super p-brane σ-models

A famous class of field theories of higher WZW type are the Green-Schwarz action functionals for **super-p-brane σ-models**. These are WZW-type models induced by the exceptional invariant super Lie algebra cocycles on the super translation Lie algebra, hence on super-Minkowski spacetime:

$$\underset{\text{super-spacetime}}{\mathbb{R}^{d;N=1}} \xrightarrow[\text{higher background field}]{\langle \Psi \wedge [E^p \wedge \Psi] \rangle} \mathbf{B}^p\mathbb{R} \; .$$

These cocycles have been **classified** in the "**old brane scan**" [1] (see [6] for an introduction with an eye towards the L_∞-perspective below, and see [4] for a comprehensive classification):

d	$p=0$	1	2	3	4	5	6	7	8	9
11			(1) $\mathfrak{m2brane}$							
10		(1) $\mathfrak{string}_{\mathrm{het}}$				(1) $\mathfrak{ns5brane}_{\mathrm{het}}$				
9					(1)					
8				(1)						
7			(1)							
6		(1) $\mathfrak{littlestring}_{\mathrm{het}}$		(1)						
5			(1)							
4		(1)	(1)							
3		(1)								

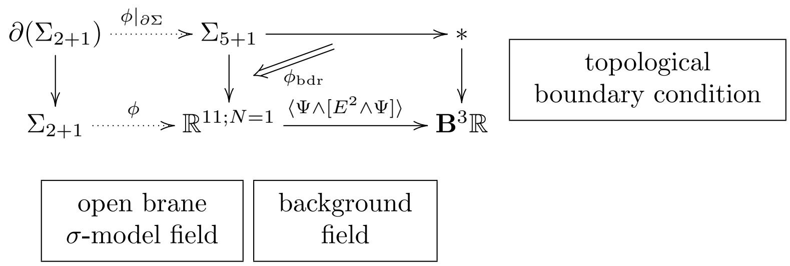
But the old brane scan is still missing many branes, for instance the M5-brane.

Where are the missing branes? They have been proposed and built by hand [3]...

...but can we **discover them as local higher WZW models**?

3 Boundary field theory and L_∞-algebra extensions

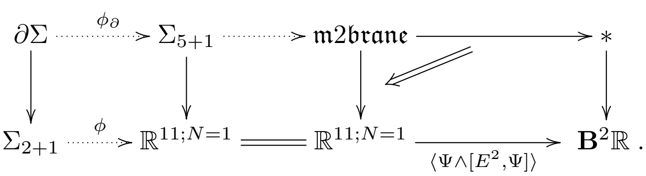
By the rules of **prequantum boundary field theory** [13] a **boundary condition for an open brane** involves a trivialization/gauging-away of its gauge coupling term on the boundary, for instance for the 3d σ-model of the M2-brane:[1]



By the universal property of the **homotopy pullback** of super L_∞-algebras, this means, that the map $\Sigma_{5+1} \to \mathbb{R}^{11|N=1}$ equivalently factors through the **homotopy fiber** super L_∞-algebras

$$\mathfrak{m2brane} := \mathrm{hfib}(\langle \Psi \wedge E^2 \wedge \Psi \rangle)$$

so that we have a factorization



Consequently:

- the M5-brane itself is a σ-model not on super-spacetime itself, but on a **higher extension super Lie 3-algebra** $\mathfrak{m2brane}$ of spacetime.

Under higher Lie integration [14] this is a higher analog of a super-orbifold:

- $\exp(\mathfrak{m2brane})$: a **higher super-orbispace** (a super-∞-stack) extension of super-spacetime.

One checks that this reproduces the proposals [3] in the literature...
and refines them as follows...

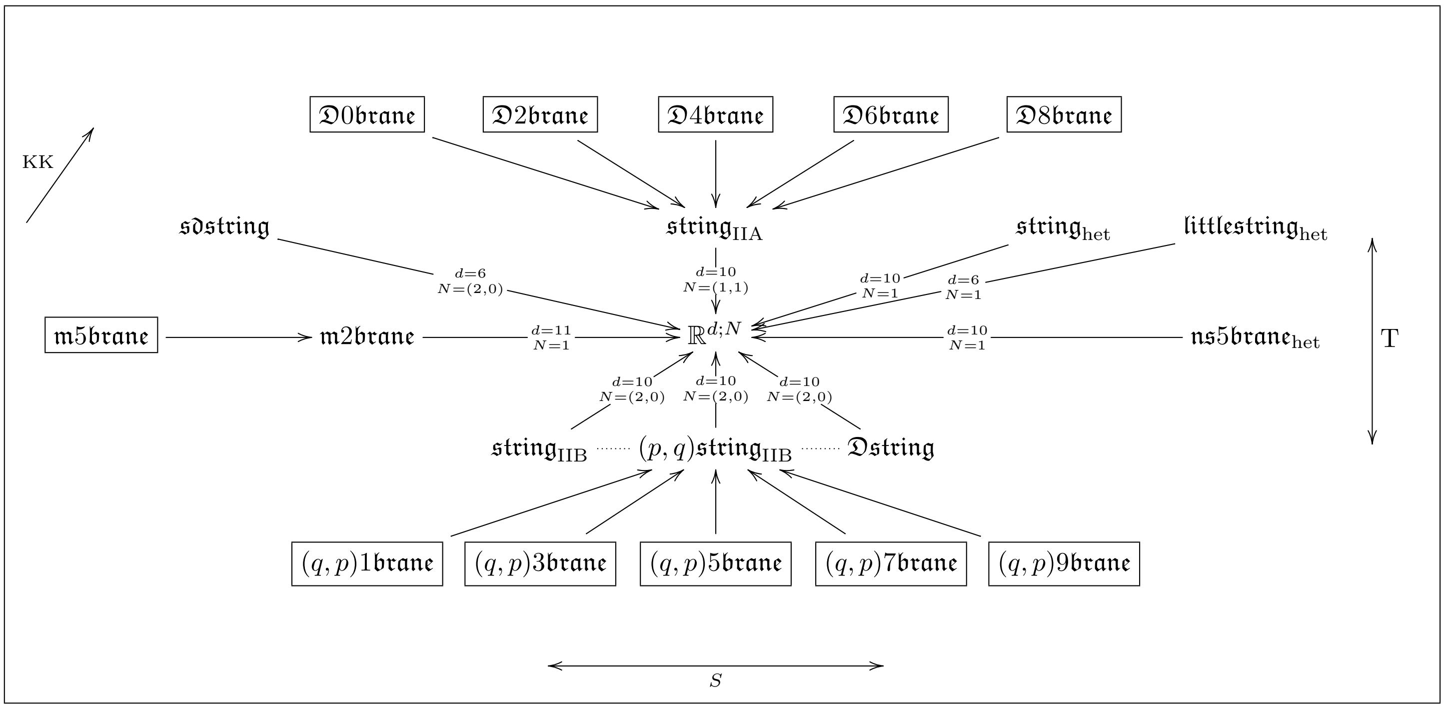
[1] Here the maps on the left are displayed by dotted arrows because strictly speaking they live in a different category, for ease of exposition. This is resolved by the formulation below in sections 5 and 6.

4 Example II: The Brane Bouquet

If we include such higher super-orbispace target spaces, then we find the following refinement of the "old brane scan"

D =	$p=0$	1	2	3	4	5	6	7	8	9
11			(1) m2brane			(1) m5brane				
10	(1,1) D0brane	(1) string$_{\text{het}}$ (1,1) string$_{\text{IIA}}$ (2,0) string$_{\text{IIB}}$ (2,0) D1brane	(1,1) D2brane	(2,0) D3brane	(1,1) D4brane	(1) ns5brane$_{\text{het}}$ (1,1) ns5brane$_{\text{IIA}}$ (2,0) ns5brane$_{\text{IIB}}$ (2,0) D5brane	(1,1) D6brane	(2,0) D7brane	(1,1) D8brane	(2,0) D9brane
9					(1)					
8				(1)						
7			(1)							
6		(1) littlestring$_{\text{het}}$ (2,0) sdstring		(1)						
5			(1)							
4		(1)	(1)							
3		(1)								

Moreover, the boundary conditions/brane intersection laws are expressed by the following diagram:
The brane bouquet.



Here each object denotes a super Lie $(p+1)$-algebra on $\mathbb{R}^{d;N}$ with (d,p,N) as in the above table, and a morphism from a p_1-brane to a p_2-brane L_∞-algebra denotes an extension of the latter by a degree-$(p_1 - p_2 + 1)$ super L_∞-cocycle.
Proof. After translation of supergravity theorist's "FDA"-notation to homotopy theory of super-L_∞-algebras as in [16, 10] this follows with section 3 of [2] for the M2/M5-brane, section 6 of [5] for the type IIA branes, section 2 of [7] for the type IIB branes, and section 6 of [4] for the self-dual string in $d=6$, $N=(2,0)$. □

Remark. This is reminiscent of a famous cartoon of "M-theory" (figure 4 in [8]), but the above diagram is a theorem in super L_∞-algebra cohomology theory.

The WZW models induced by this diagram reproduce the super-p-brane actions and their intersection laws (brane-on-brane laws) as known in string theory.

5 Higher geometric prequantum theory

To handle these examples we need a **local higher WZW-type field theory** for WZW terms **on higher super-orbispaces**.

First, to describe smooth higher supergeometry we need to pair $\boxed{\text{geometry}} + \boxed{\text{homotopy theory}}$: we say that

1. a **higher super-orbispace** (super ∞-stack) is a functor $X : \text{SuperMfd}^{\text{op}} \to \text{Top}$;
2. the homotopy theory SmoothSuper∞Grpd of higher super-orbispaces is the result of universally turning stalkwise weak homotopy equivalences ("lwhe") between such functors into actual homotopy equivalences

$$\boxed{\text{SmoothSuper}\infty\text{Grpd} := L_{\text{lwhe}}\text{Func}(\text{SmoothMfd}^{\text{op}}, \text{Top})}\,.$$

This geometric homotopy theory is "cohesive" – section 4.6 of [17] – which in particular implies the following. For every **higher super group** G (super group ∞-stack) there is

the coefficient object which modulates higher G**-principal bundles**	$\mathbf{B}G$
the coefficient object which modulates **flat** G**-principal higher connections**	$\flat\mathbf{B}G$
the coefficient object which modulates **flat** G**-valued differential forms**	$\flat_{\text{dR}}\mathbf{B}G$
the **higher Maurer-Cartan form**	$G \xrightarrow{\theta_G} \flat_{\text{dR}}\mathbf{B}G$
if $G = \mathbb{G}$ is abelian (braided), then a differential coefficient object which modulates $\mathbb{G}$**-principal higher connections** (with curvature)	$\mathbf{B}\mathbb{G}_{\text{conn}}$

So for $X \in \text{SmoothSuper}\infty\text{Grpd}$ any higher super-orbispace (super ∞-stack), a map

$$\nabla : X \to \mathbf{B}^n U(1)_{\text{conn}}$$

is equivalently

- a circle n-bundle with n-form connection on X with furvature F_∇ [14];
- a higher prequantization of the pre-n-plectic form F_∇ [9];
- a local Lagrangian/action functional for an n-dimensional local prequantum field theory with moduli stack of fields given by X [12, 13].

In the last interpretation, the σ-model induced by ∇ is the local prequantum field theory which to a closed oriented manifold Σ_k assigns the $(n-k)$-bundle with $(n-k)$-connection which is the **transgression** of ∇ to the space $[\Sigma_k, X]$ of fields on Σ [11, 12]:

$$\exp(2\pi i \int_{\Sigma_k} [\Sigma_k, \nabla]) : \; [\Sigma_k, X] \xrightarrow{[\Sigma_k,\nabla]} [\Sigma_k, \mathbf{B}^n U(1)_{\text{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_k}(-))} \mathbf{B}^{n-k}U(1)_{\text{conn}} \; .$$

For for $k = n$ we have $\mathbf{B}^0 U(1)_{\text{conn}} = U(1)$ and so in codimension 0 this is the **action functional**. In codimension 1 it is the **prequantum bundle**.

6 ∞-WZW models

By [14] we may **Lie integrate** each super Lie $(p+1)$-cocycle in the *brane bouquet* to their higher smooth orbispaces

$$\mathbf{c} : \ \mathbf{B}G \xrightarrow{\exp(\langle \Psi\wedge [E^p \wedge \Psi]\rangle)} \mathbf{B}^{p+1}U(1) \ .$$

Now we obtain the corresponding **∞-WZW term** (the "∞-WZW gerbe") $\mathbf{L}_{\mathrm{WZW}}$ by a universal construction in SmoothSuper∞Grpd as follows:

$$\begin{array}{ccccc} \tilde G & \xrightarrow{\tilde\theta_G} & \Omega_{\mathrm{flat}}(-,\mathfrak{g}) & \xrightarrow{\mathrm{CS}_{\mathbf{c}}} & \Omega^{n+1}_{\mathrm{cl}} \\ \downarrow & & \downarrow & & \downarrow \\ G & \xrightarrow{\theta_G} & \flat_{\mathrm{dR}}\mathbf{B}G & \xrightarrow{\flat_{\mathrm{dR}}\mathbf{c}} & \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \flat\mathbf{B}G & \xrightarrow{\flat\mathbf{c}} & \flat\mathbf{B}^{n+1}U(1) \end{array} \quad \simeq \quad \begin{array}{ccccc} \tilde G & \overset{\mathbf{L}_{\mathrm{WZW}}}{\dashrightarrow} & \mathbf{B}^nU(1)_{\mathrm{conn}} & \xrightarrow{F_{(-)}} & \Omega^{n+1}_{\mathrm{cl}} \\ \downarrow & & \downarrow & & \downarrow \\ G & \xrightarrow{\Omega\mathbf{c}} & \mathbf{B}^nU(1) & \xrightarrow{\theta_{\mathbf{B}^nU(1)}} & \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1) \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & * & \longrightarrow & \flat\mathbf{B}^{n+1}U(1) \end{array} ,$$

where the squares on the far right and far left are homotopy pullback squares.

This means:

- $\mathbf{L}_{\mathrm{WZW}}$ is the Lagrangian of a local σ-model prequantum field theory as above;
- defined on a higher super-orbispace $\tilde G$ which is a differential extension of the higher super group G;
- such that its curvature is the original super L_∞-cocycle, regarded as a left-invariant form on the super ∞-group;
- such that its integral class is the above integral lift of this cocycle.

Together this identifies $\mathbf{L}_{\mathrm{WZW}}$ as a higher analog of the "WZW gerbe", an n-connection whose local n-connection form is a WZW potential for the given cocycle.

Remark. That $\tilde G$ is a differential extension of G means that a σ-model on $\tilde G$ has fields which are multiplets consisting of maps from the worldvolume to G and of differential forms on the worldvolume. Hence $\tilde G$ is the target super orbispace for **tensor multiplets** on branes (notably the DBI 1-forms on the D-branes and the 2-form multiplet on the M5-brane).

With a general higher geometric prequantum theory and a general construction of higher WZW terms in hand, we can now

- formulate their higher prequantum geometry;
- formulate and compute their higher Hesenberg/Noether current Lie n-algebras and the corresponding super n-groups.

7 Application: Higher Noether current super L_∞-algebras

For gauge-coupling terms in higher prequantum geometry the fully localized version of the prequantum bundle coincides with the local action functional

$$\boxed{\text{fully localized higher prequantum bundle}} \simeq \boxed{\text{local action functional}}$$

namely the connection $(p+1)$-form

$$\mathbf{L}_{\text{WZW}} : G \to \mathbf{B}^n U(1)_{\text{conn}}$$

is the Lagrangian form and hence the transgression to codimension zero [12]is the **∞-WZW action functional** [12]

$$\exp(iS_{\text{WZW}}) = \exp(2\pi i \int_{\Sigma_{p+1}} [\Sigma_{p+1}, \mathbf{L}_{\text{WZW}}]) : \; [\Sigma_{\text{p+1}}, G] \xrightarrow{[\Sigma_{p+1}, \mathbf{L}_{\text{WZW}}]} [\Sigma_{p+1}, \mathbf{B}^n U(1)_{\text{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_{p+1}}(-))} U(1) \;.$$

Using this one can observe that

$$\boxed{\text{higher quantomorphism}} \simeq \boxed{\text{higher Noether current}}$$

Because a higher quantomorphism is [9] a transformation of the form

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{\;\simeq\;} & \tilde{G} \\ & \overset{\simeq}{\swarrow}_{\alpha} & \\ {}_{\mathbf{L}_{\text{WZW}}}\searrow & & \swarrow_{\mathbf{L}_{\text{WZW}}} \\ & \mathbf{B}^n U(1)_{\text{conn}} & \end{array}$$

and infinitesimally and locally this is

$$\mathcal{L}_{\delta\phi} L_{\text{WZW}} = d\alpha \;,$$

where $\mathcal{L}$ is the Lie derivative, given as $\mathcal{L} = d\iota + \iota d$. Hence

$$\iota_v \langle \theta \wedge \cdots \theta \rangle = d\left(\iota_{\delta\phi} L_{\text{WZW}} - \alpha\right) \;.$$

The term on the left vanishes on shell (here gauge coupling sector only) and so $J_\phi := \iota_{\delta\phi} L_{\text{WZW}} - \alpha$ is a **conserved p-form Noether current**. This gives us the corresponding **super-Lie $(p+1)$-group of exponentiated currents**

$$\mathbf{Noeth}(\mathbf{L}_{\text{WZW}}) \simeq \mathbf{Heis}(\mathbf{L}_{\text{WZW}}) \simeq \left\{ \begin{array}{ccc} \tilde{G} & \xrightarrow{\;\simeq\;} & \tilde{G} \\ & \swarrow_{\alpha} & \\ {}_{\mathbf{L}_{\text{WZW}}}\searrow & & \swarrow_{\mathbf{L}_{\text{WZW}}} \\ & \mathbf{B}^{p+1} U(1)_{\text{conn}} & \end{array} \right\} .$$

In [9] is proven that:

Theorem. For each ∞-WZW model $\mathbf{L}_{\text{WZW}}$, there is a homotopy fiber sequence of higher super-groups

$$\mathbf{B}^p U(1) \longrightarrow \mathbf{Noeth}(\mathbf{L}_{\text{WZW}}) \longrightarrow \tilde{G} \;.$$

which differentiates to an extension of the super L_∞-algebra $\mathfrak{g}$ by $\mathbf{B}^p\mathbb{R}$:[2]

$$\mathbf{B}^p\mathbb{R} \longrightarrow \mathfrak{Noether}(\mathbf{L}_{\text{WZW}}) \longrightarrow \mathfrak{g} \;.$$

For the ordinary WZW model this reproduces the String(G)-extension that motivated us back on p. 2. For the M2/M5 brane system this yields the integrated M-theory super Lie algebra and more...

[2]More details on this in the companion talk by Domenico Fiorenza.

References

[1] A. Achucarro, J. M. Evans, P. Townsend, and D. L. Wiltshire, *Super p-branes*, Phys. Lett. **B 198** (1987), 441–446.

[2] R. D'Auria and P- Fré, *Geometric supergravity in $d = 11$ and its hidden supergroup*, Nucl.Phys. **B201** (1982), 101–140; Erratum-ibid. **B206** (1982), 496.

[3] I. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. Sorokin and M. Tonin, *Covariant action for the super-fivebrane of M-theory*, Phys. Rev. Lett. **78** (1997), 4332, hep-th/9701149

[4] F. Brandt, *Supersymmetry and Lie algebra cohomology IV*, J. Math. Phys. **54** (2013), arXiv:1303.6211

[5] C. Chryssomalakos, J. A. de Azcárraga, J. M. Izquierdo, and J. C. Peŕez Bueno *The geometry of branes and extended superspaces*, Nucl. Phys. **B 567**(2000), 293–330, arXiv:hep-th/9904137

[6] J. Huerta, *Division Algebras, Supersymmetry and Higher Gauge Theory* PhD thesis (2012) arXiv:1106.3385

[7] M. Sakaguchi, *IIB-Branes and new spacetime superalgebras*, JHEP **0004** (2000), 019, arXiv:hep-th/9909143

[8] E. Witten, *Magic, Mystery and Matrix*, Notices of the AMS, volume 45, number 9 (1998) `http://www.sns.ias.edu/~witten/papers/mmm.pdf`

[9] D. Fiorenza, C. Rogers, U. Schreiber, *Higher geometric prequantum theory* arXiv:1304.0236

[10] D. Fiorenza, C. Rogers, U. Schreiber, *L_∞-algebras of local observables from higher prequantum bundles*, arXiv:1304.6292

[11] D. Fiorenza, H. Sati, U. Schreiber, *Extended higher cup-product Chern-Simons theories*, to appear in Journal of Geometry and Physics, arXiv:1207.5449

[12] D. Fiorenza, H. Sati, U. Schreiber, *A higher stacky perspective on Chern-Simons theory*, to appear in D. Calacque et al. (eds.) *New mathematical aspects of quantum field theory*, arXiv:1301.2580

[13] D. Fiorenza, U. Schreiber, et al. *Higher Chern-Simons local prequantum field theory*, `http://ncatlab.org/schreiber/show/Higher+Chern-Simons+local+prequantum+field+theory`

[14] D. Fiorenza, U. Schreiber, J. Stasheff, Čech cocycles for differential characteristic classes, Adv. Theor. Math. Phys. 16 (2012) 149-250 arXiv:1011.4735

[15] T. Nikolaus, D. Stevenson, U. Schreiber, *Principal ∞-bundles, I: General theory*, arXiv:1207.0248

[16] H. Sati, U. Schreiber, J. Stasheff *L_∞-connections* Recent Developments in QFT, Birkhuser (2009), arXiv:0801.3480

[17] U. Schreiber, *Differential cohomology in a cohesive ∞-topos* extended Habilitation thesis `http://ncatlab.org/schreiber/show/differential+cohomology+in+a+cohesive+topos`