Local pre-quantum field theory

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Abstract

We study the local ("extended", "multi-tiered") refinement of topological *pre*-quantum field theory, hence of the differential geometric data of moduli stacks of physical fields equipped with Lagrangians/action functionals, whose (path integral) quantization is supposed to yield genuine local topological quantum field theory. We observe how such local topological prequantum field theory with boundaries (branes) and with defects (domain walls) are encoded, via the cobordism theorem, by a collection of higher correspondences in an ambient ∞-topos of moduli stacks of fields, sliced over the "space of higher quantum phases". We show that if the ambient ∞-topos is *differentially cohesive* then there exist canonical families of higher prequantum field theories of universal higher topological Yang-Mills type, whose codimension-1 boundaries are the higher Chern-Simons-type prequantum field theories, whose codimension-2 corners support higher Wess-Zumino-Witten type prequantum field theories, and whose codimension-3 defects contain Wilson loop/Wilson surface field theories. This provides a systematic way of determining and organizing the boundary, corner and defect structure of higher Chern-Simons type field theories and their holographically related theories on geometric (Lagrangian) data, before genuine quantization. Further examples we find naturally from the axioms of higher local prequantum field theories are the super p-branes of string/M-theory realized as higher WZW-type prequantum theories, including their tensor multiplet worldvolume fields, their intersection laws and their anomaly cancellation conditions. For instance we find on the boundary of the M2-brane ending on a Hořava-Witten O9-plane the heterotic string as the boundary condition which is precisely the twisted String-structure of the Green-Schwarz anomaly cancellation. We close with an outlook on how this higher local prequantum data is quantized by push-forward in naturally associated generalized cohomology theories. These examples and their cohomological quantization are discussed in full detail in companion articles.

Contents

1 Introduction

The quantum field theories (QFTs) of interest, both in nature as well as theoretically, are typically not generic examples of the axioms of quantum field theory (see [5] for a survey of modern formalizations of QFT) but rather are special in two respects:

1. they arise from geometric data – the Lagrangian and action functional – via some process of quantization, and notably from "higher geometric data" (mainly higher gauge form fields), which involves gauge transformations, and higher gauge of gauge transformations;

2. they are *local* in that the spaces of configurations (states) which they assign to a piece of worldvolume/spacetime are determined from gluing the data assigned to pieces of any decomposition of the worldvolume/spacetime.

Under some conditions that are often, but not necessarily, referred to as *topological* quantum field theories TQFTs [**?**], then the above prescriptions can be systematically encoded into the quantization process that goes by the name of *geometric quantization*. Namely, in a n-dimensional TQFT, one has basically two assignments:

- to a (closed compact oriented) n-dimensional smooth manifold Σ_n is associated a complex number $Z(\Sigma_n)$, the *partition function* of the theory;

- to a (closed compact oriented) $(n-1)$-dimensional smooth manifold Σ_{n-1} is associated a Hilbert space $\mathcal{H}_{\Sigma_{n-1}}$, the space of *states* of the theory.

In geometric quantization, both of these processes [n-manifolds $\to \mathbb{C}$] and [$(n-1)$-manifolds $\to \mathbb{C}$-vector spaces] are two-step processes. More precisely, the partition function $Z(\Sigma_n)$ arises schematically as the integral

$$Z(\Sigma_n) = \int_{\phi \in \mathrm{Fields}(\Sigma_n)} \mathcal{D}\phi \; e^{2\pi i S_{\Sigma_n}(\phi)} \, ,$$

where $\mathrm{Fields}(\Sigma_n)$ is the space of field configurations on Σ_n and $S_{\Sigma_n} : \mathrm{Fields}(\Sigma_n) \to \mathbb{R}$ is the action functional. Notice that in order for the above integral to be (at least formally) defined one actually only needs that exponentiated action functional

$$\exp\left(2\pi i S_{\Sigma_n}\right) : \mathrm{Fields}(\Sigma_n) \to U(1)$$

to be defined and not necessarily its lift through the covering map $\mathbb{R} \to U(1)$. Similarly, the Hilbert space $\mathcal{H}_{\Sigma_{n-1}}$ is obtained as

$$\mathcal{H}_{\Sigma_{n-1}} = H^0(\mathrm{Fields}(\Sigma_{n-1}), L_{\Sigma_{n-1}}) \, ,$$

the space of "polarized" (e.g. holomorphic with respect to a given complex structure) sections of a complex line bundle with connection $(L_{\Sigma_{n-1}}, \nabla)$ on the space of field configurations on Σ_{n-1}. This is the prequantum line bundle of the theory, whose curvature is the symplectic 2-form on the space of field configurations on Σ_{n-1}. Since the datum of a complex line bundle (with connection) is completely equivalent to the datum of a principal $U(1)$-bundles with connection, we see that the geometric datum behind $\mathcal{H}_{\Sigma_{n-1}}$ is that of a morphism

$$\mathrm{Fields}(\Sigma_{n-1}) \to \mathbf{B}U(1)_{\mathrm{conn}}$$

from the space of field configurations on Σ_{n-1} to the stack of principal $U(1)$-bundle with connections. This way the following pattern emerges:

$$Z: \quad \{\Sigma_n\} \xrightarrow{\text{prequantization}} \left\{e^{2\pi i S_{\Sigma_n}} : \mathrm{Fields}(\Sigma_n) \to U(1)\right\} \xrightarrow{\quad \int \mathcal{D}\phi\,(-) \quad} \mathbb{C}$$

$$\mathcal{H}: \quad \{\Sigma_{n-1}\} \xrightarrow{\text{prequantization}} \{\nabla : \mathrm{Fields}(\Sigma_{n-1}) \to \mathbf{B}U(1)_{\mathrm{conn}}\} \xrightarrow{H^0} \mathbb{C}\text{-vector spaces} \, .$$

Here the second step in Z, the *path integral* $\int D\phi(-)$, is conceptually subtle and often taken as a guiding heuristic more than a precise definition.[1] On the other hand, the first step is a completely rigorous procedure and one can think of it as considering topological field theories before quantization. Therefore, following the established term "prequantum bundle" here we will refer to these field theories as *prequantum field theories*.

A little reflection shows that a prequantum field theory as defined above only approximatively matches the requirements for "field theories of geometric origin" we sketched at the beginning.

First, the spaces of field configurations only keep a vague memory of the gauge and gauge-of-gauge transformations of the theory: the gauge invariance of the action functional. One would instead like to have a space of field configurations also containing all the information about the gauge fields, and to have the invariance of the action functional as a byproduct. This is achieved by considering a (higher) *stack* (geometric groupoid) **Fields** of fields and replacing the space $\mathrm{Fields}(\Sigma)$ of field configurations with the moduli stack $[\Sigma, \mathbf{Fields}]$ of maps from Σ to the stack of fields.

The second problem is that prequantum filed theories as considered above are not local: the data for codimension-1 manifolds Σ_{n-1} does not come from deeper data for codimension-2 manifolds.[2] Imposing locality then amounts to requiring that all the data of the n-dimensional theory can be reconstructed by the data for codimension-n manifolds, hence for collections of just points. To continue the pattern that we have seen emerging in codimension-0 and 1, one sees that the natural codimension-k datum for a n-dimensional prequantum theory is that of a morphism of stacks

$$[\Sigma_{n-k}, \mathbf{Fields}] \to \mathbf{B}^{n-k}U(1)_{\mathrm{conn}} \;,$$

where on the right we have the $(n-k)$-stack of $U(1)$-$(n-k)$-bundles with connection (bundle $(n-k-1)$-gerbes with connection). Going down to codimension n and observing that if $*$ denotes the 1-point manifold then $[*, \mathbf{Fields}] \cong \mathbf{Fields}$, we see that imposing locality on a prequantum theory means that the whole theory, in any codimension, is determined by a single datum: a morphism of higher stacks of the form

$$\mathbf{L} : \mathbf{Fields} \to \mathbf{B}^n U(1)_{\mathrm{conn}} \,.$$

Notice that such an n-connection on the moduli stack of fields is at least locally given by a plain differential n-form. Moreover, this being an n-form on a stack means that for each test manifold Σ this is an n-form (locally) on Σ, depending on the field configurations on Σ. Such a form is familiar in and central to traditional prequantum field theory: it is the *Lagrangian* of the theory; whence the choice of symbol "$\mathbf{L}$".

And indeed, once such an $\mathbf{L}$ is given, all the codimension-k prequantum $(n-k)$-$U(1)$-bundles with connections on the moduli stacks $[\Sigma_{n-k}, \mathbf{Fields}]$ are naturally obtained by transgression of n-bundles (fiber integration/push-forward on cocycles in differential cohomology):

$$[\Sigma_{n-k}, \mathbf{Fields}] \xrightarrow{[\Sigma_{n-k}, \mathbf{L}]} [\Sigma_{n-k}, \mathbf{B}^n U(1)_{\mathrm{conn}}] \xrightarrow{\exp 2\pi i \int_{\Sigma_{n-k}}} \mathbf{B}^{n-k}U(1)_{\mathrm{conn}} \,.$$

The rightmost map here is fiber integration in Deligne cohomology, seen as morphism of smooth stacks, see [GT, FSS13]. In particular, for $k=0$ one recovers the action functional as

$$\exp(2\pi i S_{\Sigma_n}) = \exp(2\pi i \int_{\Sigma_n} [\Sigma_n, \mathbf{L}]) : [\Sigma_n, \mathbf{Fields}] \to \mathbf{B}^0 U(1)_{\mathrm{conn}} \simeq U(1) \,.$$

The curvature morphisms

$$\mathrm{curv} : \mathbf{B}^{n-k}U(1)_{\mathrm{conn}} \to \Omega^{n-k+1}_{\mathrm{cl}}$$

endow the moduli spaces of field configurations with canonical closed degree $n-k+1$ differential forms. For $k=1$ this is the traditional (pre-)symplectic structure on $[\Sigma_{n-1}, \mathbf{Fields}]$, so the "local prequantization" can be seen as a *de-transgression* of this presymplectic structure to a pre-n-plectic structure on the stack of fields.

[1] We will briefly come back to this in Section 4.5.

[2] In physics terminology essentially an instance of this issue is referred to as *non-covariance of canonical quantization*, referring to the explicit and non-natural choice of $(n-1)$-dimensional spatial slices Σ_{n-1} of spacetime.

In particular, an n-dimensional prequantum field theory comes with an $(n+1)$-dimensional *symplectic field theory*, an idea which has long been proposed and investigated in *multisymplectic geometry*; see [FRS13a] for a recent survey. More precisely, the relation between the prequantum symplectic theory is stated by saying that the n-dimensional prequantum theory is a boundary theory for a $(n+1)$-dimensional symplectic field theory. One of the aims of the present article is to show how this somehow vague statement can be rigorously formalized within the context of *boundary conditions* for fully extended topological field theories [L09a].

The prototypical example of this is the relation between 3d *Chern-Simons theory* and 4d *topological Yang-Mills theory*. Namely, the fact that 3d Chern-Simons theory is a theory which ultimately deals with boundaries of 4-manifolds is something coming from the very origin of the theory [CS]. In the language of smooth stacks this can be completely formalized and summarized in the following (homotopy) commutative diagram

$$\begin{tikzcd}
 & \mathbf{B}G_{\mathrm{conn}} \arrow[d, dashed, "\mathbf{cs}"] \arrow[ddl, bend right] \arrow[ddr, bend left, "\langle F_{(-)} \wedge F_{(-)} \rangle"] & \\
 & \mathbf{B}^3 U(1)_{\mathrm{conn}} \arrow[dl] \arrow[dr, "\mathrm{curv}"] & \\
 {*} \arrow[dr, "0"'] \arrow[rr, phantom, "(\mathrm{pb})"] & & \Omega^4_{\mathrm{cl}} \arrow[dl, "\exp(2\pi i S_{\mathrm{tYM}})"] \\
 & \flat \mathbf{B}^4 U(1) &
\end{tikzcd},$$

where $\mathbf{B}G_{\mathrm{conn}}$ is the stack of principal G-bundles with connection for a compact simple and simply connected Lie group G,

$$\langle F_{(-)} \wedge F_{(-)} \rangle : \mathbf{B}G_{\mathrm{conn}} \to \Omega^4_{\mathrm{cl}}$$

is the *Chern-Weil* 4-form representing the fundamental degree four characteristic class of G, and

$$\mathbf{cs} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

is the Chern-Simonsaction functional lifted to a morphism of stacks from $\mathbf{B}G_{\mathrm{conn}}$ to the 3-stack of $U(1)$-3-bundles with connection (see [FSS13] for details). In the lower part of the diagram,

$$\exp(2\pi i S_{\mathrm{tYM}}) : \Omega^4_{\mathrm{cl}} \to \flat \mathbf{B}^4 U(1)$$

is the canonical embedding of closed 4-forms into the stack of flat $U(1)$-4-bundles with connection. Here we are denoting it by the symbol $\exp(2\pi i S_{\mathrm{tYM}})$ since we are physically interpreting it as the the Lagrangian of *topological 4d Yang-Mills theory*. The lower part of the diagram is what exhibits 3d Chern-Simons as a boundary theory for 4d topological Yang-Mills. More precisely, since the lower part of the diagram is a homotopy pullback, it exhibits $\mathbf{B}^3 U(1)_{\mathrm{conn}}$ as the *universal* boundary condition for 4d topological Yang-Mills. we will come back to this in detail in Section **??**.

Finally, in a fully extended field theory, going from the bulk to the boundary is only the first step: one can go in higher codimension to boundaries of boundaries (or corners) or consider high codimension submanifolds of the bulk. For instance, in 4d topological Yang-Mills, this is the way Wess-Zumino-Witten theory and and Wilson loop actions appears as a codimension-2 corner theory and as codimension-3 defects, respectively. We will recover these as examples of more general corner and defect theories in Section **??**.

2 Classical mechanics by prequantized Lagrangian correspondences

We show and disucss here how classical mechanics (Hamiltonian mechanics, Lagranian mechanics, Hamilton-Jacobi theory, see e.g. [1]) naturally arises from and is accurately captured by "pre-quantized Lagragian correspondences". Since (pre-quantum) field theory is a refinement of classical mechanics, this serves as a blueprint for the discussion of local prequantum field by higher correspondences in the following sections.

The reader unfamiliar with classical mechanics may take the following to be a brief introduction to and indeed a systematic derivation of the central concepts of classical mechanics from correspondences in slices toposes. The reader familiar with classical mechanics may take the translation of classical mechanics into correspondences in slice toposes as the motivating example for the formalization of prequantum field theory in 3.3.16 below. The translation is summarized as a diagramatic dictionary below in 2.7.

The notion of plain Lagrangian correspondences (not pre-quantized) has been observed already in the early 1970s to usefully capture central aspects of Fourier transformation theory [4] and of classical mechanics [6], notably to unify the notion of Lagrangian subspaces of phase spaces with that of "canonical transformations", hence symplectomorphisms, betwee them. This observation has since been particularly advertized by Weinstein (e.g [7]), who proposed that some kind of *symplectic category* of symplectic manifolds with Lagrangian correspondences between them should be a good domain for a formalization of *quantization* along the lines of geometric quantization. Several authors have since discussed aspects of this idea. A recent review in the context of field theory is in [3].

But geometric quantization proper proceeds not from plain symplectic manifolds but from a lift of their symplectic form to a cocycle in differential cohomology, called a *pre-quantization* of the symplectic manifold. Therefore it is to be expected that some notion of pre-quantized Lagrangian correspondences, which put into correspondence these prequantum bundles and not just their underlying symplectic manifolds, is a more natural domain for geometric quantization, hence a more accurate formalization of pre-quantum geometry.

There is an evident such notion of prequantization of Lagrangian correspondences, and this is what we introduce and discuss in the following. While evident, it seems that it has previously found little attention in the literature, certainly not attention comparable to the fame enjoyed by Lagrangian correspondences.

The purpose of this section here is therefore twofold; on the one had to show how pre-quantized Lagrangian correspondences naturally and accurately formalize and indeed induce classical mechanics both in its main structures but also in its fine detail, and on the other hand to provide a formulation of classical mechanics which seamlessly leads over to the formulation of higher dimensional prequantum field theory by higher categories of higher correspondences.

2.1 Phase spaces and symplectic manifolds

Given a physical system, one says that its *phase space* is the space of its possible ("classical") histories or trajectories. Newton's second law of mechnanics says that trajectories of physical systems are (typically) determined by differential equations of *second* order, and therefore these spaces of trajectories are (typically) equivalent to initial value data of 0th and of 1st derivatives. In physics this data (or rather its linear dual) is referred to as the *canonical coordinates* and the *canonical momenta*, respectively, traditionally denoted by the symbols "q" and "p". Being coordinates, these are actually far from being canonical in the mathematical sense; all that has invariant meaning is, locally, the surface element $\mathbf{d}p \wedge \mathbf{d}q$ spanned by a change of coordinates and momenta.

So far this says that a physical phase space is mathematically formalized by a sufficiently smooth manifold X which is equipped with a closed and non-degenerate differential 2-form $\omega \in \Omega^2_{\mathrm{cl}}(X)$, hence by a symplectic manifold (X, ω).

Example 2.1.1. The simplest nontrivial example is the phase space $\mathbb{R}^2 \simeq T^*\mathbb{R}$ of a single particle propagating on the real line. The standard coordinates on the plane are traditionally written $q, q : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and the symplectic form is the canonical volume form $\mathbf{d}q \wedge \mathbf{d}p$.

The non-degeneracy of a symplectic form encodes the special property (as we will make explicit below in 2.4) that (time) evolution of coordinates and momenta is uniquely induced by an action functional/Hamiltonian generating the evolution. This is however famously not the case for systems with *gauge equivalences*, hence such systems which have configurations that are nominally different but nevertheless physically equivalent. Presence of such gauge equivalences is not the exception, but is the rule for physical systems, and therefore we want to include this case in our discussion.

In the presence of gauge equivalences, the phase space form ω is still a closed differential 2-form, it just need not be non-degenerate anymore. While in such a case the pair (X, ω) could just be called a *smooth manifold equipped with a closed differential 2-form*, it is traditional to call this a *pre-symplectic manifold* in order to amplify the indented use as a model for phase spaces. (Some authors demand that a pre-symplectic form be a closed form with constant rank, but here this technical condition will not be relevant and will not be considered.)

When dealing with spaces X that are equipped with extra structure, such as a closed differential 2-form $\omega \in \Omega^2_{\mathrm{cl}}(X)$, then it is useful to have a *universal moduli space* for these structures, and this will be central for our developments here. So we need a "smooth space" $\mathbf{\Omega}^2_{\mathrm{cl}}$ of sorts, characterized by the property that there is a natural bijection between smooth closed differential 2-forms $\omega \in \Omega^2_{\mathrm{cl}}(X)$ and smooth maps $X \longrightarrow \mathbf{\Omega}^2_{\mathrm{cl}}$. Of course such a universal moduli spaces of closed 2-forms does not exist in the category of smooth manifolds. But it does exist canonically if we slightly generalize the notion of "smooth space" suitably.

Definition 2.1.2. A *smooth space* or *smooth 0-type* X is

1. an assignment to each $n \in \mathbb{N}$ of a set, to be written $X(\mathbb{R}^n)$ and to be called the *set of smooth maps from $\mathbb{R}^n$ into X*,
2. an assignment to each ordinary smooth function $f : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ between Cartesian spaces of a function of sets $X(f) : X(\mathbb{R}^{n_2}) \to X(\mathbb{R}^{n_1})$, to be called the *pullback of smooth functions into X along f*;

such that

1. this assignment respects composition of smooth functions;
2. this assignment respect the covering of Cartesian spaces by open disks: for every good open cover $\{\mathbb{R}^n \simeq U_i \hookrightarrow \mathbb{R}^n\}_i$, the set $X(\mathbb{R}^n)$ of smooth functions out of $\mathbb{R}^n$ into X is in natural bijection with the set $\left\{(\phi_i)_i \in \prod_i X(U_i) \mid \forall_{i,j}\ \phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}\right\}$ of tuples of smooth functions out of the patches of the cover which agree on all intersections of two patches.

While the formulation of this definition is designed to make transparent its geometric meaning, of course equivalently but more abstractly this says the following:

Definition 2.1.3. Write CartSp for the category of Cartesian spaces with smooth functions between them, and consider it as a site by equipping it with the coverage (the Grothendieck pre-topology) of good open covers. A *smooth space* or *smooth 0-type* is a sheaf on this site. The *topos of smooth 0-types* is the sheaf category

$$\mathrm{Smooth0Type} := \mathrm{Sh}(\mathrm{CartSp}) \,.$$

In the following we will abbreviate the notation to

$$\mathbf{H} := \mathrm{Smooth0Type} \,.$$

For the discussion of presymplectic manifolds, we need the following two examples.

Example 2.1.4. Every smooth manifold $X \in \mathrm{SmoothManifold}$ becomes a smooth 0-type by the assignment

$$X : n \mapsto C^\infty(\mathbb{R}^n, X) \,.$$

This construction extends to a full and faithful embedding of smooth manifolds into smooth 0-types

$$\mathrm{SmoothManifold} \hookrightarrow \mathbf{H} \,.$$

Example 2.1.5. For $p \in \mathbb{N}$, write $\mathbf{\Omega}^p_{\mathrm{cl}}$ for the smooth space given by the assignment

$$\mathbf{\Omega}^p_{\mathrm{cl}} : n \mapsto \Omega^p_{\mathrm{cl}}(\mathbb{R}^n)$$

and by the evident pullback maps of differential forms.

This solves the moduli problem for closed smooth differential forms:

Proposition 2.1.6. *For $p \in \mathbb{N}$ and $X \in \mathrm{SmoothManifold} \hookrightarrow \mathrm{Smooth0Type}$, there is a natural bijection*

$$\mathbf{H}(X, \mathbf{\Omega}^p_{\mathrm{cl}}) \simeq \Omega^p_{\mathrm{cl}}(X)\,.$$

So a presymplectic manifold (X, ω) is equivalently a map of smooth spaces of the form

$$\omega :\; X \longrightarrow \mathbf{\Omega}^2_{\mathrm{cl}}\;\;.$$

2.2 Canonical transformations and Symplectomorphisms

An equivalence between two phase spaces, hence a re-expression of the "canonical" coordinates and momenta, is called a *canonical transformation* in physics. Mathematically this is a *symplectomorphism*.

The above formulation of pre-symplectic manifolds as maps into a moduli space of closed 2-forms yields the following formulation of symplectomorphisms, which is very simple in itself, but contains in it the seed of an important phenomenon:

Proposition 2.2.1. *Given two symplectic manifolds (X_1, ω_1) and (X_2, ω_2), a symplectomorphism ϕ : $(X_1, \omega_1) \to (X_2, \omega_2)$ is equivalently a commuting diagram of smooth spaces of the following form:*

$$\begin{tikzcd} X_1 \arrow[rr, "\phi"] \arrow[dr, "\omega_1"'] & & X_2 \arrow[dl, "\omega_2"] \\ & \mathbf{\Omega}^2_{\mathrm{cl}} & \end{tikzcd}\;.$$

Situations like this are naturally interpreted in the *slice topos*:

Definition 2.2.2. For $A \in \mathbf{H}$ any smooth space, the *slice topos* $\mathbf{H}_{/A}$ is the category whose objects are objects $X \in \mathbf{H}$ equipped with maps $X \to A$, and whose morphisms are commuting diagrams in $\mathbf{H}$ of the form

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & A & \end{tikzcd}\;.$$

Hence if we write SymplManifold for the category of smooth pre-symplectic manifolds and symplectomorphisms betwen them, then we have the following.

Proposition 2.2.3. *The construction of prop. 2.1.6 constitutes a full embedding*

$$\mathrm{SymplManifold} \hookrightarrow \mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}}$$

of pre-symplectic manifolds with symplectomorphisms between them into the slice topos of smooth spaces over the smooth moduli space of closed differential 2-forms.

2.3 Trajectories and Lagrangian correspondences

A symplectomorphism clearly puts two symplectic manifolds "in relation" to each other. It turns out to be useful to say this formally. Recall:

Definition 2.3.1. For $X, Y \in \mathrm{Set}$ two sets, a relation R between elements of X and elements of Y is a subset of the Cartesian product set

$$R \hookrightarrow X \times Y .$$

More generally, for $X, Y \in \mathbf{H}$ two objects of a topos (such as the topos of smooth spaces), then a relation R between them is a subobject of their Cartesian product

$$R \hookrightarrow X \times Y \ .$$

In particular any function induces the relation "y is the image of x":

Example 2.3.2. For $f : X \longrightarrow Y$ a function, its *induced relation* is the relation which is exhibited by *graph* of f

$$\mathrm{graph}(f) := \{(x, y) \in X \times Y \mid f(x) = y\}$$

canonically regarded as a subobject

$$\mathrm{graph}(f) \hookrightarrow X \times Y .$$

Hence in the context of classical mechanics, in particular any symplectomorphism $f : (X_1, \omega_1) \longrightarrow (X_2, \omega_2)$ induces the relation

$$\mathrm{graph}(f) \hookrightarrow X_1 \times X_2 .$$

Since we are going to think of f as a kind of"physical process", it is useful to think of the smooth space *graph*(f) here as the *space of trajectories* of that process. To make this clearer, notice that we may equivalently rewrite every relation $R \hookrightarrow X \times Y$ as a diagram of the following form:

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \quad = \quad \begin{tikzcd} & R \arrow[d] & \\ & X \times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & & Y \end{tikzcd}$$

reflecting the fact that every element $(x \sim y) \in R$ defines an element $x = i_X(x \sim y) \in X$ and an element $y = i_Y(x \sim y) \in Y$.

Then if we think of the space $R = \mathrm{graph}(f)$ of example 2.3.2 as being a space of trajectories starting in X_1 and ending in X_2, then we may read the relation as "there is a trajectory from an incoming configuration x_1 to an outgoing configuration x_2":

$$\begin{tikzcd} & \mathrm{graph}(f) \arrow[dl, "\text{incoming}"'] \arrow[dr, "\text{outgoing}"] & \\ X_1 & & X_2 \end{tikzcd} .$$

Notice here that the defining property of a relation as a subset/subobject translates into the property of classical physics that there is *at most one trajectory* from some incoming configuration x_1 to some outgoing trajectory x_2 (for a fixed and small enough parameter time interval at least, we will formulate this precisely in the next section when we genuinely consider Hamiltonian correspondences).

In a more general context one could consider there to be several such trajectories, and even a whole smooth space of such trajectories between given incoming and outgoing configurations. Each such trajectory would "relate" x_1 to x_2, but each in a possible different way. We can also say that each trajectory makes x_1 *correspond* to x_2 in a different way, and that is the mathematical term usually used:

Definition 2.3.3. For $X, Y \in \mathbf{H}$ two spaces, a correspondence between them is a diagram in $\mathbf{H}$ of the form

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

with no further restrictions. Here Z is also called the *correspondence space*.

Observe that the graph of a function $f\colon X \to Y$ is, while defined differently, in fact equivalent to just the space X, the equivalence being induced by the map $x \mapsto (x, f(x))$

$$X \xrightarrow{\simeq} \mathrm{graph}(f)\,.$$

In fact the relation/correspondence which expresses "y is the image of f under x" may just as well be exhibited by the diagram

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \quad .$$

It is clear that this correspondence with correspondence space X should be regarded as being equivalent to the one with correspondence space $\mathrm{graph}(f)$. We may formalize this as follows

Definition 2.3.4. Given two correspondences $X \longleftarrow Z_1 \longrightarrow Y$ and $X \longleftarrow Z_2 \longrightarrow Y$ between the same objects in $\mathbf{H}$, then an equivalence between them is an equivalence $Z_1 \xrightarrow{\simeq} Z_2$ in $\mathbf{H}$ which fits into a commuting diagram of the form

$$\begin{tikzcd} & Z_1 \arrow[dl] \arrow[dr] \arrow[dd, "\simeq"] & \\ X & & Y \\ & Z_2 \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Example 2.3.5. Given an function $f\colon X \longrightarrow Y$ we have the commuting diagram

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] \arrow[dd, "\simeq"] & \\ X & & Y \\ & \mathrm{graph}(f) \arrow[ul, "i_X"] \arrow[ur, "i_Y"'] & \end{tikzcd}$$

exhibiting an equivalence of the correspondence at the top with that at the bottom.

Correspondences between X any Y with such equivalences between them form a *groupoid*. Hence we write

$$\mathrm{Corr}\,(\mathbf{H})\,(X, Y) \in \mathrm{Grpd}\,.$$

Moreover, if we think of correspondences as modelling spaces of trajectories, then it is clear that their should be a notion of composition:

$$\left(\begin{tikzcd} & Y_1 \arrow[dl] \arrow[dr] & & Y_2 \arrow[dl] \arrow[dr] & \\ X_1 & & X_2 & & X_3 \end{tikzcd}\right) \mapsto \left(\begin{tikzcd} & Y_1 \circ_{X_2} Y_2 \arrow[dl] \arrow[dr] & \\ X_1 & & X_3 \end{tikzcd}\right).$$

Heuristically, the composite space of trajectories $Y_1 \circ_{X_2} Y_2$ should consist precisely of those pairs of trajectories $(f, g) \in Y_1 \times Y_2$ such that the endpoint of f is the starting point of g. The space with this property is precisely the *fiber product* of Y_1 with Y_2 over X_2, denoted $Y_1 \underset{X_2}{\times} Y_2$ (also called the *pullback* of $Y_2 \longrightarrow X_2$ along $Y_1 \longrightarrow X_2$:

$$\left(\begin{tikzcd} & Y_1 \circ_{X_2} Y_2 \arrow[dl] \arrow[dr] & \\ X_1 & & X_3 \end{tikzcd}\right) = \left(\begin{tikzcd} & & Z_1 \underset{Y}{\times} Z_2 \arrow[dl] \arrow[dr] & & \\ & Z_1 \arrow[dl] \arrow[dr] & & Z_2 \arrow[dl] \arrow[dr] & \\ X & & Y & & Z \end{tikzcd}\right).$$

Hence given a topos $\mathbf{H}$, correspondences between its objects form a category which composition the fiber product operation, where however the collection of morphisms between any two objects is not just a set, but is a groupoid (the groupoid of correspondences between two given objects and equivalences between them).

One says that correspondences form a $(2,1)$-*category*

$$\mathrm{Corr}(\mathbf{H}) \in (2,1)\mathrm{Cat}\,.$$

One reason for formalizing this notion of correspondences so much in the present context that it is useful now to apply it not just to the ambient topos $\mathbf{H}$ of smooth spaces, but also to its slice topos $\mathbf{H}_{/\mathbf{\Omega}^2_{cl}}$ over the universal moduli space of closed differential 2-forms.

To see how this is useful in the present context, notice the following

Proposition 2.3.6. *Let $\phi : (X_1, \omega_1) \to (X_2, \omega_2)$ be a symplectomorphism. Write*

$$(i_1, i_2) : \mathrm{graph}(\phi) \hookrightarrow X_1 \times X_2$$

for the graph of the underlying diffeomorphsm. This fits into a commuting diagram in $\mathbf{H}$ of the form

$$\begin{tikzcd} & \mathrm{graph}(\phi) \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ X_1 \arrow[dr, "\omega_1"'] \arrow[rr, phantom, "="] & & X_2 \arrow[dl, "\omega_2"] \\ & \mathbf{\Omega}^2_{\mathrm{cl}} & \end{tikzcd}\,.$$

Conversely, a smooth function $\phi : X_1 \to X_2$ is a symplectomorphism precisely if its graph makes the above diagram commute.

Traditionally this is formalized as follows.

Definition 2.3.7. Given a symplectic manifold (X, ω), a submanifold $L \hookrightarrow X$ is called a *Lagrangian submanifold* if $\omega|_L = 0$ and if L has dimension $dim(L) = dim(X)/2$.

Definition 2.3.8. For (X_1, ω_1) and (X_2, ω_2) two symplectic manifolds, a correspondence $X_1 \xleftarrow{p_1} Y \xrightarrow{p_2} X_2$ of the underlying manifolds is a *Lagrangian correspondence* if the map $Y \to X_1 \times X_2$ exhibits a Lagrangian submanifold of the symplectic manifold given by $(X_1 \times X_2, p_2^*\omega_2 - p_1^*\omega_1)$.

Given two Lagrangian correspondence which intersect transversally over one adjacent leg, then their *composition* is the correspondence given by the intersection.

But comparison with def. 2.2.2 shows that Lagrangian correspondences are in fact plain correspondences, just not in smooth spaces, but in the slice $\mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}}$ of all smooth spaces over the universal smooth moduli space of closed differential 2-forms:

Proposition 2.3.9. *Under the identification of prop. 2.2.3 the construction of the diagrams in prop. 2.3.6 constitutes an injection of Lagrangian correspondence between (X_1, ω_1) and (X_2, ω_2) into the Hom-space $\mathrm{Corr}\left(\mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}}\right)((X_1, \omega_1), (X_2, \omega_2))$. Moreover, composition of Lagrangian correspondence, when defined, coincides under this identification with the composition of the respective correspondences.*

Remark 2.3.10. The composition of correspondences in the slice topos is always defined. It may just happen the the composite is given by a correspondence space with is a smooth space but not a smooth manifold. Or better, one may replace in the entire discussion the topos of smooth spaces with a topos of "derived" smooth spaces, modeled not on Cartesian spaces but on Cartesian dg-manifolds. This will then automatically make composition of Lagrangian correspondences take care of "transversal perturbations". Here we will not further dwell on this possibility. In fact, the formulation of Lagrangian correspondences and later of prequantum field theory by correspondences in toposes implies a great freedom in the choice of type of geometry in which set up everything. Below in 3.1 we specify the bare minimum condition on the topos $\mathbf{H}$ which we will require (namely that it be *differentially cohesive*).

It is also useful to make the following phenomenon explicit, which is the first incarnation of a recurring theme in the following discussions.

Proposition 2.3.11. *The category $\mathrm{Corr}(\mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}})$ is naturally a symmetric monoidal category, where the tensor product is given by*

$$(X_1, \omega_1) \otimes (X_2, \omega_2) = (X_1 \times X_2, \omega_1 + \omega_2)\,.$$

The tensor unit is $(*, 0)$. With respect to this tensor product, every object is dualizable, with dual object given by

$$(X, \omega)^v = (X, -\omega)\,.$$

Remark 2.3.12. Duality induces natural equivalences of the form

$$\mathrm{Corr}\left(\mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}}\right)((X_1, \omega_1), (X_2, \omega_2),) \xrightarrow{\simeq} \mathrm{Corr}\left(\mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}}\right)((*, 0), (X_1 \times X_2, \omega_2 - \omega_1),)\,.$$

Under this equivalence an isotropic (Lagrangian) correspondences which in $\mathbf{H}$ is given by a diagram as in prop. 2.3.6 maps to the diagram of the form

$$\begin{tikzcd} & \mathrm{graph}(\phi) \arrow[dl] \arrow[dr, "{(i_1, i_2)}"] & \\ * \arrow[dr, "0"'] & & X_1 \times X_2 \arrow[dl, "\omega_2 - \omega_1"] \\ & \mathbf{\Omega}^2_{\mathrm{cl}} \arrow[u, phantom, "="] & \end{tikzcd}\,.$$

This makes the condition that the pullback of the difference $\omega_2 - \omega_1$ vanishes on the correspondence space more manifest. It is also the blueprint of a phenomenon that is important in the generalization to field theory in the sections to follow, where trajectories map to boundary conditions, and vice versa.

2.4 Hamiltonian (time evolution) trajectories and Hamiltonian correspondences

We have seen so far transformations of phase space given by "canonical transformations", hence symplectomorphisms. Of central importance in physics are of course those transformations that are part of a smooth evolution group, notably for time evolution. These are the "canonical transformations" coming from a generating function, hence the symplectomorphisms which come from a Hamiltonian function (the energy function, for time evolution), the *Hamiltonian symplectomorphisms*. Below in 2.6 we see that this notion is implied by prequantizing Lagrangian correspondences, but here it is good to recall the traditional definition.

Definition 2.4.1. Given a symplectic manifold (X, ω) and a function $H : X \to \mathbb{R}$, its *Hamiltonian vector field* is the unique $v \in \Gamma(TX)$ which satisfies *Hamilton's equation of motion*

$$\mathbf{d}H = \iota_v \omega \,.$$

The flow of this v is called the corresponding *Hamiltonian flow*. Given two functions $f, g : X \to \mathbb{R}$ with Hamiltonian vector fields v, w, respectively, their *Poisson bracket* is the function

$$\{f, g\} := \iota_w \iota_v \omega \,.$$

Since by Cartan's formula the Lie derivative of ω along v is given by $\mathcal{L}_v \omega = \mathbf{d}\iota_v\omega + \iota_v \mathbf{d}\omega = \mathbf{d}^2 H = 0$ it follows that

Proposition 2.4.2. *Every Hamiltonian flow is a symplectomorphism.*

Those symplectomorphisms arising this way are called the *Hamiltonian symplectomorphisms*. Notice that the Hamiltonian symplectomorphism depends on the Hamiltonian only up to addition of a locally constant function.

Using the Poisson bracket $\{-,-\}$ induced by the symplectic form ω and identifying the derivation $\{H, -\} : C^\infty(X) \longrightarrow C^\infty(X)$ with the corresponding Hamiltonian vector field v and the exponent notation $\exp(t\{H,-\})$ with the Hamiltonian flow for parameter "time" $t \in \mathbb{R}$, we may write these Hamiltonian symplectomorphisms as

$$\exp(t\{H,-\}) \,:\, (X, \omega) \longrightarrow (X, \omega)\,.$$

It then makes sense to say that a Lagrangian correspondence, def. 2.3.8, which is induced from a Hamiltonian symplectomorphism is a *Hamiltonian correspondences*

$$\left(\begin{tikzcd} & \mathrm{graph}\left(\exp\left(t\left\{H,-\right\}\right)\right) \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ X & & X \end{tikzcd} \right) \simeq \left(\begin{tikzcd} & X \arrow[dl, "="'] \arrow[dr, "\exp(t\{H,-\})"] & \\ X & & X \end{tikzcd} \right) .$$

Remark 2.4.3. The smooth correspondence space of a Hamiltonian correspondence is naturally identified with the space of *classical trajectories*

$$\mathrm{Fields}^{\mathrm{class}}_{\mathrm{traj}}(t) := \mathrm{graph}\left(\exp(t)\{H,-\}\right)$$

in that

1. every point in the space corresponds uniquely to a trajectory of parameter time length t characterized as satisfying the equations of motion as given by Hamilton's equations for H;

2. the two projection maps to X send a trajectory to its initial and to its final configuration, respectively.

group structure is

Remark 2.4.4. By constuction, Hamiltonian flows form a 1-parameter Lie group. By prop. 2.3.9 this group structure is preserved by the composition of the induced Hamiltonian correspondences.

It is useful to highlight this formally as follows.

Definition 2.4.5. Write $\mathrm{Bord}_1^{\mathrm{Riem}}$ for the category of 1-dimensional cobordisms equipped with Riemannian structure (hence with a real, non-negative length which is additive under composition), regarded as a symmetric monoidal category under disjoint union of cobordisms.

Then:

Proposition 2.4.6. *The Hamiltonian correspondences induced by a Hamiltonian function $H : X \longrightarrow \mathbb{R}$ are equivalently encoded in a smooth monoidal functor of the form*

$$\exp((-)\{H,-\}) : \mathrm{Bord}_1^{\mathrm{Riem}} \longrightarrow \mathrm{Corr}_1(\mathbf{H}_{/\Omega^2}),$$

where on the right we use the monoidal structure on correspondence of prop. 2.3.11.

Below the general discussion of prequantum field theory, such monoidal functors from cobordisms to correspondences of spaces of field configurations serve as the fundamental means of axiomatization. Whenever one is faced with such a functor, it is of particular interest to consider its value on *closed* cobordisms. Here in the 1-dimensional case this is the circle, and the value of such a functor on the circle would be called its (pre-quantum) *partition function.*

Proposition 2.4.7. *Given a phase space symplectic manifold (X, ω) and a Hamiltonian $H : X \longrightarrow \mathbb{R}$, them the prequantum evolution functor of prop. 2.4.6 sends the circle of circumference t, regarded as a cobordism from the empty 0-manifold to itself*

$$\begin{tikzcd} & S^1 & \\ \emptyset \arrow[ur, hook] & & \emptyset \arrow[ul, hook'] \end{tikzcd}$$

and equipped with the constant Riemannian metric of 1-volume t, to the correspondence

$$\begin{tikzcd} & \{x \in X \,|\, \exp(t\{H,-\})(x) = x\} \arrow[dl] \arrow[dr] & \\ * & & * \end{tikzcd}$$

which is the smooth space of H-Hamiltonian trajectories of (time) length t that are closed, hence that come back to their initial value, regarded canonically as a correspondence form the point to itself.

Proof. We can decompose the circle of length t as the compositon of

1. The coevaluation map on the point, regarded as a dualizable object $\mathrm{Bord}_1^{\mathrm{Riem}}$;
2. the interval of length t;
3. the evaluation map on the point.

The monoidal functor accordingly takes this to the composition of correspondences of

1. the coevaluation map on X, regarded as a dualizable object in $\mathrm{Corr}(\mathbf{H})$;
2. the Hamiltonian correspondence induced by $\exp(t\{H,-\})$;
3. the evaluation map on X.

As a diagram in $\mathbf{H}$, this is the following:

$$
\begin{tikzcd}
& X \arrow[dl] \arrow[dr, "\Delta"] & & \mathrm{graph}(\exp(t\{H,-\}))\times X \arrow[dl] \arrow[dr] & & X \arrow[dl, "\Delta"'] \arrow[dr] & \\
* & & X\times X & & X\times X & & *
\end{tikzcd}
.
$$

By the definition of composition in $\mathrm{Corr}(\mathbf{H})$, the resulting composite correspondence space is the joint fiber product in $\mathbf{H}$ over these maps. This is essentially verbatim the diagrammatic definition of the space of closed trajectories of parameter length t. □

2.5 The kinetic action, pre-squantization and differential cohomology

Given a pre-symplectic form $\omega \in \Omega^2_{\mathrm{cl}}(X)$, by the Poincaré lemma there is a good cover $\{U_i \hookrightarrow X\}_i$ and smooth 1-forms $\theta_i \in \Omega^1(U_i)$ such that $\mathbf{d}\theta_i = \omega_{|U_i}$. Physically such a 1-form is (up to a factor of 2) a choice of *kinetic energy density* called a *kinetic Lagrangian* L_{kin}:

$$\theta_i = 2L_{\mathrm{kin},i}\,.$$

Example 2.5.1. Consider the phase space $(\mathbb{R}^2,\ \omega = \mathbf{d}q\wedge\mathbf{d}p)$ of example 2.1.1. Since $\mathbb{R}^2$ is a contractible topological space we consider the trivial covering ($\mathbb{R}^2$ covering itself) since this is already a good covering in this case. Then all the $\{g_{ij}\}$ are trivial and the data of a prequantization consists simply of a choice of 1-form $\theta \in \Omega^1(\mathbb{R}^2)$ such that

$$\mathbf{d}\theta = \mathbf{d}q\wedge\mathbf{d}p\,.$$

A standard such choice is

$$\theta = -p\wedge\mathbf{d}q\,.$$

Then given a trajectory $\gamma\colon [0,1] \longrightarrow X$ which satisfies Hamilton's equation for a standard kinetic energy term, then $(p\mathbf{d}q)(\dot\gamma)$ is this kinetic energy of the particle which traces out this trajectory.

Given a path $\gamma : [0,1] \to X$ in phase space, its *kinetic action* S_{kin} is supposed to be the integral of $\mathcal{L}_{\mathrm{kin}}$ along this trajectory. In order to make sense of this in generality with the above locally defined kinetic Lagrangians $\{\theta_i\}_i$, there are to be transition functions $g_{ij} \in C^\infty(U_i\cap U_j,\mathbb{R})$ such that

$$\theta_j|_{U_j} - \theta_i|_{U_i} = \mathbf{d}g_{ij}\,.$$

If on triple intersections these functions satisfy

$$g_{ij} + g_{jk} = g_{ik} \quad \text{on } U_i\cap U_j\cap U_K$$

then there is a well defined action functional

$$S_{\mathrm{kin}}(\gamma) \in \mathbb{R}$$

obtained by dividing γ into small pieces that each map to a single patch U_i, integrating θ_i along this piece, and adding the contribution of g_{ij} at the point where one switches from using θ_i to using θ_j.

However, requiring this condition on triple overlaps as an equation between $\mathbb{R}$-valued functions makes the local patch structure trivial: if this holds then one can find a single $\theta \in \Omega^1(X)$ and functions $h_i \in C^\infty(U_i,\mathbb{R})$ such that $\theta_i = \theta|_{U_i} + \mathbf{d}h_i$. This has the superficially pleasant effect that the the action is simply the integral against this globally defined 1-form, $S_{\mathrm{kin}} = \int_{[0,1]}\gamma^* L_{\mathrm{kin}}$, but it also means that the pre-symplectic form ω is exact, which is not the case in many important examples.

On the other hand, what really matters in physics is not the action functional $S_{\mathrm{kin}} \in \mathbb{R}$ itself, but the *exponentiated* action

$$\exp\left(\tfrac{i}{\hbar}S\right) \in \mathbb{R}/(2\pi\hbar)\mathbb{Z}\,.$$

For this to be well defined, one only needs that the equation $g_{ij} + g_{jk} = g_{ik}$ holds modulo addtion of an integral multiple of $h = 2\pi\hbar$, which is *Planck's constant.* If this is the case, then one says that the data $(\{\theta_i\}, \{g_{ij}\})$ defines equivalently

- a $U(1)$-principal connection;
- a degree-2 cocycle in ordinary differential cohomology

on X, with *curvature* the given symplectic 2-form ω.

Such data is called a *pre-quantization* of the symplectic manifold (X, ω). Since it is the exponentiated action functional $\exp(\frac{i}{\hbar}S)$ that enters the quantization of the given mechanical system (for instance as the integrand of a path integral), the prequantization of a symplectic manifold is indeed precisely the data necessary before quantization.

Therefore, in the spirit of the above discussion of pre-symplectic structures, we would like to refine the smooth moduli space of closed differential 2-forms to a moduli space of prequantized differential 2-forms.

Again this does naturally exist if only we allow for a good notion of "space". An additional phenomenon to be taken care of now is that while pre-symplectic forms are either equal or not, their pre-quantizations can be different and yet be *equivalent*:

because there is still a remaining freedom to change this data without changing the exponentiated action along a *closed* path: we say that a choice of functions $h_i \in C^\infty(U_i, \mathbb{R}/(2\pi\hbar)\mathbb{Z})$ defines an equivalence between $(\{\theta_i\}, \{g_{ij}\})$ and $(\{\tilde{\theta}_i\}, \{\tilde{g}_{ij}\})$ if $\tilde{\theta}_i - \theta_i = \mathbf{d}h_i$ and $\tilde{g}_{ij} - g_{ij} = h_j - h_i$.

This means that the space of prequantizations of (X, ω) is similar to an *orbifold*: it has points which are connected by gauge equivalences: there is a *groupoid* of pre-quantum structures on a manifold X. Otherwise this space of prequantizations is similar to the spaces $\mathbf{\Omega}^2_{\mathrm{cl}}$ of differential forms, in that for each smooth manifold ther is a collection of smooth such data and it may consistently be pullback back along smooth functions of smooth manifolds.

One formalizes this by promoting $\mathbf{H}$ from the category of *smooth spaces* or *smooth 0-types* to what is called the $(2,1)$-category of *smooth stacks* or *smooth groupoids* or *smooth 1-types.* This then contains a smooth groupoid

$$\mathbf{B}U(1)_{\mathrm{conn}} \in \mathbf{H}\,,$$

to be called the *universal moduli stack of prequantizations*, which is characterized by the the following properties:

1. For X any smooth manifold, smooth functions

$$X \longrightarrow \mathbf{B}U(1)_{\mathrm{conn}}$$

   are equivalent to prequantum structure $(\{\theta_i\}, \{g_{ij}\})$ on X,

2. a homotopy

$$\begin{tikzcd} X \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \mathbf{B}U(1)_{\mathrm{conn}} \end{tikzcd}$$

   between two such maps is equivalently a gauge transformation $(\{h_i\})$ between these prequantizations.

There is then in $\mathbf{H}$ a morphism

$$F:\ \mathbf{B}U(1)_{\mathrm{conn}} \longrightarrow \mathbf{\Omega}^2_{\mathrm{cl}}$$

from this universal moduli stack of prequantizations back to the universal smooth moduli space of closed differential 2-form. This is the *universal curvature* map in that for $\nabla : X \longrightarrow \mathbf{B}U(1)_{\mathrm{conn}}$ a prequantization datum $(\{\theta_i\}, \{g_{ij}\})$, the composite

$$F_{(-)}:\ X \xrightarrow{\nabla} \mathbf{B}U(1)_{\mathrm{conn}} \xrightarrow{F_{(-)}} \mathbf{\Omega}^2_{\mathrm{cl}}$$

is the closed differential 2-form on X characterized by $\omega|_{U_i} = \mathbf{d}\theta_i$, for every patch U_i. Again, this property characterizes the map $F_{(-)}$ and may be taken as its definition.

Using this language of the $(2,1)$-topos $\mathbf{H}$ of smooth groupoids, we may then formally capture the above discussion of prequantization as follows:

Definition 2.5.2. Given a symplectic manifold (X, ω), regarded by prop. 2.2.3 as an object $(X \xrightarrow{\omega} \mathbf{\Omega}^2_{\mathrm{c}}) \in \mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}}$, then a *prequantization* of (X, ω) is a lift ∇ in the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\nabla"] \arrow[dr, "\omega"'] & \mathbf{B}U(1)_{\mathrm{conn}} \arrow[d, "F_{(-)}"] \\ & \Omega^2_{\mathrm{cl}} \end{tikzcd}$$

in $\mathbf{H}$, hence is a lift of (X, ω) through the base change/dependent sum functor

$$\sum_{F_{(-)}} \; : \; \mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}} \longrightarrow \mathbf{H}_{/\mathbf{\Omega}^2_{\mathrm{cl}}}$$

from the slice over the universal moduli stack of prequantizations to the slice over the universal smooth moduli space of closed differential 2-forms.

2.6 The classical action, the Legendre transform and Hamiltonian flows

The reason to consider Hamiltonian symplectomorphisms, prop. 2.4.2 instead of general symplectomorphisms is really because these give homomorphisms not just between plain symplectic manifolds, but between their prequantizations, def. 2.5.2. To these we turn now.

Consider a morphism

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\nabla"'] & \arrow[d, phantom, "\swArrow"] & X \arrow[dl, "\nabla"] \\ & \mathbf{B}U(1)_{conn} & \end{tikzcd} \quad ,$$

hence a morphism in the slice $\mathbf{H}_{/\mathbf{B}U(1)_{conn}}$. This has been discussed in detail in [FRS13a].

One finds that infinitesimally such morphism are given by a Hamiltonian and its Legendre transform.

Proposition 2.6.1. *Consider the phase space $(\mathbb{R}^2,\ \omega = \mathbf{d}q \wedge \mathbf{d}p)$ of example 2.1.1 equipped with its canonical prequantization by $\theta = p\mathbf{d}q$ from example 2.5.1. Then for $H \colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ a Hamiltonian, and for $t \in \mathbb{R}$ a parameter ("time"), a lift of the Hamiltonian symplectomorphism $\exp(t\{H, -\})$ from $\mathbf{H}$ to the slice topos $\mathbf{H}_{/\mathbf{B}U(1)_{conn}}$ is given by*

$$\begin{tikzcd} X \arrow[rr, "\exp(t\{H,-\})"] \arrow[dr, "\theta"'] & \arrow[d, phantom, "\swArrow \exp(iS_t)"] & X \arrow[dl, "\theta"] \\ & \mathbf{B}U(1)_{conn} & \end{tikzcd} \quad ,$$

where

- $S_t \,:\, \mathbb{R}^2 \longrightarrow \mathbb{R}$ *is the action functional of the classical trajectories induced by H,*
- *which is the integral $S_t = \int_0^t L\, dt$ of the Lagrangian $L\, dt$ induced by H,*
- *which is the Legendre transform*

$$L := p\frac{\partial H}{\partial p} - H \; : \; \mathbb{R}^2 \longrightarrow \mathbb{R}\, .$$

In particular, this induces a functor

$$\exp(iS) : \mathrm{Bord}_1^{\mathrm{Riem}} \longrightarrow \mathbf{H}_{/\mathbf{B}U(1)_{\mathrm{conn}}} .$$

Conversely, a symplectomorphism, being a morphism in $\mathbf{H}_{/\mathbf{\Omega}^2_{cl}}$ *is a Hamiltonian symplectomorphism precisely if it admits such a lift to* $\mathbf{H}_{/\mathbf{B}U(1)_{conn}}$.

This is a special case of the discussion in [FRS13a].

Proof. The canonical prequantization of $(\mathbb{R}^2, \mathbf{d}q\wedge\mathbf{d}p)$ is the globally defined connection on a bundle—connection 1-form

$$\theta := p\,\mathbf{d}q\,.$$

We have to check that on $graph(\exp(t\{H,-\}))$ we have the equation

$$p_2\mathbf{d}q_2 = p_1\mathbf{d}q_1 + \mathbf{d}S\,.$$

Or rather, given the setup, it is more natural to change notation to

$$p_t\mathbf{d}q_t = p\mathbf{d}q + \mathbf{d}S\,.$$

Notice here that by the nature of $\mathrm{graph}(\exp(t\{H,-\}))$ we can identify

$$\mathrm{graph}(\exp(t\{H,-\})) \simeq \mathbb{R}^2$$

and under this identification

$$q_t = \exp(t\{H,-\})q$$

and

$$p_t = \exp(t\{H,-\})p\,.$$

It is sufficient to check the claim infinitesimal object—infinitesimally. So let $t=\epsilon$ be an infinitesimal, hence such that $\epsilon^2 = 0$. Then the above is Hamilton's equations and reads equivalently

$$q_\epsilon = q + \frac{\partial H}{\partial p}\epsilon$$

and

$$p_\epsilon = p - \frac{\partial H}{\partial q}\epsilon\,.$$

Using this we compute

$$\begin{aligned}
\theta_\epsilon - \theta &= p_\epsilon\,\mathbf{d}q\epsilon - p\mathbf{d}q \\
&= \left(p - \frac{\partial H}{\partial q}\epsilon\right)\mathbf{d}\left(q + \frac{\partial H}{\partial p}\epsilon\right) - p\mathbf{d}q \\
&= \epsilon\left(p\mathbf{d}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\mathbf{d}q\right) \\
&= \epsilon\left(\mathbf{d}\left(p\frac{\partial H}{\partial p}\right) - \frac{\partial H}{\partial p}\mathbf{d}p - \frac{\partial H}{\partial q}\mathbf{d}q\right) \\
&= \epsilon\mathbf{d}\left(p\frac{\partial H}{\partial p} - H\right)
\end{aligned} .$$

□

Remark 2.6.2. Proposition **??** says that the slice topos $\mathbf{H}_{/\mathbf{B}U(1)_{conn}}$ unifies classical mechanics in its two incarnations as Hamiltonian mechanics and as Lagrangian mechanics. A morphism here is a diagram in $\mathbf{H}$ of the form

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & \mathbf{B}U(1)_{conn} & \end{tikzcd}$$

and which may be regarded as having two components: the top horizontal 1-morphism as well as the homotopy/2-morphism filling the slice. Given a smooth flow of these, the horizontal morphism is the flow of a Hamiltonian vector field for some Hamiltonian function H, and the 2-morphism is a $U(1)$-gauge transformation given (locally) by a $U(1)$-valued function which is the exponentiated action functional that is the integral of the Lagrangian L which is the Legendre transform of H.

So in a sense the prequantization lift through the base change/dependent sum along the universal curvature map

$$\sum_{F_{(-)}} : \mathbf{H}_{/\mathbf{B}U(1)_{conn}} \longrightarrow \mathbf{H}_{/\mathbf{\Omega}^2_{cl}}$$

is the Legendre transform which connects Hamiltonian mechanics with Lagrangian mechanics.

2.7 The classical action functional pre-quantizes Lagrangian correspondences

It is therefore natural to declare that a *prequantized Lagrangian correspondence* is

$$\begin{tikzcd} & \mathrm{graph}(\phi) \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ X_1 \arrow[dr, "\nabla_1"] \arrow[ddr, "\omega_1"'] \arrow[rr, phantom, "\Swarrow"] & & X_2 \arrow[dl, "\nabla_2"'] \arrow[ddl, "\omega_2"] \\ & \mathbf{B}U(1)_{\mathrm{conn}} \arrow[d, "F_{(-)}"] & \\ & \Omega^2_{\mathrm{cl}} & \end{tikzcd}$$

A prequantization of a Lagrangian correspondence is a prequantization of the source and target symplectic manifold by prequantum circle bundles, togther with a choice of (gauge) equivalence between thes respective pullback of these two bundles to the graph of the Hamiltonian symplectomorphism.

From prop. 2.6.1 and under the equivalence of example 2.3.5 it follows that smooth 1-parameter groups of prequantized Lagrangian correspondences are equivalently Hamiltonian flows, and that the prequantizaton of the underlying Hamiltonian correspondences is given by the classical action funtional.

In summary, the description of classical mechanics here identifies prequantized Lagrangian correspondences schematically as follows:

$$
\begin{tikzcd}
& \mathrm{graph}\left(\exp\left(t\{H,-\}\right)\right) \arrow[dl] \arrow[dr] & \\
X \arrow[dr, "\nabla_{\mathrm{in}}"'] & \arrow[d, phantom, "\swarrow \exp\left(\frac{i}{\hbar}S_t\right)=\exp\left(\frac{i}{\hbar}\int_0^t L dt\right)"] & X \arrow[dl, "\nabla_{\mathrm{out}}"] \\
& \mathbf{B}U(1)_{\mathrm{conn}} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \text{space of trajectories} \arrow[dl, "\text{initial values}"'] \arrow[dr, "\text{Hamiltonian evolution}"] & \\
\text{incoming configurations} \arrow[dr, "\text{prequantum bundle}"'] & \arrow[d, phantom, "\swarrow \text{action functional}"] & \text{outgoing configurations} \arrow[dl, "\text{prequantum bundle}"] \\
& \text{2-group of phases} &
\end{tikzcd}
$$

With this picture in mind we pass now to the general abstract formulation of local prequantum field theory in terms of higher correspondences in higher slices toposes.

3 Local prequantum field theory

Here we set up the general abstract framework of local topological boundary prequantum field theory in terms of functors from cobordisms into higher correspondences.

3.1 The ambient ∞-topos

We freely speak higher category theory and higher topos theory here. The reader unfamiliar with these can harmlessly skip to 4.2 where we review the basics concepts as we walk through the simplest examples of the general theory. A friendly introduction to the subject, for the aims of the present article can be found in [FSS13]; an exhaustive treatment is in [L06, S].

As mentioned in the Introduction, the basic idea of the higher categorical approach to prequantum filed theories is that the spaces of field configurations should be promoted to *moduli stacks of fields*, which are object of some ambient ∞-topos. For instance the basic choice of the ∞-topos ∞-**Grpd** of (geometrically discrete) ∞-groupoids as the ambient allows to model collections of fields for *discrete* theories, such as Dijkgraaf-Witten theory and its higher analogs. For the discussion of fields of Chern-Simons theory and its higher analogs, however, one needs an ∞-topos whose objects may carry genuine differential geometric structure. Therefore, our running choice of ambient ∞-topos will be the ∞-topos $\mathbf{H}$ of *simplicial sheaves* over the site of smooth manifolds. We will also refer to the objects of $\mathbf{H}$ as *smooth ∞-stacks* or *smooth ∞-groupoids*.

3.2 Morphisms from homotopy types to smooth stacks

An important feature of the ∞-topos $\mathbf{H}$ is that it has *internal homs*, i.e., for any two two objects X and Y in $\mathbf{H}$ there is an object $[X, Y]$ in $\mathbf{H}$, the "space of maps" from X to Y together with a canonical equivalence

$$\mathbf{H}(Z, [X, Y]) \cong \mathbf{H}(Z \times X, Y)$$

for any Z in $\mathbf{H}$. In particular, identifying a smooth manifold Σ with the sheaf of smooth functions with values in it, each smooth manifold is naturally an object of $\mathbf{H}$. Hence for any fixed object **Fields** in $\mathbf{H}$ one can consider the moduli stack $[\Sigma, \mathbf{Fields}]$ of field configurations on Σ.

In applications to topological field theories, we also need to consider a variant of this construction. Namely, we need to consider the moduli stack of maps form the *homotopy type* of Σ to the stack of fields. This goes as follows. To begin with, one has a natural embedding

$$\mathrm{LConst} : \infty\text{-}\mathbf{Grpd} \hookrightarrow \mathbf{H}$$

which when viewed as a discrete ∞-groupoid looks as a locally constant smooth ∞-groupoid. The crucial property of this embedding is that it is *reflective*, i.e., that it has a left adjoint

$$\Pi : \mathbf{H} \to \infty\text{-}\mathbf{Grpd}\ ,$$

see [S]. While for a general higher stack X the description of $\Pi(X)$ may be elusive, for a smooth manifold Σ the ∞-groupoid $\Pi(\Sigma)$ has a completely straightforward description: it is the simplicial set $\mathrm{Sing}(X)$ of singular simplices of Σ or, with a more evocative name which also justifies the notation Π, the ∞-*Poincaré groupoid* of Σ. It contains all the information on the homotopy type of Σ, and this leads to the following definition.

Definition 3.2.1. Let

$$\mathbf{\Pi}_\infty : \mathbf{H} \to \mathbf{H}$$

be the composition $\mathbf{\Pi}_\infty := \mathrm{LConst} \circ \Pi$. For Σ a smooth manifold and X any smooth stack, we set

$$\mathbf{Maps}^h(\Sigma, X) = [\mathbf{\Pi}_\infty(\Sigma), X]$$

and call it the *moduli stack of maps from the homotopy type* of Σ to X.

Notice that for $*$ the terminal object of $\mathbf{H}$ we have a natural equivalence $\mathbf{\Pi}_\infty(*) \cong *$. In particular, this gives a natural equivalence

$$\mathbf{Maps}^h(*, X) \cong X$$

for any object X in $\mathbf{H}$.

Remark 3.2.2. If X is a smooth manifold, seen as a smooth stack, the moduli stack of *homotopy classes of maps* from Σ to X is

$$\mathbf{Maps}^h(\Sigma, \mathbf{\Pi}_\infty(X)) \simeq [\Sigma, \mathbf{\Pi}_\infty(X)] \simeq [\mathbf{\Pi}_\infty(\Sigma), \mathbf{\Pi}_\infty(X)]\ .$$

In other words, when we consider homotopy classes of maps to X, also X appears in the form of its homotopy type.

Example 3.2.3. There is a natural equivalence $\mathbf{Maps}^h(S^1, X) \cong \mathcal{L}X$ between the moduli stack of maps from the homotopy type of the circle S^1 to X and the *free loop space object* of X. Namely, the free loop space object $\mathcal{L}X$ is defined as the homotopy pullback of its diagonal map along itself

$$\mathcal{L}X := X \underset{X\times X}{\times} X\ ,$$

i.e., as the object defined by the homotopy pullback diagram

$$\begin{tikzcd}
\mathcal{L}X \arrow[r] \arrow[d] & X \arrow[d, "\Delta_X"] \\
X \arrow[r, "\Delta_X"'] & X \times X\ .
\end{tikzcd}$$

One then notices that S^1 is obtained by gluing two segments (which are contractible) along their endpoints, which amount to saying that at the level of homotopy types we have an equivalence

$$\Pi(S^1) \simeq * \coprod_{*\coprod *} *\ ,$$

and uses the fact that $[-, X]$ preserves homotopy limits.

The above example immediately generalizes from circles to arbitrary n-spheres.

Definition 3.2.4. For X an object in $\mathbf{H}$ and for $n \in \mathbb{N}$, the *free n-sphere space object* of X is

$$\mathbf{Maps}^h(S^n, X)\,.$$

An $(n+1)$-sphere is obtained by gluing two $(n+1)$-disks along their common boundary, which is an n-sphere. Since the disks are contractible, from a homotopy type point of view, this amount to the natural equivalence

$$\Pi(S^{n+1}) \simeq * \coprod_{\Pi(S^n)} * \,.$$

Applying the internal homs to X to this equivalence and recalling that $[-, X]$ preserves homotopy limits, one obtains the following result.

Proposition 3.2.5. *For all $n \in \mathbb{N}$ we have a natural homotopy pullback square*

$$\begin{tikzcd} \mathbf{Maps}^h(S^{n+1}, X) \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & \mathbf{Maps}^h(S^n, X)\,. \end{tikzcd}$$

3.3 Local prequantum field theories

Here we introduce the formalization of *local prequantum field theory* that we want to consider and discuss some of its basic properties. We will work within the framework of fully extended TQFTs as developed in [L09a], to which we refer the reader for details on the definitions and the results of this section. See also [L09b] for the conventions used here on (∞, n)-categories.

3.3.1 The geometric background of the bulk field theory

Definition 3.3.1. For $n \in \mathbb{N}$, we write $\mathrm{Bord}_n^{\otimes}$ for the symmetric monoidal (∞, n)-category of n-dimensional framed cobordism. For $\mathcal{C}^{\otimes}$ any symmetric monoidal (∞, n)-category, a *local topological field theory* in dimension n with coefficients in $\mathcal{C}$ is a symmetric monoidal (∞, n)-functor

$$Z : \mathrm{Bord}_n^{\otimes} \to \mathcal{C}^{\otimes}\,.$$

By the cobordism hypothesis, $\mathrm{Bord}_n^{\otimes}$ is the free symmetric monoidal (∞, n)-category with full duals generated by a single object $*$. This means that an n-dimensional local topological field theory is completely determined by the object $Z(*)$ in $\mathcal{C}^{\otimes}$ and that this object is necessarily a fully dualizable object.

Remark 3.3.2. Note the slight notational difference to [L09a]: there the undecorated symbols "Bord_n" denote *framed* cobordisms.

The following definition is sketched in section 3.2 of [L09a] (there written "Fam_n" instead of "Corr_n".)

Definition 3.3.3. Write

$$\mathrm{Corr}_1 := \left\{ \begin{tikzcd} i & c \arrow[l] \arrow[r] & o \end{tikzcd} \right\}$$

for the category free on a single correspondence, i.e. consisting of three objects and two non-identity morphisms from one to the other two. For $n \in \mathbb{N}$ write

$$\mathrm{Corr}_n := (\mathrm{Corr}_1)^{\times^n}$$

for the n-fold cartesian product of this category with itself. Finally, $\mathrm{Corr}_n(\mathbf{H})$ is the ∞-groupoid of functors from Corr_n to $\mathbf{H}$.

Remark 3.3.4. Under composition of correspondences by fiber product of maps to a common face, this naturally carries the structure of an n-fold category object in ∞**-Grpd**, hence of an (∞, n)-category. Moreover, from the cartesian product in $\mathbf{H}$ the (∞, n)-category $\mathrm{Corr}_n(\mathbf{H})$ inherits a natural structure of symmetric monoidal (∞, n)-category, which we will denote $\mathrm{Corr}_n(\mathbf{H})^{\otimes}$

Example 3.3.5. By definition, Corr_n is the terminal category, so a 0-morphism (i.e., an object) in $\mathrm{Corr}_n(\mathbf{H})$ is just an object in $\mathbf{H}$. A 1-morphism in $\mathrm{Corr}_n(\mathbf{H})$ is a diagram in $\mathbf{H}$ of the form

$$A_i \longleftarrow A_c \longrightarrow A_o \ .$$

In the application to prequantum field theory such a diagram is typically interpreted as follows: A_i is a moduli stack of fields on an *i*ncoming piece of worldvolume and A_o that of field on an *o*utgoing piece. The object A_c is that of fields on a piece of worldvolume *c*onnecting these two pieces, putting them in *c*orrespondence, hence A_c is the collection of *trajectories* of field configurations from the incoming to the outgoing piece. The left map sends such a trajectory to its initial configuration, the right one to its final configuration. A 2-morphism in $\mathrm{Corr}_n(\mathbf{H})$ is a diagram in $\mathbf{H}$ of the form

$$\begin{tikzcd} A_{ii} & A_{ic} \arrow[l] \arrow[r] & A_{io} \\ A_{ci} \arrow[u] \arrow[d] & A_{cc} \arrow[l] \arrow[r] \arrow[u] \arrow[d] & A_{co} \arrow[u] \arrow[d] \\ A_{oi} & A_{oc} \arrow[l] \arrow[r] & A_{oo} \end{tikzcd} ,$$

and so on. Composition of morphisms is via homotopy fiber products in $\mathbf{H}$. For instance, the composition of the two 1-morphisms

$$X \longleftarrow Y \longrightarrow Z \qquad \text{and} \qquad Z \longleftarrow S \longrightarrow T$$

is the 1-morphism

$$X \longleftarrow Y \times_Z S \longrightarrow T \ .$$

In the above interpretation of these correspondences in prequantum field theory, this operation corresponds to gluing or concatenating trajectories of field configurations whenever they match over their outgoing/ingoing pieces of worldvolume, respectively. The compositions of higher morphisms are defined analogously.

Proposition 3.3.6. *For all $n \in \mathbb{N}$, every object $X \in \mathrm{Corr}_n(\mathbf{H})^{\otimes}$ is fully dualizable and is in fact its own full dual. The k-dimensional trace of the identity on X in $\mathrm{Corr}_n(\mathbf{H})^{\otimes}$ is its free k-sphere space object:*

$$\dim_k(X) \simeq \mathbf{Maps}^h(S^k, X) \,,$$

seen as a k-fold correspondence from the terminal object to itself.

Proof. This is essentially remark 3.2.3 in [L09a], spelled out in more detail. For $X \in \mathbf{H}$, take the co-evaluation and evaluation morphisms $\epsilon : \mathbb{I} \to X \times X$ and $\eta : X \times X \to \mathbb{I}$ to be given by the "⊂" and by the "⊃", i.e., by the correspondences

$$* \longleftarrow X \xrightarrow{\Delta_X} X \times X \qquad \text{and} \qquad X \times X \xleftarrow{\Delta_X} X \longrightarrow * \ ,$$

where Δ_X denotes the diagonal map for X. For these to exhibit a self-duality, the zig-zig-identities

$$\begin{tikzcd} X \arrow[r, "X\times\epsilon"] \arrow[rr, bend right, "\mathrm{id}_X"'] & X \times X \times X \arrow[r, "\eta\times X"] & X \end{tikzcd} , \quad \begin{tikzcd} X \arrow[r, "\epsilon\times X"] \arrow[rr, bend right, "\mathrm{id}_X"'] & X \times X \times X \arrow[r, "\eta\times X"] & X \end{tikzcd}$$

have to hold. A moment of reflection shows that indeed

$$
\begin{tikzcd}
 & & X \arrow[dl, "\Delta_X"'] \arrow[dr, "\Delta_X"] \arrow[dll, bend right, "\mathrm{id}_X"'] \arrow[drr, bend left, "\mathrm{id}_X"] & & \\
 & X \times X \arrow[dl, "p_1"'] \arrow[dr, "(\mathrm{id}_X, \Delta_X)"'] & {\scriptstyle (\mathrm{pb})} & X \times X \arrow[dl, "(\Delta_X, \mathrm{id}_X)"] \arrow[dr, "p_2"] & \\
X & & X \times X \times X & & X ,
\end{tikzcd}
$$

and similarly for the other composite. As a consequence, the trace of the identity of X

$$
\mathrm{tr}(\mathrm{id}_X) := \ \mathbb{I} \xrightarrow{\epsilon} X \times X \xrightarrow{\eta} \mathbb{I}
$$

is given by the correspondence

$$
\begin{tikzcd}
 & & \mathcal{L}X \arrow[dl] \arrow[dr] & & \\
 & X \arrow[dl] \arrow[dr, "\Delta_X"] & {\scriptstyle (\mathrm{pb})} & X \arrow[dl, "\Delta_X"'] \arrow[dr] & \\
* & & X \times X & & * .
\end{tikzcd}
$$

Therefore,

$$
\dim_1(X) \simeq \mathcal{L}X \simeq \mathbf{Maps}^h(S^1, X) ,
$$

which amounts to the pictorial identity

Ↄ ∘ C ≅ O .

Next to exhibit $(\epsilon \dashv \eta)$ as a full adjunction, we need to produce units and co-units, which again will have adjoints, and so on. Take the unit of $(\epsilon \dashv \eta)$ and its adjoint to be 2-fold correspondences given by the "bowl" and the "dome", i.e.,

$$
\begin{tikzcd}
* & * \arrow[l] \arrow[r] & * \\
* \arrow[u] \arrow[d] & X \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \mathbf{Maps}^h(S^1, X) \arrow[u] \arrow[d] \\
* & * \arrow[l] \arrow[r] & *
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
* & * \arrow[l] \arrow[r] & * \\
\mathbf{Maps}^h(S^1, X) \arrow[u] \arrow[d] & X \arrow[l] \arrow[r] \arrow[u] \arrow[d] & * \arrow[u] \arrow[d] \\
* & * \arrow[l] \arrow[r] & * ,
\end{tikzcd}
$$

take the co-unit to be the "saddle", i.e., the 2-fold correspondence

$$
\begin{tikzcd}
X \times X & X \times X \arrow[l, "\mathrm{id}_{X \times X}"'] \arrow[r, "\mathrm{id}_{X \times X}"] & X \times X \\
X \times X \arrow[u, "\mathrm{id}_{X \times X}"] \arrow[d, "\mathrm{id}_{X \times X}"'] & X \arrow[l, "\Delta_X"'] \arrow[r, "\Delta_X"] \arrow[u, "\Delta_X"] \arrow[d, "\Delta_X"] & X \times X \arrow[u, "\mathrm{id}_{X \times X}"'] \arrow[d, "\mathrm{id}_{X \times X}"] \\
X \times X & X \times X \arrow[l, "\Delta_X \circ p_1"] \arrow[r, "\Delta_X \circ p_2"'] & X \times X
\end{tikzcd}
$$

and the adjoint counit to be the "reverse saddle" corresponding to the reverse 2-fold correspondence.

Then to check the first zig-zag identity we need to observe that the composite

$$\begin{tikzcd}
* & X \arrow[l] \arrow[r, "\Delta_X"] & X \times X & & X \times X \arrow[ll, "\mathrm{id}_{X\times X}"'] \arrow[rr, "\mathrm{id}_{X\times X}"] & & X \times X \\
* \arrow[u] \arrow[d] & X \arrow[l] \arrow[r, "\Delta_X"] \arrow[u, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] & X \times X \arrow[u, "\mathrm{id}_{X\times X}"] \arrow[d, "\mathrm{id}_{X\times X}"'] & & X \arrow[ll, "\Delta_X"'] \arrow[rr, "\Delta_X"] \arrow[u, "\Delta_X"] \arrow[d, "\Delta_X"'] & & X \times X \arrow[u, "\mathrm{id}_{X\times X}"'] \arrow[d, "\mathrm{id}_{X\times X}"] \\
* \arrow[d, equal] & X \arrow[l] \arrow[r, "\Delta_X"] \arrow[d, equal] & X \times X \arrow[d, phantom, "\text{(pb)}"] & & X \times X \arrow[ll, "\Delta_X \circ p_1"'] \arrow[rr, "\Delta_X \circ p_2"] \arrow[d, equal] & & X \times X \arrow[d, equal] \\
* & X \arrow[l] & \mathbf{Maps}^h(S^1, X) \times X \arrow[l] \arrow[rr] & & X \times X \arrow[r, "p_2"] & X \arrow[r, "\Delta_X"] & X \times X \\
* \arrow[u] \arrow[d] & & X \times X \arrow[ll] \arrow[rrr, "p_2"] \arrow[u] \arrow[d, "p_2"] & & & X \arrow[r, "\Delta_X"] \arrow[u] \arrow[d, "\mathrm{id}_X"] & X \times X \arrow[u, "\mathrm{id}_{X\times X}"'] \arrow[d, "\mathrm{id}_{X\times X}"] \\
* & & X \arrow[ll] \arrow[rrr, "\mathrm{id}_X"] & & & X \arrow[r, "\Delta_X"] & X \times X
\end{tikzcd}$$

is equivalent to the "vertical identity" 2-correspondence

$$\begin{tikzcd}
* & X \arrow[l] \arrow[r, "\Delta_X"] & X \times X \\
* \arrow[u] \arrow[d] & X \arrow[l] \arrow[r, "\Delta_X"] \arrow[u, "\mathrm{id}_X"] \arrow[d, "\mathrm{id}_X"'] & X \times X \arrow[u, "\mathrm{id}_{X\times X}"'] \arrow[d, "\mathrm{id}_{X\times X}"] \\
* & X \arrow[l] \arrow[r, "\Delta_X"] & X \times X
\end{tikzcd}$$

by the universal property of the homotopy pullback enjoyed by $\mathbf{Maps}^h(S^1, X)$. Checking of the other zig-zag identities is completely analogous.

In this fashion we are to proceed by induction. The k-fold units and their adjoints will be k-fold correspondences of correspondences with tips given by

$$* \longleftarrow X \longrightarrow \mathbf{Maps}^h(S^k, X) \qquad \text{and} \qquad \mathbf{Maps}^h(S^k, X) \longleftarrow X \longrightarrow * \ .$$

By proposition 3.2.5 the k-fold trace on the identity then is

$$\begin{tikzcd}
 & & \mathbf{Maps}^h(S^{k+1}, X) \arrow[dl] \arrow[dr] & & \\
 & X \arrow[dl] \arrow[dr] & & X \arrow[dl] \arrow[dr] & \\
* & & \mathbf{Maps}^h(S^k, X) & & * \ .
\end{tikzcd}$$

□

By the classification of topological field theories [L09a], we therefore have the following

Proposition 3.3.7. *Any higher smooth stack* **Fields** *in* **H** *determines a fully extended topological quantum field theory with values in* $\mathrm{Corr}_n(\mathbf{H})$*,*

$$Z_{\mathbf{Fields}} : \mathrm{Bord}_n^{\otimes} \to \mathrm{Corr}_n(\mathbf{H})^{\otimes} ,$$

characterized by the condition $Z_{\mathbf{Fields}}(*) \cong \mathbf{Fields}$*.*

Definition 3.3.8. By abuse of notation we will write

$$\mathbf{Fields} : \mathrm{Bord}_n^{\otimes} \to \mathrm{Corr}_n(\mathbf{H})^{\otimes} .$$

for the symmetric monoidal (∞, n)-functor $Z_{\mathbf{Fields}}$ and will call it the n-dimensional *local bulk field theory* with stacks of fields **Fields**.

Remark 3.3.9. By handle decomposition of smooth manifolds it follows that the symmetric monoidal functor **Fields** sends a closed manifold Σ_k of dimension k to the smooth stack $\mathbf{Maps}^h(\Sigma_k, \mathbf{Fields})$, seen as a k-fold correspondence of correspondences between the terminal object and itself. Similarly, a manifold with boundary $\partial\Sigma_k \hookrightarrow \Sigma_k$ is mapped to the k-fold correspondence which is trivial except for its tip, which is

$$\mathbf{Maps}^h(\partial\Sigma_k, \mathbf{Fields}) \longleftarrow \mathbf{Maps}^h(\Sigma_k, \mathbf{Fields}) \longrightarrow * \; .$$

This pattern continues for boundaries of boundaries, and so on.

3.3.2 Local action functionals for the bulk field theory

In addition to field configurations, prequantum field theory encodes the local *action functionals* or *Lagrangians* on these. This involves equipping all the objects described above with maps to a given space "of phases", a suitable higher version of the group $U(1)$ in which traditional action functionals take values. For instance, in the Introduction we considered Lagrangians of the form $\mathbf{L} : \mathbf{Fields} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$, in which the space of phases was the n-stack of $U(1)$ n-bundles with connection. More generally, we will choose the space of phases to be a commutative group object **Phases** in **H**. Clearly, since we are working in a higher categorical setting, "commutative" here means "commutative up to coherent homotopies", and the same consideration applies to the group structure of the space of phases. That is, more precisely, **Phases** is an E_∞-group object in **H**.

Remark 3.3.10. The fact that here we consider **Phases** to be group object in **H** instead of a more general stack of symmetric monodical (∞, n)-categories is related to the fact that in this paper we are focusing on what in classical terms is prequantum field theory as opposed to quantum field theory. For the latter one chooses a representation $\mathbf{Phases} \to \mathcal{C}$ of the space of phases on a genuine (∞, n)-category and postcomposes the Lagrangian with this.

The general mechanism to describe local action functionals is based on the following simple observation.

Remark 3.3.11. The commutative group structure on **Phases** endows the slice topos $\mathbf{H}_{/\mathbf{Phases}}$ with a natural tensor product lifting the cartesian product of **H** by

$$\begin{bmatrix} X \\ \big\downarrow{\scriptstyle F_1} \\ B \end{bmatrix} \otimes \begin{bmatrix} Y \\ \big\downarrow{\scriptstyle F_2} \\ B \end{bmatrix} := \begin{bmatrix} X \times Y \\ \big\downarrow{\scriptstyle \pi_X^* F_1 + \pi_Y^* F_2} \\ \mathbf{Phases} \end{bmatrix} := \begin{bmatrix} X \times Y \\ \big\downarrow{\scriptstyle (\pi_X^* F_1, \pi_Y^* F_2)} \\ \mathbf{Phases} \times \mathbf{Phases} \\ \big\downarrow{\scriptstyle +} \\ \mathbf{Phases} \end{bmatrix} ,$$

where on the right we use the group structure on **Phases**. Here π_X and π_Y are the corresponding projections. The tensor unit is the unit inclusion:

$$\mathbb{I} = \left[\begin{array}{c} * \\ \downarrow{\scriptstyle 0} \\ \mathbf{Phases} \end{array} \right] .$$

We can therefore mimic definition 3.3.3 and remark 3.3.4.

Definition 3.3.12. The symmetric monoidal (∞, n)-category $\mathrm{Corr}_n(\mathbf{H}_{/\mathbf{Phases}})^{\otimes}$ is the (∞, n)-category structure on the is the ∞-groupoid of functors from Corr_n to $\mathbf{H}_{/\mathbf{Phases}}$ with compositions of correspondences by fiber product of maps to a common face and symmetric monoidal product induced by the symmetric monoidal category structure on $\mathbf{H}_{/\mathbf{Phases}}$ described in remark 3.3.11.

Notice that the forgetful morphism $\mathbf{H} \to \mathbf{H}_{/\mathbf{Phases}}$ induces a natural forgetful monoidal contravariant functor

$$\mathrm{Corr}_n(\mathbf{H}_{/\mathbf{Phases}})^{\otimes} \to \mathrm{Corr}_n(\mathbf{H})^{\otimes}.$$

Thanks to the commutative group structure on the space of phases, we have the following generalization of proposition 3.3.6.

Proposition 3.3.13. *Every object* $X \xrightarrow{F} \mathbf{Phases}$ *in* $\mathrm{Corr}_n(\mathbf{H}_{/\mathbf{Phases}})^{\otimes}$ *is fully dualizable, with dual* $-F$.

Proof. Take the co-evaluation map $\mathbb{I} \to F \otimes (-F)$ and evaluation map $F \otimes (-F) \to \mathbb{I}$ to be given by

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr, "\Delta_X"] & \\ * \arrow[dr, "0"'] & & X \times X \arrow[dl, "p_1^*F - p_2^*F"] \\ & \mathbf{Phases} & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X \arrow[dl, "\Delta_X"'] \arrow[dr] & \\ X \times X \arrow[dr, "p_1^*F - p_2^*F"'] & & * \arrow[dl, "0"] \\ & \mathbf{Phases} & \end{tikzcd} ,$$

respectively. Here p_1 and p_2 denote projection to the first and second factors, respectively, and the squares are filled by the canonical equivalence $p_1 \circ \Delta_X \cong p_2 \circ \Delta_X$. From here on the argument proceeds just as in the proof of proposition 3.3.6. □

We therefore have the following analogue of proposition 3.3.7.

Proposition 3.3.14. *Any morphism* $\mathbf{L} : \mathbf{Fields} \to \mathbf{Phases}$ *in* $\mathbf{H}$ *determines a fully extended topological quantum field theory with values in* $\mathrm{Corr}_n(\mathbf{H}_{/\mathbf{Phases}})$,

$$e^{2\pi i \mathbf{L}} : \mathrm{Bord}_n^{\otimes} \longrightarrow \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{Phases}})^{\otimes} ,$$

characterized by the condition $e^{2\pi i \mathbf{L}}(*) \cong \mathbf{L}$.

Definition 3.3.15. In view of the fully extended TQFT it defines, we will call a morphism $\mathbf{L} : \mathbf{Fields} \to \mathbf{Phases}$ a *local action functional* (or *Lagrangian*) for the TQFT with **Fields** as stack of fields.

Remark 3.3.16. Since fully extended TQFTs are completely determined by their value on the point, a local action functional for a given geometric background/stack of fields is equivalent to the datum of a symmetric monoidal lift

$$\begin{tikzcd} & \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{Phases}}) \arrow[d] \\ \mathrm{Bord}_n \arrow[ur, "\exp(2\pi i \mathbf{L})"] \arrow[r, "\mathbf{Fields}"'] & \mathrm{Corr}_n(\mathbf{H}) \end{tikzcd} ,$$

which is the perspective in [FHLT].

Example 3.3.17. The circle S^1 is mapped to the following span in $\mathrm{Corr}_n(\mathbf{H}_{/\mathbf{Phases}})$

$$\begin{tikzcd}
 & & \mathbf{Maps}^h(S^1, \mathbf{Fields}) \arrow[dl] \arrow[dr] & & \\
 & \mathbf{Fields} \arrow[dl] \arrow[dr, "\Delta_{\mathbf{Fields}}"] \arrow[rr, phantom, "\text{(pb)}"] & & \mathbf{Fields} \arrow[dl, "\Delta_{\mathbf{Fields}}"'] \arrow[dr] & \\
 * \arrow[drr, "0"'] & & \mathbf{Fields} \times \mathbf{Fields} \arrow[d, "p_1^*\mathbf{L} - p_2^*\mathbf{L}"] & & * . \arrow[dll, "0"] \\
 & & \mathbf{Phases} & &
\end{tikzcd}$$

By the universal property of the pullback, this induces a morphism

$$\mathbf{Maps}^h(S^1, \mathbf{Fields}) \to \mathbf{\Omega Phases},$$

where $\mathbf{\Omega Phases}$ is the based loop space of **Phases**, i.e., the object in **H** defined by the homotopy pullback diagram

$$\begin{tikzcd}
\mathbf{\Omega Phases} \arrow[r] \arrow[d] & * \arrow[d, "0"] \\
* \arrow[r, "0"'] & \mathbf{Phases} .
\end{tikzcd}$$

Notice that, since **Phases** is an abelian group object in **H**, then so is $\mathbf{\Omega Phases}$.

3.3.3 Boundary field theory

We now turn to the discussion of boundary data for a local prequantum field theory. We write $\mathrm{Bord}_n^{\mathrm{bdr}\otimes}$ for the symmetric monoidal (∞, n)-category of n-dimensional cobordism with (constrained) boundaries. It is freely generated by adding to $\mathrm{Bord}_n{}^{\otimes}$ the "minimal" constrained boundary, i.e.,

$$\vdash\!\!\!\longrightarrow * \;,$$

interpreted as a cobordism from the empty set to the point. In more formal terms, we are saying that, just as $\mathrm{Bord}_n{}^{\otimes}$ is the free symmetric monoidal (∞, n) category with duals on a single self-dual object, $\mathrm{Bord}_n^{\mathrm{bdr}\otimes}$ is free symmetric monoidal (∞, n) category with duals on a single morphism to the unit object to a self-dual object, see Section ?? in [L09a].

Definition 3.3.18. Let **Fields** a stack of fields for a fully extended TQFT. A *boundary condition* (or *boundary extension*) for **Fields** is a symmetric monoidal extension

$$\begin{tikzcd}
\mathrm{Bord}_n{}^{\otimes} \arrow[r, "\mathbf{Fields}"] \arrow[d] & \mathrm{Corr}_n(\mathbf{H})^{\otimes} \; , \\
\mathrm{Bord}_n^{\mathrm{bdr}\otimes} \arrow[ur, "Z^{\mathrm{bdr}}_{\mathbf{Fields}}"'] &
\end{tikzcd}$$

where the right vertical arrow is the inclusion of cobordism without (constrained) boundaries into cobordism with boundaries.

Proposition 3.3.19. *A boundary condition for* $\mathbf{L}$ *is equivalently a morphism*

$$\mathbf{Fields}_{\partial} \to \mathbf{Fields}$$

in **H**

Proof. Since $\mathrm{Bord}_n^{\mathrm{bdr}}$ is free symmetric monoidal with duals on a single morphism out of the unit object, a symmetric monoidal functor $e^{2\pi i \mathbf{L}_\partial} : \mathrm{Bord}_n^{\mathrm{bdr}\otimes} \to \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{Phases}})^\otimes$ is equivalent to the datum of a 1-morphism in $\mathrm{Corr}_n(\mathbf{H})$ out of $*$. Requiring that the morphism $Z^{\mathrm{bdr}}_{\mathbf{Fields}}$ covers **Fields** then amounts to asking that this 1-morphism in $\mathrm{Corr}_n(\mathbf{H})$ has target **Fields**, and so it is a correspondence of the form

$$* \longleftarrow \mathbf{Fields}_\partial \longrightarrow \mathbf{Fields} \ .$$

Since $*$ is the terminal object in $\mathbf{H}$, this is equivalent to the datum of the morphism $\mathbf{Fields}_\partial \to \mathbf{Fields}$. □

Corollary 3.3.20. *The* ∞-category of boundary conditions *for* **Fields** *is the slice* ∞*-topos* $\mathbf{H}_{/\mathbf{Fields}}$.

Proposition 3.3.21. *For* $\partial\Sigma \hookrightarrow \Sigma$ *a cobordism with closed marked boundary with* Σ *a* k*-dimensional manifold, the field theory with boundary conditions*

$$(\mathbf{Fields}_\partial \to \mathbf{Fields}) \ : \ (\mathrm{Bord}_n^{\mathrm{bdr}})^\otimes \to \mathrm{Corr}_n(\mathbf{H})^\otimes$$

acts as

$$(\mathbf{Fields}_\partial \to \mathbf{Fields}) \ : \ (\partial\Sigma \hookrightarrow \Sigma) \ \mapsto \ \mathbf{Maps}^h(\partial\Sigma, \mathbf{Fields}_\partial) \underset{\mathbf{Maps}^h(\partial\Sigma, \mathbf{Fields})}{\times} \mathbf{Maps}^h(\Sigma, \mathbf{Fields}) \ ,$$

seen as a k*-fold correspondence of correspondences between the terminal object and itself.*

Proof. Every cobordism Σ with marked boundary component $\partial\Sigma$ decomposes as the gluing of the cylinder $(\ \vdash\!\!\longrightarrow\!\! * \ ) \times \partial\Sigma$ with Σ regarded as a manifold with unmarked boundary. Since $\vdash\!\!\longrightarrow\!\! *$ is mapped to the 1-morphism

$$* \longleftarrow \mathbf{Fields}_\partial \longrightarrow \mathbf{Fields}$$

in $\mathrm{Corr}_n(\mathbf{H})^\otimes$ by the TQFT with boundary associated with $\mathbf{Fields}_\partial \to \mathbf{Fields}$, we find that $(\ \vdash\!\!\longrightarrow\!\! * \ ) \times \partial\Sigma$ is mapped to

$$* \longleftarrow \mathbf{Maps}^h(\partial\Sigma, \mathbf{Fields}_\partial) \longrightarrow \mathbf{Maps}^h(\partial\Sigma, \mathbf{Fields}) \ .$$

On the other hand, on the "piece" given by Σ with unmarked boundary $\partial\Sigma$ the TQFT reduces to the one associated with the stack **Fields**, and we know from remark 3.3.9 that $\partial\Sigma \hookrightarrow \Sigma$ is mapped by **Fields** to

$$\mathbf{Maps}^h(\partial\Sigma, \mathbf{Fields}) \longleftarrow \mathbf{Maps}^h(\Sigma, \mathbf{Fields}) \longrightarrow * \ .$$

The composite of these two contributions is

$$* \longleftarrow \mathbf{Maps}^h(\partial\Sigma, \mathbf{Fields}_\partial) \underset{\mathbf{Maps}^h(\partial\Sigma, \mathbf{Fields})}{\times} \mathbf{Maps}^h(\Sigma, \mathbf{Fields}) \longrightarrow * \ ,$$

as claimed. □

Remark 3.3.22. In more colloquial terms, the above proposition means that for the TQFT with boundary conditions $\mathbf{Fields}_\partial \to \mathbf{Fields}$, a field configurations on a manifold Σ with constrained boundary $\partial\Sigma$, it is the most natural among the possibilities: it is given by a bulk field configuration on Σ together with a boundary field configuration on $\partial\Sigma$ and an equivalence of the boundary field configuration with the restriction of the bulk field configuration to the boundary. These data are equivalently those of a *twisted cocycle* with local coefficient bundle $\mathbf{Fields}_\partial \to \mathbf{Fields}$, *relative* to the boundary inclusion [SSS12] [FSS13][NSS]. In particular, when $\mathbf{Fields}_\partial \simeq *$ then these are equivalently cocycles in *relative cohomology* with coefficients in **Fields**.

We now add local action functionals to the above picture.

Definition 3.3.23. Let $\mathbf{L} : \mathbf{Fields} \to \mathbf{Phases}$ be a local Lagrangian for a TQFT. A *boundary condition* (or *boundary extension*) for $\mathbf{L}$ is a symmetric monoidal extension

$$\begin{tikzcd} \mathrm{Bord}_n{}^{\otimes} \arrow[r, "e^{2\pi i \mathbf{L}}"] \arrow[d] & \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{Phases}})^{\otimes} \\ \mathrm{Bord}_n^{\mathrm{bdr}\otimes} \arrow[ur, "e^{2\pi i \mathbf{L}_\partial}"'] & \end{tikzcd} ,$$

where the vertical arrow is the inclusion of cobordism without (constrained) boundaries into cobordism with boundaries.

Proposition 3.3.24. *A boundary condition for* $\mathbf{L}$ *is equivalently a morphism*

$$\mathbf{Fields}_{\partial} \to \mathrm{fib}(\mathbf{L})$$

in $\mathbf{H}$*, where* $\mathrm{fib}(\mathbf{L})$ *is the homotopy fiber of* $\mathbf{L} : \mathbf{Fields} \to \mathbf{Phases}$ *on the zero element of the commutative group stack of phases.*

Proof. Since $\mathrm{Bord}_n^{\mathrm{bdr}}$ is free symmetric monoidal with duals on a single morphism out of the unit object, a symmetric monoidal functor $e^{2\pi i \mathbf{L}_\partial} : \mathrm{Bord}_n^{\mathrm{bdr}\otimes} \to \mathrm{Corr}_n(\mathbf{H}_{/\mathbf{Phases}})^{\otimes}$ is equivalent to the datum of a 1-morphism in $\mathrm{Corr}_n(\mathbf{H}_{/\mathbf{Phases}})$ out of $* \xrightarrow{0} \mathbf{Phases}$. Requiring that the morphism $e^{2\pi i \mathbf{L}_\partial}$ covers $e^{2\pi i \mathbf{L}}$ then amounts to asking that this 1-morphism in $\mathrm{Corr}_n(\mathbf{H}_{/\mathbf{Phases}})$ has target $\mathbf{Fields} \xrightarrow{\mathbf{L}} \mathbf{Phases}$, and so it is a homotopy commutative diagram in $\mathbf{H}$ of the form

$$\begin{tikzcd} & \mathbf{Fields}_{\partial} \arrow[dl] \arrow[dr] & \\ * \arrow[dr, "0"'] & \arrow[d, phantom, "\swarrow"] & \mathbf{Fields} \arrow[dl, "\mathbf{L}"] \\ & \mathbf{Phases} & \end{tikzcd} .$$

By the universal property of the homotopy pullback, such a homotopy commutative diagram is equivalent to a morphism $\mathbf{Fields}_{\partial} \to \mathrm{fib}(\mathbf{L})$, where $\mathrm{fib}(\mathbf{L})$ is defined by the homotopy pullback diagram

$$\begin{tikzcd} \mathrm{fib}(\mathbf{L}) \arrow[r] \arrow[d] & \mathbf{Fields} \arrow[d, "\mathbf{L}"] \\ * \arrow[r, "0"'] & \mathbf{Phases} \end{tikzcd} .$$

□

Definition 3.3.25. The ∞-*category of boundary conditions* for $\mathbf{L} : \mathbf{Fields} \to \mathbf{Phases}$ is the slice ∞-topos $\mathbf{H}_{/\mathrm{fib}(\mathbf{L})}$. The object $\mathrm{fib}(\mathbf{L})$ of $\mathbf{H}$ is the *universal boundary condition* for $\mathbf{L}$.

Proposition 3.3.26. *Here we should have the analogue of Proposition 3.3.21, but with local action functionals.*

3.3.4 Corner field theory

We now consider two different boundary conditions as above, together with a "defect" or "corner condition" that interpolates from one to the other. For this purpose, we consider the symmetric monoidal (∞, n)-category $(\mathrm{Bord}_n^{\partial\partial})^{\otimes}$ of cobordisms with constrained boundaries and corners. More precisely, here we are considering two different colors for the boundaries (i.e., two different boundary conditions), and a single possible corner (an equivalence between the given boundary conditions in codimension-2 boundaries), the one where two boundaries of different colors meet. But in principle one could consider an arbitrary number of colors for the boundaries, and different possibilities for the corners. From the physics point of view, each of these colors is a *brane* and the corners are the brane intersections. For instance one could consider a single self-intersecting brane. In that case one would have a single boundary condition and a self-equivalence of it in codimension-2. We will see an example of this phenomenon in section **??**.

Remark 3.3.27. From a purely categorical point of view, $(\mathrm{Bord}_n^{\partial\partial})^{\otimes}$ is the symmetric monoidal (∞, n)-category which is free on

1. an object $*$;
2. two morphism $(\vdash\!\!\longrightarrow *) : \emptyset \to *$ and $(\vdash\!\!\longrightarrow *) : \emptyset \to *$,
3. and a 2-morphism of the form

$$(\vdash\!\!\longrightarrow *) \times \begin{pmatrix} - \\ | \\ * \end{pmatrix} : \quad \begin{tikzcd} \emptyset \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \emptyset \arrow[d] \\ \emptyset \arrow[r] & * \end{tikzcd} .$$

As an immediate consequence, we have:

Proposition 3.3.28. *A symmetric monoidal functor*

$$(\mathrm{Bord}_n^{\partial\partial})^{\otimes} \to \mathrm{Corr}_n(\mathbf{H})^{\otimes}$$

is equivalently the datum of

1. *a moduli stack* **Fields** *of* bulk fields*;*
2. *two moduli stacks* $\mathbf{Fields}^{\partial}, \mathbf{Fields}^{\partial}$ *of boundary field configurations for two kinds of boundary conditions;*
3. *a moduli stack* $\mathbf{Fields}^{\partial\partial}$ *of* corner fields *or* ∂-∂-defect fields*;*
4. *a homotopy commutative diagram*

$$\begin{tikzcd} \mathbf{Fields}^{\partial\partial} \arrow[r] \arrow[d] & \mathbf{Fields}^{\partial} \arrow[d] \arrow[dl, Rightarrow, "\simeq"] \\ \mathbf{Fields}^{\partial} \arrow[r] & \mathbf{Fields} \end{tikzcd}$$

in $\mathbf{H}$.

Remark 3.3.29. The above proposition can be rephrased by saying that a corner extension of a TQFT with **Fields** as stack of fields is a 1-morphism in $\mathrm{Corr}_n(\mathbf{H}_{/\mathbf{Fields}})$.

Here we are missing all the part of "local lagrangians for corner field theories". To be written.

3.3.5 Defect field theory

Finally, let us sketch a few lines on topological defects. These corespond to adding another piece to the picture of framed cobordism, namely that of a punctured k-disk, seen as a morphism from the vacuum to the $(k-1)$-sphere. In more formal terms, since a k-disk is homotopically trivial, this amounts to the following.

Definition 3.3.30. Give a bulk field **Fields** in **H**, a codimension-k defect datum is a k-fold correspondence whose only nontrivial part is the tip

$$\mathbf{Fields}^{\mathrm{def}} \longleftarrow \mathbf{Fields}_{\mathrm{traj}} \longrightarrow \mathbf{Maps}^h(S^{k-1}, \mathbf{Fields}) \ .$$

Examples of such defects and further comments on how to think of them appear as example 4.3.24 and example 4.3.29 below.

Here, too, we are missing all the part of "local lagrangians for field theories with defects". To be written.

3.4 Stacks of higher $U(1)$-bundles with connections

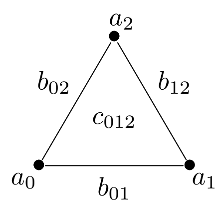
The essential tool for producing the ∞-stack **Phases** with commutative ∞-group structure is the Dold-Kan correspondence, which associates such a stack to any presheaf of complexes of abelian groups concentrated in nonegative degrees (see for instance section III.2 of [GJ] for a review). It can be briefly described as follows: given the chain complex

$$A_\bullet = \cdots \xrightarrow{\partial} A_3 \xrightarrow{\partial} A_2 \xrightarrow{\partial} A_1 \xrightarrow{\partial} A_0 \ \ ,$$

the simplicial set $\mathrm{DK}(A_\bullet)$ is defined as follows:

- the abelian group of 0-simplices of $\mathrm{DK}(A_\bullet)$ is the abelian group A_0;
- the abelian group of n-simplices of $\mathrm{DK}(A_\bullet)$ is the abelian group whose elements are standard n-simplices decorated by an element x in A_n such that ∂x equals the (oriented) sum of the decorations on the boundary $(n-1)$-simplices.

For instance, a 2-simplex in $\mathrm{DK}(A_\bullet)$ is



where

- $a_i \in A_0$;
- $b_{ij} \in A_1$ and $\partial b_{ij} = a_j - a_i$;
- $c_{012} \in A_2$ and $\partial c_{012} = b_{12} - b_{02} + b_{01}$.

All this prolongs directly to presheaves of chain complexes, so that to any such presheaf is naturally associated a presheaf of simplicial sets. By sheafifying this presheaf, one obtains a stack, and the abelian group structure on the groups in the chain complex naturally induces a commutative ∞-group structure on this stack.

Definition 3.4.1. We say that the stack **Phases** is presented via the Dold-Kan correspondence by the presheaves of chain complexes of abelian groups $\mathcal{A}_\bullet$ if it is obtained from $\mathcal{A}_\bullet$ via the procedure described above.

In the following sections we will consider 'higher versions' of the abelian group $U(1)$ where classical phases take their values. We now describe these stacks.

Definition 3.4.2. For $n \in \mathbb{N}$,

1. $\mathbf{B}^n U(1)$ is the stack presented by the complex $\underline{U}(1) \to 0 \to 0 \cdots \to 0$, with the sheaf $\underline{U}(1)$ of smooth functions with values in $U(1)$ placed in degree n;

2. $\mathbf{B}^n U(1)_{\mathrm{conn}}$ is the stack presented by the complex $\underline{U}(1) \xrightarrow{\frac{1}{2\pi i} d\log} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n$, with $\underline{U}(1)$ placed in degree n;

3. $\flat \mathbf{B}^n U(1)$ is the stack presented by the complex $\underline{U}(1) \xrightarrow{d\log} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{\mathrm{cl}}$, with $\underline{U}(1)$ placed in degree n;

4. $\flat_{\mathrm{dR}} \mathbf{B}^n U(1)$ is the stack presented by the complex $\Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_{\mathrm{cl}}$, with Ω^1 in degree $n-1$.

These stacks can be related as follows.

Proposition 3.4.3. *For $n \in \mathbb{N}$ we have a pasting diagram of homotopy pullback squares in $\mathbf{H}$ of the form*

$$\begin{tikzcd}
\mathbf{B}^n U(1)_{\mathrm{conn}} \arrow[r] \arrow[d, "\mathrm{curv}"'] & \mathbf{B}^n U(1) \arrow[r] \arrow[d] & * \arrow[d] \\
\Omega^{n+1}_{\mathrm{cl}} \arrow[r] & \flat_{\mathrm{dR}} \mathbf{B}^{n+1} U(1) \arrow[r] \arrow[d] & \flat \mathbf{B}^{n+1} U(1) \arrow[d] \\
 & * \arrow[r] & \mathbf{B}^{n+1} U(1)
\end{tikzcd},$$

where each map is the evident one.

This is discussed in [FSSt]. For the consideration in Section 4.3 below the composite morphism $\Omega^{n+1}_{\mathrm{cl}} \to \flat \mathbf{B}^{n+1} U(1)$ plays a central role.

4 Examples and Applications

We now discuss examples and applications of the general mechanism of higher local prequantum field theory.

We start in

- 4.1 – *Vacuum defects from spontaneous symmetry breaking*

with discussion of how the general abstract theory in 3.3 of correspondence spaces in higher homotopy types nicely captures the traditional notions in physics phenomenology of *spontaneous symmetry breaking vacuum defects* called *cosmic monopoles*, *cosmic strings* and *cosmic domain walls*, including the traditional rules by which these may end on each other. This discussion uses a minimum of mathematical sophistication (just some homotopy pullbacks) but may serve to nicely illustrate the interpretation of the abstract formalism in actual physics. Readers not interested in this interpretation may want to skip this section.

Then, still in a pedagogical vein, in

- 4.2 – *Higher Dijkgraaf-Witten local prequantum field theory*

we consider the formulation of discrete (e.g. finite) higher topological prequantum gauge field theories of generalized Dijkgraaf-Witten-type and use this as an occasion to review some basics of homotopy theory that is used throughout the article.

Our main example here is then

- 4.3 – *Higher Chern-Simons local prequantum field theory*

where we observe that in the ∞-topos $\mathbf{H}$ of smooth stacks there is a canonical tower of topological higher local prequantum field theories whose cascade of higher codimension defects naturally induce higher Chern-Simons type prequantum field theories and their associated theories.

Finally we observe in

- 4.4 – *Higher Wess-Zumino-Witten local prequantum field theory*

a general mechanism that induces higher local Wess-Zumino-Witten-type prequantum field theories from higher cocycles. We close by showing that, applied to exceptional higher cocycles of super L_∞-algebras extending the super translation Lie algebra, this reproduces and refines the theory of Green-Schwarz type super p-brane models in string theory and M-theory. Detailed discussion of this last example is given in a companion article.

4.1 Vacuum defects from spontaneous symmetry breaking

In particle physics phenomenology and cosmology, there is a traditional notion of *defects in the vacuum structure* of gauge field theories which exhibit spontaneous symmetry breaking, such as in the Higgs mechanism. A review of these ideas is in [VS]. A discussion of how such vacuum defects due to symmetry breaking may end on each other, and hence form a network of defects of varying codimension, is in [PV]. Here we briefly review the mechanism indicated in the latter article and then show how it is neatly formalized within the general notion of defect field theories as in Section 3.3.5. This is intended to serve as an illustration of the physical interpretation of the abstract notion of defects in field theories and of their formalization by correspondences, particularly. Readers not interested in physics phenomenology may want to skip this section.

Consider an inclusion of topological groups

$$H \hookrightarrow G\,.$$

Here we are to think of G as the gauge group (more mathematically precise: structure group) of a gauge theory and of $H \hookrightarrow G$ as the subgroup that is preserved by any one of its degenerate vacua (for instance in a Higgs-mechanism), hence the gauge group that remains after spontaneous symmetry breaking. In this case the quotient space (coset space) G/H is the moduli space of vacuum configurations, so that a vacuum configuration up to continuous deformations on a spacetime Σ is given by the homotopy class of a map from Σ to G/H.

Traditionally a *codimension-k defect in the vacuum structure* of a theory with such spontaneous symmetry breaking is a spacetime locally of the form $\mathbb{R}^n - (D^k \times \mathbb{R}^{n-k})$ with a vacuum classified locally by a the homotopy class of a map

$$S^{k-1} \simeq \mathbb{R}^n - (D^k \times \mathbb{R}^{n-k}) \to G/H ,$$

hence by an element of the $(k-1)$-st homotopy group of G/H. If this element is non-trivial, one says that the vacuum has a *codimension-k defect*. Specifically in an $(n = 4)$-dimensional spacetime Σ

- for $k = 1$ this is called a *domain wall*;
- for $k = 2$ this is called a *cosmic string*;
- for $k = 3$ this is called a *monopole*.

Next consider a sequence of inclusions of topological groups

$$H_2 \hookrightarrow H_1 \hookrightarrow H_0 = G .$$

Along the above lines this is now to be thought of as describing the breaking of a symmetry group $G = H_0$ first to H_1 at some energy scale E_1, and then a further breaking down to H_2 at some lower energy scale E_2.

So at the high energy scale the moduli space of vacuum structures is $G/H_1 = H_0/H_1$ as before. But at the low energy scale the moduli space of vacuum structures is now H_1/H_2. If there is a vacuum defect at low energy, classified by a map $S^{k-1} \to H_1/H_2$, then if it is "heated up" or rather if it "tunnels" by a quantum fluctuation through the energy barrier, it becomes instead a defect classified by a map to H_0/H_2, namely by the composite

$$S^{k-1} \to H_1/H_2 \to H_0/H_2 .$$

Here the map on the right is the fiber inclusion of the H_1-associated H_1/H_2-fiber bundle

$$H_1/H_2 \to H_0/H_2 \to H_0/H_1$$

naturally induced by the sequence of broken symmetry groups. The heated defect may be unstable, hence given by a trivial element in the $(k-1)$-st homotopy group of H_0/H_2, even if the former is not, in which case one says that the original defect is *metastable*. In terms of diagrams, metastability of the low energy defect means precisely that its classifying map $S^{k-1} \to H_1/H_2$ extends to a homotopy commutative diagram of the form

$$\begin{tikzcd} S^{k-1} \arrow[r] \arrow[d] & H_1/H_2 \arrow[d] \\ D^k \arrow[r] & H_0/H_2 \end{tikzcd} ,$$

where the left vertical arrow is the boundary inclusion $S^{k-1} \hookrightarrow D^k$. Now according to [PV], the decay of a metastable low-energy vacuum defect of codimension-k leads to the formation of a stable high-energy defect of codimension-$(k+1)$ at its decaying boundary. For instance a metastable cosmic string defect in the low energy vacuum structure is supposed to be able to end (decay) on a cosmic monopole defect in the high energy vacuum structure.

We now turn to a formalization of this story. By def. 3.3.30 and remark 3.2.2, the discussion in [PV] shows that the transition from metastable codimension-k defects in the low energy vacuum structure to stable high-energy $(k+1)$-defects should be represented by a correspondence of the form

$$\mathbf{Maps}^h(S^{k-1}, \mathbf{\Pi}_\infty(H_1/H_2)) \longleftarrow \mathbf{Maps}^h(S^k, \mathbf{\Pi}_\infty(H_0/H_1)) \longrightarrow * ,$$

exhibiting the high energy defects as boundary data for the low energy defects.

To see how to obtain this in line with the phenomenological story, observe that the heating/tunneling process as well as the decay process of the heated defects are naturall represented by the maps on the left and the right of the following diagram, respectively:

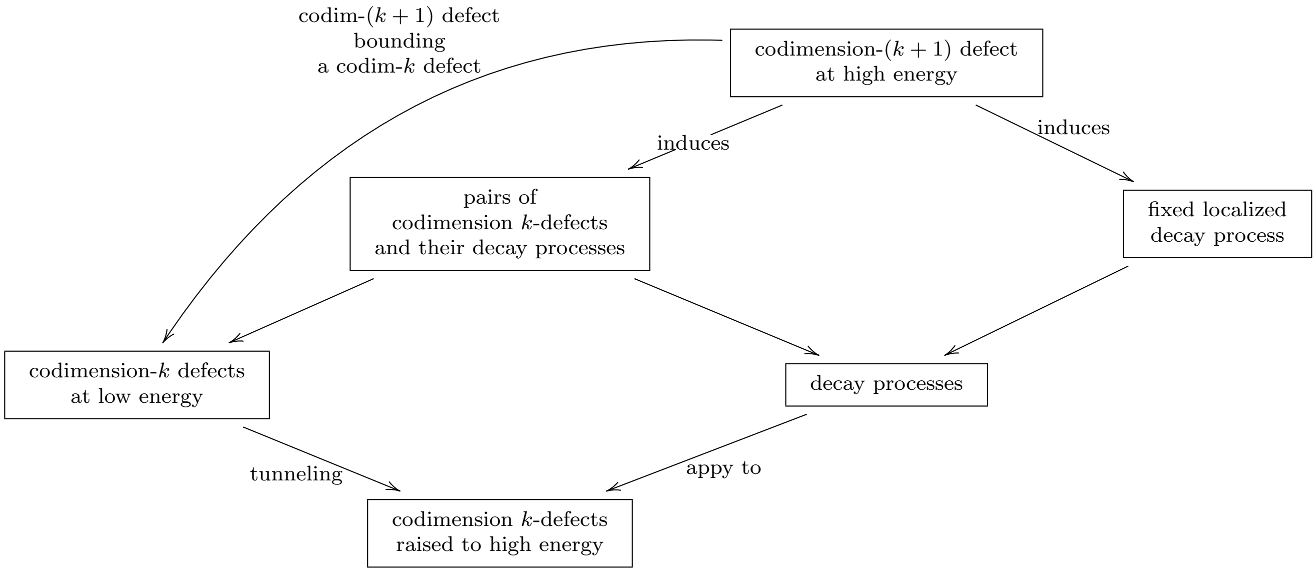
$$\begin{tikzcd}
 & & {*} \arrow[dl, "D^k \to H_0/H_2"'] \\
\mathbf{Maps}^h(S^{k-1}, \mathbf{\Pi}_\infty(H_1/H_2)) \arrow[dr, "H_1/H_2 \to H_0/H_2"'] & & \mathbf{Maps}^h(D^k, \mathbf{\Pi}_\infty(H_0/H_2)) \arrow[dl, "S^{k-1} \hookrightarrow D^k"] \\
 & \mathbf{Maps}^h(S^{k-1}, \mathbf{\Pi}_\infty(H_0/H_2)) &
\end{tikzcd}$$

The left map sends a low energy defect to its high energy version, the right map sends a high energy decay process to the field configuration which is decaying. For a specific spatially localized defect process $D^k \to H_0/H_2$ we are to pick one point in the space of defect processes, which is what the top right map reflects.

Therefore, the moduli space of decay processes of metastable low energy defects is precisely the homotopy fiber product of these two maps, namely the space of pairs consisting of a low energy defect and a localized decay process of its heated version (up to a pertinent gauge transformation that identifies the heated defect with the field configuration which decays). By the above fiber sequence of quotient spaces one finds that this homotopy pullback is $[\Pi(S^{k-1}, \Omega\Pi(H_0/H_1))]$. Hence, in conclusion, we find the desired correspondence as the top part of the following homotopy pullback diagram

$$\begin{tikzcd}
 & & & {[\Pi(S^k), \Pi(H_0/H_1)]} \arrow[d] & \\
 & & & {[\Pi(S^{k-1}), \Omega\Pi(H_0/H_1)]} \arrow[dll] \arrow[dr] & \\
 & {[S^{k-1} \to \Pi(H_1/H_2), D^k \to \Pi(H_0/H_2)]} \arrow[dl] \arrow[dr] & & \text{(pb)} & {*} . \arrow[dll] \\
{[\Pi(S^{k-1}), \Pi(H_1/H_2)]} \arrow[dr, "{[\Pi(S^{k-1}), \Pi(H_1/H_2) \to \Pi(H_0/H_2)]}"'] & \text{(pb)} & {[\Pi(D^k), \Pi(H_0/H_2)]} \arrow[dl, "{[\Pi(S^{k-1} \hookrightarrow \Pi(D^k)), \Pi(H_0/H_2)]}"] & & \\
 & {[\Pi(S^{k-1}), \Pi(H_0/H_2)]} & & &
\end{tikzcd}$$

In summary, this diagram encodes the phenomenological story of the decay of metastable defects as follows:



4.2 Higher Dijkgraaf-Witten local prequantum field theory

We discuss here aspects of higher *Dijkgraaf-Witten-type* prequantum field theories, which are those prequantum field theories whose moduli stack **Fields** is a discrete ∞-groupoid (usually required to be finite). This is a special case of the higher Chern-Simons theories discussed below in Section 4.3, and hence strictly speaking need not be discussed separately. Therefore, this section here is aimed at readers who desire more introduction and motivation to the basic topics of local prequantum field theory. Other readers should skip ahead to Section 4.3.

The original Dijkgraaf-Witten theory is that in dimension 3 (reviewed in Section 4.2.2 below), which was introduced in Section [DW] as a toy version of 3d Chern-Simons theory. A comprehensive account with first indications of its role as a local (extended, multi-tiered) field theory then appeared in [FQ], and ever since this has served as a testing ground for understanding the general principles of local field theory, e.g. [F93], independently of the subtleties of giving meaning to concepts such as the path integral when the space of fields is not finite. In section 3 of [FHLT] the general prequantum formalization as in Remark 3.3.16 is sketched for Dijkgraaf-Witten type theories, and in section 8 there the quantization of these theories to genuine local quantum field theories is sketched.

4.2.1 1d DW theory

Dijkgraaf-Witten theory in dimension 1 is what results when one regards a *group character* of a finite group G as an action functional in the sense of def. 3.3.16. We give here an expository discussion of this example in the course of which we introduce some basics of the homotopy theory of groupoids.

A group character is just a group homomorphism of the form $G \to U(1)$. In order to regard this as an action functional, we are to take G as the *gauge group* of a physical field theory. The simplest such case is a field theory such that on the point there is just a single possible field configuration, to be denoted ϕ_0. The reader familiar with basics of traditional gauge theory may think of the fields as being gauge connections ("vector potentials"), hence represented by differential 1-forms. But on the point there is only the vanishing 1-form, hence just a single field configuration ϕ_0.

Even though there is just a single such field, that G is the gauge group means that for each element $g \in G$ there is a gauge transformation that takes ϕ_0 to itself, a state of affairs which we suggestively denote by the symbols $\phi_0 \xrightarrow{g} \phi_0$. Again, the reader familiar with traditional gauge theory may think of gauge transformations as in Yang-Mills theory. Over the point these form, indeed, just the gauge group itself, taking the trivial field configuration to itself.

That the gauge group is indeed a group means that gauge transformations can be applied successively, which we express in symbols as

$$\begin{tikzcd} \phi_0 \arrow[r, "g_1"] \arrow[rr, bend right, "g_2 \cdot g_1"'] & \phi_0 \arrow[r, "g_2"] & \phi_0 \end{tikzcd} ,$$

or better, with a bit more space in between the symbols, as

$$\begin{tikzcd} & \phi_0 \arrow[dr, "g_2"] & \\ \phi_0 \arrow[ur, "g_1"] \arrow[rr, "g_2 \cdot g_1"'] & & \phi_0 \end{tikzcd} .$$

Regarded this way, we say the gauge group acting on the single field ϕ_0 forms a *groupoid*, whose single *object* is ϕ_0 and whose set of *morphisms* is G.

Definition 4.2.1. A *groupoid* is a pair of two sets $\mathcal{G}_0$ and $\mathcal{G}_1$ equipped with functions

$$\begin{tikzcd} \mathcal{G}_1 \underset{\mathcal{G}_0}{\times} \mathcal{G}_1 \arrow[r, "\circ"] & \mathcal{G}_1 \arrow[r, shift left=2, "t"] \arrow[r, shift right=2, "s"'] & \mathcal{G}_0 \arrow[l, "i" description] \end{tikzcd}$$

such that... A *homomorphism* of groupoids $f_\bullet : \mathcal{G}_\bullet \to \mathcal{K}_\bullet$ (a *functor*) is a map of components that respects this structure.

Example 4.2.2. For G a group, its *delooping groupoid* $(\mathbf{B}G)_\bullet$ has

- $(\mathbf{B}G)_0 = *$;
- $(\mathbf{B}G)_1 = G$.

For G and K two groups, group homomorphisms $G \to K$ are in natural bijection with groupoid homomorphisms $(\mathbf{B}G)_\bullet \to (\mathbf{B}K)_\bullet$. In particular a group character for G is equivalently a groupoid homomorphism

$$c_\bullet : \ (\mathbf{B}G)_\bullet \longrightarrow (\mathbf{B}U(1))_\bullet \ .$$

Example 4.2.3. The interval I is the groupoid with

- $I_0 = \{a, b\}$;
- $I_1 = \{\mathrm{id}_a, \mathrm{id}_b, \ a \longrightarrow b \ \}$.

Example 4.2.4. For Σ a topological space, its *fundamental groupoid* $\Pi_1(\Sigma)$ is

- $\Pi_1(\Sigma)_0 =$ points in X;
- $\Pi_1(\Sigma)_1 =$ continuous paths in X modulo homotopy that leaves the endpoints fixed.

Example 4.2.5. For $\mathcal{G}_\bullet$ any groupoid, there is the *path space* groupoid $\mathcal{G}_\bullet^I$ with

- $\mathcal{G}_0^I = \mathcal{G}_1$;
- $\mathcal{G}_1^I =$ commuting squares in $\mathcal{G}_\bullet$.

This comes with two canonical homomorphisms

$$\mathcal{G}^I_\bullet \underset{\mathrm{ev}_0}{\overset{\mathrm{ev}_1}{\rightrightarrows}} \mathcal{G}_\bullet$$

given by endpoint evaluation.

Definition 4.2.6. For $f_\bullet, g_\bullet : \mathcal{G}_\bullet \to \mathcal{K}_\bullet$ two morphisms between groupoids, a *homotopy* $f \Rightarrow g$ (a *natural transformation*) is a homomorphism of the form $\eta_\bullet : \mathcal{G}_\bullet \to \mathcal{K}^I_\bullet$ (with codomain as in example 4.2.5) such that it fits into the diagram as depicted here on the right:

$$\begin{tikzcd} \mathcal{G} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{K} \end{tikzcd} \quad := \quad \begin{tikzcd} & \mathcal{K}_\bullet \\ \mathcal{G}_\bullet \arrow[ur, "f_\bullet"] \arrow[r, "\eta_\bullet"] \arrow[dr, "g_\bullet"'] & \mathcal{K}^I_\bullet \arrow[u, "\mathrm{ev}_0"'] \arrow[d, "\mathrm{ev}_1"] \\ & \mathcal{K}_\bullet \end{tikzcd}$$

Here and in what follows, the convention is that we write

- $\mathcal{G}_\bullet$ when we regard groupoids with just homomorphisms between them,
- $\mathcal{G}$ when we regard groupoids with homomorphisms between them and homotopies between these.

Example 4.2.7. For X, Y two groupoids, the mapping groupoid, $[X, Y]$ or Y^X, is

- $[X, Y]_0$ = homomorphisms $X \to Y$;
- $[X, Y]_1$ = homotopies between such homomorphisms.

Definition 4.2.8. A (homotopy-)*equivalence* of groupoids is a morphism $\mathcal{G} \to \mathcal{K}$ which has a left and a right inverse up to homotopy. We write $\mathcal{G} \xrightarrow{\simeq} \mathcal{K}$ for such equivalences.

Proposition 4.2.9. *Assuming the axiom of choice in the ambient set theory, every groupoid is equivalent to a disjoint union of delooping groupoids of example 4.2.2.*

Proof. Choose one point in each connected component of G, hence a section $b : \pi_0(\mathcal{G}) \to \mathcal{G}$. Let $\mathrm{sk}(\mathcal{G})$ be the groupoid with

- $\mathrm{sk}(\mathcal{G})_0 := \pi_0(\mathcal{G})$;
- $\mathrm{sk}(\mathcal{G})_0 := \pi_0(\mathcal{G}) \underset{\mathcal{G}_0 \times \mathcal{G}_0}{\times} \mathcal{G}_1$.

□

Remark 4.2.10. The statement of Prop. 4.2.9 becomes false when we pass to groupoids that are equipped with geometric structure. This is the reason why for discrete geometry all Chern-Simons-type field theories fundamentally involve just groups (and higher groups), while for nontrivial geometry there are genuine groupoid theories, for instance the AKSZ σ-models [FRS11]. But even so, Dijkgraaf-Witten theory is usefully discussed in terms of groupoid technology, in particular since the choice of equivalence in Prop. 4.2.9 is not canonical.

Definition 4.2.11. Given two morphisms of groupoids $X \xrightarrow{f} B \xleftarrow{g} Y$, their *homotopy fiber product*

$$
\begin{tikzcd}
X \underset{B}{\times} Y \arrow[r] \arrow[d] & X \arrow[d, "f"] \arrow[dl, phantom, "\Swarrow"] \\
Y \arrow[r, "g"'] & B
\end{tikzcd}
$$

is the limiting cone

$$
\begin{tikzcd}
X_\bullet \underset{B_\bullet}{\times} B_\bullet^I \underset{B_\bullet}{\times} Y_\bullet \arrow[rr] \arrow[dd] & & X_\bullet \arrow[d, "f_\bullet"] \\
& B_\bullet^I \arrow[r, "\mathrm{ev}_1"'] \arrow[d, "\mathrm{ev}_0"] & B_\bullet \\
Y_\bullet \arrow[r, "g_\bullet"'] & B_\bullet &
\end{tikzcd} ,
$$

hence the ordinary iterated fiber product over the path space groupoid, as indicated.

Example 4.2.12. For G a group and $\mathbf{B}G$ its delooping groupoid from example 4.2.2, we have

$$G = * \underset{\mathbf{B}G}{\times} * .$$

Proof. The path space groupoid $(\mathbf{B}G)^I$ has as objects the elements of G, and morphisms starting at one such element are given by a pair of elements of G acting on the given one by left and right multiplication. The fiber product in def. 4.2.11 picks in there just those morphisms that are labeled by the trivial pair. $\square$

Example 4.2.13. We have the mapping groupoid

$$[\Pi(S^1), X] \simeq X \underset{[\Pi(S^0), X]}{\times} X .$$

Proof. Since $S^0 = * \coprod *$ we have $[\Pi(S^0), X] \simeq X \times X$. It follows that $[\Pi(S^0), X]^I$ has as objects pairs of morphisms in X, and as moprhisms pairs of homotopies between these. The defining fiber product then picks among these pairs those for which the two morphisms start and end at the same object. So the objects in $[\Pi(S^1), X]$ are diagrams in X of the form

$$
\begin{tikzcd}
x_1 \arrow[r, bend left] \arrow[r, bend right] & x_2
\end{tikzcd}
$$

and morphisms are cylinders over these. $\square$

Proposition 4.2.14. *The prequantum field theory defined by a group character*

$$
\left[
\begin{tikzcd}
\mathbf{Field} \arrow[d, "\exp(iS)"] \\
\flat \mathbf{B}U(1)
\end{tikzcd}
\right]
:=
\left[
\begin{tikzcd}
\mathbf{B}G \arrow[d, "c"] \\
\flat \mathbf{B}U(1)
\end{tikzcd}
\right]
$$

assigns to the circle the action functional which sends a field configuration $g \in G = [\Pi(S^1), \mathbf{B}G]_0$ to its value $c(g) \in U(1) = (\flat \mathbf{B} U(1))_1$:

$$
\begin{tikzcd}[column sep=small]
 & & {[\Pi(S^1), \mathbf{B}G]} \arrow[dl] \arrow[dr] & & \\
 & \mathbf{B}G \arrow[dl] \arrow[dr, "\Delta"] & \arrow[dl, phantom, "\swarrow"] & \mathbf{B}G \arrow[dl, "\Delta"'] \arrow[dr] & \\
 \mathbf{B}G \arrow[drr, "0"'] & & \mathbf{B}G \times \mathbf{B}G \arrow[d, "\exp(iS(p_1) - iS(p_2))"] & & \mathbf{B}G \arrow[dll, "0"] \\
 & & \flat \mathbf{B}^1 U(1) & &
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}[column sep=small]
 & & {[\Pi_1(S^1), \mathbf{B}G]_\bullet} \arrow[dll] \arrow[d, "S_\bullet"] \arrow[drr] & & \\
 \mathbf{B}G \arrow[dr, "0"'] & & \flat \mathbf{B} U(1)^I_\bullet \arrow[dl] \arrow[dr] & & \mathbf{B}G \arrow[dl, "0"] \\
 & \flat \mathbf{B} U(1) & & \flat \mathbf{B} U(1) &
\end{tikzcd}
$$

where $S_1 = c : G \to U(1)$ is the group character.

Proof. Use the proof of example 4.2.13. $\square$

4.2.2 3d DW theory

The group character $c : G \to U(1)$ which defines 1-dimensional prequantum Dijkgraaf-Witten theory in Section 4.2.1 is equivalently a cocycle in degree-1 group cohomology $[c] \in H_{\mathrm{Grp}}(G, U(1))$. In view of this it should be plausible that one may interpret a cocycle in degree-n group cohomology, for all $n \in \mathbb{N}$, as a higher order action functional $\mathbf{B}G \to \flat \mathbf{B}^n U(1)$ and deduce an n-dimensional local prequantum Dijkgraaf-Witten-type theory from it.

Here we review how to formalize this and then consider the example of DW theory in dimension 2.

Definition 4.2.15. A simplicial set is...

Definition 4.2.16. The *nerve* $N(\mathcal{G}_\bullet)$ of a groupoid $\mathcal{G}_\bullet$ is the simplicial set with

$$N(\mathcal{G}_\bullet)_n := \mathcal{G}_1^{\times^n_{\mathcal{G}_0}} \,.$$

Proposition 4.2.17. *The nerve construction of a full embedding of groupoids into simplicial sets: for $\mathcal{G}_\bullet$, $\mathcal{K}_\bullet$ two groupoids there is a natural bijection between groupoid homomorphisms $\mathcal{G}_\bullet \to \mathcal{K}_\bullet$ and simplicial set homomorphisms $N(\mathcal{G}_\bullet) \to N(\mathcal{K}_\bullet)$.*

Proposition 4.2.18. *The nerve of a groupoid is a simplicial set with the special property that for every horn $\Lambda^n_i \to N(\mathcal{G}_\bullet)$ there is a unique way to complete it to an n-simplex, hence a unique dashed extension in*

$$
\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & N(\mathcal{G}_\bullet) \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd} .
$$

Definition 4.2.19. A *Kan complex* or ∞-*groupoid* is a simplicial set S such that for each horn $\Lambda^n_i \to S$ there exists some dashed extension in

$$
\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & S \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd} .
$$

(...)

For X, A two Kan complexes, write

$$\infty\mathrm{Grpd}(X,A) = \mathrm{sSet}(X \times \Delta^\bullet, A)\,.$$

...

The Dold-Kan correspondence is

$$\mathrm{DK} :\ \mathrm{Ch}_{\bullet \geq 0} \xrightarrow{\simeq} \mathrm{sAb}$$

$$\mathrm{DK}(V_\bullet) : [k] \mapsto \mathrm{Hom}_{\mathrm{Ch}_\bullet}(N_\bullet(C(\Delta^k)), V_\bullet)\,.$$

Set

$$\flat \mathbf{B}^n U(1) = \mathrm{DK}(U_{\mathrm{disc}}(1)[n])\,.$$

Group cohomology is

$$H^n_{\mathrm{Grp}}(G, U(1)) \simeq \infty\mathrm{Grpd}(\mathbf{B}G, \mathbf{B}^n U(1))$$

3d DW prequantum theory is the prequantum field theory obtained from regarding a 3-cocycle $c \in H^3_{\mathrm{Grp}}(G, U(1))$ as a local action functional

$$\mathbf{B}G \to \flat \mathbf{B}^3 U(1)$$

this way.

(...)

4.3 Higher Chern-Simons local prequantum field theory

We now turn to the class of those local prequantum field theories which deserve to be termed of *Chern-Simons type*. We show that these arise rather canonically as the boundary data for the canonical differential cohomological structure of prop. 3.4.3 which is exhibited by every cohesive ∞-topos $\mathbf{H}$.

4.3.1 Survey: towers of boundaries, corners, ... and of circle reductions

We discuss in the following towers/hierarchies of iterated defects of increasing codimension of a universal topological Yang-Mills theory.

Most of these defects however are best recognized after "gluing their endpoints" after which they equivalently becomes circle-reductions/transgression to loop space of the original theory.

Restricted to the archetypical case of 3d Chern-Simons theory, the following discussion essentially goes through this diagram here:

$$\begin{tikzcd}
\text{3d CS} \arrow[r, hook] \arrow[d, "S^1"] & \text{4d tYM} \arrow[d, "S^1"] \\
\text{2d WZW} \arrow[r, hook] \arrow[d, "S^1"] & \text{3d tYM} \arrow[d, "S^1"] \\
\text{1d Wilson line} \arrow[r, hook] \arrow[d, "S^1"] & \text{2d tYM} \arrow[d, "S^1"] \\
\text{3dCS action} \arrow[r, hook] & \text{1d tYM}
\end{tikzcd}$$

This is a pattern of iterated higher codimension corners and iterated circle reductions which had long been emphasized by Hisham Sati to govern the grand structure of theories inside string/M-theory [Sa].

For instance there shoule be a tower of this kind which instead of 3d Chern-Simons theory has 11-dimensional supergravity, or something going by the name of "M-theory" as follows (notice that $11 = 3 + 8$, by this seems to be related to the previous tower by some 8-periodic phenomena which maybe one day we'll further discuss here):

$$\begin{tikzcd}
\text{het} \arrow[r, hook] \arrow[d, "S^1"] & \text{11d SuGra} \arrow[r, hook] \arrow[d, "S^1"] & \text{bounding 12d} \arrow[d, "S^1"] \\
\text{9d bdr} \arrow[r, hook] & \text{IIA} \arrow[r] & \text{bounding11}
\end{tikzcd}$$

There are more such towers in string/M-theory. For instance Edward Witten has been exploring a system of reductions [W11] which in (small) parts involves a system roughly as follows

$$\begin{tikzcd}
\text{something M5} \arrow[r, hook] \arrow[d, "S^1"] & \text{7d CS} = \mathrm{KK}_{S^4} \text{ of 11d CS} \arrow[d, "S^1"] \\
\text{5d} \arrow[r, hook] & \text{(2,0) QFT on M5} \arrow[d, "S^1"] \\
& \text{5d sYM} \arrow[d, "S^1"] \\
& \text{4d sYM} \arrow[d, "T^2"] \\
& \text{Langlands}
\end{tikzcd}$$

4.3.2 $d = n + 1$, Universal topological Yang-Mills theory

As a special case of prop. 3.3.13 we have:

Proposition 4.3.1. *For $n \in \mathbb{N}$, the morphism $\exp(iS_{\mathrm{tYM}})$ in prop. 3.4.3, regarded as an object*

$$\left[\begin{tikzcd} \Omega^{n+1}_{\mathrm{cl}} \arrow[d, "\exp(iS_{\mathrm{tYM}})"] \\ \flat \mathbf{B}^{n+1} U(1) \end{tikzcd}\right] \in \mathrm{Corr}_n(\mathbf{H}_{/\flat \mathbf{B}^n U(1)})^{\otimes}$$

is fully dualizable, with dual $-S_{\mathrm{tYM}}$.

Definition 4.3.2. For $n \in \mathbb{N}$, we call the local prequantum field theory defined by the fully dualizable object S_{tYM} of prop. 4.3.1 the *universal topological Yang-Mills* local prequantum field theory

$$\exp(iS_{\mathrm{tYM}}) : \mathrm{Bord}_{n+1} \to \mathrm{Corr}_{n+1}(\mathbf{H}_{/\flat \mathbf{B}^{n+1} U(1)}) \,.$$

This terminology is justified below in remark 4.3.5.

4.3.3 $d = n + 0$, Higher Chern-Simons field theories

We discuss now the boundary conditions of the universal topological Yang-Mills local prequanutm field theory

Remark 4.3.3. The universal boundary condition for S_{tYM} according to def. **??** is given by the top rectangle in prop. 3.4.3, naturally regarded as a correspondence in the slice:

$$\begin{tikzcd} & \mathbf{B}^n U(1)_{\mathrm{conn}} \arrow[dl] \arrow[dr, "F_{(-)}"] & \\ * \arrow[dr] & & \Omega^{n+1}_{\mathrm{cl}} \arrow[dl, "\exp(iS_{\mathrm{tYM}})"] \\ & \flat \mathbf{B}^{n+1} U(1) & \end{tikzcd} .$$

So by prop. **??** there is a natural equivalence of ∞-categories

$$\mathrm{Bdr}(\exp(iS_{\mathrm{tYM}})) \simeq \mathbf{H}_{/\mathbf{B}^n U(1)_{\mathrm{conn}}}$$

between the ∞-category of boundary conditions for the universal topological Yang-Mills theory in dimension $(n+1)$ and the slice ∞-topos of $\mathbf{H}$ over the moduli stack of $U(1)$-n-connections.

Corollary 4.3.4. *The $(\infty, 1)$-category of boundary conditions for the universal topological Yang-Mills local prequantum field theory S_{tYM} are equivalently* ∞-Chern-Simons local prequantum field theories *[FSS13]: moduli stacks $\mathbf{Fields}_\partial \in \mathbf{H}$ equipped with a prequantum n-bundle [FRS13a]*

$$\nabla_{\mathrm{CS}} : \mathbf{Fields}_\partial \to \mathbf{B}^n U(1)_{\mathrm{conn}} .$$

The automorphism ∞-group of a given boundary condition for S_{tYM} is hence equivalently the quantomorphism *∞-group of the corresponding Chern-Simons theory [FRS13a].*

Proof. This is just a special case of prop. **??**. Explicitly, by the universal property of the homotopy pullback in $\mathbf{H}$, given any boundary condition for S_{tYM}, hence by remark 3.3.24 a diagram in $\mathbf{H}$ of the form

$$\begin{tikzcd} & \mathbf{Fields}_\partial \arrow[dl] \arrow[dr, "\langle F_{(-)} \wedge \cdots \wedge F_{(-)} \rangle"] & \\ * \arrow[dr, "0"'] \arrow[rr, phantom, "\swarrow \nabla"] & & \Omega^{n+1}_{\mathrm{cl}} \arrow[dl, "\exp(iS_{\mathrm{tYM}})"] \\ & \flat \mathbf{B}^{n+1} U(1) & \end{tikzcd} ,$$

this is equivalent to the dashed morphism in

$$\begin{tikzcd} & \mathbf{Fields}_\partial \arrow[d, dashed, "\nabla"] \arrow[ddl, bend right] \arrow[ddr, bend left, "\langle F_{(-)} \wedge \cdots \wedge F_{(-)} \rangle"] & \\ & \mathbf{B}^n U(1)_{\mathrm{conn}} \arrow[dl] \arrow[dr, "F_{(-)}"] & \\ * \arrow[dr, "0"'] & & \Omega^{n+1}_{\mathrm{cl}} \arrow[dl, "\exp(iS_{\mathrm{tYM}})"] \\ & \flat \mathbf{B}^{n+1} U(1) & \end{tikzcd} .$$

□

In order to interpret this, notice the following.

Remark 4.3.5. For the special case that $\mathbf{Fields}_{\partial}$ is a moduli stack $\mathbf{B}G_{\mathrm{conn}}$ of G-principal ∞-connections for some ∞-group G, we may think of morphism

$$\langle F_{(-)} \wedge \cdots F_{(-)} \rangle : \mathbf{B}G_{\mathrm{conn}} \to \Omega^{n+1}_{\mathrm{cl}}$$

as encoding an *invariant polynomial* $\langle -, \cdots, - \rangle$ on (the ∞-Lie algebra of) G [FSSt]. By extrapolation from this case we may also speak of invariant polynomials if $\mathbf{Fields}|_{\partial\Sigma}$ is of more general form, in which case we have invariant polynomials on *smooth ∞-groupoids* [FRS11]. Restricting to the group-al case just for definiteness, notice that a boundary field configuration, which by prop. 3.3.21 is given by

$$\begin{tikzcd} \partial\Sigma \times U \arrow[r, "\nabla"] \arrow[d] & \mathbf{B}G_{\mathrm{conn}} \arrow[d] \\ \Sigma \times U \arrow[r, "\omega"] & \Omega^{n+1}_{\mathrm{cl}} \end{tikzcd} ,$$

forces the closed $(n+1)$-form ω of the bulk theory to become the ∞-Chern-Weil form of a G-principal ∞-connection with respect to the invariant polynomial $\langle -, \cdots, - \rangle$ at the boundary:

$$\omega|_{\partial\Sigma} = \langle F_{\nabla} \wedge \cdots \wedge F_{\nabla} \rangle \,.$$

For G an ordinary Lie group, this is known as the *Lagrangian for topological G-Yang-Mills theory*. More generally, for G any smooth ∞-group, we may hence think of this as the Lagrangian of a topological ∞-Yang-Mills theory.

Specifically for the *universal* boundary condition $\mathbf{Fields}_{\partial} = \mathbf{B}^n U(1)_{\mathrm{conn}}$ of remark 4.3.3 we find a field theory which assigns $U(1)$-n-connections ∇ to n-dimensional manifolds Σ_n and closed $(n+1)$-forms ω on $(n+1)$-dimensional manifolds Σ_{n+1}, such that whenever the latter bounds the former, the exponentiated integral of ω equals the *n-volume holonomy* of ∇. This is just the relation between circle n-connections and their curvatures which is captured by the axioms of *Cheeger-Simons differential characters*. Hence it makes sense to call the higher topological Yang-Mills theory which is induced from the universal boundary condition the *Cheeger-Simons theory* in the given dimension.

But corollary 4.3.4 says more: the universality of the Cheeger-Simons theory as a boundary condition for topological Yang-Mills theory means that a consistent such boundary condition is necessarily not just an invariant polynomial, but is crucially a lift of that from de Rham cocycles to differential cohomology. This means that it is a *refined ∞-Chern-Weil homomorphism* in the sense of [FSSt] and equivalently a *higher pre* of the invariant polynomial in the sense of [FRS13a]. In either case a lift ∇ in the diagram

$$\begin{tikzcd} & \mathbf{B}^n U(1)_{\mathrm{conn}} \arrow[d] \\ \mathbf{B}G_{\mathrm{conn}}\rangle \arrow[ur, "\nabla"] \arrow[r, "\langle F_{(-)}\wedge\cdots\wedge F_{(-)}"'] & \Omega^{n+1}_{\mathrm{cl}} \end{tikzcd} .$$

Example 4.3.6. For the canonical binary invariant polynomial $\langle -, - \rangle$ on a simply connected semisimple Lie group G such as $G = \mathrm{Spin}$ or $G = \mathrm{SU}$ (the *Killing form*) a consistent boundary condition as in remark 4.3.5 is provided by the differential refinement of the first fractional Pontrjagin class $\frac{1}{2}p_1$ and of the second

Chern class c_2, respectively, that have been constructed in [FSSt]:

$$
\begin{tikzcd}
 & \mathbf{B}^3U(1)_{\mathrm{conn}} \arrow[d] \\
\mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \arrow[ur, "\frac{1}{2}\widehat{\mathbf{p}_1}"] \arrow[r, "\langle F_{(-)}\wedge F_{(-)}\rangle"'] & \Omega^4_{\mathrm{cl}}
\end{tikzcd}
\quad , \qquad
\begin{tikzcd}
 & \mathbf{B}^3U(1)_{\mathrm{conn}} \arrow[d] \\
\mathbf{B}\mathrm{SU}_{\mathrm{conn}} \arrow[ur, "\widehat{\mathbf{c}}_2"] \arrow[r, "\langle F_{(-)}\wedge F_{(-)}\rangle"'] & \Omega^4_{\mathrm{cl}}
\end{tikzcd}
\quad .
$$

Furthermore, for the canonical quaternary invariant polynomial on the smooth String-2-group (see appendix of [FSS12] for a review) a consistent boundary condition as in remark 4.3.5 is provided by the differential refinement of the second fractional Pontrjagin class $\frac{1}{6}p_2$ that has also been constructed in[FSSt]:

$$
\begin{tikzcd}
 & \mathbf{B}^7U(1)_{\mathrm{conn}} \arrow[d] \\
\mathbf{B}\mathrm{String}_{\mathrm{conn}} \arrow[ur, "\frac{1}{6}\widehat{\mathbf{p}}_2"] \arrow[r, "\langle F_{(-)}\wedge F_{(-)}\wedge F_{(-)}\wedge F_{(-)}\rangle"'] & \Omega^8_{\mathrm{cl}}
\end{tikzcd}
\quad .
$$

This describes a 7-dimensional Chern-Simons theory of nonabelian 2-form connections [FSS12].

4.3.4 $d = n - 1$, Topological Chern-Simons boundaries

We now consider codimension-2 corners of the universal topological Yang-Mills theory, hence codimension-1 boundaries of higher Chern-Simons theories. These turn out to be related to Wess-Zumino-Witten like theories. (Further below in 4.4 we discuss that a natural differential variant of this type of theories also arises as ∞-Chern-Simons theories themselves.)

For characterizing the data assigned by a field theory to such corners, we will need to consider the generalization of the following traditional situation.

Example 4.3.7. For (X, ω) a symplectic manifold, $\omega \in \Omega^2_{\mathrm{cl}}(X)$, a submanifold $Y \to X$ is *isotropic* if $\omega|_Y = 0$ and *Lagrangian* if in addition $\dim(X) = 2\dim(Y)$. If (X, ω) is equipped with a *prequantum bundle*, namely a lift ∇ in

$$
\begin{tikzcd}
 & \mathbf{B}U(1)_{\mathrm{conn}} \arrow[d, "F_{(-)}"] \\
X \arrow[ur, "\nabla"] \arrow[r, "\omega"'] & \Omega^2_{\mathrm{cl}}
\end{tikzcd}
\quad ,
$$

then we may ask not only for a trivialization of ω but even of ∇ on Y:

$$
\nabla|_Y \simeq 0\,.
$$

If this exists, then traditionally Y is called a *Bohr-Sommerfeld leaf* of (X, ∇), at least when Y is one leaf of a foliation of X by Lagrangian submanifolds.

Hence we set generally:

Definition 4.3.8. Given $X \in \mathbf{H}$ and $\nabla : X \to \mathbf{B}^nU(1)_{\mathrm{conn}}$, a *Bohr-Sommerfeld isotropic space* of (X, ∇) is a diagram of the form

$$
\begin{tikzcd}
Y \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[dl, Rightarrow] \\
X \arrow[r, "\nabla"'] & \mathbf{B}^nU(1)_{\mathrm{conn}}
\end{tikzcd}
$$

in $\mathbf{H}$.

Remark 4.3.9. The *universal* Bohr-Sommerfeld isotropic space over (X, ∇) is the homotopy fiber $\mathrm{fib}(\nabla) \to X$ of ∇. In a sense this is the "maximal" Bohr-Sommerfeld isotropic space over (X, ∇), as every other one factors through this, essentially uniquely. Below we see that these are equivalently the universal codimension-2 corners of higher Chern-Simons theory. While the property of being "isotropic and maximally so" is reminiscent of Lagrangian submanifolds, it seems unclear what the notion of Lagrangian submanifold should refine to generally in higher prequantum geometry, if anything.

Proposition 4.3.10. *A corner, def. 3, for the universal topological Yang-Mills theory, def. 4.3.2, from a non-trivial to a trivial boundary condition, hence a boundary condition for an ∞-Chern-Simons theory, corollary 3.3.9, $\nabla : \mathbf{Fields}_{\partial} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$, is equivalently a* Bohr-Sommerfeld isotropic space *of boundary fields, def. 4.3.8, namely a map*

$$\mathbf{Fields}_{\partial\partial} \to \mathbf{Fields}_{\partial}$$

such that $F_{\nabla|_{\partial^2}} = 0$ and equipped with a homotopy

$$\nabla|_{\mathbf{Fields}_{\partial\partial}} \simeq 0\,.$$

Proof. The boundary condition for ∇ is a correspondence-of-correspondences from

$$\begin{tikzcd}
& \mathbf{Fields}_{\partial} \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\Swarrow \nabla"] & \\
* \arrow[dr, "0"'] & & \Omega^{n+1}_{\mathrm{cl}} \arrow[dl, "\exp(iS_{\mathrm{tYM}})"] \\
& \flat \mathbf{B}^{n+1} U(1) &
\end{tikzcd}$$

to

$$\begin{tikzcd}
& * \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\Swarrow 0"] & \\
* \arrow[dr, "0"'] & & \Omega^{n+1}_{\mathrm{cl}} \arrow[dl, "\exp(iS_{\mathrm{tYM}})"] \\
& \flat \mathbf{B}^{n+1} U(1) &
\end{tikzcd} .$$

The tip of this correspondence-of-correspondences is a correspondence of the form

$$\begin{tikzcd}
& \mathbf{Fields}_{\partial\partial} \arrow[dl] \arrow[dr] & \\
* & & \mathbf{Fields}_{\partial}
\end{tikzcd} ,$$

hence is just a map as on the right. The correspondence-of-correspondences is then filled with a second order homotopy between ∇, regarded as a homotopy, and the 0-homotopy.

Unwinding what this means in view of def. 3.4.2, one sees that this homotopy is given by a Čech-Deligne cochain

$$(\cdots, A^{\nabla_{\mathrm{bdr}}}, 0, 0)$$

such that

$$D(\cdots, A^{\nabla_{\mathrm{bdr}}}, 0, 0) = (\cdots, A^{\nabla}, 0)|_{\partial\partial}\,.$$

Where

$$D(\cdots, A^{\nabla}, 0) = (0, 0, \cdots, 0, \omega)|_{\partial}\,.$$

∎

Example 4.3.11 (topological boundary for 3d Chern-Simons theory)**.** This is pretty much what is proposed as the data on codimension-1 defects for ordinary Chern-Simons theory on p. 11 of [KSa]. They propose (somewhat implicitly in their text) that the boundary connection should be such that U-component of $\langle F_\nabla \wedge F_\nabla \rangle$ vanishes at each point of Σ. But for us the fields are $A : \Pi(\Sigma) \times U \to \mathbf{B}G_{conn}$, hence are flat along Σ, hence that component vanishes anyway. As a result, the proposal in [KSa] essentially comes down to asking that boundary fields ∇ are the maximal solution to trivializing $\langle F_\nabla \wedge F_\nabla \rangle$. If we refine this statement from de Rham cocycles to differential cohomology, we arrive at the above picture.

Remark 4.3.12. Chern-Simons theory is famously related to Wess-Zumino-Witten theory in codimension-1. However, WZW theory is not directly a "topological boundary" of Chern-Simons theory. Below in 4.3.6 we show that (the topological sector of) pre-quantum WZW theory is a codimension-1 *defect* from $\exp(iS_{\mathrm{CS}})$ to itself, via $\exp(iS_{\mathrm{tYM}})$.

Remark 4.3.13. So the *universal* boundary condition for ∞-Chern-Simons local prequantum field theory $\nabla : \mathbf{Fields} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$ (regarded itself as a boundary condition for its topological Yang-Mills theory) is the homotopy fiber of ∇.

Example 4.3.14. Let $\mathfrak{P}$ be Poisson Lie algebroid and $\nabla : \tau_1 \exp(\mathfrak{P}) \to \mathbf{B}^2(\mathbb{R}/\Gamma)_{\mathrm{conn}}$ the prequantum 2-bundle of the corresponding 2d Poisson-Chern-Simons prequantum field theory. A maximally isotropic sub-Lie algebroid $\mathfrak{C} \hookrightarrow \mathfrak{P}$ is identified in [CF] with a D-brane for the theory. See [FRS13a] (...)

Further developing example 4.3.6, we have by [FSSt] the following.

Example 4.3.15. The universal boundary condition for ordinary Spin-Chern-Simons theory regarded as a local prequantum field theory $\frac{1}{2}\widehat{\mathbf{p}}_1 : \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$ is the moduli stack of String-2-connections

$$\mathbf{B}\mathrm{String}_{\mathrm{conn}} \longrightarrow \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \xrightarrow{\frac{1}{2}\widehat{\mathbf{p}}_1} \mathbf{B}^3 U(1)_{\mathrm{conn}} \ .$$

The universal boundary condition for 7-dimensional String-Chern-Simons local prequantum field theory [FSS12] $\frac{1}{6}\widehat{\mathbf{p}}_2 : \mathbf{B}\mathrm{String}_{\mathrm{conn}} \to \mathbf{B}^7 U(1)_{\mathrm{conn}}$ is the moduli stack of Fivebrane-6-connections

$$\mathbf{B}\mathrm{Fivebrane}_{\mathrm{conn}} \longrightarrow \mathbf{B}\mathrm{String}_{\mathrm{conn}} \xrightarrow{\frac{1}{6}\widehat{\mathbf{p}}_2} \mathbf{B}^7 U(1)_{\mathrm{conn}} \ .$$

Examples 4.3.16. A rich variant of this class of examples of topological prequantum boundary conditions turns out to be the intersection laws of Green-Schwarz type super p-branes. This we discuss below in 4.4.3.

4.3.5 $d = n - k$, Holonomy defects

The higher parallel transport of an n-connection over a k-dimensional manifold with boundary takes values in sections of the transgression of the n-bundle to an $(n-k+1)$-bundle over the boundary. Here we discuss this construction at the level of moduli stacks and then observe that it is naturally interpreted in terms of defects for higher topological Yang-Mills/higher Chern-Simons theory. The Wess-Zumino-Witten defects and the Wilson line/surface defects in the following sections 4.3.6 and 4.3.7 build on this class of examples.

First observe that a particularly simple boundary condition for topological Yang-Mills theory is to take the connection to be trivial on the boundary via the following

$$
\begin{tikzcd}
 & \Omega^n \arrow[dl] \arrow[dr, "d"] & \\
* \arrow[dr, "0"'] & \arrow[d, phantom, "\swarrow"] & \Omega^{n+1}_{\mathrm{cl}} \arrow[dl, "\exp(iS^{n+1}_{\mathrm{tYM}})"] \\
 & \flat \mathbf{B}^{n+1}U(1) &
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
 & \Omega^n \arrow[d, dashed] \arrow[ddl, bend right] \arrow[ddr, bend left, "d"] & \\
 & \mathbf{B}^n U(1)_{\mathrm{conn}} \arrow[dl] \arrow[dr, "F_{(-)}"] & \\
* \arrow[dr, "0"'] & & \Omega^{n+1}_{\mathrm{cl}} \arrow[dl, "\exp(iS^{n+1}_{\mathrm{tYM}})"] \\
 & \flat \mathbf{B}^{n+1}U(1) &
\end{tikzcd},
$$

which corresponds to the inclusion

$$\Omega^n \to \mathbf{B}^n(1)_{\mathrm{conn}}$$

of globally defined differential n-forms regarded as connections on trivial n-bundles.

Let $n, k \in \mathbb{N}$ with $k \leq n$. For

$$\Sigma_k \in \mathrm{SmoothMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$$

a closed manifold equipped with an orientation, the ordinary fiber integration of differential forms

$$\textstyle\int_{\Sigma_k} : \Omega^n(\Sigma_k \times U) \longrightarrow \Omega^{n-k}(U)$$

is natural in $U \in \mathrm{CartSp} \in \mathrm{SmoothMfd}$ and hence comes from a morphism of smooth spaces

$$\textstyle\int_{\Sigma_k} : [\Sigma_k, \Omega^n] \longrightarrow \Omega^{n-k}$$

in Smooth∞Grpd. Similarly, transgression in ordinary cohomology constitues a morphism in Smooth∞Grpd. This induces a fiber integration formula also on cocoycles in $\mathbf{B}^n U(1)_{\mathrm{conn}}$. The following statement expresses this siutation in detail. This is (the image under the Dold-Kan correspondence of) theorem 3.1 of [GT].

Proposition 4.3.17. *For Σ_k a closed oriented manifold, we have horizontal morphisms making the following diagram commute*

$$
\begin{tikzcd}
{[\Sigma_k, \mathbf{B}^n U(1)_{\mathrm{conn}}]} \arrow[r, "\exp(2\pi i \int_{\Sigma_k}(-))"] \arrow[d] & \mathbf{B}^{n-k}U(1)_{\mathrm{conn}} \arrow[d] \\
{[\Sigma_k, \Omega^{n+1}_{\mathrm{cl}}]} \arrow[r, "\int_{\Sigma_k}(-)"] \arrow[d, "{[\Sigma_k, \exp(iS^{n+1}_{\mathrm{tYM}})]}"'] & \Omega^{n+1-k}_{\mathrm{cl}} \arrow[d, "\exp(iS^{n+1-k}_{\mathrm{tYM}})"] \\
{[\Sigma_k, \flat \mathbf{B}^{n+1}U(1)]} \arrow[r, "\exp(2\pi i \int_{\Sigma}(-))"] & \flat \mathbf{B}^{n+1-k}U(1)
\end{tikzcd}.
$$

Moreover, for Σ_k an oriented manifold with boundary $\partial\Sigma_k$ of dimension $(k-1)$ we have a diagram

$$
\begin{tikzcd}
 & {[\Sigma_k, \mathbf{B}^n U(1)_{\mathrm{conn}}]} \arrow[dl, "(-)|_{\partial\Sigma}"'] \arrow[dr, "\omega_\Sigma"] & \\
{[\partial\Sigma_k, \mathbf{B}^n U(1)_{\mathrm{conn}}]} \arrow[dr, "\exp(2\pi i \int_{\partial\Sigma}(-))"'] & \arrow[d, phantom, "\swarrow \exp(2\pi i \int_{\Sigma}(-))"] & \Omega^{n-k+1} \arrow[dl] \\
 & \mathbf{B}^{n-k+1}U(1)_{\mathrm{conn}} &
\end{tikzcd}
$$

such that when $\partial\Sigma_k = \emptyset$ the homotopy filling this diagram coincides with the above integration map under the identification

$$\mathbf{B}^{n-k}U(1)_{\mathrm{conn}} \simeq * \underset{\mathbf{B}^{n-k+1}U(1)_{\mathrm{conn}}}{\times} \Omega^{n-k+1}_{\mathrm{cl}} \,;$$

hence for $\partial\Sigma_k = \emptyset$ we have

$$\begin{tikzcd}
 & {[\Sigma_k, \mathbf{B}^n U(1)_{\mathrm{conn}}]} \arrow[dl] \arrow[dr, "\omega_\Sigma"] \arrow[d, phantom, "\swArrow\ \exp(2\pi i \int_\Sigma(-))"', shift left=0] & \\
* \arrow[dr] & {} & \Omega^{n-k+1} \arrow[dl] \\
 & \mathbf{B}^{n-k+1}U(1)_{\mathrm{conn}} &
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
 & {[\Sigma_k, \mathbf{B}^n U(1)_{\mathrm{conn}}]} \arrow[d, "\exp(2\pi i \int_{\Sigma_k}(-))"] \arrow[ddl, bend right] \arrow[ddr, bend left, "\omega_\Sigma"] & \\
 & \mathbf{B}^{n-k}U(1)_{\mathrm{conn}} \arrow[dl] \arrow[dr, "F_{(-)}"] & \\
* \arrow[dr, "0"'] & & \Omega^{n-k+1} \arrow[dl] \\
 & \mathbf{B}^{n-k+1}U(1)_{\mathrm{cl}} &
\end{tikzcd} \,.$$

Proof. For the first statement, we need to produce for each $U \in \mathrm{CartSp} \hookrightarrow \mathrm{SmoothMfd}$ a map

$$\mathbf{H}(U \times \Sigma_k, \mathbf{B}^n U(1)_{\mathrm{conn}}) \to \mathbf{H}(U, \mathbf{B}^{n-k}U(1)_{\mathrm{conn}})$$

such that this is natural in U. By the general discussion of $\mathbf{B}^n U(1)_{\mathrm{conn}}$, after a choice of good open cover $\mathcal{U}$ of Σ_k (inducing the good cover $\mathcal{U} \times U$ of $\Sigma_k \times U$) this is given, under the Dold-Kan correspondence $\mathrm{DK}(-)$, by a chain map of the form

$$\begin{tikzcd}
C^n(\mathcal{U} \times U, \underline{U}(1) \to \cdots \to \Omega^n) \arrow[r, "D"] \arrow[d, "\int_\Sigma"] & \cdots \arrow[r, "D"] & C^1(\mathcal{U} \times U, \underline{U}(1) \to \cdots \to \Omega^n) \arrow[r, "D"] \arrow[d, "\int_\Sigma"] & Z^0(\mathcal{U} \times U, \underline{U}(1) \to \cdots \to \Omega^n) \arrow[d, "\int_\Sigma"] \\
0 \arrow[r] & \cdots \arrow[r] & C^1(U, \underline{U}(1) \to \cdots \to \Omega^{n-k}) \arrow[r, "D"] & Z^0(U, \underline{U}(1) \to \cdots \to \Omega^{n-k})
\end{tikzcd} .$$

In [GT] a map $\int_\Sigma$ as above is defined and theorem 2.1 there asserts that it satisfies the equation

$$\int_\Sigma \circ D - (-1)^k \, D \circ \int_\Sigma \;\; = \;\; \int_{\partial\Sigma} \circ (-)|_{\partial\Sigma} \qquad\qquad (\star)$$

in (and this is important) the chain complex $C^\bullet(\mathcal{U} \times U, \underline{U}(1) \to \cdots \to \Omega^{n-k})$. For $\partial\Sigma_k = \emptyset$ this asserts that $\int_\Sigma$ is a chain map as needed for the above.

Next, for the more general statement in the presence of a boundary, we are instead in interpreting formula $(\star)$ as a chain homotopy taking place in $C^\bullet(\mathcal{U} \times U, \underline{U}(1) \to \cdots \to \Omega^{n-k+1})$

$$\begin{tikzcd}
 & \cdots \arrow[r, "D"] & C^1(\mathcal{U} \times U, \underline{U}(1) \to \cdots \to \Omega^n) \arrow[r, "D"] \arrow[dl, "\int_\Sigma"'] & Z^0(\mathcal{U} \times U, \underline{U}(1) \to \cdots \to \Omega^n) \arrow[dl, "\int_\Sigma"'] \\
C^2(U, \underline{U}(1) \to \cdots \to \Omega^{n-k+1}) \arrow[r, "D"] & C^1(U, \underline{U}(1) \to \cdots \to \Omega^{n-k+1}) \arrow[r, "D"] & Z^0(U, \underline{U}(1) \to \cdots \to \Omega^{n-k+1}) &
\end{tikzcd} .$$

The subtlety to be taken care of now is that the equation in theorem 2.1 of [GT] holds in $C^\bullet(\mathcal{U} \times U, \underline{U}(1) \to \cdots \to \Omega^{n-k})$ instead of in $C^\bullet(\mathcal{U} \times U, \underline{U}(1) \to \cdots \to \Omega^{n-k+1})$ as we need it here. But the difference is only that in the latter complex the Deligne differential of an $(n-k)$-form on single patches differs from that in the former by the de Rham differential d of that differential form, which is by definition absent in the former case. But by degree-counting this difference appears only in the map

$$D : C^1(U, \underline{U}(1) \to \cdots \to \Omega^{n-k+1}) \to Z^0(U, \underline{U}(1) \to \cdots \to \Omega^{n-k+1}) = \Omega^{n-k+1}(U)\,.$$

Therefore, we may absorb it by modifying the integration chain map in degree 0. To that end, notice that for $\mathcal{A} \in Z^0(\mathcal{U} \times U, \underline{U}(1) \to \cdots \to \Omega^{n-k})$ we have that

$$(0, \ldots, 0, (\int_{\partial\Sigma} \mathcal{A}|_{\partial\Sigma})_i - d(\int_{\Sigma} \mathcal{A})_i) \quad \in \quad Z^0(U, \underline{U}(1) \to \cdots \to \Omega^{n-k+1})$$

(hence that the difference is a globally well defined differential form), since

$$\delta(\int_{\partial\Sigma} \mathcal{A}|_{\partial\Sigma})_{ij} = \pm d(\int_{\partial\Sigma} \mathcal{A}|_{\partial\Sigma})_{ij}\,,$$

this being the (ij)-component of the identity $D(\int_{\partial\Sigma} \mathcal{A}|_{\partial\Sigma}) = 0$ given by the version of $(\star)$ without boundary applied to the boundary, and since also

$$\delta(d(\int_{\Sigma} \mathcal{A}))_{ij} = d(\delta(\int_{\Sigma} \mathcal{A})_{ij}) = d(\int_{\partial\Sigma} \mathcal{A}|_{\partial\Sigma})_{ij}\,,$$

this being the image under d of the (ij)-component of $(\star)$ applied to the cocycle $\mathcal{A}$, which gives $D\int_{\Sigma} \mathcal{A} = \int_{\partial\Sigma} \mathcal{A}|_{\partial\Sigma}$.

Therefore, there is a natural chain map

$$\begin{tikzcd}
\cdots \arrow[r, "D"] & C^1(\mathcal{U} \times U, \underline{U}(1) \to \cdots \to \Omega^n) \arrow[r, "D"] \arrow[d] & Z^0(\mathcal{U} \times U, \underline{U}(1) \to \cdots \to \Omega^n) \arrow[d, "\mathcal{A} \mapsto (\int_{\partial\Sigma} \mathcal{A}|_{\partial\Sigma})_i - d(\int_{\Sigma} \mathcal{A})_i"] \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \Omega^{n-k+1}(U) \arrow[d, "="] \\
\cdots \arrow[r, "D"] & C^1(U, \underline{U}(1) \to \cdots \to \Omega^{n-k+1}) \arrow[r, "D"] & Z^0(U, \underline{U}(1) \to \cdots \to \Omega^{n-k+1})\,,
\end{tikzcd}$$

which under $\mathrm{DK}(-)$ presents the map denoted

$$[\Sigma_k, \mathbf{B}^n U(1)_{\mathrm{conn}}] \xrightarrow{\omega_\Sigma} \Omega^{n-k+1} \longrightarrow \mathbf{B}^{n-k+1} U(1)_{\mathrm{conn}}$$

in the above statement. This is now manifestly so that adding its negative to the right of equation $(\star)$ makes this equation define a chain homotopy in $C^\bullet(\mathcal{U} \times U, \underline{U}(1) \to \cdots \to \Omega^{n-k+1})$ of the form

$$[D, \int_{\Sigma}] \;:\; \int_{\partial_\Sigma} (-)|_{\partial\Sigma} \;\;\Rightarrow\;\; \omega_\Sigma\,.$$

□

Remark 4.3.18. These maps express the relative higher *holonomy* and *parallel transport* of n-form connections, respectively. The second statement says that the parallel transport of an n-connection over a k-dimensional manifold with boundary is a section of the $\mathbf{B}^{n-k}U(1)$-principal bundle underlying the transgression of the underlying $\mathbf{B}^{n-1}U(1)$-principal connection to the mapping space out of the boundary $\partial\Sigma_k$. The section trivializes that underlying bundle and hence identifies a globally defined connection $(n-k+1)$-form. This is the form ω_Σ in the above diagram.

Example 4.3.19 (Relative Chern-Simons forms)**.** For $\Sigma = D^1$ the 1-disk (the interval $D^1 = [0,1]$), $[D^1, \mathbf{B}G_{\mathrm{conn}}]$ is the moduli stack of paths or *concordances* of G-principal connections. For G a simple Lie group with Killing form $\langle -,- \rangle$ the image of the diagram

$$
\begin{tikzcd}
 & {[D^1, \mathbf{B}G_{\mathrm{conn}}]} \arrow[d] & \\
 & {[D^1, \mathbf{B}^3 U(1)_{\mathrm{conn}}]} \arrow[dl, "(-)|_{\partial\Sigma}"'] \arrow[dr, "\omega_{D^1}"] \arrow[dd, phantom, "\swArrow\ \exp(2\pi i \int_\Sigma(-))"] & \\
{[S^0, \mathbf{B}^3 U(1)_{\mathrm{conn}}]} \arrow[dr, "\exp(2\pi i \int_{\partial\Sigma}(-))"'] & & \Omega^3 \arrow[dl] \\
 & \mathbf{B}^3 U(1)_{\mathrm{conn}} &
\end{tikzcd}
$$

under $[X, -]$, for some manifold X, expresses the classical definition of *relative Chern-Simons forms*: the top piece is the moduli stack of 1-parameter collections of G-principal connections on X, the left map is the union of the connections and the endpoints of the path, and ω_{D^1} is the relative Chern-Simons form, satisfying

$$d\omega_{D^1} = \langle F_{\nabla_1} \wedge F_{\nabla_1} \rangle - \langle F_{\nabla_0} \wedge F_{\nabla_0} \rangle \,.$$

Remark 4.3.20. If $\mathbf{Fields} \in \mathbf{H}$ is the moduli stack of field configurations of some topological field theory as in remark **??**, then $[X, \mathbf{Fields}]$ is the moduli stack of fields over the point which, by corollary 3.3.9 and the defining property of the internal hom $[-,-]$, determined the field theory that sends a k-dimensional closed manifold Σ_k to the original field configurations on $\Pi(\Sigma_k) \times X$

$$\Sigma_k \mapsto [\Pi(\Sigma_k) \times X, \mathbf{Fields}]$$

(with the right hand side is regarded as a k-fold correspondence on the point). At least for discrete theories, hence with $\mathbf{Fields} \simeq \flat\mathbf{Fields}$ such that

$$[\Pi(\Sigma_k) \times X, \mathbf{Fields}] \simeq [\Pi(\Sigma_k \times X), \mathbf{Fields}] \,,$$

this is sometimes known as the *(stacky) dimensional reduction* of the original field theory on X. This captures some – but not all – aspects of what is understood as "dimensional reduction" in the physics literature.

Definition 4.3.21. For $\exp(iS_{\mathrm{CS}}) : \mathbf{Fields} \to \mathbf{B}^n U(1)_{\mathrm{conn}}$ and $\Sigma_k \in \mathrm{SmoothMfds} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ an oriented smooth manifold of dimension $k \leq n$ with boundary, we say that the *transgression* $\exp(2\pi i \int_\Sigma [\Sigma, S_{\mathrm{CS}}])$ of ∇ to maps out of Σ is the diagram obtained by composing the mapping space construction $[\Sigma, -] : \mathbf{H} \to \mathbf{H}$ with the fiber integration $\exp(2\pi i \int_\Sigma(-))$ of prop. 4.3.17:

$$
\begin{tikzcd}[column sep=small]
 & {[\Sigma_k, \mathbf{Fields}]} \arrow[dl, "(-)|_{\partial\Sigma}"'] \arrow[dr] \arrow[dd, phantom, "\swArrow\ \exp(2\pi i \int_{\Sigma_k}[\Sigma_k, S_{\mathrm{CS}}])"] & \\
{[\partial\Sigma_k, \mathbf{Fields}]} \arrow[dr, "\exp(2\pi i \int_{\partial\Sigma_k}[\partial\Sigma_k, S_{\mathrm{CS}}])"'] & & \Omega^{n-k+1} \arrow[dl] \\
 & \mathbf{B}^{n-k+1} U(1)_{\mathrm{conn}} &
\end{tikzcd}
\;:=\;
\begin{tikzcd}[column sep=small]
 & {[\Sigma_k, \mathbf{Fields}]} \arrow[dl, "(-)|_{\partial\Sigma_k}"'] \arrow[dr, "{[\Sigma_k, \nabla]}"] & & \\
{[\partial\Sigma_k, \mathbf{Fields}]} \arrow[dr, "{[\partial\Sigma_k, \nabla]}"'] & & {[\Sigma_k, \mathbf{B}^n U(1)_{\mathrm{conn}}]} \arrow[dl, "(-)|_{\partial\Sigma}"'] \arrow[dr] \arrow[dd, phantom, "\swArrow\ \exp(2\pi i \int_\Sigma(-))"] & \\
 & {[\partial\Sigma_k, \mathbf{B}^n U(1)_{\mathrm{conn}}]} \arrow[dr, "\exp(2\pi i \int_{\partial\Sigma}(-))"'] & & \Omega^{n-k+1} \arrow[dl] \\
 & & \mathbf{B}^{n-k+1} U(1)_{\mathrm{conn}} &
\end{tikzcd}
\,.
$$

Example 4.3.22. If $X \in \mathrm{SmoothMfd} \hookrightarrow \mathrm{Smooth}\infty\mathrm{Grpd}$ is a smooth manifold and

$$\nabla : X \to \mathbf{B}^n U(1)_{\mathrm{conn}}$$

is an n-connection on X, and for Σ_n a closed oriented n-dimensional manifold, then the transgression

$$\exp(2\pi i \int_\Sigma [\Sigma, \nabla]) \ : \ [\Sigma, X] \to U(1)$$

is the *n-volume holonomy* function of ∇. For $n = 1$, hence ∇ is a $U(1)$-principal connection, and $\Sigma = S^1$, this is the traditional notion of holonomy function of a principal connection along closed curves in X.

4.3.6 $d = n - 1$, Wess-Zumino-Witten field theories

We now consider codimension 1-defects for higher Chern-Simons theories, hence codimension-2 corners for topological Yang-Mills theory.

Remark 4.3.23. In [RFFS] 2-dimensional (rational) conformal field theories of WZW type have been constructed and classified by assigning to a punctured marked surface Σ a CFT-n-point function which is induced by applying the Reshitikhin-Turaev 3d TQFT functor (hence local quantum Chern-Simons theory) to a 3-d cobordism cobounding the "double" of the marked surface. In the case that Σ is orientable and without boundary, this is the 3d cylinder $\Sigma \times [-1, 1]$ over Σ. In the language of extended QFTs with defects this construction of a 2d theory from a 3d theory may be formulated as a realization of 2d WZW theory as a codimension-1 defect in 3d Chern-Simons theory. The two chiral halves of the WZW theory correspond to the two "phases" of the 3d theory which are separated by the defect Σ. This perspective on [RFFS] has later been amplified in [KSb].

Now let

$$\exp(iS_{\mathrm{CS}}) = \mathbf{c} : \mathbf{B}G_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

be a Chern-Simons prequantum field theory.

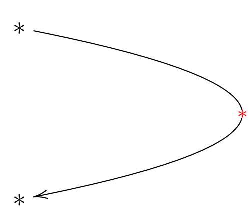
Definition 4.3.24. The *Wess-Zuminio-Witten defect* is the morphism

$$\exp(iS_{\mathrm{CS}} \circ p_1 - iS_{\mathrm{CS}} \circ p_2) \longrightarrow \exp(iS_{\mathrm{tYM}})$$

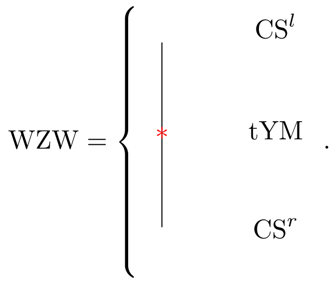
in $\mathrm{Corr}(\mathbf{H}_{/\mathbf{B}^3 U(1)_{\mathrm{conn}}})$ given in $\mathbf{H}$ by the the transgression, def. 4.3.21, of the Chern-Simons connection over the 1-disk

$$\begin{tikzcd}
 & & {[D^1, \mathbf{B}G_{\mathrm{conn}}]} \arrow[dl, "(-)|_{\partial D^1}"'] \arrow[dr] \arrow[dd, phantom, "\swArrow\ \exp(2\pi i \int_{D^1}[D^1, \mathbf{c}])"] & \\
{[S^0, \mathbf{B}G_{\mathrm{conn}}]} \simeq & \mathbf{B}G_{\mathrm{conn}} \times \mathbf{B}G_{\mathrm{conn}} \arrow[dr, "\mathbf{c}\circ p_1 - \mathbf{c}\circ p_2"'] & & \Omega^3_{\mathrm{cl}} \arrow[dl] \\
 & & \mathbf{B}^3 U(1)_{\mathrm{conn}} &
\end{tikzcd}$$

Remark 4.3.25. This is a codimension-1 defect of S^3_{tYM} according to def. 3.3.30. It may be visualized as a 1-dimensional "cap"



for a single copy of the CS-theory, whose 0-dimensional tip carries a tYM-theory. By duality we may straighten this structure and visualize it schematically as



This defect becomes a plain boundary for the tYM-theory when the left end is attached to a boundary that couples the left with the right part of the CS-theory:

Definition 4.3.26. The *Wess-Zumino-Witten codimension-2* corner in 4d topological Yang-Mills theory is the boundary

$$
\begin{tikzcd}
\mathbb{I} \arrow[r] \arrow[d] & \mathbb{I} \arrow[d] \\
\mathbb{I} \arrow[r, "S^3_{\mathrm{tYM}}"'] & S^4_{\mathrm{tYM}}
\end{tikzcd}
$$

of the boundary 3d tYM theory given as a diagram in $\mathbf{H}$ by the composite

$$
\begin{tikzcd}
 & G \arrow[d, dashed, "\exp(iS_{\mathrm{WZW}})"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & \mathbf{B}^2U(1)_{\mathrm{conn}} \arrow[dl] \arrow[dr, "F_{(-)}"] & \\
{*} \arrow[dr, "0"'] \arrow[rr, phantom, "\swarrow \nabla_{\mathrm{ChS}}"] & & \Omega^3_{\mathrm{cl}} \arrow[dl, "\exp(iS^3_{\mathrm{tYM}})"] \\
 & \mathbf{B}^3U(1)_{\mathrm{conn}} &
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
 & & G \arrow[dl] \arrow[dr] & & \\
 & {*} \arrow[dl] \arrow[dr] & & {[D^1, \mathbf{B}G_{\mathrm{conn}}]} \arrow[dl, "(-)|_{\partial D^1}"'] \arrow[dr] & \\
{*} \arrow[drr, "0"'] & & {[S^0, \mathbf{B}G_{\mathrm{conn}}]} \arrow[d, "\mathbf{c}\circ p_1 - \mathbf{c}\circ p_2"] \arrow[rr, phantom, "\swarrow \exp(2\pi i \int_{D^1}(-))"] & & \Omega^3_{\mathrm{cl}} \arrow[dll] \\
 & & \mathbf{B}^3U(1)_{\mathrm{conn}} & &
\end{tikzcd}
,
$$

where the bottom right square is that of def. 4.3.24, the bottom left square is filled with the evident equivalence, and where the map $G \to [S^1, \mathbf{B}G_{\mathrm{conn}}]$ in the top square is given by resolving the simply connected Lie group G by its based path space P_*G, regarded as a diffeological space. Then each path uniquely arises as the parallel transport of a G-principal connection on the interval and two paths with the same endpoint have a unique gauge transformation relating them.

Remark 4.3.27. Here G is the differential concretification of the pullback in the middle.

Proposition 4.3.28. *The morphism*

$$\exp(iS_{\mathrm{WZW}}) : G \to \mathbf{B}^2U(1)_{\mathrm{conn}}$$

from def. 4.3.26 is the WZW-2-connection (the "WZW gerbe"/"WZW B-field").

Proof. This follows along the lines of the discussion in [FSS13], where it was found that the composite

$$G \longrightarrow [S^1, \mathbf{B}G_{\mathrm{conn}}] \xrightarrow{[S^1, \mathbf{c}]} [S^1, \mathbf{B}^3U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{S^1}(-))} \mathbf{B}^2U(1)_{\mathrm{conn}}$$

is the (topological part of) the localized WZW action. ∎

4.3.7 $d = n-2$, Wilson loop/Wilson surface field theories

In [FSS13] a description of how Wilson loop line defects in 3d Chern-Simons theory is given by the following data.

Let $\lambda \in \mathfrak{g}$ be a regular weight, corresponding via Borel-Weil-Bott to the irreducible representation which labels the Wilson loop. Then the stabilizer subgroup $G_\lambda \hookrightarrow G$ of λ under the adjoint action is a maximal torus $G_\lambda \simeq T \hookrightarrow G$ and $G/G_\lambda \simeq \mathcal{O}_\lambda$ is the coadjoint orbit.

Integrality of λ means that pairing with λ constitutes a morphism of moduli stacks of the form

$$S_W : \Omega^1(-,\mathfrak{g})//T \xrightarrow{\langle\lambda,-\rangle} \mathbf{B}U(1)_{\mathrm{conn}} .$$

This is the local Lagrangian/the prequantum bundle of the Wilson loop theory in that there is a diagram

$$\begin{tikzcd}
\mathcal{O}_\lambda \arrow[r, "\mathrm{fib}(\mathbf{J})"] \arrow[d] & \Omega^1(-,\mathfrak{g})//T \arrow[r, "\langle\lambda,-\rangle"] \arrow[d, "\mathbf{J}"] & \mathbf{B}U(1)_{\mathrm{conn}} \\
* \arrow[r] & \mathbf{B}G_{\mathrm{conn}} \arrow[r, "\mathbf{c}"] & \mathbf{B}^3U(1)_{\mathrm{conn}} ,
\end{tikzcd}$$

whose top composite is the Kirillov prequantum bundle on the coadjoint orbit and which is such that a Chern-Simons + Wilson loop field configuration (A, g) is a diagram

$$\begin{tikzcd}
S^1 \arrow[r, "A|^g_{S^1}"] \arrow[d, hook] & \Omega^1(-,\mathfrak{g})//T \arrow[d, "\mathbf{J}"] \arrow[dl, Rightarrow, "g"] \\
\Sigma_3 \arrow[r, "A"'] & \mathbf{B}G_{\mathrm{conn}}
\end{tikzcd}$$

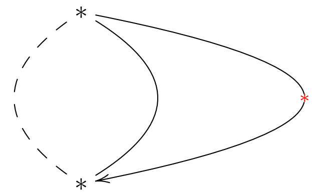
and the corresponding action functional is the product of $\langle\lambda,-\rangle$ transgressed over S^1 and $\mathbf{c}$ transgressed over Σ_3.

We now interpret this formally as a codimension-2 defect of Chern-Simons theory analogous to the WZW defect, hence as a codimension-3 structure in the ambient tYM theory.

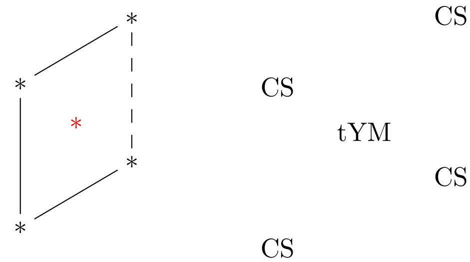
Definition 4.3.29. Let $\phi : D^2 \to S^2$ be a smooth function which on the interior of S^2 is a diffeomorphism on $S^2 - \{*\}$. The *universal Wilson line/Wilson surface defect* is, as a diagram in $\mathbf{H}$, the transgression diagram, def. 4.3.21

$$\begin{tikzcd}
 & & {[S^2, \mathbf{B}G_{\mathrm{conn}}]} \arrow[dll] \arrow[dd, "{[\phi|_{S^1}, \mathbf{B}G_{\mathrm{conn}}]}"] \arrow[drr, "{[\phi, \mathbf{B}G_{\mathrm{conn}}]}"] & & & & \\
{[\Pi(S^1), \mathbf{B}G_{\mathrm{conn}}]} \arrow[drr] & & & & {[D^2, \mathbf{B}G_{\mathrm{conn}}]} \arrow[dll, "(-)|_{\partial D^2}"'] \arrow[drr] & & \\
 & & {[S^1, \mathbf{B}G_{\mathrm{conn}}]} \arrow[drr, "\exp(2\pi i \int_{S^1}[S^1, \mathbf{c}])"'] & & \arrow[d, phantom, "\Swarrow \exp(2\pi i \int_{D^2}[D^2, \mathbf{c}])"] & & \Omega^2_{\mathrm{cl}} . \arrow[dll] \\
 & & & & \mathbf{B}^2U(1)_{\mathrm{conn}} & &
\end{tikzcd}$$

Remark 4.3.30. This is a codimension-2 defect according to 3.3.30. It may be visualized as a 2-dimensional CS theory cap



with a tYM-theory sitting at the very tip. By duality there is the corresponding straightened picture



We now define a defect that factors through the universal Wilson defect of def. 4.3.29 and reproduces the traditional Wilson line action functional. To that end, let $\nabla_{S^2} : S^2 \to \mathbf{B}T_{\mathrm{conn}}$ be a T-principal connection on the 2-sphere, where T is the maximal torus of G. We may identify the integral of its curvature 2-form over the sphere with the weight λ

$$\lambda = \int_{S^2} F_{\nabla_{S^2}} \ .$$

Then consider the morphism

$$p_1^* \nabla_{S^1} + p_2^*(-) : \ \Omega^1(-, \mathfrak{g})//T \longrightarrow [S^2, \mathbf{B}G_{\mathrm{conn}}]$$

in $\mathbf{H}$ which over a test manifold $U \in \mathrm{CartSp}$ sends a connection 1-form $A \in \Omega^1(U, \mathfrak{g})$ to

$$p_1^* \nabla_{S^2} + p_2^* A \in \mathbf{H}(S^2 \times U, \mathbf{B}G_{\mathrm{conn}}) \, .$$

This is indeed a homomorphism since T is abelian.

Proposition 4.3.31.

$$\begin{tikzcd}
 & \Omega^1(-,\mathfrak{g})//T \arrow[d, "\langle \lambda, - \rangle"] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & \mathbf{B}U(1)_{\mathrm{conn}} \arrow[dl] \arrow[dr, "F_{(-)}"] \arrow[d, phantom, "\Swarrow \nabla_{\mathrm{CS}}"] & \\
 {*} \arrow[dr, "0"'] & & \Omega^2_{\mathrm{cl}} \arrow[dl, "\exp(i S^2_{\mathrm{tYM}})"] \\
 & \mathbf{B}^2 U(1)_{\mathrm{conn}} &
\end{tikzcd}
\quad \simeq \quad
\begin{tikzcd}
 & \Omega^1(-,\mathfrak{g})//T \arrow[dl] \arrow[dr, "p_1^*\nabla_{S^2} + p_2^*(-)"] & & \\
 {*} \arrow[dr] & & {[S^2, \mathbf{B}G_{\mathrm{conn}}]} \arrow[dl, "(-)|_{\partial S^2}"'] \arrow[dr] \arrow[d, phantom, "\Swarrow \exp(2\pi i \int_{S^2}[S^2,-])"] & \\
 & {[\emptyset, \mathbf{B}G_{\mathrm{conn}}]} \arrow[dr, "\exp(2\pi i \int_{\emptyset}[\emptyset, \mathbf{c}])"'] & & \Omega^2_{\mathrm{cl}} \arrow[dl] \ . \\
 & & \mathbf{B}^2 U(1)_{\mathrm{conn}} &
\end{tikzcd}$$

Proof. By construction and since T is abelian, the component of the Chern-Simons form of $p_1^* \nabla_{S^2} + p_2^* A$ with two legs along S^2 is proportional to $\langle F_{\nabla_{S^2}} \wedge A \rangle$. Hence its fiber integral over $S^2 \times U \to U$ is

$$\int_{S^2} \langle F_{\nabla_{S^2}} \wedge A \rangle = \langle \lambda, A \rangle \, .$$

$\square$

So we set

Definition 4.3.32. The Wilson line defect is

$$
\begin{tikzcd}
 & \Omega^1(-,\mathfrak{g})//T \arrow[d, "p_1^*\nabla_{S^2} + p_2^*(-)"] & & \\
 & {[S^2, \mathbf{B}G_{\mathrm{conn}}]} \arrow[dl] \arrow[dr, "{[\phi, \mathbf{B}G_{\mathrm{conn}}]}"] & & \\
{[\Pi(S^1), \mathbf{B}G_{\mathrm{conn}}]} \arrow[dr] & & {[D^2, \mathbf{B}G_{\mathrm{conn}}]} \arrow[dl, "(-)|_{\partial D^2}"'] \arrow[dr] \arrow[d, phantom, "\swarrow \exp(2\pi i \int_{D^2}[D^2,\mathbf{c}])"] & \\
 & {[S^1, \mathbf{B}G_{\mathrm{conn}}]} \arrow[dr, "\exp(2\pi i \int_{S^1}[S^1,\mathbf{c}])"'] & {} & \Omega^2_{\mathrm{cl}} \arrow[dl] \\
 & & \mathbf{B}^2 U(1)_{\mathrm{conn}} &
\end{tikzcd}
.
$$

4.4 Higher Wess-Zumino-Witten local prequantum field theory

In the above discussion in 4.3.6 we found the traditonal Wess-Zumino-Witten term as a boundary condition of the local prequantum version of traditional Chern-Simons theory. But as the discussion there also shows, in fact the WZW term itself is of *higher local prequantum Chern-Simons type* (actually of relatively *lower* Chern-Simons type) in that it is given by a map to $\mathbf{B}^n U(1)_{\mathrm{conn}}$, for suitable n.

Here we consider higher WZW-type local prequantum field theories as examples of ∞-Chern-Simons theories, hence, by 4.3.3, as boundary conditions for the universal topological Yang-Mills theory. We find that for every cohesive ∞-group G equipped with a higher cocycle given by a universal characteristic map $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^{n+1}U(1)$ and equipped with a choice differential Chern-Simons form data $\mathrm{cs}_{\mathbf{c}}$ representing the induced de Rham hypercohomology class $\flat_{\mathrm{dR}}\mathbf{c}$, there is a canonical action functional

$$
\exp(iS_{\mathrm{WZW}}) : \; \tilde{G} \longrightarrow \mathbf{B}^n U(1)_{\mathrm{conn}}
$$

for a higher local prequantum field theory,

- whose field moduli $\tilde{G}$ are a twisted product of σ-model maps to G and higher worldvolume gauge fields

and such that

- the underlying n-bundle $G \to \mathbf{B}^n U(1)$ is the looping $\Omega\mathbf{c}$ of the original universal cocycle;
- the underlying curvature is the Maurer-Cartan form of G evaluated in the Chern-Simons form.

Restricted to the canonical 3-cocycle on a suitable Lie group this reproduces the traditional WZW model, with $\tilde{G} \simeq G$. But for higher cocycles on higher groups $\tilde{G}$ this turns out to be a genuine differential refinement of G, meaning that higher WZW models are no longer pure σ-models but rather are mixtures of σ-models and higher gauge theories. This is a phenomenon expected for the worldvolume field theories of higher super-branes in string theory and M-theory, such as the Dirac-Born-Infeld (DBI) actions of D-branes and the tensor multiplet fields on the M5-branes. Indeed, in 4.4.3 we indicate how the full super-p-brane content of string theory/M-theory naturally arises by applying this construction of higher local WZW-type prequantum field theory to exceptional cocycles in higher supergeometry.

The following is a brief survey of [SuperOrbi]

4.4.1 Motivation

In traditional literature, the "WZW term" in the action functional of the WZW-model is often introduced in a simplified fashion as follows. Let G be a simply-connected semisimple Lie group, $k \in H^3(G,\mathbb{Z}) \simeq \mathbb{Z}$ a cohomology class and Σ_2 a closed 2-dimensional smooth manifold that is the boundary of a 3-manifold Σ_3. Then the *WZW-term* is the functional

$$\exp(iS_{\mathrm{WZW}}) : [\Sigma_2, G] \to U(1) ,$$

which is given on a smooth function $\phi : \Sigma_2 \to G$ by choosing an extension $\hat{\phi} : \Sigma_3 \to G$ and then assigning

$$\exp(iS_{\mathrm{WZW}}(\phi)) := \exp\left(2\pi i \int_{\Sigma_3} \hat{\phi}^* \langle \theta \wedge [\theta \wedge \theta]\rangle\right) \in U(1) ,$$

where $\theta \in \Omega^1_{\mathrm{flat}}(G)$ is the Maurer-Cartan form on G and $\langle -,-\rangle$ is the Killing form invariant polynomial on the Lie algebra $\mathfrak{g}$ of G normalized such that the 3-form $\langle \theta \wedge [\theta \wedge \theta]\rangle \in \Omega^3(G)$ is a de Rham representative of the chosen $k \in H^3(G,\mathbb{Z})$. By this condition the above assignment is well-defined, since for $\hat{\phi}'$ any other extension of ϕ the difference

$$\int_{\Sigma_3} \hat{\phi}^* \langle \theta \wedge [\theta \wedge \theta]\rangle - \int_{\Sigma_3} (\hat{\phi}')^* \langle \theta \wedge [\theta \wedge \theta]\rangle \in \mathbb{Z} \hookrightarrow \mathbb{R}$$

is an integer and hence vanishes after exponentiation $\exp(2\pi i(-))$ to an element in $U(1)$.

But there is a more conceptual way to understand the existence of this WZW term. Indeed, notice that the condition on the normalization of $\langle -,-\rangle$ says that it constitutes a diagram of smooth stacks of the form

$$\begin{tikzcd} G \arrow[r, "\langle \theta\wedge[\theta\wedge\theta]\rangle"] \arrow[d, "k"'] & \Omega^3_{\mathrm{cl}} \arrow[d] \arrow[dl, Rightarrow, "\simeq"] \\ \mathbf{B}^2 U(1) \arrow[r, "\mathrm{curv}"'] & \flat_{\mathrm{dR}} \mathbf{B}^3 U(1) \end{tikzcd}$$

and therefore, by the universal property of the homotopy pullback, a map

$$\nabla : \; G \longrightarrow \mathbf{B}^2 U(1)_{\mathrm{conn}} \; .$$

This is a circle 2-bundle with connection (a $U(1)$-bundle gerbe with connection) on the Lie group G satisfying the following two properties

1. the underlying integral class is $\chi(\nabla) = k \in H^3(G,\mathbb{Z})$;
2. the underlying curvature 3-form is $F_\nabla = \langle \theta \wedge [\theta \wedge \theta]\rangle$.

In terms of this the above WZW term is simply the surface holonomy of this 2-connection:

$$\exp(iS^{\nabla}_{\mathrm{WZW}}) : \; [\Sigma_2, G] \xrightarrow{[\Sigma_2, \nabla]} [\Sigma_2, \mathbf{B}^2 U(1)_{\mathrm{conn}}] \xrightarrow{\exp(2\pi i \int_{\Sigma_2}(-))} U(1) \; .$$

This more intrisic description has the advantage that it applies generally, not depending on the existence of coboundaries of Σ_2 in G.

This description of the traditional WZW term by differential cohomology/bundle gerbes is known since [Ga], see for instance [ScW] for a review. Here we now discuss an even more fundamental origin of this construction which will generalize it from WZW-terms on ordinary Lie groups to WZW terms on general smooth ∞-groups.

The key to this description is to understand the above construction as a "differentially twisted delooping" in differential cohomology. This we motivate now. The formal definition follows below in Section 4.4.2.

In the above example, by assumption on G there is a universal characteristic map

$$c : BG \to B^3U(1)$$

representing a class $[c] \in H^4(BG, \mathbb{Z}) \simeq \mathbb{Z}$ which is such that the class k above is the *looping* of c:

$$k \simeq \Omega c : G \to B^2U(1)\,.$$

For instance if $G = \mathrm{Spin}$ is the Spin-group and $k = 1$ is the generator on $H^3(\mathrm{Spin}, \mathbb{Z})$, then the corresponding $c = \frac{1}{2}p_1$ is the *first fractional Pontrjagin class*.

The following digram displays the first steps in the long homotopy fiber sequence of $\frac{1}{2}p_1 : B\mathrm{Spin} \to B^3U(1)$ together with a given Spin-principal bundle $P \to X$ classified by a map $X \to B\mathrm{Spin}$. All squares are homotopy pullback squares of bare homotopy types.

$$\begin{tikzcd}
BU(1) \arrow[r] \arrow[d] & \mathrm{String} \arrow[r] \arrow[d, "{\substack{BU(1)\\ \text{bundle}}}"] & \hat{P} \arrow[r] \arrow[d, "{\substack{BU(1)\\ \text{bundle}}}"] \arrow[dd, bend right, "{\substack{\text{String}\\ \text{bundle}}}"'] & {*} \arrow[d] & \\
{*} \arrow[r] & \mathrm{Spin} \arrow[r] \arrow[d] \arrow[rr, bend right=15, "{\substack{\text{canonical}\\ 3-\text{class}}}"'] & P \arrow[r] \arrow[d, "{\substack{\text{Spin}\\ \text{bundle}}}"] & B^2U(1) \arrow[r] \arrow[d] & {*} \arrow[d] \\
 & {*} \arrow[r, "x"] & X \arrow[r, dashed] \arrow[rr, bend right=15, "{\substack{\text{classifies}\\ \text{Spin bundle}}}"'] \arrow[drr, bend right, "{\substack{\text{Pontrjagin}\\ \text{class}}}"'] & B\mathrm{String} \arrow[r] \arrow[d] & B\mathrm{Spin} \arrow[d, "\frac{1}{2}p_1"] \\
 & & & {*} \arrow[r] & B^3U(1)\,.
\end{tikzcd}$$

The topological group String which appears here as the loop space object of the homotopy fiber of $\frac{1}{2}p_1$ is the *String group*. It is a $BU(1)$-extension of the Spin-group: $BU(1) \to \mathrm{String} \to \mathrm{Spin}$. We may think of the String group here as being the total space of the *WZW 2-bundle* (WZW bundle gerbe) which is classified by $\chi : \mathrm{Spin} \to B^2U(1)$.

Or rather, it is the *geometric realization* of the total space of the smooth (stacky) total space of the WZW 2-bundle. For if X happens to be equipped with the structure of a smooth manifold, then it is natural to also equip the Spin-principal bundle $P \to X$ with the structure of a smooth bundle, and hence to lift the classifying map $X \to B\mathrm{Spin}$ to a morphism $X \to \mathbf{B}\mathrm{Spin}$ into the *smooth moduli stack* of smooth Spin-principal bundles (the morphism that not just classifies but "modulates" $P \to X$ as a smooth structure). An evident question then is: can the rest of the diagram be similarly lifted to a smooth context? This indeed turns out to be the case, if we work in the context of *higher* smooth stacks. For instance, there is a smooth moduli 3-stack $\mathbf{B}^2U(1)$ such that a morphism $\mathrm{Spin} \to \mathbf{B}^2U(1)$ not just classifies a $BU(1)$-bundle over Spin, but "modulates" a smooth *circle 2-bundle* or $U(1)$*-bundle gerbe* over Spin. One then gets the following diagram

$$
\begin{tikzcd}
\mathbf{B}U(1) \arrow[r] \arrow[d] & \mathrm{String} \arrow[r] \arrow[d, "{\substack{\mathrm{WZW} \\ 2-\mathrm{bundle}}}"] & \hat{P} \arrow[r] \arrow[d, "{\substack{\mathbf{B}U(1) \\ 2-\mathrm{bundle}}}"] \arrow[dd, bend right, "{\substack{\mathrm{String} \\ 2-\mathrm{bundle}}}"'] & * \arrow[d] & \\
* \arrow[r] & \mathrm{Spin} \arrow[r] \arrow[d] \arrow[rr, bend right, "{\substack{\mathrm{modulates} \\ \mathrm{WZW} \\ 2-\mathrm{bundle}}}"'] & P \arrow[r] \arrow[d, "{\substack{\mathrm{Spin} \\ \mathrm{bundle}}}"] & \mathbf{B}^2U(1) \arrow[r] \arrow[d] & * \arrow[d] \\
 & * \arrow[r, "x"] & X \arrow[r, dashed] \arrow[rr, bend right, "{\substack{\mathrm{modulates} \\ \mathrm{Spin\ bundle}}}"'] \arrow[drr, bend right, "{\substack{\mathrm{modulates} \\ \mathrm{Chern-Simons} \\ 3-\mathrm{bundle}}}"'] & \mathbf{B}\mathrm{String} \arrow[r] \arrow[d] & \mathbf{B}\mathrm{Spin} \arrow[d, "\frac{1}{2}\mathbf{p}_1"] \\
 & & & * \arrow[r] & \mathbf{B}^3U(1)
\end{tikzcd}
$$

where now all squares are homotopy pullbacks of smooth higher stacks.

With this smooth geometric structure in hand, one can then go further and ask for *differential* refinements: the smooth Spin-principal bundle $P \to X$ might be equipped with a principal connection ∇, and if so, this will be "modulated" by a morphism $X \to \mathbf{B}\mathrm{Spin}_{\mathrm{conn}}$ into the smooth moduli stack of Spin-connections. One of the main results in [FSSt] is that the universal first fractional Pontrjagin class can be lifted to this situation to a *differential smooth* universal morphism of higher moduli stacks, which we write $\frac{1}{2}\hat{\mathbf{p}}_1$. Inserting this into the above diagram and then forming homotopy pullbacks as before yields further differential refinements. It turns out that these now induce the Lagrangians of 3-dimensional Spin Chern-Simons theory and of the WZW theory on the group Spin. We then have the diagram

$$
\begin{tikzcd}
\mathbf{B}U(1) \arrow[r] \arrow[d] & \mathrm{String} \arrow[r] \arrow[d, "{\substack{\mathrm{WZW} \\ 2-\mathrm{bundle}}}"] & \hat{P} \arrow[d, "{\substack{\mathbf{B}U(1) \\ 2-\mathrm{bundle}}}"] \arrow[dd, bend right, "{\substack{\mathrm{String} \\ 2-\mathrm{bundle}}}"'] & & \\
* \arrow[r] & \mathrm{Spin} \arrow[r] \arrow[d] \arrow[rr, bend right, "{\substack{\mathrm{WZW} \\ \mathrm{Lagrangian}}}"'] & P \arrow[r] \arrow[d, "{\substack{\mathrm{Spin} \\ \mathrm{bundle}}}"] & \mathbf{B}^2U(1)_{\mathrm{conn}} \arrow[d] & \\
 & * \arrow[r, "x"] & X \arrow[r, dashed] \arrow[rr, bend right, "{\substack{\mathrm{Spin} \\ \mathrm{connection}}}"'] \arrow[drr, bend right, "{\substack{\mathrm{Chern-Simons} \\ \mathrm{Lagrangian}}}"'] & \mathbf{B}\mathrm{String}_{\mathrm{conn}} \arrow[r] \arrow[d] & \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \arrow[d, "\frac{1}{2}\hat{\mathbf{p}}_1"] \\
 & & & * \arrow[r] & \mathbf{B}^3U(1)_{\mathrm{conn}}\,.
\end{tikzcd}
$$

The point to notice here is that as we pass from the smooth version of the diagram to the differential version, the simple looping procedure no longer quite applies: the looping of the differentially refined universal characteristic map

$$\tfrac{1}{2}\hat{\mathbf{p}}_1 : \mathbf{B}\mathrm{Spin}_{\mathrm{conn}} \to \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

is not the WZW Lagrangian, but it instead only its *flat* and discrete version

$$\Omega(\tfrac{1}{2}\hat{\mathbf{p}}_1) : \flat\mathrm{Spin} \to \flat\mathbf{B}^2 U(1)\,.$$

The idea of the general definition of WZW terms that we turn to now is therefore to replace the looping of differentially refined universal characteristic maps by a natural "differentially twisted" version that does send higher cocycles/Chern-Simons terms to desirably *non-flat* higher WZW terms.

4.4.2 General mechanism: differentially twisted delooping of higher Chern-Simons

For $\mathbf{H}$ a differentially cohesive ∞-topos, consider in the following

1. a cohesive ∞-group $G \in \mathrm{Grp}(\mathbf{H})$;
2. a choice of cohesive universal characteristic map $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^{n+1}U(1)$.

Remark 4.4.1. The long homotopy fiber sequence of $\mathbf{c}$ is obtained by forming successive homotopy fibers as shown in the following diagram

$$\begin{tikzcd}
\hat{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {*} \arrow[d] & \\
G \arrow[r, "\mathbf{\Omega c}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{B}^n U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {*} \arrow[d] \\
{*} \arrow[r] & \mathbf{B}\hat{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{B}G \arrow[d, "\mathbf{c}"] \\
 & {*} \arrow[r] & \mathbf{B}^{n+1}U(1)\,.
\end{tikzcd}$$

Accordingly, by the universal property of the homotopy pullback, this induces a sequence of horizontal maps fitting into the following diagram

$$\begin{tikzcd}
G \arrow[r, "\Omega\mathbf{c}"] \arrow[d] & \mathbf{B}^{n-1}U(1) \arrow[d] \\
\flat_{\mathrm{dR}}\mathbf{B}G \arrow[r, "\flat_{\mathrm{dR}}\mathbf{c}"] \arrow[d] & \flat_{\mathrm{dR}}\mathbf{B}^n U(1) \arrow[d] \\
\flat\mathbf{B}G \arrow[r, "\flat\mathbf{c}"] \arrow[d] & \flat\mathbf{B}^n U(1) \arrow[d] \\
\mathbf{B}G \arrow[r, "\mathbf{c}"] & \mathbf{B}^n U(1)\,.
\end{tikzcd}$$

Notably, the map $\Omega\mathbf{c}$ here is induced from the commuting square

$$\begin{tikzcd}
G \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \flat_{\mathrm{dR}}\mathbf{B}G \arrow[r, "\flat_{\mathrm{dR}}\mathbf{c}"] \arrow[d] & \flat_{\mathrm{dR}}\mathbf{B}^n U(1) \arrow[d] \\
{*} \arrow[r] & \flat\mathbf{B}G \arrow[r, "\flat\mathbf{c}"] & \flat\mathbf{B}^n U(1)
\end{tikzcd}
\simeq
\begin{tikzcd}
G \arrow[r, "\Omega\mathbf{c}"] \arrow[d] & \mathbf{B}^{n-1}U(1) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \flat_{\mathrm{dR}}\mathbf{B}^n U(1) \arrow[d] \\
{*} \arrow[r] & {*} \arrow[r] & \flat\mathbf{B}^n U(1)\,.
\end{tikzcd}$$

Definition 4.4.2. A choice of *flat Chern-Simons form* $\mathrm{cs}_{\mathbf{c}}$ for $\mathbf{c}$ is a completion of $\flat_{\mathrm{dR}}\mathbf{c}$ to a commuting diagram of the form

$$\begin{tikzcd} \Omega(-,\mathfrak{g})_{\mathrm{flat}} \arrow[r, "\mathrm{cs}_{\mathbf{c}}"] \arrow[d] & \Omega^{n+1}_{\mathrm{cl}} \arrow[d] \\ \flat_{\mathrm{dR}}\mathbf{B}G \arrow[r, "\flat_{\mathrm{dR}}\mathbf{c}"] & \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1) \end{tikzcd},$$

where the vertical maps are such that for each manifold Σ their image under $[\Sigma,-]$ is a 1-epimorphism.

Definition 4.4.3. Given $\mathbf{c}$ and $\mathrm{cs}_{\mathbf{c}}$ as above, we say that

1. the object $\tilde{G} \in \mathbf{H}$ given by the homotopy pullback

$$\begin{tikzcd} \tilde{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega(-,\mathfrak{g})_{\mathrm{flat}} \arrow[d] \\ G \arrow[r, "\theta_G"] & \flat_{\mathrm{dR}}\mathbf{G} \end{tikzcd}$$

   is the corresponding *differential extension* of G.

2. the map

$$\exp(iS_{\mathrm{WZW}}) : \tilde{G} \longrightarrow \mathbf{B}^n U(1)_{\mathrm{conn}} ,$$

   which is defined by $\mathbf{c}$ and the universal property of the homotopy pullback, as shown in the following diagram (built using the diagrams in remark 4.4.1)

$$\begin{tikzcd} \tilde{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega(-,\mathfrak{g}) \arrow[r, "\mathrm{cs}_{\mathbf{c}}"] \arrow[d] & \Omega^{n+1}_{\mathrm{cl}} \arrow[d] \\ G \arrow[r, "\theta_G"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \flat_{\mathrm{dR}}\mathbf{B}G \arrow[r, "\flat_{\mathrm{dR}}\mathbf{c}"] \arrow[d] & \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1) \arrow[d] \\ {*} \arrow[r] & \flat\mathbf{B}G \arrow[r, "\flat\mathbf{c}"] & \flat\mathbf{B}^{n+1}U(1) \end{tikzcd} \simeq \begin{tikzcd} \tilde{G} \arrow[r, dashed, "\exp(iS_{\mathrm{WZW}})"] \arrow[d] & \mathbf{B}^n U(1)_{\mathrm{conn}} \arrow[r, "F_{(-)}"] \arrow[d, "\chi"] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega^{n+1}_{\mathrm{cl}} \arrow[d] \\ G \arrow[r, "\Omega\mathbf{c}"] \arrow[d] & \mathbf{B}^n U(1) \arrow[r, "\mathrm{curv}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1) \arrow[d] \\ {*} \arrow[r] & {*} \arrow[r] & \flat\mathbf{B}^{n+1}U(1) \end{tikzcd},$$

   is the corresponding ∞-*WZW local action functional*.

Remark 4.4.4. By the very commutativity of these diagrams we have that

1. the underlying n-bundle χ of $\exp(iS_{\mathrm{WZW}})$ is the looping of the original cocycle $\mathbf{c}$ (pulled back from G to its differential extension $\tilde{G}$);

2. the curvature $(n+1)$-form of $\exp(iS_{\mathrm{WZW}})$ is the Chern-Simons form evaluated on the globally defined representative of the canonical Maurer-Cartan form θ on the cohesive ∞-group G.

These are the two criteria that justify referring to $\exp(iS_{\mathrm{WZW}})$ as being the (higher local) *Wess-Zumino-Witten*-type action functional assigned to $(\mathbf{c}, \mathrm{cs}_{\mathbf{c}})$.

Remark 4.4.5. Comparison of these diagram with the proof of cor. 3.3.25 shows that $\exp(iS_{\mathrm{WZW}})$ is precisely a boundary condition for the universal topological Yang-Mills theory in dimension n:

$$
\begin{tikzcd}
 & \tilde{G} \arrow[dl, bend right] \arrow[d, dashed, "\exp(iS_{\mathrm{WZW}})"] \arrow[dr, bend left, "\mathrm{cs}_{\mathbf{c}}(\theta)"] & \\
 & \mathbf{B}^n U(1)_{\mathrm{conn}} \arrow[dl] \arrow[dr, "F_{(-)}"] & \\
 * \arrow[dr, "0"'] & \arrow[l, phantom, "\swarrow"] & \Omega^{n+1}_{\mathrm{cl}} \arrow[dl, "\exp(iS_{\mathrm{tYM}})"] \\
 & \flat \mathbf{B}^{n+1} U(1) &
\end{tikzcd}
.
$$

Notice that this diagram alone only fixes the curvature (pre-n-plectic form) of $\exp(iS_{\mathrm{WZW}})$, and it takes the factorization demanded by the above definition to define $\exp(iS_{\mathrm{WZW}})$.

Remark 4.4.6. Suppose we have an extended CS-theory

$$
\widehat{\mathbf{c}} : \mathbf{B}G_{\mathrm{conn}} \longrightarrow \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}
$$

Then

$$
\begin{tikzcd}
\tilde{G} \arrow[r, "\theta_{\mathrm{global}}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega(-,\mathfrak{g}) \arrow[r, "\mathrm{cs}_{\mathbf{c}}"] \arrow[d] & \Omega^{n+1}_{\mathrm{cl}} \arrow[d] \\
G \arrow[r, "\theta_G"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \flat_{\mathrm{dR}}\mathbf{B}G \arrow[r, "\flat_{\mathrm{dR}}\mathbf{c}"] \arrow[d] & \flat_{\mathrm{dR}}\mathbf{B}^{n+1}U(1) \arrow[d] \\
* \arrow[r] \arrow[d] & \flat\mathbf{B}G \arrow[r, "\flat\mathbf{c}"] \arrow[d] & \flat\mathbf{B}^{n+1}U(1) \arrow[d] \\
* \arrow[r] & \mathbf{B}G_{\mathrm{conn}} \arrow[r, "\widehat{\mathbf{c}}"] & \mathbf{B}^{n+1}U(1)_{\mathrm{conn}}
\end{tikzcd}
.
$$

Then inside this diagram we find a boundary for the Chern-Simons theory:

$$
\begin{tikzcd}
 & \tilde{G} \arrow[dl] \arrow[dr, "\theta_{\mathrm{global}}"] & \\
 * \arrow[dr] & \arrow[l, phantom, "\swarrow"] & \mathbf{B}G_{\mathrm{conn}} \arrow[dl, "\widehat{\mathbf{c}}"] \\
 & \mathbf{B}^{n+1}U(1)_{\mathrm{conn}} &
\end{tikzcd}
$$

We mention two examples, which are at opposite extremes and which are the main building blocks of the generic examples.

Example 4.4.7 (ordinary WZW model on Lie group)**.** Let $G \in \mathrm{LieGrp} \hookrightarrow \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})$ be an ordinary compact simply connected Lie group. Then there is an essentially unique smooth refinement $k\mathbf{c} : \mathbf{B}G \to \mathbf{B}^3U(1)$ of any class in $H^4(BG, \mathbb{Z})$, as constructed in [FSSt]. One finds that $\flat_{\mathrm{dR}}\mathbf{B}G \simeq \Omega^1_{\mathrm{flat}}(-,\mathfrak{g})$ is the ordinary sheaf of smooth flat differential 1-forms with values in the Lie algebra $\mathfrak{g}$ of G, and that the intrinsic Maurer-Cartan form here is identified, via the Yoneda lemma, with the standard Maurer Cartan form $\theta_G \in \Omega^1_{\mathrm{flat}}(G, \mathfrak{g})$. Consequently, the atlas-relative-to-manifolds $\Omega_{\mathrm{flat}}(-, \mathfrak{g}) \to \flat_{\mathrm{dR}}\mathbf{B}G$ is naturally chosen to be just the identity map. This implies that $\tilde{G} \simeq G$ in this case, hence that the WZW model in this case is indeed defined on G itself, not on a differential extension of G. Moreover, $\mathrm{cs}_{\mathbf{c}}$ is now the standard pairing $k\langle -, [-, -]\rangle : \Omega_{\mathrm{flat}}(-, \mathfrak{g}) \to \Omega^3_{\mathrm{cl}}$. Therefore, by the very fact that $\exp(iS_{\mathrm{WZW}})$ fits into its defining diagram as above, it follows that this is a 2-connection with underlying class $\chi(\exp(iS_{\mathrm{WZW}})) \simeq k\Omega\mathbf{c}$ and curvature $\langle \theta \wedge [\theta \wedge \theta]\rangle$. This identifies it as the tradition "WZW gerbe" on G.

Example 4.4.8 (WZW model on circle $(n+1)$-group)**.** For $n \in \mathbb{N}$ consider the circle $(n+1)$-group $G := \mathbf{B}^n U(1) \in \mathrm{Grp}(\mathrm{Smooth}\infty\mathrm{Grpd})$. This carries a tautological $(n+1)$-cocycle

$$\mathbf{DD}_n : \ \mathbf{B}(\mathbf{B}^n U(1)) \xrightarrow{\simeq} \mathbf{B}^{n+1}U(1)$$

which may be called the smooth refinement of the order-n *Dixmier-Douady class* (or order $(n+1)$ 'first Chern class'), but which is just the canonical equivalence, as indicated. Similarly, $\mathbf{cs}_{\mathbf{DD}_n}$ in this case is naturally chosen to be the identity on $\Omega^{n+1}_{\mathrm{cl}}$. With this choice it follows that the "differential extension" of the circle $(n+1)$-group, according to def. 4.4.3, is just its differential coefficient object

$$\tilde{G} \simeq \mathbf{B}^n U(1)_{\mathrm{conn}} \ ,$$

according to prop. 3.4.3. As a result, in this case the higher local WZW Lagrangian is also the identity

$$\exp(iS^{\mathbf{DD}_n}_{\mathrm{WZW}}) \simeq \mathrm{id} : \mathbf{B}^n U(1)_{\mathrm{conn}} \to \mathbf{B}^n U(1)_{\mathrm{conn}} \, .$$

Remark 4.4.9. It follows, by Postnikov decomposition, that a general higher ∞-WZW model looks roughly like a twisted combination of example 4.4.7 with example 4.4.8: its field content is a combination of

1. a σ-model field $\phi : \Sigma \to G$;
2. a gauge field $A : \Sigma \to \mathbf{B}^n U(1)_{\mathrm{conn}}$.

Remark 4.4.10. By the above discussion there is an n-dimensional local prequantum field theory of WZW type associated with a cohesive universal characteristic map $\mathbf{c} : \mathbf{B}G \to \mathbf{B}^{n+1}(\mathbb{R}/\Gamma)$ and a differential form representative of its image $\flat_{\mathrm{dR}}\mathbf{c}$ in de Rham hypercohomology. By the construction in [FSSt] one source of such data are L_∞-algebra cocycles, hence maps of L_∞-algebras of the form

$$\mu : \ \mathfrak{g} \longrightarrow \mathbf{B}^n\mathbb{R} \ \ .$$

As discussed there, these may naturally be Lie integrated to smooth cocycles

$$\mathbf{c} = \exp(\mu) : \ \mathbf{B}G \longrightarrow \mathbf{B}^{n+1}(\mathbb{R}/\Gamma)$$

for $G := \Omega\tau_{n+1}\exp(\mathfrak{g})$ and for Γ the group of periods of μ. Here $\exp(\mathfrak{g})$ is a smooth ∞-stack to be thought of as the delooping of the ∞-connected Lie integration of $\mathfrak{g}$ (its geometric realization is a contractible homotopy type), and where $\tau_{n+1} : \mathbf{H} \to \mathbf{H}$ is the internal Postnikov truncation in degree $n+1$.

In Section 4.4.3 below we indicate a class of examples of higher WZW models with such "tensor multiplet" field content of remark 4.4.9 constructed with the technology recalled in remark 4.4.10.

4.4.3 Outlook: Exceptional examples – The super p-branes of string/M-theory

We indicate here a particularly interesting class of examples of the higher local WZW-type prequantum field theories of 4.4.2, namely the *super-p-brane models* of string theory/M-theory. We just give a brief survey, a detailed discussion of this example is in the companion article [SuperOrbi].

The ambient ∞-topos of higher supergeometry. In [FSSt] the construction of smooth higher cocycles by Lie integration, as in remark 4.4.10, was described in higher smooth geometry, hence for $\mathbf{H} \simeq \mathrm{Sh}_\infty(\mathrm{SmthMfd})$. But the same mechanism immediately works relative to any base ∞-presheaf ∞-topos. In particular, therefore, by the discussion in section 4.6 of [S], we may apply this to higher *supergeometry* modeled by smooth geometry over the base ∞-topos over the category of superpoints

$$\mathrm{SmoothSuper}\infty\mathrm{Grpd} := \mathrm{Sh}_\infty(\mathrm{SuperMfd}) \underset{\Gamma}{\overset{\mathrm{Disc}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathrm{Sh}_\infty(\mathrm{SuperPoints}) \;,$$

where

$$\mathrm{SuperPoints} := (\mathrm{GrassmannAlgebra}_{/\mathbb{R}}^{\mathrm{fin-gen}})^{\mathrm{op}}$$

is the opposite category of finitely generated real Grassmann algebras, hence the function algebras on the supermanifolds of the form $\mathbb{R}^{0|q}$, for $q \in \mathbb{N}$.

Under the ∞-Yoneda embedding $\mathrm{SuperMfd} \hookrightarrow \mathrm{SmoothSuper}\infty\mathrm{Grpd}$ we have the canonical line object $\mathbb{R} = \mathbb{R}^{1|0}$. One observes (this goes back to [S84], [V84]) that an $\mathbb{R}$-module in $\mathrm{Sh}_\infty(\mathrm{SuperPoints})$ is externally equivalently a real super vector space, and that a (commutative) $\mathbb{R}$-algebra in $\mathrm{Sh}_\infty(\mathrm{SuperPoints})$ is externally equivalently a (commutative) super-algebra.

In particular, an L_∞-algebra over the base topos over superpoint is a *super L_∞-algebra*, the homotopy-theoretic generalization of the traditional notion of super Lie algebra. Super L_∞-algebras $\mathfrak{g}$ of finite type have the following simple and useful description: they form the opposite category of the category of super-graded-commutative differential graded superalgebras $\mathrm{CE}(\mathfrak{g})$ whose underlying super-graded-algebra is free on a degree-wise finite dimensional super vector space. Here $\mathrm{CE}(\mathfrak{g})$ is the *Chevalley-Eilenberg super dg-algebra* of $\mathfrak{g}$.

Super-L_∞ algebras in supergravity. It turns out that super-Chevalley-Eilenberg dg-algebras of super L_∞-algebras have secretly been used in the literature on higher dimensional supergravity since at least 1982 [D'AF], where it was observed that the subtle nature of the action functional of 11-dimensional supergravity has a natural interpretation in terms of these structures. A comprehensive textbook of these methods is [CDAF91]. [3]

We had observed this relation between higher supergravity and the homotopy theory of super L_∞-algebras in [SSS08, S]. Now we indicate how in view of higher local prequantum field theory it serves to indeed derive the brane content of higher supergravity, superstring theory and M-theory.

Super WZW-models from super L_∞-cocycles. For brevity, we now concentrate the construction of higher WZW terms in just the "rational" component of the construction in Section 4.4.2, hence its de Rham coefficients $\flat_{\mathrm{dR}}(-)$. By [S] the de Rham coefficients for the Lie integration of a super L_∞-algebra $\mathfrak{g}$ according to [FSSt] are given by the smooth homotopy type of flat L_∞-algebra valued differential forms [SSS08], namely of maps of super L_∞-algebroids

$$\omega : TX \to \mathfrak{g}\,,$$

for TX the tangent Lie algebroid of the given base space smooth manifold X. [4]

A choice of basepoint of X identifies such a map with a smooth map from X to the super ∞-group G which Lie integrates $\mathfrak{g}$. Indeed, the Chevalley-Eilenberg dual of ω is a map

$$\mathrm{CE}(\mathfrak{g}) \to \Omega^\bullet(X)$$

which we may think of as the pullback of the left-invariant forms on G along such a smooth map.

[3] In these and related physics references these super Chevalley-Eilenberg dg-algebras are called "free differential algebras" or "FDA"s for short, and one speaks of the "FDA approach to supergravity". Beware that this is a misnomer, since it is crucially only the underlying graded algebra which is free, while the differential is free only if the algebra is weakly equivalent to the trivial algebra.

[4] Strictly speaking we should write $\mathbf{B}\mathfrak{g}$ here to indicate the one-object super-L_∞-algebroid induced by $\mathfrak{g}$.

Consider now one of the simplest non-trivial super Lie algebras: the super-translation Lie algebra. Super-Minkowski space is the supermanifold $\mathbb{R}^{d;N}$ defined by having global bosonic coordinates $\{x^a\}_{a=1}^d$ and global fermionic coordinates $\{\xi^\alpha\}$ forming a suitable representation of $\mathrm{Spin}(d-1,1)$. The super-translation group structure is such that the left-invariant combination of the corresponding differential forms dx^a and $d\xi^\alpha$ is

$$\Psi^\alpha := d\xi^a$$

and

$$E^a := dx^a + \frac{i}{2}\bar{\xi}\Gamma^a(d\xi)\,.$$

Jointly (E^a, Ψ^α) is also called the canonical *super-vielbein* field on super-Minkowski spacetime. These left-invariant forms constitute the generators of the super-Chevalley-Eilenberg algebra

$$\mathrm{CE}(\mathbb{R}^{d;N}) = (\langle E^a, \Psi, \rangle\,, d_{\mathrm{CE}})$$

and the above equations imply that the non-trivial component of the differential is

$$d_{\mathrm{CE}}E^a = \langle \Psi, [t^a, \Psi]\rangle := \bar{\xi}\Gamma^a\xi\,,$$

While ordinary Minkowski space $\mathbb{R}^d$, regarded as the translation Lie algebra/Lie group has no interesting Lie algebra cohomology, super Minkowski space $\mathbb{R}^{d;N}$ has finitely many non-trivial exceptional Lorentz-invariant cocycles of the form

$$\mu = \langle \Psi, [E^{\wedge p}, \Psi]\rangle := \overline{\Psi} \wedge E^{a_1} \wedge \cdots E^{a_p}\Gamma^{a_1}\cdots\Gamma^{a_p}\Psi\,.$$

For $N = 1$ these cocycles have been classified by what is called the "old brane scan" [AETW87, AT80], summarized in the following table, which has an entry for (d,p) precisely if there is an exceptional cohomology class of degree $(p+2)$ on $\mathbb{R}^{d;N=1}$.

d	$p=0$	1	2	3	4	5	6	7	8	9
11			(1) $\mathfrak{m2brane}$							
10		(1) $\mathfrak{string}_{\mathrm{het}}$				(1) $\mathfrak{ns5brane}_{\mathrm{het}}$				
9					(1)					
8				(1)						
7			(1)							
6		(1) $\mathfrak{littlestring}_{\mathrm{het}}$		(1)						
5			(1)							
4		(1)	(1)							
3		(1)								

By the above discussion, each entry at (d,p) here corresponds to a higher Wess-Zumino-Witten type prequantum field theory describing a p-brane propagating on d-dimensional super-Minkowski spacetime. These σ-models are known as the *Green-Schwarz super p-brane* models [AT80]. Notice that each of these correspond to a known brane in string theory/M-theory, but there are *more* branes in string M-theory than appear in "old brane scan", notably for instance the $p=5$-brane in $d=11$ or all the D-branes in $d=10$. The literature does know a "new brane scan", but that is not induced by super Lie algebra cohomology and does not come with a mechanism for how to find their worldvolume action functionals. On the other hand show now that these missing branes appear once we consider genuinely higher super-WZW models (which higher stacky super target spaces, higher "super-orbispace" targets).

The super-2-brane and its boundary condition. In particular there is the 4-cocycle $\langle \Psi, [E^2, \Psi]\rangle$ which describes a 3-dimensional WZW model whose WZW term has curvature

$$\langle \Psi, [E^{\wedge 2}, \Psi]\rangle :\ \mathbb{R}^{10;N=1} \longrightarrow \mathbf{B}^3\mathbb{R}$$

This super-WZW model is well known in the physics literature: it is the Green-Schwarz formulation of the M2-brane.

So for Σ_{2+1} the worldvolume of an M2-brane, the "rational component" of a field configuration in this model is a map

$$\phi : \Sigma_{2+1} \longrightarrow \mathbb{R}^{11;N=1}$$

of smooth manifolds, which we now regard as the induced map of super L_∞-algebroids from $T\Sigma_{2+1}$.

Consider now the case that the worldvolume Σ_{2+1} has a non-empty boundary $\partial\Sigma_{2+1}$ and write

$$\partial\Sigma_{2+1} \hookrightarrow \Sigma_{2+1}$$

for the corresponding inclusion. This would be called the worldvolume of an open M2-brane. We now study the boundary conditions added to the data of a submanifold $\Sigma_{p_2+1} \to \mathbb{R}^{11;N=1}$ such that this may serve as a boundary for the open M2-brane higher WZW model, hence such that there are consistent bulk/boundary-field configurations of the form

$$\begin{tikzcd} \partial(\Sigma_{2+1}) \arrow[r, "\phi|_{\partial\Sigma}"] \arrow[d] & \Sigma_{5+1} \arrow[d] \\ \Sigma_{2+1} \arrow[r, "\phi"] & \mathbb{R}^{11;N=1} \end{tikzcd} .$$

By the general rules of boundary prequantum field theory in 3.3.3 and specifically the discussion of topological boundary condition for Chern-Simons-type actions functionals in 4.3.4, it follows that such a boundary condition for the M2-brane is a trivialization/gauging-away of its gauge coupling term on the boundary, hence (at the rational/de Rham coefficient level on which we are concentrating here for brevity) a homotopy ϕ_{bdr} of maps of super-L_∞-algebras as on the right of the following diagram:

$$\begin{tikzcd} \partial(\Sigma_{2+1}) \arrow[r, "\phi|_{\partial\Sigma}"] \arrow[d] & \Sigma_{5+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\swarrow \phi_{\mathrm{bdr}}"] & \ast \arrow[d] & \boxed{\begin{array}{c}\text{topological}\\ \text{boundary condition}\end{array}} \\ \Sigma_{2+1} \arrow[r, "\phi"] & \mathbb{R}^{11;N=1} \arrow[r, "\langle \Psi \wedge [E^2 \wedge \Psi]\rangle"] & \mathbf{B}^3\mathbb{R} & \\ & \boxed{\begin{array}{c}\text{open brane}\\ \sigma\text{-model field}\end{array}} & \boxed{\begin{array}{c}\text{background}\\ \text{field}\end{array}} & \end{tikzcd} .$$

Now as discussed generally in 4.3.4, by the universal property of the homotopy pullback of super L_∞-algebras, this means, that the map $\Sigma_{p_2+1} \to \mathbb{R}^{11|N=1}$ equivalently factors through the homotopy fiber super L_∞-algebra

$$\mathfrak{m2brane} := \mathrm{hfib}(\langle \Psi \wedge E^2 \wedge \Psi \rangle) ,$$

which is the super Lie 3-algebra extension of $\mathbb{R}^{11;N=1}$ which is classified by the 4-cocycle $\langle \Psi, E^2\Psi \rangle$

$$\mathbf{B}^2\mathbb{R} \longrightarrow \mathfrak{m2brane} \longrightarrow \mathbb{R}^{11;N=1} .$$

Using the recognition principle of L_∞-extensions in [FRS13b] it follows that $\mathfrak{m2brane}$ is characterized as being that super L_∞-algebra whose Chevalley-Eilenberg super-dg-algebra is obtained from $\mathrm{CE}(\mathbb{R}^{11;N=1})$ by adding one new generator H in cohomological degree 3 and even super-degree and extending the differential to that by setting

$$d_{\mathrm{CE}(\mathfrak{m2brane})} H = \langle \Psi, [E^2, \Psi] \rangle .$$

Comparison shows that this is precisely the super-dg-algebra which was originally considered in (3.15) of [D'AF]. [5]

[5] There this super-dg-algebra is not understood to be related to M2-branes, but fields with values in this Lie 3-algebra are found to naturally lead to the action functional of 11-dimensional supergravity. The relation between these two perspective can be understood, after we include the M5-brane below, by Ads(7)/CFT(6) duality. This we discuss in [SuperOrbi].

So we find that for a subspace $\Sigma_{p_2+1} \to \mathbb{R}^{11;N=1}$ to be a consistent topological boundary condition for the M2-brane, it must factor through the dashed morphism in

$$\begin{tikzcd}
\partial\Sigma \arrow[r, "\phi_\partial"] \arrow[d] & \Sigma_{p_2+1} \arrow[r, dashed] \arrow[d] & \mathfrak{m2brane} \arrow[r] \arrow[d] & * \arrow[d] \arrow[dll, phantom, "\Swarrow"] \\
\Sigma_{2+1} \arrow[r, "\phi"] & \mathbb{R}^{11;N=1} \arrow[r, equal] & \mathbb{R}^{11;N=1} \arrow[r, "\langle\Psi\wedge[E^2,\Psi]\rangle"'] & \mathbf{B}^2\mathbb{R}\,.
\end{tikzcd},$$

hence constitute a boundary condition diagram as in the general discussion above of the form

$$\begin{tikzcd}
& \Sigma_{p_2+1} \arrow[d, dashed] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
& \mathfrak{m2brane} \arrow[dl] \arrow[dr] & \\
* \arrow[dr, "0"'] & & \mathbb{R}^{11;N=1} \arrow[dl, "\langle\Psi,[E^2,\Psi]\rangle"] \\
& \mathbf{B}^3\mathbb{R} &
\end{tikzcd}.$$

The 5-brane and its tensor multiplet fields content. This boundary condition analysis of the M2-brane σ-model has the following interesting implication: it means that if we regard Σ_{p_2+1} itself as the worldvolume of a brane then (rationally) the correct target space for this p_2-brane is not spacetime $\mathbb{R}^{11;N=1}$ itself, but is the *higher extension* of spacetiem given by the super Lie 3-algebra $\mathfrak{m2brane}$. Heuristically, this is spacetime filled with a "condensate" of open 2-branes which couples to the p_2-brane. We may indeed derive what this coupling is like:

if also the p_2-brane worldvolume theory is supposed to be a higher super-WZW model as in 4.4.2, then it must be induced by a (p_2+2)-cocycle, but now with codomain the extension of super-spacetime:

$$\mu : \mathfrak{m2brane} \to \mathbf{B}^{p_2+1}\mathbb{R}\,.$$

This is equivalently a d_{CE}-closed element in the Chevalley-Eilenberg super-dg-algebra $\mathrm{CE}(\mathfrak{m2brane})$.

That there is indeed a non-trivial such higher super L_∞-cocycle for $p_2 = 5$ was first observed on p. 18 of [D'AF]. It is of the form

$$\mu_7 = \langle\Psi, [E^5, \Psi]\rangle + \langle\Psi, [E^2, \Psi]\rangle \wedge H\,.$$

We can read off what the corresponding higher super-WZWZ model is like

$$\begin{tikzcd}
\Sigma_{5+1} \arrow[r] & \mathfrak{m2brane} \arrow[r, "\mu_7"] \arrow[d] & \mathbf{B}^6\mathbb{R} \\
& \mathbb{R}^{11;N=1} &
\end{tikzcd}.$$

It has field content

1. super σ-model fields whose "curvature" is the super-vielbein (E, Psi);
2. a higher twisted 2-form gauge field whose 3-form curvature H satisfies the twisted Bianchi identity $dH = \langle\Psi, [E^2, \Psi]\rangle$.

and the WZW term is given by the curvature 7-form μ_7. Notice that we concentrate on the differential form data here only for the ease of exposition. Plugging this data into the full def. 4.4.3 yields a fully globally defined and localized action functional for this model whose moduli stack is the super 3-stack $\Omega_{\tau_4} \exp(\mathfrak{m2brane})$ and whose action functional is the 3-volume holonomy of a super circle 3-bundle connection (super bundle 2-gerbe).

The rational/differential form data above is the kind of data that existing string theory literature displays . Indeed, direct comparison shows that the super 5-brane WZW-model which we have derived from higher prequantum boundary field theory is given by precisely the differential form data that was proposed for the M5-brane in [2] (see around equation (8) there).

Looking back, we see in summary that the original systematic "old brane scan" could see the M5-brane because it only induced WZW models from cocycles on ordinary super Lie algebras. Here we find that the M5-brane is indeed still of the same general type, *if* we pass to super L_∞-homotopy theory and consider higher local WZW models induced from higher cocycles on higher spaces.

The full brane bouquet. By following the previous discussion, there is a super Lie 6-algebra which extends the M2-brane super Lie 3-algebra as classied by the above super L_∞-7-cocycle, and which deserves to be called $\mathfrak{m5brane}$. Hence in total we have a sequence of exceptional super L_∞-extensions of the form

$$\mathfrak{m5brane} \longrightarrow \mathfrak{m2brane} \longrightarrow \mathbb{R}^{11;N=1} \ ,$$

a kind of exceptional super L_∞-Whitehead-tower, analogous to the smooth Whitehead tower

$$\cdots \longrightarrow \mathbf{B}\mathrm{Fivebrane} \longrightarrow \mathbf{B}\mathrm{String} \longrightarrow \mathbf{B}\mathrm{Spin}$$

which controls the heterotic string anomaly cancellation [SSS12] (we come back to this below in 4.5).

Conversely, recall from the above discussion that given any such a sequence of extensions, we may read off the existence of super-p-branes and their intersection laws (their "brane-ending-on-brane-laws"):

- a super L_∞-extension of the form
$$p\mathfrak{brane} \longrightarrow \mathbb{R}^{d;N}$$
induces a higher local prequantum WZW-model that describes p-branes propagating in super Minkowski spacetime $\mathbb{R}^{d;N}$ which may possibly be open and end on othe branes;
- a super L_∞-extension of the form
$$p_2\mathfrak{brane} \longrightarrow p\mathfrak{brane}$$
corresponds to a boundary condition that signifies that the given open p-brane may end of the given p_2-brane.

Accordingly one is led to studying the classification of such exceptional towers of iterative super L_∞-extensions based on the super translation Lie algebra.

Doing so, we find the following diagram of super L_∞-algebra extensions.

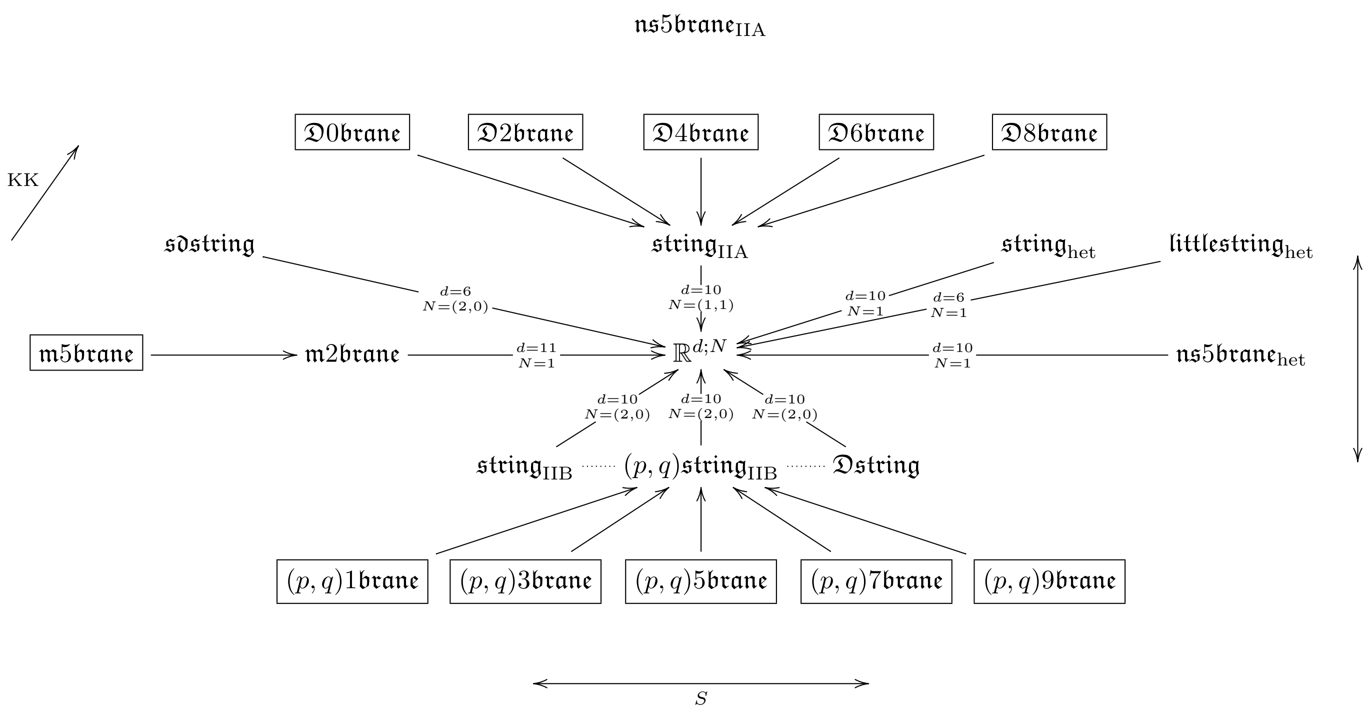
Definition 4.4.11 (the refined brane scan)**.** The refined *brane scan* is the finite subset

$$\{(D, p, N)\} \subset \{3, \cdots 11\} \times \{0, \cdots 9\} \times \{1, (1,1), (2,0)\} \, ,$$

which has as elements the entries in the following table, with the value of N in parenthesis and with the names which we give to the element next to that in fraktur font.

D =	$p=0$	1	2	3	4	5	6	7	8
11			(1) $\mathfrak{m2brane}$			(1) $\mathfrak{m5brane}$			
10	(1,1) $\mathfrak{D0brane}$	(1) $\mathfrak{string}_{\mathrm{het}}$ (1,1) $\mathfrak{string}_{\mathrm{IIA}}$ (2,0) $\mathfrak{string}_{\mathrm{IIB}}$ (2,0) $\mathfrak{D1brane}$	(1,1) $\mathfrak{D2brane}$	(2,0) $\mathfrak{D3brane}$	(1,1) $\mathfrak{D4brane}$	(1) $\mathfrak{ns5brane}_{\mathrm{het}}$ (1,1) $\mathfrak{ns5brane}_{\mathrm{IIA}}$ (2,0) $\mathfrak{ns5brane}_{\mathrm{IIB}}$ (2,0) $\mathfrak{D5brane}$	(1,1) $\mathfrak{D6brane}$	(2,0) $\mathfrak{D7brane}$	(1,1) $\mathfrak{D8b}$
9					(1)				
8				(1)					
7			(1)						
6		(1) $\mathfrak{littlestring}_{\mathrm{het}}$ (2,0) $\mathfrak{sdstring}$		(1)					
5			(1)						
4		(1)	(1)						
3		(1)							

Proposition 4.4.12 (the brane bouquet)**.** *There exists a system of higher super-Lie-n-algebra extensions of the super-translation Lie algebra $\mathbb{R}^{d;N}$ for $(d = 11, N = 1)$, $(d = 10, N = (1,1))$, for $(d = 10, N = (2,0))$ and for $(d = 6, N = (2,0))$, which is jointly given by the following diagram in* sLie$_\infty$Alg



where

- *An object in this diagram is precisely a super-Lie-$(p+1)$-algebra extension of $\mathbb{R}^{d;N}$, with (d, p, N) as given by the entries of the same name in the refined brane scan, def. 4.4.11;*
- *every morphism is a super-Lie $(p+1)$-algebra extension by an exceptional $\mathbb{R}$-valued $\mathfrak{o}(d)$-invariant super-L_∞-cocycle of degree $p+2$ on the domain of the morphism;*
- *the unboxed morphisms are hence super Lie $(p+1)$-algebra extensions of $\mathbb{R}^{d;N}$ by a super Lie algebra*

$(p+2)$-cocycle, hence are homotopy fibers in $\mathrm{sLie}_\infty\mathrm{Alg}$ *of the form*

$$\begin{tikzcd} p_1\mathfrak{brane} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ \mathbb{R}^{d;N} \arrow[r, "\text{some cocycle}"] & \mathbf{B}^{p+1}\mathbb{R} \end{tikzcd},$$

- *hence the boxed super-L_∞-algebras are super Lie $(p+1)$-algebra extensions of genuine super-L_∞-algebras (which are not plain super Lie algebras), again by $\mathbb{R}$-cocycles*

$$\begin{tikzcd} p_2\mathfrak{brane} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & * \arrow[d] \\ p_1\mathfrak{brane} \arrow[r, "\text{some cocycle}"] & \mathbf{B}^{p_2+1}\mathbb{R} \end{tikzcd}.$$

This is discussed in detail in [SuperOrbi].

Remark 4.4.13. The diagram in prop. 4.4.12 is reminiscent of a famous cartoon of M-theory (figure 4 in [W98]). As opposed to that cartoon, the above diagram is a theorem with a precise interpretation.

4.5 Outlook: Cohomological holographic quantization of higher local prequantum field theory

Here should go a brief outlook on the cohomological quantization of the prequantum field theories as discussed above, along the lines discussed in [N].

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