Nonabelian homotopical cohomology,
higher fiber bundles with connection,
and their σ-model QFTs

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Abstract

Nonabelian cohomology can be regarded as a generalization of group cohomology to the case where both the group itself as well as the coefficient object are allowed to be generalized to ∞-group*oid*s or even to general ∞-categories and even to parameterized ∞-categories: ∞-stacks. Cocycles in nonabelian cohomology in particular represent higher principal bundles (gerbes) – possibly equivariant, possibly with connection – as well as the corresponding *associated* higher vector bundles.

We formulate nonabelian cohomology and its classification of fiber bundles in a general context $\mathcal{C}$ of enriched homotopy theory independent of concrete choices for models of ∞-categories but universally giving rise to $\mathcal{C}$-internal weak ω-categories of Trimble. We discuss general issues such as lifting and extension problems in this context. We list examples and applications with enrichment over higher categories which describe higher principal bundles and higher vector bundles, possibly equivariant. If equivariant with respect to a fundamental ∞-groupoid these are higher bundles with connection incorporating and generalizing the constructions of [43].

Building on this we propose, expanding on considerations in [15, 48, 6], a systematic ∞-functorial formalization of the σ-model quantum field theory associated with a given nonabelian cocycle regarded as the background field for a brane coupled to it. We define propagation in these σ-model QFTs and recover central aspects of *groupoidification* [1, 2].

In a series of examples we show how this formalization reproduces familiar structures in σ-models with finite target spaces such as Dijkgraaf-Witten theory and the Yetter model. Applications to σ-models with smooth target spaces is developed elsewhere [31].

Contents

1 Introduction

A *σ-model* should, quite generally, be an n-dimensional quantum field theory which is canonically associated with the geometric structure given by a connection on a bundle whose fibers are n-categories – for instance a (higher) gerbe with connection.
For example for $n = 1$ a line bundle with connection over a Riemann manifold gives rise to the ordinary quantum mechanics of a charged particle. For $n = 2$ a line bundle gerbe with connection over a Lie group G gives rise to the 2-dimensional quantum field theory known as the WZW-model. For $n = 3$ a Chern-Simons 2-gerbe with connection over BG gives rise to Chern-Simons QFT.

The natural conceptual home of these higher connections appearing here is *differential nonabelian cohomology* [31], a joint generalization of sheaf cohomology, group cohomology and nonabelian group cohomology. One arrives at this general notion of cohomology for instance by first generalizing the coefficients of sheaf cohomology from complexes of abelian groups, via crossed complexes of groupoids and their equivalent ∞-groupoids [8], to general ∞-categories, and secondly by generalizing the domain spaces via orbifolds, hypercovers and their equivalent ∞-groupoids also to general ∞-categories. Therefore nonabelian cocycles are cocycles *on* ∞-categories with coefficients *in* ∞-categories. Moreover, when suitably interpreted such a cocycle is nothing but an ∞-functor from its domain to its coefficient object, hence a rather fundamental concept. In one way or other it is well known that such cocycles in particular classify fiber bundles – possibly equivariant – whose fibers are higher categories.

In a series of articles [3, 41, 42, 43] (see also [27]) it was shown for low n that by *internalizing* this notion of ∞-functorial nonabelian cocycles from the category of plain sets into a category of generalized *smooth spaces*, it yields a good notion of generalized *differential cohomology*: if the domain ∞-category is taken to be the smooth fundamental ∞-groupoid of a smooth space, then smooth ∞-functors out of it provide a higher dimensional notion of *parallel transport* and characterize higher connections on higher fiber bundles.

Indeed, regarding an ∞-functorial cocycle as a parallel transport functor generally provides a useful heuristic for the sense in which generalized cocycles are nothing but ∞-functors from their domain to their coefficient object, even if there is no smooth structure and no connection around: the ∞-functorial cocycles characterizing for instance a fiber bundle is the *fiber-assigning functor* which to each point in base space assigns the fiber sitting over that point, to each morphism in base space (be it a jump along an orbifold action, or a jump between points in the fiber of a Čech cover, or indeed a smooth path in base space) the corresponding morphisms between the fibers over its endpoints, and similarly for higher morphisms.

From this perspective much can already be learned from and achieved in finite approximations to full smooth differential cocycles. A central example is Dijkgraaf-Witten theory as a finite version of Chern-Simons theory: while Chern-Simons theory is a σ-model governed by a differential 3-cocycle on BG – in the smooth context– usually addressed as the *Chern-Simons 2-gerbe* –, Dijkgraaf-Witten theory is diagrammatically the same setup, but now internal to ∞-categories internal to Sets: the space BG is replaced by the finite groupoid $\mathbf{B}G$ with one object and Hom-set the finite group G. So among other things, ∞-functorial nonabelian cohomology, which treats group cocycles and higher bundles/higher gerbes intrinsically on the same footing, gives a precise formalization of the way in which finite group models such as Dijkgraaf-Witten theory are related to their smooth cousins such as Chern-Simons theory.

For that reason it is worthwhile to study the ∞-functorial nonabelian cohomology perspective on finite group σ-models before adding the further technical complication of working internal to smooth spaces. While discussion of differential nonabelian cohomology in the context of smooth spaces is in preparation in [31], here we develop some concepts and their applications in the simpler context of plain sets.

In sections 5.1 and 7 we set up the central concepts which we use to formalize the notion of a σ-model associated with a (differential) nonabelian cocycle. In particular we formalize in this context the notion of higher sections and higher spaces of states as indicated in [15, 48], and generalize to corresponding notions of branes and bibranes [17]. In section 7 we then go through a list of examples and applications illustrating these concepts.

2 Basic ideas

An *extended quantum field theory* is supposed to be a representation of some notion of extended cobordisms with values in some notion of higher vector spaces. For formalizing this idea, it is noteworthy that cobordisms naturally form co-spans [18] – where a cobordisms Σ with incoming boundary Σ_{in} and outgoing boundary Σ_{out} corresponds to the co-span of boundary inclusions

$$\begin{tikzcd} & \Sigma & \\ \Sigma_{\mathrm{in}} \arrow[ur, hook] & & \Sigma_{\mathrm{out}} \arrow[ul, hook'] \end{tikzcd}$$

– such that gluing of cobordisms along common boundaries is naturally the composition of their cospans [19] (by pushout) – while generalized linear maps are naturally thought of as spans [2]

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

which act on generalized vectors given by one sided-spans

$$\begin{tikzcd} & \Psi \arrow[dl] \arrow[dr] & \\ \mathrm{pt} & & X \end{tikzcd}$$

by composition of spans (by pullback). A natural way to map cospans to spans such that their composition is respected is obtained by mapping the co-spans objectwise into a fixed *target space* object P_X:

$$\left[\begin{tikzcd} & \Sigma & \\ \Sigma_{\mathrm{in}} \arrow[ur, hook, "\mathrm{in}"] & & \Sigma_{\mathrm{out}} \arrow[ul, hook', "\mathrm{out}"'] \end{tikzcd} , P_X \right] = \begin{tikzcd} & {[\Sigma, P_X]} \arrow[dl, "\mathrm{in}^*"'] \arrow[dr, "\mathrm{out}^*"] & \\ {[\Sigma_{\mathrm{in}}, P_X]} & & {[\Sigma_{\mathrm{out}}, P_X]} \end{tikzcd} .$$

This again is noteworthy as we expect large classes of quantum field theories to be *σ-models* in that their assignment to a given piece of parameter space Σ (often called "worldvolume") is precisely determined by maps – these are the very "fields" in QFT – from Σ into target space P_X. Moreover, the construction seamlessly generalizes to *extended cobordisms* – cobordisms equipped with a hierarchical system of boundary piece inclusions, making explicit boundary structure in all codimensions – if we realize that these in turn are naturally encoded by *multi-cospans*: for instance the cylinder $\Sigma \otimes I$ over the cobordism Σ from above is naturally an extended cobordism with boundary components distinguished by whether they come from Σ or from I, which is naturally captured by regarding the interval itself as a cospan

$$\begin{tikzcd} & I & \\ \mathrm{pt} \arrow[ur, hook] & & \mathrm{pt} \arrow[ul, hook'] \end{tikzcd}$$

and then tensoring term-wise to obtain the multi-cospan

$$\left(\begin{tikzcd} & \Sigma & \\ \Sigma_{\mathrm{in}} \arrow[ur, hook] & & \Sigma_{\mathrm{out}} \arrow[ul, hook'] \end{tikzcd} \right) \otimes \left(\begin{tikzcd} & I & \\ \mathrm{pt} \arrow[ur, hook] & & \mathrm{pt} \arrow[ul, hook'] \end{tikzcd} \right) = \begin{tikzcd} & & \Sigma_{\mathrm{out}} \arrow[dl] \arrow[dr] & & \\ & \Sigma \arrow[dr] & & \Sigma_{\mathrm{out}} \otimes I \arrow[dl] & \\ \Sigma_{\mathrm{in}} \arrow[ur] \arrow[dr] & & \Sigma \otimes I & & \Sigma_{\mathrm{out}} \arrow[ul] \arrow[dl] \\ & \Sigma_{\mathrm{in}} \otimes I \arrow[ur] & & \Sigma \arrow[ul] & \\ & & \Sigma_{\mathrm{in}} \arrow[ul] \arrow[ur] & & \end{tikzcd}$$

which encodes for $\Sigma \otimes I$ four boundary components (codimension 1) and four "corners" (codimension 2) connecting them.

Finally, it is well known that more generally σ-models – and in particular certain equivalences between σ-models – are not just encoded by target space object P_X, but more generally, again, by spans

$$\begin{tikzcd} & P_Q \arrow[dl] \arrow[dr] & \\ P_X & & P_Y \end{tikzcd}$$

which for $P_X = \mathrm{pt}$ are known as *branes* Q in X, whereas generally they can be addressed as *bi-branes.* In this context one finds [29, 17] what can be interpreted as that pull-push through spans

$$\left[\Sigma,\ \begin{tikzcd} & P_Q \arrow[dl, "\mathrm{in}^*"'] \arrow[dr, "\mathrm{out}^*"] & \\ P_X & & P_Y \end{tikzcd}\right] = \begin{tikzcd} & {[\Sigma, P_X]} \arrow[dl] \arrow[dr] & \\ {[\Sigma, P_X]} & & {[\Sigma, P_Y]} \end{tikzcd}$$

describes the operation of duality transformations induced by a *defect* on the fields on Σ.

To amplify the picture notice that this suggests that in general σ-model QFT has to do with mapping multi-cospans – interpreted as parameter space – into multi-spans – interpreted as target space – to produce multi-spans – interpreted as generalized linear maps. For instance in the diagram

$$\left[\begin{tikzcd} & \Sigma & \\ \Sigma_{\mathrm{in}} \arrow[ur] & & \Sigma_{\mathrm{out}} \arrow[ul] \end{tikzcd},\ \begin{tikzcd} & Q \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}\right] = \begin{tikzcd} & & {[\Sigma_{\mathrm{out}}, X]} & & \\ & {[\Sigma, X]} \arrow[ur] \arrow[dl] & & {[\Sigma_{\mathrm{out}}, Q]} \arrow[ul] \arrow[dr] & \\ {[\Sigma_{\mathrm{in}}, X]} & & {[\Sigma, Q]} \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & & {[\Sigma_{\mathrm{out}}, Y]} \\ & {[\Sigma_{\mathrm{in}}, Q]} \arrow[ul] \arrow[dr] & & {[\Sigma, Y]} \arrow[ur] \arrow[dl] & \\ & & {[\Sigma_{\mathrm{in}}, Y]} & & \end{tikzcd}$$

we might read the various sub-spans on the right as encoding (by span composition) the operation

- either first propagate a field configuration on Σ_{in} in the σ-model with target space P_X along the "worldvolume" Σ to a field configuration on Σ_{out}, and then apply a "duality" transformation by transforming to a field configuration with respect to the σ-model given instead by target space P_Y;
- or proceed the other way around, by first transforming a field configuration on Σ_{in} from one on P_X to one in P_Y and then propagate along Σ in the latter model.

We want to investigate this idea of extended σ-models in a context that makes a minimum amount of assumptions on any detailed technical implementation in concrete examples. As described in more detail in section 5 it seems reasonable to assume that our multi-cospans are internal to a category S which is enriched over a homotopical category $\mathcal{V}$ in which the multi-spans are internal that arise as coefficients (values) of QFTs. It is then useful to take target space more generally to be given by mult-spans in a category of $\mathcal{V}$-valued presheaves on S $\mathcal{C} = [S^{\mathrm{op}}, \mathcal{V}]$, equipped itself with a suitable compatibles homotopical structure. It is known in various special cases that the homotopy categories $\mathrm{Ho}_{\mathcal{C}}$ for such setups encode categories of objects that deserve to be addressed as ∞-stacks. One crucial point of this for the study of σ-model QFTs is that morphisms in $\mathcal{C}$ are *generalized nonabelian cocycles* which classify higher generalizations of fiber bundles $P_X \to X$ over target space X – possibly equivariant, possibly with connection – and that these fiber bundles are the *background fields* such that the corresponding σ-model QFT $[-, P_X]$ is supposed to describe the quantum dynamics of the fundamental branes propagating on X and coupled to P_X.

Formally this means that a background field on X is characterized by a morphism

$$\nabla :\ X \longrightarrow F\ ,$$

where F is an object whose points are the possible *fibers* of P_X over X, and where ∇ is naturally addressed as the *parallel transport* given by the background field, whose *transgression* to mapping spaces

$$[\Sigma, \nabla] : [\Sigma, X] \to [\Sigma, F]$$

computes the generalized *phase shift* which the background field effects on the *trajectories* in $[\Sigma, X]$. For every F there is a universal bundle $\mathbf{E}_{\mathrm{pt}}F \to F$ and total spaces $P_X \to X$ appearing in the above discussion arise as the pullback of this universal bundle along the cocycle ∇

$$\begin{tikzcd}
P_X \simeq \nabla^* \mathbf{E}_{\mathrm{pt}} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E}_{\mathrm{pt}} F \arrow[d] \; . \\
X \arrow[r, "\nabla"] & F
\end{tikzcd}$$

In terms of these cocycles our bi-branes are therefore spans of morphisms

$$\begin{tikzcd}
 & Q \arrow[dl] \arrow[dr] \arrow[dd, "\nabla_Q" description] & \\
X \arrow[dd, "\nabla_X" description] & & Y \arrow[dd, "\nabla_X" description] \\
 & {[\mathcal{I}, F]} \arrow[dl] \arrow[dr] & \\
F & & F
\end{tikzcd}$$

and some aspects of the theory, such as the interpretation of the internal hom $[\Sigma, \nabla]$ as transgression to mapping space, are more naturally understood at the level of cocycles than at the levl of the total spaces P_X of the bundles classified by them.

The elementary but crucial fact for translating between the two pictures is that transgressing a cocycle $\nabla : X \to F$ classifying a bundle $P_X \to X$ to the space of fields $[\Sigma, F]$ and then computing the bundle $[\Sigma, \nabla]^* \mathbf{E}_{\mathrm{pt}}[\Sigma, F]$ it classifies there yields the same result as passing to the mapping space $[\Sigma, P_X]$ as in the above discussion. This is because the covariant Hom $[\Sigma, -]$ is a *continuous* functor and hence preserves pullbacks

$$\begin{tikzcd}
{[\Sigma, \nabla^* \mathbf{E}_{\mathrm{pt}} F] \simeq [\Sigma, \nabla]^* [\Sigma, \mathbf{E}_{\mathrm{pt}} F]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\Sigma, \mathbf{E}_{\mathrm{pt}} F] \simeq \mathbf{E}_{\mathrm{pt}} [\Sigma, F]} \arrow[d] \; . \\
{[\Sigma, X]} \arrow[r, "{[\Sigma, \nabla]}"] & {[\Sigma, F]}
\end{tikzcd}$$

Remarkably, everything about σ-model QFTs thus comes down to the fact that the Hom-functor is continuous in both arguments.

... now something about the examples...

$$\left[\begin{tikzcd}
 & & \mathbf{B}\mathbb{Z} \sqcup \mathbf{B}\mathbb{Z} & & \\
 & \mathbf{B}\mathbb{Z} \arrow[ur] & & \mathbf{B}\mathbb{Z} \arrow[ul] & \\
\mathrm{pt} \arrow[ur] \arrow[drr] & & \mathrm{pt} \arrow[ul] \arrow[ur] & & \mathrm{pt} \arrow[ul] \arrow[dll] \\
 & & \mathbf{B}\mathbb{Z} \arrow[uuu, bend left] & &
\end{tikzcd} , \mathbf{B}G \right] = \begin{tikzcd}
 & & \Lambda \mathbf{B}G \times_{\mathbf{B}G} \Lambda \mathbf{B}G \arrow[dl] \arrow[dr] \arrow[ddd, bend right] & & \\
 & \Lambda \mathbf{B}G \arrow[dl] \arrow[dr] & & \Lambda \mathbf{B}G \arrow[dl] \arrow[dr] & \\
\mathbf{B}G & & \mathbf{B}G & & \mathbf{B}G \\
 & & \Lambda \mathbf{B}G \arrow[ull] \arrow[urr] & &
\end{tikzcd}$$

notice the difference between

$$\Omega \mathbf{B}G = \ker(\; [\mathcal{I}, \mathbf{B}G] \xrightarrow{d_0 \times d_1} \mathbf{B}G \times \mathbf{B}G \;) = G$$

and

$$\Lambda \mathbf{B}G = [\mathbf{B}\mathbb{Z}, \mathbf{B}G] = G//_{\mathrm{Ad}} G$$

3 Multi-(co)spans

We define a multi-cospan as the image of a poset in a category $\mathcal{C}$ with finite colimits (and dually for multi-spans). We think of such a diagram in $\mathcal{C}$ as a hierarchical cell complex where morphisms describe inclusion of boundary pieces.

Definition 3.1 (finite posets) *A finite poset is a finite category D with $D(a,b)$ either empty or the singleton set, for all $a, b \in \mathrm{Obj}(D)$, i.e. a finite category enriched over $(\{\emptyset, \mathrm{pt}\}, \times)$. Write* Posets *for the full sub-category of* $\mathsf{Categories}$ *on finite posets.*

Simple posets of relevance for the following are

- the terminal poset $\{\top\}$;
- the interval $\mathcal{I} = \{a \to \top\}$;
- the abstract cospan $$\wedge := \left\{ \begin{tikzcd} & \top & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \end{tikzcd} \right\}$$
- the cartesian product of the abstract cospan with itself $$\wedge^2 = \left\{ \begin{tikzcd} a_1 \arrow[r] \arrow[d] & b_1 \arrow[d] & a_2 \arrow[l] \arrow[d] \\ b_2 \arrow[r] & \top & b_3 \arrow[l] \\ a_3 \arrow[r] \arrow[u] & b_4 \arrow[u] & a_4 \arrow[l] \arrow[u] \end{tikzcd} \right\};$$
- the co-cone over two glued copies of the abstract cospan $$\left\{ \begin{tikzcd} & & \top & & \\ & b_1 \arrow[ur] & & b_2 \arrow[ul] & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \arrow[ur] & & a_3 \arrow[ul] \end{tikzcd} \right\}$$
- the over-categories $\Gamma \downarrow n$ of $$\Gamma = \left\{ [0] \overset{\sigma}{\underset{\tau}{\rightrightarrows}} [1] \overset{\sigma}{\underset{\tau}{\rightrightarrows}} [2] \overset{\sigma}{\underset{\tau}{\rightrightarrows}} \cdots \,|\sigma \circ \sigma = \tau \circ \sigma; \sigma \circ \tau = \tau \circ \tau \right\}$$

  for instance $$\Gamma \downarrow 3 = \left\{ \begin{tikzcd} & \top & \\ c_1 \arrow[ur, "\sigma"] & & c_2 \arrow[ul, "\tau"'] \\ b_1 \arrow[u] \arrow[urr] & & b_2 \arrow[u] \arrow[ull] \\ a_1 \arrow[u] \arrow[urr] & & a_2 \arrow[u] \arrow[ull] \end{tikzcd} \right\}$$

Definition 3.2 (terminal and coterminal object) *A terminal object $\top$ in a poset D is an object such that $\forall a \in \mathrm{Obj}(D) : (a, \top) \simeq \mathrm{pt}$. A co-terminal object b is one for which $\forall a \neq b \in \mathrm{Obj}(D) : D(a,b) \simeq \emptyset$.*

In the above examples the coterminal objects are the a_i.

Definition 3.3 *Write* $\mathsf{Posets}_{\max}$ *the category whose objects are the posets that have a terminal object and whose morphisms are morphisms of posets that preserve the terminal object.*

There is an obvious forgetful functor $U : \mathsf{Posets}_{\max} \to \mathsf{Posets}$ with a left adjoint $\bar{(\;)} : \mathsf{Posets} \to \mathsf{Posets}_{\max}$ which freely adjoins a terminal object to a given poset.

For instance
$$\overline{\left\{ \begin{tikzcd} & b_1 & & b_2 & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \arrow[ur] & & a_3 \arrow[ul] \end{tikzcd} \right\}} = \left\{ \begin{tikzcd} & & \top & & \\ & b_1 \arrow[ur] & & b_2 \arrow[ul] & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \arrow[ur] & & a_3 \arrow[ul] \end{tikzcd} \right\}$$

Definition 3.4 (multi-cospan) *A multi-cospan in a category $\mathcal{C}$ with finite colimits is a finite poset with terminal object, $D \in \mathsf{Posets}_{\max}$, and a functor $K : D \to \mathcal{C}$. The category of multi-cospans in $\mathcal{C}$ is*

$$\mathsf{MultiCospans}(\mathcal{C}) := (\mathsf{Posets}_{\max} \downarrow_{\mathsf{Categories}} \mathcal{C})^{\mathrm{op}}$$

whose morphisms $(f, \phi) : K \to K'$ are triangles

$$\begin{tikzcd} D \arrow[dr, "K"'] & \overset{\phi}{\Longrightarrow} & D' \arrow[ll, "f"'] \arrow[dl, "K'"] \\ & \mathcal{C} & \end{tikzcd}$$

of natural transformations ϕ, composition is the obvious pasting of these triangles in $\mathsf{Categories}$

Remark. For K a multi-cospan we think of the object $K(\top) \in \mathcal{C}$, for $\top$ the terminal object of D, as a single top-dimensional cell in a hierarchical complex, in that we think of any morphism $K(a \to \top)$ in $\mathcal{C}$ for every $a \in D$ as embedding a boundary piece $K(a)$ labeled by a into the total space (but there is no requirement that $K(a \to \top)$ be a monomorphism) and think of each morphism $K(b \to a)$ for $b \to a$ in D as describing a boundary piece $K(b)$ of a boundary piece $K(a)$.

3.1 Composition

The idea is that multi-cospans are composed by first gluing them along a common sub-multi-cospan, then forming the colimit cocone over that, and finally picking a sub-multi-cospan in that, containing the tip of the colimit cocone.

Definition 3.5 (multi-cospan closure) *For $D \in \mathsf{Posets}$ and $K : D \to \mathcal{C}$ a functor the multi-cospan closure of K (or rather one of the canonically isomorphic choices) is the unique multi-cospan*

$$\bar{K} : \bar{D} \to \mathcal{C}$$

such that

$$\begin{tikzcd} D \arrow[r, hook] \arrow[rr, bend right, "K"'] & \bar{D} \arrow[r, "\bar{K}"] & \mathcal{C} \end{tikzcd}$$

and

$$\begin{tikzcd} \{\top\} \arrow[r, hook] \arrow[rr, bend right, "\top \mapsto \mathrm{colim}_D K"'] & \bar{D} \arrow[r, "\bar{K}"] & \mathcal{C} \end{tikzcd}$$

For instance for $D = \left\{ \begin{tikzcd} & b_1 & & b_2 & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \arrow[ur] & & a_3 \arrow[ul] \end{tikzcd} \right\}$ we have for any $K : D \to \mathcal{C}$

$$\bar{K} : \left\{ \begin{tikzcd} & & \top & & \\ & b_1 \arrow[ur] & & b_2 \arrow[ul] & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \arrow[ur] & & a_3 \arrow[ul] \end{tikzcd} \right\} \mapsto \left\{ \begin{tikzcd} & & \mathrm{colim}_D K & & \\ & K(b_1) \arrow[ur] & & K(b_2) \arrow[ul] & \\ K(a_1) \arrow[ur] & & K(a_2) \arrow[ul] \arrow[ur] & & K(a_3) \arrow[ul] \end{tikzcd} \right\}$$

Definition 3.6 (multi-cospan composition) *For $K_1 : D_1 \to \mathcal{C}$ and $K_2 : D_2 \to \mathcal{C}$ two multi-cospans in $\mathcal{C}$, and given a diagram $U(D_1) \hookleftarrow D_{\mathrm{glue}} \hookrightarrow U(D_2)$ in* Posets *of sub-poset inclusions respecting co-terminal objects, such that*

$$\begin{tikzcd} D_{\mathrm{glue}} \arrow[r] \arrow[d] & D_1 \arrow[d, "K_1"] \\ D_2 \arrow[r, "K_2"] & \mathcal{C} \end{tikzcd}$$

and given a morphism in $\mathsf{Posets}_{\mathrm{max}}$ $D_{\mathrm{comp}} \hookrightarrow \overline{D_1 \sqcup_{\mathrm{glue}} D_2}$ *we say that* <u>*composition*</u> *of K_1 and K_2 along D_{glue} to D_{comp} is the multi-cospan (or rather any one of the canonically isomorphic choices)*

$$\begin{tikzcd} K_{\mathrm{comp}} : \ D_{\mathrm{comp}} \arrow[r, hook] & \overline{D_1 \sqcup_{\mathrm{glue}} D_2} \arrow[r, "\overline{K_1 \sqcup_{\mathrm{glue}} K_2}"] & \mathcal{C} \end{tikzcd}$$

Example: ordinary spans and cospans. We obtain ordinary cospans and their composition by taking all multi-cospan domains to be $D = \left\{ \begin{tikzcd} & \top & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \end{tikzcd} \right\}$ and $D_{\mathrm{glue}} = \{\bullet\}$, so that $D \sqcup_{\mathrm{glue}} D = \left\{ \begin{tikzcd} & b_1 & & b_2 & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \arrow[ur] & & a_3 \arrow[ul] \end{tikzcd} \right\}$ and $\overline{D \sqcup_{\mathrm{glue}} D} = \left\{ \begin{tikzcd} & & \top & & \\ & b_1 \arrow[ur] & & b_2 \arrow[ul] & \\ a_1 \arrow[ur] & & a_2 \arrow[ul] \arrow[ur] & & a_3 \arrow[ul] \end{tikzcd} \right\}$ and finally taking $D_{\mathrm{comp}} = D$ with $D_{\mathrm{comp}} \hookrightarrow \overline{D \sqcup_{\mathrm{glue}} D}$ given by $a_1 \mapsto a_1$ and $a_2 \mapsto a_3$.

Dually, we get ordinary multispans in $\mathcal{C}^{\mathrm{op}}$. But already in this case we get a little more flexibility spans for handling cospans. For definitenss, consider cospans in$\mathcal{C} = \mathsf{Sets}^{\mathrm{op}}$.

Composition of the multispan $\bar{K}_1$ which is the closure of

$$K_1 := \left\{ \begin{tikzcd} \Psi_{\mathrm{in}} \arrow[d] & \Psi_{\mathrm{out}} \arrow[d] \\ X & Y \end{tikzcd} \right\}$$

with

$$K_2 := \left\{ \begin{tikzcd} & R \arrow[d] & \\ & X \times Y \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd} \right\}$$

along the preimages of X and Y *to* $\top$ is the tip of

$$\begin{tikzcd}
 & \langle \Psi_{\mathrm{in}} | R | \Psi_{\mathrm{out}} \rangle \arrow[d] \arrow[ddl] \arrow[ddr] & \\
 & R \arrow[d] & \\
\Psi_{\mathrm{in}} \arrow[d] & X \times Y \arrow[dl] \arrow[dr] & \Psi_{\mathrm{out}} \arrow[d] \\
X & & Y
\end{tikzcd}$$

and describes the contraction of the matrix K with the vectors Ψ_{in} and Ψ_{out}.

Example: more general multi-cospans in $\mathsf{Sets}^{\mathrm{op}}$. Next, consider two matrices R_1 and R_2 given by the multispan $\bar{K}_1$ which is the closure of

$$K_1 := \left\{ \begin{tikzcd}
 & R_1 \arrow[d] & & R_2 \arrow[d] & \\
 & X \times Y \arrow[dl] \arrow[d] & & Y \times Z \arrow[d] \arrow[dr] & \\
X & Y & & Y & Z
\end{tikzcd} \right\}$$

and the consider the multispan

$$K_2 := \left\{ \begin{tikzcd}
X \arrow[ddd, dash, "="] & & X \times Z \arrow[ll] \arrow[rr] & & Z \arrow[ddd, dash, "="'] \\
 & & X \times Y \times Z \arrow[u] \arrow[ull] \arrow[urr] \arrow[dl] \arrow[dr] & & \\
 & X \times Y \arrow[dl] \arrow[dr] & & Y \times Z \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd} \right\}.$$

Then composition *along* the lower zig-zag *to* the resulting top zig-zag yields the matrix product

$$K_{\mathrm{comp}} = \left\{ \begin{tikzcd}
 & R_1 \cdot R_2 \arrow[d] & \\
 & X \times Z \arrow[dl] \arrow[dr] & \\
X & & Z
\end{tikzcd} \right\}$$

These represent $\mathbb{N}$-valued matrices. Somewhat more interestingly, let k be some field and

$$K_1 := \left\{ \begin{tikzcd}
 & R_1 \arrow[d] & & R_2 \arrow[d] & \\
 & (X \times Y) \times k \arrow[dl] \arrow[d] & & (Y \times Z) \times k \arrow[d] \arrow[dr] & \\
X & Y & & Y & Z
\end{tikzcd} \right\}$$

two k-valued matrices to be composed with the multispan

$$K_2 := \left\{ \begin{tikzcd}
 & & (X \times Z) \times k \arrow[dll] \arrow[drr] & & \\
X \arrow[ddd, dash, "="] & & & & Z \arrow[ddd, dash, "="'] \\
 & & (X \times Y \times Z) \times k \times k \arrow[uu, "(-)\cdot(-)"'] \arrow[ull] \arrow[urr] \arrow[dl] \arrow[dr] & & \\
 & (X \times Y) \times k \arrow[dl] \arrow[dr] & & (Y \times Z) \times k \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd} \right\}.$$

Then the result is (after k-valued cardinality) the product of k-valued matrices.

Example: Grandis' cubical cospans. The cubical multi-cospans in [18] are reproduced by restricting the domain posets to be of the form $\wedge^n$, $n \in \mathbb{N}$.

3.2 Trace and co-trace

Definition 3.7 (co-trace) *If for $K : D \to \mathcal{C}$ a multi-cospan in which for a collection $\{a_i \in \mathrm{Obj}(D)\}_i$ of coterminal objects we have $K(a_i) \simeq X$ for all i and for X a given object of $\mathcal{C}$ the <u>co-trace</u> of K over $\{a_i\}$ is the composition of K with the co-span*

$$K_X : \left\{ \begin{tikzcd} & & \top & & \\ a_1 \arrow[urr] & a_2 \arrow[ur] & \cdots & a_i \arrow[ul] & \cdots \end{tikzcd} \right\} \overset{\mathrm{const}_X}{\to} \mathcal{C}$$

along the obvious identifications of the a_i to the diagram D with the a_i removed.

For spans the co-trace is called the <u>trace</u>.

Examples. Let $\mathcal{C} = \mathsf{Categories}$ and $\mathcal{I} := \{a \to b\}$ be the 1-globe (the (directed) interval) regarded as a co-span

$$\begin{tikzcd} & \mathcal{I} & \\ \mathrm{pt} \arrow[ur] & & \mathrm{pt} \arrow[ul] \end{tikzcd}$$

then the co-trace of $\mathcal{I}$ over pt is the the colimit over

$$\begin{tikzcd} & \mathcal{I} & \\ \mathrm{pt} \arrow[ur] \arrow[dr] & & \mathrm{pt} \arrow[ul] \arrow[dl] \\ & \mathrm{pt} & \end{tikzcd}$$

which is $\mathbf{B}\mathbb{N}$ (the (directed) circle).

Dually, let $\mathcal{C} = \mathsf{Sets}^{\mathrm{op}}$ and consider spans of finite sets again, with

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ X & & X \end{tikzcd}$$

an $|X| \times |X|$-matrix, then the trace of this is the limit over

$$\begin{tikzcd} & R \arrow[dl] \arrow[dr] & \\ X & & X \\ & X \arrow[ul] \arrow[ur] & \end{tikzcd}$$

which is $\sqcup_{x \in X} Rx, x$, as expected.

Definition 3.8 (free loop object) *For $B \in \mathcal{F}_0$ an object in a category of fibrant objects equipped with a compatible interval object $\mathcal{I}$, the <u>free loop object</u> ΛB is the pullback*

$$\begin{tikzcd} \Lambda B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{I}, B]} \arrow[d, "d_0 \times d_1"] \\ B \arrow[r, "\mathrm{Id} \times \mathrm{Id}"] & B \times B \end{tikzcd} .$$

Lemma 3.9 *For $B = [\mathrm{pt}, B]$ the free loop object ΛB is the hom-object of maps from the cotrace on the interval obect to B:*

$$\Lambda B = [\mathrm{cotr}(\mathcal{I}), B]\,.$$

Proof. The internal hom sends colimits to limits, so

$$\left[\begin{tikzcd} \mathrm{cotr}(\mathcal{I}) \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{I} \arrow[l] \\ \mathrm{pt} \arrow[u] & \mathrm{pt} \sqcup \mathrm{pt} \arrow[l, "\mathrm{Id} \sqcup \mathrm{Id}"'] \arrow[u, "\mathrm{in} \sqcup \mathrm{out}"'] \end{tikzcd} \;,\; B \right] = \begin{tikzcd} {[\mathrm{cotr}(\mathcal{I}), B]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{I}, B]} \arrow[d, "d_0 \times d_1"] \\ B \arrow[r, "\mathrm{Id} \times \mathrm{Id}"] & B \times B \end{tikzcd}$$

□

Proposition 3.10 *In a category of fibrant objects, the homotopy trace on the identity span on B is weakly equivalent to the free loop object of B:*

$$\mathrm{hotr}\left(\begin{tikzcd}[column sep=small] & B \arrow[dl, "\mathrm{Id}"'] \arrow[dr, "\mathrm{Id}"] & \\ B & & B \end{tikzcd} \right) \overset{\simeq}{\to} \Lambda B\,.$$

Proof. By definition of span trace we have

$$\mathrm{hotr}(\mathrm{Id}_B) = \mathrm{holim}\left(B \xrightarrow{\mathrm{Id} \times \mathrm{Id}} B \times B \xleftarrow{\mathrm{Id} \times \mathrm{Id}} B \right)\,.$$

By a general fact about homotopy limits of diagrams of fibrant objects this is weakly equivalent to the ordinary limit over a weakly equivalent pullback diagram where one of the morphisms is a fibration. Such a fibrant replacement is provided by that path object factorization:

$$\begin{tikzcd} B \arrow[r, "\mathrm{Id} \times \mathrm{Id}"] \arrow[d, "\simeq"] & B \times B \arrow[d, "\mathrm{Id}"] & B \arrow[l, "\mathrm{Id} \times \mathrm{Id}"'] \arrow[d, "\mathrm{Id}"] \\ {[\mathcal{I}, B]} \arrow[r, two heads, "d_0 \times d_1"] & B \times B & B \arrow[l, "\mathrm{Id} \times \mathrm{Id}"'] \end{tikzcd} \quad .$$

Hence

$$\mathrm{hotr}(\mathrm{Id}_B) \overset{\simeq}{\to} \lim\left([\mathcal{I}, B] \overset{d_0 \times d_1}{\twoheadrightarrow} B \times B \xleftarrow{\mathrm{Id} \times \mathrm{Id}} B \right) =: \Lambda B$$

□

3.3 Pointed objects

In *abelian* homotopy theory one considers (for instance section 4 of [7]) the case that in the homotopical category exists an initial object isomorphic to the terminal one. This makes all objects uniquely *pointed* in the sense of the following definition and allows to define *loop group* objects of all objects.

To retain a nonabelian (directed) setup we generalize this to the case where there need not be an initial object isomorphic to the terminal object by instead considering objects equipped with a specified point. This leads to loop *monoids* and will allow us in particular to discuss not only higher principal bundles but also higher vector bundles.

Definition 3.11 (pointed object) *The category of pointed objects in $\mathcal{C}_0$ is the under-category $\mathcal{C}_0\backslash\mathrm{pt}$: a pointed object is a morphism $\mathrm{pt} \xrightarrow{\mathrm{pt}_F} F$ and a morphism of pointed objects is a morphism $f : E \to F$ making the diagram*

$$\begin{tikzcd}
 & \mathrm{pt} \arrow[dl, "\mathrm{pt}_E"'] \arrow[dr, "\mathrm{pt}_F"] & \\
E \arrow[rr, "f"] & & F
\end{tikzcd}$$

commute.

Lemma 3.12 *The identity* $\mathrm{Id} : \mathrm{pt} \to \mathrm{pt}$ *is both the terminal as well as the initial object of* $\mathcal{C}_0\backslash\mathrm{pt}$.

This says that $\mathcal{C}_0\backslash\mathrm{pt}$ is pointed. In the following we consider operations on pointed objects but in all of $\mathcal{C}_0$, which itself need not be pointed.

Definition 3.13 (kernel and cokernel) *The kernel* $\ker(f)$ *of a morphism* $f : A \to B$ *into a pointed object* B *is the pullback*

$$\begin{tikzcd}
\ker(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{pt} \arrow[d, "\mathrm{pt}_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd} .$$

The cokernel $\mathrm{coker}(f)$ *of a morphism* $f : B \to A$ *is the pushout*

$$\begin{tikzcd}
B \arrow[r, "f"] \arrow[d] & A \arrow[d] \\
\mathrm{pt} \arrow[r, "\mathrm{pt}_{\mathrm{coker}(f)}"] & \mathrm{coker}(f) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd} .$$

The cokernel is naturally a pointed object $\mathrm{pt} \xrightarrow{\mathrm{pt}_{\mathrm{coker}(f)}} \mathrm{coker}(f)$ as indicated, the kernel is naturally pointed if A is a pointed object and f a morphism of pointed objects, with the point $\mathrm{pt} \xrightarrow{\mathrm{pt}_{\ker(f)}} \ker(f)$ given by the universal dashed morphism in

$$\begin{tikzcd}
\mathrm{pt} \arrow[dr, dashed, "\mathrm{pt}_{\ker(f)}"] \arrow[drr, bend left] \arrow[ddr, bend right, "\mathrm{pt}_B"'] & & \\
 & \ker(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{pt} \arrow[d, "\mathrm{pt}_F"] \\
 & A \arrow[r, "f"] & B
\end{tikzcd} .$$

The kernel of a fibration of pointed objects is called its fiber.

If we have an ambient structure of a model category then by replacing limits and colimits in the above with their homotopy coherent versions, we obtain the homotopy kernel $\mathrm{hoker}(f)$ and the homotopy cokernel $\mathrm{hocoker}(f)$.

Definition 3.14 (monoid of loops) *The monoid of loops* $\Omega_{\mathrm{pt}}B$ *of a pointed object* $\mathrm{pt} \xrightarrow{\mathrm{pt}_B} B$ *is the fiber of* $[\mathcal{I}, B] \xrightarrow{d_0 \times d_1} B \times B$ *with* $B \times B$ *equipped with its canonical point* $\mathrm{pt} \xrightarrow{\mathrm{pt}_B \times \mathrm{pt}_B} B \times B$ *, i.e. the pullback*

$$\begin{tikzcd}
\Omega_{\mathrm{pt}}B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{I}, B]} \arrow[d, "d_0 \times d_1"] \\
\mathrm{pt} \arrow[r, "\mathrm{pt}_B \times \mathrm{pt}_B"] & B \times B
\end{tikzcd}$$

equipped with the structure of a a homotopy coherent (A_∞) monoid induced from the structure of a homotopy coherent co-category on $\mathcal{I}$.

Lemma 3.15 *In the case that $\mathcal{V}$ is a category of fibrant objects with initial object isomorphic to the terminal one, the image of $\Omega_{\mathrm{pt}}B$ in the homotopy category is the loop group object considered in section 4 of [7].*

Lemma 3.16 *The loop monoid is the pullback of the free loop object ΛB, definition 3.8, to the point*

$$\begin{tikzcd}
\Omega_{\mathrm{pt}}B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Lambda B \arrow[d] \\
\mathrm{pt} \arrow[r, "\mathrm{pt}_B"] & B
\end{tikzcd} .$$

3.4 Bi-pointed objects and interval objects

Definition 3.17 (category with interval object) *A category with interval object is*

- *a symmetric closed monoidal homotopical category $\mathcal{V}$;*
- *and with tensor unit I in $\mathcal{V}$ the terminal object in $\mathcal{V}$ – which we write $I \simeq \mathrm{pt}$;*
- *equipped with a co-span of the form* $\begin{tikzcd}[column sep=small] & \mathcal{I} & \\ \mathrm{pt} \arrow[ur, "\sigma"] & & \mathrm{pt} \arrow[ul, "\tau"'] \end{tikzcd}$ *in $\mathcal{V}$;*

such that

- *the pushout* $$\begin{tikzcd}[column sep=small] & \mathcal{I}^{\vee 2} & \\ \mathrm{pt} \arrow[ur, "\sigma"] & & \mathrm{pt} \arrow[ul, "\tau"'] \end{tikzcd} := \begin{tikzcd}[column sep=small] & & \mathcal{I} \sqcup_{\mathrm{pt}} \mathcal{I} & & \\ & \mathcal{I} \arrow[ur] & & \mathcal{I} \arrow[ul] & \\ \mathrm{pt} \arrow[ur, "\sigma"] & & \mathrm{pt} \arrow[ul, "\tau"'] \arrow[ur, "\sigma"] & & \mathrm{pt} \arrow[ul, "\tau"'] \end{tikzcd}$$ *exists in $\mathcal{V}$;*
- *and such that all hom $\mathcal{V}$-objects of cospans are weakly equivalent to the point,* ${}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee k}]_{\mathrm{pt}} \xrightarrow{\simeq} \mathrm{pt}$.

Here we used

Definition 3.18 (internal homs of internal spans) *For* $\begin{tikzcd}[column sep=small, row sep=small] & S & \\ x \arrow[ur, "\sigma_S"] \arrow[dr, "\sigma_T"'] & & y \arrow[ul, "\tau_S"'] \arrow[dl, "\tau_T"] \\ & T & \end{tikzcd}$ *two parallel cospans in $\mathcal{C}$ we write*

${}_x[S,T]_y \in \mathrm{Ob}(\mathcal{V})$ for the V-object of cospan morphisms, define as the pullback

$$\begin{tikzcd}
{}_x[S,T]_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{pt} \arrow[d, "\sigma_T \times \tau_T"] \\
{[S,T]} \arrow[r, "{[\sigma_S \sqcup \tau_S, T]}"] & {[\mathrm{pt} \sqcup \mathrm{pt}, T]}
\end{tikzcd}$$

in $\mathcal{C}$.

An interval object $\mathcal{I}$ can be thought of as a homotopy coherent- or A_∞-co-category internal to $\mathcal{V}$.

Lemma 3.19 *Equipped with the multi-composition*

$$\begin{tikzcd}
{}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee k}]_{\mathrm{pt}} \otimes \left({}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee r_1}]_{\mathrm{pt}} \otimes {}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee r_2}]_{\mathrm{pt}} \otimes \cdots \otimes {}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee r_k}]_{\mathrm{pt}} \right) \arrow[rr, "\circ_{k,\{r_i\}}"] \arrow[dr, "\mathrm{Ten}"'] & & {}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee \sum_i r_i}]_{\mathrm{pt}} \\
& {}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee k}]_{\mathrm{pt}} \otimes {}_{\mathrm{pt}}[\mathcal{I}^{\vee k}, \mathcal{I}^{\vee \sum_i r_i}]_{\mathrm{pt}} \arrow[ur] &
\end{tikzcd}$$

the collection of objects $\{{}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee k}]_{\mathrm{pt}} \in \mathrm{Ob}(\mathcal{V})\}_k$ is a contractible operad *over $\mathcal{V}$.*

This is just the co-endomorphism operad on $\mathcal{I}$. That it is *contractible* just means that all its morphism objects $\mathcal{I}(k)$ are weakly equivalent to the point (which is true here by definition of an interval object), which encodes the *coherence conditions* on $\mathcal{I}$ regarded as a homotopy cohoherent co-category.

The raison d'être of an interval object in $\mathcal{C}$ is that it allows to probe every object B of C for the "paths and higher cells inside it" and extract that information coherently as a *fundamental n-category* $\Pi_n(B)$ in the form of a Trimblean weak n-catgeory [39].

Definition 3.20 (fundamental Trimblean 1-category) *For $\mathcal{C}$ a category with interval object, for every object B in $\mathcal{C}$ the above induces on the span $\Pi_1(B) :=$*

$$\begin{tikzcd} & {[\mathcal{I},B]} \arrow[dl, "s:={[\mathcal{I},\sigma]}"'] \arrow[dr, "t:={[\mathcal{I},\tau]}"] & \\ B_0 := [\mathrm{pt},B] & & B_0 := [\mathrm{pt},B] \end{tikzcd}$$

a structure that can be thought of as a homotopy coherent- or A_∞-category internal to $\mathcal{C}_0$ in that

$$\left\{ \bigsqcup_{x_i \in B_0} {}_{x_0}[\mathcal{I},B]_{x_1} \otimes {}_{x_1}[\mathcal{I},B]_{x_2} \otimes \cdots \otimes {}_{x_{k-1}}[\mathcal{I},B]_{x_k} \right\}_{k \in \mathbb{N}}$$

naturally carries the structure of an algebra over the interval operad $\mathcal{I}$, where the action is just composition in $\mathcal{C}$ (or rather composition of homs in $\mathcal{V}$ with powers in $\mathcal{C}$)

$${}_{\mathrm{pt}}[\mathcal{I},\mathcal{I}^{\vee k}]_{\mathrm{pt}} \otimes \left({}_{x_0}[\mathcal{I},B]_{x_1} \otimes {}_{x_1}[\mathcal{I},B]_{x_2} \otimes \cdots \otimes {}_{x_{k-1}}[\mathcal{I},B]_{x_k} \right) \xrightarrow{\quad\circ\quad} {}_{x_0}[\mathcal{I},B]_{x_k} \;.$$

For instance for $c \in {}_{\mathrm{pt}}[\mathcal{I},\mathcal{I}^{\vee 2}]_{\mathrm{pt}}$ a map from the interval onto the double interval sitting in a diagram

$$\begin{tikzcd} & & \mathcal{I} \sqcup_{\mathrm{pt}} \mathcal{I} & & \\ & \mathcal{I} \arrow[ur] & & \mathcal{I} \arrow[ul] & \\ \mathrm{pt} \arrow[ur, "\sigma"] \arrow[drr, "\sigma"'] & & \mathrm{pt} \arrow[ul, "\tau"'] \arrow[ur, "\sigma"] & & \mathrm{pt} \arrow[ul, "\tau"'] \arrow[dll, "\tau"] \\ & & \mathcal{I} \arrow[uuu, bend right, "c"'] & & \end{tikzcd}$$

for every object B the image of this diagram under $[-,B]$ yields the corresponding composition map

$$\begin{tikzcd} & & {[\mathcal{I},B] \times_B [\mathcal{I},B]} \arrow[dl] \arrow[dr] \arrow[ddd, bend left, "c"] & & \\ & {[\mathcal{I},B]} \arrow[dl, "s"'] \arrow[dr, "t"] & & {[\mathcal{I},B]} \arrow[dl, "s"'] \arrow[dr, "t"] & \\ B & & B & & B \\ & & {[\mathcal{I},B]} \arrow[ull, "s"] \arrow[urr, "t"'] & & \end{tikzcd}$$

of two morphisms in $\Pi_1(B)$.

4 Extended QFT

Definition 4.1 (extended QFT) *For $S^{\mathrm{op}}, \mathcal{V}$ categories with finite limits an extended S-QFT with coefficients in $\mathcal{V}$ is a functor*

$$Z : \mathsf{MultiSpans}(S^{\mathrm{op}}) \to \mathsf{MultiSpans}(\mathcal{V})$$

which respects composition of multispans.

For instance for $S = \mathrm{Top}$ this is supposed to be an *extended topological QFT*.

Definition 4.2 (σ-model QFT) *If $\mathcal{V}$ is closed monoidal and given a $\mathcal{V}$-enrichment of $[S^{\mathrm{op}}, \mathcal{V}]$, for*

$$B : S \to [S^{\mathrm{op}}, \mathcal{V}]$$

a functor respecting finite colimits and $(P_X \to X) \in [S^{\mathrm{op}}, \mathcal{V}]$, the S-QFT

$$[B(-), P_X] : \mathsf{MultiSpans}(S^{\mathrm{op}}) \to \mathsf{MultiSpans}(\mathcal{V})$$

is a <u>σ-model</u> with <u>target space</u> X and background field P_X.

Consider $S = \mathrm{Top}$, $\mathcal{V} = \mathrm{Cat}$ and $\mathcal{C} = [S^{\mathrm{op}}, \mathcal{V}]$. As noticed in theorem 4.5 in [19], Brown's homotopy van Kampen theorem ensures that the fundamental groupoid assignment $\Pi_1 : \mathrm{Top} \to \mathsf{Categories}$ extends to a functor $\mathsf{MultiSpans}(\mathrm{Top}^{\mathrm{op}}) \to \mathsf{MultiSpans}(\mathsf{Categories}^{\mathrm{op}})$ which respects composition of multispans.

Hence for any $C \in [S^{\mathrm{op}}, \mathcal{V}]$ any category-valued presheaf

$$[\Pi_1(-), C] : \mathsf{MultiSpans}(\mathrm{Top}^{\mathrm{op}}) \to \mathsf{MultiSpans}(\mathsf{Categories}^{\mathrm{op}})$$

respects composition of multispans.

Examples. Let $S = \mathrm{Top}$, $\mathcal{V} = \mathsf{Groupoids}$, G a finite group, then the σ-model

$$[\Pi_1(-), \mathbf{B}G] : \mathsf{MultiSpans}(\mathrm{Top}^{\mathrm{op}}) \to \mathsf{MultiSpans}(\mathsf{Groupoids})$$

is essentially the untwisted Dijkgraaf-Witten model.

5 Axiomatic nonabelian cohomology

There are two major well-developed 1-categorical tools for handling models for (directed) spaces and higher (directed) homotopies, i.e. for ∞- or ω-categories: these are *enriched category theory* and *model category theory.* A comprehensive treatment is obtained from the combination of the two, known as *enriched homotopy theory* or *homotopy coherent category theory.*

For enriched category theory we rely on the canonical textbook [23]. For homotopy theory we mainly make use of the seminal article [7] and hence mostly require less structure than in full model category theory. The systematic study of enriched homotopy theory is much younger: we adopt the point of view of [38], which follows the textbooks [12, 21].

In section 5.1 we consider higher fiber bundles – possibly equivariant, possibly with connection – in a generic homotopical context without comitting ourserves to a concrete model for ∞-categories or ω-categories, aiming to come close to requiring a necessary minimum of structural prerequisites. The idea is to stipulate that an object $\mathbf{A}$ in a category of higher structures should be

1. a *generalized space* locally modeled on objects in a category S;
2. and equipped with a consistent notion of *homotopy* between maps into it.

The first point we read as implying that $\mathbf{A}$ is characterized by maps of test-objects in S into it, making it a presheaf on S. The second point then suggests that this presheaf takes values in a homotopical category $\mathcal{V}$ and that S is $\mathcal{V}$-enriched such that there are $\mathcal{V}$-internal spaces of morphisms. The solution to this requirement suggested by [38] is to take $\mathcal{V}$ to be a *closed monoidal homotopical category* so that $\mathcal{C} := [S^{\mathrm{op}}, \mathcal{V}]$ is $\mathcal{V}$-enriched and becomes *$\mathcal{V}$-enriched homotopical* after choosing suitable local extensions of the weak equivalences in $\mathcal{V}$ to $\mathcal{C}$. The nice consequence of these natural assumptions is that the homotopy category $\mathrm{Ho}_{\mathcal{C}}$ of $\mathcal{C}$ is naturally $\mathrm{Ho}_{\mathcal{V}}$-enriched while itself homotopical in a $\mathcal{V}$-enriched sense and thus retains information about higher homotopies and their weak inverses. This should make it an accurate enriched 1-categorical model for an ∞-category of ∞-categories modeled on S.

Two complementary useful perspectives on objects in such homotopically enriched presheaf categories $\mathcal{C}$ are familiar: as generalized homotopical spaces and as ∞-*stacks*. For instance if $S = \mathrm{Diff}$ is the site of smooth manifolds, then a higher structure *probeable* by mapping objects in S into it may be a Lie groupoid $\mathbf{A}$, a higher structure whose smoothness is modeled by how test-objects in Diff are mapped into it. By instead regarding the assignment to an object $X \in \mathrm{Diff}$ of collections $[X, \mathbf{A}] \overset{\text{Yoneda}}{\simeq} \mathbf{A}(X)$ of maps from X into $\mathbf{A}$ as primary, it appears instead equivalently as the *differentiable stack* on S presented by the Lie groupoid.

5.1 Nonabelian homotopical cohomology

We place ourselves in the context of derived $\mathcal{V}$-enriched category theory for $\mathcal{V}$ a (symmetric) *closed monoidal homotopical category* and consider a $\mathcal{V}$*-enriched homotopical category* $\mathcal{C}$ as described in [38].

Recall (sections 15 and 16 in [38]) that this means that $\mathcal{V}$ is a category equipped with a choice of weak equivalences compatible with its closed monoidal structure, and in particular that on the Sets-enriched category $\mathcal{C}_0 := \mathcal{V}-\mathrm{Cat}(I, \mathcal{C})$ underlying the $\mathcal{V}$-enriched category $\mathcal{C}$ there is a $\mathcal{V}$-valued hom-functor hom : $\mathcal{C}_0 \times \mathcal{C}_0 \to \mathcal{V}$ compatible with the action of $\mathcal{V}$ on $\mathcal{C}$ by powers $[-,-]$ and by copowers $\otimes$ which determines the $\mathcal{V}$-enrichment of $\mathcal{C}$ by $\mathcal{C}(\mathbf{X}, \mathbf{A}) \simeq \mathrm{hom}(\mathbf{X}, \mathbf{A})$ for all objects $\mathbf{Y}$, $\mathbf{A}$ in $\mathcal{C}_0$. Its right-derived functor $\mathbb{R}\mathrm{hom}$

$$\begin{tikzcd}
\mathcal{C}_0^{\mathrm{op}} \times \mathcal{C}_0 \arrow[r, "\mathrm{hom}"] \arrow[d] & \mathcal{V} \arrow[d] \\
\mathrm{Ho}_{\mathcal{C}_0}^{\mathrm{op}} \times \mathrm{Ho}_{\mathcal{C}_0} \arrow[r, "\mathbb{R}\mathrm{hom}"] & \mathrm{Ho}_{\mathcal{V}}
\end{tikzcd}$$

similarly induces the $\mathrm{Ho}_{\mathcal{V}}$-enriched category $\mathrm{Ho}_{\mathcal{C}}$ whose underlying Sets-enriched category is the homotopy category $\mathrm{Ho}_{\mathcal{C}_0}$ of $\mathcal{C}_0$ (proposition 16.2 in [38]).

In such a setup cohomology identifies with the hom-objects in the homotopy categories, and algorithms for computation of cohomology are algorithms for computation of the right derived (internal) hom-functors:

Definition 5.1 (cohomology) *For $\mathcal{C}$ a $\mathcal{V}$-enriched homotopical category of the form $[S^{\mathrm{op}}, \mathcal{V}]$ or $\mathrm{Sh}(S, \mathcal{V})$, we say for $\mathbf{A}$ any object of $\mathcal{C}$ and X a representable object that $H(X, \mathbf{A}) := \mathrm{Ho}_{\mathcal{C}_0}(X, \mathbf{A})$ is the cohomology of X with coefficients in $\mathbf{A}$. The generalized elements $c : I \to Z(X, \mathbf{A})$ of the $\mathcal{V}$-object $Z(X, \mathbf{A}) := \mathrm{Ho}_{\mathcal{C}}(X, \mathbf{A})$ are the cocycles on X with coefficients in $\mathbf{A}$. Homotopies between these are the coboundaries.*

More generally $\mathrm{Ho}_{\mathcal{C}}(-, \mathbf{A})$ can be interpreted as computing *equivariant cohomology* and generalizations thereof:

Definition 5.2 (equivariant and relative cohomology) *For $i : \mathbf{X} \hookrightarrow \underline{\mathbf{X}}$ a monomorphism in $\mathcal{C}_0$ it is often useful to address $H^i(\mathbf{X}, \mathbf{A}) := \mathrm{Ho}_{\mathcal{C}_0}(\underline{\mathbf{X}}, \mathbf{A})$ as equivariant cohomology on $\mathbf{X}$, where the kind of equivariance is controlled by i. Conversely, the kernel of i^* is relative cohomology $H_i(\underline{\mathbf{X}}, \mathbf{A}) := \ker(H(\underline{\mathbf{X}}, \mathbf{A}) \xrightarrow{i^*} H(\mathbf{X}, \mathbf{A}))$ on $\underline{\mathbf{X}}$ relative to $\mathbf{X}$.*

In terms of $\mathrm{Ho}_{\mathcal{C}}$ the familiar grading on cohomology is entirely implicit in the grading that the coefficient object $\mathbf{A}$ carries for cases that $\mathcal{V}$ may be interpreted as a category of graded or higher structures. To guarantee a consistent interpretation for which this is the case, we impose the additional condition that $\mathcal{V}$ be a *category with interval object*:

Definition 5.3 (category with interval object) *A category with interval object is*

- *a symmetric closed monoidal homotopical category $\mathcal{V}$;*
- *and with tensor unit I in $\mathcal{V}$ the terminal object in $\mathcal{V}$ – which we write $I \simeq \mathrm{pt}$;*

- *equipped with a co-span of the form* $\begin{tikzcd}[column sep=small] & \mathcal{I} & \\ \mathrm{pt} \arrow[ur, "\sigma"] & & \mathrm{pt} \arrow[ul, "\tau"'] \end{tikzcd}$ *in* $\mathcal{V}$*;*

such that

- *the pushout* $\begin{tikzcd}[column sep=small] & \mathcal{I}^{\vee 2} & \\ \mathrm{pt} \arrow[ur, "\sigma"] & & \mathrm{pt} \arrow[ul, "\tau"'] \end{tikzcd}$ $:=$ $\begin{tikzcd}[column sep=small] & & \mathcal{I} \sqcup_{\mathrm{pt}} \mathcal{I} & & \\ & \mathcal{I} \arrow[ur] & & \mathcal{I} \arrow[ul] & \\ \mathrm{pt} \arrow[ur, "\sigma"] & & \mathrm{pt} \arrow[ul, "\tau"'] \arrow[ur, "\sigma"] & & \mathrm{pt} \arrow[ul, "\tau"'] \end{tikzcd}$ *exists in* $\mathcal{V}$*;*

- *and such that all hom* $\mathcal{V}$*-objects of cospans are weakly equivalent to the point,* ${}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee k}]_{\mathrm{pt}} \xrightarrow{\simeq} \mathrm{pt}$ *.*

Here we used

Definition 5.4 (internal homs of internal spans) *For* $\begin{tikzcd}[column sep=small] & S & \\ x \arrow[ur, "\sigma_S"] \arrow[dr, "\sigma_T"'] & & y \arrow[ul, "\tau_S"'] \arrow[dl, "\tau_T"] \\ & T & \end{tikzcd}$ *two parallel cospans in* $\mathcal{C}$ *we write* ${}_x[S,T]_y \in \mathrm{Ob}(\mathcal{V})$ *for the* *<u>V-object of cospan morphisms</u>, define as the pullback*

$$\begin{tikzcd} {}_x[S,T]_y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{pt} \arrow[d, "\sigma_T \times \tau_T"] \\ {[S,T]} \arrow[r, "{[\sigma_S \sqcup \tau_S, T]}"] & {[\mathrm{pt} \sqcup \mathrm{pt}, T]} \end{tikzcd}$$

in $\mathcal{C}$*.*

An interval object $\mathcal{I}$ can be thought of as a homotopy coherent- or A_∞-co-category internal to $\mathcal{V}$.

Lemma 5.5 *Equipped with the multi-composition*

$$\begin{tikzcd} {}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee k}]_{\mathrm{pt}} \otimes \left({}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee r_1}]_{\mathrm{pt}} \otimes {}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee r_2}]_{\mathrm{pt}} \otimes \cdots \otimes {}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee r_k}]_{\mathrm{pt}}\right) \arrow[rr, "\circ_{k,\{r_i\}}"] \arrow[dr, "\mathrm{Ten}"'] & & {}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee \sum_i r_i}]_{\mathrm{pt}} \\ & {}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee k}]_{\mathrm{pt}} \otimes {}_{\mathrm{pt}}[\mathcal{I}^{\vee k}, \mathcal{I}^{\vee \sum_i r_i}]_{\mathrm{pt}} \arrow[ur] & \end{tikzcd}$$

the collection of objects $\{{}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee k}]_{\mathrm{pt}} \in \mathrm{Ob}(\mathcal{V})\}_k$ *is a* contractible operad *over* $\mathcal{V}$*.*

This is just the co-endomorphism operad on $\mathcal{I}$. That it is *contractible* just means that all its morphism objects $\mathcal{I}(k)$ are weakly equivalent to the point (which is true here by definition of an interval object), which encodes the *coherence conditions* on $\mathcal{I}$ regarded as a homotopy cohoherent co-category.

The raison d'être of an interval object in $\mathcal{C}$ is that it allows to probe every object B of C for the "paths and higher cells inside it" and extract that information coherently as a *fundamental* n*-category* $\Pi_n(B)$ in the form of a Trimblean weak n-catgeory [39].

Definition 5.6 (fundamental Trimblean 1-category) *For* $\mathcal{C}$ *a category with interval object, for every object* B *in* $\mathcal{C}$ *the above induces on the span* $\Pi_1(B) :=$ $\begin{tikzcd}[column sep=small] & {[\mathcal{I}, B]} \arrow[dl, "s := {[\mathcal{I}, \sigma]}"'] \arrow[dr, "t := {[\mathcal{I}, \tau]}"] & \\ B_0 := [\mathrm{pt}, B] & & B_0 := [\mathrm{pt}, B] \end{tikzcd}$ *a structure that can be thought of as a homotopy coherent- or* A_∞*-category internal to* $\mathcal{C}_0$ *in that*

$$\left\{ \bigsqcup_{x_i \in B_0} {}_{x_0}[\mathcal{I}, B]_{x_1} \otimes {}_{x_1}[\mathcal{I}, B]_{x_2} \otimes \cdots \otimes {}_{x_{k-1}}[\mathcal{I}, B]_{x_k} \right\}_{k \in \mathbb{N}}$$

naturally carries the structure of an algebra over the interval operad $\mathcal{I}$, where the action is just composition in $\mathcal{C}$ (or rather composition of homs in $\mathcal{V}$ with powers in $\mathcal{C}$)

$$ {}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee k}]_{\mathrm{pt}} \otimes \left({}_{x_0}[\mathcal{I}, B]_{x_1} \otimes {}_{x_1}[\mathcal{I}, B]_{x_2} \otimes \cdots \otimes {}_{x_{k-1}}[\mathcal{I}, B]_{x_k} \right) \xrightarrow{\circ} {}_{x_0}[\mathcal{I}, B]_{x_k} \; . $$

For instance for $c \in {}_{\mathrm{pt}}[\mathcal{I}, \mathcal{I}^{\vee 2}]_{\mathrm{pt}}$ a map from the interval onto the double interval sitting in a diagram

$$\begin{tikzcd}
 & & \mathcal{I} \sqcup_{\mathrm{pt}} \mathcal{I} & & \\
 & \mathcal{I} \arrow[ur] & & \mathcal{I} \arrow[ul] & \\
\mathrm{pt} \arrow[ur, "\sigma"] \arrow[drr, "\sigma"'] & & \mathrm{pt} \arrow[ul, "\tau"'] \arrow[ur, "\sigma"] & & \mathrm{pt} \arrow[ul, "\tau"'] \arrow[dll, "\tau"] \\
 & & \mathcal{I} \arrow[uuu, bend right, "c"'] & &
\end{tikzcd}$$

for every object B the image of this diagram under $[-, B]$ yields the corresponding composition map

$$\begin{tikzcd}
 & & {[\mathcal{I}, B] \times_B [\mathcal{I}, B]} \arrow[dl] \arrow[dr] \arrow[ddd, bend left, "c"] & & \\
 & {[\mathcal{I}, B]} \arrow[dl, "s"'] \arrow[dr, "t"] & & {[\mathcal{I}, B]} \arrow[dl, "s"'] \arrow[dr, "t"] & \\
B & & B & & B \\
 & & {[\mathcal{I}, B]} \arrow[ull, "s"] \arrow[urr, "t"'] & &
\end{tikzcd}$$

of two morphisms in $\Pi_1(B)$.

Definition 5.7 (directed and undirected objects) *An object B in a category with interval object is undirected if $B \xrightarrow{i} [\mathcal{I}, B]$ is a weak equivalence.*

On undirected or *groupoidal* objects the computation of cohomology is usually easier, in particular if these arrange themselves into a *category of fibrant objects* in the sense of [7].

Definition 5.8 (compatible fibrant objects) *A homotopical category with interval $\mathcal{I}$ has a compatible structure of a category of fibrant objects if it is equipped with the structure of a category of fibrant objects in the sense of [7] such that the weak equivalences of both structures coincide and such that for every object B the object $[\mathcal{I}, B]$ with its canonical structure morphisms is a path object for B.*

Since in a category of fibrant objects the morphisms $B \xrightarrow{i} [\mathcal{I}, B]$ are weak equivalences, the objects of such a category are *undirected*.

An important structure present in a category of fibrant objects is the factorization lemma.

Lemma 5.9 (factorization lemma [7]) *In a category of fibrant objects for every morphism $\mathbf{X} \longrightarrow \mathbf{A}$ there is a diagram*

$$\begin{tikzcd}
 & \mathbf{Y} \arrow[dl, two heads, "p"', "\sim"] \arrow[dr, two heads] & \\
\mathbf{X} \arrow[rr] \arrow[urr, phantom, "\Nearrow" near end] & & \mathbf{A}
\end{tikzcd}$$

such that the acyclic fibration p has a section $\sigma : \mathbf{X} \to \mathbf{Y}$.

It follows that for fixed $\mathbf{A}$ the functor $\mathrm{hom}(-,\mathbf{A}) : \mathcal{C}_0^{\mathrm{op}} \to \mathcal{V}$ is *homotopical* (respects weak equivalences) and hence already coincides on objects with its right-derived functor if it sends acyclic fibrations to weak equivalences. The objects $\mathbf{A}$ for which this is true at least for acyclic fibrations $\pi : \mathbf{Y} \xrightarrow{\simeq} X$ over representables X are called ∞-stacks. For each such π the condition that $\mathrm{hom}(-,\mathbf{A})$ be homotopical is that

$$\mathrm{hom}(\pi,\mathbf{A}) : \; \mathbf{A}(X) \simeq \mathrm{hom}(X,\mathbf{A}) \xrightarrow{\simeq} \mathrm{hom}(\mathbf{Y},\mathbf{A})$$

is a weak equivalence. This is the descent condition on $\mathbf{A}$ and $\mathrm{hom}(\mathbf{Y},\mathbf{A})$ is the $\mathcal{V}$-object of descent data of $\mathbf{A}$ along π. Conversely, this means that the passage from $\mathcal{C}_0$ to $\mathrm{Ho}_{\mathcal{C}_0}$ is to be addressed as ∞-stackification.

A $\mathcal{V}$-enriched homotopical category $\mathcal{C}$ with interval object is our general situation. A compatible structure of fibrant objects puts us in an undirected *groupoidal* context. In the following we consider inclusions $\mathcal{F}_0 \subset \mathcal{C}_0$ of subcategories, with $\mathcal{F}_0$ a category of fibrant objects compatible with the homotopical structure induced $\mathcal{C}_0$ and $\mathcal{P}_0$ a pointed compatible category of fibrant objects.

In summary, our general setup is

Definition 5.10 (context for nonabelian cohomology) *A context for nonabelian cohomology is*

- *a $\mathcal{V}$-enriched homotopical category $\mathcal{C}$;*
- *for $\mathcal{V}$ a monoidal model category;*
  - *equipped with an interval object $\mathcal{I} \in \mathcal{V}$;*
- *and fixed subcategory $\mathcal{F}_0 \subset \mathcal{C}_0$ equipped with the strucuture of a category of fibrant objects compatible with the homotopical structure on $\mathcal{C}_0$ and with the interval object.*

5.2 Universal bundles

This section takes place in the category of fibrant objects $\mathcal{F}_0$ equipped compatibly with the interval object $\mathcal{I}$.

Definition 5.11 (universal bundles) *For $f : C \longrightarrow B$ a morphism we say the morphism $p_f : \mathbf{E}_f B \longrightarrow B$ defined as the composite vertical morphism in the pullback diagram*

$$\begin{tikzcd}
\mathbf{E}_f B \arrow[r, two heads, "\simeq"] \arrow[d] \arrow[dd, bend right=60, "p_f"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\
{[\mathcal{I}, B]} \arrow[r, two heads, "d_0", "\simeq"'] \arrow[d, two heads, "\simeq d_1"] & B \\
B &
\end{tikzcd}$$

is the universal B-bundle relative to f.

Lemma 5.12 *The morphism p_f is a fibration, $\mathbf{E}_f B \overset{p_f}{\twoheadrightarrow} B$.*

Proof. The pullback diagram in definition 5.11 can be refined to the double pullback diagram

$$\begin{tikzcd}
\mathbf{E}_f B \arrow[r, two heads] \arrow[d] \arrow[dd, bend right=60, "p_f"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \times B \arrow[r, two heads, "\mathrm{pr}_1"] \arrow[d, "f \times \mathrm{id}"] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\
{[\mathcal{I}, B]} \arrow[r, two heads, "d_0 \times d_1"] \arrow[d, "d_1"] & B \times B \arrow[r, two heads, "\mathrm{pr}_1"] \arrow[dl, "\mathrm{pr}_2"] & B \\
B & &
\end{tikzcd}$$

which exhibits p_f as the composite of two fibrations. ☐

Lemma 5.13 *The morphism* $\mathbf{E}_f B \xrightarrow{\simeq} C$ *has a section* $\sigma_f : C \xrightarrow{\simeq} \mathbf{E}_f B$ *and its composite with* p_f *is* f

$$\begin{tikzcd}
\mathbf{E}_f B \arrow[d, two heads, "p_f"'] & C \arrow[l, "\sigma_f"', "\simeq"] \arrow[dl, "f"] \\
B &
\end{tikzcd} .$$

Proof. The section is the morphism induced via the universal property of the pullback by the section σ of $[\mathcal{I}, B] \xrightarrow{d_0} B$:

$$\begin{tikzcd}
C \arrow[r, "\sigma_f"] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[rr, bend left, "\mathrm{Id}"] & \mathbf{E}_f B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\
B \arrow[r, "\simeq", "\sigma"'] \arrow[rr, bend left, "\mathrm{Id}"] \arrow[dr, "\mathrm{Id}"'] & {[\mathcal{I}, B]} \arrow[r, "\simeq", "d_1"'] \arrow[d, "\simeq"', "d_0"] & B \\
& B &
\end{tikzcd}$$

☐

Lemmas 5.12 and 5.13 together constitute the *factorization lemma* 5.9.

Universal bundles can be understood as a way to realize homotopy limits $\mathrm{holim}_D F := \mathbb{R}\mathrm{lim}_D F$ by ordinary limits evaluated on fibrant replacements.

Definition 5.14 (homotopy fiber product) *The homotopy fiber product of a diagram* $D \longrightarrow B \xleftarrow{f} C$ *is the pullback*

$$D \times_{B^{\mathcal{I}}} C \;:=\; \lim \left(\begin{tikzcd}
& & C \arrow[d, "f"] \\
& {[\mathcal{I}, B]} \arrow[r, "d_0"'] \arrow[d, "d_1"] & B \\
D \arrow[r] & B &
\end{tikzcd} \right) .$$

In the context of topological spaces this is definition 2.1.10 in [25].

Lemma 5.15 *The homotopy fiber product is the fiber product with a universal bundle:*

$$D \times_{B^{\mathcal{I}}} C = D \times_B \mathbf{E}_f B \,.$$

Proof. The homotopy fiber product can be expressed as two consecutive pullbacks

$$\begin{tikzcd}
D \times_{B^{\mathcal{I}}} C \arrow[r] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E}_f B \arrow[r, two heads, "\simeq"] \arrow[d] \arrow[dd, bend right, two heads, "p_f"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\
 & {[\mathcal{I}, B]} \arrow[r, two heads, "\simeq", "d_0"'] \arrow[d, two heads, "\simeq", "d_1"'] & B \\
D \arrow[r] & B &
\end{tikzcd} ,$$

where the right pullback is a universal bundle in the sense of definition 5.11. ☐

Corollary 5.16 *The two projection morphisms out of a homotopy fiber product are fibrations:*

$$\begin{tikzcd}
D \times_{C^{\mathcal{I}}} B \arrow[r, two heads] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner\!\lrcorner", very near start] & C \arrow[d] \\
D \arrow[r] & B
\end{tikzcd} .$$

Proof. By lemma 5.12 the morphism p_f in the above proof is a fibration, hence so is its pullback $\mathrm{pr}_1 : D \times_{B^{\mathcal{I}}} C \xrightarrow{\simeq} D$. By the symmetry of the situation the same argument applies to $\mathrm{pr}_2 : D \times_{B^{\mathcal{I}}} C \xrightarrow{\simeq} C$
☐

Corollary 5.17 *For* $\begin{tikzcd}[column sep=small] C \arrow[rr, "f", "\simeq"'] \arrow[dr] & & C' \arrow[dl] \\ & B & \end{tikzcd}$ *a weak equivalence, the induced morphism*

$$D \times_{B^{\mathcal{I}}} f \ : \ D \times_{B^{\mathcal{I}}} B \xrightarrow{\simeq} D \times_{B^{\mathcal{I}}} B'$$

is a weak equivalence.

Proof. By corollary 5.16 the morphism is the pullback of a weak equivalence along a fibration:

$$\begin{tikzcd}
D \times_{B^{\mathcal{I}}} C \arrow[r] \arrow[d, "\simeq"] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\simeq"', "f"] \\
D \times_{B^{\mathcal{I}}} C' \arrow[r, two heads] & C'
\end{tikzcd} .$$

☐

Lemma 5.18 *If $\mathcal{V}$ is* symmetric *monoidal then the internal hom-functor respects homotopy fiber products in its second argument:*

$$[V, D \times_{B^{\mathcal{I}}} C] \simeq [V, D] \times_{[V,B]^{\mathcal{I}}} [V, C] \ .$$

Proof. Use that the internal hom preserves ordinary limits and that in a closed symmetric monoidal category $[V, [I, B]] \simeq [I, [V, B]]$. ☐

Definition 5.19 (fibrant replacement diagrams) *A fibrant replacement diagram for a pullback diagram* $D \to B \leftarrow C$ *is a weakly equivalent diagram* $\begin{tikzcd} D \arrow[r] \arrow[d, "\simeq"] & B \arrow[d, "\simeq"] & C \arrow[l] \arrow[d, "\simeq"] \\ D' \arrow[r] & B' & C' \arrow[l, two heads] \end{tikzcd}$ *such that* $C' \twoheadrightarrow B'$ *is a fibration, as indicated.*

Lemma 5.20 *For* $D \longrightarrow B \xleftarrow{f} C$ *any diagram the universal bundle diagram* $D \longrightarrow B \xleftarrow{p_f} \mathbf{E}_f B$ *is a fibrant replacement diagram.*

Proof. By lemma 5.13 we have a weak equivalence of diagrams

$$\begin{tikzcd} D \arrow[r] \arrow[d, "="] & B \arrow[d, "="] & C \arrow[l, "f"'] \arrow[d, "\sigma_f", "\sim"'] \\ D \arrow[r] & B & \mathbf{E}_f B \arrow[l, two heads, "p_f"'] \end{tikzcd} .$$

By lemma 5.12 this is a fibrant replacement diagram in that p_f is a fibration, as indicated. □

Corollary 5.21 *If the ambient category* $\mathcal{F}_0$ *of fibrant objects extends to the structure of a model category, the homotopy fiber product of a pullback diagram as above is weakly equivalent to the homotopy limit* $\mathbb{R}\mathrm{lim}$ *of the diagram.*

Proof. By example 4.2 of [**?**] the homotopy limit is weakly equivalent to the ordinary limit of any fibrant replacement diagram. □

Definition 5.22 (monoid of loops) *The monoid of loops* $\Omega_{\mathrm{pt}} B$ *of a pointed object* $\mathrm{pt} \xrightarrow{\mathrm{pt}_B} B$ *is the homotopy fiber product of the point with itself over* B*:*

$$\Omega_{\mathrm{pt}} B := \mathrm{pt} \times_{B^{\mathcal{I}}} \mathrm{pt} \ .$$

Notice that the monoid of loops

- is the fiber of the universal B-bundle over the point.
- is the fiber of $[\mathcal{I}, B] \overset{d_0 \times d_1}{\twoheadrightarrow} B \times B$ with $B \times B$ equipped with its canonical point $\mathrm{pt} \xrightarrow{\mathrm{pt}_B \times \mathrm{pt}_B} B \times B$, i.e. the pullback

$$\begin{tikzcd} \Omega_{\mathrm{pt}} B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{I}, B]} \arrow[d, "d_0 \times d_1"] \\ \mathrm{pt} \arrow[r, "\mathrm{pt}_B \times \mathrm{pt}_B"] & B \times B \end{tikzcd} .$$

This shows that $\Omega_{\mathrm{pt}} B$ is naturally equipped with the structure of an A_∞-monoid induced from the structure of the interval object.

Definition 5.23 *There is a natural action* $\rho : \mathbf{E}_{\mathrm{pt}} \times \Omega_{\mathrm{pt}} B \to \mathbf{E}_{\mathrm{pt}} B$ *of the monoid of loops on the universal bundle, induced from the co-category structure on* $\mathcal{I}$.

Lemma 5.24 *This action is a morphism of bundles*

$$\begin{tikzcd} \mathbf{E}_{\mathrm{pt}} B \times \Omega_{\mathrm{pt}} B \arrow[rr, "\rho"] \arrow[dr, "p \circ p_1"'] & & \mathbf{E}_{\mathrm{pt}} B \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

5.3 Cocycles and bundles

This section still takes place in $\mathcal{F}_0$. By the central theorem in [7] morphisms in the homotopy category $\mathrm{Ho}_{\mathcal{F}_0}$ are represented already by single spans whose left leg is an acyclic fibration.

Definition 5.25 (anamorphisms) *An <u>anamorphism</u> or <u>cocycle</u> on X with values in A is a span*

$$\begin{tikzcd}
\hat{X} \arrow[r] \arrow[d, two heads, "\simeq"] & A \\
X &
\end{tikzcd}.$$

Lemma 5.26 *Anamorphisms have a consistent composition induced by pullback*

$$\begin{tikzcd}
g_1^*\hat{B} \arrow[r] \arrow[d, two heads, "\simeq"] & \hat{B} \arrow[r, "g_2"] \arrow[d, two heads, "\simeq"] & C \\
\hat{A} \arrow[r, "g_1"] \arrow[d, two heads, "\simeq"] & B & \\
A & &
\end{tikzcd}.$$

which is associative and unital up to isomorphism of spans.

Definition 5.27 (bundles obtained from cocycles) *Given a cocycle* $X \xleftarrow{\simeq} \hat{X} \xrightarrow{g} B$ *into a pointed object* $\mathrm{pt} \xrightarrow{\mathrm{pt}_B} B$ *the corresponding B-bundle* $p:\ g^*\mathbf{E}_{\mathrm{pt}}B \longrightarrow X$ *is the pullback*

$$\begin{tikzcd}
g^*\mathbf{E}_{\mathrm{pt}}B \arrow[r] \arrow[d, two heads] \arrow[dd, two heads, bend right, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E}_{\mathrm{pt}}B \arrow[d, two heads] \\
\hat{X} \arrow[r, "g"] \arrow[d, two heads, "\simeq"] & B \\
X &
\end{tikzcd}$$

This bundle inherits an action $\rho : (g^*\mathbf{E}_{\mathrm{pt}}B)\times\Omega_{\mathrm{pt}}B \to g^*\mathbf{E}_{\mathrm{pt}}B$ of the monoid of loops from the commutativity of

$$\begin{tikzcd}
g^*\mathbf{E}B \times \Omega_{\mathrm{pt}}B \arrow[r] \arrow[d, "p_1"] & \mathbf{E}B \times \Omega_{\mathrm{pt}}B \arrow[r, "\rho"] \arrow[d, "p_1"] & \mathbf{E}B \arrow[ddl] \\
g^*\mathbf{E}B \arrow[r] \arrow[d] & \mathbf{E}B \arrow[d] & \\
\hat{X} \arrow[r, "g"] & B &
\end{tikzcd}$$

Lemma 5.28 *This induced action is still a morphism of bundles*

$$\begin{tikzcd}
(g^*\mathbf{E}_{\mathrm{pt}}B) \times \Omega_{\mathrm{pt}}B \arrow[rr, "\rho"] \arrow[dr, "p \circ p_1"'] & & g^*\mathbf{E}_{\mathrm{pt}}B \arrow[dl, "p"] \\
& X &
\end{tikzcd}$$

Lemma 5.29 *If X is pointed and the cocycle $X \overset{\simeq}{\twoheadleftarrow} \hat{X} \xrightarrow{g} B$ respects the point, then the fiber of $g^*\mathbf{E}_{\mathrm{pt}}B$ over the point is the loop monoid $\Omega_{\mathrm{pt}}B$*

$$\begin{tikzcd}
\Omega_{\mathrm{pt}}B \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, very near start, "\lrcorner"] & g^*\mathbf{E}_{\mathrm{pt}}B \arrow[d, two heads] \\
\mathrm{pt}_X \arrow[r, "\mathrm{pt}"] & X
\end{tikzcd}$$

Definition 5.30 (fiber bundle) *We say the morphism $P \longrightarrow X$ equipped with an action of $\Omega_{\mathrm{pt}}B$ is a B-fiber bundle if there is a cocycle $X \overset{\simeq}{\twoheadleftarrow} \hat{X} \xrightarrow{g} B$ and a weak equivalence*

$$\begin{tikzcd}
g^*\mathbf{E}_{\mathrm{pt}}B \arrow[rr, "\simeq"] \arrow[dr] & & P \arrow[dl] \\
& X &
\end{tikzcd}$$

respecting the $\Omega_{\mathrm{pt}}B$-action on both sides.

Definition 5.31 (trivial bundle) *The trivial B-bundle over an object X is the pullback of $\mathbf{E}_{\mathrm{pt}}B$ along the trivial cocycle: the one that factors through the point of B.*

Lemma 5.32 *The trivial B-bundle over X is the product of X with the monoid of loops $\Omega_{\mathrm{pt}}B$:*

$$X \times \Omega_{\mathrm{pt}}B \xrightarrow{p_1} X\ .$$

Proof. By lemma **??** we have

$$\begin{tikzcd}
X \times \Omega_{\mathrm{pt}}B \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \Omega_{\mathrm{pt}}B \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{E}_{\mathrm{pt}}B \arrow[d] \\
X \arrow[r] & \mathrm{pt} \arrow[r, "\mathrm{pt}_B"] & B
\end{tikzcd}.$$

□

Proposition 5.33 (fiber bundle trivializes over itself) *Every fiber bundle $P \to X$ is trivializable after pulled back to its own total space.*

Proof. For $X \overset{\simeq}{\twoheadleftarrow} \hat{X} \longrightarrow B$ a cocycle characterizing the bundle $P \to X$ we obtain the pullback diagram

$$\begin{tikzcd}
P \arrow[dd] & g^*\mathbf{E}_{\mathrm{pt}}B \arrow[l, two heads, "\simeq"'] \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{E}_{\mathrm{pt}}B \arrow[r, two heads] \arrow[d] & \mathrm{pt} \arrow[d] \\
& g^*B^{\mathcal{I}} \arrow[r] \arrow[d, two heads, "\simeq"] \arrow[dr, phantom, very near start, "\lrcorner"] & B^{\mathcal{I}} \arrow[r, two heads, "d_0", "\simeq"'] \arrow[d, two heads, "\simeq"', "d_1"] & B \\
X & \hat{X} \arrow[l, two heads, "\simeq"'] \arrow[r, "g"] & B &
\end{tikzcd}.$$

The cocycle g pulled back to P is represented by the morphism from $g^*\mathbf{E}_{\mathrm{pt}}B$ to the B at the bottom. The right part of the diagram says that this is homotopic to a map factoring through the point. □

Definition 5.34 (associated bundle) *Let $g^*\mathbf{E}_{\mathrm{pt}}B \to X$ be a B-bundle as above and let $\rho : B \to F$ be a morphism in $\mathcal{C}_0$ to a pointed object $\mathrm{pt} \xrightarrow{\mathrm{pt}_F} F$ not necessarily fibrant. Then we call ρ a representation of*

B on F and call the pullback

$$\begin{tikzcd}
g^*\rho^*\mathbf{E}_{\mathrm{pt}}F \arrow[r] \arrow[d] \arrow[dd, bend right] & \rho^*\mathbf{E}_{\mathrm{pt}}F \arrow[r] \arrow[d] & \mathbf{E}_{\mathrm{pt}}F \arrow[d] \\
\hat{X} \arrow[r, "g"] \arrow[d, two heads] & B \arrow[r, "\rho"] & F \\
X & &
\end{tikzcd}$$

the associated bundle, associated by ρ to P.

So in particular $\rho^*\mathbf{E}_{\mathrm{pt}}F$ is the F bundle ρ-associated to the univeral B-bundle.

5.4 Fibration sequences and Postnikov tower

Definition 5.35 (homotopy pullback [25]) *A commutative diagram* $\begin{tikzcd} W \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & B \end{tikzcd}$ *is a homotopy pullback square, denoted* $\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\!\lrcorner", very near start] & C \arrow[d] \\ D \arrow[r] & B \end{tikzcd}$*, if the induced composite morphism* $W \dashrightarrow D \times_{B^{\mathcal{I}}} B$ *to the homotopy fiber product from definition 5.14 is a weak equivalence:*

$$\begin{tikzcd}
W \arrow[r, dashed] \arrow[rr, bend right, "\simeq"'] & D \times_B C \arrow[r, dashed] & D \times_{B^{\mathcal{I}}} C
\end{tikzcd} .$$

Proposition 5.36 *In a category of fibrant objects, if* $D \twoheadrightarrow B$ *is a fibration then the ordinary pullback is a homotopy pullback*

$$\begin{tikzcd}
W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "f"] \\
D \arrow[r, two heads] & B
\end{tikzcd} \Rightarrow \begin{tikzcd}
W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\!\lrcorner", very near start] & C \arrow[d, "f"] \\
D \arrow[r] & B
\end{tikzcd} .$$

Proof. Recall from lemma 5.15 that $D \times_{B^{\mathcal{I}}} B = D \times_B \mathbf{E}_f B$. Consider the double pullback square

$$\begin{tikzcd}
D \times_B C \arrow[r] \arrow[d, dashed, "\simeq"] \arrow[dd, bend right] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "\sigma_f"', "\simeq"] \arrow[dd, bend left, "f"] \\
D \times_{B^{\mathcal{I}}} C \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{E}_f B \arrow[d, "p_f"'] \\
D \arrow[r, two heads] & B
\end{tikzcd}$$

constructed using the morphism σ_f from lemma 5.13, where the bottom square is due to lemma 5.15. Using that in a category of fibrant objects fibrations are preserved under pullback (by definition) the middle horizontal morphism is a fibration. Using that weak equivalences are preserved under pullback along fibrations (by lemma 2, p. 428 in [7]) the dashed morphism is a weak equivalence. □

Proposition 5.37 *The pasting composition of two homotopy pullback diagrams*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\!\lrcorner", very near start] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner\!\lrcorner", very near start] & E \arrow[d] \\
C \arrow[r] & D \arrow[r] & F
\end{tikzcd}$$

is itself a homotopy pullback diagram

Proof. Consider the pasting composite of the two homotopy fibrations involved:

$$\begin{tikzcd}
A \arrow[d, "\simeq"] \arrow[rrrd] & & & & & \\
C \times_{D^{\mathcal{I}}} B \arrow[rrr, two heads] \arrow[d, dashed, "\simeq"] \arrow[rd, "\simeq"] \arrow[ddddd, bend right] & & & B \arrow[d, "\simeq"] \arrow[rrd, two heads] & & \\
C \times_{F^{\mathcal{I}}} E \arrow[r, "\simeq"] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[ddd] & C \times_{D^{\mathcal{I}}} D \times_{F^{\mathcal{I}}} E \arrow[rr, two heads] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd, two heads] & & D \times_{F^{\mathcal{I}}} E \arrow[rr, two heads] \arrow[dd, two heads] & & E \arrow[d] \\
& & & & {[\mathcal{I}, F]} \arrow[r] \arrow[d] & F \\
& \mathbf{E}_f D \arrow[r] \arrow[rr, bend left, two heads, "p_f"] \arrow[d, "\simeq"] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathcal{I}, D]} \arrow[r, two heads, "d_0"'] \arrow[d, "d_1"] & D \arrow[r] & F & \\
C \arrow[ru, "\sigma_f", "\simeq"'] \arrow[r, "\mathrm{Id}"] \arrow[rrru, bend right, "f"'] & C \arrow[r, "f"] & D & & &
\end{tikzcd},$$

where in the bottom left we have inserted the universal D-bundle $\mathbf{E}_f D$ relative to f together with the section morphism σ_f to factor the left vertical morphisms to C through $\mathbf{E}_f D$. This is as in the proof of lemma 5.36 and serves to show that the morphism $C \times_{F^{\mathcal{I}}} E \xrightarrow{\simeq} C \times_{D^{\mathcal{I}}} (D \times_{F^{\mathcal{I}}} E)$ is a weak equivalence, being the pullback of the weak equivalence σ_f by a fibration, which itself is the pullback of the fibration $D \times_{F^{\mathcal{I}}} E \twoheadrightarrow D$ given by corollary 5.16.

By corollary 5.17 the morphism $C \times_{D^{\mathcal{I}}} B \xrightarrow{\simeq} C \times_{D^{\mathcal{I}}} (D \times_{F^{\mathcal{I}}}) E$ is a weak equivalence. It then follows by two-out-of-three that the dashed morphism is a weak equivalence, which finally gives the required weak equivalence $A \xrightarrow{\simeq} C \times_{F^{\mathcal{I}}} E$. □

Definition 5.38 (homotopy fibration sequence) *A sequence of morphisms*

$$\cdots \xrightarrow{f_{n+1}} C_{n+1} \xrightarrow{f_n} C_n \longrightarrow \cdots$$

of pointed objects is a homotopy fibration sequence if every morphism is the homotopy kernel of its succesor, in that it fits into a homotopy pullback diagram of the form

$$\begin{tikzcd}
C_{n+2} \arrow[r] \arrow[d, "f_{n+1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{pt} \arrow[d, "\mathrm{pt}_{C_n}"] \\
C_{n+1} \arrow[r, "f_n"] & C_n
\end{tikzcd}.$$

Proposition 5.39 *Every homotopy fiber product*

$$\begin{tikzcd}
X \times_{B^{\mathcal{I}}} \mathrm{pt} =: P \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{pt} \arrow[d, "\mathrm{pt}_B"] \\
X \arrow[r] & B
\end{tikzcd}$$

induces a long homotopy fibration sequence to the left, of the form

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega_{\mathrm{pt}}\Omega_{\mathrm{pt}}P \arrow[d, "\Omega_{\mathrm{pt}} i"] & & \\
& \Omega_{\mathrm{pt}}P \arrow[d, "\Omega_{\mathrm{pt}}"] & & \\
& \Omega_{\mathrm{pt}}X \arrow[r, "\overline{\Omega g}"] & \Omega_{\mathrm{pt}}B \arrow[d] & \\
& & P \arrow[d, "p"] & \\
& & X \arrow[r, "g"] & B
\end{tikzcd} .$$

Proof. The homotopy fibration sequence is constructed by the pasting of homotopy pullbacks:

$$\begin{tikzcd}
\vdots \arrow[r] & \Omega_{\mathrm{pt}}X \arrow[r] \arrow[d] \arrow[dr, phantom, "\rrbracket", very near start] & \mathrm{pt} \arrow[d] & \\
{} \arrow[r] & \Omega_{\mathrm{pt}}B \arrow[r] \arrow[d] \arrow[dr, phantom, "\rrbracket", very near start] & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\rrbracket", very near start] & \mathrm{pt} \arrow[d] \\
& \mathrm{pt} \arrow[r] & X \arrow[r, "g"] & B
\end{tikzcd} ,$$

where we identify the pullback objects appearing here (up to weak equivalence) by using lemma 5.37 for identifying every two consecutive pullback diagrams with their total pullback diagram.

In the first step this yields

$$\begin{tikzcd}
\mathrm{pt} \times_{X^{\mathcal{I}}} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\rrbracket", very near start] & P \arrow[r] \arrow[d] \arrow[dr, phantom, "\rrbracket", very near start] & \mathrm{pt} \arrow[d] \\
\mathrm{pt} \arrow[r] & X \arrow[r, "g"] & B
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
\Omega_{\mathrm{pt}}B \arrow[r] \arrow[d] \arrow[dr, phantom, "\rrbracket", very near start] & \mathrm{pt} \arrow[d] \\
\mathrm{pt} \arrow[r] & B
\end{tikzcd}$$

by definition 5.22. Similarly, in the next step we have

$$\begin{tikzcd}
\Omega_{\mathrm{pt}}B \times_{P^I} \mathrm{pt} \arrow[r] \arrow[d] \arrow[dr, phantom, "\rrbracket", very near start] & \mathrm{pt} \arrow[d] \\
\Omega_{\mathrm{pt}}B \arrow[r] \arrow[d] \arrow[dr, phantom, "\rrbracket", very near start] & P \arrow[d] \\
\mathrm{pt} \arrow[r] & X
\end{tikzcd}
\quad \Leftrightarrow \quad
\begin{tikzcd}
\Omega_{\mathrm{pt}}X \arrow[r] \arrow[d] \arrow[dr, phantom, "\rrbracket", very near start] & \mathrm{pt} \arrow[d] \\
\mathrm{pt} \arrow[r] & X
\end{tikzcd} ,$$

where again the homotopy pullback is identified with the loop monoid, only that now the orientation of the loops appears in the opposite order, so that the induced morphism of loop monoids is

$$\overline{\Omega_{\mathrm{pt}} g} : \; \Omega_{\mathrm{pt}}X \longrightarrow \Omega_{\mathrm{pt}}B \; .$$

And so on. □

Definition 5.40 (Postnikov tower)

$$\begin{tikzcd}
\vdots \arrow[d, two heads] & \\
g_2^*\mathbf{E}_{\mathrm{pt}}B_2 \simeq P_2 \arrow[r, "g_3"] \arrow[d, two heads] & B_3 \\
g_1^*\mathbf{E}_{\mathrm{pt}}B_1 \simeq P_1 \arrow[r, "g_2"] \arrow[d, two heads] & B_2 \\
P_0 \arrow[r, "g_1"] & B_1
\end{tikzcd}$$

5.5 Extension and lifting problem

Given a cocycle $g : X \to A$ it is of interest to ask if

- it *lifts* through a given fibration of its codomain;
- it *extends* through a given cofibration of its domain

$$\begin{tikzcd}
 & \overline{A} \arrow[d, two heads] \\
X \arrow[r, "g"] \arrow[ur, dashed, "\overline{g}"] \arrow[d, hook] & A \\
\underline{\hat{X}} \arrow[ur, dashed, "\underline{g}"'] &
\end{tikzcd} .$$

In nice cases the obstructions to lifts and extensions are computed by connecting homomorphisms in *long exact sequences in cohomology.*

5.5.1 Lifting

Definition 5.41 (good obstruction theory) *A morphism* $f : \overline{A} \longrightarrow A$ *admits a* <u>good obstruction theory</u> *if it is weakly equivalent to the* second *morphism in a fibration sequence, in that*

$$\begin{tikzcd}
\mathrm{hoker}(f) \arrow[d] & \Omega_{\mathrm{pt}}B \arrow[l, "\simeq"'] \arrow[d] & \\
\overline{A} \arrow[d, "f"] & \overline{A}' \arrow[l, "\simeq"'] \arrow[d, two heads, "p"] & \\
A & A' \arrow[l, two heads, "\simeq"'] \arrow[r] & B
\end{tikzcd} .$$

Proposition 5.42 *If* $f : \overline{A} \to A$ *admits a good obstruction theory, then for every* X *we have a long exact sequence in cohomology*

$$\cdots \longrightarrow \mathrm{Ho}_{\mathcal{C}_0}(X, \Omega_{\mathrm{pt}}B) \longrightarrow \mathrm{Ho}_{\mathcal{C}_0}(X, \overline{A}) \xrightarrow{f_*} \mathrm{Ho}_{\mathcal{C}_0}(X, A) \xrightarrow{\delta} \mathrm{Ho}_{\mathcal{C}_0}(X, B) \ ,$$

which is exact as a sequence of pointed sets.

Definition 5.43 (obstructions and twisted cohomology) *If $f : \overline{A} \to A$ admits a good obstruction theory we say that*

- *for $[g] \in \mathrm{Ho}_{\mathcal{C}_0}(X, A)$ an A-cohomology class*
  - *an element in the preimage $[\overline{g}] \in f_*^{-1}(\{[h]\}) \subset \mathrm{Ho}_{\mathcal{C}_0}(X, \overline{A})$ is a lift through f;*
  - *its image $\delta[g]$ is the obstruction to lifting;*
- *for $[c] \in \mathrm{Ho}_{\mathcal{C}_0}(X, B)$ the preimage $\mathrm{Ho}^c_{\mathcal{C}}(X, \overline{A})$ of $[c]$ under δ, i.e. the pullback*

$$\begin{tikzcd}
\mathrm{Ho}^c_{\mathcal{C}}(X, \overline{A}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{pt} \arrow[d, "{[c]}"] \\
\mathrm{Ho}_{\mathrm{C}}(X, \overline{A}) \arrow[r, "\delta"] & \mathrm{Ho}_{\mathrm{C}}(X, B)
\end{tikzcd}$$

is the twisted $\overline{A}$-cohomology of X, with twist $[c]$.

5.5.2 Local semi-trivializations

Proposition 5.33 states that every B-bundle trivializes over its own total space. If $\hat{B}$ is part of an exact sequence $A \xrightarrow{i} \hat{B} \xrightarrow{p} B$, with i the homotopy kernel of p, then there is a relative version of this statement, which states that a $\hat{B}$ bundle becomes equivalent to an A-bundle with a certain $\hat{B}$-equivariance on the total space of the underlying B-bundle.

Definition 5.44 (relative equivariance) *For $\pi : P \to X$ a B-bundle we say a cocycle $P \mathrel{-\!\!\!|\!\!\!\rightarrow} A$ on its total space is relatively equivariant with respect to the above sequence if*

- *there exists $\hat{B}$-cocycle $\underline{g}$ on X*
- *and a homotppy $\pi^* \underline{g} \Rightarrow p^* g$*
- *and such that P is classified by $p^* g$*

Proposition 5.45 (local semi-trivialization) *$\hat{B}$-cohomology on X is in bijection with relatively equivariant A-cohomology on the underlying B-bundles.*

Proof. Consider

$$\begin{tikzcd}
 & & A \arrow[dr] & & & \\
 & & & \hat{B} \arrow[dddr] & & \\
P \arrow[d] & g^*\mathbf{E}_{\mathrm{pt}}B \arrow[l, two heads, "\simeq"'] \arrow[d] \arrow[rrr] \arrow[uur, dashed] & & & \mathbf{E}_{\mathrm{pt}}B \arrow[r, two heads] \arrow[d] & \mathrm{pt} \arrow[d] \\
P//G \arrow[d, two heads, "\simeq"] & g^*[\mathcal{I}, B] \arrow[l, two heads, "\simeq"'] \arrow[d, two heads, "\simeq"] \arrow[rrr] & & & {[\mathcal{I}, B]} \arrow[r, two heads, "d_0", "\simeq"'] \arrow[d, two heads, "\simeq"', "d_1"] & B \\
X & \hat{X} \arrow[l, two heads, "\simeq"'] \arrow[rrr, "g"] \arrow[uuurr, "\hat{g}"] & & & B &
\end{tikzcd}$$

□

5.5.3 Extension

Consider extensions $\underline{g}$ of morphisms g along morphisms i

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "i"] & A \\
\underline{X} \arrow[ur, dashed, "\underline{g}"'] &
\end{tikzcd}$$

Definition 5.46 (equivariant structure and flat connection) *An extension through a morphism $i : A \hookrightarrow X$ which is an isomorphism from the points of A to the points of X*

$$[\mathrm{pt}, A] \xrightarrow[\simeq]{i_*} [\mathrm{pt}, X]$$

is called an i-equivariant structure or an i-flat connection.

Theorem 5.47 (long exact sequence for extensions) *For every object B for which there is a diagram*

$$\begin{tikzcd}
B \arrow[d, "i"] & \\
\mathbf{W}B \arrow[r, two heads, "\simeq"] \arrow[d, "p"] & \mathrm{pt} \\
\mathrm{hocoker}(i) &
\end{tikzcd}$$

and for every extension $\iota : X \to \underline{X}$ there is a morphism $\delta : H(X, B) \to H_X(\underline{X}, \mathrm{hocoker}(i))$ whose kernel is the image of ι^*. If furthermore i is the homotopy kernel of p then then image of δ is the kernel of $H_X(\underline{X}, \mathrm{hocoker}(i)) \to H(X, \mathrm{hocoker}(i))$ so that we get a semi-long exact sequence in cohomology

$$H_X(\underline{X}, B) \longrightarrow H(\underline{X}, B) \xrightarrow{i^*} H(X, B) \xrightarrow{\delta} H_X(\underline{X}, \mathrm{hocoker}(i)) \longrightarrow H(\underline{X}, \mathrm{hocoker}(i)) \longrightarrow \cdots$$

Proof. (the idea)

The connecting homomorphism δ is obtained by first constructing a diagram

$$\begin{tikzcd}
X \arrow[d, "i"] & \hat{X} \arrow[l, two heads, "\simeq"'] \arrow[r, "g"] \arrow[d, hook, "\exists \hat{\imath}"] & B \arrow[d, "i"] \\
\underline{X} & \underline{\hat{X}} \arrow[l, two heads, "\simeq"'] \arrow[r, "\exists F"] \arrow[ur, dashed] \arrow[dr, "\delta g"'] & \mathbf{W}B \arrow[d, "p"] \\
 & & \mathrm{hocoker}(i)
\end{tikzcd}$$

and then setting $\delta g := p \circ F$. Here the existence of the cofibration $\hat{\imath}$ follows from the general factorization property in a model category, while F is constructed as the dashed morphism in

$$\begin{tikzcd}
\hat{X} \arrow[r] \arrow[dd, hook, "\hat{\imath}"] & B \arrow[d, "i"] \\
 & \mathbf{W}B \arrow[d, two heads, "\simeq"] \\
\underline{\hat{X}} \arrow[r] \arrow[ur, dashed, "F"] & \mathrm{pt}
\end{tikzcd}$$

which exists by the lifting property of cofibrations. By the property of a homotopy colimit δg is precisely the obstruction for the dashed lift to exists. That every cocycle in $H(\underline{X}, \mathrm{coker}(i))$ which trivializes under i^* arises this way follows by using that $i = \mathrm{hoker}(p)$... □

Definition 5.48 (curvature and characteristic classes) *Here F is the curvature, δg the characteristic classes of g for the extension along i.*

5.6 Sections and homotopies

One way to think of a section of an ω-bundle is as a morphism from a certain trivial ω-bundle into it. The following formalizes this and then provides reformulations of this notion which are useful later on in section 7.

Definition 5.49 (section) *A section σ of an F-cocycle $\hat{X} \xrightarrow{\nabla} F$ is a directed homotopy from the trivial F-cocycle with fiber pt_F into ∇.*

$$\Gamma(\nabla) := \left\{ \begin{tikzcd} & \hat{X} \arrow[dl] \arrow[dr, "\mathrm{Id}"] & \\ \mathrm{pt} \arrow[rr, Rightarrow, dashed, "\sigma"] \arrow[dr, "\mathrm{pt}_F"'] & & \hat{X} \arrow[dl, "\nabla"] \\ & F & \end{tikzcd} \right\}$$

Proposition 5.50 *If the F-cocycle ∇ is ρ-associated to a B-cocycle $\hat{X} \xrightarrow{g} B$ then sections of ∇ are equivalently lifts of g through $\rho^*\mathbf{E}_{\mathrm{pt}}F \twoheadrightarrow \mathbf{B}G$*

$$\Gamma(\nabla) \simeq \left\{ \begin{tikzcd} & \rho^*\mathbf{E}_{\mathrm{pt}}F \arrow[d, two heads] \\ \hat{X} \arrow[ur, dashed, "\sigma"] \arrow[r, "g"] & \mathbf{B}G \end{tikzcd} \right\} .$$

Proof. First rewrite

$$\left\{ \begin{tikzcd} & \hat{X} \arrow[dl] \arrow[d, "g"] \\ \mathrm{pt} \arrow[r, Rightarrow, "\sigma"'] \arrow[dr, "\mathrm{pt}_F"'] & \mathbf{B}G \arrow[d, "\rho"] \\ & F \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} & \mathrm{pt} \arrow[r, "\mathrm{pt}_F"] & F \\ \hat{X} \arrow[ur] \arrow[rr, dashed, "\sigma"] \arrow[dr, "g"'] & & {[\mathcal{I}, F]} \arrow[u, "d_0"'] \arrow[d, "d_1"'] \\ & \mathbf{B}G \arrow[r, "\rho"] & F \end{tikzcd} \right\}$$

using the characterization of right (directed) homotopies by the (directed) path object $[\mathcal{I}, F]$. Using the universal property of $\mathbf{E}_{\mathrm{pt}}F$ as a pullback this yields

$$\cdots \simeq \left\{ \begin{tikzcd} & & \mathbf{E}_{\mathrm{pt}}F \arrow[d, two heads] \\ \hat{X} \arrow[urr, dashed, "\sigma"] \arrow[r, "g"'] & \mathbf{B}G \arrow[r, "\rho"'] & F \end{tikzcd} \right\} \simeq \left\{ \begin{tikzcd} & \rho^*\mathbf{E}_{\mathrm{pt}}F \arrow[d, two heads] \\ \hat{X} \arrow[ur, dashed, "\sigma"] \arrow[r, "g"] & \mathbf{B}G \end{tikzcd} \right\} .$$

□

A third way to think about sections comes from observing that since a directed homotopy between two cocycles $\begin{tikzcd} \hat{X} \arrow[r, bend left=50, "g_1"{name=U}] \arrow[r, bend right=50, "g_2"'{name=D}] & F \arrow[Rightarrow, from=U, to=D, "\eta"] \end{tikzcd}$ is given by a morphism $\begin{tikzcd} \hat{X} \arrow[r, "\eta"] \arrow[d, two heads, "\simeq"] & {[\mathcal{I}, F]} \\ X \end{tikzcd}$ it can itself be regarded as an $[\mathcal{I}, F]$-cocycle.

Definition 5.51 (universal $F^{\mathcal{I}}$-bundle) *The object $[\mathcal{I}, F]$ is naturally equipped with the point $\mathrm{pt}_{[\mathcal{I},F]}$ defined by*

$$\begin{tikzcd} \mathrm{pt} \arrow[r, "\mathrm{pt}_F"] \arrow[rr, bend right, "\mathrm{pt}_{[\mathcal{I},F]}"'] & F \arrow[r] & {[\mathcal{I}, F]} \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathrm{pt} \arrow[r, "\mathrm{pt}_{[\mathcal{I},F]}"] \arrow[rr, bend right, "\mathrm{pt}_F \times \mathrm{pt}_F"'] & {[\mathcal{I}, F]} \arrow[r, "d_0 \times d_1"] & F \times F \end{tikzcd} .$$

We write $\mathbf{E}_{\mathrm{pt}}([\mathcal{I}, F]) \twoheadrightarrow [\mathcal{I}, F]$ *for the corresponding universal $[\mathcal{I}, F]$-bundle according to definition 5.11.*

Notice the commutativity of the diagram

$$\begin{tikzcd} \mathbf{E}_{\mathrm{pt}}([\mathcal{I}, F]) \arrow[r] \arrow[d] & {[\mathcal{I}, [\mathcal{I}, F]]} \arrow[r, "{[\mathcal{I}, d_i]}"] \arrow[d, "\mathrm{d}_0"] & {[\mathcal{I}, F]} \arrow[d, "d_0"] \\ \mathrm{pt} \arrow[r, "\mathrm{pt}_{[\mathcal{I},F]}"] \arrow[rr, bend right, "\mathrm{pt}_F"] & {[\mathcal{I}, F]} \arrow[r, "d_i"] & F \end{tikzcd}$$

for $i = 0$ and $i = 1$. The right square commutes by the functoriality of $[\mathcal{I}, -]$, the left square and the bottom triangle by definition 5.51 of the universal $[\mathcal{I}, F]$-bundle.

Definition 5.52 *Let* $\begin{tikzcd} \mathbf{E}_{\mathrm{pt}} F & \mathbf{E}_{\mathrm{pt}}([\mathcal{I}, F]) \arrow[l, dashed, "\mathbf{E}_{\mathrm{pt}} d_0"'] \arrow[r, dashed, "\mathbf{E}_{\mathrm{pt}} d_1"] & \mathbf{E}_{\mathrm{pt}} F \end{tikzcd}$ *be the universal morphisms induced from the commutativity of the outermost rectangle of the above diagram in view of the universal property of $\mathbf{E}F$ as a pullback.*

Proposition 5.53 *The morphism $\mathbf{E}_{\mathrm{pt}} d_0 \times \mathbf{E}_{\mathrm{pt}} d_1$ covers the morphism* $[\mathcal{I}, F] \xrightarrow{d_0 \times d_1} F \times F$ *of base spaces:*

$$\begin{tikzcd} \mathbf{E}_{\mathrm{pt}} F \arrow[d, two heads] & \mathbf{E}([\mathcal{I}, F]) \arrow[l, "\mathbf{E} d_0"'] \arrow[r, "\mathbf{E} d_1"] \arrow[d, two heads] & \mathbf{E}_{\mathrm{pt}} F \arrow[d, two heads] \\ F & {[\mathcal{I}, F]} \arrow[l, "d_0"'] \arrow[r, "d_1"] & F \end{tikzcd} .$$

Proof. By inspection of the above commuting diagram. □

This is sometimes called a concordance of bundles.

In total this yields for every homotopy of F-cocycles a span of the corresponding bundles.

Definition 5.54 (associated span of bundles) *For* $\begin{tikzcd} & \hat{X} \arrow[dr, "\nabla_1"] \arrow[dd, Rightarrow] & \\ \hat{Q} \arrow[ur, "\iota"] \arrow[dr, "\iota'"'] & & F \\ & \hat{Y} \arrow[ur, "\nabla_2"'] & \end{tikzcd}$ *a directed homotopy of F-cocycles, we say that the rear of the joint pullback diagram*

$$\begin{tikzcd}
& & & \eta^*\mathbf{E}_{\mathrm{pt}}[\mathcal{I},F] \arrow[dll] \arrow[dr] \arrow[drr] \arrow[ddd] & & & \\
& \nabla_1^*\mathbf{E}_{\mathrm{pt}}F \arrow[dr] \arrow[ddd] & & & \hat{Q} \arrow[dll, "\iota"] \arrow[drr, "\iota'"] \arrow[ddd, "\eta"] & \nabla_2^*\mathbf{E}_{\mathrm{pt}}F \arrow[dr] \arrow[ddd] & \\
& & \hat{X} \arrow[ddd, "\nabla_1"] & & & & \hat{Y} \arrow[ddd, "\nabla_2"] \\
& & & \mathbf{E}_{\mathrm{pt}}[\mathcal{I},F] \arrow[dll] \arrow[dr] \arrow[drr] & & & \\
& \mathbf{E}_{\mathrm{pt}}F \arrow[dr] & & & {[\mathcal{I},F]} \arrow[dll, "d_0"] \arrow[drr, "d_1"] & \mathbf{E}_{\mathrm{pt}}F \arrow[dr] & \\
& & F & & & & F
\end{tikzcd}$$

is the associated span of ω-bundles.

From definition 5.22 notice the following observation:

Lemma 5.55 *In the case that the homotopy in question is a section* $\begin{tikzcd} & \hat{X} \arrow[dl] \arrow[dr, "\mathrm{Id}"] & \\ \mathrm{pt} \arrow[dr, "\mathrm{pt}_F"'] \arrow[rr, phantom, "\overset{\sigma}{\Rightarrow}"] & & \hat{X} \arrow[dl, "\nabla"] \\ & F & \end{tikzcd}$ *(definition 5.49) the associated span of bundles is of the form* $\begin{tikzcd} & \sigma^*\mathbf{E}_{\mathrm{pt}}[\mathcal{I},F] \arrow[dl] \arrow[dr] & \\ \Omega_{\mathrm{pt}}F & & g^*\mathbf{E}_{\mathrm{pt}}F \end{tikzcd}$ *with $\Omega_{\mathrm{pt}}F$ the* monoid of loops *from definition 5.22.*

Remark on groupoidification. In the cased that $\mathcal{F}_0$ is a category of ∞-groupoids, this realizes a section of an associated bundle as an ∞-groupoid over the total space ∞-groupoid of the associated ∞-bundle, and equipped with a map to to the ground ∞-monoid. Since this is the description of vectors in the context of *groupoidification* [1, 2] it motivates the following definition.

Definition 5.56 (generalized sections of associated ω-bundles) *Given an F-bundle $V := \nabla^*\mathbf{E}_{\mathrm{pt}}F$, its generalized sections are spans* $|\Psi\rangle :=$ $\begin{tikzcd} & \Psi \arrow[dl] \arrow[dr] & \\ \Omega_{\mathrm{pt}}F & & V \end{tikzcd}$ *and its generalized co-sections are spans* $\langle\Psi| :=$ $\begin{tikzcd} & \Psi \arrow[dl] \arrow[dr] & \\ V & & \Omega_{\mathrm{pt}}F \end{tikzcd}$ *. We write $\mathcal{H}(V)$ for the collection of all generalized sections of V.*

6 Examples and Applications

6.1 Homotopical contexts

- $\mathcal{V} = \mathsf{SimplicialSets}$: the theory of $\mathrm{Ho}_{[S^{\mathrm{op}},\mathcal{V}]}$ developed in great detail in series of articles by Toën [?].

6.2 Closed monoidal homotopical categories

- Cat with standard tensor and folk model structure
- 2Cat with Gray tensor and Lack model structure
- ωCat with Crans-Gray tensor and folk model structure (need to confirm some axioms of closed monoidal homotopical).
- ωGroupoids with Brown-Golasiski tensor and model structure (need to confirm some axioms of closed monoidal homotopical).

6.3 Pointed objects

- one-object ω-groupoids $\mathrm{pt}! \xrightarrow{\exists} \mathbf{B}G$
- category of vector spaces, point maps to ground field $\mathrm{pt} \xrightarrow{\mathrm{pt} \mapsto k} \mathrm{Vect}_k$
- similarly for higher vector spaces,

6.4 Monoid of loops

1. for $\mathrm{pt} \xrightarrow{\exists!} \mathbf{B}G$ we have $\Omega \mathbf{B}G = G$.
2. for $\mathrm{pt} \xrightarrow{\mathrm{pt} \mapsto k} \mathrm{Vect}_k$ we have $\Omega_{\mathrm{pt}}\mathrm{Vect} = k$,
3. for $\mathrm{pt} \longrightarrow 2\mathrm{Vect}_k$ we have $\Omega_{\mathrm{pt}} 2\mathrm{Vect} = \mathrm{Vect}_k$, etc.

6.5 Universal bundles

1. for $B = \mathbf{B}G$ and G a 1-group the universal B-bundle is $\mathbf{E}G := G//G$ is the action groupoid of G acting on itself and the sequence $G \longrightarrow \mathbf{E}G \longrightarrow \mathbf{B}G$ maps under nerve and topological realization to the universal G-bundle in its incarnation in topological spaces. This is discussed in [30] as a preparation for the following example.
2. for $B = \mathbf{B}G$ and G a 2-group or bi-group [4] shows that $\mathbf{E}G$ is the action bigroupoid of G acting on itself. we show that $\mathbf{E}G$ is the universal 2-bundle (sketched in [30]).

6.6 Bundles

- $\rho : \mathbf{B}G \to \mathrm{Vect}$ a representation, then $\rho^* \mathbf{E}_{\mathrm{pt}}\mathrm{Vect}$ is the action groupoid (ρ-associated vector bundle of $\mathbf{E}G$)
- pullback along G_1-cocycle g weakly equivalent to G-principal bundle;
- pullback along G_2-cocycle g weakly equivalent to G_2-principal 2-bundle (Bartels, Baković, Wockel);
- combined pullback $g^* \circ \rho^*$ weakly equivalent to associated bundle

etc. pp.

6.7 Lifting problems

6.8 Extension problem

- ; $X \hookrightarrow X//G$: ordinary equivariance under G-action; connection homomorphism computes obstruction to having equivariant structure;
- $X \hookrightarrow \Pi(X)$: flat connection, connection homomorphism computes curvature (characteristic forms);

6.9 Equivariance on semi-total spaces

- for $\mathbf{B}U(1) \to \mathrm{AUT}(U(1)) \to \mathbb{Z}_2$ this yields Jandl gerbes [40]
- for $\mathbf{B}U(1) \to \mathrm{String}(G) \to G$ this yields String bundle gerbes on the total space of a G-bundle
- etc.

7 Quantization of nonabelian cocycles to σ-models

We want to think of a ρ-associated F-cocycle $\hat{X} \xrightarrow{\nabla} F$ and the corresponding F-bundle $\nabla^*\mathbf{E}_{\mathrm{pt}}F \twoheadrightarrow X$ as a *background field* (a generalization of an electromagnetic field) on X to which a higher dimensional *fundamental brane* – such as a *particle*, a *string* or a *membrane* – propagating on X may *couple*.

We now propose a formalization in the context of homotopical cohomology of what it means to *quantize* such a background field to obtain the corresponding σ-model quantum field theory as a functorial QFT (as described in [36] and references given there). Our constructions are motivated by and supposed to implement and generalize the considerations of [15, 48] and make contact with [28].

We define a notion of *parameter space* or *worldvolume* category and a notion of *background field* over a *target space* coming from an ω-bundle. This pair of data we call a *σ-model*, We show how this data induces a functor from the parameter space category to spans in ω-groupoids. Using methods from *groupoidification* [1, 2] we show that these spans represent linear maps which deserve to be addressed as *propagation* in the quantum field theory induced by the σ-model.

7.1 σ-Models

In the following $B := \mathbf{B}G$ denotes an object of $\mathcal{F}_0$ which we think of as modelling a one object ∞-groupoid the automorphisms of whose single object form the ∞-group G. At the moment this is just notation meant to be suggestive.

Definition 7.1 (background structure) *A background structure for a σ-model is*

- *an ω-groupoid X called target space;*
- *an ω-group G, called the gauge group;*
- *a representation*
$$\begin{tikzcd} \rho^*\mathbf{E}G \arrow[r] \arrow[d] & \mathbf{E}_{\mathrm{pt}}F \arrow[d, two heads] & \\ \mathbf{B}G \arrow[r, "\rho"] & F & \mathrm{pt} \arrow[l, "\mathrm{pt}_F"'] \end{tikzcd}$$
called the matter content;
- *an F-cocycle*
$$\begin{tikzcd} \hat{X} \arrow[r, "\nabla"] \arrow[d, two heads, "\simeq"] & F \\ X & \end{tikzcd}$$
ρ-associated to a G-principal cocycle on X, $\nabla : \hat{X} \xrightarrow{g} \mathbf{B}G \xrightarrow{\rho} F$,

called the background field.

For brevity we shall indicate a background structure just as ($\hat{X} \xrightarrow{\nabla} F$), leaving the choice of representation and the target space X weakly equivalent to the hypercover $\hat{X}$ implicit.

Definition 7.2 (parameter space category) *A parameter space ω-category* Cob *is a sub ω-category of* $\mathsf{Cospans}(\omega\mathsf{Groupoids})$.

Definition 7.3 (σ-model) *A σ-model is a pair consisting of a parameter space category and a background structure for a σ-model.*

Definition 7.4 *Given a σ-model with background structure* $\nabla : \hat{X} \xrightarrow{\nabla} F$ *and with parameter space* Cob *for every object* $\Sigma \in \mathrm{Cob}$ *we say that*

- $C(\Sigma) := [\Sigma, \hat{X}]$ *is the space of fields over* Σ;
- *The* $[\Sigma, F]$*-cocycle* $[\Sigma, \nabla] : [\Sigma, \hat{X}] \to [\Sigma, F]$ *on the space of fields over* Σ *is the action functional over* Σ.

Remark. One can identify $[\Sigma, \nabla]$ with the *transgression* of the cocycle g to the mapping space $[\Sigma, X]$. Examples showing that this canonical operation indeed reproduces the ordinary notion of transgression of cocycles are in [41, 42] and in our section 8.

In section 7.3 we construct for every σ-model its corresponding quantum field theory. This involves the notion of *bibranes* discussed in section 7.2.

7.2 Branes and bibranes

From the second part of definition 5.49 one sees that spaces spaces of sections of ω-bundles are given by certain morphisms between background fields pulled back to spans/correspondences of target spaces. From the diagrammtics this has an immediate generalization, which leads to the notion of *branes* and *bibranes*.

Definition 7.5 (branes and bibranes) *A brane for a background structure* ($\hat{X} \xrightarrow{\nabla} F$) *is a morphism* $\iota : \hat{Q} \to \hat{X}$ *equipped with a section of the background field pulled back to* $\hat{Q}$*, i.e. a transformation*

$$\begin{tikzcd}
 & \hat{Q} \arrow[dl] \arrow[dr, "\iota"] & \\
\mathrm{pt} \arrow[dr, "\mathrm{pt}_F"'] \arrow[rr, Rightarrow, phantom, "\overset{V}{\Rightarrow}"] & & \hat{X} \arrow[dl, "g"] \\
 & F &
\end{tikzcd}.$$

More generally, given two background structures ($\hat{X} \xrightarrow{\nabla} F$) *and* ($\hat{X}' \xrightarrow{\nabla'} F$)*, a bibrane between them is a span*

$$\begin{tikzcd}
 & \hat{Q} \arrow[dl, "\iota"'] \arrow[dr, "\iota'"] & \\
\hat{X} & & \hat{X}'
\end{tikzcd}$$

equipped with a transformation

$$\begin{tikzcd}
 & \hat{Q} \arrow[dl, "\iota"'] \arrow[dr, "\iota'"] & \\
\hat{X} \arrow[dr, "\nabla"'] \arrow[rr, phantom, "\overset{V}{\Rightarrow}"] & & \hat{X}' \arrow[dl, "\nabla'"] \\
 & F &
\end{tikzcd}.$$

Bibranes may be composed –"fused" – along common background structures ($\hat{X} \xrightarrow{\nabla} F$): the composite or *fusion* of a bibrane V_1 on $\hat{Q}$ with a bibrane V_2 on $\hat{Q}'$ is the bibrane $V_1 \cdot V_2$ given by the diagram

$$\begin{tikzcd}
 & \hat{Q} \times_{\hat{X}'} \hat{Q} \arrow[dl] \arrow[dr] & & & & & \hat{Q} \times_{X'} \hat{Q}' \arrow[dl, "s"'] \arrow[dr, "t"] & & \\
\hat{X}_1 \arrow[rr, Rightarrow, "V_1 \cdot V_2"] \arrow[dr, "\nabla_1"'] & & \hat{X}_3 \arrow[dl, "\nabla_3"] & := & & \hat{Q} \arrow[dl] \arrow[dr] & & \hat{Q}' \arrow[dl] \arrow[dr] & \\
 & F & & & \hat{X}_1 \arrow[rr, Rightarrow, "V"] \arrow[drr, "\nabla_1"'] & & \hat{X}_2 \arrow[rr, Rightarrow, "V'"] \arrow[d, "\nabla_2"] & & \hat{X}_3 \arrow[dll, "\nabla_3"] \\
 & & & & & & F & &
\end{tikzcd}$$

Proposition 7.6 (composition of associated spans from fusion of bibranes) *The associated span of ω-groupoids corresponding, according to definition 5.54, to the fusion of two bibranes is the composition of the spans associated with each bibrane:*

$$\begin{tikzcd}
 & & & & & (\eta_1 \cdot' \eta_2)^* \mathbf{E}_{\mathrm{pt}}(F^{I \sqcup_{\mathrm{pt}} I}) \arrow[dlll] \arrow[d] \arrow[dr] & & & & . \\
 & & \eta_1^* \mathbf{E}_{\mathrm{pt}}(F^{\mathcal{I}}) \arrow[dll] \arrow[ddd] \arrow[dr] \arrow[drr] & & & \hat{Q} \times_{\hat{Y}} \hat{Q}' \arrow[dll, bend right] \arrow[dl] \arrow[drr] \arrow[ddddddd, dashed, "\eta_1 \cdot' \eta_2"] & \eta_2^* \mathbf{E}_{\mathrm{pt}}(F^{\mathcal{I}}) \arrow[ddd] \arrow[dr] \arrow[drr] & & & \\
\nabla_1^* \mathbf{E}_{\mathrm{pt}} F \arrow[dr] \arrow[ddd] & & & \hat{Q} \arrow[dll, "\iota_1"] \arrow[drr, "\iota_2"] \arrow[ddd, "\eta_1"] & \nabla_2^* \mathbf{E}_{\mathrm{pt}} F \arrow[dr] \arrow[ddd] & & & \hat{Q}' \arrow[dll, "\iota_3"'] \arrow[drr, "\iota_4"] \arrow[ddd, "\eta_2"] & \nabla_3^* \mathbf{E}_{\mathrm{pt}} F \arrow[dr] \arrow[ddd] & \\
 & \hat{X} \arrow[ddd, "\nabla_1"] & & & & \hat{Y} \arrow[ddd, "\nabla_2"] & & & & \hat{Z} \arrow[ddd, "\nabla_3"] \\
 & & \mathbf{E}_{\mathrm{pt}}(F^{\mathcal{I}}) \arrow[dll] \arrow[dr] \arrow[drr] & & & & \mathbf{E}_{\mathrm{pt}}(F^{\mathcal{I}}) \arrow[dll] \arrow[dr] \arrow[drr] & & & \\
\mathbf{E}_{\mathrm{pt}} F \arrow[dr] & & & F^{\mathcal{I}} \arrow[dll, "d_0"] \arrow[drr, "d_1"] & \mathbf{E}_{\mathrm{pt}} F \arrow[dr] & & & F^{\mathcal{I}} \arrow[dll, "d_0"'] \arrow[drr, "d_1"] & \mathbf{E}_{\mathrm{pt}} F \arrow[dr] & \\
 & F & & & & F & & & & F \\
 & & & & \mathbf{E}_{\mathrm{pt}}(F^{I \sqcup_{\mathrm{pt}} I}) \arrow[dr] \arrow[uuull] \arrow[uuurr] & & & & & \\
 & & & & & F^{I \sqcup_{\mathrm{pt}} I} \arrow[uuull] \arrow[uuurr] \arrow[d] & & & & \\
 & & & & & F^{\mathcal{I}} & & & &
\end{tikzcd}$$

Proof. By commutativity of pullbacks. □

Remark on groupoidification. Comparing with the remark above definition 5.56 we find that fusion of bibranes corresponds to composition of groupoidified linear maps.

If $\hat{Q}$ carries further structure, the fused bibrane on $\hat{Q} \times_{\hat{Y}} \hat{Q}$ may be pushed down again to $\hat{Q}$.

Definition 7.7 *Let B be a category enriched in the bicategory $\mathcal{V} := \mathsf{Spans}(\omega\mathsf{Categories})$ of spans in $\omega\mathsf{Categories}$ and let F be an ω-category. Then the category of bibranes relative to B and F is given by:*

- *objects are background structures $\hat{X} \xrightarrow{\nabla} F$ for $\hat{X}$ an object of B;*
- *morphisms are bibranes on morphisms of B;*
- *composition of morphisms is given by bibrane fusion followed by push-forward along the composition map in B.*

A simple special case is a category $\hat{Q} \underset{t}{\overset{s}{\rightrightarrows}} \hat{X}$ *internal* to ω-groupoids, equivalently a monad in the bicategory of spans internal to $\omega\mathsf{Groupoids}$, with composition operation the morphism of spans

$$\begin{tikzcd}
 & & \hat{Q} \times_{t,s} \hat{Q} \arrow[dl] \arrow[dr] \arrow[ddd, bend left, "\mathrm{comp}"] & & \\
 & \hat{Q} \arrow[dl, "s"'] \arrow[dr, "t"] & & \hat{Q} \arrow[dl, "s"] \arrow[dr, "t"] & \\
\hat{X} & & \hat{X} & & \hat{X} \\
 & & \hat{Q} \arrow[ull, "s"] \arrow[urr, "t"'] & &
\end{tikzcd} .$$

Definition 7.8 (monoidal structure on bibranes) *Given an internal category as above, and given a background structure $\nabla : \hat{X} \to F$, the composite of two bibranes*

$$\begin{tikzcd}
 & \hat{Q} \arrow[dl] \arrow[dr] & \\
\hat{X} \arrow[dr, "\nabla"'] \arrow[rr, phantom, "\Downarrow V{,}W"] & & \hat{X} \arrow[dl, "\nabla"] \\
 & \hat{F} &
\end{tikzcd}$$

on $\hat{Q}$ is the result of first forming their composite bibrane on on $\hat{Q} \times_{t,s} \hat{Q}$ and then pushing that forward along comp*:*

$$V \star W := \int_{\mathrm{comp}} (s^* V) \cdot (t^* W) \, .$$

Here for finite cases, which we concentrate on, push-forward is taken to be the right adjoint to the pullback in a proper context.

Remarks. Notice that branes are special cases of bibranes and that bibrane composition restricts to an action of bibranes on branes. Also recall that the sections of a cocycle on X are the same as the branes of this cocycle for $\iota = \mathrm{Id}_X$.

The idea of bibranes was first formulated in [17] in the language of modules for bundle gerbes. We show in section 8.1.4 how this is reproduced within the present formulation. In its smooth L_∞-algebraic version the idea also appears in [33].

7.3 Quantum propagation

Every σ-model with parameter space Cob and background structure $\hat{X} \xrightarrow{\nabla} F$ induces a functor

$$\exp(\int \nabla) : \mathrm{Cob} \to \mathrm{Spans}(\omega\mathsf{Groupoids})$$

which sends

$$\begin{tikzcd}[column sep=small]
 & & \Sigma & & & & {[\Sigma,\nabla]^*\mathbf{E}_{\mathrm{pt}}[\Sigma,F]} \arrow[dl] \arrow[dr] & \\
\exp(\int\nabla) : & \Sigma_{\mathrm{in}} \arrow[ur, "\iota"] & & \Sigma_{\mathrm{out}} \arrow[ul, "\tau"'] & \mapsto & {[\Sigma_{\mathrm{in}},\nabla]^*\mathbf{E}_{\mathrm{pt}}[\Sigma_{\mathrm{in}},\nabla]} & & {[\Sigma_{\mathrm{out}},\nabla]^*\mathbf{E}_{\mathrm{pt}}\Sigma_{\mathrm{out}},F]}
\end{tikzcd}$$

a morphism $\Sigma : \Sigma_{\mathrm{in}} \to \Sigma_{\mathrm{out}}$ in Cob to the span of ω-bundles associated to, definition 5.54, the bibrane on the span

$$\begin{tikzcd}
 & {[\Sigma,\hat{X}]} \arrow[dl, "\iota^*"'] \arrow[dr, "\tau_*"] & \\
{[\Sigma,\hat{X}]} & & {[\Sigma,\hat{X}]}
\end{tikzcd}$$

which is induced by transgression of the background field:

$$\begin{tikzcd}[column sep=small]
 & & {[\Sigma,\nabla]^*\mathbf{E}_{\mathrm{pt}}[\Sigma,F]} \arrow[dll] \arrow[ddd] \arrow[dr] \arrow[drr] & & & \\
{[\Sigma_{\mathrm{in}},\nabla]^*\mathbf{E}_{\mathrm{pt}}[\Sigma_{\mathrm{in}},F]} \arrow[ddd] \arrow[dr] & & & {[\Sigma,\hat{X}]} \arrow[dll, "\iota"'] \arrow[ddd, "{[\Sigma,\nabla]}"] \arrow[drr, "\iota'"] & {[\Sigma_{\mathrm{out}},\nabla]^*\mathbf{E}_{\mathrm{pt}}[\Sigma_{\mathrm{out}},F]} \arrow[ddd] \arrow[dr] & \\
 & {[\Sigma_{\mathrm{in}},\hat{X}]} \arrow[ddd, "{[\Sigma_{\mathrm{in}},\nabla]}"] & & & & {[\Sigma_{\mathrm{out}},\hat{X}]} \arrow[ddd, "{[\Sigma_{\mathrm{out}},\nabla]}"] \\
 & & \mathbf{E}_{\mathrm{pt}}[\Sigma,F] \arrow[dll] \arrow[dr] \arrow[drr] & & & \\
\mathbf{E}_{\mathrm{pt}}[\Sigma_{\mathrm{in}},F] \arrow[dr] & & & F^{\Sigma} \arrow[dll, "d_0"] \arrow[drr, "d_1"] & \mathbf{E}_{\mathrm{pt}}[\Sigma_{\mathrm{out}},F] \arrow[dr] & \\
 & {[\Sigma_{\mathrm{in}},F]} & & & & {[\Sigma_{\mathrm{out}},F]}
\end{tikzcd}$$

A state of the σ-model over Σ is a generalized section, definition 5.56 of $[\Sigma_a,\nabla]$ in $\mathcal{H}_a := \mathcal{H}([\Sigma_a,\nabla])$ and the propagation of along a morphism Σ is the map

$$\int_{\hom(\Sigma,X)} \exp(\int\nabla) : \mathcal{H}_{\Sigma_{\mathrm{in}}} \longrightarrow \mathcal{H}_{\Sigma_{\mathrm{out}}}$$

induced by pull-push through the span $\exp(\int\nabla)(\Sigma)$.

An example is spelled out in section 8.1.6.

8 Examples and applications

We start with some simple applications to illustrate the formalism and then exhibit some useful constructions in the context of finite group quantum field theory.

8.1 General examples

8.1.1 Ordinary vector bundles

Let G be an orinary group, hence a 1-group, and denote by $F :=$ Vect the 1-category of vector spaces over some chosen ground field k. A linear representation ρ of G on a vector space V is indeed the same thing as a functor $\rho : \mathbf{B}G \to \mathrm{Vect}$ which sends the single object of $\mathbf{B}G$ to V.

The canonical choice of point $\mathrm{pt}_F : \mathrm{pt} \to \mathrm{Vect}$ is the ground field k, regarded as the canonical 1-dimensional vector space over itself. Using this we find

- from definition **??** that the *ground ω-monoid* in this case is just the ground field itself, $K = k$,
- from definition **??** that the *universal* Vect-*bundle* is $\mathbf{E}_{\mathrm{pt}}\mathrm{Vect} = \mathrm{Vect}_*$, the category of *pointed* vector spaces with $\mathrm{Vect}_* \twoheadrightarrow \mathrm{Vect}$ the canonical forgetful functor;
- from definition **??** that the ρ-associated vector bundle to the universal G-bundle is $V//G \twoheadrightarrow \mathbf{B}G$, where $V//G := \left(V \times G \overset{p_1}{\underset{\rho}{\rightrightarrows}} V \right)$ is the *action groupoid* of G acting on V, the weak quotient of V by G;
- From definition 5.49 that for $g : X \xrightarrow{g} \mathbf{B}G$ a cocycle describing a G-principal bundle and for V the corresponding ρ-associated vector bundle according to definition **??**, that sections $\sigma \in \Gamma(V)$ are precisely sections of V in the ordinary sense.

8.1.2 Group algebras and category algebras from bibrane monoids

In its simplest version the notion of monoidal bibranes from section 7.2 reproduces the notion of *category algebra* $k[C]$ of a category C, hence also that of a *group algebra* $k[G]$ of a group G. Recall that the category algebra $k[C]$ of C is defined to have as underlying vector space the span of C_1, $k[C] = \mathrm{span}_k(C_1)$, where the product is given on generating elements $f, g \in C_1$ by

$$f \cdot g = \begin{cases} g \circ f & \text{if the composite exists} \\ 0 & \text{otherwise} \end{cases}$$

To reproduce this as a monoid of bibranes in the sense of section 7.2, take the category of fibers in the sense of section **??** to be $F = \mathrm{Vect}$ as in section 8.1.1. Consider on the space (set) of objects, C_0, the trivial line bundle given as an F-cocycle by $i : C_0 \longrightarrow \mathrm{pt} \xrightarrow{\mathrm{pt}_k} \mathrm{Vect}$. An element in the monoid of bibranes for this trivial line bundle on the span given by the source and target map

$$\begin{tikzcd} & C_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ C_0 & & C_0 \end{tikzcd}$$

is a transformation of the form

$$\begin{tikzcd} & C_1 \arrow[dl, "s"'] \arrow[dr, "t"] & \\ C_0 \arrow[dr, "i"'] \arrow[rr, Rightarrow, "V"] & & C_0 \arrow[dl, "i"] \\ & \mathrm{Vect} & \end{tikzcd}$$

. In terms of its components this is canonically identified with a function $V : C_1 \to k$ from the space (set) of morphisms to the ground field and every such function gives such a transformation. This identifies the C-bibranes with functions on C_1.

Given two such bibranes V, W, their product as bibranes is, according to definition 7.8, the push-forward along the composition map on C of the function on the space (set) of composable morphisms

$$C_1 \times_{t,s} C_1 \to k$$

$$(\xrightarrow{f}\xrightarrow{g}) \mapsto V(f) \cdot W(g)\,.$$

This push-forward is indeed the product operation on the category algebra.

8.1.3 Monoidal categories of graded vector spaces from bibrane monoids

The straightforward categorification of the discussion of group algebras in section 8.1.2 leads to bibrane monoids equivalent to monoidal categories of graded vector spaces.

Let now $F := 2\mathrm{Vect}$ be a model for the 2-category of 2-vector spaces. For our purposes and for simplicity, it is sufficient to take $F := \mathbf{B}\mathrm{Vect} \hookrightarrow 2\mathrm{Vect}$, the 2-category with a single object, vector spaces as morphims with composition being the tensor product, and linear maps as 2-morphisms. This can be regarded as the full sub-2-category of 2Vect on 1-dimensional 2-vector spaces. And we can assume $\mathbf{B}\mathrm{Vect}$ to be strictified.

Notice from definition **??** that the ground ω-monoid in this case is the monoidal category $K = \mathrm{Vect}$.

Then bibranes over G for the trivial 2-vector bundle on the point, i.e. transformations of the form

$$\begin{tikzcd}
 & G \arrow[dl] \arrow[dr] & \\
\mathrm{pt} \arrow[dr] \arrow[rr, Rightarrow] & & \mathrm{pt} \arrow[dl] \\
 & \mathbf{B}\mathrm{Vect} &
\end{tikzcd}$$

canonically form the category Vect^G of G-graded vector spaces. The fusion of such bibranes reproduces the standard monoidal structure on Vect^G.

8.1.4 Twisted vector bundles

The ordinary notion of a brane in string theory is: for an abelian gerbe $\mathcal{G}$ on target space X a map $\iota : Q \to X$ and a $\mathrm{PU}(n)$-principal bundle on Q whose lifting gerbe for a lift to a $U(n)$-bundle is the pulled back gerbe $\iota^*\mathcal{G}$. Equivalently: a twisted $U(n)$-bundle on Q whose twist is $\iota^*\mathcal{G}$. Equivalently: a gerbe module for $\iota^*\mathcal{G}$.

We show how this is reproduced as a special case of the general notion of branes from definition 7.5, see also [43].

The bundle gerbe on X is given by a cocycle $g : \; X \longrightarrow \mathbf{B}\mathbf{B}U(1)$. The coefficient group has a canonical representation $\rho : \mathbf{B}^2U(1) \to F := \mathbf{B}\mathrm{Vect} \hookrightarrow 2\mathrm{Vect}$ on 2-vector spaces (as in section 8.1.3) given by

$$\rho : \begin{tikzcd}
\bullet \arrow[r, bend left=50, "\mathrm{Id}", ""{name=U, below}] \arrow[r, bend right=50, "\mathrm{Id}"', ""{name=D, above}] & \bullet
\arrow[from=U, to=D, Rightarrow, "c\in U(1)"]
\end{tikzcd}
\;\mapsto\;
\begin{tikzcd}
\bullet \arrow[r, bend left=50, "\mathbb{C}", ""{name=U, below}] \arrow[r, bend right=50, "\mathbb{C}"', ""{name=D, above}] & \bullet
\arrow[from=U, to=D, Rightarrow, "\cdot c"]
\end{tikzcd}\;.$$

See also [43, 36].

By inspection one indeed finds that branes in the sense of diagrams

$$\begin{tikzcd}
 & Q \arrow[dl] \arrow[dr, "\iota"] & \\
\mathrm{pt} \arrow[dr, "\mathrm{pt}_F"'] \arrow[rr, Rightarrow, "V"] & & X \arrow[dl, "\rho\circ g"] \\
 & \mathbf{B}\mathrm{Vect} &
\end{tikzcd}$$

are canonically identified with twisted vector bundles on Q with twist given by the ι^*g: the naturality condition satisfied by the components of V is

$$\begin{tikzcd}
 & \mathbb{C} \arrow[dr, "\mathbb{C}"] \arrow[d, Rightarrow, "\mathrm{Id}"] & \\
\mathbb{C} \arrow[ur, "\mathbb{C}"] \arrow[rr, "\mathbb{C}"] \arrow[dd, "(\pi_1^*E)_y"] & {} & \mathbb{C} \arrow[dd, "\pi_3^*E_y"] \arrow[ddll, Rightarrow, "\pi_{13}^*g_{\mathrm{tw}}(y)"] \\
 & & \\
\mathbb{C} \arrow[rr, "\mathbb{C}"] & & \mathbb{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & \mathbb{C} \arrow[dr, "\mathbb{C}"] \arrow[dd, "(\pi_2^*E)_y"] \arrow[dddl, Rightarrow, "\pi_{12}^*g_{\mathrm{tw}}(y)"] & \\
\mathbb{C} \arrow[ur, "\mathbb{C}"] \arrow[dd, "\pi_1^*E_y"] & & \mathbb{C} \arrow[dd, "(\pi_3^*E_y)"] \arrow[dl, Rightarrow, "\pi_{23}^*g_{\mathrm{tw}}(y)"] \\
 & \mathbb{C} \arrow[dr, "\mathbb{C}"] \arrow[d, Rightarrow, "\cdot g(y)"] & \\
\mathbb{C} \arrow[ur, "\mathbb{C}"] \arrow[rr, "\mathbb{C}"] & {} & \mathbb{C}
\end{tikzcd}\;,$$

for all $y \in Y \times_X Y \times_X Y \times_X Y$ in the triple fiber product of a local-sections admitting map $\pi : Y \to X$ whose simplicial nerve $Y^\bullet$, regarded as an ω-category, provides the cover for the ω-anafunctor $X \overset{\simeq}{\twoheadleftarrow} Y^\bullet \xrightarrow{g} \mathbf{B}^2U(1)$ representing the gerbe. See [43] for details. $E \to Y$ is the vector bundle on the cover encoded by the transformation V. The above naturality diagram says that its transition function g_{tw} satisfies the usual cocycle condition for a bundle only up to the twist given by the gerbe g: if $Y \to X$ is a cover by open subsets $Y = \sqcup_i U_i$, then the above diagram is equivalent to the familiar equation

$$(g_{\mathrm{tw}})_{ij}(g_{\mathrm{tw}})_{jk} = (g_{\mathrm{tw}})_{ik} \cdot g_{ijk} \,.$$

In this functorial cocyclic form twisted bundles on branes were described in [35, 43].

8.1.5 2-Hilbert spaces

Let B be the category internal to spans in ω-categories given by all product spans in Sets with composition morphisms the canonical morphisms. Let $F = \mathbf{B}\mathrm{Vect}$ as before. Then the 2-category of (B, F)-bibranes is the 2-category of 2-Hilbert spaces as in [6].

8.1.6 The path integral

We unwrap the notion of propagation in a σ-model form section 7.3 for the case that the background field is an ordinary vector bundle (with connection), i.e. for the case $F = \mathrm{Vect}$. This can be regarded in terms of the quantization of the charged 1-particle as well as, after transgression, as the top-dimensional propagation in higher dimensional theories. We shall re-encounter this example in the discussion of Dijkgraaf-Witten theory in section 8.2.

Let for the present example the parameter space Cob consist just of a single edge

$$\mathrm{Cob} = \left\{ \begin{tikzcd} & \Sigma := \{a \to b\} & \\ \Sigma_{\mathrm{in}} := \{a\} \arrow[ur] & & \Sigma_{\mathrm{in}} := \{b\} \arrow[ul] \end{tikzcd} \right\} .$$

Recall from section 8.1.1 that for $F = \mathrm{Vect}$ and $\rho : \mathbf{B}G \to \mathrm{Vect}$ a linear representation, we have $\rho^* \mathbf{E}_{\mathrm{pt}} F = V//G$ is the action groupoid of G acting on the representation space V.

Write $\nabla := \rho \circ g$ for the background field. It follows that the ω-bundle over X is given by the groupoid $\nabla^* \mathbf{E}_{\mathrm{pt}} F$ with morphisms

$$(\nabla^* \mathbf{E}_{\mathrm{pt}} F)_1 = \left\{ (x_1, v_1) \xrightarrow{\gamma} (x_2, v_2) \,|(x \xrightarrow{\gamma} y) \in X, v_1, v_2 \in V, v_2 = \rho(g(\gamma)) \right\}$$

with the obvious composition operation.

So a *state* in $\mathcal{H}_{\Sigma_a}$, a groupoid $v : \Psi \to \nabla^* V//G$ over $\nabla^* V//G$, is over each point $x \in X$ a groupoid over V. By the yoga of groupoid cardinality [1, 2] we can hence identify a state $v : \Psi \to \nabla^* V//G$ with a V-valued function on $\mathrm{Obj}(X)$.

The objects of the transgressed background bundle $(\nabla^\Sigma)^* \mathbf{E}_{\mathrm{pt}}(F^\Sigma)$ are the morphisms of $\nabla^* \mathbf{E}_{\mathrm{pt}} F$.

The pull-push propagation map

$$\int_{\hom(\Sigma, X)} \exp(\int \nabla) : \mathcal{H}_a \to \mathcal{H}_b$$

reproduces the path integral in this setup as described in [37].

8.2 Dijkgraaf-Witten model: target space $\mathbf{B}G_1$

Dijkgraaf-Witten theory [16] is the σ-model which in our terms is specified by the data

- target space $X = \mathbf{B}G$, the one-object groupoid corresponding to an ordinary 1-group G;
- background field $\alpha : \mathbf{B}G \to \mathbf{B}^3 U(1)$, a group 3-cocycle on G.

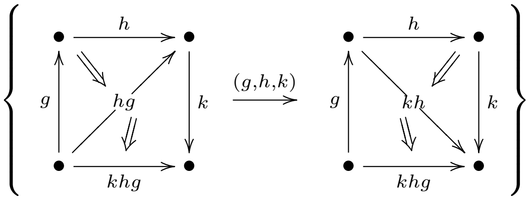
8.2.1 The 3-cocycle

Indeed, we can understand group cocycles precisely as ω-anafunctors $\mathbf{B}G \overset{\simeq}{\twoheadleftarrow} Y \xrightarrow{\alpha} \mathbf{B}^n U(1)$. This is described in [8]. Here it is convenient to take Y to be essentially the free ω-category on the nerve of $\mathbf{B}G$, i.e. $Y := F(N(\mathbf{B}G))$, but with a few formal inverses thrown in to ensure that we have an acyclic fibration to $\mathbf{B}G$:

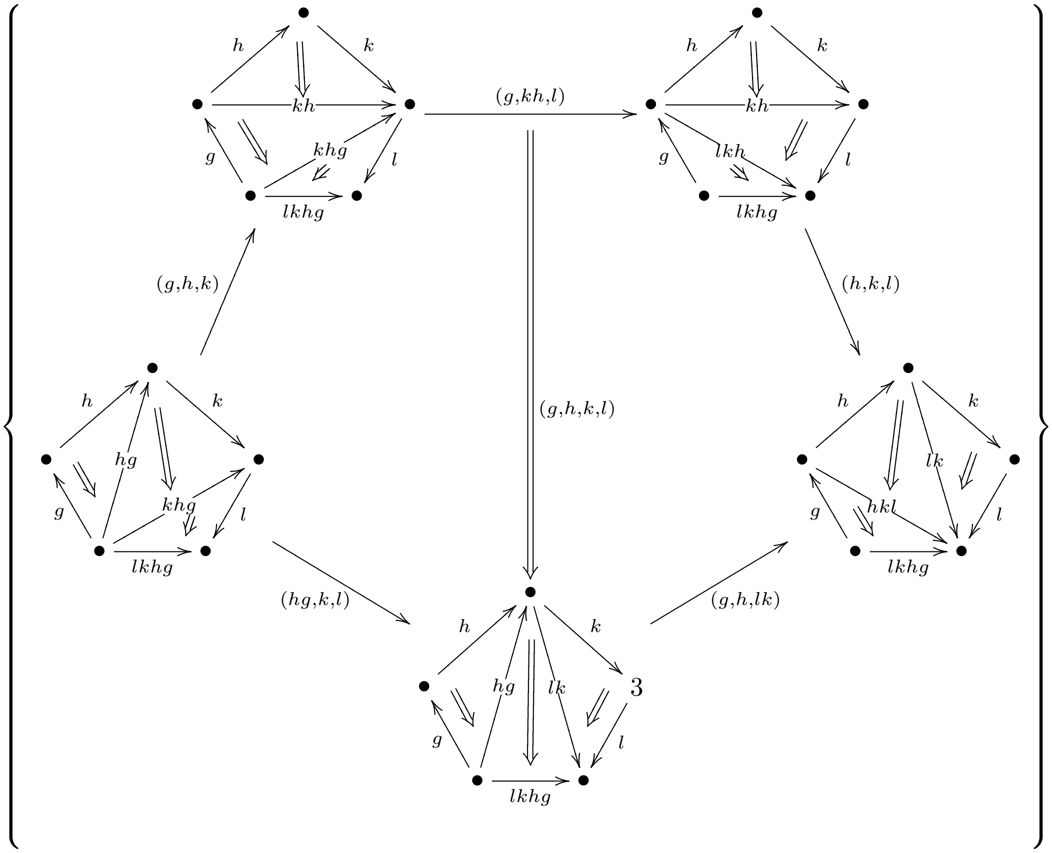
the 1-morphisms of Y are given by finite sequences of elements of G, its 2-morphisms are freely generated as pasting diagrams from 2-morphisms of the form

$$\left\{ \begin{tikzcd} & \bullet \arrow[dr, "h"] & \\ \bullet \arrow[ur, "g"] \arrow[rr, "hg"'] & \Downarrow & \bullet \end{tikzcd} \right\}$$

together with their formal inverses. Its 3-morphisms are freely generated as pasting diagrams from 3-morphisms of the form



together with their formal inverses. Its 4-morphisms are freely generated from pasting diagrams of 4-morphisms of the form



together with their formal inverses.

The ω-functor $\alpha : Y \to \mathbf{B}^3U(1)$ has to send the generating 3-morphisms (g,h,k) to a 3-morphism in $\mathbf{B}^3U(1)$, which is an element $\alpha(g,h,k) \in U(1)$. In addition, it has to map the generating 4-morphisms between pasting diagrams of these 3-morphisms to 4-morphisms in $\mathbf{B}^3U(1)$. Since there are only identity 4-morphisms in $\mathbf{B}^3U(1)$ and since composition of 3-morphisms in $\mathbf{B}^3U(1)$ is just the product in $U(1)$, this says that α has to satisfy the equations

$$\forall g,h,k,l \in G : \ \alpha(g,h,k)\alpha(g,kh,l)\alpha(h,k,l) = \alpha(hg,k,l)\alpha(g,h,lk)$$

in $U(1)$. This identifies the ω-functor α with a group 3-cocycle on G. Conversely, every group 3-cocycle gives rise to such an ω-functor and one can check that coboundaries of group cocycles correspond precisely to transformations between these ω-functors. Notice that α uniquely extends to the additional formal inverses of cells in Y which ensure that $Y \xrightarrow{\simeq} \mathbf{B}G$ is indeed an acyclic fibration. For instance the 3-cell

$$\left\{\begin{tikzcd}
\bullet \arrow[r, "h"] & \bullet \arrow[d, "k"] & {} & \bullet \arrow[r, "h"] \arrow[dr, "kh" description] & \bullet \arrow[d, "k"] \\
\bullet \arrow[u, "g"] \arrow[ur, "hg" description] \arrow[r, "khg"'] & \bullet & {} & \bullet \arrow[u, "g"] \arrow[r, "khg"'] & \bullet
\arrow[from=2-1, to=1-1, phantom, shift left=4ex, "\Uparrow"]
\arrow[from=2-1, to=2-2, phantom, shift left=3ex, "\Uparrow"]
\arrow[from=1-5, to=2-5, phantom, shift left=4ex, "\Uparrow"]
\arrow[from=2-4, to=2-5, phantom, shift left=3ex, "\Uparrow"]
\arrow[from=1-3, to=2-3, phantom, "\xrightarrow{(g,h,k)'}"]
\end{tikzcd}\right\}$$

has to go to $\alpha(g,h,k)^{-1}$.

8.2.2 Chern-Simons theory

In this article we do not want to get into details of the discussion of ω-categories internal to smooth spaces, but in light of the previous section 8.2 it should be noted that in terms of nonabelian cocycles the appearance of Chern-Simons theory is formally essentially the same as that of Dijkgraaf-Witten theory:

if we take BG to be a smooth model of the classifying space of G-principal bundles, then a smooth cocycle $BG \longrightarrow\!\!\!\!\!\mapsto \mathbf{B}^3U(1)$, i.e. an ω-anafunctor internal to (suitably generalized) smooth spaces is precisely the cocycle for a 2-gerbe, i.e. a line 3-bundle. In nonabelian cohomology, the difference between group cocycles and higher bundles is no longer a conceptual difference, but just a matter of choice of target "space" ω-groupoid.

8.2.3 Transgression of DW theory to loop space

Proposition 8.1 *The background field α of Dijkgraaf-Witten theory transgressed according to definition 7.4 to the mapping space of parameter space $\Sigma := \mathbf{B}\mathbb{Z}$ – a combinatorial model of the circle –*

$$\tau_{\mathbf{B}\mathbb{Z}}\alpha := \hom(\mathbf{B}\mathbb{Z}, \alpha)_1 : \Lambda G \to \mathbf{B}^2U(1)$$

is the groupoid 2-cocycle known as the twist of the Drinfeld double, as recalled for instance on the first page of [48]:

$$(\tau_{\mathbf{B}\mathbb{Z}}\alpha) : (\ x \xrightarrow{g} gxg^{-1} \xrightarrow{h} (hg)x(hg)^{-1}\ ) \mapsto \frac{\alpha(x,g,h)\ \alpha(g,h,(hg)x(hg)^{-1})}{\alpha(h,gxg^{-1},g)} .$$

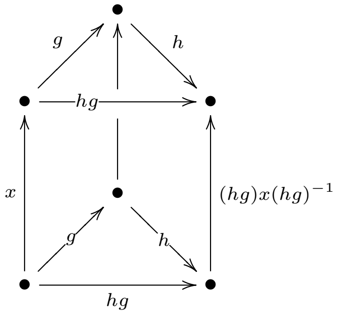
Proof. According to definition A.13 the transgressed functor is obtained on 2-cells as the composition of ω-anafunctors $\mathbf{B}\mathbb{Z} \xrightarrow{(x,g,)} \mathbf{B}G \xrightarrow{\alpha} \mathbf{B}^3U(1)$, given by

$$\begin{tikzcd}
(x,g,h)^*Y \arrow[r] \arrow[d, "\simeq"] & Y \arrow[r, "\alpha"] \arrow[d, "\simeq"] & \mathbf{B}^3U(1) \ , \\
\mathbf{B}\mathbb{Z} \otimes O([2]) \arrow[r, "{(x,g,h)}"] & \mathbf{B}G &
\end{tikzcd}$$

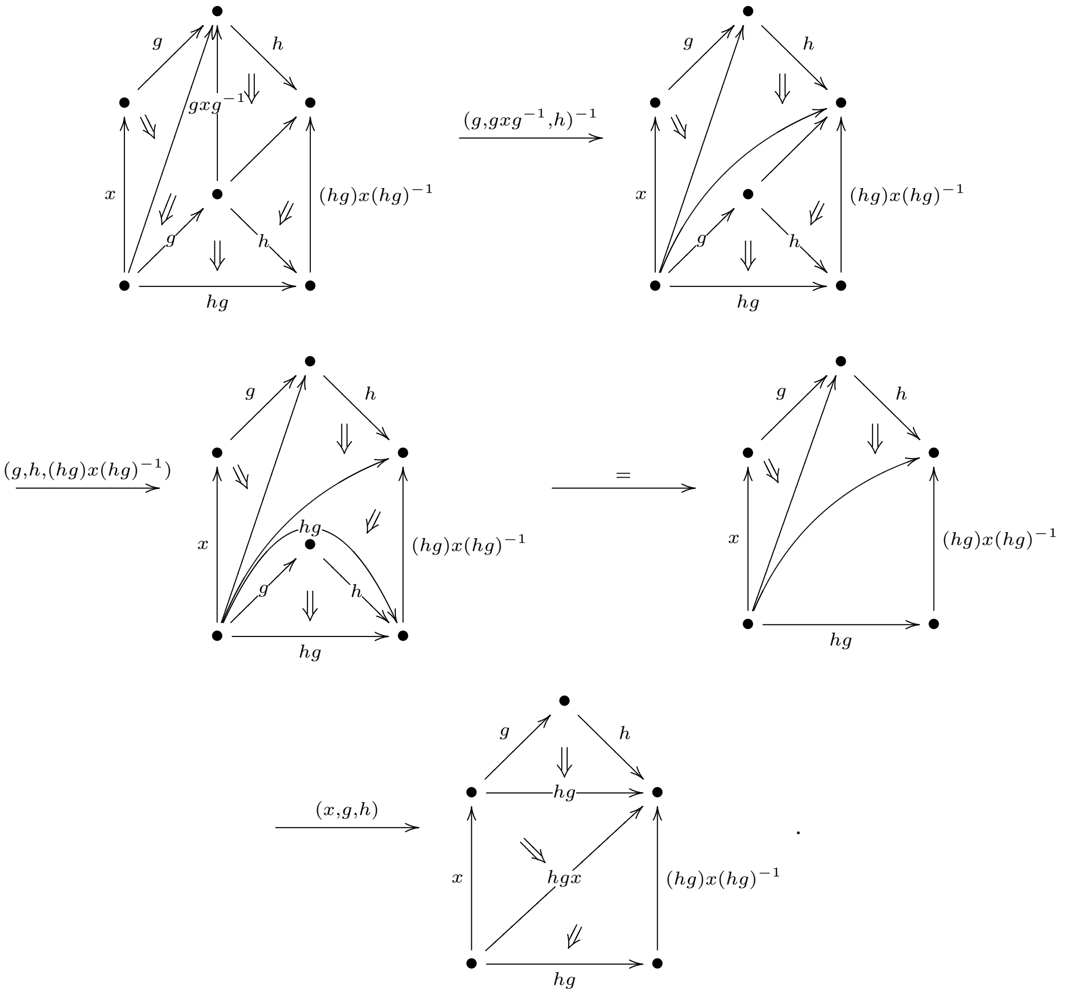
where (x, g, h) denotes a 2-cell in ΛG

$$\begin{tikzcd}
 & gxg^{-1} \arrow[dr, "h"] & \\
x \arrow[ur, "g"] \arrow[rr, "hg"'] & & (hg)x(hg)^{-1}
\end{tikzcd}$$

which comes from a prism



in $\mathbf{B}G$. The 2-cocycle $\tau_{\mathbf{B}\mathbb{Z}}\alpha$ evidently sends this to the evaluation of α on a 3-morphism in the cover Y filling this prism. One representation of such a 3-morphism, going from the back and rear to the top and front of this prism, is



Here the first step follows by 2-dimensional whiskering of the standard 3-morphism:

$$\begin{tikzcd}
\bullet \arrow[rr, "gxg^{-1}"] \arrow[dr, phantom, "\Downarrow"] & & \bullet \arrow[d, "h"] \\
\bullet \arrow[u, "g"] \arrow[urr, "gx"] \arrow[urr, bend left=90, "gx"] \arrow[rr, "hgx"] \arrow[dr, "g"'] \arrow[rr, phantom, shift right=3, "\Downarrow"] & & \bullet \\
& \bullet \arrow[ur, "hgxg^{-1}"'] &
\end{tikzcd}
\xrightarrow{(g,gxg^{-1},h)}
\begin{tikzcd}
\bullet \arrow[rr, "gxg^{-1}"] \arrow[drr, "hgxg^{-1}"] \arrow[dr, phantom, "\Downarrow"] & & \bullet \arrow[d, "h"] \\
\bullet \arrow[u, "g"] \arrow[urr, bend left=90, "gx"] \arrow[rr, "hgx"] \arrow[dr, "g"'] \arrow[rr, phantom, shift right=3, "\Downarrow"] & & \bullet \\
& \bullet \arrow[ur, "hgxg^{-1}"'] &
\end{tikzcd}
.$$

This manifestly yields the cocycle as claimed. ☐

8.2.4 The Drinfeld double modular tensor category from DW bibranes

Let again $\rho : \mathbf{B}^2U(1) \to 2\mathrm{Vect}$ be the representation of $\mathbf{B}U(1)$ from section 8.1.3 and let $\tau_{\mathbf{B}\mathbb{Z}}\alpha : \Lambda G \to \mathbf{B}^2U(1)$ be the 2-cocycle obtained in section 8.2.3 from transgression of a Dijkgraaf-Witten line 3-bundle on $\mathbf{B}G$ and consider the the ρ-associated 2-vector bundle $\rho \circ \tau_{\mathbf{B}\mathbb{Z}}\alpha$ corresponding to that. Its sections according to definition 5.49 form a category $\Gamma(\tau_{\mathbf{B}\mathbb{Z}}\alpha)$.

Corollary 8.2 *The category $\Gamma(\tau_{\mathbf{B}\mathbb{Z}}\alpha)$ is canonically isomorphic to the representation category of the α-twisted Drinfeld double of G.*

Proof. Follows by inspection of our definition of sections applied to this case and using the relation established in 8.2.3 between nonabelian cocycles and the ordinary appearance of the Drinfeld double in the literature. ☐

In the case that α is trivial, the representation category of the twisted Drinfeld double is well known to be a modular tensor category. We now show how the fusion tensor product on this category is reproduced from a monoid of bibranes on ΛG.

Consider any 2-group $\mathbf{B}G_2 := (\, G \ltimes H \underset{(\mathrm{Id}\cdot\delta)}{\overset{p_1}{\rightrightarrows}} G \longrightarrow \mathrm{pt} \,)$.

Pullback to the single object of $\mathbf{BE}\mathbb{Z}$ yields a canonical morphism from the *disk-space* $DG_2 := \hom(\mathbf{BE}\mathbb{Z}, \mathbf{B}G_2)$ to $\mathbf{B}G$, $p : DG_2 \to \mathbf{B}G$ which inherits from the 2-group the structure of a category internal to groupoids in that on the span
$$\begin{tikzcd}
& DG_2 \arrow[dl, "p"'] \arrow[dr, "p"] & \\
\mathbf{B}G & & \mathbf{B}G
\end{tikzcd}$$
there is induced the structure of a monad from the horizontal composition in G_2. Notice that DG_2 is very similar to but in general slightly different from the action groupoid $H//G$ obtained from the canonical action of G on H in a 2-group. Both coinide in the special case that $G_2 = \mathbf{E}G$, so that $H = G$. In this case the morphism p exhibits DG_2 as the action groupoid (as in section 8.1.1) of G acting on itself by the adjoint action.

For $\mathbf{B}G \to 2\mathrm{Vect}$ the trivial gerbe, the transformations
$$\begin{tikzcd}
& DG_2 \arrow[dl] \arrow[dr] & \\
\mathbf{B}G \arrow[rr, Rightarrow] \arrow[dr] & & \mathbf{B}G \arrow[dl] \\
& 2\mathrm{Vect} &
\end{tikzcd}$$
are representations of DG_2 on vector spaces. In the case that $H = G$ and the boundary map is the identity we have $DG_2 = \Lambda G$, so that, by the above, bibranes on DG_2 become representations of ΛG.

One checks in this case that the fusion product of bibranes using the internal category structure on DG_2 according to 7.8 does reproduce the familiar fusion tensor product on representations of ΛG, hence of the Drinfeld double.

8.2.5 The DW path integral

Let S_{in} and S_{out} be two oriented surfaces and let V be an oriented 3-manifold with boundary $\partial V = S_{\mathrm{in}} \sqcup \bar{S}_{\mathrm{out}}$.

Forming fundamental groupoids yields a co-span

$$\begin{tikzcd}
& \Sigma := \Pi_1(V) & . \\
\Sigma_{\mathrm{in}} := \Pi_1(S_{\mathrm{in}}) \arrow[ur] & & \Sigma_{\mathrm{out}} := \Pi_1(S_{\mathrm{out}}) \arrow[ul]
\end{tikzcd}$$

Notice that the space of fields on V in DW theory $\mathrm{hom}(\Sigma, \mathbf{B}G)$ is equivalent to the groupoid of G-principal bundles on V. This implies that pull-push quantum propagation in the sense of section 7.3 reproduces the right DW path integral.

8.3 Yetter-Martins-Porter model: target space $\mathbf{B}G_2$

The Yetter-Martins-Porter model is a σ-model with target space $X = \mathbf{B}G$ for G a 2-group.

Here, too, our quantization reproduces the right combinatorial path integral factor [**?**].

A ω-Categories and their Homotopy Theory

An ∞-category is a combinatorial model for higher directed homotopies, a combinatorial model for a *directed space*. The fact that it is *directed* means that not all cells in this space are necessarily *reversible*. If they are, the ∞-category is an ∞-groupoid, a combinatorial model for an ordinary space.

There are various definitions of ∞-categories and ∞-groupoids [24]. Most of them model ∞-categories as conglomerates of n-dimensional cells of certain shape, for all $n \in \mathbb{N}$, equipped with certain structure and certain properties.

Conglomerates of cells. A "conglomerate of n-dimensional cells of certain shape" technically means a presheaf on a category of basic cells.

Simplicial sets and $(\infty, 1)$-categories. The most familiar example is the simplicial category Δ whose objects are the standard cellular simplices and presheaves on which are simplicial sets. A popular model for ∞-groupoids are simplicial sets with the *Kan property*: *Kan complexes.* The Kan property can be interpreted as ensuring that for all adjacent simplices in the Kan simplicial set there exists a composite simplex and that for all simplices there exists a reverse simplex. Replacing the Kan property on simplicial sets by a slightly weaker property called the *weak Kan property* generalizes Kan complexes to a model of ∞-categories called *weak Kan complexes* or *quasicategories* or *$(\infty, 1)$-categories*: the weak Kan condition ensures just that for all n-simplices for $n \geq 2$ there exists a reverse simplex. A further weakening of the Kan condition such as to ensure only the existence of composites without any restriction on reversibility leads to a definition of weak ∞-categories based on simplicial sets with extra properties proposed by Ross Street []. This is very general but also somewhat unwieldy.

Two other basic shapes of relevance besides simplices are globes and cubes.

Globular sets and ω-categories.

Cubical sets and n-fold categories.

∞-Categories in terms of 1-categories. A general strategy to handle ∞-categories in practice is to regard them as categories (i.e. 1-categories) with extra bells and whistles. This notably involves the tools of *enriched category theory* and of *model category theory.*

Enriched categories. The definition of a category *enriched* over a monoidal category $\mathcal{V}$ [23] is like that of an ordinary category, but with the requirement that there is a *set* of morphisms between any two objects replaced by the requirement that there is an *object of* C for any two objects. If the enriching category $\mathcal{V}$ is a category of higher structures, such as simplicial sets, $\mathcal{V}$-enriched categories are models for ∞-categories. In practice the advantage of conceiving ∞-categories as suitably enriched categories is that enriched category theory is a well-developed subject with a supply of powerful general tools.

Model categories. From the modern perspective, a model category (Quillen model category), is the 1-categorical truncation of an $(\infty, 1)$-category, remembering which of the 1-morphisms retained used to be like isomorphisms, monomorphisms and epimorphisms up to higher coherent cells, in the original $(\infty, 1)$-category: in a model category these special 1-morphisms are, respectively, called *weak equivalences*, *cofibrations* and *fibrations* and satisfy a couple of properties.

One says that model categories are *presentations* of $(\infty, 1)$-categories in that they provide a convenient re-packaging of the information contained in an $(\infty, 1)$-category in purely 1-categorical terms. In practical computation the model category structure on a 1-category is in particular used to generalize morphisms between given objects to morphisms between suitable weakly equivalent *replacements* of these objects.

Our approach. The ∞-vector bundles which we want to describe are given by cocycles with values in ∞-categories (of models for ∞-vector spaces) which are not ∞-groupoids and are not $(\infty, 1)$-categories in that in general they have non-reversible cells in all degrees.

Among the simplicial models for ∞-categories this would force one to use models such as Street's weak ∞-categories. This model, however, we find unwieldy for our applications.

Among the remaining choices of models for ∞-categories for our developments in sections 5.1 and 7 we choose one which combines the "folk" model category structure [14] on $\omega\mathsf{Categories}$ with the enrichment of $\omega\mathsf{Categories}$ over itself [11]. For most considerations in section 7 and 8 this means effectively that we work in the 1-category $\omega\mathsf{Categories}$ while making use of the internal hom-functor and using the freedom to replace ω-categories by weakly equivalent replacements.

For handling $\omega\mathsf{Categories}$ the different shapes – globes, simplices, cubes – are useful for different purposes. Globular sets have the simplest boundary structure, simplicial sets provide powerful computational tools, cubical sets provide the important monoidal structure. In the following all three models of shapes are combined: following [8, 11] we conceive ω-categories as globular sets for general purposes and make use of their incarnation as cubical sets for describing their biclosed monoidal structure. Moreover, following [44, 8] we use cosimplicial ω-categories such as the orientals to pass between $\omega\mathsf{Categories}$ and $\mathsf{SimplicialSets}$, mostly for the purpose of constructing weakly equivalent replacements of ω-categories.

A.1 Shapes for ∞-cells

Three types of basic shapes are used frequently: globes, simplices and cubes. These are modeled, respectively, by the $\underline{\text{globular category}}$ G, the $\underline{\text{simplicial category}}$ Δ and the $\underline{\text{cubical category}}$ C. These three categories have as objects the integers, $n \in \mathbb{N}$, thought of as the standard cellular n-globe G^n, the standard cellular n-simplex Δ^n and the standard cellular n-cube C^n, respectively. Morphisms are all maps between these standard cellular shapes which respect the cellular structure.

Definition A.1 (globular category) *The* $\underline{\textit{globular category}}$ G *is the category whose objects are the integers* $\mathbb{N}$ *and whose morphisms are generated from morphisms*

$$\sigma_n, \tau_n : [n] \to [n+1]$$

subject to the relations

$$\begin{tikzcd}
{[n]} \arrow[r, "\sigma_n"] \arrow[d, "\sigma_n"'] & {[n+1]} \arrow[d, "\sigma_n"] & , & {[n]} \arrow[r, "\tau_n"] \arrow[d, "\tau_n"'] & {[n+1]} \arrow[d, "\sigma_n"] \\
{[n+1]} \arrow[r, "\tau_n"'] & {[n+2]} & & {[n+1]} \arrow[r, "\tau_n"'] & {[n+2]}
\end{tikzcd}$$

for all $n \in \mathbb{N}$.

Definition A.2 (simplicial category) *The simplicial category Δ is the full subcategory of* Categories *on categories which are freely generated from connected linear graphs. Equivalently, Δ is the category with totally ordered finite sets as objects and order-preserving maps as morphisms.*

Definition A.3 (cubical category) *The cubical category C is defined ... section 2 of [11]*

Definition A.4 (monoidal structure on the cubical category) *section 2 of [11]*

A.2 ω-Categories

Recall the following standard facts:

- The category Sets is symmetric monoidal with respect to the standard cartesian product.
- For $\mathcal{V}$ a symmetric monoidal category, the category $\mathcal{V}-\mathrm{Cat}$ of $\mathcal{V}$-enriched categories is naturally itself symmetric monoidal.

Definition A.5 (strict globular n-category, [13]) *The category of 0-categories is* $0\mathsf{Categories} := \mathsf{Sets}$*. For $n \in \mathbb{N}$, $n \geq 1$ the category of ("strict, globular") n-categories is defined inductively as the category*

$$n\mathsf{Categories} := (n-1)\mathsf{Categories}-\mathrm{Cat}$$

of categories enriched over $(n-1)\mathsf{Categories}$.

One notices that for all $n \in \mathbb{N}$ there is a canonical inclusion $n\mathsf{Categories} \hookrightarrow (n+1)\mathsf{Categories}$.

Definition A.6 (ω-category, [45]) *The category of ω-categories is the direct limit over this chain of inclusions*

$$\omega\mathsf{Categories} := \lim_{\rightarrow n\in\mathbb{N}} n\mathsf{Categories}\,.$$

Unwrapping this definition shows that ω-categories are globular sets equipped with compatible structures of a strict 2-category on all sub-globular sets of length two:

Definition A.7 (globular set) *A globular set S is a presheaf on the globular category G, i.e. a functor $S : G^{\mathrm{op}} \to \mathsf{Sets}$.*

We write $S(\ [n] \xrightarrow{\sigma_n,\tau_n} [n+1]\ ) := \ S_{n+1} \xrightarrow{s_n,t_n} S_n$ and call S_n the set of n-globes, s_n the n-source map and t_n the n-target map of S. The identities $s_n \circ s_{n+1} = s_n \circ t_{n+1}$ and $t_n \circ s_{n+1} = t_n \circ t_{n+1}$, called the globular identities, ensure that for all $n, k \in \mathbb{N}$ there are unique maps $S_{n+k} \xrightarrow{s,t} S_n$ themselves satisfying analogous globular identities.

Proposition A.8 (ω-category, [44]) *An ω-category C is a globular set $C : G^{\mathrm{op}} \to \mathsf{Sets}$ equipped for all $n, k \in \mathbb{N}$ the structure of a category extending $C_{n+k} \underset{t}{\overset{s}{\rightrightarrows}} C_n$ such that this makes for all $n, k, l \in \mathbb{N}$ $C_{n+k+l} \underset{t}{\overset{s}{\rightrightarrows}} C_{n+k} \underset{t}{\overset{s}{\rightrightarrows}} C_n$ into a strict 2-category.*

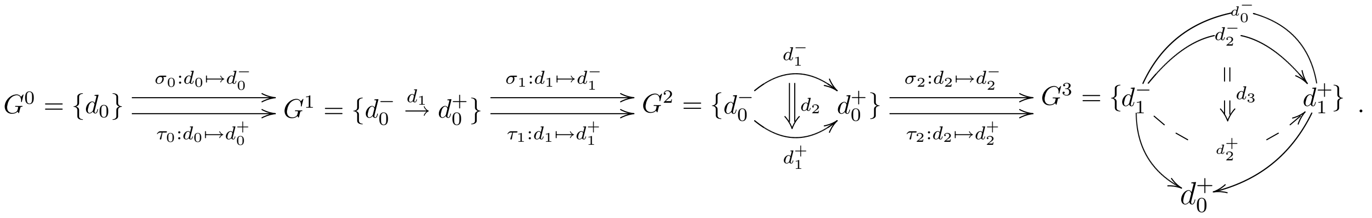
The elements in C_k are called <u>k-morphisms</u>. The composition in $C_{n+k} \overset{s}{\underset{t}{\rightrightarrows}} C_n$ is called composition of $n+k$-morphisms along n-morphisms. A morphism between ω-categories, called an ω-functor, is a morphism of the underlying globular sets respecting all the additional structure.

Definition A.9 (standard globular globes) *The globular set G_n represented by $n \in \mathbb{N}$, $G_n := \mathrm{Hom}_G(-, [n])$ is the standard globular globe. There is a unique structure of an ω-category on G_n. This yields co-globular ω-category $G_\bullet$, i.e. a functor $G^\bullet : G \to \omega\mathsf{Categories}$.*

We also write

- $\emptyset =: G^{-1} =: \mathcal{I}^{-1}$ for the ω-category on the empty globular set (the initial object in $\omega\mathsf{Categories}$);
- $\mathrm{pt} =: I =: \mathcal{I}^0 := G^0 = \{\bullet\}$ for the ω-category with a single object and no nontrivial morphisms (the terminal object in $\omega\mathsf{Categories}$ and the tensor unit with respect to the Crans-Gray tensor product $\otimes$ described below);
- $\mathcal{I} =: \mathcal{I}^1 := G^1 = \{\, a \longrightarrow b \,\}$ for the ω-category with two objects and a single nontrivial morphism connecting them.

The first few n-globes can be depicted as follows:



A.3 ω-Groupoids

...ω-groupoids and crossed complexes...

A.4 Cosimplicial ω-categories

We can translate back and forth between simplicial sets and ω-categories by means of a fixed cosimplicial ω-categeory, i.e. a functor $O : \Delta \to \omega\mathsf{Categories}$ from the simplicial category Δ: from any such we obtain an ω-nerve functor $N : \omega\mathsf{Categories} \to \mathsf{SimplicialSets}$ by

$$N(C) : \Delta^{\mathrm{op}} \xrightarrow{O^{\mathrm{op}}} \omega\mathsf{Categories}^{\mathrm{op}} \xrightarrow{\mathrm{Hom}(-,C)} \mathsf{Sets}$$

and its left adjoint $F : \mathsf{SimplicialSets} \to \omega\mathsf{Categories}$ given by the coend formula

$$F(S^\bullet) := \int^{[n]\in\Delta} S^n \cdot O([n])\,.$$

Ross Street defined such a cosimplicial ω-category called the orientals [44], for which $O([n])$ is the ω-category free on a single n-morphism of the shape of an n-simplex. To obtain more inverses, we can alternatively use the unorientals, for which $O([n])$ is the ω-category with n-objects, with 1-morphisms finite sequences of these objects, 2-morphisms finite sequences of such finite sequences, and so on.

A.5 Monoidal biclosed structure on ωCategories

The category ωCategories is is equipped with the Crans-Gray tensor product [11], which is the extension to ω-categories of the tensor product on cubical sets which in turn is induced via Day convolution from the canonical tensor product on the cube category, which finally comes from addition of natural numbers. This means that the Crans-Gray tensor product is dimension raising in a way analogous to the cartesian product on topological spaces:

for instance the tensor product of the interval ω-category $I = \{ a \longrightarrow b \}$ with itself is the ω-category free on a single directed square

$$I \otimes I = \left\{ \begin{tikzcd} (a,a) \arrow[r] \arrow[d] & (a,b) \arrow[d] \arrow[dl, Rightarrow] \\ (b,a) \arrow[r] & (b,b) \end{tikzcd} \right\} .$$

Moreover, ωCategories is biclosed with respect to this monoidal structure.

Definition A.10 (internal hom) *For ω-categories C and D the ω-category $[C,D]$ is given by the globular set* $\mathrm{Hom}(G_{[-]} \otimes C, D) : G^{\mathrm{op}} \to \mathsf{Sets}$ *on which the composition of k-morphisms along an l-morphism is defined as the image of the diagram which glues two standard k-globes along a common l-globe*

$$\begin{tikzcd}
 & & G_k \sqcup_{G_l} G_k & & \\
 & G_k \arrow[ur, "\iota"] & & G_k \arrow[ul, "\tau"'] & \\
G_l \arrow[ur, "\sigma"] \arrow[drr, "\sigma"'] & & G_l \arrow[ul, "\tau"'] \arrow[ur, "\sigma"] & & G_l \arrow[ul, "\tau"'] \arrow[dll, "\tau"] \\
 & & G_k \arrow[uuu, bend right, "c_{k,l}"'] & &
\end{tikzcd}$$

under $\mathrm{Hom}(G_{[-]} \otimes C, D)$*:*

$$\begin{tikzcd}
 & & \mathrm{Hom}((G_k \sqcup_{G_l} G_k) \otimes C, D) \arrow[dl, "\iota^*"'] \arrow[dr, "\tau^*"] \arrow[ddd, bend left, "\mathrm{comp}:=c^*_{k,l}"] & & \\
 & \mathrm{Hom}(G_k \otimes C, D) \arrow[dl, "s"'] \arrow[dr, "t"] & & \mathrm{Hom}(G_k \otimes C, D) \arrow[dl, "s"'] \arrow[dr, "t"] & \\
\mathrm{Hom}(G_l \otimes C, D) & & \mathrm{Hom}(G_l \otimes C, D) & & \mathrm{Hom}(G_l \otimes C, D) \\
 & & \mathrm{Hom}(G_k \otimes C, D) \arrow[ull, "s"] \arrow[urr, "t"'] & &
\end{tikzcd}$$

Remarks. Notice that everything in this definition works by abstract nonsense – for instance that the contravariant Hom takes colimits to limits – except the existence of the maps $c_{k,l}$, which encodes genuine information about pasting of standard globes [10]. For instance $G_2 \sqcup_{G_0} G_2 =$

$$\left\{ \begin{tikzcd} a \arrow[r, bend left=50, ""{name=U1, below}] \arrow[r, bend right=50, ""{name=D1}] & b \arrow[r, bend left=50, ""{name=U2, below}] \arrow[r, bend right=50, ""{name=D2}] & c \arrow[Rightarrow, from=U1, to=D1] \arrow[Rightarrow, from=U2, to=D2] \end{tikzcd} \right\}$$

while $G_2 \sqcup_{G_1} G_2 = \left\{ \begin{tikzcd} a \arrow[rr, bend left=50] \arrow[rr] \arrow[rr, bend right=50] & \Downarrow \\[-2em] & \Downarrow & b \end{tikzcd} \right\}$, where the right sides denote the free ω-categories on the indicated pasting diagram [10].

Proposition A.11 *For every* $C \in \omega\mathsf{Categories}$ *this extends to a functor* $[C, -] : \omega\mathsf{Categories} \to \omega\mathsf{Categories}$ *which is right adjoint to* $- \otimes C : \omega\mathsf{Categories} \to \omega\mathsf{Categories}$.

Of particular interest to us are the internal hom-ω-categories of the form $A^{\mathcal{I}} := [\mathcal{I}, A]$ which satisfy $\mathrm{Hom}(I \otimes X, A) \simeq \mathrm{Hom}(X, A^{\mathcal{I}})$, where the set in question here is the set of lax transformations

$$\begin{tikzcd} X \arrow[rr, bend left=50, "f"{name=U}] \arrow[rr, bend right=50, "g"'{name=D}] & & A \arrow[from=U, to=D, Rightarrow, "\eta"] \end{tikzcd} \Leftrightarrow$$

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[rr, bend left=40, "f \times g"] & A^{\mathcal{I}} \arrow[r, "d_0 \times d_1"] & A \times A \end{tikzcd}$$

or *directed right homotopies* between ω-functors from X to A.

A.6 Model structure on $\omega\mathsf{Categories}$

That the 1-category $\omega\mathsf{Categories}$ is really an ∞-structure itself is remembered by a *model category structure* carried by it, due to [14], with respect to which the acyclic fibrations or hypercovers $f : C \xrightarrow{\simeq} D$ are those ω-functors which are k-surjective for all $k \in \mathbb{N}$, meaning that the universal dashed morphism in

$$\begin{tikzcd} C_{k+1} \arrow[dr, dashed] \arrow[drr, bend left=20, "f_{k+1}"] \arrow[ddr, bend right=20, "s \times t"'] & & \\ & (f_k \times f_k)^* C_{k+1} \arrow[r] \arrow[d] & D_{k+1} \arrow[d, "s \times t"] \\ & C_k \times C_k \arrow[r, "f_k \times f_k"'] & D_k \times D_k \end{tikzcd}$$

is epi, for all k. The weak equivalences $f : C \xrightarrow{\simeq} D$ are those ω-functors where these dashed morphisms become epi after projecting onto ω-equivalence classes of $(k+1)$-morphisms.

Using this we define an ω-anafunctor from an ω-category X to an ω-category A to be a span

$$(g : X \longrightarrow\!\!\!\!\!\mapsto A) := \begin{tikzcd} \hat{X} \arrow[r, "g"] \arrow[d, two heads, "\simeq"] & A \\ X & \end{tikzcd}$$

whose left leg is a hypercover. (This terminology follows [26, 5].) One finds [7] that in the context of $\omega\mathsf{Groupoids}$ such ω-anafunctors represent morphisms in the homotopy category $[g] \in \mathrm{Ho}(X, A)$ which allows us to regard g as a cocycle in nonabelian cohomology on the ω-groupoid X with coefficients in the ω-groupoid A. Cocycles are regarded as distinct only up to refinements of their covers. This makes their composition by pullbacks

$$\begin{tikzcd} & & & g^*\hat{A} \arrow[r] \arrow[d, two heads, "\simeq"] & \hat{A} \arrow[r, "r"] \arrow[d, two heads, "\simeq"] & A' \\ (X \arrow[r, "g", "|" marking] & A \arrow[r, "r", "|" marking] & A') := & \hat{X} \arrow[r, "g"] \arrow[d, two heads, "\simeq"] & A & \\ & & & X & & \end{tikzcd}$$

well defined (noticing that acyclic fibrations are closed under pullback) and associative.

Definition A.12 *We write* **Ho** *for the corresponding category of ω-anafunctors,*

$$\mathbf{Ho}(C,D) := \mathrm{colim}_{\hat{C}\in \mathsf{Hypercovers}(C)} \mathrm{Hom}(\hat{C}, D)\,.$$

(This is to be contrasted with the true homotopy category Ho, which is obtained by further dividing out homotopies.)

While cocycles in nonabelin cohomology are morphisms in **Ho**, coboundaries should be morphisms between these morphisms. Hence **Ho** is to be thought of as enriched over $\omega\mathsf{Categories}$.

Definition A.13 *Define a functor* $\mathrm{hom} : \mathbf{Ho}^{\mathrm{op}} \times \mathbf{Ho} \to \omega\mathsf{Categories}$ *by* $\mathrm{hom}(C,D) := F(\mathrm{Hom}(C \otimes O([\bullet]), D)))\,.$

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