Hisham Sati[1] *and* *Urs Schreiber*[2]

Geometric Orbifold Cohomology

[1]Mathematics, Division of Science, and Center for Quantum and Topological Systems (CQTS), NYUAD Research Institute, New York University Abu Dhabi, UAE, `hsati@nyu.edu`

[2]Mathematics, Division of Science, and Center for Quantum and Topological Systems (CQTS), NYUAD Research Institute, New York University Abu Dhabi, UAE, `us13@nyu.edu`

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Foreword

The mathematical theory presented here — geometric cohomology formulated modally in cohesive ∞-toposes — goes back to [SSS12][Sc13] and has seen a number contributions since (a recent expository digest may be found in [Sc25]), but a published monograph had been missing.

The nucleation seed of this book were developments in 2019, when we started finding evidence [FSS20][FSS21] for the hypothesis ("Hypothesis H") that something called *flux-quantization of 11D super-gravity* (reviewed in [SS25a]) on spacetime manifolds should take place in the generalized cohomology theory named *tangentially twisted unstable 4-Cohomotopy*. But we also found [SS20] that in the neighbourhood of orbifold-singularities the relevant cohomology theory should really be *proper equivariant unstable 4-Cohomotopy in the RO-degree given by the singularity's isotropy representation*. The proof that the latter is indeed a special case of the former, when the notion of "tangential twist" is generalized from manifolds to orbifolds, became the seed text (now Thm. 6.2.6, see Fig. 1.7 on p. 21) that eventually grew into this book.

However, while a first draft of the book circulated — which (as referees rightly remarked) did not dwell on more traditional orbifold cohomology theories such as notably K-theory — we became absorbed with first discovering and then developing the relation of Hypothesis H to quantum materials relevant for topological quantum computing – this via orbifold K-theory [SS23a][SS23b] whose natural dicussion via cohesion in ∞-toposes meanwhile turned into a monograph of its own [SS25d].

Concretely, Hypothesis H has led (applied to *M5-brane probes of orbifolds* [SS25b][SS25c]) to an understanding [SS25e][SS25g] of symmetry-protected anyonic topological order in quantum materials known as *crystalline fractional Chern insulators*. Connecting this back to the now traditional K-theory classification of topological phases of matter requires analyzing the Boardman homomorphism from unstable Cohomotopy to K-theory, but generalized to an operation between orbifold cohomology theories twisted by the corresponding crystallographic point group. Establishing this is now the content of §xy below.

The eventual ambition of these applications — to resolve beyond the equivariant topology also the more fine-grained geometric (Riemannian, conformal, supersymmetric, ...) aspects of orbifolds and their differential cohomology — motivates the detailed development of modal orbifold *geometry* in §5. The eponymous paradigm of this is book is that (differential, Cartan) geometry formulated "synthetically" via modalities in cohesive ∞-toposes *automatically* generalizes from manifolds to orbifolds (and further to étale ∞-stacks) and thereby provides a powerful and conceptually neat unification of orbifold geometry and orbifold cohomology with classical differential geometry and classical differential cohomology.

Preface

Both the geometry and the cohomology of orbifolds have received fair attention, but less so the systematic combination of these two aspects. This book provides a unified framework of geometric orbifold cohomology, naturally based on the principle of *cohesion* in higher toposes, and develops key examples.

Although generally abstract in itself, the topic has strong motivation from contemporary (experimental) physics: First, configuration spaces $\mathbf{X}$ of matter systems are (super-geometric) *manifolds* by default and generically *orbifolds* when subject to symmetries, a key example being the spaces of quasi-momenta of electrons in a crystal. On this backdrop, the (effective) force fields (like Berry connections) in their full *global* (non-perturbative, solitonic) guise are represented by maps $\mathbf{X} \xrightarrow{\Phi} \mathbf{A}$ to more general *classifying spaces* (or rather *moduli stacks*) of which only their gauge equivalence classes $[\Phi]$ are physically discernible; these classes constitute the geometric $\mathbf{A}$-*cohomology* of the orbifold $\mathbf{X}$!

From this perspective, geometric orbifold cohomology is about the very foundations of physical systems *including* their oft-neglected global "topological" aspects, and the book concludes with outlook on application to contemporary questions in the study of both quantum materials and high-energy physics.

In pure mathematics, this principle of (functorially) assigning cohomology classes $[\Phi]$ to their base spaces $\mathbf{X}$ is of course the hallmark of algebraic topology, which in conjunction with homotopy theory is understood as systems of homotopy classes of maps $\mathbf{X} \xrightarrow{\Phi} \mathbf{A}$ — and this book emphasizes our perspective that all manner of generalized and adjective-laden cohomology theories (hyper, sheaf, étale, extraordinary, twisted, equivariant, differential,...) are uniformly to be understood as *represented* by objects $\mathbf{A}$ of suitable (cohesive) higher toposes. As a prime example the book lays out a useful model of orbifold K-theory this way, relates it to the orbifold *Cohomotopy* and indicates applications to the physics of topological quantum materials.

The technical task then is to give precise meaning and tractable operability to the symbols $\big[\mathbf{X} \xrightarrow{\Phi} \mathbf{A}\big]$ in view of the joint geometric, orbifolded and classifying nature of the spaces involved, and this book offers a neat and natural way to do so, tying together separate traditional discussions to provide a unified and practical basis for discussing geometric orbifold cohomology.

1

Introduction

1.1 Motivation

1.1.1 Orbifolds

Where a *manifold* is a space that looks locally like a Cartesian space $\mathbb{R}^n$ (cf. [Lee12]), so an *orbifold* ([Sa56][Sa57][Th80][Hae84], review in [MM03][Ka08, §6][BG08, §4][IKZ10]) is, more generally, a space that looks locally like the quotient (suitably understood) of an $\mathbb{R}^n$ by the action of a finite group G of diffeomorphisms. Here the G-action may have fixed-points which in the quotient become *singular* points, such as a crease in a piece of paper or the tip of a cone, cf. Fig. 1.1:

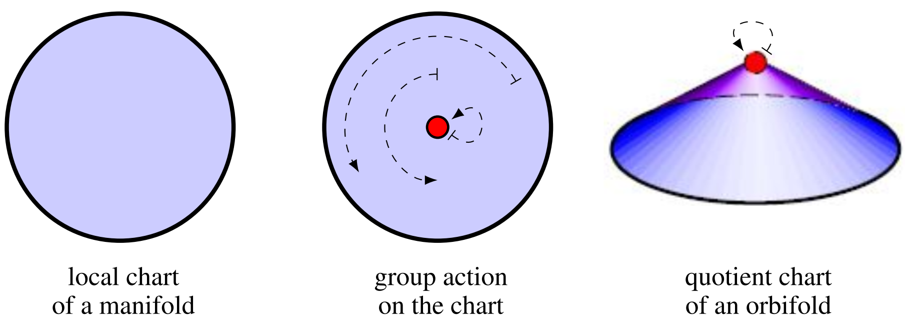


Figure 1.1 – Orbifold charts.

The purpose of orbifold structure is to generalize the differentiable structure of manifolds to allow for singular loci of such form: *orbi-singularities*.

A basic class of examples is provided by the orbifold quotients $\mathbb{T}^n /\!\!/ \mathbb{Z}_2$ of the n-torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ by the $\mathbb{Z}_2$-action that swaps the signs of the canonical coordinates:

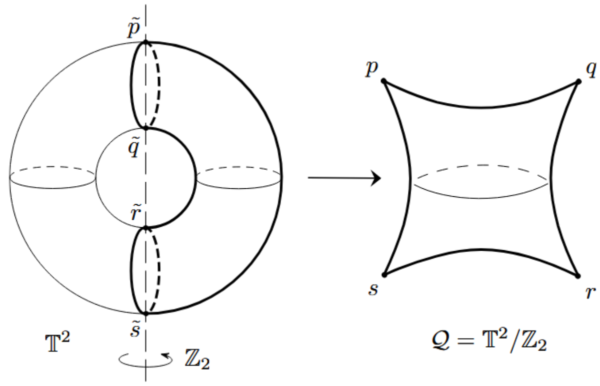


Figure 1.2 – The pillowcase orbifold $\mathbb{T}^2 \mathbin{/\!\!/} \mathbb{Z}_2$ (graphics from [Dr11, Fig. 2.6]).

- $\mathbb{T}^0 \mathbin{/\!\!/} \mathbb{Z}_2$ is – in a sense which we will discuss in detail – the *classifying* or *delooping* groupoid $\mathbf{B}\mathbb{Z}_2$ (1.14): A single point but equipped with a non-trivial involution.
- $\mathbb{T}^1 \mathbin{/\!\!/} \mathbb{Z}_2$ plays a central role in Hořava-Witten theory [HW96].
- $\mathbb{T}^2 \mathbin{/\!\!/} \mathbb{Z}_2$ is known as the *pillowcase orbifold* (cf. Fig. 1.2);
- $\mathbb{T}^3 \mathbin{/\!\!/} \mathbb{Z}_2$ does not have a special name but appears as the space of effective crystal momenta for quantum materials with time-reversal symmetry (cf. §xy).
- $\mathbb{T}^4 \mathbin{/\!\!/} \mathbb{Z}_2$ is the time-honored *Kummer surface* (cf. [Do20] and Fig. 1.3).
- $\mathbb{T}^5 \mathbin{/\!\!/} \mathbb{Z}_2$ is the background of *MO5-planes M-theory* (cf. [Wi96][SS20]).

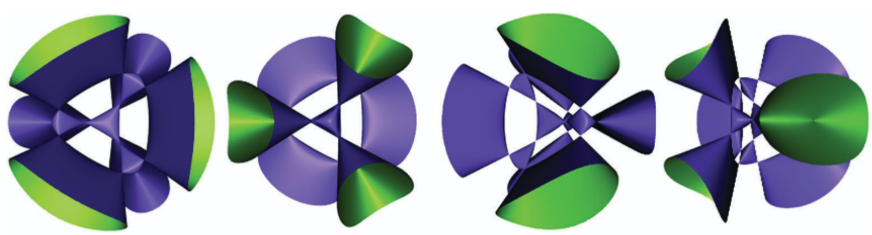


Figure 1.3 – The Kummer orbifold $\mathbb{T}^4 \mathbin{/\!\!/} \mathbb{Z}_2$. Shown is a sequence of projections to 2d of the 4-dimensional structure (graphics from [Do20]).

Thus, orbifolds have become commonplace in mathematics (e.g. [BLP05][Rat06, §13][JY11]), and play a decisive role in theoretical physics (see [AMR02]), notably so in string/M theory ([DHVW85][DHVW86][BL99] [SS20]) and in solid state physics ([JBD96][Jo99][GT19][SS23b][SS22b]) — in fact it is via "geometric engineering" of quantum field theories [KKV97] (cf. [Ka98, §1.4.1][DZ23]), on the branes of string/M-theory, probing orbi-singularities, that modern descriptions [SS25e] of strongly-correlated quantum materials arise [SS25b][SS25c]. This book is to provide rigorous but practical mathematical framework and tools notably for such applications.

This is not, a priori, immediate: Definition of the *geometric homotopy theory* (cf.

§I) thus of the *geometric cohomology* (§II) of orbifolds may appear subtle and elusive, as witnessed by the convoluted history of the concept (cf. [Le08, Intro.][IKZ10, §1]). In fact, the issue had remained somewhat open, as we proceed to recall:

1.1.2 Orbifolds as étale stacks?

A proposal popular among Lie theorists [MP97] (see [Mo02][Le08][Am12]) is to regard an orbifold with local charts $G_i \circlearrowright U_i$ (2.13) as

- the étale groupoid; in particular: Lie groupoid (see [MM03][TX06]) or topological groupoid (see [CPRST14]);
- equivalently, the étale geometric stack; in particular: differentiable or topological stack ([Ca11][Ca19][Gi13])

obtained by gluing the corresponding *homotopy quotient stacks* $U_i /\!\!/ G_i$ (1.15).

This proposal is directly modeled (explictly so in [Jo12, §8]) on the concept of Deligne-Mumford stacks in algebraic geometry ([DM69], review in [Kr09]) and extends to a concept of general étale ∞-stacks [Ca20][Ca16]. It relies on the fact that étale stacks, in their role as homotopy-theoretic generalizations of sheaves, fully capture geometric aspects (via generalized sheaf cohomology [Br73], see [NSS12a]), while in their role as geometric refinements of classifying spaces they support *Borel equivariant cohomology* (see [Tu11]). However, Borel cohomology is coarser than the *proper equivariant cohomology* that is generally relevant in theory and in applications:

1.1.3 Proper equivariant cohomology

Proper equivariant cohomology [1] formulated in equivariant homotopy theory (review in [Blu17][May96]), is obtained by refining the purely homotopy-theoretic nature of Borel cohomology by the geometric ("cohesive", see §1.2) nature of fixed loci (see Ex. 4.2.26) of topological group actions – hence by the characteristic nature of orbifold geometry – as encoded in the category of orbits of the equivariance group (recalled in §2.2). The proper equivariant version of ordinary cohomology is known as *Bredon cohomology* [Br67a][Br67b] (review in [Blu17, §1.4][tD79, §7]); beyond that, there is a wealth of proper equivariant generalized cohomology theories (Def. 2.2.6 below) such as *equivariant K-theory* [Se68][AS69] (which is proper equivariant by [AS04, §6 & A3.2][FHT07, A.5][DL98]) and *equivariant Cohomotopy theory* [Se71][tD79, §8][SS20][BSS19].

However, if orbifolds are modeled just by étale stacks, then their proper equivariant cohomology remains, by and large, invisible. This is true even for Chen-Ruan orbifold cohomology:

[1] We follow [DHLPS19] with the terminology "proper equivariant cohomology", see Remark 5.2.47 below, using it to distinguish from *naïve* or *Borel* equivariance.

1.1.4 Traditional orbifold cohomology and its shortcomings

Given an orbifold $\mathscr{X}$, we write (see §1.2) $\cup\mathscr{X}$ for the étale stack underlying it, and[2] $\int\cup\mathscr{X}$ for its geometric realization or classifying space (often denoted $B\mathscr{X}$). In the case that $\mathscr{X}$ is the global quotient orbifold of a G-space X, this is the homotopy type of the *Borel construction*; so that we may generally call $\int\cup\mathscr{X}$ the *Borel space* of the orbifold. Now, traditional orbifold cohomology is [ALR07, p. 38] just the ordinary cohomology (e.g. singular cohomology) of this Borel space, hence is *Borel cohomology*:

$$\overset{\substack{\text{traditional}\\ \text{orbifold cohomology}}}{H^\bullet_{\mathrm{trad}}\big(\mathscr{X}, A\big)} := \overset{\text{Borel cohomology}}{\underset{\substack{\text{singular}\\ \text{cohomology}}}{H^\bullet_{\mathrm{sing}}}\big(\underset{\substack{\text{Borel}\\ \text{space}}}{\int\cup\mathscr{X}}, A\big)}. \tag{1.1}$$

This can be considered with any kind of coefficients A, notably in the generality of local coefficient systems [MP99], but it always remains an invariant of just the Borel space. Moreover, for a coefficient ring that inverts the order of the isotropy groups of $\mathscr{X}$, hence in particular for rational, real and complex number coefficients $A \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$, the purely torsion cohomology of the orbifold's finite isotropy groups becomes invisible, and traditional orbifold cohomology reduces further (e.g. [ALR07, Prop. 2.12]) to an invariant of just the shape $\int\vee\mathscr{X}$ of the singular quotient space $\vee\mathscr{X}$ (the "coarse moduli space") underlying the orbifold (often denoted $|\mathscr{X}|$):

$$\overset{\substack{\text{traditional rational}\\ \text{orbifold cohomology}}}{H^\bullet_{\mathrm{trad}}\big(\mathscr{X}, \mathbb{Q}\big)} \simeq \overset{\text{ordinary cohomology}}{\underset{\substack{\text{singular}\\ \text{cohomology}}}{H^\bullet_{\mathrm{sing}}}\big(\underset{\substack{\text{naïve/coarse}\\ \text{quotient}\\ \text{space}}}{\int\vee\mathscr{X}}, \mathbb{Q}\big)}. \tag{1.2}$$

It is in this form that orbifold cohomology was originally introduced (in [Sa56, Thm. 1], following [Ba54], reviewed in [ALR07, 2.1]).

Of course it did not go unnoticed that this coarse notion of orbifold cohomology is insensitive to the actual nature of orbifolds. In reaction (and motivated by algebraic constructions [DHVW85][DHVW86] on 2d conformal field theories interpreted as describing strings propagating on orbifold spacetimes), Chen and Ruan famously proposed a new orbifold cohomology theory in [CR04]. But in fact Chen-Ruan cohomology of an orbifold is (see [Cl14, p. 4,7] for review) just Satake's coarse cohomology (1.2), but applied to the corresponding "inertia orbifold" (cf. [LU04b][SS24]) $\mathbf{Map}\big(\int S^1, \mathscr{X}\big)$ of maps from the shape of the circle:

$$\overset{\substack{\text{Chen-Ruan}\\ \text{orbifold cohomology}}}{H^\bullet_{\mathrm{CR}}\big(\mathscr{X}\big)} \simeq \overset{\text{traditional orbifold cohomology}}{H^\bullet_{\mathrm{trad}}\big(\underset{\text{inertia orbifold}}{\mathbf{Map}(\int S^1, \mathscr{X})}, \mathbb{C}\big)}. \tag{1.3}$$

Still, it turns out that, for global G-quotient orbifolds $\mathscr{X} = \prec(X /\!\!/ G)$, Chen-Ruan cohomology is equivalent to a proper equivariant cohomology theory, namely to Bredon

[2] The "esh"-symbol "ʃ" stands for *shape* [Sc13, 3.4.5][Sh15, 9.7], following [Bo75], which for well-behaved topological spaces is another term for their *homotopy type* [Lu09a, 7.1.6][Wa17, 4.6]; see Ex. 4.1.18.

cohomology with coefficient system given specifically by:

$$A_{\mathrm{CR}} \;:\; G/H \longmapsto \mathrm{ClassFunctions}(H,\mathbb{C})\,. \tag{1.4}$$

This was observed in [Mo02, p. 18], using [Ho90, Thm. 5.5] with [Ho88, Prop. 6.5 b)]:

$$\underset{\substack{\text{global quotient}\\ \text{orbifold}}}{\overset{\text{Chen-Ruan cohomology}}{\mathbf{H}^\bullet_{\mathrm{CR}}\big(\prec\!(X/\!\!/G)\,,\mathbb{C}\big)}} \;\simeq\; \underset{\substack{\text{specific system}\\ \text{of coefficients (1.4)}}}{\overset{\text{Bredon cohomology}}{H^\bullet_G\big(X\,,A_{\mathrm{CR}}\big)}}\,. \tag{1.5}$$

Thus the success of Chen-Ruan cohomology (surveyed in [ALR07, §4,5]) highlights the relevance of proper equivariance in orbifold cohomology. At the same time, this means that to detect the full proper equivariant homotopy type of orbifolds, one needs an orbifold cohomology theory that induces Bredon coefficient systems more general than (1.4); and, in fact, one that subsumes also generalized equivariant cohomology theories such as equivariant K-theory. In [AR01] the authors *define* orbifold K-theory to be the equivariant K-theory of any global quotient presentation (see also [ARZ06][BU09][HW11]):

$$\underset{\substack{\text{global quotient}\\ \text{orbifold}}}{\overset{\substack{\text{traditional}\\ \text{orbifold K-theory}}}{K^\bullet\big(\prec\!(X/\!\!/G)\big)}} \;:=\; \overset{\substack{\text{equivariant}\\ \text{K-theory}}}{K^\bullet_G(X)}\,. \tag{1.6}$$

This has been justified for this specific case of K-theory by checking explicitly [PS10, Prop. 4.1] that the right hand side of (1.6) is independent of the choice of global quotient presentation on the left. However, in general, this approach of circumventing an intrinsic definition of orbifold cohomology by just defining it to be equivariant cohomology of global quotient presentations is, besides being somewhat unsatisfactory, in need of justification:

1.1.5 Orbifolds in global equivariant homotopy theory?

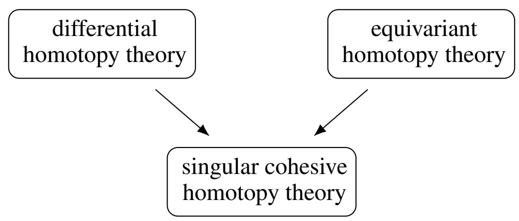
That orbifold cohomology should also capture proper equivariant cohomology was suggested explicitly in [PS10]. However, the fundamental issue remained that a quotient presentation $\mathscr{X} \simeq \prec\!(X/\!\!/G)$ of an orbifold is not intrinsic to the orbifold, similarly to a choice of coordinate atlas, while in equivariant cohomology theory the equivariance group G is traditionally taken to be fixed. But this suggests [Schw17, Intro.][Schw18, p. ix-x] (details in [Ju20]) that the right context for orbifold cohomology is "global" equivariant homotopy theory [Schw18] (following [HG07] and originally motivated from patterns seen in genuine equivariant stable homotopy theory [Se71][LMS86]) where the equivariance group G is allowed to vary in a prescribed class of groups. On the other hand, plain global homotopy theory retains no geometric information!

1.1.6 The open problem

The open problem is thus to set up a mathematical theory of *proper orbifold cohomology* which unifies:

(i) the higher *geometric* and *differential* aspects of orbifolds captured by geometric/differential homotopy theory; and

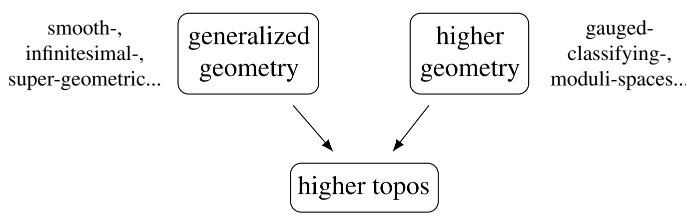
(ii) the *singular* (equivariant) aspects of orbifolds captured by proper and global equivariant homotopy theory.



To achieve this, we turn to higher topos theory ([TV05][Lu09a][Re10], see §3.1) as the ambient foundational homotopy theory in which we formulate higher orbi-geometry by means of systems of cohesive modalities ([Sc13][Sc19][Sc25][SS25d], see §4, §5):

1.1.7 Higher toposes, where higher geometry takes place

For our purposes, a higher topos is a universe of generalized & higher geometric spaces (exposition in [Sc25]):



(a) Here "**generalized geometry**" refers to what Grothedieck called *functorial geometry* [Gr65] (review in [DG80]), which he urged in [Gr73] should supersede any point-set (locally ringed) definition of geometric spaces (further amplified by Lawvere, cf. [La86][La91]): The idea is to define spaces operationally – much as envisioned in physics – by how they may be *probed* by a small category of *probe spaces* (*affine spaces* or *charts*, see [FSS14][JSSW19][Sc25][GS25] and Def. 4.1.9 below), such as

$$\mathrm{Chrt} = \left\{ \begin{array}{ll} \mathrm{CrtSpc} & \text{(Def. 3.1.5)} \\ \mathrm{JetCrtSp} & \text{(Def. 4.1.22)} \\ \mathrm{SuperCrtSp} & \text{(Def. 4.1.41)} \\ \mathrm{Snglrt} & \text{(Def. 4.2.1)} \\ \mathrm{SnglrChrt} & \text{(Lem. 4.2.15)} \\ \ldots, & \end{array} \right. \tag{1.7}$$

and then to encode a would-be generalized ("target"-)space $\mathscr{X}$ by assigning to each $\Sigma \in \mathrm{Chrt}$ the collection

$$\underset{\textbf{probe space}}{\Sigma} \;\longmapsto\; \underset{\substack{\textbf{collection of probinges of}\\ \textbf{generalized space } \mathscr{X} \textbf{ by } \Sigma}}{\mathscr{X}(\Sigma) := \big\{ \text{“}\, \Sigma \to \mathscr{X} \,\text{”} \big\}} \tag{1.8}$$

of geometric (e.g. smooth, super-geometric, etc.) maps into $\mathscr{X}$; where the quotation marks indicate that, at this point of bootstrapping $\mathscr{X}$ into existence, the category in which these *probings* are actual maps is yet to be specified. To that end, one observes that a minimal set of consistency conditions on such an abstract assignment (1.8) to be anything like collections of maps into a space $\mathscr{X}$ are:

(1) Functoriality of probes. For every map ϕ of Chrt there is an operation of "pre-composition of probe maps by ϕ":

$$\begin{array}{ccc} \substack{\textbf{map of}\\ \textbf{probe spaces}} & \substack{\textbf{pre-composition operation}\\ \textbf{on collections of probes}} & \\ \Sigma_1 & \mathscr{X}(\Sigma_1) & \text{such that} \\ \Big\downarrow{\scriptstyle \phi} & \Big\uparrow{\scriptstyle \mathscr{X}(\phi) = \text{“}(-)\circ\phi\text{”}} & \begin{array}{c}\mathscr{X}(\phi_2)\circ\mathscr{X}(\phi_1)\\ \simeq \\ \mathscr{X}(\phi_2\circ\phi_1)\,.\end{array} \\ \Sigma_2 & \mathscr{X}(\Sigma_2) & \end{array} \tag{1.9}$$

(2) Gluing of probes. If $\{ U_i \twoheadrightarrow \Sigma \}_{i \in I}$ is a cover of $\Sigma \in \mathrm{Chrt}$ by several $U_i \in \mathrm{Chrt}$, then probingses of $\mathscr{X}$ by Σ should be equivalent to those tuples of probinges by the U_i which are coherently identified on intersections:

$$\mathscr{X}(\Sigma) \;\simeq\; \left\{ \begin{array}{c} \text{tuples of probes } U_i \longrightarrow \mathscr{X} \\ \text{identified on intersections } U_i \cap U_j \\ \text{compatibly on } U_i \cap U_j \cap U_k \\ \text{etc.} \end{array} \right\}. \tag{1.10}$$

In the jargon of topos theory (see [MLM92][Joh02]), condition (1.9) says that the collection $\mathscr{X}(-)$ of probes of $\mathscr{X}$ is a *pre-sheaf* on Chrt, while condition (1.10) says that this is in fact a *sheaf*. Hence the category of generalized geometric spaces probeable by Chrt is the category of sheaves (the *Grothendieck topos*) on Chrt:

$$\underset{\substack{\textbf{topos of generalized}\\ \textbf{geometric spaces}}}{\mathrm{GnrlzdSpc}} \;:=\; \underset{\substack{\textbf{category of sheaves}\\ \textbf{on site of charts}}}{\mathrm{Shv}\big(\mathrm{Chrt}\big)} \tag{1.11}$$

Now, every $\Sigma \in \mathrm{Chrt}$ is itself canonically regarded as a generalized space $y(\Sigma) \in \mathrm{GnrlzdSpc}$, by taking its probes to be those given by morphisms of Chrt (this is the *Yoneda embedding*[3], recalled as Prop. 3.1.37 below):

$$\underset{\substack{\textbf{chart regarded as}\\ \textbf{generalized space}}}{y(\Sigma)} \;:\; \Sigma' \longmapsto \underset{\substack{\textbf{collection of its}\\ \Sigma'\textbf{-shaped probes}}}{\big\{\Sigma' \to \Sigma\big\}} \;=:\; \mathrm{Chrt}(\Sigma', \Sigma) \tag{1.12}$$

Hence we have completed the bootstrap construction of generalized spaces $\mathscr{X}$ in (1.8) if we may remove the quotation marks there, hence if for $\mathscr{X} \in \mathrm{GnrlzdSpc}$ there

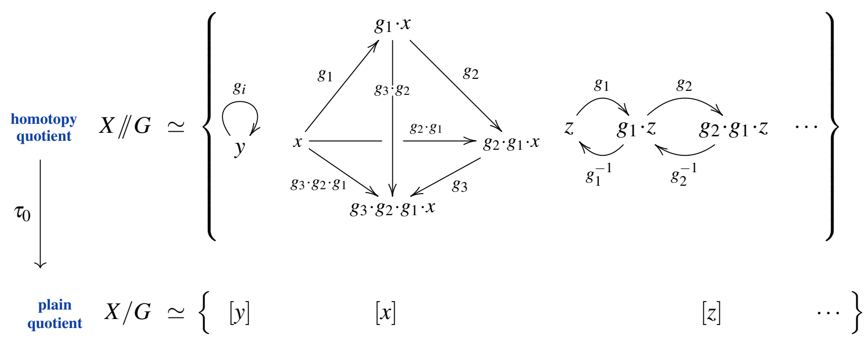
[3] Shown here for sub-canonical Grothendieck topologies on Chrt, which is the case in all examples of interest here.

is a natural equivalence

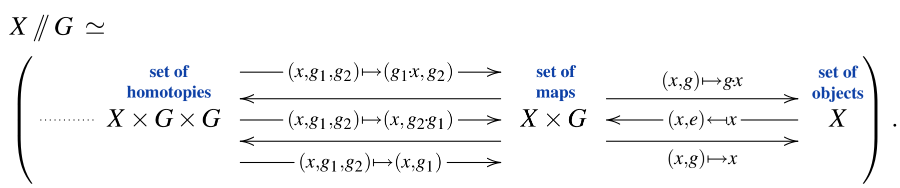
$$\underset{\text{probes of }\mathscr{X}}{\overset{\Sigma\text{-shaped}}{\mathscr{X}(\Sigma)}} \quad \simeq \quad \overset{\text{actual maps from } y(\Sigma) \text{ to } \mathscr{X}}{\big\{ y(\Sigma) \to \mathscr{X} \big\} \;:=\; \mathrm{GnrlzdSpc}\big(y(\Sigma), \mathscr{X}\big)} \,. \tag{1.13}$$

That this is indeed the case is the statement of the *Yoneda lemma* (recalled as Prop. 3.1.38 below), which thus implies consistency and existence of generalized geometry!

(b) On the other hand, "**higher geometry**" (see [FSS14][FSS19][JSSW19] for exposition and applications) refers to the refinement of the above theory of generalized geometric spaces, where the collection of probes (1.8) of a generalized space is not necessarily just a set, but may be a set equipped with equivalences between its elements (a *gauged set*), and with higher order equivalences (higher gauge transformations) between these, etc. – called an ∞-*groupoid* (typically modeled as a Kan simplicial set, see [GJ99, I.3]). For example, for $X \in \mathrm{Set}$ and G a discrete group acting on X, the corresponding *action groupoid* (Ex. 3.1.15 below) consists of the elements $x \in X$, but equipped with an equivalence between x_1 and x_2 for every group element whose action takes x_1 to x_2:



This is a model for the *homotopy quotient* of X by G, which resolves the plain quotient X/G (the set of equivalence classes) by remembering not only *that* but *how* pairs of elements are equivalent. More precisely, the action groupoid remembers the *graph* and *syzygies* of the G-action, encoded in its Kan simplicial *nerve* (Ex. 3.1.69 below):



In particular, if an element $y \in X$ is fixed by the group action, then in the homotopy quotient it appears as the one-object *delooping groupoid* of G:

$$\mathbf{B}G \;\equiv\; * /\!\!/ G \;=\; \Big\{ \overset{g}{\underset{*}{\circlearrowright}} \;\Big|\; g \in G \Big\} \,. \tag{1.14}$$

More generally, if $X \in \mathrm{Chrt}$ in the list (1.7) is equipped with the action of a discrete group G, then we obtain a higher generalized space $\mathscr{X} := X /\!\!/ G$ whose ∞-groupoid of Σ-shaped probes (1.8) is the action groupoid of the induced action on the set of Σ-shaped probes of X (the following formula is for contractible charts, Lemma 4.1.12):

$$\underset{\substack{\text{global quotient}\\ \text{orbifold}}}{X /\!\!/ G} \;:\; \Sigma \;\longmapsto\; \overbrace{\big(X /\!\!/ G\big)(\Sigma) \;:=\; X(\Sigma) /\!\!/ G \;=\; \mathrm{Chrt}(\Sigma, X) /\!\!/ G}^{\text{groupoid of its }\Sigma\text{-shaped probes}}\,. \tag{1.15}$$

Such a higher generalized space with collections of probes (1.8) being groupoids, and satisfying the appropriate gluing condition (1.10), may be called a 2-*sheaf* or *sheaf of groupoids* [Br93] on Chrt, in generalization of (1.11), but is commonly known as a *stack* [DM69][Gi72][Ja01][Ho08], following *champ* [Gi66]. Generally, a higher generalized space with ∞-groupoids of probes is thus an ∞-*sheaf* or ∞-*stack* on Chrt, in generalization of (1.11):

$$\underset{\infty\text{-topos}}{\mathbf{H}} \;:=\; \underset{\substack{\text{higher category of}\\ \text{higher generalized spaces}}}{\mathrm{HigherGnrlzdSpc}} \;:=\; \overset{\infty\text{-category of }\infty\text{-stacks}}{\mathrm{Shv}_\infty\big(\underset{\substack{\infty\text{-site of}\\ \text{probe spaces}}}{\mathrm{Chrt}}\big)}\,. \tag{1.16}$$

1.1.8 Higher topos theory

The theory of ∞-stacks originates with [Br73], developed in [Ja87][Ja96] (survey in [Ja15]) and brought into the more abstract form in [TV05][Lu09a][Re10] (introduction in [Re19]). While the theory has a reputation of being intricate, this is really a reflection of its simplicial models and hence of the richness of its implications, while – on the contrary — finitary constructions internal to ∞-toposes behave so very well that they may naturally be formulated [Sh19] in a kind of programming language now known as *homotopy type theory* [UFP13]. While we will not dwell on this here, we do focus on elegant internal constructions. For some of these, a homotopy type-theoretic formulation has already been explored in the literature, cf. Table 1.1.

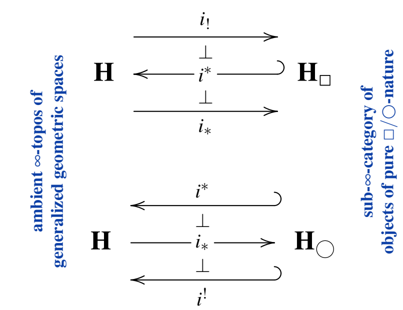
Theory internal to an ∞-topos	**Formulation in ordinary math**	**Formulation in ho-type theory**
Galois theory	§3.2 [NSS12a]	[BvDR18]
modalities & cohesion	§4.1 [SSS12][Sc13]	[RSS17][Sh15]
étale ∞-stacks	§5.2 [KS17]	[Ch24][CRi20]
cohomology	§6 [SSS12][NSS12a]	[Cav15][BH18]

Table 1.1 – Existing formalizations.

In particular a key aspect of our treatment here is that we capture (orbi-)geometry in terms of systems of adjoint *modal operators* on the ambient higher topos:

1.1.9 Dual modalities in an ∞-Topos

In view of the above every ∞-topos $\mathbf{H}$ may be thought of as a homotopy theory of generalized geometric spaces of a a certain nature. In order to narrow back in, among these very generalized spaces, onto those which are relatively tame, we may, in the spirit of [La91][La94][La07], axiomatize qualities of geometric spaces (such as being *discrete*, *smooth*, *étale*, *reduced*, *bosonic*, *singular*, etc.) via the systems of (co-)reflective sub-∞-categories $\mathbf{H}_{\bigcirc}, \mathbf{H}_{\Box}, \cdots \subset \mathbf{H}$, that the objects with these properties (should) form inside $\mathbf{H}$ [SSS12][Sc13]:



(1.17)

These *reflections* induce systems of adjoint (co-)projection operators $\bigcirc \dashv \Box : \mathbf{H} \to \mathbf{H}$, the associated *idempotent (co-)monads*:

$$\bigcirc := i^* \circ i_! \,,\quad \Box := i^* \circ i_* \qquad \text{or} \qquad \bigcirc := i^* \circ i_* \,,\quad \Box := i^! \circ i_* \,, \tag{1.18}$$

to which we refer as *modal operators* or just *modalities* [Sc13][RSS17][Co20]. These are idempotent (Prop. 3.1.29),

$$\bigcirc\bigcirc X \simeq \bigcirc X \,, \qquad \Box\Box X \simeq \Box X \,, \tag{1.19}$$

which means that they act like *projecting out* certain qualitative aspects of generalized spaces, while them being adjoint means that they project out an *opposite pair* of such qualities. Therefore, their (co-)unit transformations $\eta^{\Box}$ (3.44) and $\varepsilon^{\bigcirc}$ (3.45) exhibit every $X \in \mathbf{H}$ as carrying a quality intermediate to these two opposite extreme aspects [LR03, p. 245]:

$$\underset{\substack{\text{pure}\\ \bigcirc\text{-aspect}}}{\bigcirc X} \xrightarrow{\;\varepsilon^{\bigcirc}_X\;} \underset{\substack{\text{generalized}\\ \text{geometric space}}}{X} \xrightarrow{\;\eta^{\Box}_X\;} \underset{\substack{\text{pure}\\ \Box\text{-aspect}}}{\Box X} \,. \tag{1.20}$$

It turns out that by axiomatizing, this way, that every space X has a pair of opposite extreme aspects $\bigcirc X$ and $\Box X$ to it, the spaces X themselves are forced to behave like carrying the kind of extra geometric structure which may be in between these opposites.

1.1.10 Differential topology in an ∞-topos

For example, any adjoint modality $\flat \dashv \sharp$ (see Def. 4.1.1 below) that contains the initial modality $\varnothing \dashv *$ (which globally projects to the initial and the terminal object, respectively) acts like projecting out *discrete* and *purely continuous* (co-discrete, chaotic) aspects of a space. Consequently, the existence of such a modality on $\mathbf{H}$ exhibits each space $X \in \mathbf{H}$ as carrying quality intermediate to these extremes, hence, in this example, as equipped with a kind of *topology* (see [Sh15, §3], following [La94]).

We observe here that extending this basic example to a larger system of adjoint modalities allows to abstractly encode the presence of differential geometric structure (Def. 4.1.21 below) and of super-geometric structure (Def. 4.1.40 below) in a powerful abstract way.

1.1.11 Generalized cohomology in an ∞-topos

At the same time, ∞-toposes may be understood as naturally embodying the ultimate notion of *generalized cohomology theories* (following [SSS12][NSS12a][Sc13]) subsuming and combining all of the examples listed in Table 1.2 on p. 12.

Namely, all these cohomology theories become "representable" in ∞-topos theory, meaning that their cohomology classes are simply the (homotopy) equivalence classes of maps in the ∞-topos to a given *classifying object* $A \in \mathbf{H}$:

Concretely, for $X, A \in \mathbf{H}$ a pair of objects, with X regarded as a domain "space" and A as the "coefficients" of cohomology, then *A-cohomology of X* is embodied by the morphisms from X to A:

(i) a morphism $X \xrightarrow{c} A$ is a *cocycle*;

(ii) a homotopy $X \underset{c_2}{\overset{c_1}{\rightrightarrows}} A$ (with $\Downarrow$ between c_1 and c_2) is a *coboundary*;

(iii) the homotopy groups of the cocycle space

$$H^{-n}(X,A) := \pi_n \mathbf{H}(X,A) \simeq \pi_0 \mathbf{H}(X, \Omega^n A) \tag{1.21}$$

are the *cohomology sets* of X with coefficients in A. (Here $\Omega^n(-)$ is the n-fold based looping operation.)

This is the *intrinsic cohomology theory* of the ∞-topos $\mathbf{H}$ — we discuss various examples below in §6.

Flavor of cohomology	**realized in ∞-toposes**	
Sheaf hyper-cohomology	non-discrete ∞-toposes	[Br73]
Non-abelian cohomology	general ∞-toposes	[SSS12][NSS12a, 3]
Twisted non-abelian cohomology	slice ∞-toposes	Prp. 3.1.46, Rem. 3.2.21
Twisted extraordinary cohomology	tangent ∞-toposes	Ex. 3.1.51, Rem. 3.2.23
Differential cohomology	cohesive ∞-toposes	Def. 4.1.1, Rem. 4.1.20
Étale cohomology	elastic ∞-toposes	Def. 4.1.21, Def. 6.2.1
Superspace cohomology	solid ∞-toposes	Def. 4.1.40, Rem. 4.1.44
Proper equivariant cohomology	singular ∞-toposes	Def. 4.2.3, Rem. 6.1.4, Thm. 6.1.9

Table 1.2 – Generalized cohomology and ∞-toposes

1.1.12 Geometric orbifold cohomology in cohesive ∞-toposes

In summary, this suggests that the otherwise thorny question of how to conceive of

(i) orbifold geometry with its delicate interplay of differential geometric with equivariant and homotopy theoretic aspects

(ii) orbifold cohomology sensitive to this peculiar orbi-geometry while unconstrained by it in generality

finds a natural and powerful solution when orbifolds are understood as objects of ∞-toposes equipped with systems of adjoint geometric modalities. This approach we lay out in the present book.

1.2 Results

With this machinery in hand, we develop the following results.

1.2.1 Axiomatic orbifold geometry in modal homotopy theory

By the above, to formulate proper orbifold cohomology we ask for ∞-toposes (1.16) equipped with a system of adjoint modalities (1.18) that capture both aspects of proper orbifold cohomology:

(i) the *geometric* (differential, étale) aspect

(ii) the *singular* (proper equivariant) aspect.

τ_n **n-groupoidal**		
$\int$ **shaped**	$\flat$ **discrete**	$\sharp$ **continuous**
$\Re$ **reduced**	$\Im$ **étale**	$\mathscr{L}$ **locally constant**
$\rightrightarrows$ **even**	$\rightsquigarrow$ **bosonic**	Rh **rheonomic**
$⩗$ **singular**	$\cup$ **smooth**	$\curlyvee$ **orbi-singular**

Table 1.3 – Modalities for Singular Super-Geometry (§4). The table shows the symbols used for the modalities in the main text, and indicates the modal geometric quality which they express. For example a space X is *discrete* or *bosonic* or *singular* iff it is equivalent to its purely discrete aspect $X \simeq \flat X$, or its purely bosonic aspect, $X \simeq \overset{\rightsquigarrow}{X}$, or its purely singular aspect $X \simeq ⩗X$, respectively.
The modality τ_n expresses that an object contains higher gauge transformation only of degree $\leq n$.

1. The geometric aspect of orbifold theory. In order to formulate, in suitable ∞-toposes, the **(a)** differential topology, **(b)** differential geometry, and **(c)** super-geometry of orbifolds (hence of manifolds, super-manifolds, super-orbifolds, etc.) in their smooth guise as étale ∞-stacks (1.16), we consider a corresponding progression of adjoint modalities (1.18), which starts out in the form of the "axiomatic cohesion" of [La07], on to a second layer that contains a "de Rham shape" operation $\Im$ as considered in [Si96][ST97], and then to a third layer which captures super-geometry in a new axiomatic way [Sc13]:

$$
\begin{array}{ccccccccccc}
\mathrm{id} & \dashv & \mathrm{id} & & & & & & & & \\
\vee & & \vee & & & & & & & & \\
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \mathrm{Rh} & & & & & & \text{for super-geometry in solid } \infty\text{-toposes (Def. 4.1.40)} \\
& & \vee & & \vee & & & & & & \\
& & \Re & \dashv & \Im & \dashv & \mathscr{L} & & & & \text{for differential geometry in elastic } \infty\text{-toposes (Def. 4.1.21)} \\
& & & & \vee & & \vee & & & & \\
& & & & \int & \dashv & \flat & \dashv & \sharp & & \text{for differential topology in cohesive } \infty\text{-toposes (Def. 4.1.1)} \\
& & & & & & \vee & & \vee & & \\
& & & & & & \varnothing & \dashv & \ast & &
\end{array}
$$

Table 1.4 – Progression of the geometric modalities.

The key observation then is how this system of geometric cohesion goes along with a parallel system of cohesive modalities axiomaizing orbi-singular structure [Re14]:

2. The singular aspect of orbifold theory. Envision the picture of an orbifold singularity $\prec$ and a mathematical magnifying glass held over the singular point. Under this magnification, one sees resolved the singular point as a *fuzzy fattened point*, to be denoted $\prec^{G}$. Removing the magnifying glass, what one sees with the bare eye depends on how one squints:

(i) The physicists (like [BL99, §1.3]) and the classical geometers (like [IKZ10][Wat15]) say that they see an actual singular point, such as the tip of a cone $\vee$. This is the *plain quotient* $\vee^{G} := \ast/G = \ast$, a point.

(ii) The higher geometers (like [MP97] [CPRST14]) say that still they see the smooth G-action on that point, hence a smooth (stacky) geometry $\cup$. This is the *homotopy quotient* $\cup^{G} := \ast /\!\!/ G = \mathbf{B}G$ (1.14), a groupoid.

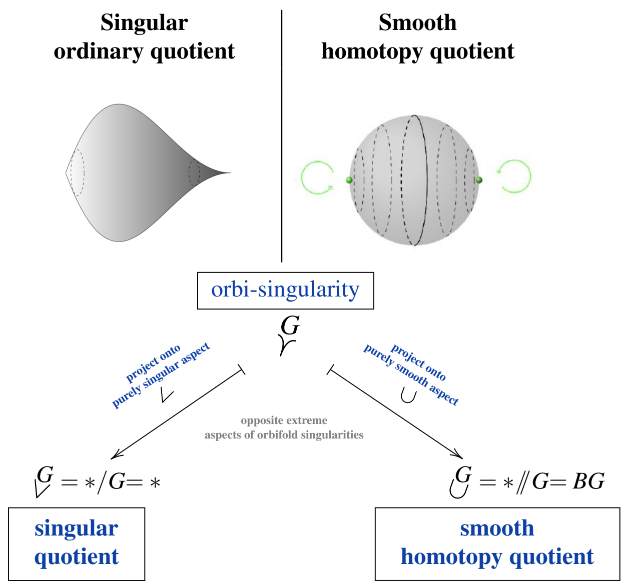


Figure 1.4 – Two opposite aspects of orbi-singularities.

We observe in §4.2 that just this pair of dual perspectives is captured by the cohesive structure on global equivariant homotopy theory that had been found in [Re14], but whose conceptual interpretation had remained open [Re14, Footnote 8]. We find that the resulting system of modalities $\vee \dashv \cup \dashv \curlyvee$ serves to neatly axiomatize the nature of orbi-singularities, hence of orbispaces.

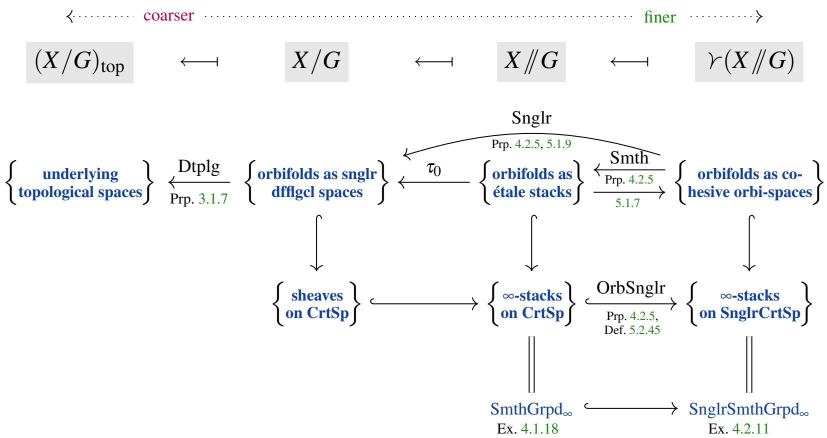
In combination with the previous geometric modalities, we thereby obtain the axiomatic infrastructure for our theory of geometric orbifold cohomology. This general abstract backdrop we lay out in Part I.

Using this general cohesive axiomatics, in part II we begin (§5) by (re-)developing the differential geometry of orbifolds in this language, starting in the broader generality of "∞-orbifolds", namely of étale ∞-stacks.

1.2.2 Differential geometry of étale ∞-stacks

We present, in §5.2, a general theory of higher differential geometry formulated internally to these elastic ∞-toposes (§4.1). This deals with étale ∞-stacks locally modeled on any group ∞-stack V ("V-folds", Def. 5.2.1). For the special case $V = (\mathbb{R}^n, +)$, this subsumes ordinary manifolds (Ex. 5.2.4) and ordinary étale Lie groupoids (Ex. 5.2.5). For V a super-symmetry group (4.126), this produces a theory of super-orbifolds (Ex. 5.2.7), capturing, for instance, those that appear as target spaces in superstring theory (e.g. [PR04][GIR08]) and M-theory [HSS18], or those that appear as moduli spaces of super-Riemann surfaces [Ra87][LBR88][Wi12][CV17].

The different incarnations of geometric orbifolds in the resulting modal language are summarized by the following diagram:



1.2.3 Bundles and Gerbes on étale ∞-stacks

With orbifolds, in their incarnation as étale stacks, thus embedded into a fully-fledged ∞-topos, the general theory of (equivariant) ∞-bundles [SS25d][NSS12a][NSS12b] (see §3.2 below) applies to provide the theory of *fiber bundles* on orbifolds (cf. [LGTX04][Se06][BG08]) and of *gerbes* on orbifolds (cf. [LU04a][Ca10][BX][TT14]) naturally generalized to higher, to non-abelian and to twisted gerbes on orbifolds.

1.2.4 Differential cohomology of étale ∞-stacks

Moreover, since the intrinsic cohomology theory of cohesive ∞-toposes is differential cohomology (Rem. 4.1.20), this realization of étale ∞-stacks within cohesive ∞-toposes immediately provides their (generalized, nonabelian) *differential cohomology theory* (via [FSS23, §9], reviewed in [SS25f], following [SSS12][FSS14][Sc13][BNV13]). This includes, in particular (as made explicit in [PR19]), the Borel-equivariant/orbifold Deligne cohomology considered in [KT18] (which, for finite groups, coincides with [LU03][Gom05]), given, in low degrees, by:

(i) gerbes with connection, hence including what in string theory is known as the *discrete torsion* classification of the B-field on orbifolds [Va86][VW95][Sha00a] [Sha00c][Sha02][Ru03]; and

(ii) 2-gerbes with connection, hence including what in M-theory is known as the *discrete torsion* classification of the C-field on orbifolds [Sha00b][Se01][dB$^+$02, §4.6.2].

1.2.5 Proper equivariant enhancement of geometric homotopy theory

We enhance all of the above to a theory of properly orbi-singular spaces, formulated internally to "singular-elastic" ∞-toposes (§4.2), where a natural notion of orbi-singularization $\curlyvee$ (Prop. 4.2.5) promotes (Def. 5.2.45) any such ∞-category of étale

∞-groupoids faithfully to its *proper* orbifold version (Rem. 5.2.47). This detects geometric fixed point spaces (Def. 4.2.24) in the sense of proper equivariant homotopy theory. We show (Prop. 5.1.2, Lem. 5.1.7) that the cohesive shape (Def. 4.1.1) of the orbi-singularization of an étale groupoid is its incarnation as an orbispace in global equivariant homotopy theory, in the sense of [HG07][Re14][Kö16][Schw17] (Rem. 5.1.1).

1.2.6 The proper 2-category of orbifolds

One model for the axioms of singular-cohesive homotopy theory is the ∞-topos of *singular-smooth ∞-groupoids* (established as Ex. 4.1.18, 4.2.11 below). In this model, the proper 2-category (Rem. 5.2.47) of orbifolds $\mathscr{X}$, either Lie theoretically (Ex. 5.1.10) or topologically (Ex. 5.1.11), is equivalent, via passage to

(i) their purely smooth aspect $\cup \mathscr{X}$, to the traditional 2-category of étale stacks (Prop. 5.1.9),

(ii) while their purely singular aspect $\vee \mathscr{X}$ gives the underlying singular coarse quotient space (Prop. 5.1.6).

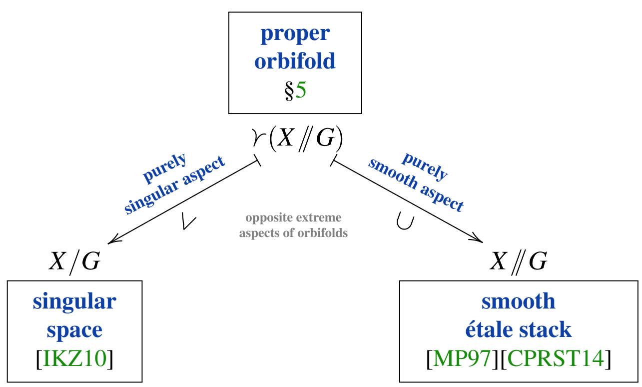


Figure 1.5 – Two opposite aspects of orbifolds.

1.2.7 Cartan geometry of étale ∞-stacks

In this higher cohesive topos-theoretic formulation we find all fundamental phenomena of differential geometry naturally generalized to étale ∞-stacks, hence in particular to orbifolds, as indicated in Table 1.5 (p. 19).

1.2.8 Orbifold étale cohomology

Based on this, we give a natural general definition of *étale cohomology of V-étale ∞-stacks* (Def. 6.2.1) hence in particular of *orbifold étale cohomology*, which is sensitive to the above (integrable) G-structures, and hence to geometry/special holonomy on orbifolds. For example, in the case of complex structure, this orbifold étale cohomology subsumes traditional notions of complex-geometric orbifold cohomology such as orbifold Dolbeault cohomology [Ba54][Ba56][CR04][Fe03] and orbifold Bott-Chern cohomology [An12][Ma05]. Abstractly, orbifold étale cohomology is the intrinsic cohomology (1.21) of integrably G-structured étale ∞-stacks when regarded in the slice ∞-topos (Prop. 3.1.46) over the G-Haefliger stack (Def. 5.2.32) via the classifying map of their G-structure (Prop. 5.2.34). As such, orbifold étale cohomology is the progenitor of *tangentially twisted* (proper) orbifold cohomology (Def. 6.2.3, Def. 6.2.5), to which we turn next.

1.2.9 Proper equivariant cohomology

While the proper 2-category of orbifolds is equivalent to the traditional one of orbifolds as étale stacks, its full embedding into an ambient singular-cohesive ∞-topos (§4.2) provides for more general coefficient objects, each of which is guaranteed to produce a proper orbifold Morita-class invariant (Rem. 5.2.47). Our **first main Theorem** 6.1.9 shows that the intrinsic cohomology (1.21) of orbifolds, regarded in singular-cohesive homotopy theory (Def. 4.2.3), subsumes all proper G-equivariant cohomology theories: Bredon cohomology with any coefficient system, as well as proper equivariant generalized cohomology theories.

1.2.10 Traditional orbifold cohomology

In particular, Prop. 5.1.2 and Theorem 6.1.9 imply, via [Ju20] (Remark 5.1.1), that proper orbifold cohomology in singular-cohesive homotopy theory subsumes Chen-Ruan orbifold cohomology, via (1.5), and Adem-Ruan orbifold K-theory, via (1.6). Hence it also subsumes Freed-Hopkins-Teleman orbifold K-theory [FHT07] (reviewed in [Nu13, §3.2.2]) and Jarvis-Kaufmann-Kimura's "full orbifold K-theory" [JKK05][GHHK08] for orbifolds with global quotient presentations (by [FHT07, Prop. 3.5] and [JKK05, 3], respectively). Moreover, singular-cohesion provides a natural transformation $\cup \mathscr{X} \xrightarrow{\varepsilon^{\cup}_{\mathscr{X}}} \curlyvee \mathscr{X}$ which restricts this proper orbifold cohomology to the underlying étale stack, where it reduces to traditional Borel orbifold cohomology (1.1) and, in particular, to Satake cohomology (1.2) (see also, e.g., [ADG11][BNSS18]).

	§5.2	**Cartan geometry for étale ∞-stacks**	Literature for ordinary orbifolds
(i)	Def. 5.2.13	***Frame bundles***	[MM03, p. 42]
(ii)	Def. 5.2.23	***G-structures***	[Wo16][Zh06][BZ03]
	Def. 5.2.29,	*-locally integrable*	
	Def. 5.2.30	*-globally integrable*	
(ii.a)	Ex. 5.2.31	***Geometric structures***	[Ap00, §1.8][Wo16]
		- Riemannian structure	[Bo92][HM04][Rat06] [BZ07][He09a] [He09b][Ak12][Kan13] [BDP17][Lan18]
		- Flat structure	[BDP17][Ref06] [IU12, §8][SS20]
		- Complex structure	[SW99][FS07]
		- Symplectic structure	[Ve00][Go01][DE05] [HM12][CP14][Ch17] [RC19]
		- Lorentzian structure	[HS91][Ne02][LMS02a] [LMS02b][BR07] [ZR12]
		- Pseudo-Riemannian structure	[Me09][Zh18][BZ19]
		- Conformal structure	[Ap98][Ap00]
		- CR-structure	[DM02]
		- Hypercomplex structure	[BGM98]
(ii.b)	Ex. 5.2.31	***Special holonomy***	[Jo00][CT05]
		- Kähler structure	[Fu83][Je97][Ab01] [BBFMT16]
		- Calabi-Yau structure	[Ro91][Jo98][Jo99a] [Jo99b][Jo00, §6.5.1] [St10][RZ11][CDR16]
		- Quaternionic Kähler	[GL88][Jo00, §7.5.2]
		- Hyper-Kähler struc.	[BD00]
		- G_2*-structure*	[Jo00, §11][Rei15]
		- $\mathrm{Spin}(7)$*-structure*	[Jo00, §13][Ba07]
(iii)	Def. 5.2.28	***Local isometries***	[BZ07]
(iv)	Def. 5.2.32	***Haefliger stacks***	[Hae71][Hae84] [Ca19, §2.2, §3][Ca16]).
(v)	Def. 5.2.35	***Tangential structures***	[Wee18][Pa20]
(v.a)	Ex. 5.2.39	***Higher Spin-structures***	
		- Orientation	[Dr94]
		- Spin structure	[Ve96][Ac01] [BGR07][DLM02]
		- Spin^c *structure*	[Du96, §14]
		- String structure	[PW88][LU04a][LU06]
		- Fivebrane structure	[BL12] (cf. [SSS09][SSS12])

Table 1.5 – Differential orbi-geometry in differential cohesive ∞-toposes

1.2.11 Twisted orbifold cohomology

All these cohomology theories generalize to their twisted versions (e.g., local coefficients for ordinary cohomology, as in [MP99], or twisted K-theory, as in [AR01]), by passage to slices of the ambient singular-cohesive ∞-topos (Remark 3.2.21). In particular, slicing of orbifolds over $\prec^{\mathbb{Z}_2}$ via their orientation bundle promotes them (Ex. 6.1.10) to *orientifolds* [DFM11][FSS15, 4.4][SS20].

1.2.12 Revisiting twisted orbifold K-theory

For illustration, we redevelop twisted orbifold K-theory in this language, using the result of its stacky representability from [SS25d, Ex. 4.5.4].

1.2.13 Proper orbifold étale cohomology

Finally, we promote (Def. 6.2.5) orbifold étale cohomology, in its guise as tangentially twisted cohomology, to a *proper* orbifold cohomology theory in the above sense (Rem. 5.2.47). Our **second main theorem** 6.2.6 shows that this *proper orbifold étale cohomology* unifies:

(i) ($\cup$) étale cohomology (Def. 6.2.1) of smooth V-folds (Def. 5.2.1),

(ii) ($\flat$) proper equivariant cohomology (Def. 6.1.2) of flat orbifolds (Def. 5.2.40), i.e., of their flat frame bundles (Prop. 5.2.41).

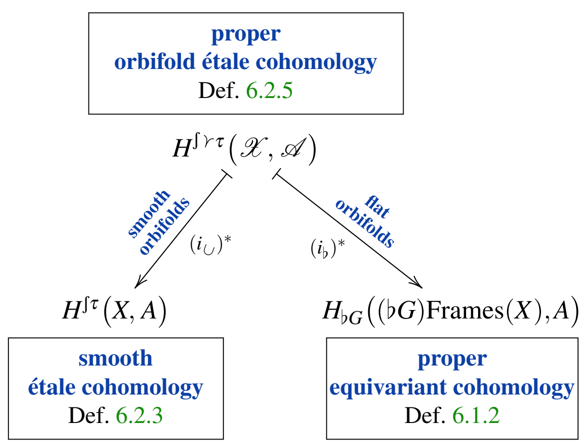


Figure 1.6 – Two opposite aspects of étale cohomology.

1.2.14 Tangentially twisted orbifold Cohomotopy

We construct a fundamental class of examples of such proper orbifold étale cohomology theories, which we call *tangentially twisted* (Def. 6.2.18). Their coefficients are *Tate spheres* (Def. 6.2.9), in the sense of (unstable) motivic homotopy theory (Ex.

6.2.10), with twisting via an intrinsic *Tate J-homomorphism* (Def. 6.2.14). Specified to ordinary orbifolds (Ex. 6.2.19), Theorem 6.2.6 shows that tangentially twisted orbifold Cohomotopy subsumes:

(i) ($\cup$) tangentially twisted Cohomotopy theory of smooth but curved spaces, as introduced in [FSS20][FSS21].

(ii) ($\flat$) RO-graded equivariant Cohomotopy theory of flat orbifolds, as discussed in [SS20][BSS19].

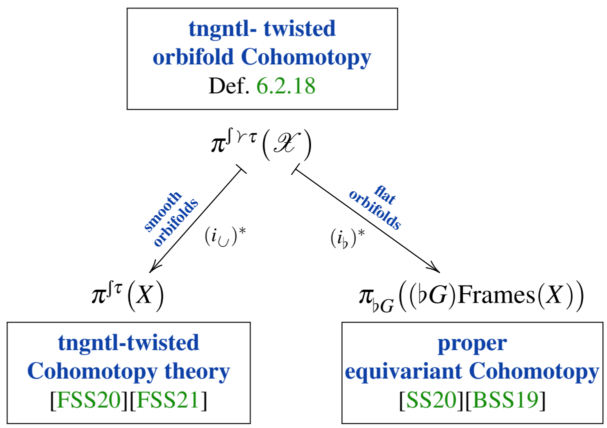


Figure 1.7 – Two aspects of tangentially twisted orbifold Cohomotopy. Theorem 6.2.6 shows that the natural notion of tangential twist of Cohomotopy, when generalized to orbifolds reduces near orbi-singularities to proper equivariant Cohomotopy in the RO-degree of the singularity's isotropy representation.

We conclude in Remark 6.2.20 on the impact of this unification.

1.2.15 Outlook on differential orbifold cohomology

While

(i) generalized differential cohomology on smooth manifolds [HS05] is well-understood (see [Bu12]) and

(ii) plain global equivariant cohomology has been established [Schw18] and understood to provide proper orbifold cohomology ([Ju20], see Remark 5.1.1 below),

their combination to (generalized, global) *proper equivariant differential cohomology* has remained elusive. Explicit constructions have been explored for the case of equivariant/orbifold differential K-theory [SV07][BS09] [Or09], but even these theories do not seem to be well-understood yet [BS12, p. 47]. What has been missing is a coherent theoretical framework for proper equivariant differential cohomology: Since

(a) differential cohomology is the intrinsic cohomology (1.21) of cohesive ∞-toposes (by Remark 4.1.20) and

(b) proper equivariant cohomology is the intrinsic cohomology of ∞-toposes over a (global) orbit category (by Remark 6.1.4),

proper equivariant differential cohomology should be the intrinsic cohomology of ∞-toposes that combine these two properties. This is exactly what our notion of singular-cohesive ∞-toposes expresses (Def. 4.2.3), as confirmed by Theorem 6.1.9. For example, in singular-cohesive ∞-toposes there exists the (global) proper equivariant version of twisted differential non-abelian cohomology [FSS23], now given by homotopy fiber products parametrized over Snglrt (Def. 4.2.1). Hence singular-cohesive ∞-toposes constitute a coherent framework in which to discuss proper equivariant/orbifold differential cohomology in general. We will develop this elsewhere.

We briefly comment on another related approach in the literature:

1.2.16 Proper ∞-categories of general étale ∞-stacks

Another general theory of étale ∞-stacks has been presented in [Ca20], generalizing an elegant characterization of étale 1-stacks due to [Ca19] by following the discussion of derived Deligne-Mumford stacks conceived as structured ∞-toposes in [Lu09b]. This approach proceeds externally via characterizing the *sites* (recalled below as Prop. 3.1.41) which present ∞-toposes of étale ∞-stacks; and is thus complementary to the internal perspective proceeding from inside an ambient ∞-topos which we are presenting here. We briefly indicate the relation between the two:

- The approach in [Ca20] is to pick an ∞-site of PrbSpc (denoted "$\mathscr{L}$" there) which is equipped with a suitable notion of which of its 1-morphisms qualify as being étale maps (the external version of our notion Def. 4.1.26). The inclusion i of the wide subcategory on these étale morphisms induces, by left Kan extension, a pair of adjoint ∞-functors $(i_! \dashv i^*)$ between the corresponding ∞-stack ∞-toposes, and the étale ∞-stacks are then characterized as those in the essential image of the left adjoint $i_!$. This is shown on the right of the following diagram:

$$
\begin{array}{ccccc}
\mathrm{Shv}_\infty\big(\mathrm{Chrt}\times\mathrm{Snglrt}\big) & \underset{\xleftarrow{\ \mathrm{OrbSinglr}\ }}{\overset{\xrightarrow{\ \mathrm{Smth}\ }}{\bot}} & \mathrm{Shv}_\infty\big(\mathrm{Chrt}\big) & \underset{\xleftarrow{\ i_!\ }}{\overset{\xrightarrow{\ i^*\ }}{\top}} & \mathrm{Shv}_\infty\big(\mathrm{Chrt}^{\text{ét}}\big) \\
\uparrow & & \uparrow & \swarrow & \\
\mathrm{OrbSinglr}\big(\text{ÉtStcks}_\infty\big) & \underset{\simeq}{\overset{\textbf{Prop. 4.2.5}}{\longleftarrow}} & \text{ÉtaleStacks}_\infty & & \\
\substack{\textbf{proper } \infty\textbf{-category} \\ \textbf{of higher orbifolds} \\ \textbf{(Remark 5.2.47)}} & & \substack{\infty\textbf{-category} \\ \textbf{of étale } \infty\textbf{-stacks} \\ \textbf{[Ca20]}} & &
\end{array}
\tag{1.22}
$$

- Following Remark 5.2.47, we may and should enhance this construction to the *proper ∞-category of higher orbifolds* Def. 4.2.3, Def. 5.2.45, as shown on the left in (1.22).
- In fact, the archetypical example of PrbSpc considered in [Ca20] is SmoothManifolds (Def. 3.1.9), in which case the left hand side of (1.22) is the singular-cohesive ∞-topos of our Ex.s 4.1.18, 4.2.11, containing the proper (Def. 5.2.45) ∞-category of orbi-$\mathbb{R}^n$-folds in our Ex. 5.2.5.

- ∘ On the other hand, a general ∞-topos $\mathrm{Shv}_\infty(\mathrm{PrbSpc})$ is not going to be cohesive (Def. 4.1.1) or even elastic (Def. 4.1.21). This means that various nice geometric properties, which we derive here, of objects in the proper ∞-category of higher orbifolds, are not guaranteed to exist in the general setup of [Ca20]. Notably the theory of frame bundles on orbifolds, according to Prop. 5.2.13, and the main theorem on the induced étale cohomology of orbifolds (Theorem 6.2.6) crucially uses the internal modal logic of singular-cohesive and singular-elastic ∞-toposes as in §4, which may not exist, or not exist completely, for any given site of PrbSpc as in [Ca20].

2

Background

We have to assume a basic fluency of the reader in abstract and equivariant homotopy theory. This section recalls some basics and provides pointers to the literature.

2.1 Model toposes

We recall some basics of model categories (e.g. [GJ99, 2]) of simplicial presheaves ([Ja87][Ja96][Ja15]) as presentations of ∞-toposes ([Lu09a, A.2, A.3]).

Model categories of simplicial presheaves.

Definition 2.1.1 (Model category of simplicial presheaves)**.** Let $\mathscr{C}$ be a site. We write

(i)

$$\mathrm{sPShv}(\mathscr{C})_{\mathrm{loc}} \;\in\; \mathrm{HomotopicalCategories} \tag{2.1}$$

for the category of simplicial presheaves on $\mathscr{C}$, regarded as a homotopical category with weak equivalences the local weak homotopy equivalences of simplicial sets.

(ii)

$$\mathrm{sPShv}(\mathscr{C})_{{}^{\mathrm{inj}/}_{\mathrm{proj}},\mathrm{loc}} \;\in\; \mathrm{ModelCategories} \tag{2.2}$$

for the same category regarded as either the corresponding injective or projective model category.

(iii)

$$\mathrm{sPShv} \xrightarrow{\;\ell\;} L_{\mathrm{lwhe}}\mathrm{sPShv}_{\mathrm{loc}} \;=:\; \mathbf{H} \tag{2.3}$$

for the corresponding simplicial localization.

Lemma 2.1.2 (Cofibrancy in projective model structure [Du01, Cor. 9.4])**.** *Let $\mathscr{C}$ be a site. For a simplicial presheaf $X_\bullet \in \mathrm{sPShv}(\mathscr{C})_{\mathrm{proj,loc}}$ in the projective model structure (2.2) to be cofibrant it is sufficient that $X_\bullet$ is degreewise*
(i) *a coproduct of representables, such that*
(ii) *the degenerate cells split off as a direct summand.*

Lemma 2.1.3 (Simplicial presheaf represents its own hocolim [DHI04, 2.1][Sc13, 2.3.21])**.** *Let $\mathscr{C}$ be a site and $X_\bullet \in \mathrm{sPShv}(\mathscr{C})$ a simplicial presheaf (Def. 2.1.1). Then* *its image under simplicial localization* (2.3) *is equivalently the simplicial homotopy colimit over the images of its component presheaves:*

$$\ell(X_\bullet) \simeq \varinjlim (\ell X)_\bullet \in \mathbf{H}.$$

Topological mapping stacks

Example 2.1.4 (Model category presentation of smooth ∞-groupoids)**.** Let $\mathscr{C} =$ CrtSpc (Def. 3.1.5). Then the simplicial localization (2.3) of $\mathrm{sPShv}(\mathscr{C})_{\mathrm{loc}}$ (2.2) is $\mathrm{SmthGrpd}_\infty$ (Ex. 4.1.18):

$$L_{\mathrm{lwhe}} \mathrm{sPShv}(\mathrm{CrtSpc})_{\mathrm{loc}} \simeq \mathrm{SmthGrpd}_\infty .$$

Lemma 2.1.5 (Mapping stack from delooping of discrete group to topological stack)**.** *In* $\mathrm{SmthGrpd}_\infty$ *(Ex. 4.1.18) consider*

(i) *a finite group embedded via* (4.154)

$$G \in \mathrm{Grp} \xrightarrow{\mathrm{Disc}} \mathrm{Grp}\big(\mathrm{SmthGrpd}_\infty\big), \tag{2.4}$$

(ii) *a topological groupoid, embedded via* (4.53)

$$\begin{array}{ccc} \mathrm{TopGrpd} & \xrightarrow{\mathrm{Cdfflg}} & \mathrm{SmthGrpd}_\infty \\ \mathscr{X}_{\mathrm{top}} & \mapsto & \mathscr{X}_{\cup} \end{array} \tag{2.5}$$

Then the mapping stack (3.54) *formed in* $\mathrm{SmthGrpd}_\infty$ *is the degreewise image under* Cdfflg (3.12) *of the topological groupoid representing the mapping stack of topological groupoids (which exists by [No10] since G is finite, hence compact):*

$$\mathbf{Map}\big(\mathbf{B}G, \mathscr{X}_{\cup}\big) \simeq \mathrm{Cdfflg}\,\mathbf{Map}\big(\mathbf{B}G, \mathscr{X}_{\mathrm{top}}\big). \tag{2.6}$$

Proof. Since (by Ex. 4.1.18)

$$\mathrm{SmthGrpd}_\infty \simeq \mathrm{Shv}_\infty(\mathrm{CrtSpc}) \underset{\perp}{\overset{L}{\rightleftarrows}} \mathrm{PShv}_\infty(\mathrm{CrtSpc})$$

it is sufficient to show that we have an equivalence of ∞-presheaves of the form

$$\mathbb{R}^n \longmapsto \begin{array}{l} \mathrm{PShv}_\infty\big(\mathrm{CrtSpc}\big)(\mathbb{R}^n \times \mathbf{B}G, \mathscr{X}_{\cup}) \\ \simeq \mathrm{PShv}_\infty\big(\mathrm{CrtSpc}\big)(\mathbb{R}^n, \mathrm{Cdfflg}\,\mathbf{Map}(\mathbf{B}G, \mathscr{X}_{\mathrm{top}})) \end{array} \tag{2.7}$$

By Ex. 2.1.4, we may model this in the global projective model structure on simplicial presheaves over CrtSpc:

$$\mathrm{sPShv}(\mathrm{CrtSpc}) \xrightarrow{\ell} L_{\mathrm{lwhe}} \mathrm{sPShv}(\mathrm{CrtSpc})_{\mathrm{proj}} \simeq \mathrm{Shv}_\infty(\mathrm{CrtSpc}) \tag{2.8}$$

by the following models (Lemma 2.1.3):

- **(a)** A model under ℓ (2.8) of the Cartesian product $\mathbb{R}^n \times \mathbf{B}G$ with the delooping $\mathbf{B}G \simeq \varinjlim G^{\times \bullet}$ (3.124), is given by the simplicial presheaf

$$\mathbb{R}^n \times G^{\times \bullet} \in \mathrm{sPShv}(\mathrm{CrtSpc})_{\mathrm{proj}} . \tag{2.9}$$

- **(b)** A model under ℓ (2.8) for the image (2.5) of a topological groupoid $\mathscr{X}_{\mathrm{top}}$ is given by its nerve regarded as a simplicial presheaf, componentwise via (4.52)

$$N_\bullet(\mathscr{X}_{\mathrm{top}}) \in \mathrm{sPShv}(\mathrm{CrtSpc})_{\mathrm{proj}} . \tag{2.10}$$

Moreover:

- The object (2.9) is projectively cofibrant, by Lemma 2.1.2, as is its Cartesian product with a k simplex $\Delta[k]$.
- The object (2.10) is projectively fibrant (objectwise a Kan complex) by the groupoid property of $\mathscr{X}_{\mathrm{top}}$.

Therefore, to get (2.7) it is, in turn, sufficient to exhibit for $\mathbb{R}^n \in \mathrm{CrtSpc}$ a natural isomorphism of simplicial sets of the form

$$\begin{aligned} &\int_{[k]\in\Delta} \mathrm{PShv}\big(\mathbb{R}^n \times \big(G^{\times_k} \times \Delta(k,\bullet)\big)\,,\mathrm{Cdfflg}(N_k(\mathscr{X}_{\mathrm{top}}))\big) \\ &\simeq\ \mathrm{PShv}\left(\mathbb{R}^n\,,\mathrm{Cdfflg}\left(\int_{[k]\in\Delta} N_k(\mathscr{X}_{\mathrm{top}})^{(G^{\times_k}\times\Delta(k,\bullet))}\right)\right)\,, \end{aligned} \tag{2.11}$$

where the *end* $\int_{[k]\in\Delta}(-)$ expresses the limit that computes the morphism of simplicial sets as a subset of the product of the function spaces of components. We obtain this as the following composite of natural isomorphisms:

$$\begin{aligned} &\int_{[k]\in\Delta} \mathrm{PShv}\Big(\mathbb{R}^n \times \big(G^{\times_k} \times \Delta(k,\bullet)\big)\,,\mathrm{Cdfflg}\big(N_k(\mathscr{X}_{\mathrm{top}})\big)\Big) \\ &\simeq \int_{[k]\in\Delta} \mathrm{PShv}\left(\mathbb{R}^n\,,\big(\mathrm{Cdfflg}(N_k(\mathscr{X}_{\mathrm{top}}))\big)^{(G^{\times_k}\times\Delta(k,\bullet))}\right) \\ &\simeq \int_{[k]\in\Delta} \mathrm{PShv}\left(\mathbb{R}^n\,,\mathrm{Cdfflg}\left(\big(N_k(\mathscr{X}_{\mathrm{top}})\big)^{(G^{\times_k}\times\Delta(k,\bullet))}\right)\right) \\ &\simeq \mathrm{PShv}\left(\mathbb{R}^n\,,\int_{[k]\in\Delta} \mathrm{Cdfflg}\left(\big(N_k(\mathscr{X}_{\mathrm{top}})\big)^{(G^{\times_k}\times\Delta(k,\bullet))}\right)\right) \\ &\simeq \mathrm{PShv}\left(\mathbb{R}^n\,,\mathrm{Cdfflg}\left(\int_{[k]\in\Delta} \big(N_k(\mathscr{X}_{\mathrm{top}})\big)^{(G^{\times_k}\times\Delta(k,\bullet))}\right)\right)\,. \end{aligned}$$

Here the first step is the definition of function spaces $(-)^{(-)}$, the second step uses that Cdfflg, being a right adjoint, preserves products (Prop. 3.1.26). The third step uses that the Hom-functor preserves limits (hence ends) in its second argument, while the fourth step uses that Cdfflg, being a right adjoint, preserves limits (hence ends), again by Prop. 3.1.26. □

2.2 Equivariant homotopy

For reference, we recall some basics of unstable equivariant homotopy theory (see [May96][Blu17]). We focus here on finite groups, for simplicity and since this is what we need in the main text (Remark 4.2.19), but all statements in the following, notably Elmendorf's theorem (Prop. 2.2.10 below) generalize to compact Lie groups.

Definition 2.2.1 (Topological G-spaces)**.** Let $G \in \mathrm{Grp}^{\mathrm{fin}}$ be a finite group.

(i) We write

$$G\mathrm{DTopSpc} \hookrightarrow G\mathrm{TopSpc} \;\in\; \mathrm{Cat} \tag{2.12}$$

for the categories whose objects

$$G \circlearrowright X \;:=\; \big(X,\, G \times X \xrightarrow{\rho} X\big) \tag{2.13}$$

are topological spaces X (as in Def. 3.1.2) or specifically D-topological spaces (as in Def. 3.1.2), respectively, equipped with continuous left G-actions ρ, and whose morphisms are the G-equivariant continuous functions:

$$\begin{aligned} &G\mathrm{TopSpc}\big(G \circlearrowright X_1,\, G \circlearrowright X_2\big) \\ &:= \left\{ X_1 \xrightarrow[\text{continuous}]{f} X_2 \;\middle|\; \begin{array}{ccc} G \times X_1 & \xrightarrow{\rho_1} & X_1 \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow \\ G \times X_2 & \xrightarrow[\rho_2]{} & X_2 \end{array} \right\}. \end{aligned} \tag{2.14}$$

(ii) For $G \circlearrowright X_1$ a (D-)topological G-space and $H \overset{\iota}{\hookrightarrow} G$ a subgroup, we write

$$X^H \;:=\; \Big\{ x \in X \;|\; \underset{h \in H \subset G}{\forall}\; \rho(h,x) = x \Big\} \tag{2.15}$$

for the topological subspace of H-fixed points (which, if X is D-topological, is itself again D-topological, by Prop. 3.1.4).

(iii) For $G \circlearrowright X_1$ and $G \circlearrowright X_2$ two (D-)topological G-spaces, the mapping space (3.6) between their underlying (D-)topological spaces canonically becomes a G-space via the conjugation action and the corresponding fixed point space (2.15)

$$\mathrm{Map}(X_1,X_2)^G \hookrightarrow \mathrm{Map}(X_1,X_2) \tag{2.16}$$

is the subspace on the G-equivariant functions (2.14).

Example 2.2.2 (G-cells)**.** For $G \in \mathrm{Grp}_{\mathrm{fin}}$, $H \subset G$ a subgroup and $n \in \mathbb{N}$ we have the G-spaces (Def. 2.2.1)

$$\big(G/H\big) \times D^n \;,\; \big(G/H\big) \times S^{n-1} \;\in\; G\mathrm{DTopSpc}$$

being the product spaces of the discrete orbit spaces with the standard topological unit disk and unit circle, respectively, the latter equipped with the trivial G-action. The boundary inclusions $\partial D^n = S^{n-1} \overset{\iota_n}{\hookrightarrow} D^n$ induce G-equivariant maps

$$\iota_{n,H} :\quad \big(G/H\big) \times S^{n-1} \xhookrightarrow{(\mathrm{id},\iota_n)} \big(G/H\big) \times D^n \tag{2.17}$$

for all $n \in \mathbb{N}$, $H \subset G$.

Definition 2.2.3 (G-CW-complexes)**.**

(i) A *G-CW-complex* X is a D-topological G-space (Def. 2.2.1) which is equipped with the realization as a colimit

$$X \;\simeq\; \varinjlim_n X_n \;\in G\mathrm{DTopSpc}$$

over a sequence

$$X_{-1} \longrightarrow X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \quad \in G\mathrm{DTopSpc}\,,$$

where $X_{-1} = \varnothing$ and where each $X_n \to X_{n-1}$ is given by a set of attachments of G-cells along (2.17), hence by a pushout of the form:

$$\begin{array}{ccc} \coprod\limits_{\substack{H\subset G \\ i\in I_n}} G/H \times S^{n-1} & \longrightarrow & X_{n-1} \\ {\scriptstyle (\iota_{n,H})_{n,H}} \downarrow & \text{(po)} & \downarrow \\ \coprod\limits_{\substack{H\subset G \\ i\in I_n}} G/H \times D^{n} & \longrightarrow & X_n \end{array}$$

(ii) Write

$$G\mathrm{Sets} \hookrightarrow G\mathrm{CWCmplx} \hookrightarrow G\mathrm{DTopSpc} \tag{2.18}$$

for the full subcategories on those D-topological G-spaces which admit the structure of G-CW-complexes.

Definition 2.2.4 (Homotopy theory of D-topological G-spaces)**.** The *homotopy theory of topological G-spaces* is the ∞-category

$$G\mathrm{Grpd}_\infty \;\in\; \mathrm{Cat}_\infty \tag{2.19}$$

which has the same objects as GCWCmplx (Def. 2.2.3), and with ∞-groupoids the topological shapes (Def. 3.1.13) of the mapping spaces (2.16) of G-equivariant maps:

$$G\mathrm{Grpd}_\infty\big(G \circlearrowright X_1, G \circlearrowright X_2\big) \;:=\; \mathrm{Shp}_{\mathrm{Top}}\Big(\mathrm{Map}\big(X_1, X_2\big)^G\Big)\,. \tag{2.20}$$

Definition 2.2.5 (Shape of G-topological spaces)**.** **(i)** We write

$$\mathrm{Shp}_{G\mathrm{Top}} \;:\; G\mathrm{CWCmplx} \longrightarrow G\mathrm{Grpd}_\infty \tag{2.21}$$

for the canonical ∞-functor (topologically enriched functor) from the 1-category of G-CW-complexes (Def. 2.2.3) to the ∞-category of G-∞-groupoids (Def. 2.2.4), which is the identity on objects and which on Hom-spaces is the continuous map given by the identity fuction from the discrete set of G-equivariant maps (2.14) to the topological space of G-equivariant maps (2.20).

(ii) For any choice of G-CW-approximation functor

$$G\mathrm{TopSpc} \xrightarrow{(-)_{\mathrm{cof}}} G\mathrm{CWComplex}$$

we get the corresponding shape functor on all of GTopSpc (Def. 2.2.1) and hence on GDTopSpc, which we denote by the same symbol:

$$\mathrm{Shp}_{G\mathrm{Top}} \;:\; G\mathrm{TopSpc} \xrightarrow{(-)_{\mathrm{cof}}} G\mathrm{CWCmplx} \xrightarrow{\mathrm{Shp}_{G\mathrm{top}}} G\mathrm{Grpd}_\infty\,. \tag{2.22}$$

Definition 2.2.6 (Proper G-equivariant generalized cohomology of topological G-spaces)**.** For $G \in \mathrm{Grp}^{\mathrm{fin}}$, we say that the *proper G-equivariant cohomology* of a topological G-space (Def. 2.2.1) $X \in G\mathrm{TopSpc}$ with coefficients in a (pointed)

G-∞-groupoid (Def. 2.2.4), $A \in G\mathrm{Grpd}_\infty$, is

$$H_G^{-n}(X,A) \;:=\; \pi_n\Big(G\mathrm{Grpd}\big(\mathrm{Shp}_{G\mathrm{Top}}(X)\,,\,A\big)\Big)\,,$$

where on the right we have the nth homotopy group (at the given basepoint) of the hom-∞-groupoid (2.20) from the G-topological shape of X (2.22) to A.

Elmendorf's theorem.

Definition 2.2.7 (Orbit of action of a finite group)**.** Let G be a finite group. If $G \circlearrowright S$ is a set equipped with an action by G, then an *orbit* of G in S is a subset of points $\{g(s) | g \in G\} \subset S$ obtained from any single point $s \in S$ by acting on it with all elements of G.

Definition 2.2.8 (Orbit category of a finite group)**.** The *category of G-orbits* or *orbit category of G*

$$G\mathrm{Orb} \hookrightarrow G\mathrm{Sets} \;\in\; \mathrm{Cat}$$

is the category whose objects correspond to subgroup inclusions $H \overset{\iota}{\hookrightarrow} G$ and whose morphisms are G-equivariant functions, hence morphisms of G-sets (2.18), between the corresponding coset spaces $G/H_1 \longrightarrow G/H_2$.

Example 2.2.9 (Systems of fixed point spaces)**.** Consider a topological space equipped with a G-action $G \circlearrowright X \;\in\; G\mathrm{DTopSpc}$ (Def. 2.2.1) and $H \subset G$ a subgroup. Then a G-equivariant function $G/H \xrightarrow{f} X$ from the corresponding G-orbit (Def. 2.2.8) is determined by its image $f\big([e]\big) \in X$ of the class of the neutral element, and that image has to be fixed by the action of $H \subset G$ of X. Therefore, the corresponding G-equivariant mapping spaces (2.16)

$$\mathrm{Map}\big(G/H,X\big)^G \;\simeq\; X^H \;:=\; \Big\{ x \in X \,\big|\, \underset{h \in H \subset G}{\forall} (h(x) = x) \Big\} \;\subset X$$

are the topological subspaces of H-fixed points inside X (2.15). By functoriality of the mapping space construction, these fixed point spaces are exhibited as arranging into a topological presheaf on the G-orbit category (Def. 2.2.8):

$$X^{(-)} \;:\; G\mathrm{Orb}^{\mathrm{op}} \xrightarrow{\mathrm{Map}(-,X)^G} \mathrm{TopSpc}$$

Proposition 2.2.10 (Elmendorf's theorem [El83][DwKa84, §1.2, 1.7 & Thm. 3.1], see [Blu17, Thm. 1.3.6 and 1.3.8])**.** *Let G be a finite group. The functor which sends a G-space $G \circlearrowright X$ (Def. 2.2.1) to its system of H-fixed point spaces (Ex. 2.2.9) constitutes an equivalence of ∞-categories*

$$\begin{aligned} G\mathrm{Grpd}_\infty &\xrightarrow{\;\;\simeq\;\;} \mathrm{Shv}_\infty\big(G\mathrm{Orb}\big) \\ G \circlearrowright X &\longmapsto X^{(-)} = \mathrm{Map}\big(-,X\big)^G\,. \end{aligned} \tag{2.23}$$

Part I

Cohesion

3

Higher geometry

We recall basics of higher topos theory in §3.1 and lay out in §3.2 the *internal formulation*, in ∞-toposes, of group actions and the classification of fiber bundles.

3.1 Topos theory

We briefly record basics of ∞-topos theory [TV05][Lu09a][Re10] (review is in [Re19], exposition with an eye towards differential geometric applications is in [FSS14]). This is to set up our notation and to highlight some less widely used aspects that we need further below.

3.1.1 Categories

We make free use of the language and the basic facts of category theory and homotopy theory (see [GJ99][Rie14][Ri20]) as well as of ∞-category theory (see [Joy08a][Joy08a][Lu09a][Rie14][Ci19]).

(i) We write Cat_∞ for the ("very large") ∞-category of (large) ∞-categories [Re98][Be05][Lu09a, Ch. 3], though we only use this for declaring ∞-categories. Inside Cat_∞, there is the sequence of full sub-∞-categories (Def. 3.1.1) of n-categories (i.e.: $(n,1)$-categories) as well as of n-groupoids (see Def. 3.1.12) for all $n \in \mathbb{N}$, denoted thus:

$$\begin{array}{ccccccccc} \mathrm{Sets} & \hookrightarrow & \mathrm{Cat}_1 & \hookrightarrow & \mathrm{Cat}_2 & \hookrightarrow & \cdots & \hookrightarrow & \mathrm{Cat}_\infty \\ \| & & \uparrow & & \uparrow & & & & \uparrow \dashv \downarrow \mathrm{Core} \\ \mathrm{Sets} & \hookrightarrow & \mathrm{Grpd}_1 & \hookrightarrow & \mathrm{Grpd}_2 & \hookrightarrow & \cdots & \hookrightarrow & \mathrm{Grpd}_\infty \end{array} \tag{3.1}$$

(ii) Here $\mathrm{Core}(\mathscr{C})$ denotes the maximal ∞-groupoid inside an ∞-category $\mathscr{C}$.

(iii) For $\mathscr{C} \in \mathrm{Cat}_\infty$ and for $X, Y \in \mathscr{C}$ a pair of objects, we write

$$\mathscr{C}(X,Y) \;:=\; \mathrm{Hom}_{\mathscr{C}}(X,Y) \;\in\; \mathrm{Grpd}_\infty \tag{3.2}$$

for the *hom-∞-groupoid*, i.e. the ∞-groupoid of morphisms between them, and higher homotopies between these (see [Lu09a, 1.2.2][DS09]). This is well-defined, up to equivalence of ∞-groupoids, independently of which model for ∞-categories is used, since these are all equivalent to each other [Be06][Be14].

We have no need to specify any particular model for ∞-categories (except for the construction of examples, in §2.1).

Definition 3.1.1 (Fully faithful functor [Lu09a, 1.2.10])**.** For $\mathscr{C}, \mathscr{D} \in \mathrm{Cat}_\infty$ (3.1), a functor $F : \mathscr{C} \longrightarrow \mathscr{D}$ is called *fully faithful*, to be denoted

$$\mathscr{C} \overset{F}{\hookrightarrow} \mathscr{D}, \tag{3.3}$$

if it is an equivalence on all hom-∞-groupoids (3.2):

$$\underset{X,Y \in \mathscr{C}}{\forall} \quad \mathscr{C}(X, Y) \xrightarrow[\simeq]{F_{X,Y}} \mathscr{D}\big(F(X), F(Y)\big). \tag{3.4}$$

In this case we also say that (3.3) exhibits a *full sub-∞-category inclusion*.

3.1.2 Topology

The category of Δ-*generated* or *D-topological* spaces (Remark 3.1.3) is both: a convenient foundation for homotopy theory (Prop. 3.1.4) as well as pivotal for our key example context (Example 4.1.18):

Definition 3.1.2 (Topological spaces)**.** We write

$$\mathrm{CWCmplx} \hookrightarrow \mathrm{DTopSpc} \hookrightarrow \mathrm{TopSpc} \quad \in \mathrm{Cat}_1 \tag{3.5}$$

for (from right to left):

(i) the category of all topological spaces with continuous functions between them;

(ii) the full subcategory on those spaces whose topology coincides with the final topology on the set of continuous functions out of a Euclidean space $\mathbb{R}^n$, hence whose open subsets coincide with those subsets whose pre-images under all continuous functions $\mathbb{R}^n \to X$ are open in $\mathbb{R}^n$, for all $n \in \mathbb{N}$;

(iii) the further full subcategory on those that admit the structure of a CW-cell complex, hence that are homeomorphic to topological spaces which are obtained, starting with the empty space, by gluing on standard n-disks along their $(n-1)$-sphere boundaries, iteratively for $n \in \mathbb{N}$.

Remark 3.1.3 (D-topological is Δ-generated)**.**

(i) Since the topological n-simplex Δ^n_{top} is a retract of the Euclidean space $\mathbb{R}^n$, the condition on $X \in \mathrm{TopSpc}$ of being D-topological (Def. 3.1.2) is equivalent to being Δ-*generated*, in that the open subsets of X are precisely those whose pre-images under all continuous functions of the form $\Delta^n_{\mathrm{top}} \to X$ are open.

(ii) The concept of Δ-generated spaces is due to [Sm][Dug03]; and independently due to [SYH10], where they are called *numerically generated*.

(iii) We say *D-topological* to better bring out their conceptual role, in view of Prop. 3.1.7 below.

Proposition 3.1.4 (D-topological spaces are convenient)**.** *The category of* DTopSpc *(Def. 3.1.2) is a convenient category of topological spaces in the sense of [St67] in that it:*

(i) contains all CW-complexes (3.5) [SYH10, Cor. 4.4];

(ii) has all small limits and colimits [SYH10, Prop. 3.4];

(iii) is locally presentable [FR07, Cor. 3.7];

(iv) is Cartesian closed [SYH10, Cor. 4.6]: the mapping space between $X, Y \in$ DTopSpc is the reflection (3.12) of the internal mapping space **Map** *(3.54) of* DiffSp *[SYH10, Prop. 4.7]:*

$$\mathrm{Map}(X,Y) \;=\; \mathrm{Dtplg}\Big(\mathbf{Map}\big(\mathrm{Cdfflg}(X),\, \mathrm{Cdfflg}(Y)\big)\Big). \tag{3.6}$$

3.1.3 Differential topology

D-topological spaces lend themselves to differential topology via their joint (co-)reflection (Prop. 3.1.7) both into all topological spaces and into diffeological spaces (Def. 3.1.6):

Definition 3.1.5 (Cartesian spaces)**.** We write

$$\mathrm{CrtSpc} \hookrightarrow \mathrm{SmthMfd} \quad \in \mathrm{Cat}_1 \tag{3.7}$$

for the category whose objects are the natural numbers $n \in \mathbb{N}$, thought of as representing the Cartesian spaces $\mathbb{R}^n$, and whose morphisms are the *smooth* functions between these. We regard this category as equipped with the coverage (Grothendieck pre-topology) whose covers are the differentially good open covers (i.e., such that all non-empty finite intersections of patches are *diffeomorphic* to a Cartesian space [FSS12, 6.3.9]).

Definition 3.1.6 (Diffeological spaces)**.**

(i) The category of *diffeological spaces* ([So80][So84][IZ85], see [BH08] [IZ13]) is the full subcategory of sheaves on CrtSpc (Def. 3.1.5)

$$\mathrm{DiffSp} \hookrightarrow \mathrm{Shv}(\mathrm{CrtSpc}) \tag{3.8}$$

on those $X \in \mathrm{Shv}(\mathrm{CrtSpc})$ which are *concrete sheaves* [Du79b] supported on their *underlying set*

$$X_s \;:=\; \mathrm{Shv}(\mathrm{SmthMfd})(*, X) \tag{3.9}$$

in that the canonical function

$$X(U) \hookrightarrow (U_s, X_s) \tag{3.10}$$

is an injection, for all $U \in \mathrm{CrtSpc}$, with U_s denoting their underlying set of U.

(ii) We call

$$X(U) \underset{\text{Prop. 3.1.38}}{\simeq} \mathrm{DiffSp}(U, X) \;\in \tag{3.11}$$

the set of U*-plots* of the diffeological space X.

Proposition 3.1.7 (Topological/diffeological adjunction)**. (i)** *There is an adjunction [SYH10, Prop. 3.1]*

$$\mathrm{TopSpc} \underset{\mathrm{Cdfflg}}{\overset{\mathrm{Dtplg}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{DiffSp} \tag{3.12}$$

between the categories of topological spaces (Def. 3.1.2) and of diffeological spaces (Def. 3.1.6), where

- *the right adjoint* Cdfflg *sends a topological space to the same underlying set equipped with the* **topological diffeology** *whose plots* (3.11) *are precisely the continuous functions;*
- *the left adjoint* Dtplg *sends a diffeological space to the same underlying set equipped with the* **diffeological topology** *("D-topology" [IZ13, 2.38][CSW13]), which is the final topology with respect to all plots* (3.11)*, hence such that a subset is open precisely if its pre-image under all plots is open.*

(ii) *The fixed points* $X \in \mathrm{TopSpc}$ *of this adjunction are the D-topological spaces (Remark 3.1.3)*

$$X \text{ is D-topological} \quad \Leftrightarrow \quad \mathrm{Dtplg}\big(\mathrm{Cdfflg}(X)\big) \xrightarrow[\simeq]{\varepsilon_X} X \,. \tag{3.13}$$

(iii) *The adjunction is idempotent [SYH10, Lemma 3.3], hence factors through the category of D-topological spaces, exhibiting them as a co-reflective subcategory of* TopSpc *and a reflective subcategory of* DiffSp*:*

$$\mathrm{TopSpc} \underset{\mathrm{Cdfflg}}{\overset{\longleftarrow\!\!\hookleftarrow}{\underset{\longrightarrow}{\bot}}} \mathrm{DTopSpc} \underset{}{\overset{\mathrm{Dtplg}}{\underset{\hookrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{DiffSp} \,. \tag{3.14}$$

The following Prop. 3.1.8 is due to [Har13, Thm. 3.3].

Proposition 3.1.8 (Model structure on D-topological spaces)**.**
(i) *The standard cell inclusions define a cofibrantly generated model category structure on* DTopSpc *(Def. 3.1.2).*
(ii) *With respect to this model structure and the standard model structure on* TopSpc*, the co-reflection* (3.12) *becomes a Quillen equivalence:*

$$\mathrm{TopSpc} \underset{\mathrm{Cdfflg}}{\overset{\hookleftarrow}{\underset{\longrightarrow}{\simeq_{\mathrm{Quillen}}}}} \mathrm{DTopSpc} \,. \tag{3.15}$$

Differential geometry.

Definition 3.1.9 (Smooth Manifolds)**.** We write

$$\mathrm{SmthMfd} \in \mathrm{Cat} \tag{3.16}$$

for the category of finite-dimensional paracompact smooth manifolds with smooth functions between them. We regard this as a site with the Grothendieck topology of open covers.

Proposition 3.1.10 (Cartesian spaces are dense in the site of manifolds)**.** *With respect to the coverages in Def. 3.1.9 and Def. 3.1.5, the inclusion* $\mathrm{CrtSpc} \overset{i}{\hookrightarrow} \mathrm{SmthMfd}$ *is a dense sub-site, in that it induces an equivalence of categories of sheaves*

$$\mathrm{Shv}(\mathrm{CrtSpc}) \underset{i_*}{\overset{i^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} \mathrm{Shv}(\mathrm{SmthMfd}) \,. \tag{3.17}$$

Proposition 3.1.11 (Smooth manifolds inside diffeological spaces)**.** *Every* $X \in$ SmthMfd (3.16) *becomes a diffeological space (Def. 3.1.6) on its underlying set by taking its plots* (3.11) *of shape* $U \in$ CrtSpc *to be the ordinary smooth functions:*

$$X(U) \;:=\; \mathrm{SmthMfd}(U, X)\,. \tag{3.18}$$

More generally, every possibly infinite-dimensional Fréchet manifold (e.g. [KS17, 2.2]) becomes a diffeological space this way. Moreover, this constitutes fully faithfull embeddings (Def. 3.1.1) into the category of Diffeological spaces [Lo94, Thm. 3.1.1]:

$$\underset{\text{finite-dimensional}}{\mathrm{SmthMfd}} \hookrightarrow \underset{\substack{\text{possibly}\\ \text{infinite-dimensional}}}{\mathrm{FréSmthMfd}} \hookrightarrow \mathrm{DiffSp}\,. \tag{3.19}$$

3.1.4 Homotopy theory

Definition 3.1.12 (∞-Groupoids)**.**

(i) We write

$$\mathrm{Grpd}_\infty \;\in\; \mathrm{Cat}_\infty \tag{3.20}$$

for the ∞-category which is presented by the topologically enriched category whose objects are the CW-complexes (3.5) and whose hom-spaces are the mapping spaces (3.6).

(ii) The full sub-∞-category (Def. 3.1.1) on the homotopy n-types is that of *n-groupoids*

$$\mathrm{Grpd}_n \hookrightarrow \mathrm{Grpd}_\infty\,. \tag{3.21}$$

Definition 3.1.13 (Topological shape)**.**

(i) We write

$$\mathrm{Shp}_{\mathrm{Top}} \;:\; \mathrm{CWCmplx} \longrightarrow \mathrm{Grpd}_\infty \tag{3.22}$$

for the ∞-functor from the 1-category of CW-complexes (3.5) to the ∞-category of ∞-groupoids (Def. 3.1.12) which, as a topologically enriched functor, is the identity on objects, and is on hom-spaces the continuous map given by the identity function from the discrete set of continuous maps to the mapping space (3.6).

(ii) For any choice of CW-approximation functor

$$\mathrm{TopSpc} \xrightarrow{\;(-)_{\mathrm{cof}}\;} \mathrm{CWCmplx} \tag{3.23}$$

we get the corresponding functor on all topological spaces (Def. 3.1.2), hence on D-toplogical spaces (Def. 3.1.2) which we denote by the same symbol:

$$\mathrm{Shp}_{\mathrm{Top}} \;:\; \mathrm{TopSpc} \xrightarrow{\;(-)_{\mathrm{cof}}\;} \mathrm{CWCmplx} \xrightarrow{\;\mathrm{Shp}_{\mathrm{Top}}\;} \mathrm{Grpd}_\infty\,. \tag{3.24}$$

Example 3.1.14 (Delooping groupoids)**.** For $G \in \mathrm{Grp}^{\mathrm{fin}}$, consider the groupoid with a single object $*$, and with G as its set of morphisms, whose composition is given by the product in the group:

$$\begin{array}{ccccc} & & * & & \\ & {}^{g_1}\nearrow & & \searrow^{g_2} & \\ * & & \xrightarrow[g_2 \cdot g_1]{} & & * \end{array} \tag{3.25}$$

This groupoid is the topological shape (3.1.13) of the Eilenberg-MacLane space $K(G,1)$ as well as (since G is assumed to be finite) the classifying space BG. More intrinsically, this groupoid is, equivalently, the homotopy quotient of the point by the trivial G-action:

$$* /\!\!/ G \;\in\; \mathrm{Grpd}_1 \hookrightarrow \mathrm{Grpd}_\infty\,. \tag{3.26}$$

More generally:

Example 3.1.15 (Action groupoids)**.** For $G \in \mathrm{Grp}^{\mathrm{fin}}$ a finite group and for $X \in$ a set equipped with a G-action

$$\begin{array}{ccc} G \times X & \xrightarrow{\;\rho\;} & X \\ (g,x) & \longmapsto & g \cdot x \end{array} \tag{3.27}$$

the corresponding *action groupoid* has as objects the elements of X and its morphisms and their composition are given as follows:

$$x \xrightarrow{g_1} g_1 \cdot x \xrightarrow{g_2} g_2 \cdot g_1 \cdot x, \qquad x \xrightarrow{g_2 \cdot g_1} g_2 \cdot g_1 \cdot x \tag{3.28}$$

This action groupoid is a model for the homotopy quotient of X by its G-action

$$X /\!\!/ G \;\in\; \mathrm{Grpd}_1 \hookrightarrow \mathrm{Grpd}_\infty\,. \tag{3.29}$$

The following elementary example plays a pivotal role in later constructions (Lem 5.1.7):

Example 3.1.16 (Hom-groupoid into action groupoid)**.** Let $G \in \mathrm{Grp}^{\mathrm{fin}}$, $X \in$ equipped with a G-action (3.27), hence with action groupoid/homotopy quotient $X /\!\!/ G \in \mathrm{Grpd}_1$ (Example 3.1.15). Let $K \in \mathrm{Grp}^{\mathrm{fin}}$ be any finite group, with $* /\!\!/ K \in \mathrm{Grpd}_1$ its delooping groupoid (Example 3.1.14). Then the hom-groupoid (functor groupoid) of morphisms (functors) $* /\!\!/ K \longrightarrow X /\!\!/ G$ is, equivalently, the action groupoid of G acting on the set of pairs consisting of a group homomorphism $\phi : K \to G$ and a point in X fixed by the image of ϕ:

$$\mathrm{Grpd}_1\big(* /\!\!/ K,\, X /\!\!/ G\big) \;\simeq\; \Bigg(\coprod_{\phi \in \mathrm{Grp}(K,G)} X^{\phi(K)}\Bigg) /\!\!/ G\,. \tag{3.30}$$

Here

- $\phi(K) \subset G$ denotes the subgroup of G which is image of the group homomorphism $\phi : K \to G$;
- $X^{\phi(K)} = \Big\{ x \in X \;\Big|\; \underset{h \in \phi(K)}{\forall}\, h \cdot x = x \Big\}$ denotes the $\phi(K)$-fixed-point set in X;
- the G-action by which the homotopy quotient is taken is the conjugation action on ϕ, hence $g \cdot \phi \;:=\; \mathrm{Ad}_g \circ \phi$, and the given G-action on $x \in X$.

This follows by direct unwinding of the definition of functors and of natural transformations between the groupoids (3.25) and (3.28).

Definition 3.1.17 (Simplicial-topological shape)**.** Let

$$X_\bullet \;:\; \Delta^{\mathrm{op}} \longrightarrow \mathrm{TopSpc} \tag{3.31}$$

be a simplicial topological space, for instance the nerve of a topological groupoid. Then we say that its *simplicial-topological shape* is the homotopy colimit (Prop. 3.1.36) of its degreewise topological shape (Def. 3.1.13):

$$\mathrm{Shp}_{\mathrm{sTop}}\big(X_\bullet\big) \;:=\; \underset{\longrightarrow}{\lim}\big(\mathrm{Shp}_{\mathrm{Top}}(X)\big)_\bullet \;\in \mathrm{Grpd}_\infty\,. \tag{3.32}$$

The following Prop. 3.1.18 appears as [Wa18, 4.3,. 4.4]:

Proposition 3.1.18 (Simplicial-topological shape of degreewise cofibrant spaces is fat geometric realization)**.** *If $X_\bullet$ is a simplicial topological space* (3.31) *which degreewise admits the structure of a retract of a cell complex (for instance: degreewise a CW-complex* (3.5)*), then its simplicial topological shape* (3.1.17) *is equivalent to its fat geometric realization $\|-\|$ [Se74] (see [HG07, 2.3]):*

$$\underset{\substack{\text{degreewise cofibrant}\\ \text{simplicial topological spaces}}}{X_\bullet \;\in\; \big(\mathrm{TopSpc}_{\mathrm{cof}}\big)^{\Delta_{\mathrm{op}}}} \quad\Rightarrow\quad \underset{\substack{\text{simplicial}\\ \text{topological shape}}}{\mathrm{Shp}_{\mathrm{sTop}}\big(X_\bullet\big)} \;\simeq\; \underset{\substack{\text{fat geometric}\\ \text{realization}}}{\|X_\bullet\|}\,. \tag{3.33}$$

Definition 3.1.19 (Diffeological simplices)**.**

(i) We write

$$\begin{array}{ccc} \Delta & \xrightarrow{\;\Delta^\bullet_{\mathrm{smth}}\;} & \mathrm{DiffSp} \\ [n] & \longmapsto & \Big\{\vec{x}\in\mathbb{R}^{n+1}\,\big\vert\,\sum_i x^i = 1\Big\} \end{array} \tag{3.34}$$

for the *diffeological extended simplicies*, hence for the simplicial object in diffeological spaces (Def. 3.1.6) (in fact in smooth manifolds, under Prop. 3.1.11) which in degree n is the extended n-simplex in $\mathbb{R}^{n+1}$, regarded with its sub-diffeology, and whose face and degeneracy maps are the standard ones (see [CW14, Def. 4.3][BEBP19, p. 1]).

(ii) The induced nerve/realization construction is a pair of adjoint functors (Def. 3.1.24)

$$\mathrm{DiffSp} \underset{\mathrm{Sing}_{\mathrm{diff}}}{\overset{|-|_{\mathrm{diff}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{SimplicialSets} \tag{3.35}$$

between the categories of simplicial sets and of diffeological spaces (Def. 3.1.6), where the right adjoint $\mathrm{Sing}_{\mathrm{diff}}$ sends $X \in \mathrm{DiffSp}$ to its *smooth singular simplicial set*

$$\mathrm{Sing}_{\mathrm{diff}}(X)_\bullet \;:=\; \mathrm{DiffSp}\big(\Delta^\bullet_{\mathrm{diff}}, X\big)\,. \tag{3.36}$$

The following Prop. 3.1.20 is due to [CW14, Prop. 4.14]:

Proposition 3.1.20 (Diffeological singular simplicial set of continuous Diffeology)**.** *For all $X_{\mathrm{top}} \in \mathrm{TopSpc}$ there is a weak homotopy equivalence between the diffeological singular simplicial set (Def. 3.1.19) of its continuous diffeology (Def. 3.1.7) and its ordinary singular simplicial set:*

$$\mathrm{Sing}\big(X_{\mathrm{top}}\big) \;\simeq_{\mathrm{whe}}\; \mathrm{Sing}_{\mathrm{diff}}\big(\mathrm{Cdfflg}(X_{\mathrm{top}})\big)\,. \tag{3.37}$$

Equivalently this means, in the terminology to be introduced in a moment, that the topological shape (3.1.13) *of topological spaces is equivalent to the cohesive shape*

(Def. 4.1.1) of their incarnation as continuous-diffeological spaces (see Example 4.1.18 below):

$$\mathrm{Shp}_{\mathrm{Top}}\big(X_{\mathrm{top}}\big) \;\simeq\; \mathrm{Shp}\big(\mathrm{Cdfflg}(X_{\mathrm{top}})\big) \quad \in \mathrm{Grpd}_\infty\,. \tag{3.38}$$

3.1.5 Universal constructions

All diagrams we consider now are homotopy-coherent, even if we do not notationally indicate the higher cells, unless some are to be highlighted. Similarly, all universal constructions we consider now are ∞-categorical, even if this is not further pronounced by the terminology. In particular, we say "colimit" $\underrightarrow{\lim}$ for "homotopy colimit", "limit" $\underleftarrow{\lim}$ for "homotopy limit" (see Prop. 3.1.36), "Cartesian square" for "homotopy Cartesian square", etc.:

Notation 3.1.21 (Cartesian squares)**.** We say a square in an ∞-category is *Cartesian*, to be denoted

$$\begin{array}{ccc} X \times_B Y & \longrightarrow & Y \\ {\scriptstyle f^*g}\downarrow & {\scriptstyle (\mathrm{pb})} & \downarrow{\scriptstyle g} \\ X & \xrightarrow{\;\;f\;\;} & B \end{array} \tag{3.39}$$

if it is an limit cone over the diagram consisting of f and g. We also say this is the *pullback square* of g along f.

Example 3.1.22 (Pullback of equivalence is equivalence)**.** Let $\mathscr{C} \in \mathrm{Cat}_\infty$. Then a square in $\mathscr{C}$ whose right vertical morphism is an equivalence is Cartesian (Notation 3.1.21) precisely if the left vertical morphism is also an equivalence:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & {\scriptstyle (\mathrm{pb})} & \downarrow{\scriptstyle \simeq} \\ C & \longrightarrow & D \end{array} \qquad \Leftrightarrow \qquad \begin{array}{c} A \\ \downarrow{\scriptstyle \simeq} \\ C \end{array} \tag{3.40}$$

hence precisely if $C \to D$ is equivalent to $A \to B$ in $\mathscr{C}^{\Delta^1}$.

Proposition 3.1.23 (Pasting law [Lu09a, Lemma 4.4.2.1])**.** *In any ∞-category, consider a diagram of the form*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\ D & \longrightarrow & E & \longrightarrow & F \end{array} \tag{3.41}$$

such that the right square is Cartesian (Notation 3.1.21). Then the left square is Cartesian if and only if the total rectangle is Cartesian.

Definition 3.1.24 (Adjoint ∞-functors [Lu09a, 5.2.2.7, 5.2.2.8][RV13, 4.4.4])**.** Let $\mathscr{C}, \mathscr{D} \in \mathrm{Cat}_\infty$ (3.1) and $L : \mathscr{C} \longleftrightarrow \mathscr{D} : R$ two functors between them, back and forth. This is an *adjoint pair* with L *left adjoint* and R *right adjoint*, to be denoted $(L \dashv R)$:

$$\mathscr{D} \underset{R}{\overset{L}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathscr{C} \tag{3.42}$$

if there is a natural equivalence of hom-∞-groupoids (3.2) of the form

$$\mathscr{D}\big(L(-)\,,\,-\big) \quad \simeq \quad \mathscr{C}\big(-\,,\,R(-)\big) \tag{3.43}$$

(This is unique when it exists [Lu09a, Prop. 5.2.1.3, 5.2.6.2]). In this case, one says:

(i) The *adjunction unit* is the natural transformation

$$X \xrightarrow{\eta_X} R \circ L(X) \tag{3.44}$$

which is the (pre-)image under (3.43) of the identity on $R(X)$.

(ii) The *adjunction co-unit* is the natural transformation

$$L \circ R(X) \xrightarrow{\varepsilon_X} X \tag{3.45}$$

which is the image under (3.43) of the identity on $L(X)$.

As in the classical situation of 1-category theory, it follows that:

Proposition 3.1.25 (Triangle identities)**.** *Let* $\mathscr{D} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{C}$ *be a pair of adjoint ∞-functors (Def. 3.1.24). Then their adjunction unit* η *(3.44) and counit* ε *(3.45) satisfy the following natural equivalences:*

(i) *for all* $c \in \mathscr{C}$*,*

$$\begin{array}{ccccc} & & L\circ R\circ L(c) & & \\ & \overset{L(\eta_c)}{\nearrow} & & \overset{\varepsilon_{L(c)}}{\searrow} & \\ L(c) & = & = & = & L(c)\,; \end{array} \tag{3.46}$$

(ii) *for all* $d \in \mathscr{D}$*,*

$$\begin{array}{ccccc} & & R\circ L\circ R(d) & & \\ & \overset{\eta_{R(d)}}{\nearrow} & & \overset{R(\varepsilon_d)}{\searrow} & \\ R(d) & = & = & = & R(d)\,. \end{array} \tag{3.47}$$

Proposition 3.1.26 (Right/left adjoints preserve limits/colimits [Lu09a, 5.2.3.5])**.** *Let* $\mathscr{D} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{C}$ *be a pair of adjoint ∞-functors (Def. 3.1.24) and let* $\mathscr{I} \in \mathrm{Cat}_\infty$ *.*

(i) *If* $X_\bullet : \mathscr{I} \to \mathscr{D}$ *is a diagram whose limit exists, then this limit is preserved by the right adjoint R:*

$$R\big(\varprojlim X_\bullet\big) \;\simeq\; \varprojlim RX_\bullet \tag{3.48}$$

(ii) *If* $X_\bullet : \mathscr{I} \to \mathscr{C}$ *is a diagram whose colimit exists, then this colimit is preserved by the left adjoint L:*

$$L\big(\varinjlim X_\bullet\big) \;\simeq\; \varinjlim LX_\bullet \tag{3.49}$$

Conversely:

Proposition 3.1.27 (Adjoint ∞-functor theorem [Lu09a, 5.5.2.9])**.** *Let* $\mathscr{C}_{1,2} \in \mathrm{Cat}_\infty$ *be presentable (e.g. ∞-toposes, Def. 3.1.30), then an ∞-functor* $\mathscr{C}_1 \longrightarrow \mathscr{C}_2$ *is a:*

(i) *right adjoint (i.e., has a left adjoint, Def. 3.1.24) precisely if it preserves limits (3.48);*

(ii) *left adjoint (i.e., has a right adjoint, Def. 3.1.24) precisely if it preserves colimits (3.49).*

Proposition 3.1.28 (Fully faithful adjoints [Lu09a, 5.2.7.4])**.** *For adjoint ∞-functors (Def. 3.1.24)* $\mathscr{D} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{C}$ *,*

(i) *L is fully faithful* $\mathscr{D} \overset{L}{\hookleftarrow} \mathscr{C}$ *(Def. 3.1.1) iff the adjunction unit* η *(3.44) is an equivalence:* $\mathrm{id} \underset{\simeq}{\overset{\eta}{\longrightarrow}} R \circ L$ *;*

(ii) *R is fully faithful* $\mathscr{D} \overset{R}{\hookrightarrow} \mathscr{C}$ *(Def. 3.1.1) iff the adjunction counit* ε *(3.45) is an equivalence* $L \circ R \underset{\simeq}{\overset{\varepsilon}{\longrightarrow}} \mathrm{id}$ *.*

Proposition 3.1.29 (Idempotent Monads and Comonads)**.** *For* $\mathscr{D} \underset{R}{\overset{L}{\rightleftarrows}} \mathscr{C}$ *a pair of adjoint ∞-functors (Def. 3.1.24):*

(i) *If R is fully faithful (Def. 3.1.1) then* $\bigcirc := R \circ L$ *is idempotent, exhibited by the* $\bigcirc$*-image of the adjunction unit* η *(3.44):*

$$\bigcirc(c) \xrightarrow[\simeq]{\bigcirc(\eta_{L(c)})} \bigcirc \circ \bigcirc(c) \,. \tag{3.50}$$

(ii) *If L is fully faithful (Def. 3.1.1) then* $\square := L \circ R$ *is idempotent, exhibited by the* $\square$*-image of the adjunction counit* η *(3.45):*

$$\square \circ \square(d) \xrightarrow[\simeq]{\square(\varepsilon_{R(d)})} \square(d) \,. \tag{3.51}$$

Proof. Consider case **(i)**, the other case is formally dual. Since R is fully faithful, by assumption, the condition that $\bigcirc(\eta_{L(c)}) := R \circ L(\eta_{L(c)})$ is an equivalence is equivalent to $L(\eta_{L(c)})$ being an equivalence. But, by the triangle identity (Prop. 3.1.25), we have that the composite $\varepsilon_{L(L(c))} \circ L(\eta_{L(c)})$ is an equivalence, while by Prop. 3.1.28 the counit ε is a natural equivalence. By cancellation, this implies that $L(\eta_{L(c)})$ is an equivalence. □

∞-Toposes. For our purposes, we take the following characterization to be the definition of ∞-toposes. This is due to Rezk and Lurie [Lu09a, 6.1.6.8]; we follow the presentation in [NSS12a, Prop. 2.2]:

Definition 3.1.30 (∞-Topos)**.** An ∞-topos $\mathbf{H}$ is a presentable ∞-category with the following properties:

(i) **Universal colimits.** For all morphisms $f : X \longrightarrow B$ and all small diagrams $A : I \longrightarrow \mathbf{H}_{/B}$, there is an equivalence:

$$\underset{\substack{\longrightarrow \\ i}}{\lim} f^* A_i \simeq f^*\big(\underset{\substack{\longrightarrow \\ i}}{\lim} A_i\big) \tag{3.52}$$

between the pullback (3.39) of the colimit and the colimit over the pullbacks of its components.

(ii) **Univalent universes.** For every sufficiently large regular cardinal κ, there exists a morphism $\widehat{\mathrm{Objects}}_\kappa \longrightarrow \mathrm{Objects}_\kappa$ in $\mathbf{H}$, such that for every object $X \in \mathbf{H}$, pullback (3.39) along morphisms $X \longrightarrow \mathrm{Objects}_\kappa$ constitutes an equivalence

$$\begin{array}{ccc} \mathrm{Core}\big(\mathbf{H}_{/_\kappa X}\big) & \simeq & \mathbf{H}\big(X, \mathrm{Objects}_\kappa\big) \\ E & \longmapsto & \vdash E \end{array} \qquad \begin{array}{ccc} E & \longrightarrow & \widehat{\mathrm{Objects}_\kappa} \\ \downarrow & \text{(pb)} & \downarrow \\ X & \xrightarrow[\vdash E]{} & \mathrm{Objects}_\kappa \end{array} \tag{3.53}$$

between the ∞-groupoid core (3.1) of bundles (Notation 3.1.45) which are κ-small over X, and the hom-∞-groupoid (3.2) of morphisms from X to the *object classifier* $\mathrm{Objects}_\kappa$.

Example 3.1.31 (Internal mapping space in an ∞-topos)**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30) and $X \in \mathbf{H}$ an object. As a special case of universality of colimits (3.52), we have that the functor $X \times (-)$ of Cartesian product with X preserves all colimits. Hence, by the adjoint ∞-functor theorem (Prop. 3.1.27), this functor has a right adjoint, to be denoted $\mathbf{Map}(X,-)$, the *internal hom-* or *internal mapping space-* or *mapping stack*-functor:

$$\mathbf{H} \underset{\underset{\text{internal mapping space}}{\mathbf{Map}(X,-)}}{\overset{X\times(-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{H}\,. \tag{3.54}$$

By adjointness, the probes of the internal mapping space over any $U \in \mathbf{H}$ are given by

$$\mathbf{H}\big(U, \mathbf{Map}(X,Y)\big) \simeq \mathbf{H}\big(U \times X\,, Y\big). \tag{3.55}$$

Proposition 3.1.32 (Colimits and equifibered transformations [Lu09a, 6.1.3.9(4)][Re10, 6.5])**.** *Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30), $\mathscr{I}$ a small ∞-category, $X_\bullet, Y_\bullet : \mathscr{I} \to \mathbf{H}$ two $\mathscr{I}$-shaped diagrams.*

(i) *If $X_\bullet \overset{f_\bullet}{\Rightarrow} Y_\bullet$ is a natural transformation which is equifibered [Re10, p. 9], in that its value on all morphisms $i_1 \xrightarrow{\phi} i_2$ in $\mathscr{Y}$ is a Cartesian square (Notation 3.1.21), then the value of $\varinjlim f_\bullet$ on all colimit component morphisms is also Cartesian:*

$$\underset{i_1 \xrightarrow{\phi} i_2}{\forall} \quad \begin{array}{ccc} X_{i_1} & \xrightarrow{f_{i_1}} & Y_{i_1} \\ {\scriptstyle X_\phi}\downarrow & \text{(pb)} & \downarrow{\scriptstyle Y_\phi} \\ X_{i_2} & \xrightarrow[f_{i_2}]{} & Y_{i_2} \end{array} \quad \Rightarrow \quad \underset{i}{\forall} \quad \begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_{i_1} \\ {\scriptstyle q_{X_i}}\downarrow & \text{(pb)} & \downarrow{\scriptstyle q_{Y_i}} \\ \varinjlim X_\bullet & \xrightarrow[\varinjlim f_\bullet]{} & \varinjlim Y_\bullet \end{array} \tag{3.56}$$

(ii) *Let $X_\bullet^{\triangleright} : \mathscr{I}^{\triangleright} \to \mathbf{H}$ be a cocone under $X_\bullet$, with tip $\mathscr{X} \in \mathbf{H}$, and let $Y_\bullet^{\triangleright} : \mathscr{I}^{\triangleright} \to \mathbf{H}$ denote the colimiting cocone under $Y_\bullet$ with tip $\varinjlim Y_\bullet$. If $X_\bullet^{\triangleright} \overset{f_\bullet^{\triangleright}}{\Longrightarrow} Y_\bullet^{\triangleright}$ is a natural transformation of cocone diagrams which is* *equifibered, then $X_\bullet^{\triangleright}$ is a colimiting cocone:*

$$\underset{i_1 \xrightarrow{\phi} i_2}{\forall} \quad \begin{array}{ccc} X_{i_1} & \xrightarrow{f_{i_1}} & Y_{i_1} \\ {\scriptstyle X_\phi}\downarrow & \text{(pb)} & \downarrow{\scriptstyle Y_\phi} \\ X_{i_2} & \xrightarrow[f_{i_2}]{} & Y_{i_2} \end{array} \quad \text{and} \quad \underset{i}{\forall} \quad \begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_{i_1} \\ {\scriptstyle q_{X_i}}\downarrow & \text{(pb)} & \downarrow{\scriptstyle q_{Y_i}} \\ \mathscr{X} & \xrightarrow[\underrightarrow{\lim} f_\bullet]{} & \underrightarrow{\lim}\, Y_\bullet \end{array} \tag{3.57}$$

$$\Rightarrow \qquad \mathscr{X} \simeq \underrightarrow{\lim} X_\bullet .$$

Example 3.1.33 (Initial object in ∞-topos is empty object [Re19, p. 16])**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30). Applying the implication (3.57) in Prop. 3.1.32 to the colimit over the empty diagram, which is the initial object, shows that any object with a morphism to the initial object is itself equivalent to the initial object. Hence if we write

$$\varnothing \in \mathbf{H} \quad \text{s.t.} \quad \underset{X \in \mathbf{H}}{\forall} \big(\mathbf{H}(\varnothing, X) \simeq * \big) \tag{3.58}$$

for the initial object, this means that

$$X \xrightarrow{\exists} \varnothing \qquad \Rightarrow \qquad X \simeq \varnothing . \tag{3.59}$$

Proposition 3.1.34 (Tensoring of ∞-toposes over ∞-groupoids)**.** *Let* $\mathbf{H}$ *be an* ∞*-topos (Def. 3.1.30) with inverse base geometric morphism (Prop. 3.1.43) denoted* $\Delta : \mathrm{Grpd}_\infty \longrightarrow \mathbf{H}$. *Then, for* $S \in \mathrm{Grpd}_\infty$ *and* $X, Y \in \mathbf{H}$*, there is a natural equivalence of* ∞*-groupoids*

$$\mathbf{H}\big(\Delta(S) \times X, Y\big) \simeq \mathrm{Grpd}_\infty\big(S, \mathbf{H}(X, Y)\big). \tag{3.60}$$

Proof. By [Lu09a, Cor. 4.4.4.9] we have, for $S \in \mathrm{Grpd}_\infty \hookrightarrow \mathrm{Cat}_\infty$ and $X, Y \in \mathbf{H}$, natural equivalences

$$\begin{aligned} &\underset{S}{\underrightarrow{\lim}}\, \mathrm{const}_* \simeq S \qquad \text{and} \\ &\mathbf{H}\Big(\underset{S}{\underrightarrow{\lim}}\, \mathrm{const}_X, Y\Big) \simeq \mathrm{Grpd}_\infty\big(S, \mathbf{H}(X,Y)\big). \end{aligned} \tag{3.61}$$

This implies the statement in the form (3.60) by using **(a)** that Δ preserves all colimits as well as finite limits (Prop. 3.1.43) and **(b)** that Cartesian products may be taken inside colimits, as a special case of (3.52):

$$\begin{aligned} \mathbf{H}\big(\Delta(S) \times X, Y\big) &\simeq \mathbf{H}\big(\Delta\big(\underset{S}{\underrightarrow{\lim}}\, *\big) \times X, Y\big) \simeq \mathbf{H}\big(\big(\underset{S}{\underrightarrow{\lim}}\, \underbrace{\Delta(*)}_{\simeq *}\big) \times X, Y\big) \\ &\simeq \mathbf{H}\Big(\big(\underset{S}{\underrightarrow{\lim}}\, \underbrace{(* \times X)}_{\simeq X}\big), Y\Big) \simeq \mathrm{Grpd}_\infty\big(S, \mathbf{H}(X,Y)\big). \end{aligned} \tag{3.62}$$

The composite equivalence is (3.60). ☐

Sheaves.

Notation 3.1.35 (∞-Presheaves)**.** For $\mathscr{C}$ a small ∞-category, we write

$$\mathrm{PShv}_\infty(\mathscr{C}) := \mathrm{Func}_\infty\big(\mathscr{C}^{\mathrm{op}}, \mathrm{Grpd}_\infty\big) \tag{3.63}$$

for the ∞-category of ∞-presheaves on $\mathscr{C}$. More generally, if $\mathbf{H}$ is any ∞-topos (Def. 3.1.30) we also write

$$\mathrm{PShv}_\infty\big(\mathscr{C},\mathbf{H}\big) \;:=\; \mathrm{Func}_\infty\big(\mathscr{C}^{\mathrm{op}},\mathbf{H}\big). \tag{3.64}$$

Proposition 3.1.36 (Limits and colimits in an ∞-topos [Lu09a, Lem. 4.2.4.3])**.** *Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30) and $\mathscr{C}$ a small ∞-category. Then the ∞-functor which sends an object in $\mathbf{H}$ to the $\mathbf{H}$-valued presheaf (3.64) constant on this object has a right- and a left-adjoint (Def. 3.1.24), given by the limit and colimit construction, respectively:*

$$\mathrm{Func}_\infty\big(\mathscr{C},\mathbf{H}\big) \quad \begin{matrix} \xrightarrow{\;\varinjlim\;} \\ \bot \\ \xleftarrow{\;\mathrm{const}\;} \\ \bot \\ \xrightarrow[\;\varprojlim\;]{} \end{matrix} \quad \mathbf{H} \tag{3.65}$$

Proposition 3.1.37 (∞-Yoneda embedding [Lu09a, Lemma 5.5.2.1])**.** *Let $\mathscr{C}$ be an ∞-category. Then the ∞-functor from $\mathscr{C}$ to its ∞-presheaves (3.63) which assigns representable presheaves*

$$\begin{aligned} \mathscr{C} &\xhookrightarrow{\;y\;} \mathrm{PShv}_\infty(\mathscr{C}) \\ c &\longmapsto \mathscr{C}(-,c) \end{aligned} \tag{3.66}$$

is fully faithful (Def. 3.1.1).

Proposition 3.1.38 (∞-Yoneda lemma [Lu09a, Lemma 5.5.2.1])**.** *Let $\mathscr{C}$ be an ∞-category. Then for $X \in \mathrm{PShv}_\infty(\mathscr{C})$ (3.63) and $c \in \mathscr{C}$, there is a natural equivalence*

$$\mathrm{PShv}_\infty\big(y(c),X\big) \;\simeq\; X(c)\,, \tag{3.67}$$

where y is the Yoneda embedding (3.66) from Prop. 3.1.37.

Proposition 3.1.39 ((Co-)Limits of presheaves are computed objectwise [Lu09a, Cor. 5.1.2.3])**.** *Let $\mathbf{H}$ be an ∞-topos, let $\mathscr{C}$ and $\mathscr{D}$ be small ∞-categories, and let*

$$I \;:\; \mathscr{D} \longrightarrow \mathrm{PShv}_\infty(\mathscr{C},\mathbf{H}) \tag{3.68}$$

be a diagram of $\mathbf{H}$-valued ∞-presheaves over $\mathscr{C}$. Then the limit and colimit over I exist and are given objectwise over $c \in \mathscr{C}$ by the limit and colimit of the components in Grpd_∞*:*

$$\begin{aligned} \big(\varinjlim I\big) \;&:\; c \longmapsto \big(\varinjlim I_c\big)\,, \\ \big(\varprojlim I\big) \;&:\; c \longmapsto \big(\varprojlim I_c\big)\,. \end{aligned}$$

Lemma 3.1.40 (Colimit of representable functor is contractible)**.** *Let $\mathscr{C}$ be a small ∞-category, and consider an ∞-functor $yC : \mathscr{C}^{\mathrm{op}} \longrightarrow \mathrm{Grpd}_\infty$ to the ∞-category of ∞-groupoids (3.20), which is representable, hence which is in the essential image of the ∞-Yoneda embedding (3.66). Then the colimit of this functor is contractible:*

$$\varinjlim_{\mathscr{C}}(yC) \;\simeq\; *\,. \tag{3.69}$$

Proof. The terminal $* \in \mathrm{Grpd}_\infty$ is characterized by the fact that for $S \in \mathrm{Grpd}_\infty$ there is a natural equivalence

$$S \;\simeq\; \mathrm{Grpd}_\infty\big(*\,,S\big)\,. \tag{3.70}$$

Hence it is sufficient to see that $\varinjlim(yC)$ satisfies the same property. But we have the following sequence of natural equivalences:

$$\begin{aligned}\mathrm{Grpd}_\infty\Big(\varinjlim(yC)\,,S\Big) &\simeq \mathrm{Func}_\infty(\mathscr{C}^{op})\big(yC\,,\mathrm{const}\big)\\ &\simeq (\mathrm{const}\,S)(C) \simeq S\,.\end{aligned} \tag{3.71}$$

Here the first step is the adjunction (3.65), while the second step is the ∞-Yoneda lemma (Prop. 3.1.38). □

Proposition 3.1.41 (Topos is accessibly lex reflective in presheaves over site [Lu09a, 6.1.0.6])**.** *Let* $\mathbf{H}$ *be an ∞-topos (Def. 3.1.30).*

(i) *Then there exists an ∞-site for* $\mathbf{H}$*, namely a small* $\mathscr{C} \in \mathrm{Cat}_\infty$ *equipped with a pair of adjoint ∞-functors (Def. 3.1.24) between* $\mathbf{H}$ *and* $\mathrm{PShv}_\infty(\mathscr{C})$ *(Notation 3.1.35):*

$$\mathbf{H} \underset{\hookrightarrow}{\overset{L}{\underset{\bot}{\longleftarrow}}} \mathrm{PShv}_\infty\big(\mathscr{C}\big) \tag{3.72}$$

such that **(a)** *the right adjoint is accessible and fully faithful (Def. 3.1.1) and* **(b)** *the left adjoint preserves finite limits (in addition to preserving all colimits, by Prop. 3.1.26).*

(ii) *Conversely, any such accessibly embedded lex reflective sub-∞-category of an ∞-category of ∞-presheaves is an ∞-topos.*

Definition 3.1.42 (Sheaf ∞-topos [Lu09a, 6.2])**.** An ∞-topos $\mathbf{H}$ (Def. 3.1.30) is called an *∞-category of ∞-sheaves* or *of ∞-stacks*, or just a *sheaf topos* for short, to be denoted

$$\mathbf{H} \simeq \mathrm{Shv}_\infty\big(\mathscr{C}\big) \tag{3.73}$$

if there exists a *site* $\mathscr{C}$, namely a small $\mathscr{C} \in \mathrm{Cat}_\infty$ with a reflection $(L\mathrm{const} \dashv \Gamma)$ (3.72) as in Prop. 3.1.41, such that $L\mathrm{const}$ exhibits localization at a set

$$\Big\{\underbrace{U \hookrightarrow y(c)}_{\text{covering sieves}}\Big\} \subset \coprod_{c\in\mathscr{C}} \mathrm{SubObjects}\big(y(c)\big) \tag{3.74}$$

of monomorphisms (Def. 3.1.59) into representable presheaves (3.66).

Proposition 3.1.43 (Base geometric morphism [Lu09a, 6.3.4.1])**.** *Let* $\mathbf{H}$ *be an ∞-topos (Def. 3.1.30). There is an essentially unique pair of adjoint ∞-functors (Def. 3.1.24) between* $\mathbf{H}$ *and* Grpd_∞ *(Def. 3.1.12)*

$$\mathbf{H} \underset{\Gamma}{\overset{L\mathrm{const}}{\underset{\bot}{\rightleftarrows}}} \mathrm{Grpd}_\infty \tag{3.75}$$

such that the left adjoint $L\mathrm{const}$ *preserves finite limits (in addition to preserving all colimits, by Prop. 3.1.26).*

Example 3.1.44 (Base geometric morphism via site)**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30) and $\mathscr{C}$ a site (Prop. 3.1.41). Then the composite of pairs of adjoint ∞-functors

(Def. 3.1.24)

$$\mathbf{H} \underset{\longhookrightarrow}{\overset{L}{\underset{\bot}{\longleftarrow}}} \mathrm{PShv}_\infty\big(\mathscr{C}\big) \underset{\underset{\longleftarrow}{\lim}}{\overset{\mathrm{const}}{\underset{\bot}{\longleftarrow}}} \mathrm{Grpd}_\infty \tag{3.76}$$

of **(a)** the reflection into presheaves over the site (Prop. 3.1.41) with **(b)** the limit-construction on presheaves (Prop. 3.1.36) is such that the composite left adjoint Lconst preserves finite limits (since L does by Prop. 3.1.41 and const does by Prop. 3.1.26 with Prop. 3.1.36). Hence, by the essential uniqueness of Prop. 3.1.43, the composite (3.76) is a factorization of the base geometric morphism of $\mathbf{H}$.

Bundles.

Notation 3.1.45 (Bundles and slicing.)**.** Let $\mathbf{H}$ an ∞-topos (Def. 3.1.30) and $X \in \mathbf{H}$ an object. We write:

(i) $(X,p) \in \mathbf{H}_{/X}$ for objects in the slice ∞-category of $\mathbf{H}$ over X, corresponding to morphisms p to X in $\mathbf{H}$ (*bundles* over X):

$$\begin{array}{c} E \\ \big\downarrow{\scriptstyle p} \\ X \end{array} \tag{3.77}$$

(ii) $(f,\alpha) \in \mathbf{H}_{/X}\big((E_1,p_1),(E_2,p_2)\big)$ for morphisms in the slice ∞-category, corresponding to diagrams in $\mathbf{H}$ of the form

$$\begin{array}{ccc} E_1 & \xrightarrow{\quad f \quad} & E_2 \\ & {\scriptstyle p_1}\searrow \;\overset{\alpha}{\Rightarrow}\; \swarrow{\scriptstyle p_2} & \\ & X & \end{array} \tag{3.78}$$

Proposition 3.1.46 (Slice ∞-topos [Lu09a, Prop. 6.3.5.1 (1)])**.** *Let* $\mathbf{H}$ *be an ∞-topos (Def. 3.1.30) and $X \in \mathbf{H}$ an object. Then the slice ∞-category $\mathbf{H}_{/X}$ (Notation 3.1.45) is also an ∞-topos.*

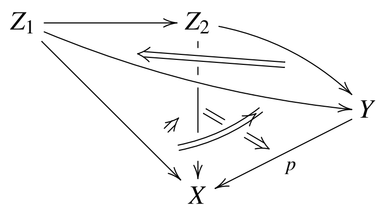
Example 3.1.47 (Iterated slice ∞-topos)**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30), $X \in \mathbf{H}$ and $(Y,p) \in \mathbf{H}_{/X}$ an object in the slice, hence (Notation 3.1.45) a morphism $Y \xrightarrow{p} X$. Then $\mathbf{H}_{/X}$ is itself an ∞-topos, by Prop. 3.1.46, and we may slice again to obtain the iterated slice ∞-topos

$$\big(\mathbf{H}_{/X}\big)_{/(Y,p)} \in \mathrm{Cat}_\infty \,. \tag{3.79}$$

(i) an object in (3.79) is a diagram in $\mathbf{H}$ of this form:

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ & \searrow \;\Rightarrow\; \swarrow{\scriptstyle p} & \\ & X & \end{array} \tag{3.80}$$

(ii) a morphism in (3.79) is a diagram in $\mathbf{H}$ of this form (which is furthermore filled by a 3-morphism, that we notationally suppress, for readability):



(3.81)

Proposition 3.1.48 (Hom-∞-groupoids in slices [Lu09a, Prop. 5.5.5.12])**.** *Let* $\mathbf{H}$ *be an* ∞*-topos (Def. 3.1.30) and* $B \in \mathbf{H}$ *an object. Then for* $(X_1, p_1), (X_2, p_2) \in \mathbf{H}_{/B}$ *two objects in the slice over* B *(Prop. 3.1.46) the hom-*∞*-groupoid between them is given by the following homotopy fiber-product of hom-*∞*-groupoids of* $\mathbf{H}$*:*

$$\mathbf{H}_{/B}\big((X_1, p_1), (X_2, p_2)\big) \;\simeq\; \{p_1\} \underset{\mathbf{H}(X_1, B)}{\times} \mathbf{H}(X_1, X_2) \tag{3.82}$$

hence by the ∞*-groupoid given by the following Cartesian square (Notation 3.1.21):*

$$\begin{array}{ccc} \mathbf{H}_{/B}\big((X_1, p_1), (X_2, p_2)\big) & \longrightarrow & \mathbf{H}(X_1, X_2) \\ \big\downarrow & \text{(pb)} & \big\downarrow {\scriptstyle p_2 \circ (-)} \\ * & \underset{\vdash p_1}{\longrightarrow} & \mathbf{H}(X_1, B)\,. \end{array} \tag{3.83}$$

Proposition 3.1.49 (Base change [Lu09a, HTT 6.3.5])**.** *Let* $\mathbf{H}$ *be an* ∞*-topos (Def. 3.1.30). Then for every morphism* $X \xrightarrow{f} Y$ *in* $\mathbf{H}$ *there is an induced base change adjoint triple (Def. 3.1.24) between the corresponding slice* ∞*-toposes (Prop. 3.1.46):*

$$\mathbf{H}_{/X} \;\underset{f_*}{\overset{f_!}{\underset{\bot}{\overset{\bot}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xleftarrow{\;f^*\;}}}}}}}\; \mathbf{H}_{/Y} \tag{3.84}$$

where, in $\mathbf{H}$*,* $f_!$ *is given by postcomposition with* f *while* f^* *is given by pullback along* f*.*

Example 3.1.50 (Bundle morphisms covering base morphisms)**.** For $\mathbf{H}$ an ∞-topos (Def. 3.1.30), the system of all its slice ∞-toposes (Prop. 3.1.46)

$$\begin{array}{ccc} \mathbf{H}^{\mathrm{op}} & \xrightarrow{\mathbf{H}_{/(-)}} & \mathrm{Cat}_\infty \\ X & \longmapsto & \mathbf{H}_{/X} \end{array} \tag{3.85}$$

related via contravariant base change (3.84) arranges into the "arrow ∞-topos" [Lu09a, 2.4.7.12]

$$\mathrm{Bundles}(\mathbf{H}) \;:=\; \int_X \mathbf{H}_{/X} \;\simeq\; \mathbf{H}^{\Delta[1]}\,, \tag{3.86}$$

which, in view of Notation 3.1.45, may be thought of as the ∞-category of bundles in $\mathbf{H}$, but now with bundle morphisms allowed to cover non-trivial base morphisms.

Example 3.1.51 (Spectral bundles and tangent ∞-topos)**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30). Instead of the system (3.85) of its plain slices, consider the corresponding system of *stabilized* slices (stabilized under the suspension/looping adjunction on pointed objects, e.g. [Lu07, 1.4]):

$$\begin{array}{ccc} \mathbf{H}^{\mathrm{op}} & \xrightarrow{\mathrm{Stab}\left(\mathbf{H}_{/(-)}\right)} & \mathrm{Cat}_\infty \\ X & \longmapsto & \mathrm{Stab}\left(\mathbf{H}_{/X}\right) \end{array} \tag{3.87}$$

The resulting total ∞-category

$$\mathrm{SpectralBundles}(\mathbf{H}) := \int_X \mathrm{Stab}\left(\mathbf{H}_{/X}\right), \tag{3.88}$$

is that of *bundles of spectra* in $\mathbf{H}$ (parametrized spectrum objects). Remarkably, this is itself an ∞-topos [Joy08a, 35.5][Lu17, 6.1.1.11], also called the *tangent ∞-topos* $T\mathbf{H}$ of $\mathbf{H}$ (e.g. [Lu07][BM19]).

Example 3.1.52 (Base change along terminal morphism)**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30) and $X \in \mathbf{H}$ any object. With $\mathbf{H} \simeq \mathbf{H}_{/*}$ regarded as its own slice (Prop. 3.1.46) over the terminal object, base change (Prop. 3.1.49) along the terminal morphism $X \to *$ is of the form

$$\mathbf{H}_{/X} \; \underset{\longrightarrow}{\overset{\xrightarrow{\mathrm{dom}}}{\underset{\bot}{\overset{\bot}{\xleftarrow{X\times(-)}}}}} \; \mathbf{H} \tag{3.89}$$

where **(a)** the top functor sends a morphism $Y \to X$ to its domain object Y, and **(b)** the middle functor is Cartesian product with X. In particular, it follows that:

(i) The base geometric morphism (Prop. 3.1.24) of the slice ∞-topos $\mathbf{H}_{/X}$ (Prop. 3.1.46) is given by

$$\left(\Delta \dashv \Gamma\right) \simeq \left((X \to *)^* \dashv (X \to *)_*\right) \tag{3.90}$$

(since $(X \to *)^*$ is a left adjoint that also preserves finite limits, as it is also a right adjoint, Prop. 3.1.26).

(ii) The forgetful functor $\mathrm{dom} : \mathbf{H}_{/X} \to \mathbf{H}$ is a left adjoint $(X \to *)_!$ and hence preserves all colimits (Prop. 3.1.26).

While dom (3.89) does not preserve all limits, it does preserve fiber products:

Proposition 3.1.53 (Fiber products in slice ∞-toposes)**.** *Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30), $B \in \mathbf{H}$, $\mathbf{H}_{/B}$ the slice ∞-topos (Prop. 3.1.46) and $\mathbf{H}_{/B} \xrightarrow{\mathrm{dom}} \mathbf{X}$ its forgetful functor* (3.89) *from Example 3.1.52.*

(i) *Given a cospan $(Y,\phi_Y) \to (X,\phi_X) \leftarrow (Z,\phi_Z)$ in $\mathbf{H}_{/B}$, the underlying object of its fiber product is the fiber product of its underlying objects:*

$$\mathrm{dom}\left((Y,\phi_Y) \underset{(X,\phi_X)}{\times} (Z,\phi_Z) \right) \simeq Y \underset{X}{\times} Z. \tag{3.91}$$

(ii) *In particular, since (X,id_X) is the terminal object in $\mathbf{H}_{/X}$, so that the plain product in the slice is*

$$(Y,\phi_Y) \times (Z,\phi_Z) = (Y,\phi_Y) \underset{(X,\mathrm{id}_X)}{\times} (Z,\phi_Z), \tag{3.92}$$

we have the that product in $\mathbf{H}_{/X}$ is given by the fiber product over X in $\mathbf{H}$:

$$\mathrm{dom}\Big((Y,\phi_Y)\times(Z,\phi_Z)\Big) \;\simeq\; Y \underset{X}{\times} Z\,. \tag{3.93}$$

Proof. Generally, limits in $\mathbf{H}_{/X}$ are given by limits in $\mathbf{H}$ over the underlying co-cone diagram. Specifically: for $Y:\ \mathscr{I} \to \mathbf{H}$ we have $\mathrm{dom}\big(\varprojlim Y_\bullet\big) \simeq \varprojlim \big(Y/X\big)_\bullet$. With this, the claim follows via [Lu09a, Prop. 4.1.1.8] from the fact that the canonical inclusion of diagram categories

$$\Big\{\; y \longrightarrow b \longleftarrow z \;\Big\} \;\hookrightarrow\; \left\{ \begin{array}{ccccc} y & \longrightarrow & b & \longleftarrow & z \\ & \searrow & \downarrow & \swarrow & \\ & & t & & \end{array} \right\} \tag{3.94}$$

is an initial functor (i.e., under $(-)^{\mathrm{op}}$ it is a final functor), as one finds by direct inspection from [Lu09a, Prop. 4.1.3.1]. □

Proposition 3.1.54 (Terminal right base change of bare ∞-groupoids)**.** *In the base ∞-topos $\mathbf{H} = \mathrm{Grpd}_\infty$ (3.20), the right base change along the terminal morphism (Example 3.1.52) of an object $X \in \mathrm{Grpd}_\infty$ is given by the hom-∞-groupoid out of X, regarded as the terminal object in the slice:*

$$(X \to *)_* \;\simeq\; \mathbf{H}_{/X}\big(X,-\big) \;\;:\;\; \big(\mathrm{Grpd}_\infty\big)_{/X} \longrightarrow \mathrm{Grpd}_\infty\,. \tag{3.95}$$

Proof. We have the following chain of natural equivalences:

$$\begin{aligned} &\mathrm{Grpd}_\infty\big(A,(\mathrm{Grpd}_\infty)_{/X}(X,B)\big) \;\simeq\; (\mathrm{Grpd}_\infty)_{/X}\big(\Delta(A)\times_X X, B\big) \\ &\simeq\; (\mathrm{Grpd}_\infty)_{/X}\big(\Delta(A),B\big) \;\simeq\; (\mathrm{Grpd}_\infty)_{/X}\big((X\to *)^*(A),B\big). \end{aligned} \tag{3.96}$$

Here the first step observes that the slice $(\mathrm{Grpd}_\infty)_{/X}$ is itself an ∞-topos by Prop. 3.1.46, so that the tensoring equivalence of Prop. 3.1.34 applies. The second step uses the fact that X is regarded as the terminal object in its own slice, so that forming Cartesian product with it is equivalently the identity operation. The last step observes that for the slice ∞-topos $\Delta \simeq (X \to *)^*$ (3.90) by Example 3.1.52. In summary, the total equivalence of (3.96) is the hom-equivalence that characterizes $\mathbf{H}_{/X}(X,-)$ as a right adjoint to $(X \to *)^*$. □

Proposition 3.1.55 (Conservative base change along effective epi [NSS12a, 3.15])**.** *Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30). For $Y \twoheadrightarrow X$ an effective epimorphism (Def. 3.1.63) in $\mathbf{H}$, the induced base change (Prop. 3.1.49)*

$$\mathbf{H}_{/X} \xrightarrow{\;i^*\;} \mathbf{H}_{/Y} \tag{3.97}$$

is a conservative ∞-functor, meaning that a morphism $f \in \mathbf{H}_{/X}$ is an equivalence if its base change $i^*(f)$ in $\mathbf{H}_{/Y}$ is an equivalence.

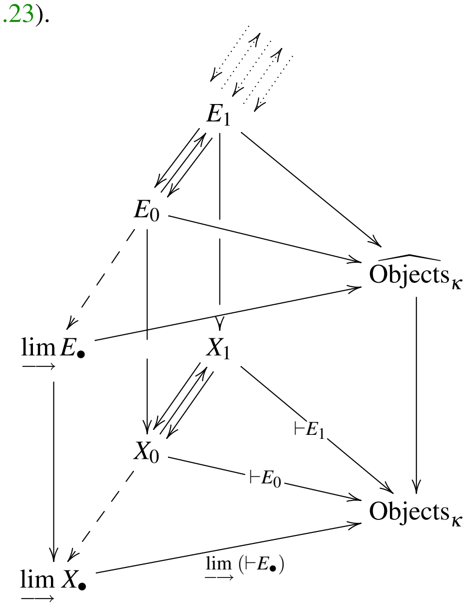
Proposition 3.1.56 (Colimits of classifying maps are classifying maps of colimits)**.** *Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30), $\mathscr{I}$ a small ∞-category, $X_\bullet : \mathscr{I} \to \mathbf{H}$ a diagram and $(\vdash E)_\bullet : X_\bullet \to \mathrm{const}_{\mathrm{Objects}_\kappa}$ a transformation to the diagram constant on the object classifier (3.53), thus classifying a diagram $E_\bullet : \mathscr{I} \to \mathbf{H}$ of bundles over $X_\bullet$. Then* *the colimit of* $(\vdash E)_\bullet$ *formed in the slice* $\mathbf{H}_{/\mathrm{Objects}_\kappa}$ *(Prop. 3.1.46) is the colimit of* $X_\bullet$ *equipped with the classifying map for the colimit of* $E_\bullet$*:*

$$\varinjlim (\vdash E)_\bullet \;\simeq\; \vdash \big(\varinjlim E_\bullet\big)\,. \tag{3.98}$$

Proof. Since underlying the colimit $\varinjlim (\vdash E)_\bullet$ in the slice ∞-topos $\mathbf{H}_{/\mathrm{Objects}_\kappa}$ is the colimit $\varinjlim X_\bullet$ in $\mathbf{H}$ (by Example 3.1.52) we are dealing with a situation as shown in the diagram on the right (where a simplicial diagram shape is shown just for definiteness of illustration). We need to demonstrate that the front square in this diagram is Cartesian. Observe that

(a) the vertical squares over each $\vdash E_i$ are Cartesian by assumption, whence

(b) also the solid vertical squares over each $X_i \to X_j$ are Cartesian, by the pasting law (Prop. 3.1.23).



(3.99)

This means that the assumption of Prop. 3.1.32 is satisfied for the left part of the diagram (regarded as a transformation of diagrams from top to bottom) implying that the dashed square is Cartesian.

This implies, together with **(a)**, that the front square is Cartesian, again by the pasting law (Prop. 3.1.23). □

n-Truncation.

Definition 3.1.57 (*n*-truncated objects [Lu09a, Def. 5.5.6.1])**.** Let $n \in \{-2,-1,0,1,2,\cdots\}$.

(i) An ∞-groupoid is called *n-truncated* for $n \geq 0$ if all its homotopy groups of degree $> n$ are trivial. It is called (-1)*-truncated* if it is either empty or contractible, and *(-2)-truncated* if it is (non-empty and) contractible.

(ii) Let $\mathscr{C}$ be an ∞-category. Then an object $X \in \mathscr{C}$ is *n-truncated* if for all objects $U \in \mathscr{C}$ the hom-∞-groupoid $\mathscr{C}(U,X)$ is n-truncated, in the above sense.

Definition 3.1.58 (n-truncated morphisms [Lu09a, Def. 5.5.6.8]). Let $n \in \{-2,-1,0,1,2,\cdots\}$.

(i) A morphism of ∞-groupoids is called *n-truncated* if all its homotopy fibers are n-truncated ∞-groupoids according to Def. 3.1.57.

(ii) Let $\mathscr{C}$ be an ∞-category. A morphism $X \xrightarrow{f} Y$ in $\mathscr{C}$ is called *n-truncated* if for all objects $U \in \mathscr{C}$ the induced morphism of hom-∞-groupoids $\mathscr{C}(U,X) \xrightarrow{\mathscr{C}(U,f)} \mathscr{C}(U,Y)$ is n-truncated in the above sense.

Definition 3.1.59 (Monomorphisms). A (-1)-truncated morphism (Def. 3.1.58) is also called a *monomorphism*, to be denoted

$$X \hookrightarrow Y\,. \tag{3.100}$$

Proposition 3.1.60 (Monomorphisms are preserved by pushout [Re19, p. 21]). *Let* $\mathbf{H}$ *be an ∞-topos (Def. 3.1.30). Then the class of monomorphisms in* $\mathbf{H}$ *(Def. 3.1.59) is closed under* **(i)** *pullback and* **(ii)** *composition.*

Definition 3.1.61 (Poset of subobjects). Let $\mathbf{H}$ be an ∞-topos and $X \in \mathbf{H}$ any object. Then the *poset of subobjects* of X is the sub-∞-category (Def. 3.1.62) of (-1)-truncated objects of the slice over X:

$$\mathrm{SubObjects}(X) \hookrightarrow \mathbf{H}_{/X} \tag{3.101}$$

whose objects are equivalently the monomorphisms (Def. 3.1.59) $U \hookrightarrow X$.

Proposition 3.1.62 (n-Trucation modality [Lu09a, 5.5.6.18]). *If* $\mathbf{H}$ *is an ∞-topos (Def. 3.1.30), for all $n \in \{-1,0,1,2,\cdots\}$, its full sub-$\infty$-category (Def. 3.1.1) of n-truncated objects (Def. 3.1.57) is reflective, in that the inclusion functor has a left adjoint (Def. 3.1.24):*

$$\underset{\infty\text{-topos}}{\mathbf{H}} \underset{i_n}{\overset{\tau_n}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \underset{\substack{\text{sub-}\infty\text{-category}\\ \text{of } n\text{-truncated objects}}}{\mathbf{H}_{\leq n}} \tag{3.102}$$

We write for the induced n-truncation modality (1.18)*:*

$$\Big(\underset{\text{"}n\text{-truncated"}}{\boldsymbol{\tau}_n} := i_n \circ \tau_n\Big) \;:\; \mathbf{H} \longrightarrow \mathbf{H}\,. \tag{3.103}$$

Definition 3.1.63 (Effective epimorphisms [Lu09a, Cor. 6.2.3.5]). Let $\mathbf{H}$ be an ∞-topos. A morphism in $\mathbf{H}$ is called an *effective epimorphism*, to be denoted

$$Y \overset{f}{\twoheadrightarrow} Z \tag{3.104}$$

if, when regarded as an object of the slice over X (Prop. 3.1.46), its (-1)-truncation (Prop. 3.1.62) is the terminal object

$$\tau_{(-1)}(f) \;\simeq\; * \quad \in \mathbf{H}_{/X}\,. \tag{3.105}$$

We write

$$\mathrm{EffEpi}(\mathbf{H}) \;\subset\; \mathbf{H}^{(0\to 1)} \;\in\; \mathrm{Cat}_\infty \tag{3.106}$$

for the full sub-∞-category (Def. 3.1.1) of the arrow-category of $\mathbf{H}$ on those that are effective epimorphisms.

Definition 3.1.64 (n-Connected morphisms [Lu09a, Prop. 6.5.1.12])**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30) and $n \in \{-1, 0, 1, 2 \cdots\}$. Then a morphism $X \xrightarrow{f} Y$ in $\mathbf{H}$ is called *n-connected* if, regarded as an object in the slice over X (Prop. 3.1.46), its n-truncation (Def. 3.1.62) is the terminal object:

$$Y \xrightarrow{f} X \quad \text{is } n\text{-truncated} \qquad \Leftrightarrow \qquad \tau_n(f) \simeq * \in \mathbf{H}_{/X} \,. \tag{3.107}$$

Hence the (-1)-connected morphisms are equivalently the effective epimorphisms (Def. 3.1.63).

Lemma 3.1.65 (Effective epimorphisms are preserved by pullback [Lu09a, 6.2.3.15])**.** *Let* $\mathbf{H}$ *be an* ∞*-topos (Def. 3.1.30). Then the class of effective epimorphisms in* $\mathbf{H}$ *(Def. 3.1.63) is closed under* **(i)** *pullback and* **(ii)** *composition.*

***n*-Image factorization.**

Proposition 3.1.66 (Connected/truncated factorization system [Lu09a, Ex. 5.2.8.16][Re10, Prop. 5.8])**.** *Let* $\mathbf{H}$ *be an* ∞*-topos. Then, for all* $n \in \{-1, 0, 1, 2, \cdots\}$*, the pair of classes of n-connected/n-truncated morphisms (Def. 3.1.64, Def. 3.1.58) forms an orthogonal factorization system:*

(i) *every morphism* f *in* $\mathbf{H}$ *factors essentially uniquely as*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle n\text{-connected}} \searrow \quad \nearrow {\scriptstyle n\text{-truncated}} & \\ & \mathrm{im}_n(f) & \end{array} \tag{3.108}$$

(ii) *every commuting square as follows has an essentially unique dashed lift:*

$$\begin{array}{ccc} X & \longrightarrow & A \\ {\scriptstyle n\text{-connected}} \downarrow & \nearrow & \downarrow {\scriptstyle n\text{-truncated}} \\ Y & \longrightarrow & B \end{array} \tag{3.109}$$

Example 3.1.67 (Epi/mono factorization)**.** For $n = -1$, the connected/truncated factorization system (Prop. 3.1.66) has as left class the effective epimorphisms (Def. 3.1.63) and as right class the monomorphisms (Def. 3.1.59). Hence, with the notation from (3.104) and (3.100):

(i) the (-1)-image factorization (3.108) reads:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow \quad \hookrightarrow & \\ & \mathrm{im}_{-1}(f) & \end{array} \tag{3.110}$$

(ii) the lifting property (3.109) for $n = -1$ reads:

$$\begin{array}{ccc} X & \longrightarrow & A \\ \downarrow & \nearrow & \hookdownarrow \\ Y & \longrightarrow & B \end{array} \tag{3.111}$$

Groupoids and Stacks.

Definition 3.1.68 (Groupoids internal to an ∞-topos [Lu09a, 6.1.2.7])**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30).

(i) A *groupoid in* $\mathbf{H}$ is a simplicial diagram

$$X_\bullet : \Delta^{\mathrm{op}} \longrightarrow \mathbf{H} \tag{3.112}$$

which satisfies the *groupoidal Segal condition*: For all $n \in \mathbb{N}$ and for all partitions of the set of $n+1$ elements by two subsets that share a unique element, the corresponding image under $X_\bullet$ is a Cartesian square (Notation 3.1.21):

$$\begin{array}{ccccccc} & \{0,1,\cdots,n\} & & & & X_n & \\ S_1 \hookrightarrow & \text{(po)} & \hookleftarrow S_2 & \overset{X_\bullet}{\longmapsto} & X_{|S_1|-1} \leftarrow & \text{(pb)} & \rightarrow X_{|S_2|-1} \\ & * & & & & X_0 & \end{array} \tag{3.113}$$

[(**ii**)] We write

$$\mathrm{Grpd}(\mathbf{H}) \hookrightarrow \mathbf{H}^{(\Delta^{\mathrm{op}})} \in \mathrm{Cat}_\infty \tag{3.114}$$

for the full sub-∞-category of that of simplicial diagrams in $\mathbf{H}$ on those that are groupoids.

Example 3.1.69 (Nerves)**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30) and $X \xrightarrow{f} \mathscr{X}$ a morphism in $\mathbf{H}$. Its *nerve* is the simplicial diagram of its iterated homotopy fiber products:

$$\begin{aligned} \mathrm{Nerve}_\bullet(f) \ : \ \Delta^{\mathrm{op}} & \longrightarrow \mathbf{H} \\ [n] & \longmapsto \underbrace{X \times_{\mathscr{X}} X \times_{\mathscr{X}} \cdots \times_{\mathscr{X}} X}_{n \text{ factors}} \end{aligned} \tag{3.115}$$

with face maps the projections and degeneracy maps the diagonals. This is evidently a groupoid object according to Def. 3.1.68:

$$\mathrm{Nerve}_\bullet(f) \in \mathrm{Grpd}(\mathbf{H}). \tag{3.116}$$

Proposition 3.1.70 (Groupoids equivalent to stacks with atlases [Lu09a, 6.2.3.5])**.** *Let* $\mathbf{H}$ *be an ∞-topos (Def. 3.1.30). Then the ∞-functor sending* $X_\bullet \in \mathrm{Grpd}(\mathbf{H})$ *(Def. 3.1.68) to the* X_0*-component of its colimiting cocone*

(i) *lands in effective epimorphisms* (3.106) *and*

(ii) *constitutes an equivalence of ∞-categories whose inverse is given by the construction of nerves (Example 3.1.69):*

$$\begin{aligned} \mathrm{Grpd}(\mathbf{H}) & \xrightarrow{\simeq} \mathrm{EffEpi}(\mathbf{H}) \\ X_\bullet & \longmapsto \big(X_0 \twoheadrightarrow \varinjlim X_\bullet\big) \\ \mathrm{Nerve}_\bullet(a) & \longleftarrow \big(X \overset{a}{\twoheadrightarrow} \mathscr{X}\big) \end{aligned} \tag{3.117}$$

$$
\begin{array}{ccccl}
\vdots & & \vdots & & \\
X_0 \times_{\mathscr{X}} X_0 & \simeq & X_1 & & \text{“groupoid”} \\
{\scriptstyle \mathrm{pr}_1} \downarrow \uparrow{\scriptstyle \Delta} \downarrow {\scriptstyle \mathrm{pr}_2} & & {\scriptstyle s} \downarrow \uparrow{\scriptstyle e} \downarrow {\scriptstyle t} & & \\
X_0 & = & X_0 & & \\
{\scriptstyle a}\downarrow & & \downarrow & & \text{“atlas”} \\
\mathscr{X} & \simeq & \varinjlim X_\bullet & & \text{“stack”}
\end{array}
\qquad (3.118)
$$

Remark 3.1.71 (Internal groupoids with prescribed properties)**.** Often one considers $X_\bullet \in \mathrm{Grpd}(\mathbf{H})$ (Def. 3.1.68) whose simplicial component diagram (3.112) is inside a chosen sub-∞-category of $\mathbf{H}$. Key examples are étale groupoids (Def. 4.1.35 below) and V-étale groupoids (Remark 5.2.2 below).

Remark 3.1.72 (Morita morphisms of groupoids)**.** A morphism between stacks $\mathscr{X} := \varinjlim X_\bullet$ underlying groupoids $X_\bullet$ (according to Prop. 3.1.70) *without* (i.e., disregarding) the corresponding atlas is also known as a *Morita morphism* (in particular, a *Morita equivalence* if it is an equivalence), or a *Hilsum-Skandalis morphism* [HS87][Pr89], or a *groupoid bibundle* [Bl07][Nu13, Prop. 2.2.34] between the corresponding groupoids:

$$
\begin{array}{lccccccl}
& \mathrm{Grpd}(\mathbf{H}) & \xrightarrow{\simeq} & \mathrm{EffEpi}(\mathbf{H}) & \xrightarrow{\mathrm{codom}} & \mathbf{H} & & \\
\text{groupoid} & X_\bullet & \longmapsto & (X_0 \twoheadrightarrow \mathscr{X}) & \longmapsto & \mathscr{X} & \text{“stack”} & \\
& & & & & {\scriptstyle f}\downarrow & \text{morphism of underlying stacks = “Morita morphism” of groupoids} & \\
\text{groupoid} & Y_\bullet & \longmapsto & (Y_0 \twoheadrightarrow \mathscr{Y}) & \longmapsto & \mathscr{Y} & \text{“stack”} &
\end{array}
\qquad (3.119)
$$

Hence whether or not there is a conceptual distinction between “geometric groupoids” and “stacks” depends on whether morphisms of groupoids are taken to be their plain morphisms or their Morita morphisms. In practice, one is typically interested in the latter case. Indeed, the groupoid atlas of a stack, whose preservation restricts Morita morphisms to plain morphisms of groupoids, by Prop. 3.1.70, is, in practice, typically required to exist with a certain property, but not required to be preserved by morphisms (this is so notably for V-étale groupoids, Remark 5.2.2 below). In particular, the $\mathrm{SmthGrpd}_\infty$ of Example 4.1.18 and the $\mathrm{JetsOfSmoothGroupoids}_\infty$ of Example 4.1.24 below are ∞-groupoids with Morita morphisms understood, hence could also be called (*jets of*) *smooth ∞-stacks*.

Proposition 3.1.73 (Equifibered morphisms of groupoids)**.** *Let* $\mathbf{H}$ *be an ∞-topos (Def. 3.1.30) and* $X_\bullet, Y_\bullet \in \mathrm{Grpd}(\mathbf{H})$ *(Def. 3.1.68). Then, under the equivalence (3.117) between groupoids and their stacks with atlases (Prop. 3.1.70), we have that equifibered morphisms of groupoids correspond to Cartesian squares between their* *atlases:*

$$
X_\bullet \xRightarrow{\;f_\bullet\;} Y_\bullet \quad \textit{such that} \quad \underset{[n_1]\xrightarrow{\phi}[n_2]}{\forall} \quad
\begin{array}{ccc}
X_{n_1} & \xrightarrow{\;f_{n_1}\;} & Y_{n_1} \\
{\scriptstyle X_\phi}\big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle Y_\phi} \\
X_{n_2} & \xrightarrow[\;f_{n_2}\;]{} & Y_{n_2}
\end{array}
$$

$$
\Leftrightarrow \qquad
\begin{array}{ccc}
X_0 & \xrightarrow{\;f_0\;} & Y_0 \\
{\scriptstyle a_X}\big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle a_Y} \\
\mathscr{X} & \xrightarrow[\;\underrightarrow{\lim}\, f_\bullet\;]{} & \mathscr{Y}
\end{array}
\tag{3.120}
$$

Proof. From right to left this follows by the pasting law (Prop. 3.1.23), while from left to right this is Prop. 3.1.32. □

3.2 Galois theory

We discuss here the *internal* formulation in ∞-toposes of the theory of *groups*, *group actions*, and *fiber bundles*, following [NSS12a][SSS12] (see [FSS14] for exposition). Externally, these concepts are known as *grouplike A_∞-algebras* or equivalently: *grouplike E_1-algebras* (here: in ∞-stacks) and as their *A_∞-modules* etc., and are traditionally presented by simplicial techniques [May72][Lu17]. But internally the theory becomes finitary and elementary, with all concepts emerging naturally from pastings of a few Cartesian squares. Accordingly, much of the following constructions may readily be expressed fully formally in homotopy type theory [BvDR18] (see p. 9). Thus, the following elegant characterizations of

- groups (Prop. 3.2.1),
- group actions (Prop. 3.2.6),
- principal bundles (Prop. 3.2.15),
- fiber bundles (Prop. 3.2.19),

in an ∞-topos **H** may be taken to be the *definition* of these notions for all purposes of internal constructions.

Groups. The following characterization of group ∞-stacks (Prop. 3.2.1) is the time-honored *May recognition theorem* [May72] generalized from Grpd_∞ to general ∞-toposes [Lu09a, 7.2.2.11][Lu17, 6.2.6.15]:

Proposition 3.2.1 (A_∞-Group recognition theorem [NSS12a, Thm. 2.19])**.** *Let* **H** *be an ∞-topos (Def. 3.1.30). Then the operation of sending an ∞-group G to the* *homotopy quotient of its action on a point constitutes an equivalence of ∞-categories:*

$$\begin{array}{ccc} \mathrm{Grp}(\mathbf{H}) & \underset{\mathbf{B}}{\overset{\Omega}{\underset{\longrightarrow}{\overset{\longleftarrow}{\simeq}}}} & \mathbf{H}^{*/}_{\geq 1} \\ G & \longmapsto & * /\!\!/ G \end{array} \tag{3.121}$$

between the ∞-category of ∞-group objects and the ∞-category of pointed and connected objects in **H**. *The inverse equivalence is given by forming the loop space object*

$$\begin{array}{ccc} G \simeq \Omega \mathbf{B} G & \longrightarrow & * \\ \downarrow & \text{(pb)} & \downarrow \\ * & \longrightarrow & \mathbf{B}G \end{array} \tag{3.122}$$

Example 3.2.2 (Point in delooping is an effective epi)**.** For $G \in \mathrm{Grp}(\mathbf{H})$, the morphism that exhibits its delooping as a pointed object (Prop. 3.2.1)

$$* \twoheadrightarrow \mathbf{B}G\,, \tag{3.123}$$

is an effective epimorphism (Def. 3.1.62). Thus, Prop. 3.1.70 says here that

(i) groups in **H** are, equivalently, the groupoids in **H** (Def. 3.1.68) that admit an atlas by the point and,

(ii) with (3.122), we have

$$\mathbf{B}G \simeq \varinjlim G^{\times \bullet} \in \mathbf{H}\,. \tag{3.124}$$

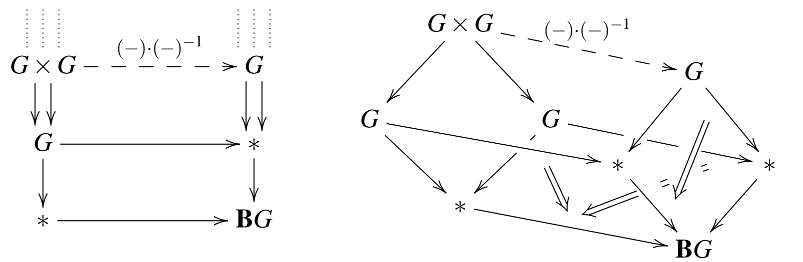
Example 3.2.3 (Neutral element)**.** Let **H** be an ∞-topos. Given a group $G \in \mathrm{Grp}(\mathbf{H})$ in the form of a pointed connected object $* \to \mathbf{B}G$, according to Prop. 3.2.1, its *neutral element* $* \xrightarrow{e} G$ is the diagonal morphism into the defining homotopy fiber product (3.122), hence the canonical morphism induced by the universal property of the homotopy fiber product from the equivalence with itself of the point inclusion into $\mathbf{B}G$ (3.123).

$$\begin{array}{ccccc} & & * & & \\ & & \downarrow{\scriptstyle e} & & \\ & & G & & \\ & \swarrow & \text{(pb)} & \searrow & \\ * & & & & * \\ & \searrow & & \swarrow & \\ & & \mathbf{B}G & & \end{array} \tag{3.125}$$

Example 3.2.4 (Group division/shear map)**.** Let **H** be an ∞-topos. Given a group $G \in \mathrm{Grp}(\mathbf{H})$ in the form of a pointed connected object $* \longrightarrow \mathbf{B}G$, according to Prop. 3.2.1, the group division operation

$$G \times G \xrightarrow{(-)\cdot(-)^{-1}} G \tag{3.126}$$

is exhibited by the universal morphism shown dashed in the following diagram:



(3.127)

On the left, we are showing this as part of a morphism of Čech nerve augmented simplicial diagrams. On the right, the situation is shown in more detail: Here the right and the two bottom squares are all the looping relation (3.122), while the left square exhibits the plain product of G with itself. With this, the universal property of the right square implies the essentially unique dashed morphism making the total diagram homotopy-commute. To note:

- **(i)** The two top squares are also Cartesian: This follows from the pasting law (Prop. 3.1.23) using, for the top front square, that the left and right and the bottom rear squares are Cartesian; and similarly for the top rear square.
- **(ii)** The total homotopy filling the top and the right faces in (3.127) is, by commutativity, equivalent to the total homotopy filling the left and the bottom faces. But, in performing the composition this way, the direction of one of the two bottom homotopies gets reversed. This is why this construction gives the division map $(-)\cdot(-)^{-1}$ (shear map) instead of the plain group product.

Proposition 3.2.5 (Mayer-Vietoris sequence [Sc13, Prop. 3.6.142])**.** *Let* $\mathbf{H}$ *be an* ∞*-topos (Def. 3.1.30),* $G \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1) and* $(X,f),(Y,g) \in \mathbf{H}_{/G}$ *two objects in the slice (Prop. 3.1.46) over the underlying object of G. Then their homotopy fiber product*

$$\begin{array}{ccc} X \underset{G}{\times} Y & \xrightarrow{\mathrm{pr}_Y} & Y \\ {\scriptstyle \mathrm{pr}_X}\big\downarrow & {\scriptstyle (\mathrm{pb})} & \big\downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & G \end{array} \qquad (3.128)$$

is equivalently exhibited by the following Mayer-Vietoris homotopy fiber sequence

$$\begin{array}{ccccc} X \times_G Y & \longrightarrow & & \longrightarrow & \ast \\ {\scriptstyle (\mathrm{pr}_X,\mathrm{pr}_Y)}\big\downarrow & & {\scriptstyle (\mathrm{pb})} & & \big\downarrow \\ X \times Y & \xrightarrow{(f,g)} & G \times G & \xrightarrow{(-)\cdot(-)^{-1}} & G, \end{array} \qquad (3.129)$$

with the composite bottom morphism $X \times Y \xrightarrow{f\cdot g^{-1}} G$,

where the morphism on the bottom right is the group division map (3.127)*.*

Group actions.

Proposition 3.2.6 (Group actions [NSS12a, 4.1])**.** *Let* $\mathbf{H}$ *an* ∞*-topos and* $G \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1).*

(i) *An action* (X, ρ) *of* G *is an object* $X \in \mathbf{H}$ *and homotopy fiber sequence in* $\mathbf{H}$ *of the form*

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{fib}(\rho)} & X /\!\!/ G \\ & & \downarrow{\scriptstyle \rho} \\ & & \mathbf{B}G\,, \end{array} \tag{3.130}$$

where $\mathbf{B}G$ *is the delooping of* G (3.2.1)*.*

(ii) *The object* $X /\!\!/ G$ *appearing in* (3.130) *is, equivalently, the homotopy quotient of the action of* G *on* V*:*

$$X /\!\!/ G \;\simeq\; \varinjlim \Big(\;\cdots\; X \times G \times G \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X \times G \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; X \;\Big)\,. \tag{3.131}$$

(iii) *Hence the* ∞*-category of* G*-actions is, equivalently, the slice* ∞*-topos (Prop. 3.1.46) of* $\mathbf{H}$ *over* $\mathbf{B}G$*:*

$$G\mathrm{Act}(\mathbf{H}) \;\simeq\; \mathbf{H}_{/\mathbf{B}G} \;\;\in \mathrm{Cat}_\infty\,. \tag{3.132}$$

We record the following immediate but important aspect of this characterization:

Lemma 3.2.7 (Homotopy quotient maps are effective epimorphisms)**.** *Let* $\mathbf{H}$ *be an* ∞*-topos,* $G \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1), and* $(X, \rho) \in G\mathrm{Act}(\mathbf{H})$ *(Prop. 3.2.6). Then the quotient morphism from* X *to its homotopy quotient* (3.131) *is an effective epimorphism (Def. 3.1.63):*

$$X \xrightarrow{\;\mathrm{fib}(\rho)\;} \!\!\!\!\to X /\!\!/ G\,. \tag{3.133}$$

Proof. By (3.130) in Prop. 3.2.6, the quotient map sits in a homotopy pullback square of the form

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{fib}(\rho)} & X /\!\!/ G \\ \downarrow & \text{(pb)} & \downarrow{\scriptstyle \rho} \\ * & \longrightarrow & \mathbf{B}G \end{array} \tag{3.134}$$

The bottom morphism is an effective epimorphism (Example 3.2.2). Since these are preserved by pullback (Lemma 3.1.65), the claim follows. □

Example 3.2.8 (Left multiplication action)**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30) and $G \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1). The defining looping relation (3.122) exhibits, by comparison with (3.130), an action of G on itself:

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{fib}(\rho_\ell)} & * \\ & & \downarrow{\scriptstyle \rho_\ell} \\ & & \mathbf{B}G \end{array} \tag{3.135}$$

This is the *left multiplication action* with $G /\!\!/ G \simeq *$.

Example 3.2.9 (Adjoint action)**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30) and $G \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1). Then the free loop space object $\mathscr{L}\mathbf{B}G$ of the delooping $\mathbf{B}G$ (3.121), defined by the Cartesian square

$$\begin{array}{ccc} \mathscr{L}\mathbf{B}G & \longrightarrow & \mathbf{B}G \\ {\scriptstyle \rho_{\mathrm{ad}}}\big\downarrow & {\scriptstyle (\mathrm{pb})} & \big\downarrow{\scriptstyle \Delta} \\ \mathbf{B}G & \underset{\Delta}{\longrightarrow} & \mathbf{B}G \times \mathbf{B}G \end{array} \tag{3.136}$$

sits in a homotopy fiber sequence of the form

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{fib}(\rho_{\mathrm{ad}})} & \mathscr{L}\mathbf{B}G \\ & & \big\downarrow{\scriptstyle \rho_{\mathrm{ad}}} \\ & & \mathbf{B}G\,. \end{array} \tag{3.137}$$

By comparison with (3.130), this exhibits an action of G on itself. This is the *adjoint action* with $G/\!\!/_{\mathrm{ad}}G \simeq \mathscr{L}\mathbf{B}G$.

Definition 3.2.10 (Equivariant maps)**.** By the functoriality/universality of the homotopy fiber construction in (3.130) and using the equivalence (3.132), we have the ∞-functor that assigns the underlying objects of the G-actions in Def. 3.2.6:

$$G\mathrm{Act}(\mathbf{H}) \simeq \mathbf{H}_{/\mathbf{B}G} \xrightarrow{\mathrm{fib}} \mathbf{H}\,. \tag{3.138}$$

With two G-actions (X_i, ρ_i) given, we say that a morphism $X_1 \to X_2 \ \in \mathbf{H}$ between their underlying objects *is equivariant* if it lifts through this functor, hence if it is the image of a morphism $(X_1, \rho_1) \to (X_2, \rho_2) \ \in \mathrm{GAct}(\mathbf{H})$.

Example 3.2.11 (Group division is equivariant under diagonal left and adjoint action)**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30) and $G \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1). Then the group division operation (Example 3.2.4) is equivariant (Def. 3.2.10) with respect to the diagonal left multiplication action ρ_ℓ (Example 3.2.8) on its domain and the adjoint action ρ_{ad} (Example 3.2.9) on its codomain:

$$(G, \rho_\ell) \times (G, \rho_\ell) \xrightarrow{(-)\cdot(-)^{-1}} (G, \rho_{\mathrm{ad}}) \quad \in G\mathrm{Act}(\mathbf{H})\,. \tag{3.139}$$

Proof. Observe the following pasting of Cartesian squares:

$$\begin{array}{ccccc} G \times G & \xrightarrow{(-)\cdot(-)^{-1}} & G & \longrightarrow & * \\ {\scriptstyle (-)^{-1}\cdot(-)\circ\sigma}\big\downarrow & & \big\downarrow & & \big\downarrow \\ G & \longrightarrow & \mathscr{L}\mathbf{B}G & \longrightarrow & \mathbf{B}G \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \Delta} \\ * & \longrightarrow & \mathbf{B}G & \xrightarrow{\Delta} & \mathbf{B}G \times \mathbf{B}G \end{array} \tag{3.140}$$

The middle horizontal composite, regarded as a morphism in the slice over $\mathbf{B}G$ and hence as a morphism of G-actions (3.130), gives (3.139). □

Proposition 3.2.12 (Restricted and induced group actions)**.** *Let* $\mathbf{H}$ *be an* ∞*-topos. Then, for* $\phi : H \to G$ *a morphism in* $\mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1), there is a triple of adjoint*

∞-functors (Def. 3.1.24) between the corresponding ∞-categories of group actions (Prop. 3.2.6)

$$H\mathrm{Act}(\mathbf{H}) \underset{\underset{\text{"right-induced"}}{\mathbf{B}\phi_*}}{\overset{\overset{\text{"left-induced"}}{\mathbf{B}\phi_!}}{\underset{\bot}{\overset{\bot}{\longleftarrow \mathbf{B}\phi^* \longrightarrow}}}} G\mathrm{Act}(\mathbf{H}) \tag{3.141}$$

such that $\mathbf{B}\phi^*$ *preserves the object being acted on ("restricted action").*

Proof. By (3.132) in Prop. 3.2.6, an adjoint triple (Def. 3.1.24) of the form (3.141) is given by base change (Prop. 3.1.49) of homotopy quotients (3.131) along the delooped morphism $\mathbf{B}\phi$ (Prop. 3.2.1). This means that $\mathbf{B}\phi^*$ is given by sending the homotopy fiber sequence (3.130) corresponding to a G-action to the following homotopy pullback (Prop. 3.2.1):

$$\begin{array}{ccccc} X & \xrightarrow{\mathrm{fib}(\phi^*\rho)} & X /\!\!/ H & \longrightarrow & X /\!\!/ G \\ & & \downarrow{\scriptstyle \phi^*\rho} & \text{(pb)} & \downarrow{\scriptstyle \rho} \\ & & \mathbf{B}H & \xrightarrow{\mathbf{B}\phi} & \mathbf{B}G \end{array} \qquad X \xrightarrow{\mathrm{fib}(\phi)} X /\!\!/ G \tag{3.142}$$

That this preserves the object X being acted on, as indicated, follows by the pasting law (Prop. 3.1.23). □

Definition 3.2.13 (Automorphism group)**.** Let $\mathbf{H}$ be an ∞-topos and $F \in \mathbf{H}$ an object. Then the *automorphism group* $\mathrm{Aut}(F) \in \mathrm{Grp}(\mathbf{H})$ of F is the looping (Prop. 3.2.1) of the (-1)-image (3.108) of the classifying map (3.53) of F:

$$* \xrightarrow{\text{(-1)-conn.}} \mathbf{B}\mathrm{Aut}(F) \xhookrightarrow{\text{(-1)-trunc.}} \mathrm{Objects}_\kappa\,. \qquad * \xrightarrow{\vdash F} \mathrm{Objects}_\kappa \tag{3.143}$$

The canonical action of this group (Prop. 3.2.6) on V is exhibited, via (3.130), by the left square of the following pasting composite of Cartesian squares:

$$\begin{array}{ccccc} F & \xrightarrow{\mathrm{fib}(\rho_{\mathrm{Aut}})} & F /\!\!/ \mathrm{Aut}(F) & \longrightarrow & \widehat{\mathrm{Objects}}_\kappa \\ \downarrow & \text{(pb)} & \downarrow{\scriptstyle \rho_{\mathrm{Aut}}} & \text{(pb)} & \downarrow \\ * & \longrightarrow & \mathbf{B}\mathrm{Aut}(F) & \hookrightarrow & \mathrm{Objects}_\kappa\,, \end{array} \qquad * \xrightarrow{\vdash F} \mathrm{Objects}_\kappa \tag{3.144}$$

where we use the pasting law (Prop. 3.1.23) to identify F as the homotopy fiber of ρ_{Aut}.

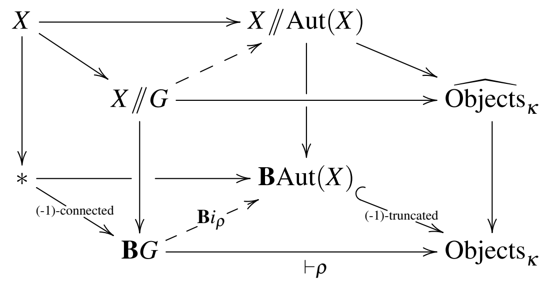
Proposition 3.2.14 (Automorphism group is universal)**.** *Let* $\mathbf{H}$ *be an ∞-topos,* $G \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1), and* $(X,\rho) \in G\mathrm{Act}(\mathbf{H})$ *(Def. 3.2.6). Then there is a group* *homomorphism from G to the automorphism group (Def. 3.2.13)*

$$G \xrightarrow{i_\rho} \mathrm{Aut}(X) \tag{3.145}$$

such that the action ρ *is the restricted action (Prop. 3.2.12) along* i_ρ *of the canonical automorphism action* (3.144)*, i.e., such that there is a Cartesian square of this form:*

$$\begin{array}{ccc} X /\!\!/ G & \longrightarrow & X /\!\!/ \mathrm{Aut}(X) \\ {\scriptstyle \rho}\downarrow & \text{(pb)} & \downarrow {\scriptstyle \rho_{\mathrm{Aut}}} \\ \mathbf{B}G & \xrightarrow[\mathbf{B}i_\rho]{} & \mathbf{B}\mathrm{Aut}(X) \end{array} \tag{3.146}$$

Proof. Let κ be a regular cardinal such that X is κ-small, and consider the following solid diagram of classifying maps (3.53) for ρ, ρ_{Aut} and for X:



(3.147)

Here the bottom square homotopy-commutes by the essential uniqueness of the classifying map $\vdash X$ (3.53). Hence the dashed lift exists essentially uniquely (3.109), by the connected/truncated factorization system (Prop. 3.1.66). □

Principal bundles.

Proposition 3.2.15 (Principal bundles [NSS12a, Thm. 3.17])**.** *Let* $\mathbf{H}$ *be an* ∞*-topos,* $X \in \mathbf{H}$*, and* $G \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1). Then G-principal* ∞*-bundles* $P \to X$ *over X are, equivalently, given by classifying maps* $\vdash P : X \to \mathbf{B}G$*. Forming their homotopy fibers*

$$\begin{array}{ccc} P & & \\ {\scriptstyle \mathrm{fib}(\vdash P)}\downarrow & & \\ X & \xrightarrow[\vdash P]{} & \mathbf{B}G \end{array} \tag{3.148}$$

constitutes an equivalence of ∞*-groupoids:*

$$\begin{array}{ccc} G\mathrm{Bundles}_X(\mathbf{H}) & \xleftarrow[\simeq]{\mathrm{fib}} & \mathbf{H}(X, \mathbf{B}G)\,. \\ P & \longmapsto & \vdash P \end{array} \tag{3.149}$$

Remark 3.2.16 (Principal base spaces are homotopy quotients)**.** Comparison of the abstract characterization of **(i)** group actions (Prop. 3.2.6) and **(ii)** principal bundles (Prop. 3.2.15), reveals that these are about one and the same abstract concept, just viewed from two different perspectives: In an ∞-topos, every G-principal bundle is a G-action whose homotopy quotient is the given base space; and, conversely, every G-action is that of a principal bundle over its homotopy quotient:

$$\begin{array}{ccccc} \text{principal } G\text{-bundle} & P \circlearrowleft G & & \text{G-action} \\ & \downarrow & & \\ \text{base space} & X \simeq & P /\!\!/ G & \text{homotopy quotient} \end{array} \tag{3.150}$$

Notice (see [NSS12a, 3.1] for exposition) that it is the higher geometry inside an ∞-topos that makes this work.

Definition 3.2.17 (Atiyah groupoid)**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30), $X \in \mathbf{H}$, $G \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1), and $P \in G\mathrm{Bundles}_X$ (Prop. 3.2.15). Then the *Atiyah groupoid* of P is the groupoid $\mathrm{At}_\bullet(P) \in \mathrm{Grpd}(\mathbf{H})$ (Def. 3.1.68) whose corresponding stack with atlas (via Prop. 4.1.36) is the (-1)-image projection (Example 3.1.67) of the bundle's classifying map $\vdash P$ (3.149):

$$X \twoheadrightarrow \mathscr{A}t(P) \hookrightarrow \mathbf{B}G\,, \qquad X \xrightarrow{\ \vdash P\ } \mathbf{B}G \tag{3.151}$$

Fiber bundles.

Definition 3.2.18 (Fiber bundle)**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30).

(i) morphism $Y \xrightarrow{p} X$ in $\mathbf{H}$ is a *fiber bundle* with *typical fiber* $F \in \mathbf{H}$ if there exists an effective epimorphism $U \overset{i}{\twoheadrightarrow} X$ (Def. 3.1.63) and a Cartesian square (Notation 3.1.21) of the form

$$\begin{array}{ccc} U \times F & \longrightarrow & Y \\ \downarrow & \text{(pb)} & \downarrow{\scriptstyle p} \\ U & \underset{i}{\twoheadrightarrow} & X \end{array} \tag{3.152}$$

(ii) We write

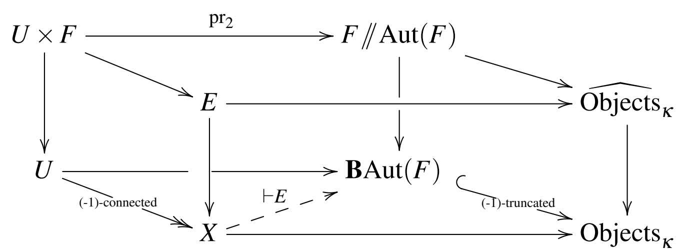
$$F\mathrm{FiberBundles}_X(\mathbf{H}) \subset \mathrm{Core}\big(\mathbf{H}_{/X}\big) \in \mathrm{Grpd}_\infty \tag{3.153}$$

for the full ∞-groupoid of the core (3.1) of the slice $\mathbf{H}_{/X}$ over X (Prop. 3.1.46) on the F-fiber bundles.

Proposition 3.2.19 (Classification of fiber bundles [NSS12a, Prop. 4.10])**.** *Let* $\mathbf{H}$ *be an* ∞*-topos (Def. 3.1.30) and* $X, F \in \mathbf{H}$*. Then fiber bundles over* X *(Def. 3.2.18) with typical fiber* F *are equivalent to morphisms* $X \longrightarrow \mathbf{B}\mathrm{Aut}(F)$ *from* X *to the delooping (Prop. 3.2.1) of the automorphism group (Def. 3.2.13) of* F*:*

$$\begin{array}{ccc} F\mathrm{FiberBundles}_X(\mathbf{H}) & \xrightarrow{\ \simeq\ } & \mathbf{H}\big(X\,, \mathbf{B}\mathrm{Aut}(F)\big) \\ E & \longmapsto & \vdash E \end{array} \tag{3.154}$$

Proof. Let κ be a regular cardinal such that F is κ-small. Then, by assumption, we have the following solid diagram of classifying maps (3.53):



(3.155)

Now the (-1)-connected/(-1)-truncated factorization system (Prop. 3.1.66) implies that the dashed morphism exists essentially uniquely (3.109).

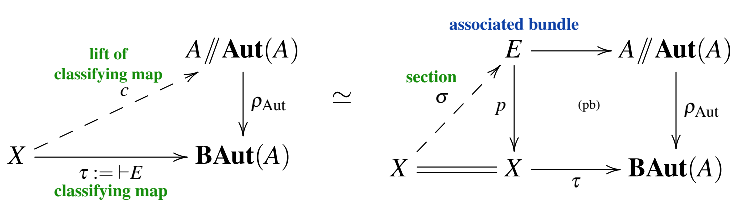
It just remains to see that this assignment is independent of the choice of U: For $U' \twoheadrightarrow X$ any other effective epimorphism with $(\vdash E)'$ the associated classifying map as above, observe that the fiber product $U \times_X U' \twoheadrightarrow X$ is again an effective epimorphism, since the class of effective epimorphisms is closed under pullbacks as well as under composition (Lemma 3.1.65). Therefore $\vdash E$ and $(\vdash E)'$ are jointly lifts in a diagram as above but with $U \times_X U'$ in the top left. Hence, by the essential uniqueness of lifts in the connected/truncated orthogonal factorization system, they are equivalent, $(\vdash E) \simeq (\vdash E)'$, in an essentially unique way. □

Notation 3.2.20 (Associated bundles)**.** We say that

(i) the morphism $\vdash E$ in (3.154) is the *classifying map* of E and

(ii) that E is *associated* to the $\mathrm{Aut}(F)$-principal bundle which is classified by $\vdash E$ according to Prop. 3.2.15.

Remark 3.2.21 (Twisted cohomology in slice ∞-toposes)**.** Prop. 3.2.19 implies (together with the universal property of the pullback) that sections σ of A-fiber bundles E over some X are, equivalently, lifts c of the classifying map $c := \vdash E$ (3.154) through ρ_{Aut} (3.144):



(3.156)

(i) If A is regarded here as a coefficient object for A-cohomology (1.21), then such a section σ is a locally A-valued cocycle, which is "twisted" over X according to the classifying map τ. Hence such a σ is a cocycle in (non-abelian) τ-*twisted cohomology* [NSS12a, 4.2]. But the left hand side of (3.156) is, equivalently, a morphism (3.78) in the slice ∞-topos (Prop. 3.1.46) $\mathbf{H}_{/\mathbf{BAut}(A)}$. It follows that *twisted cohomology is the intrinsic cohomology* (1.21) *of slice ∞-toposes*:

$$
\underset{\text{cohomology}}{\overset{\tau\text{-twisted}}{H^\tau\big(X\,,A\big)}} \;:=\; \pi_0\,\mathbf{H}_{/\mathbf{BAut}(A)}\Big(\big(X,\tau\big)\,,\,\big(A /\!\!/ \mathbf{Aut}(A),\rho_{\mathrm{Aut}}\big)\Big)
$$
$$
\simeq \left\{ \begin{array}{ccc} X & \overset{\text{cocycle}}{\underset{c}{\dashrightarrow}} & A /\!\!/ \mathbf{Aut}(A) \\ & \Leftarrow & \\ {\scriptstyle \tau}\searrow & & \swarrow{\scriptstyle \rho_{\mathrm{Aut}}} \\ & \mathbf{BAut}(A) & \end{array} \right\}_{\big/\sim} \tag{3.157}
$$

(ii) By the universality of $\mathbf{Aut}(A)$ (Prop. 3.2.14), this holds for slicing over *any* pointed connected object $\mathbf{B}G$ (3.121).

(iii) If the base object is not connected, the intrinsic cohomology of its slice may be thought of as a mixture of twisted and parametrized cohomology. We encounter an example of this in Def. 6.2.1 below.

Remark 3.2.22 (Twisted cohomology as global sections)**.** The ∞-groupoid of sections of the associated bundle $E := \tau^*(A /\!\!/ G) \xrightarrow{p} X$ in (3.156), is equivalently its image $\Gamma_X(E)$ under the base geometric morphism (Prop. 3.1.43)

$$
\mathbf{H}_{/X} \underset{\Gamma_X}{\overset{\Delta_X}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Grpd}_\infty \tag{3.158}
$$

of the slice ∞-topos $\mathbf{H}_X$ (Prop. 3.1.46), in that (by Prop. 3.1.34) $\Gamma_X(E) \simeq \mathbf{H}_X\big(\mathrm{id}_X\,,p\big)$. Hence the τ-twisted cohomology (3.157) of X is equivalently the set of connected components of the ∞-groupoid of global sections:

$$
\mathbf{H}^\tau\big(X;A\big) \;\simeq\; \pi_0\,\Gamma_X\big(\tau^*(A /\!\!/ G)\big)\,. \tag{3.159}
$$

Remark 3.2.23 (Twisted abelian cohomology in tangent ∞-toposes)**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30).

(i) Notice that the intrinsic cohomology (1.21) of Bundles($\mathbf{H}$) (Example 3.1.50) is still twisted cohomology as in Remark 3.2.21, just up to a change in perspective: now the twisting τ is encoded not in the domain object, but in the cocycles on these (a morphism of the form $\mathrm{id}_X \to \rho_{\mathrm{Aut}}$ in Bundles($\mathbf{H}$) is still manifestly given by the diagrams in (3.156)).

(ii) Therefore, similarly, the intrinsic cohomology (1.21) in the tangent ∞-topos SpectralBundles($\mathbf{H}$) (Example 3.1.51) is twisted cohomology with local coefficients being spectra [Sc13, 4.1][ABGHR14][GS19a][GS19b], hence is *twisted abelian cohomology*.

(iii) In the case that $\mathbf{H} = \mathrm{Grpd}_\infty$, the base tangent ∞-topos

$$
T\mathrm{Grpd}_\infty \;=\; \mathrm{SpectralBundles}\big(\mathrm{Grpd}_\infty\big) \tag{3.160}
$$

is the topic of traditional parametrized stable homotopy theory [Jam95][MSi06]

[ABGHR14, 2][BM19] and its intrinsic cohomology theory (1.21) is traditional twisted generalized cohomology [Do05][ABG10].

Fixed points and fixed loci.

Definition 3.2.24 (Fixed points and fixed loci)**.** Let $\mathbf{H}$ be an ∞-topos, $G \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1) and $(X, \rho) \in G\mathrm{Act}(\mathbf{H})$ (Prop. 3.2.6).

(i) A *fixed point* of (X, ρ) is an element $* \xrightarrow{x} X$ induced from a section $x /\!\!/ G$ of ρ in (3.130), as shown on the right (where we are using the pasting law, Prop. 3.1.23, and Example 3.1.22 to identify the top square as Cartesian).

$$\begin{array}{ccc} * & \longrightarrow & \mathbf{B}G \\ {\scriptstyle x}\downarrow & \text{(pb)} & \downarrow{\scriptstyle x/\!\!/G} \\ X & \xrightarrow{\mathrm{fib}(\rho)} & X /\!\!/ G \\ \downarrow & \text{(pb)} & \downarrow{\scriptstyle \rho} \\ * & \longrightarrow & \mathbf{B}G , \end{array} \tag{3.161}$$

(ii) The *G-fixed locus* of (X, ρ) is the object

$$X^G \; := \; \mathbf{B}(G \to *)_* \big((X, \rho) \big) \quad \in 1\mathrm{Action}(\mathbf{H}) \; \simeq \; \mathbf{H} , \tag{3.162}$$

that is right induced (Prop. 3.2.12) along the unique morphism to the trivial group.

Example 3.2.25 (Global points of fixed loci are homotopy fixed points)**.** The global points of a homotopy-fixed locus X^G (3.162) are indeed, equivalently, the fixed points (3.161). By the adjunction (3.141), we have the hom-equivalence (3.43)

$$\big(* \longrightarrow X^G = \mathbf{B}(G \to 1)_* (X, \rho) \; \big) \quad \leftrightarrow \quad \big(\mathbf{B}(G \to 1)^* (*) \longrightarrow (X, \rho) \; \big) \tag{3.163}$$

and, by Prop. 3.2.6, the latter morphisms are equivalent to homotopy-commuting diagrams of the form

$$\begin{array}{ccc} \mathbf{B}G & \xrightarrow{x /\!\!/ G} & X /\!\!/ G \\ & {\scriptstyle \mathbf{B}(G \to 1)^*(*)} \searrow \;\; \simeq \;\; \swarrow {\scriptstyle \rho} & \\ & \mathbf{B}G & \end{array} \tag{3.164}$$

This is just the type of diagram characterizing homotopy fixed points. as seen vertically on the right in (3.161).

Example 3.2.26 (Fixed loci in ∞-groupoids)**.** Consider $\mathbf{H} := \mathrm{Grpd}_\infty$, $G \in \mathrm{Grp}(\mathrm{Grpd}_\infty)$ and $(X, \rho) \in G\mathrm{Act}\big(\mathrm{Grpd}_\infty\big)$. Then the G-fixed locus (Def. 3.2.24) is given (due to Prop. 3.1.54) by

$$X^G \; \simeq \; \mathbf{H}_{/ * /\!\!/ G} \big(* /\!\!/ G , X /\!\!/ G \big) \quad \in \mathrm{Grpd}_\infty . \tag{3.165}$$

Definition 3.2.27 (Pointed-automorphism group)**.** Let $\mathbf{H}$ be an ∞-topos and $\big(X^{\swarrow *} \big) \in \mathbf{H}^{*/} \hookrightarrow \mathbf{H}^{\Delta[1]}$ a pointed object in $\mathbf{H}$, equivalently regarded as an object in the ∞-topos

$\mathbf{H}^{\Delta[1]}$ of ∞-functors from $\Delta[1] \coloneqq \{0 \to 1\}$ to $\mathbf{H}$. Noticing the evaluation functors (3.65)

$$\mathbf{H}^{\Delta[1]} \; \begin{matrix} \xrightarrow{\;\mathrm{ev}_1\;} \\ \bot \\ \xleftarrow{\;\mathrm{const}\;} \\ \bot \\ \xrightarrow{\;\mathrm{ev}_0\;} \end{matrix} \; \mathbf{H} \tag{3.166}$$

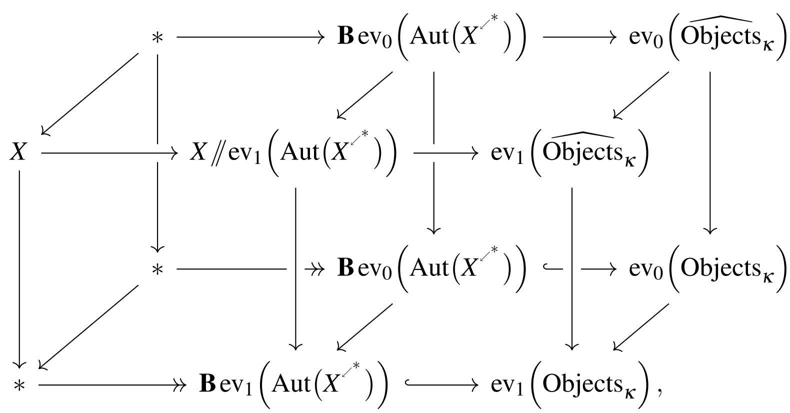
and that these preserve all ∞-limits and ∞-colimits (by Prop. 3.1.39), hence all group objects and their deloopings (by (3.130) in Prop. 3.2.6) we say that the *pointed-automorphism group* of X is the image under ev_0 of its automorphism group, according to Def. 3.2.13, formed in the arrow ∞-topos $\mathbf{H}^{\Delta[1]}$:

$$\mathrm{Aut}_*(X) \;\coloneqq\; \mathrm{ev}_0\Big(\mathbf{Aut}\big(X^{\swarrow^*}\big)\Big) \;\in\; \mathrm{Grp}(\mathbf{H}). \tag{3.167}$$

This pointed-automorphism groups comes with a canonical pointed action on X as follows: From the defining factorization (3.143)

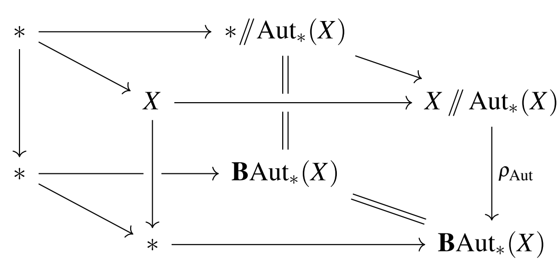
$$\mathrm{const}(*) \longrightarrow\!\!\!\!\!\rightarrow \mathbf{B}\mathrm{Aut}\big(X^{\swarrow^*}\big) \hookrightarrow \mathrm{Objects}_\kappa \tag{3.168}$$

(where now $\mathrm{Objects}_\kappa$ denotes the κ-small object classifier (3.53) of $\mathbf{H}^{\Delta}[1]$), and using again that the evaluation functors (3.166) preserves ∞-limits, hence in particular homotopy pullbacks, it follows that the front and rear faces of the following diagram are Cartesian (Ntn. 3.1.21)



(3.169)

so that pullback along the bottom diagonal morphisms shows that the pointed automorphism ∞-group (3.167) sits in a diagram in $\mathbf{H}$ of the following form:



(3.170)

Here the cartesian front face exhibits the action of the pointed-automorphism group of X on X and the Cartesian rear face exhibits its trivial action on the base point. With this and noticing that also the bottom face is Cartesian (by Example 3.1.22) the pasting law (Prop. 3.1.23) implies that also the top square is Cartesian, exhibiting the given base point as a homotopy fixed point (Def. 3.2.24) of the pointed-automorphism action.

Definition 3.2.28 (Group-automorphism group)**.** Let $\mathbf{H}$ be an ∞-topos and $G \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1). Then the group of group-automorphisms of G is the group of pointed-automorphisms (Def. 3.2.27) of its delooping $\mathbf{B}G$ (3.121):

$$\mathrm{Aut}_{\mathrm{Grp}}(G) \;:=\; \mathrm{Aut}_{*}(\mathbf{B}G) \quad \in \mathrm{Grp}(\mathbf{H})\,. \tag{3.171}$$

Proposition 3.2.29 (Canonical action of group-automorphism group)**.** *Let $\mathbf{H}$ be an ∞-topos and $G \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1). The group-automorphism group of G (Def. 3.2.28) has a canonical action (Prop. 3.2.6)*

$$\big(G, \rho_{\mathrm{Aut}_{\mathrm{Grp}}}\big) \;\in\; \mathrm{Aut}_{\mathrm{Grp}}(G)\mathrm{Act}(\mathbf{H}) \tag{3.172}$$

on the underlying object $G \in \mathbf{H}$, which is such that

(i) *The neutral element $* \xrightarrow{e} G$ (Example 3.2.3) is a fixed point of the action (Def. 3.2.24).*

(ii) *The homotopy quotient $G /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(G)$ carries the structure of a group object* (3.121) *in the slice* (3.132)

$$G /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(G) \;\in\; \mathrm{Grp}\big(\mathbf{H}_{/\mathbf{B}\mathrm{Aut}_{\mathrm{Grp}}}\big)\,, \tag{3.173}$$

whose delooping (3.121) *is the homotopy quotient of the defining action* (3.171) *on the delooping $\mathbf{B}G$ of G:*

$$\mathbf{B}\big(G /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(G)\big) \;\simeq\; (\mathbf{B}G) /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(G)\,. \tag{3.174}$$

Proof. First consider item **(ii)**: Write $G /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(G)$ for the homotopy fiber product in the following pullback square

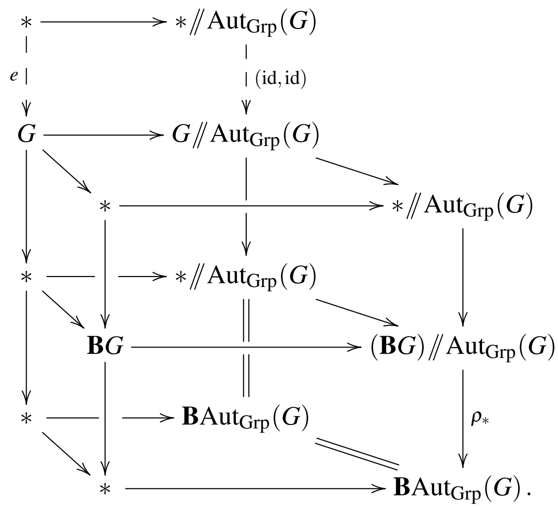
$$\begin{array}{ccc} G /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(G) & \longrightarrow & * /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(G) \\ \big\downarrow & {\scriptstyle (\mathrm{pb})} & \big\downarrow \\ * /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(G) & \longrightarrow & (\mathbf{B}G) /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(G)\,. \end{array} \tag{3.175}$$

Since this is the looping (3.122) in the slice (Prop. 3.1.46):

$$G /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(G) \;=\; \Omega\big((\mathbf{B}G) /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(G)\big) \quad \in \mathbf{H}_{/\mathbf{B}\mathrm{Aut}_{\mathrm{Grp}}(G)}\,, \tag{3.176}$$

the looping/delooping equivalence (3.121) implies the claim (3.174) as soon as we show (in view of Prop. 3.2.6) that the homotopy fiber of the left morphism in (3.175) is indeed G, in that it makes the total solid rear rectangle of the following diagram be

Cartesian:



(3.177)

Here:

- the bottom part is the diagram (3.170) (for $X = \mathbf{B}G$) which exhibits the pointed-automorphism action on $\mathbf{B}G$;
- the top front square is Cartesian and exhibits the base point being a homotopy-fixed point, as in (3.170);
- the top left square is Cartesian and exhibits the looping/delooping relation (3.122);
- the top right square is (3.175) and hence Cartesian by definition.

Hence the pasting law (Prop. 3.1.23) implies that also the solid top rear square is Cartesian.

Finally to see item **(i)**: Observe that there is the dashed morphism shown in the top right of (3.177), this being the diagonal morphism induced from the Cartesian property of the top right square, by the above. This means, by construction, that the total vertical morphism on the right is an equivalence. Now define the dashed top square to be a pullback square. Then, by the pasting law (Prop. 3.1.23), the pullback object in the top left of the dashed square is equivalently the pullback of the total rear diagram, hence the pullback of an equivalence to a point, hence is itself equivalent to the point, as shown. Since the point is terminal, the top left dashed morphism is thus a cone over the Cartesian square on the top left. By the universal property of the homotopy fiber product, this means that the top left dashed morphism must be the neutral element (Example 3.2.3). The top dashed square hence exhibits this as a homotopy fixed point. □

Proposition 3.2.30 (Group division is equivariant under group-automorphisms)**.** *Let* $\mathbf{H}$ *be an* ∞*-topos and* $G \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1). Then the group division morphism* $G \times G \xrightarrow{(-)\cdot(-)^{-1}} G$ *(Example 3.2.4) is equivariant (Def. 3.2.10) with respect to* *the canonical group-automorphism action (Prop. 3.2.29) of the group-automorphism group* $\mathrm{Aut}_{\mathrm{Grp}}(G)$ *(Def. 3.2.28) acting on all three copies of G:*

$$(G, \rho_{\mathrm{Aut}_{\mathrm{Grp}}}) \times (G, \rho_{\mathrm{Aut}_{\mathrm{Grp}}}) \xrightarrow{(-)\cdot(-)^{-1}} (G, \rho_{\mathrm{Aut}_{\mathrm{Grp}}}) \quad \in \mathrm{Aut}_{\mathrm{Grp}}(G)\mathrm{Act}(\mathbf{H}) \tag{3.178}$$

Proof. By (3.127) the group division morphism is a universal morphism induced from pasting of copies of the looping square (3.122). Thus the claim follows by Prop. 3.2.29. □

4

Singular geometry

Here we establish axiomatic foundations of a geometric homotopy theory of orbifolds, unifying:

(i) §4.1 – the cohesive geometric homotopy theory due to [SSS12][Sc19], which reflects the *geometric aspects* of orbifolds;

(ii) §4.2 – the cohesive global-equivariant homotopy theory due to [Re14], understood as reflecting the *singular aspects* of orbifolds, as in Figure D.

This is to provide, in §5 below, a general abstract theory of geometric aspects of orbi-singular spaces and of étale ∞-stacks.

4.1 Geometry

4.1.1 Differential Topology

We present a formulation of differential topology internal to ∞-toposes which we call *cohesive* [Sc13]. In ∞-categorical generalization of [La94][La07], this involves an abstract *shape* operation $\int$ that relates higher geometric spaces to their bare underlying homotopy type.

Definition 4.1.1 (Cohesive ∞-topos)**. (i)** An ∞-topos $\mathbf{H}$ (Def. 3.1.30) is called *cohesive* if its base geometric morphism (Prop. 3.1.43), to be denoted $\mathrm{Pnts} : \mathbf{H} \to \mathrm{Grpd}_\infty$, is part of an adjoint quadruple of ∞-functors (Def. 3.1.24)

$$\begin{array}{rccccc}
\textbf{“shape”} & & \times\!\!\longrightarrow & \mathrm{Shp} & \longrightarrow & \\
& & & \bot & & \\
\textbf{“discrete”} & & \longleftarrow & \mathrm{Disc} & \hookleftarrow & \\
& & & \bot & & \\
\textbf{“points”} & \mathbf{H} & \longrightarrow & \mathrm{Pnts} & \longrightarrow & \mathbf{B} \\
& & & \bot & & \\
\textbf{“chaotic”} & & \longleftarrow & \mathrm{Chtc} & \hookleftarrow & \\
& \begin{array}{c}\textbf{cohesive}\\ \textbf{∞-topos}\end{array} & & & & \begin{array}{c}\textbf{discrete}\\ \textbf{sub-topos}\end{array}
\end{array} \tag{4.1}$$

such that **(a)** Disc and Chtc are fully faithful (Def. 3.1.1), and **(b)** such that Shp preserves finite products.

(ii) We write

$$\begin{array}{c} \big(\int := \mathrm{Disc} \circ \mathrm{Shp} \big) \\ \textbf{“shape”} \\ \perp \\ \big(\flat := \mathrm{Disc} \circ \mathrm{Pnts} \big) \\ \textbf{“discrete”} \\ \perp \\ \big(\sharp := \mathrm{Chtc} \circ \mathrm{Pnts} \big) \\ \textbf{“continuous”} \end{array} \quad : \mathbf{H} \to \mathbf{H} \tag{4.2}$$

for the induced adjoint triple (Def. 3.1.24) of modalities (1.18) (*cohesive modalities*).

The following direct consequence may serve to illustrate how these axioms are put to work:

Proposition 4.1.2 (Composite cohesive modalities)**.** *The cohesive modalities (Def. 4.1.1) satisfy:*

$$\int \circ \flat \simeq \flat \quad \textit{and} \quad \flat \circ \sharp \simeq \flat\,. \tag{4.3}$$

Proof. That Disc and Chtc in (4.1) are fully faithful means, equivalently (Prop. 3.1.28), that the co-unit morphisms (3.45)

$$\mathrm{Shp} \circ \mathrm{Disc} \xrightarrow{\simeq} \mathrm{id}\ , \qquad \mathrm{Pnts} \circ \mathrm{Chtc} \xrightarrow{\simeq} \mathrm{id} \tag{4.4}$$

are natural equivalences. Hence the image under $\mathrm{Disc} \circ (-) \circ \mathrm{Pnts}$ of the first of these is a natural equivalence of the form

$$\int \circ \flat \;=\; \mathrm{Disc} \circ \mathrm{Shp} \circ \mathrm{Disc} \circ \mathrm{Pnts} \xrightarrow{\simeq} \mathrm{Disc} \circ \mathrm{Pnts} \;=\; \flat\,. \tag{4.5}$$

while the image of the second is of the form

$$\flat \circ \sharp \;=\; \mathrm{Disc} \circ \mathrm{Pnts} \circ \mathrm{Chtc} \circ \mathrm{Pnts} \xrightarrow{\simeq} \mathrm{Disc} \circ \mathrm{Pnts} \;=\; \flat\,. \tag{4.6}$$

□

Lemma 4.1.3 (Only the empty object has empty shape)**.** *Let* $\mathbf{H}$ *be a cohesive* ∞*-topos (Def. 4.1.1). Then* $X \in \mathbf{H}$ *is empty, i.e., equivalent to the initial object* $\varnothing$ (3.58)*, precisely if its shape* (4.2) *is empty:*

$$X \simeq \varnothing \qquad \Leftrightarrow \qquad \int X \simeq \varnothing\,. \tag{4.7}$$

Proof. In one direction, assume that $X \simeq \varnothing$. Noticing that $\varnothing$ is the initial colimit and that colimits are preserved by $\int$, this being a left adjoint (Prop. 3.1.26), it follows that $\int(\varnothing) \simeq \varnothing$.

In the other direction, assume that the shape of X is empty. Then the shape unit (3.44) is a morphism of the form

$$X \xrightarrow{\eta^{\int}_X} \int X \simeq \varnothing \tag{4.8}$$

and thus $X \simeq \varnothing$ follows as in (3.59), by universality of colimits (Example 3.1.33). □

4.1.2 Cohesive ∞-group actions

The condition that Shp preserves finite products implies the following properties.

Proposition 4.1.4 (Shape preserves groups, actions and their homotopy quotients)**.** *Let* $\mathbf{H}$ *be a cohesive* ∞*-topos (Def. 4.1.1),* $G \in \mathrm{Grp}(\mathbf{H})$ (3.121) *and and* $(X,\rho) \in G\mathrm{Actions}(\mathbf{H})$ *(Prop. 3.2.6).*

(i) *Then the shape* $\int X$ (4.2) *of* X *is equipped with an induced* $\int G$*-action, such that the shape of the homotopy quotient* (3.131) *is the homotopy quotient of the shapes. The analogous statement holds for* $\flat$ (4.2)*:*

$$\int\big(X /\!\!/ G\big) \;\simeq\; \big(\int X\big) /\!\!/ \big(\int G\big) \quad \text{and} \quad \flat\big(X /\!\!/ G\big) \;\simeq\; \big(\flat X\big) /\!\!/ \big(\flat G\big)\,. \tag{4.9}$$

(ii) *In particular, both* $\int$ *and* $\flat$ *preserve group objects and their deloopings (Prop. 3.2.1):*

$$\int \mathbf{B}G \;\simeq\; \mathbf{B}\int G \quad \text{and} \quad \flat\mathbf{B}G \;\simeq\; \mathbf{B}\flat G\,. \tag{4.10}$$

Proof. The homotopy quotient of X by G is, equivalently, a colimit over a simplicial diagram of finite Cartesian products of copies of X and G (3.131). Hence the statement follows for every ∞-functor that commutes with simplicial colimits and with finite products. But, since $\int$ is a left adjoint, it commutes with all colimits (Prop. 3.1.26) and also with finite products, by assumption on Shp and since Disc is a right adjoint. Similarly, $\flat$ is both left and right adjoint, and hence preserves all colimits and all limits (again Prop. 3.1.26). That preservation of homotopy quotients implies preservation of ∞-groups follows by the delooping theorem (Prop. 3.2.1). □

Lemma 4.1.5 (Cohesive shape preserves some homotopy fiber products)**.** *In a cohesive* ∞*-topos* $\mathbf{H}$ *(Def. 4.1.1), the shape functor* Shp (4.1) *preserves homotopy fiber products over cohesively discrete objects. That is, for* $B \in \mathbf{B} \xhookrightarrow{\mathrm{Disc}} \mathbf{H}$ *and* $X, Y \in \mathbf{H}_{/B}$*, we have a natural equivalence*

$$\mathrm{Shp}\big(X \underset{B}{\times} Y\big) \;\simeq\; \mathrm{Shp}(X) \underset{B}{\times} \mathrm{Shp}(Y)\,. \tag{4.11}$$

Proof. This is proven in [Sc13, Thm. 3.8.19] under the assumption that $\mathbf{H}$ admits an ∞-cohesive site of definition. This assumption was shown to be unnecessary in [BP22, Lemma 3.10]. □

Lemma 4.1.6 (Shape of $\eta^{\int}$-induced action)**.** *Let* $\mathbf{H}$ *be a cohesive* ∞*-topos (Def. 4.1.1),* $G \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1) and* $(X,\rho) \in G\mathrm{Actions}(\mathbf{H})$ *(Prop. 3.2.6).*

(i) *The left-induced action (Prop. 3.2.12)*

$$(\widetilde{X}, \widetilde{\rho}) \;:=\; \mathbf{B}\big(\eta^{\int}_G\big)_!\,(X,\rho) \;\in\; (\int G)\mathrm{Actions}(\mathbf{H}) \tag{4.12}$$

along the shape unit morphism (3.44) $G \xrightarrow{\eta^{\int}_G} \int G$ *acts on an object whose shape* (4.2) *is that of* X*:*

$$\int \widetilde{X} \;\simeq\; \int X\,, \tag{4.13}$$

whence

$$\big(\int X\,,\, \int \rho\big) \;\in\; \big(\int G\big)\mathrm{Actions}(\mathbf{H})\,. \tag{4.14}$$

(ii) *Similarly, the restricted-induced action (Prop. 3.2.12)*

$$(\widetilde{X}, \widetilde{\rho}) \;:=\; \mathbf{B}\big(\int \varepsilon^{\flat}\big)^{*} \circ \mathbf{B}\big(\eta^{\int}_G\big)_!\,(X,\rho) \;\in\; (\flat G)\mathrm{Actions}(\mathbf{H}) \tag{4.15}$$

along the pair of group homomorphisms (using Prop. 4.1.4)

$$G \xrightarrow{\eta^{\int}_G} \int G \xleftarrow{\int \varepsilon^{\flat}_G} \flat G \tag{4.16}$$

acts on an object whose shape (4.2) is that of X:

$$\int \widetilde{X} \;\simeq\; \int X\,. \tag{4.17}$$

Proof. By Prop. 3.2.6 and Prop. 3.2.12, the object $\widetilde{X}$ sits in a diagram of Cartesian squares (Notation 3.1.21) as shown on the left in the following (the full square in case **(i)**, the pasting decomposition for case **(ii)**):

$$\begin{array}{ccccc} \widetilde{X} & \longrightarrow & \widetilde{X}/\!\!/\flat G & \longrightarrow & X/\!\!/G \\ & & & & \downarrow{\scriptstyle \rho} \\ \downarrow & {\scriptstyle (pb)} & \downarrow & {\scriptstyle (pb)} & \mathbf{B}G \\ & & & & \downarrow{\scriptstyle \mathbf{B}\eta^{\int}_G} \\ * & \longrightarrow & \mathbf{B}\flat G & \xrightarrow[\mathbf{B}\int\varepsilon^{\flat}_G]{} & \mathbf{B}\int G \end{array} \quad \overset{\int}{\longmapsto} \quad \begin{array}{ccccc} \int\widetilde{X} & \longrightarrow & \big(\int\widetilde{X}\big)/\!\!/\big(\flat G\big) & \longrightarrow & \big(\int X\big)/\!\!/\big(\int G\big) \\ \downarrow & {\scriptstyle (pb)} & \downarrow & {\scriptstyle (pb)} & \downarrow{\scriptstyle \int\rho} \\ * & \longrightarrow & \mathbf{B}\flat G & \xrightarrow[\mathbf{B}\int\varepsilon^{\flat}_G]{} & \mathbf{B}\int G \end{array} \tag{4.18}$$

But, since the objects in the bottom row $\mathbf{B}\int G \simeq \int \mathbf{B}G$ and $\mathbf{B}\flat G \simeq \flat\mathbf{B}G$ (equivalences by Prop. 4.1.4) are both cohesively discrete, Lemma 4.1.5 says that the image of these squares under shape are still Cartesian. This is shown on the right in (4.18), where we have identified the shape of the various objects by using Prop. 4.1.4 and idempotency of the modality (Prop. 3.1.29). With this, the pasting law (Prop. 3.1.23) implies that the outer right square in (4.18) is itself Cartesian, hence that $\int\widetilde{X}$ is the homotopy fiber of $\int\rho$. This implies the claim, by Prop. 3.2.6. □

Proposition 4.1.7 (Automorphisms along shape-unit)**.** *Let $\mathbf{H}$ be a cohesive ∞-topos (Def. 4.1.1), $G \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1) and $(X,\rho) \in G\mathrm{Actions}(\mathbf{H})$ (Prop. 3.2.6). There is a canonical homomorphism*

$$\mathrm{Aut}(X) \xrightarrow{\mathrm{Aut}\left(\eta^{\int}_X\right)} \mathrm{Aut}(\textstyle\int X) \tag{4.19}$$

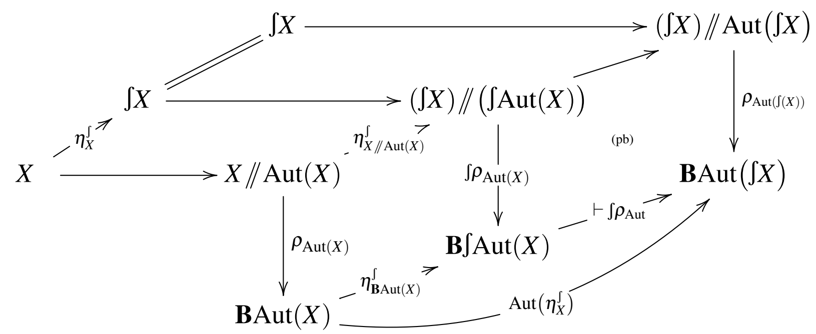
from the automorphism group (Def. 3.2.13) of X to that of the shape (4.2) of X, which is such that the shape unit $\eta^{\int}_X$ (3.44) is equivariant (Def. 3.2.10) with respect to the canonical automorphism action (3.144) on X and the restriction (Prop. 3.2.12) along this morphism (4.1.7) of the canonical automorphism action on $\int X$:

$$\big(X, \rho_{\mathrm{Aut}(X)}\big) \xrightarrow{\eta^{\int}_X} \mathrm{Aut}(\eta^{\int}_X)^*\big(\textstyle\int X, \rho_{\mathrm{Aut}(\int X)}\big) \\ \in \mathrm{Aut}(X)\mathrm{Actions}(\mathbf{H})\,. \tag{4.20}$$

Proof. Take the morphism (4.1.7) to be the composite

$$\begin{array}{ccc} \mathrm{Aut}(X) & \xrightarrow{\qquad\mathrm{Aut}\left(\eta^{\int}_X\right)\qquad} & \mathrm{Aut}\big(\int X\big) \\ & {\scriptstyle \eta^{\int}_{\mathrm{Aut}(X)}}\searrow \quad \int\big(\mathrm{Aut}(X)\big) \quad \nearrow{\scriptstyle \Omega \vdash \int\rho_{\mathrm{Aut}}} & \end{array} \tag{4.21}$$

where **(a)** the left morphism is the shape unit (3.44), using Prop. 4.1.4, while **(b)** the right morphism is that which exhibits, via Prop. 3.2.14, the $\int \mathrm{Aut}(X)$-action $\int \rho_{\mathrm{Aut}}$ (4.14) on $\int X$ from Lemma 4.1.6. Then consider the following diagram of homotopy fiber sequences:



(4.22)

Here **(i)** the fiber squence in the middle is that from the right of (4.18), **(ii)** the right part is the defining pullback from Prop. 3.2.14, while **(iii)** the left part exists by the naturality of $\eta^{\int}$. By the commutativity of the total front square it factors through the coresponding pullback square, thus implying the claim. □

4.1.3 Concrete cohesive objects

Definition 4.1.8 (Concrete objects)**.** Let $\mathbf{H}$ be a cohesive ∞-topos (Def. 4.1.1).

(i) For $X \in \mathbf{H}_0 \hookrightarrow \mathbf{H}$ 0-truncated (Def. 3.1.57), we say that X is a *concrete object* or *concrete cohesive space* if the unit $\eta^{\sharp}_X$ (3.44) of the $\sharp$-modality (4.2) is (-1)-truncated (Def. 3.1.58), hence a monomorphism. By the 0-image factorization (3.108),

$$\underset{\text{unit morphism of } \sharp\text{-modality}}{X \xrightarrow[\eta^{\sharp}_X]{\text{(-1)-conn.}} \overset{\text{image factorization}}{\sharp_1 X} \xhookrightarrow{\text{(-1)-trunc.}} \sharp X} \tag{4.23}$$

this means equivalently that X is equivalent to its 0-image under the $\sharp$-unit (3.44):

$$X \in \mathbf{H}_0 \;:\; \begin{array}{c} X \text{ is concrete} \\ \Leftrightarrow \;\; X \overset{\eta^{\sharp}_X}{\hookrightarrow} \sharp X \\ \Leftrightarrow \;\; \sharp_1 X \simeq X\,. \end{array} \tag{4.24}$$

(ii) We write

$$\mathbf{H}_{0,\sharp_1} \hookrightarrow \mathbf{H}_0 \hookrightarrow \mathbf{H} \tag{4.25}$$

for the full subcategory of the 0-truncated objects on those which are concrete.

(iii) Moreover, for $n \in \mathbb{N}$ we define, recursively, full sub-∞-categories of concrete $(n+1)$-truncated objects (Def. 3.1.57)

$$\mathbf{H}_{n+1,\sharp_1} \hookrightarrow \mathbf{H}_{n+1} \hookrightarrow \mathbf{H} \tag{4.26}$$

by declaring that $X \in \mathbf{H}_{n+1}$ is *concrete* if:

– it admits a *concrete atlas*, namely an effective epimorphism out of a concrete 0-truncated object (4.24),

– such that the homotopy fiber product of the atlas with itself (which is an n-truncated object) is a concrete:

$$\begin{aligned} & X \in \mathbf{H}_{n+1} \;:\quad X \text{ is concrete} \\ \Leftrightarrow & \underset{X_0 \in \mathbf{H}_{0,\sharp_1}}{\exists} \;:\quad X_0 \xrightarrow{(-1)\text{-trunc.}} X \quad \text{and} \quad X_0 \underset{X}{\times} X_0 \;\in\; \mathbf{H}_{n,\sharp_1}\,. \end{aligned} \tag{4.27}$$

4.1.4 Cohesive charts

Definition 4.1.9 (Charts)**.** Let $\mathbf{H}$ be a cohesive ∞-topos (Def. 4.1.1). We say that an ∞-*category of cohesive charts* for $\mathbf{H}$ is an ∞-site Chrt for $\mathbf{H}$ (Prop. 3.1.41)

$$\mathbf{H} \underset{\hookrightarrow}{\overset{L}{\underset{\bot}{\longleftarrow}}} \mathrm{PShv}_\infty(\mathrm{Chrt}) \tag{4.28}$$

all of whose objects (under the ∞-Yoneda embedding y, Prop. 3.1.37) have contractible shape (4.2):

$$\begin{array}{ccccc} \mathrm{Chrt} & \overset{y}{\hookrightarrow} & \mathbf{H} & \xrightarrow{\mathrm{Shp}} & \mathrm{Grpd}_\infty \\ U & \longmapsto & U & \longmapsto & \mathrm{Shp}(U) \simeq * \end{array}$$
$$\Leftrightarrow \quad \begin{array}{ccccc} \mathrm{Chrt} & \overset{y}{\hookrightarrow} & \mathbf{H} & \xrightarrow{\int} & \mathbf{H} \\ U & \longmapsto & U & \longmapsto & \int(U) \simeq * \end{array} \tag{4.29}$$

Lemma 4.1.10 (Charts are cohesively connected)**.** *Let* $\mathbf{H}$ *be a cohesive* ∞*-topos (Def. 4.1.1) with a site of* Chrt *(Def. 4.1.9). Then, for* $U \in \mathrm{Chrt}$ *and* $\{X_i \in \mathbf{H}\}_{i \in I}$ *an indexed set of objects of* $\mathbf{H}$*, we have that every morphism from* U *into the coproduct of the* X_i *factors through one of the* X_i*:*

$$U \xrightarrow{f} \coprod_{i\in I} X_i \qquad \Leftrightarrow \qquad \underset{i_0 \in I}{\exists} \;\; U \dashrightarrow X_{i_0} \xrightarrow{q_{X_{i_0}}} \coprod_{i\in I} X_i\,,\quad (\text{composite } = f). \tag{4.30}$$

Proof. Consider the pullbacks $U_i \xrightarrow{q_{U_i}} U$ along f of the canonical inclusions of the X_i into their coproduct, given by these Cartesian squares (Notation 3.1.21):

$$\begin{array}{ccc} U_i & \longrightarrow & X_i \\ {\scriptstyle q_{U_i}}\downarrow & \text{(pb)} & \downarrow{\scriptstyle q_{X_i}} \\ U & \xrightarrow[f]{} & \coprod_{i\in I} X_i \end{array} \tag{4.31}$$

By Prop. 3.1.32, this is such that

$$U \simeq \coprod_{i \in I} U_i\,. \tag{4.32}$$

The image of (4.32) under shape (4.2) is

$$* \simeq \int U \simeq \coprod_{i \in I} \int U_i \qquad \in \mathrm{Grpd}_\infty \xhookrightarrow{\mathrm{Disc}} \mathbf{H} \,, \tag{4.33}$$

where on the left we used the defining property (4.29) of charts and on the right we used that the shape operation, being a left adjoint, preserves coproducts (Prop. 3.1.26). But, since $* \in \mathrm{Grpd}_\infty$ is connected, this implies that there is $i_0 \in I$ with

$$\int U_i \simeq \begin{cases} \varnothing & | \quad i \neq i_0 \\ * & | \quad i = i_0 \end{cases} \tag{4.34}$$

From this, Lemma 4.1.3 implies that $U_i \simeq \varnothing$ for $i \neq i_0$ and, with (4.32), this implies

$$U_{i_0} \xrightarrow[\simeq]{q_{U_{i_0}}} U \,. \tag{4.35}$$

Using this in (4.31) gives the desired factorization. □

Lemma 4.1.11 (Quotient by cohesively discrete ∞-group)**.** *Let* $\mathbf{H}$ *be a cohesive ∞-topos (Def. 4.1.1) which admits a site of* Chrt *(Def. 4.1.9). Then, for*

$$G \in \mathrm{Grp}(\mathrm{Grpd}_\infty) \xhookrightarrow{\mathrm{Disc}} \mathrm{Grp}(\mathbf{H}) \tag{4.36}$$

a cohesively discrete ∞-group (3.121) *and* $U \in \mathrm{Chrt}$*, we have an equivalence*

$$\mathbf{H}(U, *\!\sslash\! G) \simeq *\!\sslash\! G \quad \in \mathrm{Grpd}_\infty \,. \tag{4.37}$$

Proof. Since Disc is both a left and a right adjoint, it preserves (Prop. 3.1.26) the homotopy quotient that corresponds to the effective epimorphism $* \twoheadrightarrow *\!\sslash\! G$ (Prop. 3.1.70) so that

$$*\!\sslash\! G \in \mathrm{Grpd}_\infty \xhookrightarrow{\quad \mathrm{Disc} \quad} \mathbf{H} \tag{4.38}$$

is a cohesively discrete object. With this, we have the following sequence of natural equivalences:

$$\mathbf{H}\big(U, *\!\sslash\! G\big) \simeq \mathbf{H}\big(U, \mathrm{Disc}(*\!\sslash\! G)\big) \simeq \mathrm{Grpd}_\infty\big(\mathrm{Shp}(U), *\!\sslash\! G\big) \simeq \mathrm{Grpd}_\infty\big(*, *\!\sslash\! G\big) \simeq *\!\sslash\! G \,, \tag{4.39}$$

where the second step is the hom-equivalence (3.43) of the Shp ⊣ Disc-adjunction and the third step is the condition that the chart U has contractible shape. □

Lemma 4.1.12 (Homming Charts into quotients by discrete groups)**.** *Let* $\mathbf{H}$ *be a cohesive ∞-topos (Def. 4.1.1) which admits* Chrt *(Def. 4.1.9). Then, for* $X \in \mathbf{H}$ *an object equipped with an ∞-action (Prop. 3.130) by a geometrically discrete ∞-group* G (4.36)*, the homotopy quotient* $X \sslash G$ (3.131) *is given as an ∞-sheaf on* Chrt*, by assigning to* $U \in \mathrm{Chrt}$ *the homotopy quotient of the ∞-groupoid of* U*-shapes plots of* X*:*

$$X \sslash G : U \longmapsto \mathbf{H}(U, X) \sslash G \,. \tag{4.40}$$

Proof. Consider the image of the homotopy fiber sequence that characterizes the given ∞-action (Prop. 3.2.6) under homming the chart U into it:

$$
\begin{array}{ccc}
X \xrightarrow{\mathrm{fib}(p)} X /\!\!/ G & & \\
\quad\quad\quad\quad \downarrow{\scriptstyle p} & \overset{\mathbf{H}(U,-)}{\longmapsto} & \\
\quad\quad\quad\quad * /\!\!/ G & &
\end{array}
$$

$$
\begin{array}{c}
\mathbf{H}(U,X) \xrightarrow{\mathrm{fib}(\mathbf{H}(U,p))} \mathbf{H}(U,X) /\!\!/ G \;\simeq\; \mathbf{H}\big(U, X /\!\!/ G\big) \\
\quad\quad\quad\quad\quad\quad \downarrow{\scriptstyle \mathbf{H}(U,p)} \\
\quad\quad\quad\quad\quad\quad * /\!\!/ G \;\simeq\; \mathbf{H}(U, * /\!\!/ G)
\end{array}
\tag{4.41}
$$

Since the hom-functor $\mathbf{H}(U,-)$ preserves limits, the result is again a homotopy fiber sequence, as shown on the right of (4.41). Moreover, by the assumption that G is geometrically discrete and that U is geometrically contractible, we have the equivalence (4.41) shown on the bottom right. This means that the fiber sequence on the right of (4.41) exhibits $\mathbf{H}(U, X /\!\!/ G)$ as the homotopy quotient $\mathbf{H}(U,X) /\!\!/ G$ of an ∞-action by G on $\mathbf{H}(U,X)$. □

Lemma 4.1.13 (Fixed locus in 0-truncated objects for discrete groups)**.** *Let* $\mathbf{H}$ *be a cohesive* ∞*-topos (Def. 4.1.1) with a site of* Chrt *(Def. 4.1.9). Let* $G \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1) be discrete* $G \simeq \flat G$ *and 0-truncated,* $G \simeq \tau_0 G$*, and let* $(X,\rho) \in G\mathrm{Actions}(\mathbf{H})$ *(Prop. 3.2.6) with* $X \simeq \tau_0 X$ *also 0-truncated. Then the G-fixed locus* $X^G \in \mathbf{H}$ *(Def. 3.2.24) is itself 0-truncated and such that, for* $U \in$ Chrt*, we have a natural equivalence*

$$
\mathbf{H}\big(U, X^G\big) \;\simeq\; \mathbf{H}(U,X)^G \;:=\; \Big\{ \phi \in \mathbf{H}(U,X) \;\big|\; \underset{g \in G}{\forall}\, g \cdot \phi = \phi \Big\}
\tag{4.42}
$$

between **(a)** *the hom-set from* U *to* X^G *and* **(b)** *the naive set of fixed points in the hom-set from* U *to* X*, with respect to the restriction (Prop. 3.2.12) along* $K \hookrightarrow G$ *of the induced G-action* (4.41) *on the latter.*

Proof. We claim that we have the following sequence of natural equivalences:

$$
\begin{aligned}
\mathbf{H}(U,X^G) &= \mathbf{H}\big(U, \mathbf{B}(G \to *)_*\big((X,\rho)\big)\big) \\
&\simeq \mathbf{H}_{/\mathbf{B}G}\big(\mathbf{B}(G \to *)_*\big(U\big), X /\!\!/ G\big) \\
&\simeq \mathbf{H}_{/\mathbf{B}G}\big((* /\!\!/ G) \times U \,,\, X /\!\!/ G\big) \\
&\simeq \mathbf{H}\big((* /\!\!/ G) \times U \,,\, X /\!\!/ G\big) \underset{\mathbf{H}((* /\!\!/ G) \times U \,,\, * /\!\!/ G)}{\times} \big\{\mathrm{pr}_1\big\} \\
&\simeq \mathrm{Grpd}\Big(* /\!\!/ G \,,\, \mathbf{H}\big(U \,,\, X /\!\!/ G\big)\Big) \underset{\mathrm{Grpd}\left(* /\!\!/ G,\, \mathbf{H}(U\,,\, * /\!\!/ G)\right)}{\times} \big\{\widetilde{\mathrm{pr}_1}\big\} \\
&\simeq \mathrm{Grpd}\Big(* /\!\!/ G \,,\, \mathbf{H}\big(U \,,\, X\big) /\!\!/ G\Big) \underset{\mathrm{Grpd}(* /\!\!/ G,\, * /\!\!/ G)}{\times} \big\{\mathrm{id}\big\} \\
&\simeq \mathbf{H}(U,X)^G .
\end{aligned}
\tag{4.43}
$$

Here the first three lines are the definition of fixed loci (3.162) and the hom-equivalences (3.43) of the resulting adjunction (3.89). The fourth line is the characterization (3.82) of hom-∞-groupoids in slices (Prop. 3.1.48), the fifth line uses the tensoring (3.60) of $\mathbf{H}$ over Grpd_∞ (Prop. 3.1.34), and the sixth line follows by Prop. 4.1.12.

To see the last step in (4.43), use the explicit presentation of the groupoid $\mathbf{H}(U,X)/\!\!/G$ as an action groupoid, by Example 3.1.15. This way the projection map in the fiber product in the sixth line in (4.43) is presented by a Kan fibration, whence this homotopy fiber product may be computed equivalently as a 1-categorical fiber product of sets of objects and of sets of morphisms, separately. Moreover, since $\{\mathrm{id}\}$ has no non-trivial morphisms and since the projection functor itself is faithful, there are in fact no non-trivial morphisms in this fiber product, which is hence just the set whose elements are precisely those functors of action groupoids which are equal to the identity on labels in G:

$$
\begin{aligned}
&\mathrm{Grpd}\Big(*/\!\!/G,\, \mathbf{H}\big(U,X\big)/\!\!/G\Big) \underset{\mathrm{Grpd}(*/\!\!/G,\, */\!\!/G)}{\times} \{\mathrm{id}\} \\
\simeq\;& \left\{ \begin{array}{ccc} */\!\!/G & \longrightarrow & \mathbf{H}(U,X)/\!\!/G \\ * & \longmapsto & \phi \\ {\scriptstyle g\in G}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ * & \longmapsto & g\cdot\phi \end{array} \right\} \\
\simeq\;& \mathbf{H}(U,X)^G.
\end{aligned}
\tag{4.44}
$$

□

Lemma 4.1.14 (*n*-Truncated morphisms via *n*-truncated homotopy fibers)**.** *Let* $\mathbf{H}$ *be an ∞-topos which is cohesive (Def. 4.1.1). Let* G *be a finite group in* $\mathbf{H}$ (4.154)*. Then, for every* $n \in \{-2,-1,0,1,\cdots\}$ *and for any morphism in* $\mathbf{H}$ *to its delooping groupoid (Example 3.1.14)* $\mathscr{X} \xrightarrow{p} */\!\!/G$*, the following are equivalent*

- **(i)** *p is an n-truncated morphism (Def. 3.1.58);*
- **(ii)** *the homotopy fiber of p (over the canonical point of $*/\!\!/G$) is an n-truncated object (Def. 3.1.57).*

Proof. Let $U \in \mathrm{Chrt}$ and consider homming it into the homotopy fiber sequence in question:

$$
\begin{array}{ccc} X & \longrightarrow & \mathscr{X} \\ \big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle p} \\ * & \longrightarrow & */\!\!/G \end{array}
\quad\Rightarrow\quad
\begin{array}{ccc} \mathbf{H}(U,X) & \longrightarrow & \mathbf{H}(U,\mathscr{X}) \\ \big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle \mathbf{H}(U,p)} \\ * & \longrightarrow & \mathbf{H}(U,*/\!\!/G) \simeq */\!\!/G \end{array}
\tag{4.45}
$$

Since the hom-functor $\mathbf{H}(U,-)$ preserves limits, the square on the right is again a homotopy pullback. Since U is a chart and G is discrete, we have the equivalence (4.37) shown on the bottom right. Since $*/\!\!/G$ has an essentially unique point, the square on the right exhibits the essentially unique homotopy fiber of the morphism $\mathbf{H}(U,p)$.

Since the charts U are generators of $\mathbf{H}$ (objects of an ∞-site of definition), the morphism p is n-truncated (Def. 3.1.58) precisely if for each chart U the homotopy fiber of $\mathbf{H}(U,p)$ is n-truncated. But the square on the right shows that this homotopy fiber is $\mathbf{H}(U,X)$, and hence this means, equivalently, that X is an n-truncated object (according to Def. 3.1.57). □

4.1.5 Examples of cohesive ∞-toposes

We indicate some examples of cohesive ∞-toposes (Def. 4.1.1), following [Sc13]. For more details of the constructions see spring[SS25d].

Example 4.1.15 (Discrete cohesion)**.** The base ∞-topos Grpd_∞ is trivially a cohesive ∞-topos (Def. 4.1.1) with all operations being identities:

$$\mathrm{Grpd}_\infty \;\; \begin{array}{c} \xrightarrow{\;\mathrm{id}\;} \\ \bot \\ \xleftarrow{\;\mathrm{id}\;} \\ \bot \\ \xrightarrow{\;\mathrm{id}\;} \\ \bot \\ \xleftarrow{\;\mathrm{id}\;} \end{array} \;\; \mathrm{Grpd}_\infty \tag{4.46}$$

For emphasis we also call this the ∞-topos of *geometrically discrete ∞-groupoids*.

Definition 4.1.16 (Site for homotopical cohesion)**.** A small ∞-site (3.73) is an *∞-site for homotopical cohesion* if

(i) its Grothendieck topology is trivial and

(ii) the underlying ∞-category has finite products, i.e., has a terminal object and binary Cartesian products.

Example 4.1.17 (Homotopical cohesion)**.** The ∞-topos of ∞-sheaves (Def. 3.1.42) over an ∞-site $\mathscr{C}$ for homotopical cohesion (Def. 4.1.16) is cohesive (Def. 4.1.1):

$$\mathbf{H} := \mathrm{Shv}_\infty(\mathscr{C}) \;\; \begin{array}{c} \xrightarrow{\;\varinjlim\;} \\ \bot \\ \xleftarrow{\;\mathrm{const}\;} \\ \bot \\ \xrightarrow{\;\varprojlim\;} \\ \bot \\ \xleftarrow{\;\mathrm{Chtc}\;} \end{array} \;\; \mathrm{Grpd}_\infty \tag{4.47}$$

(i) The operation $\mathrm{Pnts} \simeq \varprojlim$ forms the limit of ∞-presheaves regarded as ∞-functors on $\mathscr{C}^{\mathrm{op}}$ (by Prop. 3.1.36); but since $\mathscr{C}$ is assumed to have a terminal object, this is equivalently just the evaluation on that object:

$$\mathrm{Pnts}(X) \;\simeq\; X(*) \;\simeq\; \mathbf{H}(*,X)\,, \tag{4.48}$$

where on the right we used the ∞-Yoneda lemma (Prop. 3.1.38). This makes manifest how $\mathrm{Pnts}(X)$ is the "underlying ∞-groupoid of points of X".

(ii) The operation $\mathrm{Shp} \simeq \varinjlim$ is the colimit of ∞-presheaves regarded as ∞-functors (by Prop. 3.1.36). Since the colimit of any representable functor is the point (Lemma 3.1.40)

$$\mathscr{C} \xhookrightarrow{\;y\;} \mathrm{Shv}_\infty(\mathscr{C}) \xrightarrow{\;\mathrm{Shp}\;} \mathrm{Grpd}_\infty\,, \qquad (\mathscr{C} \xrightarrow{\;\mathrm{const}_*\;} \mathrm{Grpd}_\infty) \tag{4.49}$$

this means that $\mathscr{C}$ serves itself as a category of Chrt in this case (Def. 4.1.9).

Example 4.1.18 (Smooth cohesion)**.** The ∞-sheaf ∞-topos (Def. 3.1.42) over the site of SmthMfd (Def. 3.1.9, see [FSS12, App.]), which we call the ∞-topos of *smooth ∞-groupoids*

$$\mathrm{SmthGrpd}_\infty \;:=\; \mathrm{Shv}_\infty(\mathrm{SmthMfd})\,, \tag{4.50}$$

is cohesive (Def. 4.1.1): The adjoint quadruple (4.1) arises as in Example 4.1.17, which here happens to descend from ∞-presheaves to ∞-sheaves.

In this case we have:

(i) A category of Chrt (Def. 4.1.9) is given (Prop. 3.1.10) by CrtSpc (Def. 3.1.5)

$$\mathrm{CrtSpc} \xhookrightarrow{\;y\;} \mathrm{Shv}_\infty(\mathrm{CrtSpc}) \xrightarrow{\quad\simeq\quad} \mathrm{SmthGrpd}_\infty \qquad \textstyle\int(y(\mathbb{R}^n)) \;\simeq\; * \tag{4.51}$$

(ii) The concrete 0-truncated objects (Def. 4.1.8) are equivalently the *diffeological spaces* (Def. 3.1.6), including the *D-topological spaces*[1] (Def. 3.1.2) as well as smooth and possibly infinite-dimensional Fréchet manifolds (Prop. 3.1.11) as further full subcategories (3.12):

$$\begin{array}{c} \mathrm{TopSpc} \xrightarrow{\mathrm{Cdfflg}} \mathrm{DTopSpc} \hookrightarrow \\ \mathrm{FréMfd} \hookrightarrow \end{array} \mathrm{DiffSp} \xhookrightarrow{\substack{\text{concrete} \\ \text{0-truncated} \\ \text{objects}}} \mathrm{SmthGrpd}_\infty \tag{4.52}$$

(iii) The concrete 1-truncated objects (Def. 4.1.8) form the $(2,1)$-category of diffeological groupoids with Morita/Hilsum-Skandalis morphisms (Remark 3.1.72) between them, which includes, by (4.52), the $(2,1)$-categories of D-topological groupoids and of (possibly infinite-dimensional Fréchet-)Lie groupoids:

$$\begin{array}{c} \mathrm{TopGrpd} \xrightarrow{\mathrm{Cdfflg}} \mathrm{DTopGrpd} \hookrightarrow \\ \mathrm{FréLieGrpd} \hookrightarrow \end{array} \mathrm{DiffGrpd} \xhookrightarrow{\substack{\text{concrete} \\ \text{1-truncated} \\ \text{objcts}}} \mathrm{SmthGrpd}_\infty \tag{4.53}$$

(iv) (**Smooth Oka principle** [SS25d, Thm. 3.3.53]) The cohesive shape (4.2) is given equivalently [BEBP19][Pav20][Bunk20, §3] by the *smooth ∞-path ∞-groupoid*:

$$\begin{array}{lcccc} \textstyle\int X & \simeq & \varinjlim \mathbf{Map}\big(\Delta^\bullet_{\mathrm{smth}}, X\big) & \in & \mathrm{SmthGrpd}_\infty\,, \\ \text{hence } \;\mathrm{Shp}(X) & \simeq & \varinjlim X(\Delta^\bullet_{\mathrm{smth}}) & \in & \mathrm{Grpd}_\infty\,, \end{array} \tag{4.54}$$

where $\Delta^\bullet_{\mathrm{smth}}$ is the simplicial smooth manifold of extended simplices (Def. 3.1.19) and $\mathbf{Map}(-,-)$ denotes the internal hom (3.54) in $\mathrm{SmthGrpd}_\infty$.

(v) The cohesive shape (4.2) of **(a)** any topological space and **(b)** any finite-dimensional smooth manifold regarded, respectively, as smooth ∞-groupoids

[1]These are the Δ-*generated spaces* of [Sm][Dug03]; see Remark 3.1.3.

via (4.52) is equivalently (by (4.54) with Prop. 3.1.20, and by [Sc13, 4.3.29], respectively) its standard topological homotopy type $\mathrm{Shp}_{\mathrm{Top}}$ (3.1.13):

$$\textbf{(a)} \quad \mathrm{TopSpc} \xrightarrow{\mathrm{Cdfflg}} \mathrm{DiffSp} \hookrightarrow \mathrm{SmthGrpd}_\infty \xrightarrow{\mathrm{Shp}} \mathrm{Grpd}_\infty, \quad \Downarrow \simeq \quad \mathrm{TopSpc} \xrightarrow{\mathrm{Shp}_{\mathrm{Top}}} \mathrm{Grpd}_\infty \tag{4.55}$$

$$\textbf{(b)} \quad \mathrm{SmthMfd} \hookrightarrow \mathrm{DiffSp} \hookrightarrow \mathrm{SmthGrpd}_\infty \xrightarrow{\mathrm{Shp}} \mathrm{Grpd}_\infty, \quad \Downarrow \simeq \quad \mathrm{SmthMfd} \xrightarrow{\mathrm{Shp}_{\mathrm{Top}} \circ \mathrm{Dtplg}} \mathrm{Grpd}_\infty \tag{4.56}$$

(vi) The cohesive shape (4.2) of a topological groupoid, when regarded, via its coreflection (3.12), as a D-topological groupoid and hence as a smooth ∞-groupoid (4.53) is equivalently (by (4.55), and since $\int$ is left adjoint and hence preserves homotopy colimits, Prop. 3.1.26) its simplicial-topological shape (Def. 3.1.17):

$$\mathrm{TopGrpd} \xrightarrow{\mathrm{Cdfflg}} \mathrm{DiffGrpd} \hookrightarrow \mathrm{SmthGrpd}_\infty \xrightarrow{\mathrm{Shp}} \mathrm{Grpd}_\infty, \quad \Downarrow \simeq \quad \mathrm{TopGrpd} \xrightarrow{\mathrm{Shp}_{\mathrm{sTop}}} \mathrm{Grpd}_\infty \tag{4.57}$$

Example 4.1.19 (Spectral cohesion)**.** Let $\mathbf{H}$ be a cohesive ∞-topos (Def. 4.1.1). Then its tangent ∞-topos $T\mathbf{H} = \mathrm{SpectralBundles}(\mathbf{H})$ (Example 3.1.51) is cohesive [Sc13, 4.1.9] over the base tangent ∞-topos (3.160):

$$T\mathbf{H} \begin{array}{c} \xrightarrow{\;T\mathrm{Shp}\;} \\ \perp \\ \xleftarrow{\;T\mathrm{Disc}\;} \\ \perp \\ \xrightarrow{\;T\mathrm{Pnts}\;} \\ \perp \\ \xleftarrow{\;T\mathrm{Chtc}\;} \end{array} T\mathrm{Grpd}_\infty \tag{4.58}$$

Remark 4.1.20 (Differential cohomology in cohesive ∞-toposes)**.** The intrinsic cohomology theory (1.21) of a cohesive ∞-topos (Def. 4.1.1) is *differential cohomology* [Sc13].

(i) In the case when $\mathbf{H} := \mathrm{SmthGrpd}_\infty$ (Example 4.1.18), this is a non-abelian differential cohomology theory generalizing the theory of Cartan-Ehresmann connections on smooth fiber bundles to ∞-connections on smooth ∞-bundles [SSS12][FSS12][NSS12a].

(ii) In the case when $\mathbf{H} := T\mathrm{SmthGrpd}_\infty$ is the cohesive tangent ∞-topos (Example 4.1.19) to that of smooth ∞-groupoids (Example 4.1.18), the intrinsic cohomology furthermore subsumes abelian Hopkins-Singer differential cohomology theories and variants [BNV13], as well as the twisted versions of these (Remark 3.2.23), such as twisted differential KU-theory [GS19a] and twisted differential KO-theory [GS19b].

4.1.6 Differential Geometry

We present a formulation of differential geometry internal to ∞-toposes which we call *elastic* [Sc13][Sc18], adjoining to the plain *shape* operation $\int$ of §4.1.1 a *de Rham shape* operation $\Im$, in generalization of [Si96][ST97].

Definition 4.1.21 (Elastic ∞-topos)**.**

(i) An *elastic ∞-topos* over $\mathbf{B} = \mathrm{Grpd}_\infty$ is an ∞-topos $\mathbf{H}$ (Def. 3.1.30) whose base geometric morphism (Prop. 3.1.43), to be denoted $\mathrm{Pnts} : \mathbf{H} \to \mathrm{Grpd}_\infty$, is equipped with a factorization as follows, having adjoints (Def. 3.1.24) as shown:

$$
\begin{array}{lccccc}
\text{“reduced”} & & \xleftarrow{\;\mathrm{Rdcd}\;} & & & \\
& & \bot & & & \\
\text{“infnt shape”} \ \mathrm{Shp}: & & \xrightarrow{\;\mathrm{Shp}_{\mathrm{inf}}\;} & & \xrightarrow{\;\mathrm{Shp}_{\Re}\;} & \\
& & \bot & & \bot & \\
\text{“infnt discrete”} & \mathbf{H} & \xleftarrow{\;\mathrm{Disc}_{\mathrm{inf}}\;} & \mathbf{H}_{\Re} & \xleftarrow{\;\mathrm{Disc}_{\Re}\;} & \mathbf{B} \quad : \mathrm{Disc} \\
& & \bot & & \bot & \\
\text{“infnt points”} \ \mathrm{Pnts}: & & \xrightarrow{\;\mathrm{Pnts}_{\mathrm{inf}}\;} & & \xrightarrow{\;\mathrm{Pnts}_{\Re}\;} & \\
& & & \bot & & \\
& & \longleftarrow & \mathrm{Chtc} & \longrightarrow & \\
& \text{elastic } \infty\text{-topos} & & \text{reduced sub-topos} & & \text{discrete sub-topos}
\end{array}
\tag{4.59}
$$

(ii) Hence an elastic ∞-topos $\mathbf{H}$ is, in particular, a cohesive ∞-topos over $\mathbf{B}$, according to Def. 4.1.1, and so is its sub-∞-topos $\mathbf{H}_{\Re}$ of reduced objects.

(iii) We write

$$
\begin{array}{c}
\big(\Re \;:=\; \mathrm{Rdcd} \circ \mathrm{Shp}_{\mathrm{inf}} \big) \\
\text{“reduced”} \\
\bot \\
\big(\Im \;:=\; \mathrm{Disc}_{\mathrm{inf}} \circ \mathrm{Shp}_{\mathrm{inf}} \big) \qquad : \ \mathbf{H} \to \mathbf{H} \\
\text{“étale”} \\
\bot \\
\big(\mathscr{L} \;:=\; \mathrm{Disc}_{\mathrm{inf}} \circ \mathrm{Pnts}_{\mathrm{inf}} \big) \\
\text{“locally constant”}
\end{array}
\tag{4.60}
$$

for the further induced modalities (1.18) (*elastic modalities*), accompanying the cohesive modalities of (4.2).

Examples of elastic ∞-toposes. We indicate some examples of elastic ∞-toposes (Def. 4.1.21), following [Sc13][Sc18].

Definition 4.1.22 (Jets of Cartesian spaces)**.** Let $k \in \mathbb{N}$.

(i) We write

$$
\begin{array}{ccc}
k\mathrm{JetCrtSp} & \xhookrightarrow{\;C^\infty(-)\;} & \mathrm{CAlg}_{\mathbb{R}}^{\mathrm{op}} \\
\mathbb{R}^n \times \mathbb{D}_W & \longmapsto & C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} (\mathbb{R} \oplus W)
\end{array}
\tag{4.61}
$$

for the full subcategory of that of commutative $\mathbb{R}$-algebras on those which are tensor products of **(a)** the algebra of real-valued smooth functions on a Cartesian space $\mathbb{R}^n$, with **(b)** a finite-dimensional real algebra with a maximal ideal W that is nilpotent of order $k+1$, in that $W^{k+1} = 0$.

(ii) We write

$$\begin{array}{ccccc} \infty\mathrm{JetCrtSp} & := & \bigcup_{k\in\mathbb{N}} k\mathrm{JetCrtSp} & \xhookrightarrow{C^\infty(-)} & \mathrm{CAlg}_{\mathbb{R}}^{\mathrm{op}} \\ \mathbb{R}\times\mathbb{D}_W & \longmapsto & & & C^\infty\big(\mathbb{R}^n\big)\otimes_{\mathbb{R}} W \end{array} \tag{4.62}$$

for the analogous full subcategory where each W is (finite dimensional and) nilpotent of some finite order.

(iii) We regard these categories as equipped with the coverage (Grothendieck pre-topology) whose covers are the families of morphisms of the form

$$\begin{array}{l} \big\{\mathbb{R}^n\times\mathbb{D} \xrightarrow{f_i\times\mathrm{id}} \mathbb{R}^n\times\mathbb{D}\big\}_{i\in I} \\ \text{such that } \big\{\mathbb{R}^n \xrightarrow{f_i} \mathbb{R}^n\big\}_{i\in I} \text{ is a cover in CrtSpc (Def. 3.1.5).} \end{array} \tag{4.63}$$

Lemma 4.1.23 (Coreflections of jets of Cartesian spaces)**.** *Regarding the category* kJetCrtSp *from Def. 4.1.22:*

(i) *For $k=0$, this is equivalently the category of plain Cartesian spaces of Def. 3.1.5:*

$$0\mathrm{JetCrtSp} \simeq \mathrm{CrtSpc}\,. \tag{4.64}$$

(ii) *For any $k\in\mathbb{N}$, the evident full inclusion of* kJetCrtSp *into* $(k+1)$JetCrtSp *is co-reflective*

$$\infty\mathrm{JetCrtSp} \underset{\mathrm{Shp}_{\mathrm{inf},\infty}}{\overset{\mathrm{Rdcd}_\infty}{\rightleftarrows}} \cdots \underset{\mathrm{Shp}_{\mathrm{inf},2}}{\overset{\mathrm{Rdcd}_2}{\rightleftarrows}} 2\mathrm{JetCrtSp} \underset{\mathrm{Shp}_{\mathrm{inf},1}}{\overset{\mathrm{Rdcd}_1}{\rightleftarrows}} 1\mathrm{JetCrtSp} \underset{\mathrm{Shp}_{\mathrm{inf}}}{\overset{\mathrm{Rdcd}}{\rightleftarrows}} \mathrm{CrtSp} \tag{4.65}$$

with

$$C^\infty\big(\mathrm{Shp}_{\mathrm{inf},k}(\mathbb{R}^n\times\mathbb{D}_W)\big) \simeq C^\infty(\mathbb{R}^n)\otimes_{\mathbb{R}}(\mathbb{R}\oplus W)/W^{k+1}. \tag{4.66}$$

Proof. Statement **(i)** follows as a special case of the general fact, sometimes known as *Milnor's exercise* (since the key idea is hinted at in [MSt74, Prob. 1-C]), that passage to their real algebras of smooth functions embeds smooth manifolds fully faithfully into the opposite or real algebras (a general proof is in [KMS93, 35.10], see also [Gr05]; for general perspective see [Nes03, 6]) :

$$\mathrm{SmthMfd} \xhookrightarrow{C^\infty(-)} \mathrm{CAlg}_{\mathbb{R}}^{\mathrm{op}}\,. \tag{4.67}$$

Statement **(ii)** follows readily from the definition, using the fact that algebra homomorphisms preserve order of nilpotency. □

Example 4.1.24 (Jets of smooth ∞-groupoids)**.** For $k\in\mathbb{N}\sqcup\{\infty\}$, the ∞-sheaf ∞-topos (Def. 3.1.42) over the site of k-jets of Cartesian spaces (Def. 4.1.22)

$$k\mathrm{JetSmthGrpd}_\infty := \mathrm{Shv}_\infty(k\mathrm{JetCrtSp}) \tag{4.68}$$

is elastic (Def. 4.1.21), with $(\mathrm{Rdcd}\dashv\mathrm{Shp}_{\mathrm{inf}})$ in (4.59) given by Kan extension of the co-reflections of sites from Lemma 4.1.23:

$$
\begin{array}{ccccc}
 & \xleftarrow{\quad \mathrm{Rdcd} \quad} & & & \\
 & \xrightarrow{\quad \mathrm{Shp_{inf}} \quad} & & \xrightarrow{\quad \mathrm{Shp}_{\Re} \quad} & \\
k\mathrm{JetSmthGrpd}_\infty & \xleftarrow{\quad \mathrm{Disc_{inf}} \quad} & \mathrm{SmthGrpd}_\infty & \xleftarrow{\quad \mathrm{Disc}_{\Re} \quad} & \mathrm{Grpd}_\infty \\
 & \xrightarrow{\quad \mathrm{Pnts_{inf}} \quad} & & \xrightarrow{\quad \mathrm{Pnts}_{\Re} \quad} & \\
 & & \xleftarrow{\qquad\qquad \mathrm{Chtc} \qquad\qquad} & &
\end{array}
\tag{4.69}
$$

(i) Here for $k = 1$ we will, for short, abbreviate

$$
\mathrm{JetSmthGrpd}_\infty := 1\mathrm{JetSmthGrpd}_\infty . \tag{4.70}
$$

(ii) For the case $k = \infty$, the underlying 1-topos is the "Cahiers topos" [Du79a][Ko86][KS17].

(iii) For any k, we have:

(a) The full sub-∞-topos of reduced objects (4.59) is (by Lemma 4.1.23) that of smooth ∞-groupoids from Example 4.1.18

$$
k\mathrm{JetSmthGrpd}_\infty \xleftarrow{\ \mathrm{Disc_{inf}}\ } \mathrm{SmthGrpd}_\infty \tag{4.71}
$$

(b) the 0-truncated concrete objects (Def. 4.1.8) are still equivalently the *diffeological spaces* (Def. 3.1.6) as was the case in (4.52)

$$
\begin{array}{ccccc}
\mathrm{DTopSpc} & \searrow & & & \\
 & & \mathrm{DiffSp} & \xrightarrow[\text{objects}]{\text{0-truncated concrete}} & k\mathrm{JetSmthGrpd}_\infty \\
\mathrm{FréMfd} & \nearrow & & &
\end{array}
\tag{4.72}
$$

and, more generally, the 1-truncated concrete objects are still the *diffeological groupoids*, as was the case in (4.53):

$$
\begin{array}{ccccc}
\mathrm{DTopGrpd} & \searrow & & & \\
 & & \mathrm{DiffGrpd} & \xrightarrow[\text{objects}]{\text{1-truncated concrete}} & k\mathrm{JetSmthGrpd}_\infty \\
\mathrm{FréLieGrpd} & \nearrow & & &
\end{array}
\tag{4.73}
$$

(c) A category of charts (Def. 4.1.9) for $\mathrm{JetSmthGrpd}_\infty$ is given by $k\mathrm{JetCrtSp}$ (Def. 4.1.22) itself.

4.1.7 Étale geometry

Definition 4.1.25 (Étale-over-X modality)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21) and $X \in \mathbf{H}$ an object. We say that the *étale-over-X* modality on the slice ∞-topos over X (Def. 3.1.46) is the ∞-functor

$$
\begin{array}{ccc}
\mathbf{H}_{/X} & \xrightarrow{\ \Im_X\ } & \mathbf{H}_{/X} \\
 & & \\
\begin{array}{c} Y \\ {\scriptstyle f}\downarrow \\ X \end{array} & \longmapsto & \begin{array}{c} Y \times_{\Im X} \Im Y \\ \downarrow {\scriptstyle (\eta^{\Im}_X)^*(\Im f)} \\ X \end{array}
\end{array}
\qquad
\begin{array}{ccccc}
X & & \xrightarrow{\qquad \eta^{\Im}_X \qquad} & & \\
 & \searrow & & & \\
 & & X \times_{\Im X} \Im Y & \longrightarrow & \Im X \\
{\scriptstyle f} & & \downarrow & \text{(pb)} & \downarrow {\scriptstyle \Im f} \\
 & & Y & \xrightarrow{\ \eta^{\Im}_Y\ } & \Im Y
\end{array}
\tag{4.74}
$$

which sends any morphism f into X to the pullback of its image under the plain étale modality $\Im$ (4.60) along its unit morphism (3.44), hence to the left vertical morphism in the Cartesian square shown on the right.

Definition 4.1.26 (Local diffeomorphism)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21). We say that a morphism $Y \xrightarrow{f} X$ in $\mathbf{H}$ is a *local diffeomorphism* if it is étale-over-X (Def. 4.1.25)

$$\Im_X(f) \simeq X\,, \tag{4.75}$$

hence (see Prop. 4.1.32 for this implication) if the naturality square of the unit (3.44) of the $\Im$-modality (4.60) is a Cartesian square:

$$\begin{array}{c} Y \\ {\scriptstyle f}\big\downarrow{\scriptstyle \text{ét}} \\ X \end{array} \quad\Leftrightarrow\quad \begin{array}{ccc} Y & \xrightarrow{\eta^{\Im}_Y} & \Im Y \\ {\scriptstyle f}\big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle \Im f} \\ X & \xrightarrow[\eta^{\Im}_X]{} & \Im X \end{array} \tag{4.76}$$

Lemma 4.1.27 (Closure of class of local diffeomorphisms)**.** *Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21). The class of local diffeomorphisms in $\mathbf{H}$ (Def. 4.1.26)*

(i) *satisfies left-cancellation: given a pair of composable morphisms f, g where g is a local diffeomorphism, then f is so precisely if the composite $g \circ f$ is:*

$$\begin{array}{ccc} Z & \xrightarrow{f} & Y \\ & {\scriptstyle g\circ f}\searrow \quad \swarrow{\scriptstyle g}^{\text{ét}} & \\ & X & \end{array} \quad\Rightarrow\quad \Big(f \text{ is local diffeo} \quad\Leftrightarrow\quad g\circ f \text{ is local diffeo}\Big). \tag{4.77}$$

(ii) *is closed under pullbacks: if in a Cartesian square the right vertical morphism is a local diffeomorphism, then so is the left morphism*

$$\begin{array}{ccc} Y' \times_X Y & \longrightarrow & Y \\ {\scriptstyle g^* f}\big\downarrow & \text{(pb)} & {\scriptstyle \text{ét}}\big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & X \end{array} \quad\Rightarrow\quad g^* f \text{ is a local diffeo.} \tag{4.78}$$

Proof. This is a routine argument:

(i) For two composable morphisms, consider the pasting of their $\eta^{\Im}$-naturality squares

$$\begin{array}{ccc} Z & \xrightarrow{\eta^{\Im}_Z} & \Im Z \\ {\scriptstyle f}\big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle \Im f} \\ Y & \xrightarrow{\eta^{\Im}_Y} & \Im Y \\ {\scriptstyle g}\big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle \Im g} \\ X & \xrightarrow[\eta^{\Im}_X]{} & \Im X \end{array} \tag{4.79}$$

By the functoriality of $\Im$, the total rectangle is the $\eta^{\Im}$-naturality square of $g \circ f$. But, by the pasting law (Prop. 3.1.23) and the assumption that the bottom square is Cartesian, the total rectangle is Cartesian precisely if so is the top square.

(ii) For two morphisms with the same codomain, consider the pasting of their pullback square with the $\eta^{\Im}$-naturality square of one of them, as shown on the left here:

$$
\begin{array}{ccccc}
Y' \times_X Y & \xrightarrow{\;f^* g\;} & Y & \xrightarrow{\;\eta^{\Im}\;} & \Im Y \\
{\scriptstyle g^* f}\big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle f} & \text{(pb)} & \big\downarrow{\scriptstyle \Im f} \\
Y' & \xrightarrow[\;g\;]{} & X & \xrightarrow[\;\eta^{\Im}_X\;]{} & \Im X
\end{array}
$$

$$
\simeq \quad
\begin{array}{ccccc}
Y \times_X Y' & \xrightarrow{\;\eta^{\Im}_{(Y \times_X Y')}\;} & \Im(Y \times_X Y') & \xrightarrow{\;\Im(f^* f)\;} & \Im Y' \\
{\scriptstyle g^* f}\big\downarrow & & {\scriptstyle \Im(g^* f)}\big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle \Im g} \\
Y' & \xrightarrow[\;\eta^{\Im}_{Y'}\;]{} & \Im Y' & \xrightarrow[\;\Im(f')\;]{} & \Im X
\end{array}
\tag{4.80}
$$

By the naturality of $\eta^{\Im}$, this pasting diagram on the left is equivalent to that shown on the right. Moreover, if f is a local diffeomorphisms, it follows that three of the squares are pullbacks (the rightmost one by using that $\Im$ is right adjoint and thus preserves pullbacks, Prop. 3.1.26), as shown. With that, the pasting law (Prop. 3.1.23) implies, first, that the total rectangle on the left is a pullback, hence also that on the left, and then that the remaining square on the right is a pullback. This means that $g^* f$ is a local diffeomorphism.

□

Definition 4.1.28 (Local neighborhood)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21). For $Y \xrightarrow{f} X$ a morphism in $\mathbf{H}$, we say that the corresponding *local neighborhood* of Y in X is the purely étale aspect of f, hence is the object $N_f X \in \mathbf{H}_{/X}$ given by $\Im_{/X}(f) \simeq (\eta^{\Im}_X)^*(\Im f)$, hence given by the following homotopy pullback square:

$$
\begin{array}{ccc}
N_f X & \longrightarrow & \Im X \\
{\scriptstyle \Im_{/X}(f)}\big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle \Im f} \\
Y & \xrightarrow[\;\eta^{\Im}_X\;]{} & \Im Y
\end{array}
\tag{4.81}
$$

Definition 4.1.29 (Tangent bundle)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21). Then for $X \in \mathbf{H}$ any object, we say that its *infinitesimal tangent bundle* is

$$
TX \;:=\; X \underset{\Im X}{\times} X \;\in\; \mathbf{H}_{/X}\,,
\tag{4.82}
$$

hence the left morphism in this Cartesian square:

$$\begin{array}{ccc} TX & \longrightarrow & X \\ {\scriptstyle (\eta^{\Im}_X)^*(\eta^{\Im}_X)_!(\mathrm{id}_X)}\big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle \eta^{\Im}_X} \\ X & \underset{\eta^{\Im}_X}{\longrightarrow} & \Im X \end{array} \tag{4.83}$$

Example 4.1.30 (Local neighborhood of a point)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21). For $X \in \mathbf{H}$ any object and $* \xrightarrow{x} X$ any point, the homotopy fiber of the tangent bundle (Def. 4.1.29) over x is equivalent to the local neighborhood of x (Def. 4.1.28):

$$T_x X \;\simeq\; N_x X\,. \tag{4.84}$$

This follows immediately from the definitions, by the pasting law (Prop. 3.1.23):

$$\begin{array}{ccccc} N_x X \simeq T_x X & \longrightarrow & TX & \longrightarrow & X \\ \big\downarrow & \text{(pb)} & \big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle \eta^{\Im}_X} \\ * & \underset{x}{\longrightarrow} & X & \underset{\eta^{\Im}_X}{\longrightarrow} & \Im X \end{array} \tag{4.85}$$

Proposition 4.1.31 (Pullback along local diffeomorphisms preserves tangent bundles)**.** *In an elastic ∞-topos (Def. 4.1.21), pullback along a local diffeomorphism $Y \underset{\text{ét}}{\xrightarrow{f}} X$ (Def. 4.1.26) preserves tangent bundles (Def. 4.1.29) in that*

$$f^*(TX) \;\simeq\; TY \qquad \textit{via:} \qquad \begin{array}{ccc} TY & \xrightarrow{Tf} & TX \\ \big\downarrow & \text{(pb)} & \big\downarrow \\ Y & \underset{f}{\overset{\text{ét}}{\longrightarrow}} & X \end{array} \tag{4.86}$$

Proof. Consider the pasting of the defining Cartesian squares, shown on the left here:

$$\begin{array}{ccccc} f^*TX & \longrightarrow & TX & \longrightarrow & X \\ \big\downarrow & \text{(pb)} & \big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle \eta^{\Im}_X} \\ Y & \underset{f}{\overset{\text{ét}}{\longrightarrow}} & X & \underset{\eta^{\Im}_X}{\longrightarrow} & \Im X \end{array}$$
$$\simeq \quad \begin{array}{ccccc} TY & \longrightarrow & Y & \xrightarrow{f} & X \\ \big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle \eta^{\Im}_X} & \text{(pb)} & \big\downarrow{\scriptstyle \eta^{\Im}_X} \\ Y & \underset{\eta^{\Im}_X}{\longrightarrow} & \Im Y & \underset{\Im f}{\longrightarrow} & \Im X \end{array} \tag{4.87}$$

By the pasting law (Prop. 3.1.23), the total rectangle on the left is itself Cartesian. Moreover, the bottom composite morphism on the left is equivalent to the bottom composite morphism on the right, by the naturality of $\eta^{\Im}_X$. Therefore, using again the pasting law (Prop. 3.1.23) the total rectangle on the left is equivalent to the pasting of the two consecutive Cartesian squares shown on the right. These identify, in the top row, the middle object Y by (4.76) and thus the left object TY by (4.83). $\square$

4.1.8 Étale toposes

Definition 4.1.32 (Étale topos)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21) and $X \in \mathbf{H}$. Then we say that the *étale ∞-topos* of X, to be denoted $\mathbf{\acute{E}t}_X$, is the full sub-∞-category (Def. 3.1.1) of the slice ∞-topos over X (Prop. 3.1.46) on those morphisms that are local diffeomorphisms (Def. 4.1.26):

$$\mathbf{\acute{E}t}_X \;:=\; \big(\mathbf{H}_{/X}\big)_{\Im_X} \hookrightarrow \mathbf{H}_{/X}\,. \tag{4.88}$$

Proposition 4.1.33 (Reflections of étale toposes)**.** *Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21) and $X \in \mathbf{H}$ an object. Then the étale topos $\mathbf{\acute{E}t}_X$ from Def. 4.1.32:*

(i) *is indeed an ∞-topos (Def. 3.1.30);*

(ii) *its defining full inclusion (4.88) has both a left- and a right-adjoint (Def. 3.1.24):*

$$\mathbf{\acute{E}t}_X \underset{\underset{\mathrm{LcllCnstnt}_X}{\longleftarrow}}{\overset{\overset{\mathrm{Etl}_X}{\longleftarrow}}{\underset{\bot}{\overset{\bot}{\xhookrightarrow{\;i_X\;}}}}} \mathbf{H}_{/X} \tag{4.89}$$

(iii) *whose induced adjoint modality (1.18)*

$$\begin{array}{c} \big(\underset{\text{“étale over }X\text{”}}{\Im_X \;:=\; i_X \circ \mathrm{\acute{E}tl}_X}\big) \\ \bot \\ \big(\underset{\text{“locally constant over }X\text{”}}{\mathscr{L}_X \;:=\; i_X \circ \mathrm{LcllCnstnt}_X}\big) \end{array} \;:\; \mathbf{H}_{/X} \to \mathbf{H}_{/X} \tag{4.90}$$

is on the left that of Def. 4.1.25:

$$\mathrm{\acute{E}tl}_X \;:\; \begin{array}{c} Y \\ \downarrow{\scriptstyle p} \\ X \end{array} \;\longmapsto\; \begin{array}{c} (\eta^{\Im}_X)^*(\Im Y) \\ {\scriptstyle (\eta^{\Im}_X)^*(\Im p)}\downarrow \\ X \end{array}$$

$$\text{i.e.:}\quad \begin{array}{ccc} (\eta^{\Im}_X)^*(\Im Y) & \xrightarrow{(\Im p)^*(\eta^{\Im}_X)} & \Im Y \\ {\scriptstyle (\eta^{\Im}_X)^*(\Im p)}\downarrow & {\scriptstyle (pb)} & \downarrow{\scriptstyle \Im p} \\ X & \xrightarrow[\eta^{\Im}_X]{} & \Im X\,. \end{array} \tag{4.91}$$

Proof. First to see that (4.91) is well-defined as a functor to $\mathbf{\acute{E}t}_X$ (this proceeds as in [CHM85, 3.3][CJKP97, 3][CRi20, 7.3]): We need to check that $(\eta^{\Im}_X)^*(\Im p)$ is a local diffeomorphism (Def. 4.1.26). For this, it is sufficient to have equivalences

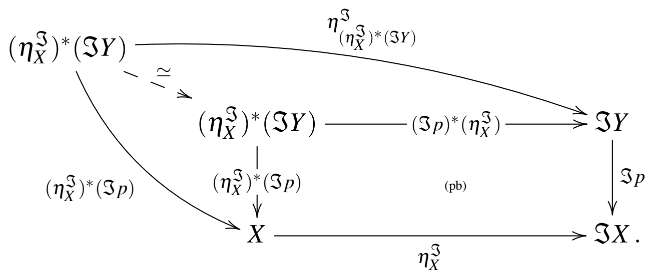
$$\Im\big((\eta^{\Im}_X)^*(\Im p)\big) \;\simeq\; \Im p\,, \tag{4.92}$$

and

$$(\Im p)^*(\eta^{\Im}_X) \;\simeq\; \eta^{\Im}_X \tag{4.93}$$

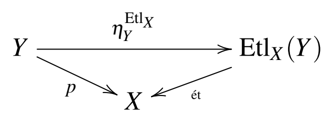
because then the Cartesian square on the right of (4.91) exhibits this property.

But (4.92) follows by applying $\Im$ to the square on the right of (4.91), by idempotency (Prop. 3.1.29) and since equivalences are preserved by pullback (Example 3.1.22). With this, (4.93) follows from the naturality of the $\Im$-unit, by the universal factorization shown dashed in the following diagram:



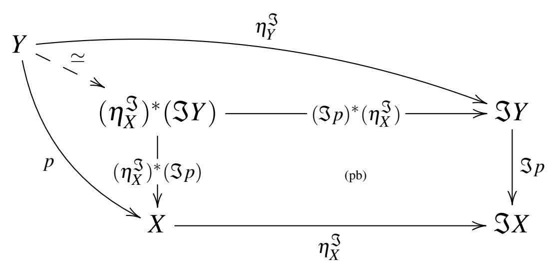
(4.94)

Notice that, similarly, there is a natural transformation



(4.95)

induced as the universal factorization shown dashed in the following diagram:



(4.96)

and notice that this in an $\Im$-equivalence:

$$\Im\left(\eta^{\mathrm{Etl}_X}_{Y_1}\right) \quad \text{is an equivalence}. \tag{4.97}$$

Condition (4.97) follows by applying $\Im$ to the whole left part of the diagram on the right of (4.98), using idempotency (Prop. 3.1.29) and that equivalences are preserved by pullback (Example 3.1.22).

Second, to see that (4.91) defines a left adjoint to the inclusion: We need to check the corresponding hom-equivalence (3.43), shown on the left here:

$$\begin{array}{ccc} \text{Étl}_X(Y_1) & \xrightarrow{\;\widetilde{f}\;} & Y_2 \\ & {\scriptstyle \text{Étl}_X(p)}\searrow \quad \swarrow{\scriptstyle \text{ét}} & \\ & B & \end{array} \quad\Leftrightarrow\quad \begin{array}{ccc} Y_1 & \xrightarrow{\;f\;} & Y_2 \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \text{ét}} & \\ & B & \end{array}$$

$$\simeq \quad \begin{array}{ccccc} & \overset{\eta^{\Im}_{Y_1}}{\nearrow} & \Im Y_1 & \xrightarrow{\;\Im f\;} & \Im Y_2 \\ & & \uparrow{\scriptstyle \eta^{\Im}_{(\eta^{\Im}_X)^*(\Im X)}} & \nearrow{\scriptstyle \eta^{\Im}_{Y_2}} & \\ Y_1 & \xrightarrow{\;\eta^{\text{Etl}_X}_{Y_1}\;} & (\eta^{\Im}_X)^*(\Im X) & \overset{\widetilde{f}}{\dashrightarrow} & Y_2 \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle \text{Etl}_X(p)} & \swarrow{\scriptstyle \text{ét}} & \searrow \\ & & X & \xrightarrow{\;\eta^{\Im}_X\;} & \Im X \end{array} \tag{4.98}$$

On the bottom of (4.98) we show an induced factorization: The square sub-diagram on the right of (4.98) is Cartesian by the assumption that we are homming into a local diffeomorphism, while the square in the middle is Cartesian by (4.94). Thus, given f, the morphism $\widetilde{f}$ is induced by the universal property of the right Cartesian square. Conversely, given $\widetilde{f}$, precomposition with the $\eta^{\text{Etl}_X}_{Y_1}$ (4.96) gives a morphism f. To see that this correspondence is an equivalence, we just need to observe that $\Im(\widetilde{f}) \simeq \Im f$. This follows by (4.97).

Thus we have established the existence of the left adjoint Étl_X. With this, to see the right adjoint LcllCnst_X as well as the fact that **Ét** is an ∞-topos, it is now sufficient to show that $\mathbf{Ét}_X \overset{i_X}{\hookrightarrow} \mathbf{H}_{/X}$ preserves colimits: Because, by the reflection Étl_X this implies, first, that $\mathbf{Ét}_X$ is a presentable ∞-category, in fact an ∞-topos (by Prop. 3.1.41, since it is then an accessibly embedded reflective subcategory of the slice $\mathbf{H}_{/X}$, which is an ∞-topos by Prop. 3.1.46); and thus, second, the existence of the right adjoint by the adjoint ∞-functor theorem (Prop. 3.1.27).

So to see that i_X preserves colimits, consider any small $\mathscr{I} \in \text{Cat}_\infty$ and a diagram

$$Y_\bullet : \; \mathscr{I} \longrightarrow \mathbf{Ét}_X \overset{i_X}{\hookrightarrow} \mathbf{H}_{/X} \,. \tag{4.99}$$

Since i_X is fully faithful by construction, it is sufficient to show that the colimit of this diagram formed in $\mathbf{H}_{/X}$ is itself in the image of i_X. This colimit, in turn, is computed in $\mathbf{H}$ (by Example 3.1.52) with its morphism q to X universally induced, and this we need to show to be a local diffeomorphism (Def. 4.1.26). Hence we need to show that the following square on the left is Cartesian:

$$
\begin{array}{ccc}
\underset{\longrightarrow}{\lim} Y_\bullet & \xrightarrow{\eta^{\Im}_{\underset{\longrightarrow}{\lim} Y_\bullet}} & \Im\big(\underset{\longrightarrow}{\lim} Y_\bullet\big) \\
{\scriptstyle q}\big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle \Im q} \\
X & \xrightarrow[\eta^{\Im}_X]{} & \Im X
\end{array}
$$

$$
\Leftrightarrow \quad
\begin{array}{ccc}
\underset{\longrightarrow}{\lim} Y_\bullet & \xrightarrow{(\eta^{\Im}_{Y_\bullet})} & \underset{\longrightarrow}{\lim}\big(\Im Y_\bullet\big) \\
{\scriptstyle q}\big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle \Im q} \\
X & \xrightarrow[\eta^{\Im}_X]{} & \Im X
\end{array}
\tag{4.100}
$$

$$
\Leftrightarrow \quad \underset{i \in \mathscr{I}}{\forall}
\begin{array}{ccc}
Y_i & \xrightarrow{\eta^{\Im}_{Y_i}} & \Im Y_i \\
{\scriptstyle q_i}\big\downarrow & \text{(pb)} & \big\downarrow{\scriptstyle \Im q_i} \\
X & \xrightarrow[\eta^{\Im}_X]{} & \Im X
\end{array}
$$

But, since $\Im$ is a left adjoint and hence preserves colimits (Prop. 3.1.26), this is equivalent to the square on in middle being Cartesian. Finally, by universality of colimits (3.52) in the ∞-topos $\mathbf{H}$, this is equivalent to all the squares on the right being Cartesian. This is the case, by the assumption (4.99). □

Remark 4.1.34 (Local and global ∞-section functors.)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21) and $X \in \mathbf{H}$. Then we may think of the étale ∞-topos $\mathbf{\acute{E}t}_X$ (Def. 4.1.32, Prop. 4.1.33) as the internal construction of the ∞-topos of ∞-*sheaves over* X. Under this interpretation:

- **(i)** the ∞-functor LcllCnst (4.89) has the interpretation of sending any ∞-bundle $E \to X$ (Notation 3.1.45) to its ∞-*sheaf of local sections* $\underline{E} := \mathrm{LcllCnst}_X(E)$;
- **(ii)** the direct image of the base geometric morphism (3.75) has the interpretation of sending any ∞-sheaf to its ∞-groupoid of global sections:

$$
\underset{\text{over } X}{\overset{\infty\text{-bundles}}{\mathbf{H}_{/X}}}
\underset{\underset{\text{form } \infty\text{-sheaf of local sections}}{\underline{(-)} := \mathrm{LcllCnstnt}_X}}{\overset{i_X}{\underset{\longrightarrow}{\overset{\hookleftarrow}{\bot}}}}
\underset{\text{on } X}{\overset{\infty\text{-sheaves}}{\mathbf{\acute{E}t}_X}}
\underset{\underset{\text{form } \infty\text{-groupoid of global sections}}{\Gamma_X}}{\overset{\Delta_X}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}}
\mathrm{Grpd}_\infty ,
\qquad \Gamma_X : \mathbf{H}_{/X} \longrightarrow \mathrm{Grpd}_\infty
\tag{4.101}
$$

Notice that the global sections of the ∞-sheaf of local sections of an ∞-bundle E is the global sections of that ∞-bundle (as in Remark 3.2.21):

$$
\Gamma_X\big(\underline{E}\big) \;\simeq\; \Gamma_X(E) \tag{4.102}
$$

(by the essential uniqueness of the base geometric morphism (Prop. 3.1.43) and the fact that the base geometric morphism on ∞-bundles forms global sections, Remark 3.2.22).

4.1.9 Étale groupoids

Definition 4.1.35 (Étale groupoid)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21).

(i) We say that $X_\bullet \in \mathrm{Grpd}(\mathbf{H})$ (Def. 3.1.68) is an *étale groupoid* if all its face maps are local diffeomorphisms (Def. 4.1.26):

$$X_\bullet \text{ is étale groupoid} \quad \Leftrightarrow \quad \underset{\substack{n\in\mathbb{N} \\ 0\leq i\leq n}}{\forall} \; X_{n+1} \xrightarrow[\text{ét}]{d_i} X_n\,. \tag{4.103}$$

(ii) We write

$$\text{ÉtGrpd}(\mathbf{H}) \hookrightarrow \mathrm{Grpd}(\mathbf{H}) \quad \in \mathrm{Cat}_\infty \tag{4.104}$$

for the full sub-∞-category of that of all groupoids (3.114) on those that are étale groupoids.

As a variant of Prop. 3.1.70 we have:

Proposition 4.1.36 (Étale groupoids are equivalent to stacks with étale atlases)**.** *Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21) and $X_\bullet \in \mathrm{Grpd}(\mathbf{H})$ (Def. 3.1.68). Then the following conditions are equivalent:*

(i) *The groupoid $X_\bullet$ is an étale groupoid (Def. 4.1.35).*

(ii) *The associated atlas $X_0 \xrightarrow{a} \mathscr{X}$ (via Prop. 3.1.70) is a local diffeomorphism (Def. 4.1.26).*

$$\begin{array}{ccccl}
\vdots & & \vdots & & \\
X\times_{\mathscr{X}} X & \simeq & X_1 & & \textbf{“étale groupoid”} \\
{\scriptstyle \text{ét}}\downarrow\uparrow{\scriptstyle \text{ét}}\downarrow{\scriptstyle \text{ét}} & & {\scriptstyle s}\downarrow\uparrow{\scriptstyle e}\downarrow{\scriptstyle t} & & \\
X_0 & = & X_0 & & \\
{\scriptstyle a}\downarrow{\scriptstyle \text{ét}} & & \downarrow & & \textbf{“étale atlas”} \\
\mathscr{X} & \simeq & \varinjlim X_\bullet & & \textbf{“étale stack”}
\end{array} \tag{4.105}$$

Proof. By definition of local diffeomorphisms, we need to demonstrate the logical equivalence shown on the left:

$$
\begin{array}{ccc}
 & \begin{array}{ccc} X_{n_1} & \xrightarrow{\eta^{\Im}_{X_{n_1}}} & \Im X_{n_1} \\ {\scriptstyle X_\phi}\downarrow & \text{(pb)} & \downarrow{\scriptstyle \Im X_\phi} \\ X_{n_2} & \xrightarrow[\eta^{\Im}_{X_{n_2}}]{} & \Im X_{n_2} \end{array} & \\
\underset{n_1 \xrightarrow{\phi} n_2}{\forall} & & \\
\Leftrightarrow & \begin{array}{ccc} X_0 & \xrightarrow{\eta^{\Im}_{X_0}} & \Im X_0 \\ {\scriptstyle a}\downarrow & \text{(pb)} & \downarrow{\scriptstyle \Im a} \\ \underset{\longrightarrow}{\lim} X_\bullet & \xrightarrow[\eta^{\Im}_{\underset{\longrightarrow}{\lim} X_\bullet}]{} & \Im \underset{\longrightarrow}{\lim} X_\bullet \end{array} & \\
\Leftrightarrow & \begin{array}{ccc} X_0 & \xrightarrow{\eta^{\Im}_{X_0}} & \Im X_0 \\ {\scriptstyle a}\downarrow & \text{(pb)} & \downarrow{\scriptstyle \Im a} \\ \underset{\longrightarrow}{\lim} X_\bullet & \xrightarrow[\underset{\longrightarrow}{\lim}\, \eta^{\Im}_{X_\bullet}]{} & \underset{\longrightarrow}{\lim} \Im X_\bullet \end{array} &
\end{array}
\tag{4.106}
$$

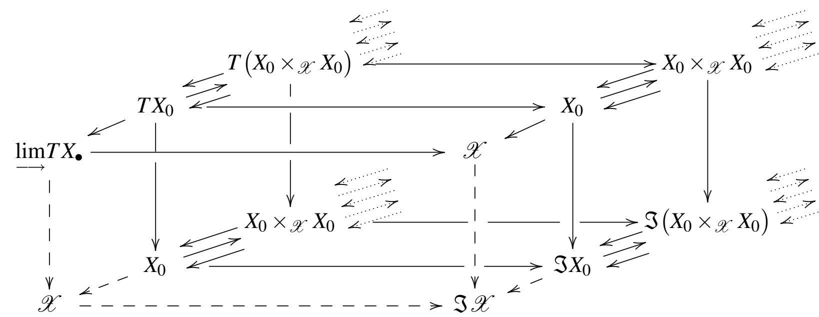
But since $\Im$ preserves all limits and colimits (being a left and a right adjoint, Prop. 3.1.26), we have **(a)** also the logical equivalence shown on the right of (4.106); and **(b)** that $\Im X_\bullet$ is itself a groupoid with atlas $\Im a$, and that $X_\bullet \xrightarrow{\eta^{\Im}_{X_\bullet}} \Im X_\bullet$ is a morphism in $\mathrm{Grpd}(\mathbf{H})$ (3.114). By **(a)**, it is now sufficient to prove the composite logical equivalence in (4.106). By **(b)**, this follows with Prop. 3.1.73. □

Proposition 4.1.37 (Tangent stacks)**.** *Let* $\mathbf{H}$ *be an elastic* ∞*-topos (Def. 4.1.21) and* $X_\bullet \in \text{ÉtaleGroupoids}(\mathbf{H})$ *(Def. 4.1.35) with étale atlas* $X \xrightarrow{\text{ét}} \mathscr{X}$ *(via Prop. 4.1.36). Then:*

- **(i)** *the system of tangent bundles* $TX_\bullet$ *(Def. 4.1.29) is itself an étale groupoid (Def. 4.1.35), the tangent groupoid;*
- **(ii)** *its atlas (under Prop. 4.1.36) is the differential* $TX_0 \xrightarrow{Ta} T\mathscr{X}$ *of the given atlas, hence the tangent stack is:*

$$T\mathscr{X} \simeq \underset{\longrightarrow}{\lim} TX_\bullet \tag{4.107}$$

Proof. **(i)** That $TX_\bullet$ is itself a groupoid (Def. 3.1.68) follows because both the tangent bundle construction $T(-)$ (4.83) as well as the groupoid Segal conditions (3.113) are pullback constructions, hence limits, which commute over each other. To see that $TX_\bullet$ is an étale groupoid, consider the following diagram:



(4.108)

Here the simplicial sub-diagram in the top right consists of local diffeomorphism by the assumption that $X_\bullet$ is étale. But this implies that all the horizontal squares in the top of (4.108) are Cartesian, by Prop 4.1.31, hence that also all morphisms of the simplicial sub-diagram in the top left are local diffeomorphisms, by Lemma 4.1.27.

(ii) To see (4.107) we need to show that the front square in (4.108) is Cartesian. Observe:

(a) All horizontal squares in (4.108) are Cartesian: the top ones by the above argument for **(i)**, the bottom ones by the assumption that $X_\bullet$ is étale.

(b) All solid vertical squares in (4.108) are also Cartesian, by definition (4.83) of tangent bundles.

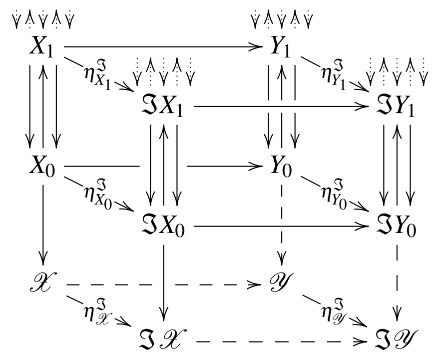
(c) The object $\mathscr{X}$ in the bottom front left of (4.108) is not just the colimit of the simplicial sub-diagram in the bottom left, but in fact of the full left sub-diagram (because of the colimit of the top left sub-diagram in the front top left). Similarly, the object $\Im \mathscr{X}$ is in fact the colimit over the full right sub-diagram in (4.108) (using that $\Im$ preserves colimits, being a left adjoint, Prop. 3.1.26).

Now **(a)** and **(b)** verify the assumption of Prop. 3.1.32 applied to the diagram (4.108), regarded as a natural transformation from its left part to its right part; and with **(c)**, the conclusion of Prop. 3.1.32 says that the front square in (4.108) is Cartesian. □

Lemma 4.1.38 (Degreewise local diffeomorphisms of étale groupoids)**.** *Let* $\mathbf{H}$ *be an elastic* ∞*-topos (Def. 4.1.21) and* $X_\bullet, Y_\bullet \in$ *ÉtaleGroupoids*$(\mathbf{H})$ *(Def. 4.1.35). If a morphism* $X_\bullet \xrightarrow{f_\bullet} Y_\bullet$ *is such that for all* $n \in \mathbb{N}$*, the component* $X_n \xrightarrow{f_n} Y_n$ *is a local diffeomorphism (Def. 4.1.26), then induced morphism on stacks* $\mathscr{X} \xrightarrow{\varinjlim f_\bullet} \mathscr{Y}$ *is also a local diffeomorphism (Def. 4.1.36).*

Proof. Consider the following diagram:

(4.109)



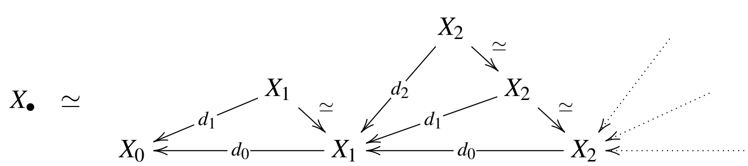
Observe that:

(a) all solid $\eta^{\Im}$-naturality squares in this diagram are Cartesian, by the assumption that the rear part of the diagram is a degreewise local diffeomorphism of étale groupoids.

(b) $\mathscr{Y}$ is not just the colimit of the partial diagram $Y_\bullet$ in the rear right, but in fact is also the colimit of the full non-dashed rear part of the diagram (using that $\mathscr{X}$ is the colimit of the rear left part). Similarly, $\Im\mathscr{Y}$ is the colimit of the non-dashed front part of the diagram (using that $\Im$ preserves limits and colimits, being a left and a right adjoint, Prop. 3.1.26).

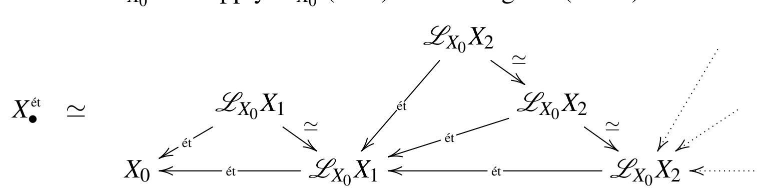
Hence if we regard the diagram as a natural transformation from its rear to its front part, then Prop. 3.1.32 applies and says that also the bottom dashed square is Cartesian, and hence that $\mathscr{X} \to \mathscr{Y}$ is a local diffeomorphism. ☐

Definition 4.1.39 (Étalification of groupoids)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21) and $X_\bullet \in \mathrm{Grpd}(\mathbf{H})$ (Def. 3.1.68). Notice that, by Prop. 3.1.70 for all $n \in \mathbb{N}$ we have for all $0 \leq i \leq n$ that all face maps $X_{n+1} \xrightarrow{d_i} X_n$ are in fact equivalent to each other, being related by an automorphism of X_{n+1} given by permutation of fiber product factors (3.115)



(4.110)

(and similarly for the degeneracy maps). Therefore, we may regard $X_\bullet$ as a diagram in the slice $\mathbf{H}_{X_0}$. and apply $\mathscr{L}_{X_0}$ (4.90) to this diagram (4.110) to obtain



(4.111)

Observe that:

(a) the simplicial diagram (4.111) is again a groupoid, since the right adjoint functor $\mathscr{L}_{X_0}$ preserves the characterizing fiber products (3.113) (by Prop. 3.1.26);

(b) this groupoid is étale (Def. 4.1.35), since the morphisms of the form $\mathscr{L}_{X_0} X_n \longrightarrow X_0$ in (4.111) are local diffeomorphisms by construction, whence all other morphisms $\mathscr{L}_{X_0} X_{n_1} \longrightarrow \mathscr{L}_{X_0} X_{n_2}$ are local diffeomorphisms by the left-cancellation property (4.77).

Hence we say that:

(i) The simplicial diagram (4.111) is the *étalification* of the groupoid $X_\bullet$.

$$X_\bullet^{\text{ét}} \in \text{ÉtGrpd}(\mathbf{H})\,. \tag{4.112}$$

(ii) If the corresponding atlas of $X_\bullet$ (via Prop. 3.1.70) is denoted $X_0 \twoheadrightarrow \mathscr{X}$, then we write

$$X_0 \overset{\text{ét}}{\twoheadrightarrow} \mathscr{X}^{\text{ét}} \tag{4.113}$$

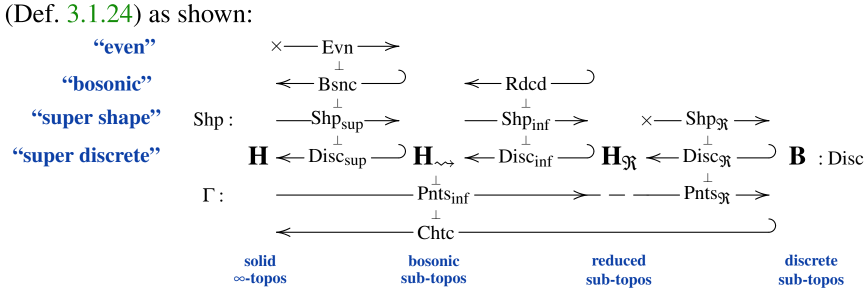
for the corresponding étale atlas (via Prop. 4.1.36) of the étalified groupoid (4.112).

4.1.10 Super Geometry

We present a formulation of super-geometry internal to ∞-toposes which we call *solid* [Sc13][Sc18].

4.1.11 Super-geometry

Definition 4.1.40 (Solid ∞-topos). **(i)** An ∞-topos $\mathbf{H}$ (Def. 3.1.30) over $\mathbf{B} = \text{Grpd}_\infty$ is a *solid ∞-topos* if its base geometric morphism (Prop. 3.1.24), to be called $\text{Pnts} : \mathbf{H} \to \mathbf{B}$, is equipped with a factorization as follows, with adjoints (Def. 3.1.24) as shown:



(4.114)

(ii) In particular, a solid ∞-topos is also an elastic ∞-topos (Def. 4.1.21), as is its sub-∞-topos $\mathbf{H}_{\rightsquigarrow}$ of bosonic objects.

(iii) We write

$$\begin{array}{c} \big(\ \rightrightarrows\ :=\ \mathrm{Bsn}\circ\mathrm{Evn}\ \big) \\ \text{“even”} \\ \bot \\ \big(\ \rightsquigarrow\ :=\ \mathrm{Bsn}\circ\mathrm{Shp}_{\mathrm{sup}}\ \big) \\ \text{“bosonic”} \\ \bot \\ \big(\ \mathrm{Rh}\ :=\ \mathrm{Disc}_{\mathrm{sup}}\circ\mathrm{Shp}_{\mathrm{sup}}\ \big) \\ \text{“rheonomic”} \end{array} \quad : \mathbf{H}\to\mathbf{H} \tag{4.115}$$

for the further induced modalities (1.18) (*solid modalities*) accompanying the elastic modalities (4.60) and the cohesive modalities (4.2).

4.1.12 Examples of solid ∞-toposes

We indicate an example of a solid ∞-topos (Def. 4.1.40). In generalization of Def. 4.1.22 we have the following:

Definition 4.1.41 (∞-Jets of super Cartesian spaces)**.**

(i) Write

$$\begin{array}{ccc} \infty\mathrm{JetSuperCrtSp} & \xhookrightarrow{C^\infty(-)} & \mathrm{CommutativeSuperAlgebras}_{\mathbb{R}}^{\mathrm{op}} \\ \mathbb{R}^{n|q}\times\mathbb{D}_W & \longmapsto & C^\infty\big(\mathbb{R}^n\big)\ \otimes_{\mathbb{R}}\ \wedge^\bullet_{\mathbb{R}}\big(\mathbb{R}^q\big)\ \otimes_{\mathbb{R}}\ (\mathbb{R}\oplus \mathrm{W}) \end{array} \tag{4.116}$$

for (as in [KS97][KS00]) the full subcategory of the opposite of super-commutative super-algebras over the real numbers on those which are tensor products of

(a) algebras $C^\infty(\mathbb{R}^n)$ of smooth functions on a Cartesian space $\mathbb{R}^n$, for $d\in\mathbb{N}$;

(b) Grassmann algebras $\wedge^\bullet_{\mathbb{R}}\mathbb{R}^q$ on $q\in\mathbb{N}$ generators in odd degree;

(c) finite dimensional $\mathbb{R}\oplus W\ \in$ CAlg with a single nilpotent maximal ideal W.

(ii) We regard this as a site via the the coverage (i.e., a Grothendieck pre-topology) whose covers are of the form

$$\Big\{\underbrace{\mathbb{R}^n\times\mathbb{R}^{0|q}}_{\mathbb{R}^{n|q}}\times\mathbb{D}\xrightarrow{f_i\times\mathrm{id}\times\mathrm{id}}\mathbb{R}^n\times\mathbb{R}^{0|q}\times\mathbb{D}\Big\}_{i\in I} \tag{4.117}$$

such that $\Big\{\mathbb{R}^n\xrightarrow{f_i}\mathbb{R}^n\Big\}_{i\in I}$ is a cover in CrtSpc (Def. 3.1.5).

Lemma 4.1.42 (Reflections of super-commutative algebras into commutative algebras)**.** *The canonical inclusion of* ∞JetCrtSp *(Def. 4.1.22) into* ∞JetSuperCrtSp *(Def.*

4.1.41) has a left and a right adjoint (Def. 3.1.24)

$$\infty\mathrm{JetSuperCrtSp} \; \underset{\underset{\mathrm{Shp_{sup}}}{\longrightarrow}}{\overset{\overset{\mathrm{Evn}}{\longrightarrow}}{\underset{\bot}{\overset{\bot}{\xleftarrow{\;\mathrm{Bsnc}\;}}}}} \; \infty\mathrm{JetCrtSp} \tag{4.118}$$

where:

(i) *The left adjoint* Evn *in* (4.118) *is given in terms of super-algebras of smooth functions* (4.116) *by passage to the sub-algebra of even-graded elements:*

$$\begin{aligned} C^\infty\Big(\mathrm{Evn}\big(\mathbb{R}^{n|q}\times\mathbb{D}\big)\Big) &\simeq C^\infty\big(\mathbb{R}^{n|q}\times\mathbb{D}\big)_{\mathrm{even}} \\ &\simeq C^\infty\big(\mathbb{R}^{n}\times\mathbb{D}\big)\otimes_{\mathbb{R}} C^\infty\big(\mathbb{R}^{0|q}\big)_{\mathrm{even}}\,. \end{aligned} \tag{4.119}$$

(ii) *The right adjoint* $\mathrm{Shp_{sup}}$ *in* (4.118) *is given in terms of super-algebras of smooth functions* (4.116) *by passage to the quotient algebra by the ideal of odd-graded elements:*

$$\begin{aligned} C^\infty\Big(\mathrm{Shp_{sup}}\big(\mathbb{R}^{n|q}\times\mathbb{D}\big)\Big) &\simeq C^\infty\big(\mathbb{R}^{n|q}\times\mathbb{D}\big)/C^\infty\big(\mathbb{R}^{n|q}\times\mathbb{D}\big)_{\mathrm{odd}} \\ &\simeq C^\infty\big(\mathbb{R}^{n}\times\mathbb{D}\big)\otimes_{\mathbb{R}} \underbrace{C^\infty\big(\mathbb{R}^{0|q}\big)/C^\infty\big(\mathbb{R}^{0|q}\big)_{\mathrm{odd}}}_{\simeq\mathbb{R}} \\ &\simeq C^\infty\big(\mathbb{R}^{n}\times\mathbb{D}\big) \end{aligned} \tag{4.120}$$

and hence directly by

$$\mathrm{Shp_{sup}}\big(\mathbb{R}^{n|q}\times\mathbb{D}\big) \;\simeq\; \mathbb{R}^n\times\mathbb{D}\,. \tag{4.121}$$

Proof. By regarding the situation under the defining embedding as being in $\mathrm{CommutativeSuperAlgebras}_{\mathbb{R}}$ (Def. 4.1.41), it is equivalent to the statement that the canonical inclusion of commutative algebras into super-commutative super-algebras has a right and a left adjoint given by passage to the even sub-algebra and to the quotient by the odd ideal, respectively:

$$\mathrm{CommutativeSuperAlgebras}_{\mathbb{R}} \; \underset{\underset{A\,\mapsto\,A_{\mathrm{even}}}{\longrightarrow}}{\overset{\overset{A\,\mapsto\,A/A_{\mathrm{odd}}}{\longrightarrow}}{\underset{\bot}{\overset{\bot}{\longleftarrow}}}} \; \mathrm{CAlg}_{\mathbb{R}}\,. \tag{4.122}$$

This follows readily by inspection from the fact that homomorphisms of super-algebras preserve super-degree, by definition. One place where this adjoint triple has been made explicit before is [CR12, below Example 3.18]. ☐

Example 4.1.43 (Jets of super-geometric ∞-groupoids)**.** The ∞-category of ∞-sheaves (Def. 3.1.42)

$$\infty\mathrm{JetSuperGrpd}_\infty \;:=\; \mathrm{Shv}_\infty\big(\infty\mathrm{JetSuperCrtSp}\big) \tag{4.123}$$

over the site from Def. 4.1.41 is a solid ∞-topos (Def. 4.1.40).

(i) Its bosonic (4.115) sub-∞-topos is that of ∞JetSmthGrpd (Example 4.1.24) and its reduced (4.59) sub-∞-topos that of $\mathrm{SmthGrpd}_\infty$ (Example 4.1.18):

$$
\begin{array}{ccccccc}
 & \xrightarrow{\;\mathrm{Evn}\;} & & & & & \\
 & \bot & & & & & \\
 & \xleftarrow{\;\mathrm{Bsnc}\;} & & & & & \\
 & \bot & & & & & \\
 & \xrightarrow{\;\mathrm{Shp}_{\mathrm{sup}}\;} & & & & & \\
 & \bot & & & & & \\
\infty\mathrm{JetSuperGrpd}_\infty & \xleftarrow{\;\mathrm{Disc}_{\mathrm{sup}}\;} & \infty\mathrm{JetSmthGrpd}_\infty & \xleftarrow{\;\mathrm{Disc}_{\mathrm{inf}}\;} & \mathrm{SmthGrpd}_\infty & \xleftarrow{\;\mathrm{Disc}\;} & \mathrm{Groupoids}_\infty \\
 & & \uparrow & & & & \\
 & & \vdots & & & & \\
 & & \uparrow & & & & \\
 & & 2\mathrm{JetSmthGrpd}_\infty & & & & \\
 & & \uparrow & & & & \\
 & & \mathrm{JetSmthGrpd}_\infty & & \hookleftarrow\!\!\text{(from } \mathrm{SmthGrpd}_\infty\text{)} & &
\end{array}
\tag{4.124}
$$

where the adjoint triple $\left(\mathrm{Evn} \dashv \mathrm{Bsnc} \dashv \mathrm{Shp}_{\mathrm{sup}}\right)$ arises by left Kan extension from that of Lemma 4.1.42.

(ii) The full inclusion of SmthMfd, inherited from (4.52), extends to a full inclusion of super-manifolds (as in [CCF11, 4.6][HKST11, 2]):

$$
\mathrm{SmthMfd} \xhookrightarrow{\;\mathrm{Disc}_{\mathrm{sup}}\;} \mathrm{SuperManifolds} \hookrightarrow \infty\mathrm{JetSuperGrpd}_\infty
\tag{4.125}
$$

(iii) Accordingly, super-Lie groups (e.g. [Ya93][CCF11, 7]) embed faithfully into all group objects (Prop. 3.2.1):

$$
\underset{\text{Lie groups}}{\mathrm{Grp}\big(\mathrm{SmthMfd}\big)} \xhookrightarrow{\;\mathrm{Disc}_{\mathrm{sup}}\;} \underset{\text{super Lie groups}}{\mathrm{Grp}\big(\mathrm{SuperManifolds}\big)} \hookrightarrow \mathrm{Grp}\big(\infty\mathrm{JetSuperGrpd}_\infty\big)
\tag{4.126}
$$

(iv) In particular, for $d \in \mathbb{N}$ and $\mathbf{N} \in \mathrm{Spin}(d,1)\mathrm{Representations}_{\mathbb{R}}$, the corresponding *supersymmetry* groups, i.e., the *super-Poincaré group* and its underlying translational *super-Minkowski group* (e.g. [Fr99, §3]) are group objects

$$
\underset{\substack{\text{super-Minkowski}\\ \text{super Lie group}}}{\mathbb{R}^{d,1|\mathbf{N}}} \hookrightarrow \underset{\substack{\text{super-Poincaré}\\ \text{super Lie group}}}{\mathrm{Iso}\big(\mathbb{R}^{d,1|\mathbf{N}}\big)} \twoheadrightarrow \mathrm{Spin}(d,1) \quad \in \mathrm{Grp}\big(\infty\mathrm{JetSuperGrpd}\big).
\tag{4.127}
$$

Remark 4.1.44 (Superspace cohomology theory in solid ∞-toposes)**.** The intrinsic cohomology (1.21) in the solid ∞-topos of $\infty\mathrm{JetSuperGrpd}_\infty$ (Example 4.1.43)

(i) includes the super-rational cohomology of super-Minkowski spacetimes (4.127) that governs the fundamental (κ-symmetric) super p-brane sigma-models of string/M-theory [FSS15a][FSS17][FSS18], review in [FSS19].

(ii) Its enhancement to *twisted* super-rational cohomology of super-Minkowski spacetimes (4.127), which happens (by Remark 3.2.23) in the intrinsic cohomology of the tangent ∞-topos $T\left(\infty\mathrm{JetSuperGrpd}_\infty\right)$ (Example 3.1.51), encodes the double dimensional reduction from fundamental M-branes to D-branes [BSS18].

(iii) Its enhancement to *proper equivariant* super-rational cohomology of super-

Minkowski spacetimes (4.127), which happens (by Remark 6.1.4 and Theorem 6.1.9 below) in the intrinsic cohomology of the singular-solid ∞-topos Snglr∞JetSuperGrpd$_\infty$ (Example 4.2.2 below), encodes also the black (solitonic) super p-branes [HSS18].

Lemma 4.1.45 (In super-geometric groupoids étale implies bosonic)**.**

In the solid ∞-topos of ∞JetSuperGrpd *(Ex. 4.1.41) we have a natural equivalence*

$$\rightsquigarrow \circ \Im \;\simeq\; \Im \tag{4.128}$$

saying that $\Im$*-modal objects* (4.60) *are bosonic* (4.115).

Proof. Observe that on ∞JetSuperCrtSp $\overset{y}{\hookrightarrow}$ ∞JetSuperGrpd$_\infty$ (Def. 4.1.41), we have a natural equivalence

$$\Re \circ \rightrightarrows \;\simeq\; \Re \tag{4.129}$$

saying that the reduction (4.60) of the even aspect (4.115) of the space is equivalently the reduced aspect.

To see this, consider $\mathbb{R}^{n|q} \times \mathbb{D}_W \in$ ∞JetSuperCrtSp and use, by Example 4.1.43 with Lemma 4.1.42, the operation $\Re \circ \rightrightarrows$ is given in terms of the defining superalgebras of functions (4.1.41) by passage to the reduced algebra of the even subalgebra

$$\begin{aligned}
C^\infty\Big(\Re \circ \rightrightarrows \big(\mathbb{R}^{n|q} \times \mathbb{D}_W\big)\Big) &\simeq \Big(\big(C^\infty\big(\mathbb{R}^n\big) \otimes_{\mathbb{R}} \big(\wedge^\bullet_{\mathbb{R}} \mathbb{R}^q\big) \otimes_{\mathbb{R}} (\mathbb{R} \oplus W)\big)_{\text{even}}\Big)_{\text{red}} \\
&\simeq \Big(C^\infty\big(\mathbb{R}^n\big) \otimes_{\mathbb{R}} \underbrace{\big(\wedge^\bullet_{\mathbb{R}} \mathbb{R}^q\big)_{\text{even}}}_{\simeq \mathbb{R} \oplus \wedge^2 \mathbb{R}^q \oplus \wedge^4 \mathbb{R}^q \oplus \cdots} \otimes_{\mathbb{R}} (\mathbb{R} \oplus W)\Big)_{\text{red}} \\
&\simeq \Big(C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \big(\mathbb{R} \oplus (W \oplus \wedge^2 \mathbb{R}^q \oplus \wedge^4 \mathbb{R}^q \oplus \cdots)\big)\Big)_{\text{red}} \\
&\simeq C^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \underbrace{\big(\mathbb{R} \oplus (W \oplus \wedge^2 \mathbb{R}^q \oplus \wedge^4 \mathbb{R}^q \oplus \cdots)\big)_{\text{red}}}_{\simeq \mathbb{R}} \\
&\simeq C^\infty(\mathbb{R}^n)\,.
\end{aligned} \tag{4.130}$$

Here in the last step we used that every non-unit element in the Grassmann algebra is nilpotent. But, by (4.120) and (4.66), we also have

$$\begin{aligned}
C^\infty\big(\Re(\mathbb{R}^{n|q} \times \mathbb{D}_W)\big) &\simeq C^\infty\big(\mathrm{Shp}_{\mathrm{inf}} \circ \mathrm{Shp}_{\mathrm{sup}}(\mathbb{R}^{n|q} \times \mathbb{D}_W)\big) \\
&\simeq C^\infty\big(\mathrm{Shp}_{\mathrm{inf}}(\mathbb{R}^n \times \mathbb{D}_W)\big) \\
&\simeq C^\infty\big(\mathbb{R}^n\big)\,,
\end{aligned} \tag{4.131}$$

where in the first step we used the elastic structure (4.114) $\Re := \mathrm{Bsnc} \circ \mathrm{Rdcd} \circ \mathrm{Shp}_{\mathrm{inf}} \circ \mathrm{Shp}_{\mathrm{sup}}$ leaving the two full embeddings on the left notationally implicit. Since all these equivalences are natural, this implies (4.129). With this, we have the following sequence of natural equivalences for general $X \in \mathbf{H} := \infty\mathrm{JetSuperGrpd}_\infty$:

$$
\begin{aligned}
\mathbf{H}\big(\mathbb{R}^{n|q}\times\mathbb{D}\,,\, \rightsquigarrow \circ \Im(X)\big) &\simeq \mathbf{H}\Big(\Re \circ \rightrightarrows \big(\mathbb{R}^{n|q}\times\mathbb{D}\big)\,,\, X\Big) \\
&\simeq \mathbf{H}\Big(\Re\big(\mathbb{R}^{n|q}\times\mathbb{D}\big)\,,\, X\Big) \\
&\simeq \mathbf{H}\big(\mathbb{R}^{n|q}\times\mathbb{D}\,,\, \Im X\big)\,,
\end{aligned}
\tag{4.132}
$$

where the first and the last steps are the defining hom-equivalences (3.43) while the middle step is (4.129). Thus the statement (4.128) follows, by the ∞-Yoneda lemma (Prop. 3.1.38). □

4.2 Singularities

Given a cohesive ∞-topos $\mathbf{H}_{\cup}$ as in §4.1.1, we construct here a new ∞-topos $\mathbf{H}$ (Def. 4.2.3 below), to be called *singular-cohesive*, with the following properties:

(i) $\mathbf{H}$ contains ((4.151) below) for each finite group G, an object $\overset{G}{\prec} \in \mathbf{H}$, to be thought of as the generic G-orbi-singularity (*Figure D*).

(ii) $\mathbf{H}$ carries (Prop. 4.2.5 below) an adjoint triple of modalities (1.18) to be read as follows

$$
\begin{array}{c}
\vee \\ \textbf{“singular”} \\ \bot \\ \cup \\ \textbf{“smooth”} \\ \bot \\ \prec \\ \textbf{“orbi-singular”}
\end{array}
\;:\; \mathbf{H} \to \mathbf{H}\,,
\tag{4.133}
$$

with $\mathbf{H}_{\cup}$ being the full sub-∞-category of smooth objects in $\mathbf{H}$,

(iii) such that (Prop. 4.2.17 below):

$\vee\big(\overset{G}{\prec}\big) \simeq *$	“The purely singular aspect of an orbi-singularity is the quotient of a point, hence a point.”
$\cup\big(\overset{G}{\prec}\big) \simeq *\!/\!\!/G$	“The purely smooth aspect of an orbi-singularity is the homotopy quotient of a point.”
$\prec\big(\overset{G}{\prec}\big) \simeq \overset{G}{\prec}$	“An orbi-singularity is purely orbi-singular.”

Essentially this list of conditions might completely characterize $\mathbf{H}$ to be as in Def. 4.2.3 below. Here we leave a fully axiomatic characterization of singular cohesion as an open problem and are content with making the following definitions:

4.2.1 Singular cohesive geometry

Definition 4.2.1 (The 2-site of singularities)**.**

(i) We write

$$\mathrm{Snglrt} := \mathrm{Grpd}_{\leq 1,\mathrm{cn},\mathrm{fin}} \hookrightarrow \mathrm{Grpd}_\infty \tag{4.134}$$

for the full sub-∞-category of ∞-groupoids on the connected 1-truncated objects whose π_1 is finite.

(ii) A skeleton of this $(2,1)$-category has, of course, as objects the delooping groupoids (Example 3.1.14) $* /\!\!/ G$ that are presented by a single object and a *finite* group G of automorphisms of that object.

(iii) When regarded as objects of Snglrt in (4.134), we will denote these by "$\prec$" attached to the symbol for the group:

$$\begin{array}{ccc} \overset{G}{\prec} & \in & \mathrm{Snglrt} \\ \downarrow & & \downarrow \\ * /\!\!/ G & \in & \mathrm{Grpd}_\infty \end{array} \tag{4.135}$$

(iv) The hom-∞-groupoids between these singularities are, equivalently, the action groupoids (Example 3.1.15) whose objects are group homomorphisms and whose morphisms are conjugation actions on these:

$$\begin{aligned} \mathrm{Snglrt}\big(\overset{G_1}{\prec}, \overset{G_2}{\prec}\big) &:= \mathrm{Grpd}_\infty\big(*/\!\!/G_1, */\!\!/G_2\big) \\ &\simeq \mathrm{Grp}(G_1, G_2) /\!\!/_{\mathrm{conj}} G_2 \end{aligned} \tag{4.136}$$

(v) We regard Snglrt as an ∞-site with trivial Grothendieck topology, so that ∞-sheaves on Snglrt are ∞-presheaves (3.64).

Remark 4.2.2 (The global orbit category)**.** The category Snglrt in Def. 4.2.1 is sometimes known in the literature as the "global orbit category" (though at other times this term is used for its wide but non-full subcategory on the faithful morphisms). It has elsewhere been denoted: "Orb case ①" (in [HG07, 4.1]), "Glob" (in [Re14, 2.2]), "Orb" (in [Kö16, 2.1][Ju20, 3.2]) and (up to equivalence) "$\mathbf{O}_{\mathrm{gl}}$" (in [Schw17][Kö16, 2.2]). The terminology in Def. 4.2.1 is meant to be more suggestive of the role this category plays in the theory, from the perspective of cohesive homotopy theory. In fact, the (global) orbit category is often taken to contain not just all finite groups, but all compact Lie groups, with the hom-spaces then being the geometric realization of the topological mapping groupoids. We restrict to discrete groups (hence finite if compact) for reasons explained in Remark 4.2.19 below. This restriction is also amplified in [DHLPS19].

Definition 4.2.3 (Singular-cohesive ∞-topos)**.** Consider a cohesive ∞-topos (Def. 4.1.1), now to be denoted with "$\cup$"-subscripts

$$\mathbf{H}_{\cup} \; \begin{array}{c} \xrightarrow{\;\mathrm{Shp}\;} \\ \bot \\ \xleftarrow{\;\mathrm{Dsc}\;} \\ \bot \\ \xrightarrow{\;\mathrm{Pnts}\;} \\ \bot \\ \xleftarrow{\;\mathrm{Chtc}\;} \end{array} \; \mathbf{B}_{\cup} := \mathrm{Grpd}_\infty \tag{4.137}$$

and assumed to have a site of Chrt (Def. 4.1.9). The corresponding *singular-cohesive* ∞-topos is that of $\mathbf{H}_{\cup}$-valued ∞-sheaves (3.64) over the site of Snglrt (Def. 4.2.1):

$$
\begin{array}{ccc}
\mathbf{H} := \mathrm{Shv}_\infty\big(\mathrm{Snglrt}, \mathbf{H}_{\cup}\big) & \underset{\mathrm{Smth}}{\overset{\mathrm{NnOrbSnglr}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} & \mathbf{H}_{\cup} \\
{\scriptstyle \mathrm{Disc}}\uparrow \dashv \downarrow{\scriptstyle \mathrm{Pnts}} & & {\scriptstyle \mathrm{Disc}}\uparrow \dashv \downarrow{\scriptstyle \mathrm{Pnts}} \\
\mathbf{B} := \mathrm{Shv}_\infty\big(\mathrm{Snglrt}, \mathbf{B}_{\cup}\big) & \underset{\mathrm{Smth}}{\overset{\mathrm{NnOrbSnglr}}{\underset{\longrightarrow}{\overset{\hookleftarrow}{\bot}}}} & \mathbf{B}_{\cup}\,,
\end{array}
\tag{4.138}
$$

where horizontally we are showing the base geometric morphisms (Prop. 3.1.43) of sheaves over the site Snglrt, while vertically we are showing the base geometric morphism (4.1) of $\mathbf{H}_{\cup}$ over $\mathbf{B}_{\cup}$ extended objectwise over Snglrt, by functoriality.

Lemma 4.2.4 (Singularities is 2-site for homotopical cohesion)**.** *The 2-site* Snglrt *(Def. 4.2.1) is an ∞-site for homotopical cohesion, in the sense of Def. 4.1.16.*

Proof. It is immediately checked that

(i) the terminal object is given by the trivial group:
$$
* \simeq \prec^{1}
\tag{4.139}
$$

(ii) Cartesian product is direct product of groups:
$$
\prec^{G_1} \times \prec^{G_2} \;\simeq\; \prec^{G_1 \times G_2}\,.
\tag{4.140}
$$
□

Proposition 4.2.5 (Singular cohesion)**.** *A singular-cohesive ∞-topos (Def. 4.2.3)*

$$
\begin{array}{ccccc}
 & & \mathbf{H} & & \\
 & {\scriptstyle \mathrm{Smth}}\swarrow & & \searrow{\scriptstyle \mathrm{Pnts}} & \\
\mathbf{H}_{\cup} & & & & \mathbf{B} \\
 & {\scriptstyle \mathrm{Pnts}}\searrow & & \swarrow{\scriptstyle \mathrm{Smth}} & \\
 & & \mathbf{B}_{\cup} & &
\end{array}
\tag{4.141}
$$

is itself cohesive (Def. 4.1.1) in two ways:

(i) *over the singular-base ∞-topos $\mathbf{B}$ by the cohesion of $\mathbf{H}_{\cup} \to \mathbf{B}_{\cup}$ (4.1) applied object-wise over all* Snglrt

$$
\begin{array}{lccc}
\text{“shape”} & & \times\!\!\longrightarrow \mathrm{Shp} \longrightarrow & \\
 & & \bot & \\
\text{“discrete”} & & \longleftarrow \mathrm{Disc} \hookleftarrow & \\
 & & \bot & \\
\text{“points”} & \mathbf{H} & \longrightarrow \mathrm{Pnts} \longrightarrow & \mathbf{B}\,; \\
 & & \bot & \\
\text{“chaotic”} & & \longleftarrow \mathrm{Chtc} \hookleftarrow &
\end{array}
\tag{4.142}
$$

(ii) *over the non-singular cohesive base ∞-topos $\mathbf{H}_{\cup}$ (Def. 4.1.1) in that the global section geometric morphism $\mathbf{H} \xrightarrow{\mathrm{Smth}} \mathbf{H}_{\cup}$ of (4.138) is part of a cohesive adjoint quadruple, to be denoted*

$$
\begin{array}{lccc}
\text{“singular”} & & \times\!\!\longrightarrow \mathrm{Snglr} \longrightarrow & \\
 & & \bot & \\
\text{“not orbi-singular”} & & \longleftarrow \mathrm{NnOrbSnglr} \hookleftarrow & \\
 & & \bot & \\
\text{“smooth”} & \mathbf{H} & \longrightarrow \mathrm{Smth} \longrightarrow & \mathbf{H}_{\cup}\,. \\
 & & \bot & \\
\text{“orbi-singular”} & & \longleftarrow \mathrm{OrbSnglr} \hookleftarrow &
\end{array}
\tag{4.143}
$$

Proof. The first statement is immediate. The second statement follows via Lemma 4.2.4 by Example 4.1.17. ☐

Notation 4.2.6 (Singular-elastic/solid ∞-topos)**.** Let $\mathbf{H}$ be a singular-cohesive ∞-topos (Def. 4.2.3) with underlying smooth cohesive ∞-topos $\mathbf{H}_{\rotatebox[origin=c]{70}{$\subset$}} \hookrightarrow \mathbf{H}$. Then

- **(i)** if $\mathbf{H}_{\rotatebox[origin=c]{70}{$\subset$}}$ is in fact an elastic ∞-topos (Def. 4.1.21), we say that $\mathbf{H}$ is a *singular-elastic ∞-topos*;
- **(ii)** if $\mathbf{H}_{\rotatebox[origin=c]{70}{$\subset$}}$ is in fact a solid ∞-topos (Def. 4.1.40), we say that $\mathbf{H}$ is a *singular-solid ∞-topos*.

Definition 4.2.7 (Singular-cohesive modalities)**.** Given a singular cohesive ∞-topos (Def. 4.2.3), with its singular cohesion from Prop. 4.2.5, we write

$$
\begin{array}{c}
\big(\underset{\text{“singular”}}{\rotatebox[origin=c]{70}{$<$}} := \mathrm{NnOrbSnglr} \circ \mathrm{Snglr} \big) \\
\bot \\
\big(\underset{\text{“smooth”}}{\rotatebox[origin=c]{70}{$\subset$}} := \mathrm{NnOrbSnglr} \circ \mathrm{Smth} \big) \\
\bot \\
\big(\underset{\text{“orbi-singular”}}{\rotatebox[origin=c]{70}{$\prec$}} := \mathrm{OrbSnglr} \circ \mathrm{Smth} \big)
\end{array}
\quad : \mathbf{H} \to \mathbf{H}
\tag{4.144}
$$

for the adjoint triple of modalities $\mathbf{H} \to \mathbf{H}$ induced (1.18) via (4.143); accompanying the cohesive modalities (4.2) induced via (4.142).

This above terminology reflects the difference (see Figure D) between a plain singularity $\rotatebox[origin=c]{70}{$<$}$ (singular but not orbi-singular) as opposed to its enhancement to an actual orbifold singularity $\rotatebox[origin=c]{70}{$\prec$}$. We record the following elementary but important consequence:

Proposition 4.2.8 (Smooth orbi-singular is smooth)**.** *The singularity modalities (Def. 4.2.7) satisfy:*

$$
\rotatebox[origin=c]{70}{$<$} \circ \rotatebox[origin=c]{70}{$\subset$} \simeq \rotatebox[origin=c]{70}{$\subset$} \quad \textit{and} \quad \rotatebox[origin=c]{70}{$\subset$} \circ \rotatebox[origin=c]{70}{$\prec$} \simeq \rotatebox[origin=c]{70}{$\subset$} \,.
\tag{4.145}
$$

Proof. As in Prop. 4.1.2. ☐

Lemma 4.2.9 (Objectwise application of singularity modalities)**.** *The singular-modalities in* (4.143) *are computed objectwise over* Chrt*, as in Example 4.1.17, followed by ∞-sheafification L_{Chrt}* (3.72)*:*

$$
\mathrm{Shv}_\infty(\mathrm{Snglrt} \times \mathrm{Chrt}) \hookrightarrow \mathrm{PShv}_\infty(\mathrm{Snglrt} \times \mathrm{Chrt}) \underset{\mathrm{const}_{\mathrm{Snglrt}}}{\overset{\underset{\mathrm{Snglrt}}{\underrightarrow{\lim}}}{\underset{\bot}{\rightleftarrows}}} \mathrm{PShv}_\infty(\mathrm{Chrt}) \underset{\bot}{\overset{L_{\mathrm{Chrt}}}{\rightleftarrows}} \mathrm{Shv}_\infty(\mathrm{Chrt})
$$

$$
\mathrm{Snglr} : \mathrm{Shv}_\infty(\mathrm{Snglrt} \times \mathrm{Chrt}) \to \mathrm{Shv}_\infty(\mathrm{Chrt}), \qquad \mathrm{NnOrbSnglr} : \mathrm{Shv}_\infty(\mathrm{Chrt}) \to \mathrm{Shv}_\infty(\mathrm{Snglrt} \times \mathrm{Chrt})
\tag{4.146}
$$

Proof. By essential uniqueness of adjoints (3.43). □

4.2.2 Examples of singular-cohesive ∞-toposes

Example 4.2.10 (Singular ∞-groupoids)**.** For $\mathbf{H}_{\cup} := \mathrm{Grpd}_{\infty}$ the base ∞-topos of plain ∞-groupoids (3.20), the singular-cohesive ∞-topos from Def. 4.2.3

$$\mathrm{SingularGroupoids}_{\infty} \;:=\; \mathrm{Shv}_{\infty}\big(\mathrm{Snglrt}, \mathrm{Grpd}_{\infty}\big) \tag{4.147}$$

is that of traditional unstable global homotopy theory [Schw18, §1s], as discussed in this form in [Re14, §4.1] (here with evaluation on all finite groups instead of all compact Lie groups).

Example 4.2.11 (Singular-smooth ∞-groupoids)**.** **(i)** We call the singular-cohesive ∞-topos (Def. 4.2.3) over those of smooth ∞-groupoids (Example 4.1.18) the ∞-topos of *singular-smooth ∞-groupoids*:

$$\begin{aligned} \mathrm{SnglrSmthGrpd}_{\infty} \;&:=\; \mathrm{Shv}_{\infty}\big(\mathrm{Snglrt}, \mathrm{SmthGrpd}_{\infty}\big) \\ &\simeq\; \mathrm{Shv}_{\infty}\big(\mathrm{CrtSpc} \times \mathrm{Snglrt}\big)\,. \end{aligned} \tag{4.148}$$

(ii) We call the singular-elastic ∞-topos (Def. 4.2.6) over $\mathrm{JetSmthGrpd}_{\infty}$ (Example 4.1.24)

$$\begin{aligned} \mathrm{SingularJetSmthGrpd}_{\infty} \;&:=\; \mathrm{Shv}_{\infty}\big(\mathrm{Snglrt}, \mathrm{JetSmthGrpd}_{\infty}\big) \\ &\simeq\; \mathrm{Shv}_{\infty}\big(\mathrm{JetCrtSp} \times \mathrm{Snglrt}\big)\,. \end{aligned} \tag{4.149}$$

(iii) We call the singular-solid ∞-topos (Def. 4.2.6) over $\infty\mathrm{JetSuperGrpd}_{\infty}$ (Example 4.1.43)

$$\begin{aligned} \mathrm{Snglr}\infty\mathrm{JetSuperGrpd}_{\infty} &:= \mathrm{Shv}_{\infty}\big(\mathrm{Snglrt}, \infty\mathrm{JetSuperGrpd}_{\infty}\big) \\ &\simeq \mathrm{Shv}_{\infty}\big(\infty\mathrm{JetSuperCrtSp} \times \mathrm{Snglrt}\big)\,. \end{aligned} \tag{4.150}$$

For the second lines of (4.148), (4.148), and (4.150), see Lemma 4.2.15.

4.2.3 Basic properties of singular cohesion

Definition 4.2.12 (Orbi-singularities)**.** Let $\mathbf{H}$ be singular-cohesive ∞-topos (Def. 4.2.3).

(i) We regard the objects $\rotatebox[origin=c]{70}{$\prec$}^{G} \in \mathrm{Snglrt}$ (4.135) as objects of $\mathbf{H}$ under the ∞-Yoneda-embedding (Prop. 3.1.37) and the inclusion (4.138) of discrete objects:

$$\rotatebox[origin=c]{70}{$\prec$}^{G} \,\in\, \mathrm{Snglrt} \overset{y}{\hookrightarrow} \mathrm{Shv}_{\infty}\big(\mathrm{Snglrt}, \mathbf{B}_{\cup}\big) \overset{\mathrm{Disc}}{\hookrightarrow} \mathrm{Shv}_{\infty}\big(\mathrm{Snglrt}, \mathbf{H}_{\cup}\big) \;=\; \mathbf{H}\,. \tag{4.151}$$

(ii) More generally, for

$$G \,\in\, \mathrm{Grp}(\mathrm{Grpd}_{\infty}) \xrightarrow{\;\mathrm{Grp(Disc)}\;} \mathrm{Grp}(\mathbf{H}_{\cup}) \tag{4.152}$$

any discrete ∞-group (4.142), we also write

$$\rotatebox[origin=c]{70}{$\prec$}^{G} \;:=\; \rotatebox[origin=c]{70}{$\prec$}(\mathbf{B}G) \,\in\, \mathbf{H} \tag{4.153}$$

for the orbi-singularization (4.143) of its delooping (3.121).

Lemma 4.2.16 shows that the two notations in Def. 4.2.12 are consistent with each other.

Remark 4.2.13 (Finite groups in singular cohesion)**.** Given a singular-cohesive ∞-topos (Def. 4.2.3), the images of a finite group G under the following sequence of inclusions are naturally all denoted by the same symbol:

$$\begin{array}{ccccccccc} \mathrm{Grp}^{\mathrm{fin}} & \hookrightarrow & \mathrm{Grp}(\mathrm{Set}) & \hookrightarrow & \mathrm{Grp}(\mathrm{Grpd}_\infty) & \xhookrightarrow{\mathrm{Grp}(\mathrm{Disc})} & \mathrm{Grp}\big(\mathbf{H}_{\cup}\big) & \xhookrightarrow{\mathrm{Grp}(\mathrm{NnOrbSnglr})} & \mathrm{Grp}\big(\mathbf{H}\big) \\ G & \longmapsto & G & \longmapsto & G & \longmapsto & G & \longmapsto & G \end{array} \tag{4.154}$$

With this understood, we also have identifications as follows (where now the ambient ∞-categories are implicit from the context):

$$\ast /\!\!/ G \simeq \mathrm{Disc}(\ast /\!\!/ G) \quad \text{and} \quad \overset{G}{\curlyvee} \simeq \mathrm{Disc}\big(\overset{G}{\curlyvee}\big) \tag{4.155}$$

where on the right we are recalling the definition (4.151).

Similarly:

Remark 4.2.14 (Smooth charts in singular cohesion)**.** Consider a singular-cohesive ∞-topos (Def. 4.2.3) with an ∞-site Chrt of charts (Def. 4.1.9). Then images of the charts $U \in \mathrm{Chrt}$ under the ∞-Yoneda embedding (Prop. 3.1.37), and further under NnOrbSnglr (4.138), are naturally denoted by the same symbol:

$$\begin{array}{ccccc} \mathscr{S} & \xrightarrow{\;\;y\;\;} & \mathbf{H}_{\cup} & \xrightarrow{\mathrm{NnOrbSnglr}} & \mathbf{H} \\ U & \longmapsto & U & \longmapsto & U \end{array} \tag{4.156}$$

Lemma 4.2.15 (∞-Yoneda on product site)**.** *Consider a singular-cohesive ∞-topos* $\mathbf{H}$ *(Def. 4.2.3) with an ∞-site* Chrt *of cohesive charts (Def. 4.1.9) for* $\mathbf{H}_{\cup}$.

(i) *Then a site (Def. 3.1.42) for the full singular-cohesive* $\mathbf{H}$ *is the Cartesian product site*

$$\mathrm{SingularCharts} := \mathrm{Chrt} \times \mathrm{Snglrt} \tag{4.157}$$

in that

$$\mathbf{H} \simeq \mathrm{Shv}_\infty\big(\mathrm{Chrt} \times \mathrm{Snglrt}\big). \tag{4.158}$$

(ii) *Under the ∞-Yoneda embedding (Prop. 3.1.37) objects in the product site map to the Cartesian product of their prolonged Yoneda embeddings (in the sense of Remark 4.2.13 and Remark 4.2.14):*

$$\begin{array}{ccc} \mathrm{Chrt} \times \mathrm{Snglrt} & \xrightarrow{\;\;\;y\;\;\;} & \mathbf{H} \\ \big(U, \overset{G}{\curlyvee}\big) & \longmapsto & U \times \overset{G}{\curlyvee}, \end{array} \tag{4.159}$$

where on the right we are using the abbreviated notation from (4.151) *and* (4.156).

Proof. On the one hand, we have a natural equivalence

$$\mathbf{H}\Big(y\big(U_1, \overset{G_1}{\curlyvee}\big)\,,\, y\big(U_2, \overset{G_2}{\curlyvee}\big)\Big) \simeq \mathrm{Chrt}(U_1, U_2) \times \mathrm{Snglrt}\big(\ast /\!\!/ G_1\,,\, \ast /\!\!/ G_2\big) \tag{4.160}$$

by fully-faithfulness of the ∞-Yoneda embedding (Prop. 3.1.37) and by the definition of product sites. On the other hand, we have a sequence of natural equivalences

$$\begin{aligned}
&\mathbf{H}\Big(y\big(U_1, \overset{G_1}{\curlyvee}\big)\,,\, U_2 \times \overset{G_2}{\curlyvee}\Big) \\
&\simeq \mathbf{H}\Big(y\big(U_1, \overset{G_1}{\curlyvee}\big), \mathrm{NnOrbSnglr}(U_2) \times \mathrm{Disc}\big(\overset{G_2}{\curlyvee}\big)\Big) \\
&\simeq \mathbf{H}\Big(y\big(U_1, \overset{G_1}{\curlyvee}\big), \mathrm{NnOrbSnglr}(U_2)\Big) \times \mathbf{H}\Big(y\big(U_1, \overset{G_1}{\curlyvee}\big), \mathrm{Disc}\big(\overset{G_2}{\curlyvee}\big)\Big) \\
&\simeq \mathbf{H}_{\cup}\Big(\mathrm{Snglr}\big(y\big(U_1, \overset{G_1}{\curlyvee}\big)\big), U_2\Big) \times \mathbf{B}\Big(\textstyle\int\big(y\big(U_1, \overset{G_1}{\curlyvee}\big)\big),\ \overset{G_2}{\curlyvee}\Big) \\
&\simeq \mathbf{H}_{\cup}\big(U_1, U_2\big) \times \mathbf{B}\big(\overset{G_1}{\curlyvee},\ \overset{G_2}{\curlyvee}\big) \\
&\simeq \mathrm{Chrt}\big(U_1, U_2\big) \times \mathrm{Snglrt}\big(\overset{G_1}{\curlyvee},\ \overset{G_2}{\curlyvee}\big).
\end{aligned} \tag{4.161}$$

Here the first step is by definition, the second step is the universal property of the Cartesian product, and the third step is the hom-equivalence (3.43) of the adjunctions $\mathrm{Snglr} \dashv \mathrm{NnOrbSnglr}$ and $\int \dashv \mathrm{Disc}$, respectively. In the fourth step, we use (4.29) and (4.163), respectively. The last step is the fully-faithfulness of the ∞-Yoneda embedding (Prop. 3.1.37). Since both (4.160) and (4.161) are natural in $\big(U', (\ast /\!\!/ G)_{\curlyvee}\big)$, and since their right hand sides coincide, it follows by the ∞-Yoneda embedding (Prop. 3.1.37) that also the representatives of the left hand sides coincide: $y\big(U_2, \overset{G_2}{\curlyvee}\big) \simeq U_2 \times \overset{G_2}{\curlyvee}$. □

Lemma 4.2.16 (Images and pre-images of orbi-singularities)**.** *Let* $\mathbf{H}$ *be a singular-cohesive ∞-topos (Def. 4.2.3). Then the images and pre-images of the generic singularities* $\overset{G}{\curlyvee}$ (4.151) *under the functors* (4.143) *exhibiting the singular cohesion are as follows (see Figure D):*

$$\begin{array}{ccccccc}
 & & \overset{G}{\curlyvee} & & & \ast /\!\!/ G & \in \mathbf{H} \\
 & \swarrow^{\mathrm{Snglr}} & & \overset{\mathrm{OrbSnglr}}{\nwarrow}\ \underset{\mathrm{Smth}}{\searrow} & & \nearrow_{\mathrm{NnOrbSnglr}} & \\
\ast = \ast/G & & & & \ast /\!\!/ G & & \in \mathbf{H}_{\cup}
\end{array} \tag{4.162}$$

Proof. By the singular cohesion established in the proof of Prop. 4.2.5 we have that:

(i) the functor $\mathrm{Snglr} \simeq \underset{\longrightarrow}{\lim}$ is the colimit functor (Prop. 3.1.36),

(ii) the functor $\mathrm{Smth} \simeq \mathrm{Snglrt}\big(\overset{1}{\curlyvee}, -\big)$ is the hom-functor (3.2) out of the terminal object (4.139).

Using this, we deduce the claim:

(i) Since colimits of representable ∞-functors are equivalent to the point (Lemma 3.1.40) we have

$$\mathrm{Snglr}\big(\overset{G}{\curlyvee}\big) \simeq \ast \simeq \ast/G. \tag{4.163}$$

(ii) Observing that (4.136) reduces to $\mathrm{Snglrt}\big(\overset{1}{\curlyvee}, \overset{G}{\curlyvee}\big) \simeq \ast /\!\!/ G$ we have

$$\mathrm{Smth}\big(\overset{G}{\curlyvee}\big) \simeq \ast /\!\!/ G. \tag{4.164}$$

(iii) With this and by the various adjunctions we have, for $U \in \mathbf{H}_{\cup}$ any geometically contractible generator (4.29) and $K \in \mathrm{Grp}^{\mathrm{fin}}$ any finite group, the following sequence of natural equivalences:

$$
\begin{aligned}
\mathbf{H}\Big(U \times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, \mathrm{OrbSnglr}(\ast /\!\!/ G)\Big) &\simeq \mathbf{H}_{\cup}\Big(\underbrace{\mathrm{Smth}\big(U \times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}}\big)}_{\simeq\, U \times \mathrm{Smth}(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}})}, \ast /\!\!/ G\Big) \\
&\simeq \mathbf{H}_{\cup}\big(U \times (\ast /\!\!/ K), \underbrace{\ast /\!\!/ G}_{\simeq \mathrm{Disc}(\ast /\!\!/ G)}\big) \\
&\simeq \mathrm{Grpd}_{\infty}\big(\underbrace{\mathrm{Shp}(U)}_{\simeq \ast} \times (\ast /\!\!/ K), \ast /\!\!/ G\big) \\
&\simeq \mathrm{Snglrt}\big(\overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big) \\
&\simeq \mathbf{B}\Big(\underbrace{\mathrm{Shp}(U)}_{\simeq \ast} \times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\Big) \\
&\simeq \mathbf{B}\Big(\mathrm{Shp}\big(U \times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}}\big), \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\Big) \\
&\simeq \mathbf{H}\Big(U \times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, \mathrm{Disc}\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big)\Big) \\
&\simeq \mathbf{H}\Big(U \times \overset{K}{\rotatebox[origin=c]{70}{$\prec$}}, \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\Big),
\end{aligned}
\tag{4.165}
$$

where in several steps we recognized geometric discreteness, by (4.155) in Remark 4.2.13.

But, by Lemma 4.2.15, this chain of natural equivalences in total is a natural equivalence of the form

$$
\mathbf{H}\Big(y\big(U, \overset{K}{\rotatebox[origin=c]{70}{$\prec$}}\big), \mathrm{OrbSnglr}\big(\ast /\!\!/ G\big)\Big) \simeq \mathbf{H}\Big(y\big(U, \overset{K}{\rotatebox[origin=c]{70}{$\prec$}}\big), \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\Big). \tag{4.166}
$$

From this, the ∞-Yoneda embedding (Prop. 3.1.37) implies that $\mathrm{OrbSnglr}\big(\ast /\!\!/ G\big) \simeq \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}$. ☐

It is useful to re-express this in terms of the modalities:

Proposition 4.2.17 (Orbi-singularities are orbi-singular)**.** *Let* $\mathbf{H}$ *be a singular-cohesive* ∞*-topos (Def. 4.2.3) and consider a finite group* $G \in \mathrm{Grp}^{\mathrm{fin}}$ (4.154)*. Then the images of the generic orbi-singularity* $\overset{G}{\rotatebox[origin=c]{70}{$\prec$}} \in \mathbf{H}$ (4.151) *under the modalities* (4.144) *are (see Figure D):*

$$
\rotatebox[origin=c]{70}{$<$}\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big) \simeq \ast, \qquad \cup\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big) \simeq \ast /\!\!/ G, \qquad \rotatebox[origin=c]{70}{$\prec$}\big(\overset{G}{\rotatebox[origin=c]{70}{$\prec$}}\big) \simeq \overset{G}{\rotatebox[origin=c]{70}{$\prec$}}. \tag{4.167}
$$

Proof. This follows directly with Lemma 4.2.16 and the definition (4.144). For example:

$$\curlyvee\big(\underset{\curlyvee}{G}\big) \simeq \underbrace{\mathrm{OrbSnglr} \circ \overbrace{\mathrm{Smth}\big(\underset{\curlyvee}{G}\big)}^{\simeq *\sslash G}}_{\simeq \underset{\curlyvee}{G}} \tag{4.168}$$

□

In the same vein, we also have the following immediate but important property:

Proposition 4.2.18 (Orbi-singularities are geometrically discrete)**.** *Let* $\mathbf{H}$ *be a singular-cohesive* ∞*-topos (Def. 4.2.3) and consider a finite group* $G \in \mathrm{Grp}^{\mathrm{fin}}$ *(4.154).*

(i) *Then the basic orbi-singularity* $\underset{\curlyvee}{G} \in \mathbf{H}$ *(4.151) is geometrically discrete (4.2) and thus also pure shape:*

$$\flat \underset{\curlyvee}{G} \simeq \underset{\curlyvee}{G}, \quad \int \underset{\curlyvee}{G} \simeq \underset{\curlyvee}{G}. \tag{4.169}$$

(ii) *The same is true for* $\mathrm{Smth}(*\sslash G)_{\curlyvee} \simeq *\sslash G$*:*

$$\flat(*\sslash G) \simeq *\sslash G, \quad \int(*\sslash G) \simeq *\sslash G. \tag{4.170}$$

Proof. Both statements follow immediately from the definitions and the fact that G is finite and hence geometrically discrete (4.154). □

Remark 4.2.19 (Need for discrete/finite groups in Snglrt)**.** It is to make Lemma 4.2.16 and hence Prop. 4.2.17 true that Def. 4.2.1 requires the global orbit category Snglrt to consist of finite groups, instead of more general compact Lie groups (Remark 4.2.2): If Snglrt were to contain non-discrete compact Lie groups G, then the same argument as in Lemma 4.2.16 would give in (4.167) the following more general formula:

$$\cup \underset{\curlyvee}{G} \simeq *\sslash \flat G \tag{4.171}$$

(where on the right we think of the Lie group G as being cohesive via (4.52)). Since the condition $G \simeq \flat G$ characterizes discrete groups, this would break Prop. 5.1.2 below, in that then the shape of the orbi-singularization of a topological groupoid would take non-traditional values on non-discrete groups in the global orbit category.

The following lemma further illustrates the nature of orbi-singular cohesion:

Lemma 4.2.20 (Smooth 0-truncated objects are orbi-singular)**.** *Let* $\mathbf{H}$ *be a singular-cohesive* ∞*-topos (Def. 4.2.3). Then if* $X \in \mathbf{H}_{\cup,0}$ *is smooth (4.144) and 0-truncated (Def. 3.1.57), it is also orbi-singular (4.144):*

$$\tau_0(X) \simeq X \;\; \text{and} \;\; \cup(X) \simeq X \qquad \Rightarrow \qquad \curlyvee(X) \simeq X. \tag{4.172}$$

Proof. Since X is smooth, there exists $X_\cup \in \mathbf{H}_\cup$ such that $X \simeq \mathrm{Smth}(X_\cup)$. Observe that X being 0-truncated implies that $X_\cup$ is 0-truncated, (by using in Def. 3.1.57 the hom-equivalence (3.43) of the right adjoint Smth).

Now let $\mathscr{S}$ be any site (3.72) for $\mathbf{H}_\cup$. Then, for $U \in \mathscr{S} \hookrightarrow \mathbf{H}_\cup$ and $G \in \mathrm{Grp}^{\mathrm{fin}}$, we have the following sequence of natural equivalences, using the various adjoint functors, their idempotency and respect for products:

$$
\begin{aligned}
(\prec X)\big(\mathrm{Smth}(U) \times \underset{\prec}{G}\big) & \simeq (\prec \mathrm{Smth}(X_{\cup}))\big(\mathrm{Smth}(U) \times \underset{\prec}{G}\big) \\
& \simeq \mathbf{H}\big(\mathrm{Smth}(U) \times \underset{\prec}{G}, \prec \mathrm{Smth}(X_{\cup})\big) \\
& \simeq \mathbf{H}\big(\mathrm{Smth}(U) \times \cup(\underset{\prec}{G}), \mathrm{Smth}(X_{\cup})\big) \\
& \simeq \mathbf{H}\big(\mathrm{Smth}(U \times (* /\!\!/ G)), \mathrm{Smth}(X_{\cup})\big) \\
& \simeq \mathbf{H}_{\cup}\big(U \times (* /\!\!/ G), X_{\cup}\big) \\
& \simeq \mathrm{Grpd}_{\infty}\Big(* /\!\!/ G, \mathbf{H}_{\cup}\big(U, X_{\cup}\big)\Big) \\
& \simeq \mathrm{Grpd}_{\infty}\Big(*, \mathbf{H}_{\cup}\big(U, X_{\cup}\big)\Big) \\
& \simeq \mathbf{H}_{\cup}\big(U, X_{\cup}\big) \\
& \simeq \mathbf{H}\big(\mathrm{Smth}(U), \mathrm{Smth}(X_{\cup})\big) \\
& \simeq \mathbf{H}\big(\mathrm{Smth}(U) \times \underset{\prec}{G}, \mathrm{Smth}(X_{\cup})\big) \\
& \simeq \mathbf{H}\big(\mathrm{Smth}(U) \times \underset{\prec}{G}, X\big) \\
& \simeq X\big(\mathrm{Smth}(U) \times \underset{\prec}{G}\big).
\end{aligned}
\tag{4.173}
$$

Here the first and the last step use the ∞-Yoneda embedding (Prop. 3.1.37), while the middle step uses the fact that $X_{\cup}$ is 0-truncated, hence that $\mathbf{H}_{\cup}(U, X_{\cup})$ is 0-truncated (i.e. a set), to find that there is in fact no dependency on G. Hence the claim follows by the ∞-Yoneda embedding (Prop. 3.1.37), in view of Lemma 4.2.15. □

Remark 4.2.21 (Degenerate case of orbi-singular)**.** The natural language statement of Lemma 4.2.20 shows that the modality $\prec$ "orbi-singular" (4.143) really means: "All singularities *that are present* are orbi-singularities.", which becomes a trivially satisfied condition when there are no singularities, such as for smooth and 0-truncated objects.

4.2.4 Interplay between geometric and singular cohesion

Lemma 4.2.22 (Smooth commutes with shape)**.** *In a singular-cohesive ∞-topos (Def. 4.2.3) the smooth-modality* (4.144) *commutes with all three cohesive modalities* (4.2) *(as per Prop. 4.2.5):*

$$
\cup \circ \int \simeq \int \circ \cup\,, \qquad \cup \circ \flat \simeq \flat \circ \cup\,, \qquad \cup \circ \sharp \simeq \sharp \circ \cup\,. \tag{4.174}
$$

Proof. Under the defining identification $\mathbf{H} \simeq \mathrm{Shv}_{\infty}\big(\mathrm{Snglrt}, \mathbf{H}_{\cup}\big)$, let $\mathscr{X} \in \mathbf{H}$ be any object regarded as a $\mathbf{H}_{\cup}$-valued ∞-presheaf on Snglrt:

$$
\mathscr{X} \;:\; \underset{\prec}{K} \longmapsto \mathscr{X}(\underset{\prec}{K}) \;\in\; \mathbf{H}_{\cup}\,. \tag{4.175}
$$

Observe then (by Example 4.1.17 via Lemma 4.2.4) that $\cup$ turns such a presheaf into the constant presheaf on its value at the terminal object $\underset{\prec}{1}$:

$$
\big(\cup \mathscr{X}\big) \;:\; \underset{\prec}{K} \longmapsto \mathscr{X}(\underset{\prec}{1})\,. \tag{4.176}
$$

On the other hand, the geometric modalities operate objectwise over Snglrt (Remark 4.2.9):

$$\big(\int \mathscr{X}\big) \;:\; \prec^{K} \longmapsto \int\big(\mathscr{X}(\prec^{K})\big)\,. \tag{4.177}$$

With this, we have the following sequence of natural equivalences for $\mathscr{X} \in \mathbf{H}$ and $\prec^{K} \in$ Snglrt:

$$\begin{aligned} \big(\cup \int \mathscr{X}\big)(\prec^{K}) &\simeq \big(\int \mathscr{X}\big)(\prec^{1}) \\ &\simeq \int\big(\mathscr{X}(\prec^{1})\big) \\ &\simeq \int\big((\cup \mathscr{X})(\prec^{K})\big) \\ &\simeq \big(\int \cup \mathscr{X}\big)(\prec^{K})\,. \end{aligned} \tag{4.178}$$

Hence the claim follows by the ∞-Yoneda embedding (Prop. 3.1.37). The argument for $\flat$ and $\sharp$ is analogous. □

Remark 4.2.23 (Dichotomy between naive and proper orbifold cohomology via singular-cohesion)**.** In contrast to Lemma 4.2.22, the orbi-singular modality $\prec$ (4.144) does *not* commute with the cohesive shape modality $\int$ (4.2), in general. This phenomenon is the very source of the *proper equivariant* structure seen in singular-cohesive ∞-toposes, reflected in the following dichotomy between geometric- and homotopy fixed points of an orbi-space and in the distinction between proper- and Borel-equivariant cohomology:

	$\prec \circ \int$	$\int \circ \prec$	
Def. 4.2.24 **(i)**	Homotopy fixed-points	Geometric fixed-points	Def. 4.2.24 **(ii)**
Def. 6.1.1	Borel-equivariant cohomology	Proper equivariant cohomology	Def. 6.1.2
Def. 6.2.3	Tangentially twisted cohomology	Tangentially twisted proper orbifold cohomology	Def. 6.2.5

Definition 4.2.24 (Geometric- and homotopy-fixed points)**.** Let $\mathbf{H}$ be a singular-cohesive ∞-topos (Def. 4.2.3), $G \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1) being discrete $G \simeq \flat G$ and 0-truncated $G \simeq \tau_0 G$, and $(X,\rho) \in G\mathrm{Actions}(\mathbf{H})$ (Prop. 3.2.6) with smooth $X \simeq \cup X$, hence

$$X \;\in\; \mathbf{H}_{\cup} \xhookrightarrow{\;\mathrm{NnOrbSnglr}\;} \mathbf{H}\,. \tag{4.179}$$

For any subgroup $K \subset G$, the ∞-groupoid of $\prec^{K}$-points in the slice (Prop. 3.1.46) over $\prec^{G}$ (4.153)...

(i) ...of the orbi-singularization (4.143) of the shape (4.142) of $X /\!\!/ G$ is the *homotopy fixed point space* of X

$$\mathrm{HmtpFxdPntSpc}^{K}(X) \;:=\; \mathbf{H}_{/\prec^{G}}\Big(\prec^{K},\; \prec \int (X /\!\!/ G)\Big)\,. \tag{4.180}$$

(ii) ...of the shape (4.142) of the orbi-singularization (4.143) of $X /\!\!/ G$ is the *geometric fixed point space* of X

$$\mathrm{GmtrcFxdPntSpc}^{K}(X) \;:=\; \mathbf{H}_{/\prec^{G}}\Big(\prec^{K}, \int \prec (X /\!\!/ G)\Big)\,. \tag{4.181}$$

On the right we are using Prop. 4.2.17 and Prop. 4.2.18 to see that both expressions indeed live in the slice over $\overset{G}{\curlyvee}$.

Proposition 4.2.25 (Homotopy-fixed point spaces are fixed loci in shapes)**.** *The homotopy-fixed point spaces* (4.180) *of the G-space X in Def. 4.2.24 are, equivalently, the fixed-loci (Def. 3.2.24) of the shape* $\mathrm{Shp}(X) \in \mathrm{Grpd}_\infty$ (4.1) *of X :*

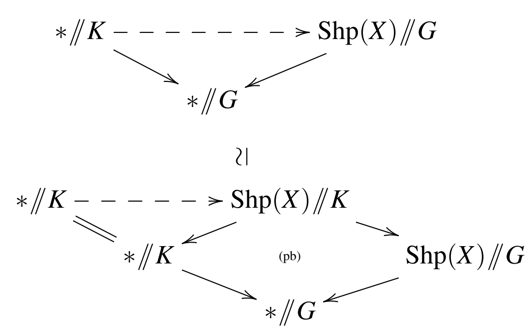
$$\mathrm{HmtpFxdPntSpcs}^K(X) \simeq \big(\mathrm{Shp}(X)\big)^K \quad \in \mathrm{Grpd}_\infty \tag{4.182}$$

with respect to the induced $G \simeq \int G$-action (using Prop. 4.1.4, discreteness of G and cohesion in the form of Prop. 4.1.2).

Proof. We claim a sequence of natural equivalences as follows:

$$\begin{aligned}
&\mathrm{HmtpFxdPntSpc}^K(X) \\
&\equiv \mathbf{H}_{/\overset{G}{\curlyvee}}\Big(\overset{K}{\curlyvee},\ \curlyvee \textstyle\int (X /\!\!/ G)\Big) \\
&\simeq \mathbf{H}_{/\overset{G}{\curlyvee}}\Big(\overset{K}{\curlyvee},\ \curlyvee (\textstyle\int X) /\!\!/ G\Big) \\
&\simeq \mathbf{H}_{/\mathrm{OrbSnglr}(\ast /\!\!/ G)}\Big(\mathrm{OrbSinglr}\big(\ast /\!\!/ K\big),\ \mathrm{OrbSnglr}\big((\textstyle\int X) /\!\!/ G\big)\Big) \\
&\simeq \big(\mathbf{H}_{\cup}\big)_{/\ast /\!\!/ G}\big(\ast /\!\!/ K,\ (\textstyle\int X) /\!\!/ G\big) \\
&\simeq \big(\mathrm{Grpd}_\infty\big)_{/\ast /\!\!/ G}\big(\ast /\!\!/ K,\ \mathrm{Shp}(X) /\!\!/ G\big) \\
&\simeq \big(\mathrm{Grpd}_\infty\big)_{/\ast /\!\!/ K}\big(\ast /\!\!/ K,\ \mathrm{Shp}(X) /\!\!/ K\big) \\
&\simeq \big(\mathrm{Shp}(X)\big)^K .
\end{aligned} \tag{4.183}$$

Here the first step is the definition (4.180), and the second step uses Prop. 4.1.4, discreteness of G and cohesion in the form of Prop. 4.1.2. In the third step we observe with $\overset{K}{\curlyvee} \simeq \curlyvee(\ast /\!\!/ K)$ (Lemma 4.2.16) and $\curlyvee := \mathrm{OrbSinglr} \circ \mathrm{Smth}$ (4.144) that all objects and morphisms are in the image of OrbSnglr, and in the fourth step we use that this functor is fully faithful, by Prop. 4.2.5. In the fifth step, we similarly observe that all objects and morphisms are, in fact, furthermore in the image of Disc (by assumption on G and by definition of $\int := \mathrm{Disc} \circ \mathrm{Shp}$ (4.2)), which is fully faithful by the axioms of cohesion (4.1). The sixth step observes the universal factorization through the pullback



(4.184)

The pullback, in turn, is the homotopy quotient of the restricted action, as shown, by Prop. 3.2.12. With this, the last step follows by Example 3.2.26. In summary, the composite of the sequence of equivalences (4.183) gives the statement (4.182). □

Example 4.2.26 (Geometric fixed points generally differ from homotopy fixed points)**.** As in Example 4.2.11, let $\mathbf{H} := \mathrm{SnglrSmthGrpd}_\infty$. For $n \in \mathbb{N}$, $n \geq 1$, consider the Cartesian space $\mathbb{R}^n \in \mathrm{SmthMfd} \hookrightarrow \mathbf{H}$, via (4.52), and regard it as equipped with the additive translation action of $\mathbb{Z}^n$ induced from the left action of the additive group $(\mathbb{R}^n, +)$ on itself, under the canonical inclusion $(\mathbb{Z}^n, +) \hookrightarrow (\mathbb{R}^n, +)$:

$$(\mathbb{R}^n, \rho_\ell) \;\in\; \mathbb{Z}^n\mathrm{Actions}(\mathbf{Hs})\,. \tag{4.185}$$

So the quotient of this action $\mathbb{R}^n /\!\!/ \mathbb{Z}^n \;\simeq\; \mathbb{R}^n/\mathbb{Z}^n \;\simeq\; \mathbb{T}^n \in \mathrm{SmthMfd} \hookrightarrow \mathbf{H}$ is the standard n-torus. We then have for the two notions of fixed-point spaces from Def. 4.2.24:

(i) The *Homotopy-fixed point space* (4.180) of the action (4.185) is equivalently the point (by Prop. 4.2.25 and (4.51)):

$$\mathrm{HmtpFxdPntSpc}^{\mathbb{Z}^n}(\mathbb{R}^n) \simeq \big(\underbrace{\textstyle\int \mathbb{R}^n}_{\simeq *}\big)^{\mathbb{Z}^n} \simeq * \tag{4.186}$$

(ii) The *geometric fixed point space* (4.181) of the action (4.185) is empty

$$\mathrm{GmtrcFxdPntSpc}^{\mathbb{Z}^n}(\mathbb{R}^n) \simeq \big(\mathbb{R}^n\big)^{\mathbb{Z}^n} \simeq \varnothing\,. \tag{4.187}$$

This follows by Lemma 5.1.7, using that no element of the set underlying $\mathbb{R}^n$ is fixed by the action of $\mathbb{Z}^n$.

Part II

Orbifolds

5

Orbifold geometry

Within an ambient context of singular-cohesive homotopy theory (§4), we now formulate the two geometric aspects of orbifolds:

- §5.1 – as cohesive spaces with orbi-singularities,
- §5.2 – as cohesive spaces locally equivalent to a given model space.

In the end, we combine both aspects to form the *proper ∞-categories of orbifolds*: this is Def. 5.2.45 below.

5.1 Orbispaces

We observe (Prop. 5.1.2) that the shape of the orbi-singularization of a topological groupoid, regarded in singular-smooth homotopy theory (Ex. 4.2.11), is the corresponding *orbispace* in global equivariant homotopy theory.

Remark 5.1.1 (Orbispaces in topology and in global equivariant homotopy theory). **(i)** **Orbispaces in topology.** The term *orbispace* was originally introduced [Hae90] to mean the topological version of orbifolds, i.e., Satake's original concept [Sa56] but disregarding any differentiable structure. From the perspective of étale groupoids/stacks, this means to consider topological groupoids/stacks instead of Lie groupoids/differentiable stacks. So this usage of the term "orbispace" serves to complete the following table:

Smooth manifold	Topological manifold	
orbifold	orbispace	(*geometric sense*)
Lie groupoid	topological groupoid	
differentiable stack	topological stack	

In this sense, orbispaces have been discussed, e.g., in [Hae84][Hae91, §5][Ch01][He01].

(ii) **Orbispaces in global equivariant homotopy theory.** In [HG07] it was suggested to change perspective and to instead regard these topological groupoids $\mathscr{X}_{\mathrm{top}}$ via the systems of homotopy types of all their geometric fixed point spaces, by the following formula [HG07, 4.2] (beware the differing conventions, as per Remark 4.2.2):

$$G \longmapsto \underbrace{\Big\| \overbrace{\mathbf{Map}\big(\mathbf{B}G, \mathscr{X}_{\mathrm{top}}\big)}^{} \Big\|}_{} \qquad \begin{array}{c}\text{orbispace} \\ (\textit{equivariant homotopical sense})\end{array} \tag{5.1}$$

(Annotations: "homotopy type of (fat) geometric realization of" above; "topological mapping groupoid" below.)

This is a global-equivariant version of how topological G-spaces are incarnated in G-equivariant homotopy theory via Elmendorf's theorem (recalled as Prop. 2.2.10), and has served to motivate the development of global equivariant homotopy theory [Schw18].

In the course of this development, homotopy theorists adopted the term "orbispace" to refer not to the topological groupoid $\mathscr{X}_{\mathrm{top}}$ (as [Hae90] originally did) but rather to the global equivariant homotopy type that is represented via (5.1). Usage of the term *orbispace* in this sense of global homotopy theory is, after [HG07], in [Re14][Kö16][Schw17][Lu19, 3][Ju20]. In [Ju20, 3.15] formula 5.1 is used (following suggestions in [Schw17, Introd.][Schw18, p. ix-x]) to define (abelian, non-geometric) cohomology of orbifolds with coefficients in global equivariant spectra.

Our Prop. 5.1.2 below shows that these two different meanings of the term "orbispace" in the literature are disentangled as well as unified by the notion of singular cohesion (Def. 4.2.3), in that orbispaces in the sense **(ii)** are the shape $\int$ (4.1) of the orbi-singularization $\curlyvee$ (4.144) of the topological groupoids in **(i)**:

$$\begin{array}{ccc} \mathrm{TopGrpd} & \xrightarrow{\ \int \circ \curlyvee\ } & \mathrm{Orbispaces} \\ \mathscr{X}_{\mathrm{top}} & \longrightarrow & \Big(\curlyvee^{G} \mapsto \big\|\mathbf{Map}\big(\mathbf{B}G, \mathscr{X}_{\mathrm{top}}\big)\big\| \Big) \end{array} \tag{5.2}$$

Hence Prop. 5.1.2 below means that, before passing to their pure shape, we may think of the orbi-singularizations of objects in singular-cohesive ∞-toposes as *cohesive orbispaces*, lifting the concept of plain orbispaces in the sense **(ii)** from plain homotopy theory to geometric (differential, étale) homotopy theory, hence back to sense **(i)** and beyond.

The crucial fact underlying the phenomenon (5.2), both in Prop. 5.1.2 and in Lemma 5.1.7 below, is that the probe of an orbi-singular object $\curlyvee \mathscr{X}_{\cup}$ by a generic orbi-singularity $\curlyvee^{K}$ (4.135) is, by adjunction (4.144), equivalently the probe of the underlying smooth object by the smooth aspect of $\curlyvee^{K}$, hence is, by (4.167) in Prop. 4.2.17, the geometric G-fixed locus in $\mathscr{X}_{\cup}$:

$$\curlyvee^{G} \longrightarrow \curlyvee \mathscr{X}_{\cup} \quad \overset{(4.144)}{\Leftrightarrow} \quad \cup \curlyvee^{G} \longrightarrow \mathscr{X}_{\cup} \quad \overset{(4.167)}{\Leftrightarrow} \quad * /\!\!/ G \longrightarrow \mathscr{X}_{\cup}\,. \tag{5.3}$$

Equivalently, since $\curlyvee^{G} \simeq \curlyvee(*/\!\!/G)$ (Lemma 4.2.16) the composite corespondence (5.3) is fully-faithfulness of $\curlyvee$.

Ex.: Topological groupoids as cohesive orbispaces

Proposition 5.1.2 (Shape of orbi-singularized topological groupoid is orbispace)**.** *Let* $\mathbf{H} := \mathrm{SnglrSmthGrpd}_\infty$ *(Ex. 4.2.11), and let*

$$\begin{array}{ccccc} \mathrm{TopGrpd} & \xrightarrow{\mathrm{Cdfflg}} & \mathrm{SmthGrpd}_\infty & \xrightarrow{\mathrm{NnOrbSnglr}} & \mathbf{H} \\ \mathscr{X}_{\mathrm{top}} & \longmapsto & & \longmapsto & \mathscr{X}_{\cup} \end{array} \tag{5.4}$$

be a topological groupoid, regarded via the embeddings (4.53) *and* (4.143)*. If* $\mathscr{X}_{\cup}$ *is such that both its space of objects and of morphisms are retracts of cell complexes (for instance: both are CW-complexes* (3.5)*) then the shape* (4.142) *of its orbi-singularization* (4.144) *is, as an* ∞*-presheaf* (4.138) *of* ∞*-groupoids on* Snglrt (4.2.1) *(i.e., on the global orbit category, Remark 4.2.2)*

$$\textstyle\int \curlyvee \mathscr{X}_{\cup} \;\in\; \mathrm{Sh}_\infty\big(\mathrm{Snglrt}\big) \xhookrightarrow{\;\;\mathrm{Disc}\;\;} \mathbf{H} \tag{5.5}$$

given by the assignment (5.3)

$$\textstyle\int \curlyvee \mathscr{X}_{\cup} \;:\; \underset{\curlyvee}{G} \longmapsto \big\| \mathbf{Map}\big(\mathbf{B}G, \mathscr{X}_{\mathrm{top}}\big) \big\| \,, \tag{5.6}$$

where on the right we have the fat geometric realization of the topological functor groupoid [Se74] (see [HG07, 2.3]), with $\mathbf{B}G \simeq * /\!\!/ G$ *(Ex. 3.1.14) regarded as a finite topological groupoid.*

Proof. Recall from (4.51) in Ex. 4.1.18 that Chrt := CrtSpc (Def. 3.1.5) is a site of cohesive charts (Def. 4.1.9) for $\mathrm{SmthGrpd}_\infty$. We claim that for $\mathbb{R}^n \in \mathrm{CrtSpc}$ and $\underset{\curlyvee}{G} \in \mathrm{Snglrt}$ (Def. 4.2.1), hence $\mathbb{R}^n \times \underset{\curlyvee}{G} \in \mathrm{CrtSpc} \times \mathrm{Snglrt}$ (Lemma 4.2.15), we have the following sequence of natural equivalences:

$$\begin{aligned} \mathbf{H}\big(\mathbb{R}^n \times \underset{\curlyvee}{G},\, \curlyvee \mathscr{X}_{\cup}\big) &= \mathbf{H}\big(\mathbb{R}^n \times \underset{\curlyvee}{G}, \mathrm{OrbSnglr}\big(\mathscr{X}_{\cup}\big)\big) \\ &\simeq \mathbf{H}_{\cup}\big(\underbrace{\mathrm{Smth}\big(\mathbb{R}^n \times \underset{\curlyvee}{G}\big)}_{\simeq \mathbb{R}^n \times \mathbf{B}G} ,\, \mathscr{X}_{\cup}\big) \\ &\simeq \mathbf{H}_{\cup}\Big(\mathbb{R}^n , \mathbf{Map}\big(\mathbf{B}G,\, \mathscr{X}_{\cup}\big)\Big) \\ &\simeq \mathbf{H}_{\cup}\Big(\mathbb{R}^n , \mathrm{Cdfflg}\mathbf{Map}\big(\mathbf{B}G,\, \mathscr{X}_{\mathrm{top}}\big)\Big). \end{aligned} \tag{5.7}$$

Here the first step is (5.12), the second is the hom-equivalence (3.43) of the adjunction $\mathrm{Smth} \dashv \mathrm{OrbSnglr}$ (4.143) and using under the brace that Smth preserves products (by Prop. 3.1.26), that $\mathbb{R}^n$ is already smooth, and that $\mathrm{Smth}\big(\underset{\curlyvee}{G}\big) \simeq (* /\!\!/ G)$ by (4.162). The third step is Lemma 2.1.5.

Since also the composite of all these natural equivalences is thus natural, the ∞-Yoneda lemma (Prop. 3.1.38) implies that

$$\curlyvee \mathscr{X}_{\cup} \;:\; \underset{\curlyvee}{K} \;\longmapsto\; \mathrm{Cdfflg}\,\mathrm{Map}\big(\mathbf{B}G,\, \mathscr{X}_{\cup}\big)\,. \tag{5.8}$$

Now, since $\int$ acts objectwise over $\underset{\curlyvee}{K}$ (4.142), we find from this that

$$\begin{aligned} \textstyle\int \curlyvee \mathscr{X}_{\cup} \;:\; \underset{\curlyvee}{K} \;\; &\longmapsto \;\; \textstyle\int \mathrm{Cdfflg}\,\mathbf{Map}\big(\mathbf{B}G,\, \mathscr{X}_{\mathrm{top}}\big) \\ &\simeq \;\; \mathrm{Shp}_{\mathrm{sTop}}\Big(\mathbf{Map}\big(\mathbf{B}G,\, \mathscr{X}_{\mathrm{top}}\big)\Big) \\ &\simeq \;\; \big\|\mathrm{Map}\big(\mathbf{B}G,\, \mathscr{X}_{\mathrm{top}}\big)\big\| \;. \end{aligned}$$

Here the first step is (4.57) and the last step follow by Prop. 3.1.18. □

5.1.1 Cohesive G-orbispaces

We now discuss in more detail the analogue of Prop. 5.1.2 in **(a)** the special case of global quotient stacks $\mathscr{X}_{\cup} \simeq X /\!\!/ G$ by a discrete group G, but **(b)** in the full generality of X being any 0-truncated cohesive space (not necessarily a topological space, but for instance a smooth manifold or diffeological space (4.52) or even a non-concrete object).

Remark 5.1.3 (Good orbifolds and good cohesive orbispaces)**.** The traditional orbifolds that arise as global quotients $\mathscr{X}_{\cup} \simeq X /\!\!/ G$ of a smooth manifold X by the action of a *discrete* group G are called *good orbifolds* (e.g. [Ka08, 6]). Therefore, the cohesive G-orbispaces discussed now (Def. 5.1.4) could be called (after forgetting their slicing over $\underset{\prec}{G}$) the *good cohesive orbispaces*.

Definition 5.1.4 (Cohesive G-orbispace)**.** Let $\mathbf{H}$ be a singular-cohesive ∞-topos (Def. 4.2.3) and $G \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1) discrete $G \simeq \flat G$. We say that a *cohesive G-orbispace* is an object

$$\begin{array}{c} \mathscr{X} \\ \downarrow{\scriptstyle p} \\ \underset{\prec}{G} \end{array} \;\in\; \mathbf{H}_{/\underset{\prec}{G}} \tag{5.9}$$

in the slice over the G-orbi-singularity (4.153) that is:

$$\begin{array}{llll} \textbf{(a)} & \text{orbi-singular:} & \prec(p) \simeq p & \text{(Def. 4.2.7)}\,, \\ \textbf{(b)} & \text{0-truncated:} & (\tau_0)_{/\underset{\prec}{G}}(p) \simeq p & \text{(Def. 3.1.57)}\,. \end{array} \tag{5.10}$$

Definition 5.1.5 (Universal covering space of a G-orbi-singular space)**.** Given a Cohesive G-orbispace $\mathscr{X} \in \mathbf{H}_{/\underset{\prec}{G}}$ (Def. 5.1.4), we say that its *universal covering space* $X \in \mathbf{H}$ the homotopy fiber of the defining morphism to $\underset{\prec}{G}$ over its essentially unique point:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{fib}(p)} & \mathscr{X} \\ & & \downarrow{\scriptstyle p} \\ & & \underset{\prec}{G} \end{array} \tag{5.11}$$

Proposition 5.1.6 (Properties of universal covering spaces)**.** *Let* $\mathbf{H}$ *be a singular-cohesive ∞-topos (Def. 4.2.3). Given a G-orbi-singular space $\mathscr{X} \in \mathbf{H}_{/\underset{\prec}{G}}$ (Def. 5.1.4), its universal covering space X (Def. 5.1.5)*

(i) *is:*
- **(a)** *0-truncated:* $\tau_0(X) \simeq X$ *(Def. 3.1.57)*,
- **(b)** *smooth:* $\cup(X) \simeq X$ *(Def. 4.1.1)*,

(ii) *and is equipped with a G-action (Prop. 3.2.6) such that $\mathscr{X}$ is the orbi-singularization* (4.144) *of the corresponding homotopy quotient:*

$$\mathscr{X} \;\simeq\; \prec\big(X /\!\!/ G\big)\,. \tag{5.12}$$

Proof. **(i)** That X is **(a)** 0-truncated follows from the condition that p is 0-truncated and using Lemma 4.1.14. To see that X is **(b)** smooth, observe that by the defining assumption (5.10) that p is orbi-singular, it is the image under OrbSnglr (4.143) of a morphism $p_{\cup}$ in $\mathbf{H}_{\cup}$:

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{fib}(p)} & \mathscr{X} \\ & & \downarrow{\scriptstyle p} \\ & & \curlyvee^{G} \end{array} \quad \simeq \quad \mathrm{OrbSnglr}\left(\begin{array}{ccc} X_{\cup} & \xrightarrow{\mathrm{fib}(p_{\cup})} & \mathscr{X}_{\cup} \\ & & \downarrow{\scriptstyle p_{\cup}} \\ & & * /\!\!/ G \end{array} \right). \tag{5.13}$$

We claim that in fact $X \simeq \mathrm{NnOrbSinglr}(X_{\cup})$, whence $X \simeq \cup(X)$: First, since OrbSnglr is a right adjoint it preserves homotopy fibers (Prop. 3.1.26), $\mathrm{fib}(p) \simeq \mathrm{OrbSnglr}\big(\mathrm{fib}(p_{\cup})\big)$, hence we have $X \simeq \mathrm{OrbSnglr}(X_{\cup})$. It follows, in particular, that $X_{\cup}$ is 0-truncated, since $X \simeq \mathrm{OrbSnglr}(X_{\cup})$ is 0-truncated by part **(a)**, and using that OrbSnglr is fully faithful. From this it follows that $\mathrm{OrbSnglr}(X_{\cup}) \simeq \mathrm{NnOrbSinglr}(X_{\cup})$, by Lemma 4.2.20. Together this gives the claim **(b)**.

With this, part **(ii)** now follows by comparison with (3.130). □

Shape of Cohesive *G*-orbispaces. We derive the following formula (5.15) in Prop. 6.1.6 which generalizes the embedding of G-spaces into global equivariant homotopy theory, discussed in [Re14, p. 7][Lu19, 3.2.17], from topological G-spaces to general cohesive G-spaces. Below in §6.1 this serves to prove that the intrinsic cohomology of good cohesive orbispaces subsumes proper equivariant cohomology (Theorem 6.1.9).

Lemma 5.1.7 (Shape of Cohesive G-orbispaces)**.** *Let* $\mathbf{H}$ *be a singular-cohesive ∞-topos (Def. 4.2.3). (3.20),* $G \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1) be a 0-truncated* $G \simeq \tau_0 G$ *and discrete* $G \simeq \flat G$ *and let* $X \in \mathbf{H}$ *be smooth* $X \simeq \cup X$ *and 0-truncated* $X \simeq \tau_0 X$ *and equipped with a G-action* $(X, \rho) \in G\mathrm{Actions}(\mathbf{H})$ *(Prop. 3.2.6).*

(i) *Then the orbi-singularization* (4.143) *of the corresponding homotopy quotient* (3.130)
$$\mathscr{X} \;:=\; \curlyvee\big(X /\!\!/ G\big) \;\in\; \mathbf{H} \;:=\; \mathrm{Shv}_{\infty}\big(\mathrm{Snglrt}, \mathbf{H}_{\cup}\big)\,, \tag{5.14}$$
when regarded as an $\mathbf{H}_{\cup}$*-valued ∞-presheaf on* Snglrt (4.138)*, assigns to a singularity* $\curlyvee^{K}$ (4.151) *the disjoint union of fixed loci* $X^{\phi(K)}$ *(Def. 3.2.24) of the smooth covering space X (Def. 5.1.5) for all group homomorphisms* $\phi : K \to G$ *homotopy-quotiented* (3.131) *by the residual G-action (Prop. 5.1.6):*
$$\mathscr{X} \;:\; \curlyvee^{K} \longmapsto \bigg(\coprod_{\phi \in \mathrm{Grp}(K,G)} X^{\phi(K)} \bigg) /\!\!/ G\,. \tag{5.15}$$

(ii) *Moreover, its shape* (4.142)
$$\mathrm{Shp}\Big(\curlyvee\big(X /\!\!/ G\big)\Big) \;\in\; \mathrm{SingularGroupoids} \;:=\; \mathrm{Shv}_{\infty}\big(\mathrm{Snglrt}\big) \tag{5.16}$$
assigns to a singularity $\curlyvee^{K}$ (4.151) *the cohesive shape* (4.1) *of these disjoint unions of fixed loci (Def. 3.2.24) of the smooth covering space X (Def. 5.1.5) homotopy-quotiented by its G-action (Prop. 5.1.6):*
$$\mathrm{Shp}\big(\mathscr{X}\big) \;:\; \curlyvee^{K} \longmapsto \mathrm{Shp}\bigg(\coprod_{\phi \in \mathrm{Grp}(K,G)} X^{\phi(K)} \bigg) /\!\!/ G\,. \tag{5.17}$$

Proof. We claim that for $U \in \mathrm{Chrt}$ (Def. 4.1.9) and $\prec^{K} \in \mathrm{Snglrt}$ (Def. 4.2.1), hence $U \times \prec^{K} \in \mathrm{Chrt} \times \mathrm{Snglrt}$ (Lemma 4.2.15), we have the following sequence of natural equivalences:

$$
\begin{aligned}
\mathbf{H}\big(U \times \prec^{K}, \mathscr{X}\big) &= \mathbf{H}\big(U \times \prec^{K}, \mathrm{OrbSnglr}(X /\!\!/ G)\big) \\
&\simeq \mathbf{H}_{\cup}\big(\underbrace{\mathrm{Smth}(U \times \prec^{K})}_{\simeq U \times (\ast /\!\!/ K)}, X /\!\!/ G\big) \\
&\simeq \mathrm{Grpd}_{\infty}\big((\ast /\!\!/ K), \mathbf{H}_{\cup}(U, X /\!\!/ G)\big) \\
&\simeq \mathrm{Grpd}_{1}\big((\ast /\!\!/ K), \mathbf{H}_{\cup}(U, X) /\!\!/ G\big) \\
&\simeq \Big(\coprod_{\phi \in \mathrm{Grp}(K,G)} \mathbf{H}_{\cup}(U, X)^{\phi(K)}\Big) /\!\!/ G \\
&\simeq \Big(\coprod_{\phi \in \mathrm{Grp}(K,G)} \mathbf{H}_{\cup}\big(U, X^{\phi(K)}\big)\Big) /\!\!/ G \\
&\simeq \Big(\mathbf{H}_{\cup}\Big(U, \coprod_{\phi \in \mathrm{Grp}(K,G)} X^{\phi(K)}\Big)\Big) /\!\!/ G \\
&\simeq \mathbf{H}_{\cup}\Big(U, \Big(\coprod_{\phi \in \mathrm{Grp}(K,G)} X^{\phi(K)}\Big) /\!\!/ G\Big).
\end{aligned}
\tag{5.18}
$$

Here the first step is (5.12), the second is the adjunction $\mathrm{Smth} \dashv \mathrm{OrbSnglr}$ (4.143) and using under the brace that Smth preserves products (by Prop. 3.1.26), that U is already smooth by assumption, and that $\mathrm{Smth}(\prec^{K}) \simeq (\ast /\!\!/ K)$ by (4.162). The third step is the tensoring of $\mathbf{H}$ over ∞-groupoids (Prop. 3.1.34) (using the geometric discreteness $(\ast /\!\!/ K) \simeq \mathrm{Disc}(\ast /\!\!/ K)$ by Remark 4.2.13) The fourth step uses the geometric contractibility of U and the discreteness of G to identify $\mathbf{H}_{\cup}(U, X /\!\!/ G) \simeq \mathbf{H}_{\cup}(U, X) /\!\!/ G$ (Lemma 4.1.12). The fifth is the general observation of Ex. 3.1.16 about hom-groupoids between quotient groupoids of sets. The sixth step uses Prop. 4.1.13 to find that the fixed points in the set of maps are the maps into the fixed point locus. After this key step, we just re-organize term: The seventh step uses the connectedness of U (Lemma 4.1.10) to find that a coproduct of homs out of U is a hom into the coproduct. Finally, the eighth step uses again Lemma 4.1.12.

(i) The composite equivalence (5.7) implies the first claim (5.15) by the ∞-Yoneda embedding (Prop. 3.1.37), using Lemma 4.2.15.

(ii) From this, the second claim (5.17) follows, using that Shp acts objectwise over Snglrt (4.142), and preserves homotopy quotients by discrete groups (Prop. 4.1.4).

□

Remark 5.1.8 (Relevance of 0-truncated orbi-singular spaces)**.**

(i) The crucial assumption that makes the proof of Lemma 5.1.7 work is, **(a)** that G is discrete and **(b)** that X is 0-truncated. This is what yields 1-groupoidal homs in the middle step of (5.7) and thus the form of the expression in the next step, as on the right hand side of (3.30).

(ii) Without the assumption of $\mathscr{X}$ being 0-truncated over $* /\!\!/ G$, the proof of Lemma 5.1.7 would proceed verbatim up to that middle step, but then would break as the nontrivial morphisms present in $\mathscr{X}$ would then mix with those of the action by G.

(iii) Lemma 5.1.7 shows that this subltety is closely related to the cohesive nature of the problem: We either have a space which is 0-truncated but carries cohesive (i.e. geometric) structure, or we turn it into its cohesive shape which is untruncated but geometrically discrete.

Singular quotient of Cohesive G-orbispaces.

Proposition 5.1.9 (Singular quotient of G-orbi-singular space)**.** *Let* $\mathbf{H}$ *be a singular-cohesive* ∞*-topos (Def. 4.2.3),* $G \in \mathrm{Grp}(\mathbf{H})$ *being discrete* $G \simeq \flat G$ *and 0-truncated* $G \simeq \tau_0 G$*. For* $\mathscr{X}$ *be a G-orbi-singular space (Def. 5.1.4) with universal covering space* $X \in \mathbf{H}_{\cup,0} \hookrightarrow \mathbf{H}$ *equipped with its induced G-action (Def. 5.1.5, Prop. 5.1.6). Then the singularization* (4.143) *of* $\mathscr{X}$ *is the plain G-quotient of X*

$$\mathrm{Snglr}\big(\mathscr{X}\big) \simeq X/G \quad \in \mathbf{H}_{\cup,0} \longrightarrow \mathbf{H}_{\cup} \tag{5.19}$$

(i.e., the quotient of the G-action formed in the 1-topos $\mathbf{H}_{\cup,0}$ *of 0-truncated objects).*

Proof. For $U \in \mathrm{Chrt}$, write

$$\mathbf{H}_{\cup}\big(U,X\big) /\!\!/ G \;\simeq\; \coprod_{c} \big(* /\!\!/ H_c\big) \quad \in \mathrm{Grpd}_1 \tag{5.20}$$

for the essentially unique decomposition of the groupoid on the left into its connected components

$$c \in \pi_0\Big(\mathbf{H}_{\cup}\big(U,X\big) /\!\!/ G\Big) \simeq \mathbf{H}_{\cup}\big(U,X\big)/G, \tag{5.21}$$

each of which is equivalent to the delooping groupoid (Ex. 3.1.14) of its fundamental group

$$H_c := \pi_1\Big(\mathbf{H}_{\cup}\big(U,X\big) /\!\!/ G, c\Big) \in \mathrm{Grp}. \tag{5.22}$$

Now, by Lemma 5.1.7 and re-instantiating the last few manipulations in (5.7), we have that over each $U \in \mathrm{Chrt}$ the incarnation of the G-orbi-singular space $\mathscr{X}$ as an ∞-presheaf on Snglrt is given by:

$$\begin{aligned}
\mathscr{X}(U) \;:\; \overset{K}{\curlyvee} \longmapsto \quad & \mathrm{Grpd}_1\Big(* /\!\!/ K, \mathbf{H}_{\cup}\big(U,X\big) /\!\!/ G\Big) \\
& \simeq \mathrm{Grpd}_1\Big(* /\!\!/ K, \coprod_c \big(* /\!\!/ H_c\big)\Big) \\
& \simeq \coprod_c \mathrm{Grpd}_1\Big(* /\!\!/ K, * /\!\!/ H_c\Big) \\
& \simeq \coprod_c \mathrm{Snglrt}\Big(\overset{K}{\curlyvee}, \overset{H_c}{\curlyvee}\Big).
\end{aligned} \tag{5.23}$$

Here the first step is (5.20), the second step uses that the delooping groupoids $*/\!\!/K$ are connected and the last step observes the definition of Snglrt (Def. 4.2.1). By the ∞-Yoneda embedding (Prop. 3.1.37) over the site of Snglrt (4.134) this means that

$$\mathscr{X}(U) \simeq \coprod_c \overset{H_c}{\prec} \;\in\; \mathrm{Shv}_\infty\big(\mathrm{Snglrt}\big)\,. \tag{5.24}$$

With this, we find that $\mathrm{Snglr}(\mathscr{X}) \in \mathrm{PreSheaves}_\infty(\mathrm{Chrt})$ is given by

$$\begin{aligned} \mathrm{Snglr}\big(\mathscr{X}\big) \;:\; U \longmapsto\;\; & \mathrm{Snglr}\big(\mathscr{X}(U)\big) \\ \simeq\; & \mathrm{Snglr}\Big(\coprod_c \overset{H_c}{\prec}\Big) \\ \simeq\; & \coprod_c \mathrm{Snglr}\Big(\overset{H_c}{\prec}\Big) \\ \simeq\; & \coprod_c * \\ \simeq\; & \pi_0\Big(\mathbf{H}_{\cup}\big(U,X\big)/\!\!/G\Big) \\ \simeq\; & \mathbf{H}_{\cup}\big(U,X\big)/G\,. \end{aligned} \tag{5.25}$$

Here the first line is the object-wise application of Snglr (Remark 4.2.9), while the next line is (5.24). From there we use that Snglr, being a left adjoint, preserves coproducts (Prop. 3.1.26) and then that it takes the elementary singularies to points, by Lemma 4.2.16. Finally, we identify (5.21). But this resulting assignment is just that of $X/G \in \mathrm{PreSheaves}(\mathrm{Chrt})$:

$$X/G \;:\; U \longmapsto \mathbf{H}(U,X)/G \tag{5.26}$$

and hence the claim follows. □

5.1.2 Examples of Cohesive G-orbispaces

We make explicit two classes of examples of cohesive G-orbispaces (Def. 5.1.4): Fréchet-smooth orbispaces and topological orbispaces.

Example 5.1.10 (Fréchet smooth G-orbispaces)**.** Consider

$$X \in \mathrm{FréchetManifolds} \hookrightarrow \mathrm{SmthGrpd}_\infty \tag{5.27}$$

a (possibly infinite-dimensional Fréchet-)smooth manifold regarded as a 0-truncated concrete smooth ∞-groupoid (4.52). Given a $G \in \mathrm{Grp}(\mathbf{H})$ (4.154) being discrete $G \simeq \flat G$, a smooth action ρ of G on X is equivalently a homotopy fiber sequence in $\mathrm{SmthGrpd}_\infty$ of this form (Prop. 3.2.6):

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{fib}(\rho)} & X/\!\!/G \\ & & \downarrow{\scriptstyle\rho} \\ & & */\!\!/G \end{array}\,. \tag{5.28}$$

Here the homotopy quotient (3.130)

$$X/\!\!/G \in \mathrm{LieGroupoids} \hookrightarrow \mathrm{SmthGrpd}_\infty \tag{5.29}$$

is the corresponding (possibly infinite-dimensional Fréchet-)Lie groupoid, regarded as a smooth ∞-groupoid via the embedding (4.53). Its orbi-singularization (4.143) is a G-orbi-singular space, in the sense of Def. 5.1.4, in the ∞-topos $\mathrm{SnglrSmthGrpd}_\infty$ (4.148):

$$\begin{array}{c}\mathscr{X}\\ \downarrow\\ \prec^{G}\end{array} \;:=\; \mathrm{OrbSnglr}\begin{pmatrix} X /\!\!/ G \\ \downarrow \\ * /\!\!/ G \end{pmatrix}. \tag{5.30}$$

This orbi-singular smooth groupoid (5.30) what we suggest is the proper incarnation of the quotient orbifold that is presented by the smooth manifold X with its G-action. Notice that (see *Figure G*):

(i) its *purely smooth aspect* is the Lie groupoid

$$\cup(\mathscr{X}) \simeq X /\!\!/ G \in \mathrm{LieGroupoids} \hookrightarrow \mathrm{SnglrSmthGrpd}_\infty, \tag{5.31}$$

(by Prop. 5.1.6) which is the incarnation of this orbifold, according to [MP97][PS10]

(ii) its *purely singular aspect* is the diffeological space

$$\vee(\mathscr{X}) \simeq X/G \in \mathrm{DiffeologicalSpaces} \hookrightarrow \mathrm{SingularSmoothGroupoid}_\infty \tag{5.32}$$

(by Prop. 5.1.9) which is the incarnation of this orbifold, according to [IKZ10].

However, it is only the full orbi-singular object $\mathscr{X}$ which is structured enough to have proper (Bredon-)equivariant cohomology. This is the content of Theorem 6.1.9 below.

Example 5.1.11 (Topological G-orbispaces)**.** For G a finite group, let $G \circlearrowright X_{\mathrm{top}}$ be a topological G-space (Def 2.2.1) with Borel construction

$$\begin{array}{ccc} X_{\mathrm{top}} & \longrightarrow & X \underset{G}{\times} EG \\ & & \downarrow \\ & & BG \end{array} \tag{5.33}$$

Via its continuous diffeology (3.12), this is equivalently a 0-truncated (and concrete) object in $\mathbf{H}_{\cup} := \mathrm{SmthGrpd}_\infty$ (Ex. 4.1.18)

$$X := \mathrm{Cdfflg}(X_{\mathrm{top}}) \in \mathbf{H}_{\cup,0} \tag{5.34}$$

equipped with a smooth G-action (Prop. 3.2.6)

$$\begin{array}{ccc} X & \longrightarrow & X /\!\!/ G \\ & & \downarrow \\ & & * /\!\!/ G. \end{array} \tag{5.35}$$

The orbi-singularization (4.143) of the corresponding homotopy quotient is a G-orbi-singular space (Def. 5.1.4)

$$\begin{array}{c}\mathscr{X}\\ \downarrow\\ \prec^{G}\end{array} \;:=\; \mathrm{OrbSnglr}\begin{pmatrix} \mathrm{Cdfflg}(X_{\mathrm{top}}) /\!\!/ G \\ \downarrow \\ * /\!\!/ G \end{pmatrix}. \tag{5.36}$$

Proposition 5.1.12 (Shape of good orbifolds)**.** *Consider a finite-dimensional smooth G-orbifold, as in Ex. 5.1.10 (a good orbifold, Remark 5.1.3)*

$$\mathscr{X} := \mathrm{OrbSnglr}\big(X /\!\!/ G\big). \tag{5.37}$$

Then its cohesive shape (4.143) $\mathrm{Shp}\big(\mathscr{X}\big) \in \mathrm{Shv}_\infty\big(\mathrm{Snglrt}\big)$ *is, over any singularity* $\prec^{K}$ (4.135)*, the topological shape* (3.1.13) *of the G-Borel construction on the disjoint union of all K-fixed subspaces* $X_{\mathrm{top}}^{\phi(K)} \subset X_{\mathrm{top}}$ (2.15) *in the underlying* (3.12) *D-topological G-space (Def. 2.2.1):*

$$\mathrm{Shp}\big(\mathscr{X}\big) \;:\; \prec^{K} \longmapsto \quad \mathrm{Shp}_{\mathrm{Top}}\Bigg(\bigg(\coprod_{\phi \in \mathrm{Grp}(K,G)} \big(\mathrm{Dtplg}(X)\big)^{\phi(K)}\bigg) \underset{G}{\times} EG\Bigg). \tag{5.38}$$

Proof. With Lemma 5.1.7, the task is reduced to showing that, for $\phi(K) \subset G$ any specified subgroup, we have an equivalence

$$\mathrm{Shp}\big(X^{\phi(K)}\big) \simeq \mathrm{Shp}_{\mathrm{Top}}\big((\mathrm{Dtplg}(X))^{\phi(K)}\big) \quad \in \mathrm{Grpd}_\infty \tag{5.39}$$

between the cohesive shape (4.1) of the orbi-singular homotopy quotient of X by G and the ordinary topological shape (3.1.13) of the D-topological space underlying X. But this is (4.56) in Ex. 4.1.18, given by [Sc13, 4.3.29]. □

Proposition 5.1.13 (Shape of topological G-orbi spaces)**.** *Consider the topological G-orbi-singular space, as in Ex. 5.1.11,*

$$\mathscr{X} := \mathrm{OrbSnglr}\big(\mathrm{Cdfflg}(X_{\mathrm{top}}) /\!\!/ G\big). \tag{5.40}$$

Then its cohesive shape (4.143) $\mathrm{Shp}\big(\mathscr{X}\big) \in \mathrm{Shv}_\infty\big(\mathrm{Snglrt}\big)$ *is, over any singularity* $\prec^{K}$ (4.135)*, the topological space* (3.1.13) *of the G-Borel construction on the disjoint union of all K-fixed subspaces* $X_{\mathrm{top}}^{\phi(K)} \subset X_{\mathrm{top}}$ (2.15)*:*

$$\mathrm{Shp}\big(\mathscr{X}\big) \;:\; \prec^{K} \longmapsto \quad \mathrm{Shp}_{\mathrm{Top}}\Bigg(\bigg(\coprod_{\phi \in \mathrm{Grp}(K,G)} X_{\mathrm{top}}^{\phi(K)}\bigg) \underset{G}{\times} EG\Bigg). \tag{5.41}$$

Proof. With Lemma 5.1.7, the task is reduced to showing that, for $\phi(K) \subset G$ any specified subgroup, we have an equivalence

$$\mathrm{Shp}\big(\mathrm{Cdfflg}(X_{\mathrm{top}})^{\phi(K)}\big) \simeq \mathrm{Shp}_{\mathrm{Top}}\big(X_{\mathrm{top}}^{\phi(K)}\big) \quad \in \mathrm{Grpd}_\infty \tag{5.42}$$

between the cohesive shape (4.1) of the orbi-singular homotopy quotient by G of the continuous-diffeological space and the ordinary topological shape (3.1.13) But this is item (4.55) in Ex. 4.1.18, given by combining the result (4.54) of [BEBP19][Bunk20, §3] with Prop. 3.1.20 from [CW14]. □

5.2 Orbifolds

We introduce a general theory of orbi-singular spaces, whose underlying smooth cohesive groupoid is locally diffeomorphic to a fixed local model space V. Since, for $V = \mathbb{R}^n \in \mathrm{JetSmthGrpd}_\infty$, these are ordinary n-folds (i.e., ordinary n-dimensional manifolds for any n, see Ex. 5.2.4), or, more generally, étale ∞-groupoids with atlases by n-folds (Ex. 5.2.5), including ordinary orbifolds, we generally speak of *V-folds*, with a hat tip to [Sa56]. Externally these are V-étale ∞-stacks (Remark 5.2.2) but their theory internal to the ambient elastic ∞-topos (such as the construction of their frame bundles in Prop. 5.2.13) is elegant and finitary and lends itself to full formalization in homotopy type theory [Ch24] (see p. 9). The *proper* incarnation (see Remark 5.2.47) of these V-folds as orbifolds is via their orbi-singularization (Def. 5.2.45, Remark 5.2.47).

5.2.1 V-folds and V-étale groupoids

Definition 5.2.1 (V-folds)**.** Let $\mathbf{H}$ be an elastic ∞-topos $\mathbf{H}$ (Def. 4.1.21).

(i) Given $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1), we say that an object $X \in \mathbf{H}$ is a *V-fold* if there exists a correspondence between V and X

$$\begin{array}{ccccc} & & U & & \\ & \overset{\text{ét}}{\swarrow} & & \overset{\text{ét}}{\searrow} & \\ V & & & & X \end{array} \tag{5.43}$$

such that

(a) both morphisms are local diffeomorphisms (Def. 4.1.26) and

(b) the right one is, in addition, an effective epimorphism (Def. 3.1.63), then called a *V-atlas of X* (3.118).

(ii) We write

$$V\mathrm{Folds}(\mathbf{H}) \subset \mathbf{H} \tag{5.44}$$

for the full sub-∞-category of V-folds in $\mathbf{H}$ and we write

$$V\mathrm{Folds}(\mathbf{H})^{\text{ét}} \subset \mathbf{H} \tag{5.45}$$

for its wide subcategory on those morphisms which are local diffeomorphisms (Def. 4.1.26).

Remark 5.2.2 (V-folds and V-étale groupoids)**.** By Prop. 4.1.36, a V-fold (Def. 5.2.1) is a stack (3.118) whose choice of V-atlas (5.43) realizes it as an étale groupoid (Def. 4.1.35) with space of objects locally diffeomorphic over V:

$$\begin{array}{cccccl} & \vdots & & \vdots & & \\ & U \times_X U & \simeq & U_1 & & \text{“}V\text{-étale groupoid”} \\ & {\scriptstyle \mathrm{pr}_1}\downarrow \uparrow{\scriptstyle \Delta} \downarrow{\scriptstyle \mathrm{pr}_2} & & {\scriptstyle s}\downarrow \uparrow{\scriptstyle e} \downarrow{\scriptstyle t} & & \\ V \xleftarrow{\text{ét}} & U & = & U_0 & & \\ & {\scriptstyle a}\downarrow{\scriptstyle \text{ét}} & & \downarrow & & \text{“}V\text{-atlas”} \\ & X & \simeq & \varinjlim U_\bullet & & \text{“}V\text{-fold”} \end{array} \tag{5.46}$$

Example 5.2.3 (V is a V-fold)**.** Let $\mathbf{H}$ be an elastic ∞-topos $\mathbf{H}$ (Def. 4.1.21) and $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1). Then the underlying object $V \in \mathbf{H}$ itself is a V-fold (Def. 5.2.1): A V-atlas (5.43) is given by the identity morphisms

$$V \xleftarrow[\text{ét}]{\mathrm{id}} V \xrightarrow[\text{ét}]{\mathrm{id}} V\,. \tag{5.47}$$

Example 5.2.4 (Smooth manifolds are $\mathbb{R}^n$-folds)**.** For $k \in \mathbb{N}$ with $k \geq 1$, let $\mathbf{H} = k\mathrm{JetSmthGrpd}_\infty$ (Ex. 4.1.24). Then, for every $n \in \mathbb{N}$, the object

$$V := \mathbb{R}^n \in \mathrm{CrtSpc} \hookrightarrow k\mathrm{JetSmthGrpd}_\infty \tag{5.48}$$

canonically carries the structure of a group object $(\mathbb{R}^n, +) \in \mathrm{Grp}(\mathbf{H})$, via addition in $\mathbb{R}^n$ regarded as a vector space. Now every smooth manifold

$$X \in \mathrm{SmthMfd} \hookrightarrow k\mathrm{JetSmthGrpd}_\infty \tag{5.49}$$

of dimension n is a V-fold, hence an $\mathbb{R}^n$-fold in the sense of Def. 5.2.1: For any choice of atlas in the traditional sense of manifold theory, namely an open cover

$$\Big\{ U_j \xrightarrow{\phi_i} X \Big\}_{j \in J} \tag{5.50}$$

by local diffeomorphisms ϕ_i from open subsets of Cartesian space

$$U_j \overset{\iota_j}{\hookrightarrow} \mathbb{R}^n\,, \tag{5.51}$$

a V-atlas (5.43) is obtained by setting:

$$\mathbb{R}^n \xleftarrow[\text{ét}]{(\iota_j)_{j\in J}} \coprod_{j \in J} U_j \xrightarrow[\text{ét}]{(\phi_j)_{j \in J}} X \tag{5.52}$$

Example 5.2.5 (Differentiable étale stacks are $\mathbb{R}^n$-folds [Sc13, Prop. 4.5.56])**.** Let $\mathbf{H} = \mathrm{JetSmthGrpd}_\infty$ (Ex. 4.1.24) and take $V = (\mathbb{R}^n, +)$ as in Ex. 5.2.4. Then a diffeological groupoid $X \in \mathbf{H}$ (4.72) is a V-fold (Def. 5.2.1) for $V = \mathbb{R}^n$ (5.48) if it is an n-dimensional *differentiable étale stack* in that:

(i) it admits an atlas (effective epimorphism) $X_0 \twoheadrightarrow X$ from a smooth n-manifold X_0 (via (4.52) and (4.71))

(ii) its source and target morphisms with respect to this atlas are local diffeomorphisms.

Generally, a smooth ∞-groupoid presented by a Kan simplicial smooth manifold is an $\mathbb{R}^n$-fold in the sense of Def. 5.2.1 if it presents an *étale ∞-groupoid* in that all its simplicial face maps are local diffeomorphisms.

Examples 5.2.6 (Super-manifolds are $\mathbb{R}^{n|q}$-folds)**.** Let $\mathbf{H} = \infty\mathrm{JetSuperGrpd}_\infty$ (Ex. 4.1.24). Then, for every $n, q \in \mathbb{N}$, the super-Cartesian space (Def. 4.1.41)

$$V := \mathbb{R}^{n|q} \in \infty\mathrm{JetSuperCrtSp} \hookrightarrow \infty\mathrm{JetSuperGrpd}_\infty \tag{5.53}$$

carries the structure of a group object, whose bosonic aspect (4.114) is (5.48). The corresponding V-folds (Def. 5.2.1) are the $(n|q)$-dimensional supermanifolds (4.125).

Example 5.2.7 (General super étale ∞-stacks)**.** Let $\mathbf{H} = \infty\mathrm{JetSuperGrpd}_\infty$ (Ex. **??**). Then for any $V \in \mathrm{Grp}(\mathbf{H})$ the corresponding V-étale ∞-stacks (Remark 5.2.2) realize a flavor of *super étale ∞-stacks*, locally modeled on V. Lemma 4.1.45 implies that, generally, the bosonic part $\overset{\rightsquigarrow}{X}$ of a super étale ∞-stack is a bosonic étale ∞-stack locally modeled on the bosonic part $\overset{\rightsquigarrow}{V}$ of V:

$$
\begin{array}{cccc}
 & V\mathrm{Folds}(\mathbf{H}) & \xrightarrow{\;\rightsquigarrow\;} & \overset{\rightsquigarrow}{V}\mathrm{Folds}(\mathbf{H}) & \\
\text{supergeometric étale } \infty\text{-stack} & X & \longmapsto & \overset{\rightsquigarrow}{X} & \text{underlying bosonic étale } \infty\text{-stack}
\end{array}
\tag{5.54}
$$

5.2.2 Quotients of V-folds

Proposition 5.2.8 (Orbifolding of a V-fold is a V-fold)**.** *Let* $\mathbf{H}$ *be an elastic ∞-topos* $\mathbf{H}$ *(Def. 4.1.21),* $V, G \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1) with* $G \simeq \flat G$ *discrete, and* $(X,\rho) \in G\mathrm{Actions}(\mathbf{H})$ *(Prop. 3.2.6). Then if* X *is a* V*-fold (Def. 5.2.1) so is its homotopy quotient* $X /\!\!/ G$ (3.131)*. Specifically, if* $U \xrightarrow{\text{ét}} X$ *is a* V*-atlas for* X (5.43)*, then a* V*-atlas for* $V /\!\!/ G$ *is given by composition with the homotopy fiber inclusion map* $\mathrm{fib}(\rho)$ (3.130)*:*

$$
V \xleftarrow{\text{ét}} U \xrightarrow{\text{ét}} X \xrightarrow{\mathrm{fib}(\rho)} X /\!\!/ G\,.
\tag{5.55}
$$

Proof. We need to show that the composite morphism on the right of (5.55) is **(a)** an effective epimorphism and **(b)** a local diffeomorphism. Since both of these classes of morphisms are closed under composition (Lemma 3.1.65 and Lemma 4.1.27), it is sufficient to show that $\mathrm{fib}(\rho)$ itself has these two properties.

For **(a)** observe that, by definition of homotopy fibers (3.130), we have a Cartesian square

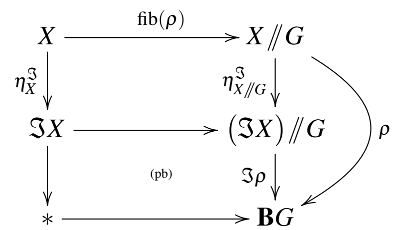
$$
\begin{array}{ccc}
X & \xrightarrow{\mathrm{fib}(\rho)} & X /\!\!/ G \\
\downarrow & \text{(pb)} & \downarrow{\scriptstyle \rho} \\
* & \longrightarrow & \mathbf{B}G
\end{array}
\tag{5.56}
$$

Here the bottom morphism is an effective epimorphism (Ex. 3.2.2). Since these are preserved by homotopy pullback, also $\mathrm{fib}(\rho)$ is an effective epimorphism.

For **(b)** consider the image of this square (5.56) under $\Im$. Since $\Im$ is both a right and a left adjoint it preserves Cartesian squares and homotopy quotients (by Prop. 3.1.26), while it preserves discrete objects by elasticity (4.59) and idempotency (Prop. 3.1.28, Prop. 3.1.29). Therefore

$$
\begin{array}{ccc}
\Im X & \xrightarrow{\Im\mathrm{fib}(\rho) \simeq \mathrm{fib}(\Im\rho)} & (\Im X) /\!\!/ G \\
\downarrow & \text{(pb)} & \downarrow{\scriptstyle \Im\rho} \\
* & \longrightarrow & \mathbf{B}G
\end{array}
\tag{5.57}
$$

is Cartesian. Consider finally the pasting composite of this second square (5.57) with the naturality square of $\eta^{\Im}$ on $\mathrm{fib}(\rho)$:



(5.58)

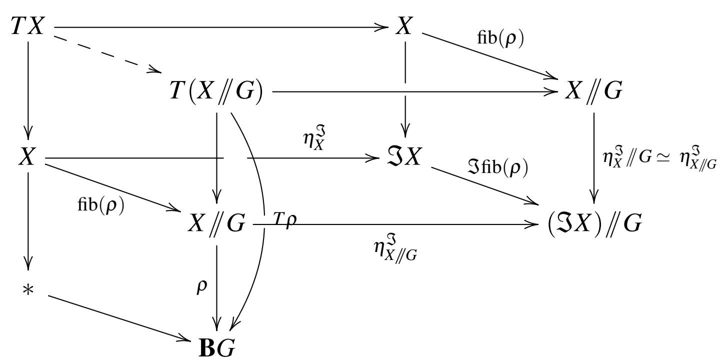
Here the composite morphism on the right is equivalent to ρ, as shown, by the naturality of $\eta^{\Im}$ and using that the object $\mathbf{B}G$, being discrete, is $\Im$-modal. Therefore, the total outer rectangle of (5.58) is Cartesian by (5.56). Moreover, the bottom square of (5.58) is Cartesian by (5.57). Therefore the pasting law (Prop. 3.1.23) implies that the top square of (5.58) is Cartesian. But this means (4.76) that $\mathrm{fib}(\rho)$ is a local diffeomorphism. □

Proposition 5.2.9 (Induced G-action on the tangent bundle)**.** *Let* $\mathbf{H}$ *be an elastic* ∞*-topos* $\mathbf{H}$ *(Def. 4.1.21),* $V, G \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1), with* $G \simeq \flat G$ *discrete,* $(X, \rho) \in G\mathrm{Actions}(\mathbf{H})$ *(Prop. 3.2.6) and* $X \in V\mathrm{Folds}(X)$ *(Def. 5.2.1). Then the tangent bundle* TX *(Def. 4.1.29) carries an essentially unique* G*-action* $T\rho$ *such that:*

- **(i)** *the defining projection* $TX \to X$ *is* G*-equivariant (Def. 3.2.10);*
- **(ii)** *the homotopy quotient of* TX *is the tangent bundle of the orbifolded* V*-fold* $X /\!\!/ G$ *(Prop. 5.2.8):*

$$(TX)/\!\!/G \simeq T(X/\!\!/G) \quad \in \mathbf{H}_{/X/\!\!/G}. \tag{5.59}$$

Proof. Consider the following diagram:



(5.60)

Here the bottom left square is that characterizing the G-action on X, by (3.130); while the bottom and right squares are both the naturality square of $\eta^{\Im}$ on the morphism $\mathrm{fib}(\rho)$ (where we use that $\Im$ commutes with taking the homotopy quotient by the discrete group G). Now observe that:

- **(a)** The bottom and right squares are pullback squares since $\mathrm{fib}(\rho)$ is a local diffeomorphism (Def. 4.1.26) by Prop. 5.2.8.

(b) The front and back squares are pullback squares by the definition of tangent bundles (Def. 4.1.29).

In particular, the solid part of the diagram is homotopy-commutative, so that, by the universal property of the front pullback square, the dashed morphism exists, essentially uniquely, such as to make the top and the top left square homotopy-commutative. Further observe, by repeatedly applying the pasting law (Prop. 3.1.23), that:

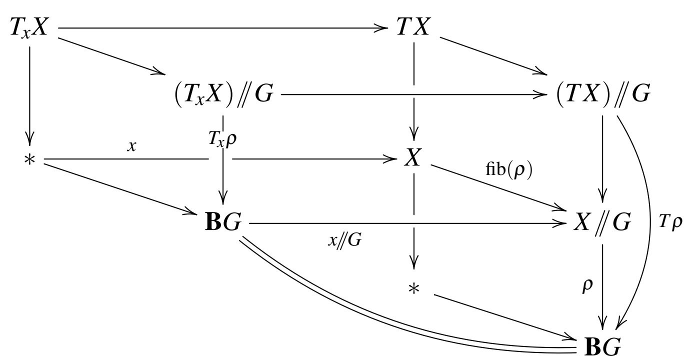
(c) The top left square is a homotopy pullback since the back, right and front squares are pullbacks by (a) and (b).

(d) The total left rectangle is a pullback, since the top one is so, by (c), and the bottom one is so, by the action property (3.130).

Thus, again by the action property (3.130), the total left rectangle exhibits a G-action on TX whose homotopy quotient is as claimed (5.59), and its factorization into two pullback squares as shown exhibits the projection $TX \to X$ as a homomorphism of G-actions, hence as being G-equivariant (Def. 3.2.10). ☐

Proposition 5.2.10 (Induced G-action on local neighborhood of fixed point)**.** *Let* $\mathbf{H}$ *be an elastic* ∞*-topos* $\mathbf{H}$ *(Def. 4.1.21),* $V, G \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1), with* $G \simeq \flat G$ *discrete,* $(X, \rho) \in G\mathrm{Actions}(\mathbf{H})$ *(Prop. 3.2.6) with* $X \in V\mathrm{Folds}(X)$ *(Def. 5.2.1) and* $* \xrightarrow{x} X$ *a homotopy fixed point (Def. 3.2.24). Then the induced G-action* $T\rho$ *on the tangent bundle* TX*, from Prop. 5.2.9, restricts to a G-action* $T_x\rho$ *on the local neighborhood* T_xX *(Ex. 4.1.30) of the homotopy fixed point x.*

Proof. Consider the following diagram:



(5.61)

Here the squares on the right are from (5.60) and are thus both homtopy Cartesian. The rear square is the homotopy pullback square defining the tangent fiber, and we define the front square to be a homotopy pullback, giving us the object denoted $(T_xX)/\!\!/G$. We need to show that this object really is the homotopy quotient of the restricted action. But the bottom horizontal square homotopy-commutes, exhibiting the homotopy fixed point by (3.161), so that, by applying the pasting law (Prop. 3.1.23) to the top vertical squares, it follows that also the top left square is Cartesian. This already identifies $(T_xX)/\!\!/G$ as the homotopy quotient of some G-action on T_xX, by Prop. 3.2.6. To see that this is indeed the restricted action, observe that the front triangle commutes, again by (3.161), so that the total diagram exhibits the fiber inclusion $T_x X \to TX$ as being a homomorphism G-actions $T_x\rho \to T_\rho$ (by Prop. 3.2.6). □

5.2.3 Frame bundles

Definition 5.2.11 (Structure group of V-folds)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21) and $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1), to be regarded as the local model space of V-folds (Def. 5.2.1). Then:

(i) We say that the automorphism group (Def. 3.2.13) of the local neighborhood (Ex. 4.1.30) of the neutral element $* \xrightarrow{e} V$ (Ex. 3.2.3)

$$\mathrm{Aut}(T_e V) \;\; \in \mathrm{Grp}(T_e V) \tag{5.62}$$

is the *structure group of V-folds*.

(ii) We write

$$(T_e V,\, \rho_{\mathrm{Aut}}) \in \mathrm{Aut}(T_e V)\mathrm{Actions}(\mathbf{H}) \tag{5.63}$$

for its canonical action (3.144).

Example 5.2.12 (Ordinary general linear group)**.** Let $\mathbf{H} = \mathrm{JetSmthGrpd}_\infty$ (Ex. 4.1.24) and let

$$V := (\mathbb{R}^n, +) \in \;\; \mathrm{Grp}(\mathrm{SmthMfd}) \hookrightarrow \mathrm{Grp}(\mathbf{H}) \tag{5.64}$$

via the full inclusion (4.72), with $\mathbb{R}^n$ regarded as a group under addition of tuples of real numbers. Then the structure group of $\mathbb{R}^n$-folds, according to Def. 5.2.11, is the traditional general linear group, regarded as a Lie group:

$$\mathrm{Aut}(T_0 \mathbb{R}^n) \;\simeq\; \mathrm{GL}(n)\,. \tag{5.65}$$

Proposition 5.2.13 (Frame bundle)**.** *Let $\mathbf{H}$ be an elastic ∞-topos $\mathbf{H}$ (Def. 4.1.21), $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1) and $X \in \mathbf{H}$ a V-fold (Def. 5.2.1). Then the tangent bundle of X (Def. 4.1.29) is a fiber bundle (Def. 3.2.18) with typical fiber the local neighborhood $T_e V$ (Def. 4.1.28) of the neutral element $* \xrightarrow{e} V$, hence is the associated bundle of an $\mathrm{Aut}(T_e V)$-principal (5.62) bundle (Prop. 3.2.15), to be called the* frame bundle *of X:*

$$\begin{array}{ccc} \overset{\text{tangent bundle}}{TX} & \longrightarrow & (T_e V) /\!\!/ \mathrm{Aut}(T_e V) \\ \big\downarrow & \text{(pb)} & \big\downarrow \\ X & \xrightarrow[\vdash \mathrm{Frm}(X)]{} & \mathbf{B}\mathrm{Aut}(T_e V) \end{array}$$

$$\begin{array}{ccc} \overset{\text{frame bundle}}{\mathrm{Frm}(X)} & \longrightarrow & * \\ \big\downarrow & \text{(pb)} & \big\downarrow \\ X & \xrightarrow[\vdash \mathrm{Frm}(X)]{} & \underset{\text{structure group}}{\mathbf{B}\mathrm{Aut}(T_e V)}\,. \end{array} \tag{5.66}$$

Proof. By Prop. 4.1.31 the tangent bundles over any V-atlas (5.43) for X form two Cartesian squares as follows:

$$
\begin{array}{ccccc}
 & & TU & & \\
 & \swarrow & & \searrow & \\
V \times T_e V \simeq TV & & \downarrow & & TX \\
\downarrow & (\mathrm{pb}) & U & (\mathrm{pb}) & \downarrow \\
 & \swarrow_{\text{ét}} & & \searrow^{\text{ét}} & \\
V & & & & X
\end{array}
\tag{5.67}
$$

Moreover, by Prop. 5.2.19 the tangent bundle of V is trivial, as shown on the left. Since Cartesian products are preserved by homotopy pullback, the left square implies that also $TU \simeq U \times T_e V$ is trivial. But with this the existence of the right square is the defining characterization for TX to be a $T_e V$-fiber bundle. □

Remark 5.2.14 (Frame bundles are well-defined)**.** The frame bundle (Def. 5.2.13) of a V-fold (Def. 5.2.1) is independent, up to a contractible space of equivalences, of the choice of V-atlas (5.43) in the construction (5.67): This follows as a special case of the essential independence of classifying maps of fiber bundles from the choice of trivializing cover, as in Prop. 3.2.19, using that not only the class of effective epimorphisms but also that of local diffeomorphisms is closed under pullback and composition (Lemma 4.1.27).

Proposition 5.2.15 (V-fold is $\mathrm{Aut}(T_e V)$-quotient of its frame bundle)**.** *Let* $\mathbf{H}$ *be an elastic* ∞*-topos* $\mathbf{H}$ *(Def. 4.1.21),* $V \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1) and* $X \in V\mathrm{Folds}(\mathbf{H})$ *(Def. 5.2.1). Then* X *is equivalent to the homotopy quotient* (3.131) *of its own frame bundle (Prop. 5.2.13) by* $\mathrm{Aut}(T_e V)$*:*

$$X \;\simeq\; \mathrm{Frm}(X) /\!\!/ \mathrm{Aut}(T_e V)\,. \tag{5.68}$$

Proof. This is immediate from the equivalence between principal bundles and homotopy qotient projections (Remark 3.2.16) applied to the frame bundle (5.66). □

Example 5.2.16 (Frame bundles on smooth manifolds)**.** Let $\mathbf{H} = \mathrm{JetSmthGrpd}_\infty$ (Ex. 4.1.24) and $X \in \mathrm{SmthMfd} \hookrightarrow \mathbf{H}$ a smooth manifold (4.72) regarded as an $\mathbb{R}^n$-fold according to Ex. 5.2.4.

- **(i)** Then its frame bundle, according to Prop. 5.2.13, is the $\mathrm{GL}(n)$-principal bundle on X which is the frame bundle in the traditional sense of differential geometry.
- **(ii)** For the same manifold but regarded in $\mathbf{H} = k\mathrm{JetSmthGrpd}_\infty$ with $k \geq 1$ we instead get the corresponding jet version of the frame bundle (see e.g. [KMS93, 12.12]).

5.2.4 Framed V-folds

Definition 5.2.17 (Framing)**.** Let $\mathbf{H}$ be an elastic ∞-topos $\mathbf{H}$ (Def. 4.1.21). A *framing* of an objext $X \in \mathbf{H}$ is a trivialization of its tangent bundle Def. 4.1.29, hence an equivalence

$$TX \;\simeq\; X \times T_x X \;\; \in \mathbf{H}_{/X} \tag{5.69}$$

for $* \xrightarrow{x} X$ any point.

Remark 5.2.18 (Framing on a V-fold)**.** If X is a V-fold (Def. 5.2.1) then a *framing* on V in the sense of Def. 5.2.17 is equivalent, by Prop. 5.2.13, to a trivialization of the frame bundle, hence to a trivialization of its classifying map (5.66):

$$
\begin{array}{ccc}
TX & \xrightarrow[\simeq]{\ \mathrm{fr}\ } & X \times T_e V \\
& \searrow \quad \swarrow & \\
& X &
\end{array}
\quad \Leftrightarrow \quad
\begin{array}{ccccc}
 & & * & & \\
 & \nearrow & \simeq \Uparrow \vdash \mathrm{fr} & \searrow & \\
X & & \xrightarrow[\vdash \mathrm{Frm}(X)]{} & & \mathbf{B}\mathrm{Aut}(T_e V)
\end{array}
. \tag{5.70}
$$

Proposition 5.2.19 (Groups carry canonical framings by left-translation)**.** *In an elastic ∞-topos $\mathbf{H}$ (Def. 4.1.21) every group object $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1) carries a canonical framing (Def. 5.2.17), which we call the framing by left translation:*

$$
TV \xrightarrow[\simeq]{\ \mathrm{fr}_\ell\ } V \times T_e V \qquad \in \mathbf{H}_{/V} . \tag{5.71}
$$

Proof. Since $\Im$ preserves group structure (as in Prop. 4.1.4), the defining homotopy fiber product of the tangent bundle of V (4.83) sits in a Mayer-Vietoris sequence (Prop. 3.2.5) as shown in the first square of the following:

$$
\begin{array}{ccccc}
TV & \longrightarrow & \longrightarrow & \longrightarrow & * \\
\downarrow & & \text{(pb)} & & \downarrow \vdash e \\
V \times V & \xrightarrow[(\eta^{\Im}_V, \eta^{\Im}_V) = \eta^{\Im}_{V \times V}]{} & \Im V \times \Im V & \xrightarrow[(-)\cdot(-)^{-1}]{} & \Im V
\end{array}
$$
$$
\simeq \quad
\begin{array}{ccccc}
TV & \longrightarrow & T_e V & \longrightarrow & * \\
\downarrow & \text{(pb)} & \downarrow & \text{(pb)} & \downarrow \vdash e \\
V \times V & \xrightarrow[(-)\cdot(-)^{-1}]{} & V & \xrightarrow[\eta^{\Im}_V]{} & \Im V
\end{array}
\tag{5.72}
$$

Using that $\Im$ preserves products (by Prop. 3.1.26) and using the naturality of its unit transformation $\eta^{\Im}$ (3.44), this Cartesian square on top is equivalent to the total rectangle shown at the bottom. By the pasting law (Prop. 3.1.23), this is the pasting of two Cartesian squares, the right one of which exhibits the local neighborhood $T_e V$ (Def. 4.1.28) as shown. To see what the Cartesian property of the left square on the right says, consider pasting to it the top square appearing in the diagram (3.127) which exhibits the group division $(-) \cdot (-)^{-1}$ in Ex. 3.2.4:

$$
\begin{array}{ccc}
TV & \longrightarrow & T_e V \\
\downarrow & \text{(pb)} & \downarrow \\
V \times V & \xrightarrow{(-)\cdot(-)^{-1}} & V \\
{\scriptstyle \mathrm{pr}_1} \downarrow & \text{(pb)} & \downarrow \\
V & \longrightarrow & *
\end{array}
\tag{5.73}
$$

Since both squares are Cartesian, the pasting law (Prop. 3.1.23) says that the total rectangle is Cartesian. This is the equivalence (5.71). □

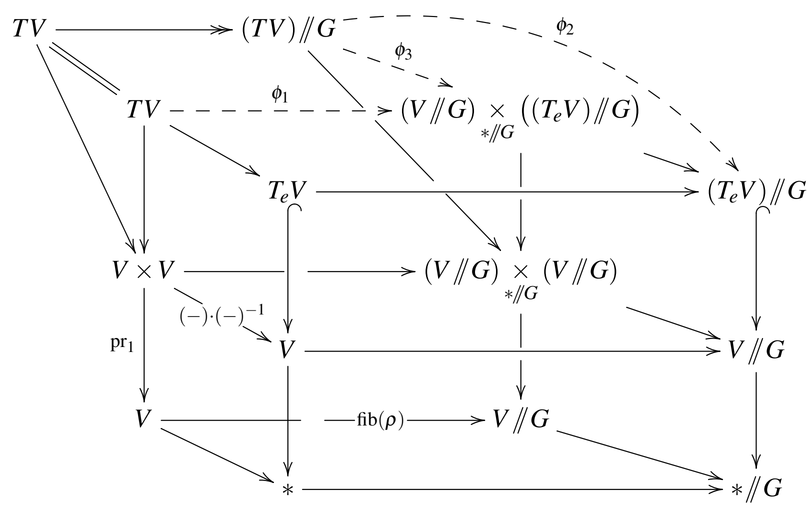
Proposition 5.2.20 (Canonical framing on group is equivariant under group automorphisms)**.** *Consider an elastic ∞-topos $\mathbf{H}$ (Def. 4.1.21), $V, G \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1).*

with 0-truncated $V \simeq \tau_0 V$ and $(V, \rho_G) \in G\mathrm{Actions}(\mathbf{H})$ (Prop. 3.2.6) acting by group-automorphisms (Prop. 3.2.29) hence by restriction $\rho_G = \mathbf{B}i^* \rho_{\mathrm{Aut}_{\mathrm{Grp}}}$ (Prop. 3.2.12) along a group homomorphism $G \xrightarrow{\;i\;} \mathrm{Aut}_e(V)$, to the group-automorphism group $\mathrm{Aut}_{\mathrm{Grp}}(V)$ (Def. 3.2.28). Then the canonical framing fr_ℓ on V from Prop. 5.2.19 is G-equivariant (Def. 3.2.10), in that it lifts to a morphism of G-actions (Prop. 3.2.6) of the form

$$(TV, T\rho) \xrightarrow{\;\mathrm{fr}_\ell\;} (V, \rho) \times (T_e V, T_e \rho) \quad \in G\mathrm{Actions}(\mathbf{H}), \tag{5.74}$$

where $T\rho$ is the induced action on TV from Prop. 5.2.9, and $T_e\rho$ is the induced action on T_eV from Prop. 5.2.10 (which exists since group-automorphisms of V are in particular pointed automorphisms of V (Def. 3.2.27).

Proof. Consider the following diagram:



(5.75)

Here

- the bottom square is the Cartesian square (3.130) which exhibits the action on V,
- the middle horizontal square is the Cartesian square which exhibits the equivariance under group-automorphisms of the group division operatoin (Prop. 3.2.30),
- the total left rectangle is the Cartesian square from (5.73) which exhibits the canonical framing,
- the total front face is the pasting of
  - on the bottom: the Cartesian square (3.130) which exhibits the action on V,
  - on the top: the Cartesian square which is the pasting of the top and the top-right squares in (5.61) equibiting the action on T_eV

and hence is itself Cartesian,

- the bottom and the total right squares are the defining Cartesian squares of the fiber products, and hence, by the pasting law, also their pasting to the total right square is Cartesian,
- the total vertical rear square, with the dashed morphism ϕ_1 on top, is the one thus induced from the universal property of the fiber product, and is itself Cartesian, by the pasting law (Prop. 3.1.23), (using, by the above items, that the left, right and front squares are Cartesian and that the diagram of squares commutes)
- the slanted square in the rear is the pasting of the Cartesian square on the left of (5.60), that exhibits the induced G-action on TV, with the diagonal square on $\mathrm{fib}(\rho)$.

Now observe that inside this big diagram (5.75) we find the following solid homotopy-commutative sub-diagram

$$\begin{array}{ccc} TV & \longrightarrow & (T_eV)/\!\!/G \\ \downarrow & \overset{\phi_2}{\nearrow} & \downarrow \\ (TV)/\!\!/G & \longrightarrow & V/\!\!/G. \end{array} \tag{5.76}$$

Here the left morphism is an effective epimorphism (by Lemma 3.2.7) and the right morphism is (-1)-truncated by the assumption that V is 0-truncated (Lemma xyz). Therefore, the connected/truncated factorization system (Prop. 3.1.66) implies an essentially unique lift ϕ_2, as shown. This, in turn, implies the morphism ϕ_3 in (5.75), again by the universal property of the homotopy fiber product.

Now, since both the slanted as well as the vertical total rear squares are Cartesian, the diagram (5.75) shows that the contravariant base change (Prop. 3.1.49) of ϕ_3 along $\mathrm{fib}(\rho)$ is an equivalence. But since $\mathrm{fib}(\rho)$ is an effective epimorphism (Lemma 3.1.55) , base change along it is conservative (Prop. 3.1.55), and hence it follows that ϕ_3 itself is already an equivalence.

With that identification, the total cube in (5.75) exhibits the G-equivariance of the framing. □

Proposition 5.2.21 (Orbifolding of framed V-folds)**.** *Let* $\mathbf{H}$ *be an elastic* ∞*-topos* $\mathbf{H}$ *(Def. 4.1.21),* $V, G \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1) with* $G \simeq \flat G$ *discrete, and* $(X, \rho_X), (T_eV, \rho_{T_eV}) \in G\mathrm{Actions}(\mathbf{H})$ *(Prop. 3.2.6) for* X *a* V*-fold (Def. 5.2.1) equipped with a framing* fr *(Def. 5.2.17). Then the following are equivalent:*

(i) *The framing is* G*-equivariant (Def. 3.2.10) with respect to the induced action on* TX *(from Prop. 5.2.9) and the product action* $\rho_X \times \rho_{T_eV}$ *on* $X \times T_eV$*, hence lifts to a morphism*

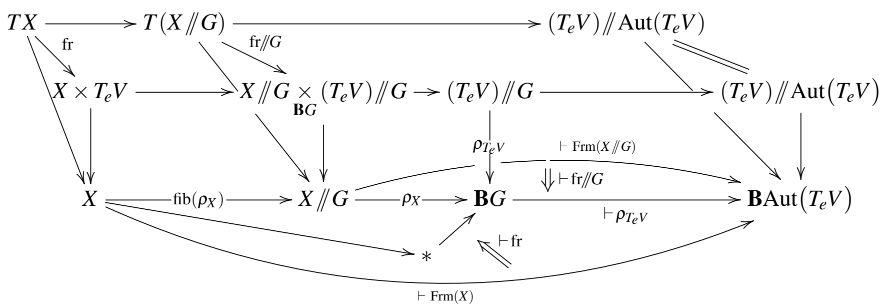
$$(TX, \rho_{TX}) \xrightarrow[\simeq]{\mathrm{fr}} (X, \rho_X) \times (T_eV, \rho_{T_eV}) \in G\mathrm{Actions}(\mathbf{H}) \tag{5.77}$$

(ii) *The classifying map* (5.66) *of the frame bundle (Def. 5.2.13) of the orbifolded*

V-fold $X /\!\!/ G$ (Prop. 5.2.8) factors through $\mathbf{B}G$ as

$$\begin{array}{c} \overset{\vdash \mathrm{Frm}(X /\!\!/ G)}{\Downarrow} \\ X /\!\!/ G \xrightarrow{\ \rho_X\ } \mathbf{B}G \xrightarrow{\ \vdash \rho_{T_e V}\ } \mathbf{B}\mathrm{Aut}(T_e V) \end{array} \tag{5.78}$$

Proof. Consider the following diagram:



(5.79)

Note that here:

(a) The total outer part of the diagram exhibits the given framing fr via its classifying homotopy $\vdash$ fr, according to Remark 5.2.18.

(b) The front squares in the middle and on the right are the pullback squares that defines the diagonal G-action and the classification of the $\rho_{T_e V}$-action respectively. Hence also their pasting composite is a pullback, by the pasting law (Prop. 3.1.23).

(i) First to see that G-equivariance of fr implies the factorization (5.78): By the characterization of G-actions (3.130) G-equivariance of fr means, equivalently, that fr is the morphism on homotopy fibers over $\mathbf{B}G$ induced from an equivalence fr$/\!\!/G$ on homotopy quotients. But, by **(b)** and Prop. 5.2.9, such an equivalence is classified by a homotopy of the form (5.78).

(ii) Now to see that, conversely, the existence of a homotopy "$\vdash$ fr$/\!\!/G$" of the form (5.78) implies the existence of a G-equivariant framing fr (quotation marks now since we yet have to show that the two are related in this way). For this, we have to show that the morphism on homotopy fibers induced by fr$/\!\!/G$ is a framing fr. But, by the nature of the G-action on TX from Prop. 5.2.9, the nature of the diagonal G-action exhibited by the middle front square, and using the pasting law (Prop. 3.1.23), this means to show that the left front and rear squares are homotopy pullbacks. For the front left square this follows by the factorization of $\rho_X \circ \mathrm{fib}(\rho_X)$ through the point, using **(a)**, **(b)** and the pasting law (Prop. 3.1.23). For the rear left square, this follows by Prop. 4.1.31, since $\mathrm{fib}(\rho)$ is a local diffeomorphism by Prop. 5.2.8. □

5.2.5 G-Structures

Definition 5.2.22 (G-Structure coefficients)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21) and $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1). Then a *coefficient for G-structure*

$$(G, \phi) \;\; \in \; \mathrm{Grp}(\mathbf{H})_{/\mathrm{Aut}(T_e V)} \tag{5.80}$$

is a group G equipped with a homomorphism of groups $G \xrightarrow{\phi} \mathrm{Aut}(T_e V)$ to the structure group (Def. 5.2.11) of V-folds. Under delooping (3.121) this is equivalently a morphism in $\mathbf{H}$ of the form $\mathbf{B}G \xrightarrow{\mathbf{B}\phi} \mathbf{B}\mathrm{Aut}(T_e V)$.

Definition 5.2.23 (G-structures on V-folds)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21), $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1), $(G,\phi) \in \mathrm{Grp}(\mathbf{H})_{/\mathrm{Aut}(T_e V)}$ (Def. 5.2.22) and $X \in V\mathrm{Folds}(\mathbf{H})$ (Def. 5.2.1).

(i) We say that

- *a* (G,ϕ)*-structure on* X (often just G*-structure* if ϕ is understood),
- or (G,ϕ)*-structure on its frame bundle* (Def. 5.2.13),
- or *reduction of the structure group* (5.2.11) *along* ϕ

is a lift (τ, g) of the frame bundle classifying map (5.66) through $\mathbf{B}\phi$:

$$\begin{array}{ccccc} & & \overset{G\text{-structure}}{} & \mathbf{B}G & \\ & & \overset{\tau}{\nearrow} \;\; \Uparrow g & \downarrow{\scriptstyle \mathbf{B}\phi} & \\ V\text{-fold} & X & \xrightarrow[\vdash \mathrm{Frm}(X)]{} & \mathbf{B}\mathrm{Aut}(T_e V) & \begin{array}{l}\text{structure group} \\ \text{of frame bundle}\end{array} \end{array} \tag{5.81}$$

(ii) We say that the *G-frame bundle* $G\mathrm{Frm}(X)$ of a V-fold X equipped with such a (G,ϕ)-structure is the G-principal bundle which is classified (via Prop. 3.2.15): by τ, hence the object in the following diagram:

$$\begin{array}{ccccc} G\mathrm{Frm}(X,\tau) & \longrightarrow & * & & \\ \downarrow & \text{(pb)} & | & & \\ \mathrm{Frm}(X) & \longrightarrow & \longrightarrow & \longrightarrow & * \\ \downarrow & \text{(pb)} & \downarrow & & \downarrow \\ X & \xrightarrow{\tau} & \mathbf{B}G & \xrightarrow{\mathbf{B}\phi} & \mathbf{B}\mathrm{Aut}(T_e V) \end{array} \qquad X \xrightarrow{\vdash \mathrm{Frm}(X)} \mathbf{B}\mathrm{Aut}(T_e V) \tag{5.82}$$

(iii) We write

$$(G,\phi)\mathrm{Structures}_X(\mathbf{H}) \; := \; \mathbf{H}_{/\mathbf{B}\mathrm{Aut}(T_e V)}\big(\vdash \mathrm{Frm}(X)\,,\, \mathbf{B}\phi\big) \;\in\; \mathrm{Grpd}_\infty \tag{5.83}$$

for the ∞-groupoid of (G,ϕ)-structures on the V-fold X.

In direct generalization of Prop. 5.2.15 we have:

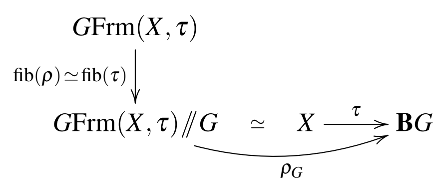
Proposition 5.2.24 (G-structured V-fold is G-quotient of its G-frame bundle)**.** *Let* $\mathbf{H}$ *be an elastic* ∞*-topos (Def. 4.1.21),* $V \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1),*

$(G,\phi) \in \mathrm{Grp}(\mathbf{H})_{/\mathrm{Aut}(T_eV)}$ *(Def. 5.2.22), $X \in V\mathrm{Folds}(\mathbf{H})$ (Def. 5.2.1) and* $(\tau, g) \in (G,\phi)\mathrm{Structures}_X(\mathbf{H})$ *(Def. 5.2.23). Then:*

(i) *X is equivalently the homotopy quotient (3.131) of its G-frame bundle (5.82) by G:*
$$X \simeq G\mathrm{Frm}(X,\tau) /\!\!/ G. \tag{5.84}$$

(ii) *the classifying map of the G-frame bundle on X exhibits the action of G on $G\mathrm{Frm}(X,\tau)$ according to (3.130).*

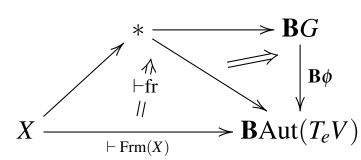
Proof. This is immediate from the equivalence between principal bundles and homotopy quotient projections (Remark 3.2.16) applied to the G-frame bundle (5.82):



(5.85)

□

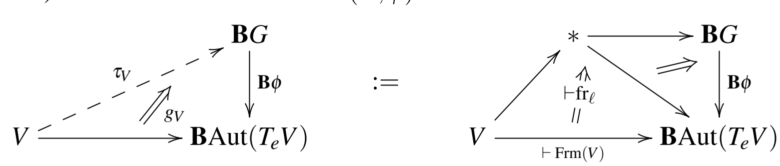
Example 5.2.25 (G-structure induced from framing)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21), $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1) and $X \in V\mathrm{Folds}(\mathbf{H})$ (Def. 5.2.1). Then a framing on X (Def. 5.2.17) induces a (G,ϕ)-structure (Def. 5.2.23) for any $(G,\phi) \in \mathrm{Grp}(\mathbf{H})_{/\mathrm{Aut}(T_eV)}$, given by the pasting



(5.86)

of the homotopy $\vdash \mathrm{fr}$ (5.70) which classifies the framing (Remark 5.2.18) with the homotopy that exhibits the group homomorphism ϕ as a morphism of pointed objects (Prop. 3.2.1).

Example 5.2.26 (Canonical G-structure)**.** Let $\mathbf{H}$ be an elastic ∞-topos $\mathbf{H}$ (Def. 4.1.21), and $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1). Then V itself, regarded as a V-fold by Ex. 5.2.3, carries a (G,ϕ)-structure (Def. 5.2.23) for any $(G,\phi) \in \mathrm{Grp}(\mathbf{H})_{/\mathrm{Aut}(T_eV)}$, induced via Ex. 5.2.25 from its canonical framing fr_ℓ (5.71) via left-translation (Prop. 5.2.19). We call this the *canonical (G,ϕ)-structure* on V:



(5.87)

5.2.6 Local isometries

Lemma 5.2.27 (G-structures pull back along local diffeomorphisms)**.** *Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21), $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1) and $(G,\phi) \in \mathrm{Grp}(\mathbf{H})_{/\mathrm{Aut}(T_eV)}$*

(Prop. 3.2.1, Def. 3.2.13, Ex. 4.1.30). Then pre-composition constitutes a contravariant ∞-functor ("pullback of (G,ϕ)-structures")

$$
\begin{array}{ccccc}
\big(V\mathrm{Folds}(\mathbf{H})^{\text{ét}}\big)^{\mathrm{op}} & \longrightarrow & \mathrm{Grpd}_\infty & & \\
X_1 & \longmapsto & (G,\phi)\mathrm{Structures}_{X_1}(\mathbf{H}) & \ni & \tau\circ f \\
{\scriptstyle \text{ét}}\big\downarrow f & & \big\uparrow f^* & & \big\uparrow \\
X_2 & \longmapsto & (G,\phi)\mathrm{Structures}_{X_2}(\mathbf{H}) & \ni & \tau
\end{array}
\tag{5.88}
$$

from the ∞-category (5.45) of V-folds and local diffeomorphisms, which assigns to any V-fold its ∞-groupoid (5.83) of (G,ϕ)-structres (Def. 5.2.23).

Proof. We need to show that for (τ,g) a (G,ϕ)-structure on X_2, the composite

$$
X_1 \xrightarrow[f]{\text{ét}} X_2 \; \begin{array}{c} \xrightarrow{\tau} \mathbf{B}G \\ \Downarrow g \quad \downarrow \mathbf{B}\phi \\ \xrightarrow[\vdash \mathrm{Frm}(X_2)]{} \mathbf{B}\mathrm{Aut}(T_eV) \end{array}
\tag{5.89}
$$

is a (G,ϕ)-structure on X_1. For this we need to exhibit a natural equivalence

$$
\big(\vdash \mathrm{Frm}(X_2)\big)\circ f_1 \;\simeq\; \vdash \mathrm{Frm}(X_1)
\tag{5.90}
$$

so that

$$
\begin{array}{ccccc}
X_1 & \xrightarrow[\text{ét}]{f} & X_2 & \xrightarrow{\tau} & \mathbf{B}G \\
 & {\scriptstyle \Downarrow \simeq} \searrow^{\vdash \mathrm{Frm}(X_1)} & \big\downarrow {\scriptstyle \vdash \mathrm{Frm}(X_2)} & {\scriptstyle \Downarrow g} \swarrow_{\mathbf{B}\phi} & \\
 & & \mathbf{B}\mathrm{Aut}(T_eV) & &
\end{array}
\tag{5.91}
$$

But this exists by Prop. 4.1.31. □

Definition 5.2.28 (Local isometries between G-structured V-folds)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21), $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1) and $(G,\phi)\in \mathrm{Grp}(\mathbf{H})_{/\mathrm{Aut}(T_eV)}$ (Prop. 3.2.1, Def. 3.2.13, Ex. 4.1.30).

(i) For $X_1, X_2 \in V\mathrm{Folds}$ (Def. 5.2.1) and $(\tau_i, g_i)\in (G,\phi)\mathrm{Structures}_{X_i}(\mathbf{H})$ (5.83), we say a *local isometry*, to be denoted

$$
\big(X_1,(\tau_1,g_1)\big) \xrightarrow[\text{met}]{(f,\sigma)} \big(X_2,(\tau_2,g_2)\big)
\tag{5.92}
$$

is a pair

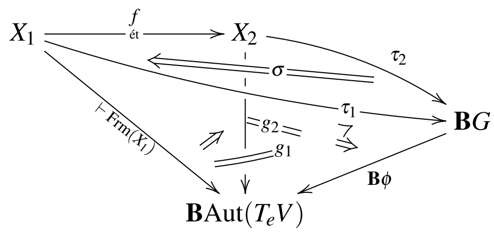
$$
X_1 \xrightarrow[\text{ét}]{f} X_2 \quad , \qquad f^*(\tau_2,g_2) \xrightarrow[\simeq]{\sigma} (\tau_1,g_1)\,,
\tag{5.93}
$$

consisting of a local diffeomorphism (Def. 4.1.26) and an equivalence of (G,ϕ)-structures (5.83) between that on its domain V-fold and the pullback (5.88) of the (G,ϕ)-structure on its codomain V-fold.

(ii) Equivalently, by (5.88), a local isometry (5.93) is a morphism between (G,ϕ)-structured V-folds regarded as objects in the iterated slice ∞-topos (Ex. 3.1.47)

(a) over $\mathbf{B}\mathrm{Aut}(T_eV)$ via their classifying maps of their frame bundles (5.66)

(b) over $\big(\mathbf{B}G, \mathbf{B}\phi\big)$ via their (G,ϕ)-structure (5.81) of this form:



$$\in \quad \big(\mathbf{H}_{/\mathbf{B}\mathrm{Aut}(T_eV)}\big)_{/(\mathbf{B}G,\mathbf{B}\phi)}\Big(\big(X_1,(\tau_1,g_1)\big)\,,\,\big(X_2,(\tau_2,g_2)\big)\Big). \tag{5.94}$$

(iii) Hence we write

$$(G,\phi)\mathrm{StrctrdVFolds}(\mathbf{H}) \longrightarrow \big(\mathbf{H}_{/\mathbf{B}\mathrm{Aut}(T_eV)}\big)_{/\mathbf{B}G} \;\in \mathrm{Cat}_\infty \tag{5.95}$$

for the sub-∞-category of this iterated slice on 1-morphisms of the form (5.94).

5.2.7 Integrability of G-structures

Definition 5.2.29 (Integrable G-structure)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21), $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1), $(G,\phi) \in \mathrm{Grp}(\mathbf{H})_{/\mathrm{Aut}(T_eV)}$ (Def. 5.2.22).

(i) Given $(X,(\tau_X,g_X)) \in (G,\phi)\mathrm{Strctrd}V\mathrm{Folds}(\mathbf{H})$ (Def. 5.2.28), we say that (τ,g) is an *integrable* (G,ϕ)-structure on the V-fold X if there exists a correspondence of local isometries (5.93) between V equipped with its canonical (G,ϕ)-structure (τ_V,g_V) (Def. 5.2.26) to $(X,(\tau_X,g_X))$:

$$\big(V,(\tau_V,g_V)\big) \xleftarrow{\;\mathrm{met}\;} \big(U,(\tau_U,g_U)\big) \xrightarrow{\;\mathrm{met}\;} \big(X,(\tau_X,g_X)\big) \tag{5.96}$$

such that the right left is, in addition, an effective epimorphism (Def. 3.1.63), then called a $(V,(\tau_V,g_V))$*-atlas of* $(X,(\tau_X,g_X))$ (3.118). (Underlying this, forgetting the (G,ϕ)-structures, is a V-atlas (5.43).)

(ii) We write

$$\mathrm{Intgrbl}(G,\phi)\mathrm{Strctrd}V\mathrm{Folds}(\mathbf{H}) \hookrightarrow (G,\phi)\mathrm{Strctrd}V\mathrm{Folds}(\mathbf{H}) \;\in \mathrm{Cat}_\infty \tag{5.97}$$

for the full sub-∞-category of that of (G,ϕ)-structured V-folds (5.95) on those that are integrable.

Definition 5.2.30 (Locally integrable G-structure)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21), $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1), $(G,\phi) \in \mathrm{Grp}(\mathbf{H})_{/\mathrm{Aut}(T_eV)}$ (Def. 5.2.22), $X \in V\mathrm{Folds}(\mathbf{H})$ (Def. 5.2.1) and $(\tau,g) \in (G,\phi)\mathrm{Structures}_X(\mathbf{H})$ (Def. 5.2.23). We say that (τ,g) is a *locally integrable* (G,ϕ)-structure if, for each point $* \xrightarrow{x} X$, there is a local diffeomorphism ϕ_x of the local neighborhood (Def. 4.1.28) of $* \xrightarrow{e} V$ onto a local neighborhood of x such that the restriction of (τ,g) along ϕ is equivalent to the canonical (G,ϕ)-structure (Def. 5.2.26) on T_eV:

$$\underset{* \xrightarrow{x} X}{\forall} \quad \underset{\begin{array}{ccc} T_eV & \underset{\text{ét}}{\xrightarrow{\phi_x}} & X \\ & {}_{e}\nwarrow \quad \nearrow_{x} & \\ & * & \end{array}}{\exists} \quad : \quad \phi_x^*(\tau, g) \simeq (\tau_{T_eV}, g_{T_eV}). \tag{5.98}$$

Another way to say this: We have a correspondence of local isometries as in (5.96), but with the right leg required to be an effective epimorphism only under $\flat$.

Example 5.2.31 (*G*-Structures on smooth manifolds and orbifolds)**.**

(i) Let $\mathbf{H} = \text{JetSmthGrpd}_\infty$ (Ex. 4.1.24) $G \in$ LieGroups $\hookrightarrow \text{Grp}(\mathbf{H})$ (see (4.72)) and $X \in$
SmthMfd $\hookrightarrow \mathbf{H}$ regarded as an $\mathbb{R}^n$-fold according to Ex. 5.2.4. In this case, the structure group of X (Def. 5.2.11) is the ordinary general linear group $\text{GL}_{\mathbb{R}}(n)$ (Ex. 5.2.12). Therefore, a G-structure on X in the sense of Def. 5.2.23 is (by Ex. 5.2.16) a G-structure in the traditional sense of differential geometry [St64, VII][Kob72][Mol77]; and it is integrable according to Def. 5.2.29 if it is "flat" in the traditional sense of [Gu65] and locally integrable according to Def. 5.2.30 precisely if it is "uniformly 1-flat" in the traditional sense of [Gu65], namely if it is torsion-free (review in [Lot01]). Examples are shown in Table 5.1 on p. 143.

(ii) For $k > 1$ and $\mathbf{H} = k\text{JetSmthGrpd}_\infty$ (Def. 4.1.24) the local integrability condition of Def. 5.2.30 is of the form of the "uniformly k-flatness"-condition of [Gu65]. But beware that, according to Def. 5.2.23 but in contrast to [Gu65], in this case the G-structure itself is not on the plain frame bundle but on the order-k jet frame bundle (by Ex. 5.2.16).

$G \xrightarrow{\phi} \mathrm{GL}_{\mathbb{R}}(n)$	(G,ϕ)**-structure**	**Locally integrable**	**Integrable**	see
$\mathrm{Sp}_{\mathbb{R}}(n) \hookrightarrow \mathrm{GL}_{\mathbb{R}}(n)$	almost symplectic	symplectic	symplectic	[St64, VII.2]
$\mathrm{GL}_{\mathbb{C}}(n/2) \hookrightarrow \mathrm{GL}_{\mathbb{R}}(n)$	almost complex	complex	complex	
$\mathrm{O}(n) \hookrightarrow \mathrm{GL}_{\mathbb{R}}(n)$	Riemannian	torsion-free Riemannian	flat Riemannian	
$\mathrm{O}(n-1,1) \hookrightarrow \mathrm{GL}_{\mathbb{R}}(n)$	Lorentzian	torsion-free Lorentzian	flat Lorentzian	[LPZ13]
$\mathrm{O}(n) \times \mathbb{R} \hookrightarrow \mathrm{GL}_{\mathbb{R}}(n)$	$\mathrm{CO}(n)$-structure	conformal	flat conformal	[AG98]
$\mathrm{CR}(\frac{n}{2}-1) \hookrightarrow \mathrm{GL}_{\mathbb{R}}(n)$	$\mathrm{CR}(n)$-structure	Cauchy-Riemann	flat Cauchy-Riemann	[DT06]
$\mathrm{GL}_{\mathbb{H}}(n/4) \hookrightarrow \mathrm{GL}_{\mathbb{R}}(n)$	$\mathrm{GL}_{\mathbb{H}}(\frac{n}{4})$-structure	hypercomplex	flat hypercomplex	[Jo95]
$\mathrm{U}(n/2) \hookrightarrow \mathrm{GL}_{\mathbb{R}}(n)$	hermitian almost complex	Kähler	Kähler	[Mor07, 11.1]
$\mathrm{SU}(n/2) \hookrightarrow \mathrm{GL}_{\mathbb{R}}(n)$	$\mathrm{SU}(n)$-structure	Calabi-Yau	Calabi-Yau	[Pri15, 1.3]
$\mathrm{Sp}(\frac{n}{4})\cdot\mathrm{Sp}(1) \hookrightarrow \mathrm{GL}_{\mathbb{R}}(n)$	almost unimodular quaternionic	quaternionic Kähler	flat quaternionic Kähler	[AM93a] [AM93b]
$\mathrm{Sp}(n/4) \hookrightarrow \mathrm{GL}_{\mathbb{R}}(n)$	almost Hyperkähler	Hyperähler	flat Hyperkähler	
$G_2 \hookrightarrow \mathrm{GL}_{\mathbb{R}}(7)$	G_2-structure	torsion-free G_2-structure	flat/interable G_2-structure	[Br05]
$\mathrm{Spin}(7) \hookrightarrow \mathrm{GL}_{\mathbb{R}}(8)$	$\mathrm{Spin}(7)$-structure	torsion-free $\mathrm{Spin}(7)$-structure	flat $\mathrm{Spin}(7)$-structure	[Br87] [Jo01]

Table 5.1 – Examples of G-structures (cf. Ex. 5.2.31)

5.2.8 Haefliger groupoids

Definition 5.2.32 (Haefliger groupoid)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21) and $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1).

(i) With no further structure,

(a) The V-*Haefliger groupoid* is the étale groupoid (Def. 4.1.35)

$$\mathrm{Haef}_{\bullet}(V) \;\in\; \mathrm{\acute{E}taleGroupoids}(\mathbf{H}), \tag{5.99}$$

which is the étalification (Def. 4.1.39) of the Atiyah groupoid (Def. 3.2.17) of the frame bundle (Def. 5.2.11) of V regarded as a V-fold (Ex. 5.2.3):

$$\mathrm{Haef}_{\bullet}(V) \;:=\; \mathrm{At}^{\text{ét}}_{\bullet}\big(\mathrm{Frm}(V)\big). \tag{5.100}$$

(b) The V-*Haefliger stack* of V is the corresponding V-fold (according to Remark 5.2.2):

$$\mathscr{H}\mathrm{aef}(V) \;:=\; \mathscr{A}\mathrm{t}^{\text{ét}}\big(\mathrm{Frm}(V)\big) \;\in\; V\mathrm{Folds}. \tag{5.101}$$

(ii) Given, in addition, $(G,\phi) \in \mathrm{Grp}(\mathbf{H})_{/\mathrm{Aut}(T_e V)}$ (Def. 5.2.22), with $G\mathrm{Frm}(V) \to V$ denoting the G-frame bundle (5.82) corresponding to the canonical (G,ϕ)-structure on V (Def. 5.2.26), we say

(a) the $\big(V,(G,\phi)\big)$-*Haefliger groupoid* is the étale groupoid (Def. 4.1.35)

$$\mathrm{Haef}_\bullet\big(V,(G,\phi)\big) \;\in\; \text{ÉtaleGroupoids}(\mathbf{H}) \tag{5.102}$$

which is the étalification (Def. 4.1.39) of the Atiyah groupoid (Def. 3.2.17) of the G-frame bundle (5.82):

$$\mathrm{Haef}_\bullet\big(V,(G,\phi)\big) \;:=\; \mathrm{At}^{\text{ét}}_\bullet\big(G\mathrm{Frm}(V)\big)\,. \tag{5.103}$$

(b) The $\big(V,(G,\phi)\big)$-*Haefliger stack* of V is the corresponding V-fold (according to Remark 5.2.2):

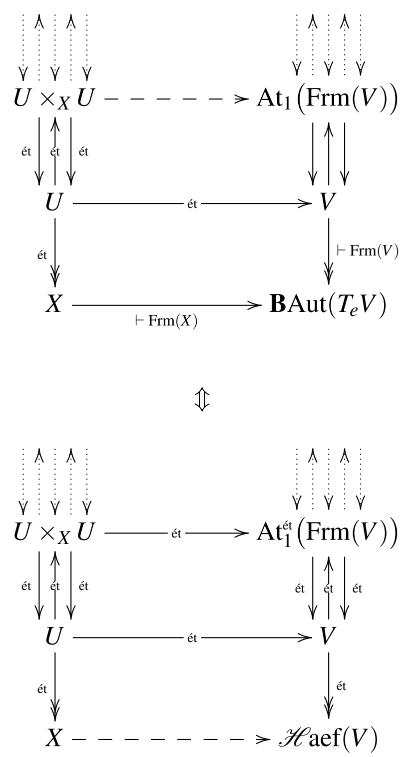
$$\mathscr{H}\mathrm{aef}\big(V,(G,\phi)\big) \;:=\; \mathscr{A}\mathrm{t}^{\text{ét}}\big(G\mathrm{Frm}(V)\big) \;\in\; V\mathrm{Folds}\,. \tag{5.104}$$

Proposition 5.2.33 (Haefliger stack represents V-fold structure)**.** *Let* $\mathbf{H}$ *be an elastic* ∞*-topos (Def. 4.1.21)* $V \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1) and* $X \in \mathbf{H}$*. Then the following are equivalent:*

(i) *X is a V-fold (Def. 5.2.1);*

(ii) *X admits a local diffeomorphism to the V-Haefliger stack (Def. 5.2.32).*

Proof. First consider the implication **(i)** $\Rightarrow$ **(ii)**: Assuming X is a V-fold, consider a V-atlas (5.43) $V \xleftarrow{\text{ét}} U \xrightarrow{\text{ét}} X$. By Prop. 4.1.31 (and as in the proof of Prop. 5.2.13) the pullbacks of the frame bundles of V and of X along this V-atlas to U coincide there, which means that we have a homotopy-commutative square of their classifying maps (5.66) as shown on the bottom left of the following diagram:



(5.105)

By passing to nerves (Ex. 3.1.69) of the vertical morphisms, this induces a morphism of groupoids as shown on the top left. But $U_\bullet$ is an étale groupoid (by Prop. 4.1.36), and $U \longrightarrow V$ is a local diffeomorphism by definition of V-atlases, so that the top left part of the left diagram in (5.105) is in the étale slice over V (Def. 4.1.32). Therefore, the adjunction (4.89) of Prop. 4.1.33 implies that the top part of the diagram on the left of (5.105) factors through the étalification (Def. 4.1.39) as shown in the top part on the right. With this we get the dashed morphism on the right by passing to colimits over the vertical simplicial diagrams (as in Prop. 4.1.36).

It only remains to see that the dashed morphism on the right is itself a local diffeomorphism. For this observe that al the horizontal morphisms are local diffeomorphisms, using the assumptions and then left-cancellability (Lemma 4.1.27). Therefore the statement follows with Lemma 4.1.38.

For the converse implication **(ii)** $\Rightarrow$ **(i)**: Given a local diffeomorphism as shown dashed on the right of (5.105), we need to produce a V-atlas for X. So now define the bottom square on the right of (5.105) to be the pullback of the étale atlas of the Haefliger stack along the griven morphism. This does make the top left span of the square a V-atlas by the fact that the classes of local diffeomorphisms and of effective epimorphisms are both closed under pullback (by Lemma 3.1.65 and Lemma 4.1.27). □

Proposition 5.2.34 (G-Structured Haefliger stack represents integrable G-structure)**.** *Let* $\mathbf{H}$ *be an elastic* ∞*-topos (Def. 4.1.21),* $V \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1),* $(G,\phi) \in \mathrm{Grp}(\mathbf{H})_{/\mathrm{Aut}(T_e V)}$ *(Def. 5.2.22). The* $\big(V,(G,\phi)\big)$*-Haefliger groupoid (Def. 5.2.32), carries a canonical integrable* (G,ϕ)*-structure (Def. 5.2.29)*

$$(\tau_{\mathscr{H}}, g_{\mathscr{H}}) \in (G,\phi)\mathrm{Structures}_{\mathscr{H}\mathrm{Haef}(V)}(\mathbf{H}) \tag{5.106}$$

such that the operation of pullback of (5.88) *along local diffeomorphism (Lemma 5.2.27) constitutes a natural bijection*

$$\begin{array}{ccc} \pi_0\,\mathrm{Intgrbl}(G,\phi)\mathrm{Strc}V\mathrm{Folds}(\mathbf{H}) & \simeq & \pi_0\,\acute{\mathrm{E}}\mathrm{t}_{\mathscr{H}\mathrm{aef}(V,(G,\phi))} \\ \big(X,(\tau,g)\big) & \longmapsto & \Big(X \xrightarrow{\vdash(\tau,g)} \mathscr{H}\mathrm{aef}\big(V,(G,\phi)\big)\Big) \end{array} \tag{5.107}$$

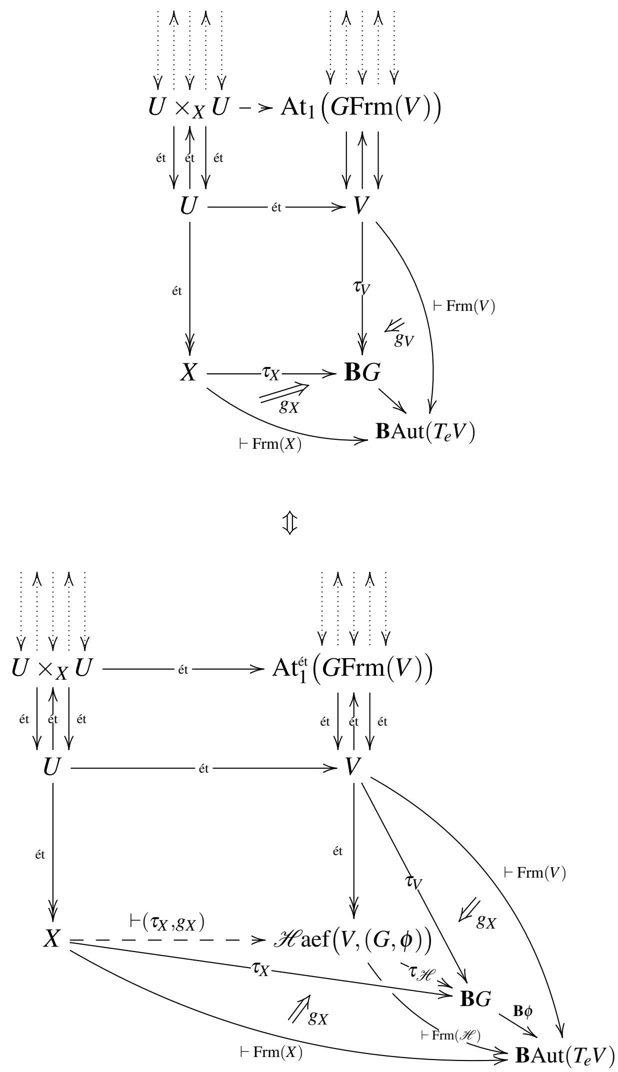
between the sets of equivalence classes of:

- **(i)** *integrably* (G,ϕ)*-structured* V*-folds (Def. 5.2.29),*

(ii) *local diffeomorphisms into the* $\big(V,(G,\phi)\big)$*-Haefliger stack, hence objects in its étale topos (Def. 4.1.32).*

Proof. We proceed as in the proof of Prop. 5.2.33, but lifting the diagram there from $\mathbf{H}$ to the iterated slice $\big(\mathbf{H}_{/\mathbf{B}\mathrm{Aut}(T_e V)}\big)_{/\mathbf{B}G}$ (5.94).

(i) First consider an integrably G-structured V-fold $\big(X,(\tau,g)\big)$. We describe the construction of a local diffeomorphism into the Haefliger stack from this: Pick any $\big(V,(\tau_V,g_V)\big)$-atlas $\big(V,(\tau_V,g_V)\big) \overset{\mathrm{met}}{\longleftarrow} \big(U,(\tau_U,g_U)\big) \overset{\mathrm{met}}{\twoheadrightarrow} \big(X,(\tau_X,g_X)\big)$ (5.96). By Def. 5.2.23, this is equivalently a choice of equivalence between the pullbacks to U of the G-structures on V and on X. Regarded in the iterated slice (5.94), this equivalently means that we have a square in $\big(\mathbf{H}_{/\mathbf{B}\mathrm{Aut}(T_e V)}\big)_{/\mathbf{B}G}$ (5.94), as shown inthe following:



(5.108)

Now we proceed as follows:

- **(a)** Observing (with Prop. 3.1.53) that fiber products in the iterated slice are actually given by the plain fiber products in $\mathbf{H}$ equipped with canonical morphisms to the slicing objects, we find that passing to nerves (Ex. 3.1.69) of the vertical morphisms on the left of (5.108) yields a morphism from the étale groupoid induced by the given V-cover of X to the Atiyah groupoid of $G\mathrm{Frm}(X)$ (Def. 3.2.17) – just as in (5.105), but now equipped with coherent maps to $\mathbf{B}\phi$.

- **(b)** Therefore, we obtain the factorization through the $\big(V,(G,\phi)\big)$-Haefliger groupoid (the étalification of the Atiyah groupoid of the G-frame bundle shown on the top right of (5.108)) just as in (5.105), but now, in addition, coherently equipped with maps to $\mathbf{B}\phi$.

- **(c)** After this étalification we may identify these maps: Since those on V remain unchanged by étalification over V, these still give the canonical (G,ϕ)-structure (τ_V, g_V), as shown on the far right of (5.108). But since now the vertical simplicial morphisms are all local diffeomorphisms, pullback along which preserves (G,ϕ)-structure (by Lemma 5.2.27) and in particular preserves tangent- and frame bundles (by Prop. 4.1.31) it follows that all stages of the $\big(V,(G,\phi)\big)$-Haefliger groupoid in the top right are now equipped with the classifying map of their frame bundles.

- **(d)** Since colimits in the slice are given by colimits in the underlying topos (by Ex. 3.1.52), the colimit over the simplicial sub-diagram on the far right of (5.108) still yields the $\big(V,(\tau,g)\big)$-Haefliger stack (5.104), as shown, now equipped with canonical maps to $\mathbf{B}\phi$.

- **(e)** We claim that the induced map from the Haefliger stack to $\mathbf{B}\mathrm{Aut}(T_eV)$, denoted $\vdash \mathrm{Frm}(\mathscr{H})$ in (5.108), is indeed the classifying map of the frame bundle of the Haefliger stack:

$$\vdash \mathrm{Frm}(\mathscr{H}) \;\simeq\; \vdash \mathrm{Frm}\Big(\mathscr{H}\mathrm{aef}\big(V,(G,\phi)\big)\Big). \tag{5.109}$$

  This follows because:

  - by **(c)** above, the component maps of the colimiting map classify the frame bundles of the stages of the simplicial nerve;
  - therefore, the colimiting map classifies the colimit of the frame bundles of the simplicial nerve, by Prop. 3.1.56,
  - but the colimit of the tangent bundles of the étale cover is the tangent bundle of the corresponding étale stack, by Prop. 4.1.37.

- **(f)** In particular, this implies that the induced homotopy which fills the bottom right part of (5.108):

$$\vdash \mathrm{Frm}\big(\mathscr{H}\big) \xRightarrow{\;g_{\mathscr{H}}\;} \mathbf{B}\phi \circ \tau_{\mathscr{H}}\,, \tag{5.110}$$

  canonically given by the colimit construction in the iterated slice, constitutes a (G,ϕ)-structure on the $\big(V,(G,\phi)\big)$-Haefliger stack.

- **(g)** In conclusion, the dashed morphism on the right of (5.108) exists and is a local diffeomorphism, as in the proof of Prop. 5.2.33; but, by construction in the iterated slice, it is now exhibited as a local isometry to the Haefliger stack equipped with the induced (G,ϕ)-structure (5.110).

(ii) The converse construction is now immediate: Given a local diffeomorphism of the form shown dashed on the right of (5.108), pulling back the étale atlas of the

Haefliger stack along it yields a V-atlas for X (just as in the proof of this converse step in Prop. 5.2.33) and pulling (via Lemma 5.2.27) the (G,ϕ)-structure (5.110) around the resulting Cartesian square makes this a $\big(V,(G,\phi)\big)$-atlas that exhibits X as equipped with an integrable (G,ϕ)-structure. This construction is clearly injective on equivalence classes, by ∞-functoriality of the pullback construction (5.88) of (G,ϕ)-structures; and it is surjective on equivalence classes by item (i) above. Hence it is a bijection on equivalence classes, as claimed.

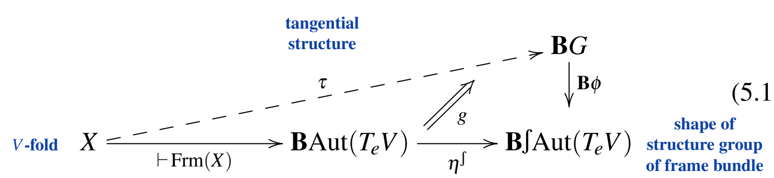
□

5.2.9 Tangential structures

Closely akin to G-structures (Def. 5.2.23) are *tangential structures* (Def. 5.2.35 below) where not the structure group itself is lifted, but only its shape:

Definition 5.2.35 (Tangential structure)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21), $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1), $(G,\phi) \in \mathrm{Grp}(\mathbf{H})_{/\int \mathrm{Aut}(T_e V)}$ (Def. 5.2.22) and $X \in V\mathrm{Folds}(\mathbf{H})$ (Def. 5.2.1).

(i) We say that a *tangential* (G,τ)*-structure* on X is a lift (τ, g) through $\mathbf{B}\phi$ of the composite of the frame bundle classifying map (5.66) with the shape-unit (3.44):



(5.111)

(ii) We write

$$\mathrm{Tangential}(G,\phi)\mathrm{Structures}_X(\mathbf{H}) \;:=\; \mathbf{H}_{/\mathbf{B}\int\mathrm{Aut}(T_e V)}\big(\eta^{\int}\circ \vdash \mathrm{Frm}(X)\,,\, \mathbf{B}\phi\big) \quad (5.112)$$

for the ∞*-groupoid of* (G,ϕ)*-tangential structures* on the V-fold X.

Example 5.2.36 (Tangential structures on smooth manifolds)**.** Let $\mathbf{H} = \mathrm{JetSmthGrpd}_\infty$ (Ex. 4.1.24) $G \in \mathrm{LieGroups} \hookrightarrow \mathrm{Grp}\big(\mathbf{H}\big)$ (see (4.72)) and $X \in \mathrm{SmthMfd} \hookrightarrow \mathbf{H}$ regarded as an $\mathbb{R}^n$-fold according to Ex. 5.2.4. In this case, the structure group of X (Def. 5.2.11) is the ordinary general linear group $\mathrm{GL}_{\mathbb{R}}(n)$ (Ex. 5.2.12). Hence here tangential structure in the general sense of Def. 5.2.35 is tangential structure in the traditional sense of differential topology (popularized under this name in [GMTW06, 5], originally introduced as "(B,f)-structure" [La63][St68, II], review in [Ko96, 1.4]).

Example 5.2.37 (Cohesive refinement of tangential structure)**.** Every (G,ϕ)-structure (Def. 5.2.23) induces tangential $(\int G, \int \phi)$-structure (Def. 5.2.35) by composition with the naturality square of $\eta^{\int}$ on $\mathbf{B}\phi$:

$$
\begin{array}{ccccc}
 & \textbf{(G,ϕ)-structure} & \mathbf{B}G & \xrightarrow{\eta^{\int}_{\mathbf{B}G}} & \mathbf{B}\int G \\
 & \overset{\tau}{\nearrow} & \Uparrow g \quad \downarrow{\scriptstyle \mathbf{B}\phi} & & \downarrow {\scriptstyle \mathbf{B}\int\phi} \\
\textbf{V-fold} \quad X & \xrightarrow[\vdash \mathrm{Frm}(X)]{} & \mathbf{B}\mathrm{Aut}(T_eV) & \xrightarrow[\eta^{\int}_{\mathbf{B}\mathrm{Aut}(T_eV)}]{} & \mathbf{B}\int\mathrm{Aut}(T_eV) \quad \textbf{shape of structure group of frame bundle}
\end{array}
\tag{5.113}
$$

Conversely, realizing a tangent structure as obtained from a G-structure this way means to find a geometric (differential) refinement.

Example 5.2.38 (Orientation structure)**.** Let $\mathbf{H} = \mathrm{JetSmthGrpd}_\infty$ (Ex. 4.1.24) and $X \in \mathbf{H}$ an $\mathbb{R}^n$-fold (Def. 5.2.1) hence an ordinary manifold (Ex. 5.2.4) or, more generally, an ordinary étale Lie groupoid (Ex. 5.2.5). With the general linear and the (special) orthogonal group regarded as smooth groups via (4.72)

$$
\mathrm{SO}(n) \xrightarrow{i_{\mathrm{SO}}} \mathrm{O}(n) \xrightarrow{i_O} \mathrm{GL}(n) \;\in\; \mathrm{Grp}(\mathrm{SmthMfd}) \longrightarrow \mathrm{Grp}(\mathbf{H}) \tag{5.114}
$$

we have:

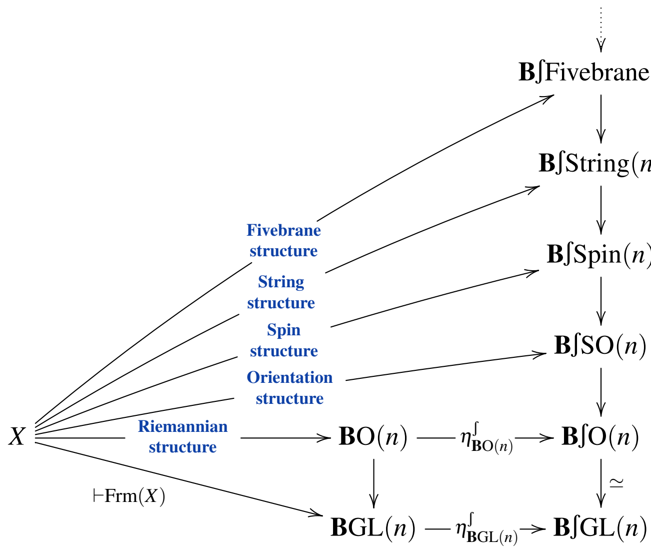
(i) an $\mathrm{O}(n)$-structure (Def. 5.2.23) on X is equivalently a Riemannian structure (Ex. 5.2.31);

(ii) but a tangential $\int\mathrm{O}(n)$-structure (Def. 5.111) is equivalently *no structure*, since $\int\mathrm{O}(n) \xrightarrow[\simeq]{\int i_{\mathrm{O}}} \int\mathrm{GL}(n)$ is an equivalence of underlying shapes (since $\mathrm{O}(n)$ is the maximal compact subgroup of $\mathrm{GL}(n)$),

(iii) while a tangential $\int\mathrm{SO}(n)$-structure (Def. 5.111) is an *orientation* of X.

(iv) A differential refinement, in the sense of Ex. 5.2.37, of such an orientation structure is an oriented Riemannian structure (via its induced volume form).

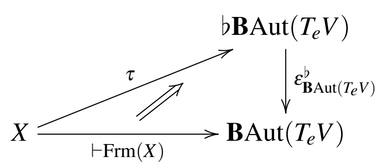
Example 5.2.39 (Higher Spin structure [SSS09][SSS12])**.** Let $\mathbf{H} = \mathrm{JetSmthGrpd}_\infty$ (Ex. 4.1.24) and $X \in \mathbf{H}$ an $\mathbb{R}^n$-fold (Def. 5.2.1) hence an ordinary manifold (Ex. 5.2.4) or, more generally, an ordinary étale Lie groupoid (Ex. 5.2.5). The sequence of groups (5.114) in Ex. 5.2.38 is, under shape, the beginning of the *Whitehead tower* of $\int O(n) \simeq \int\mathrm{GL}(n)$. The tangential structures (Def. 5.2.35, Ex. 5.2.36) corresponding to the stages in this tower are the *Spin structure* and its higher analogues:



(5.115)

5.2.10 Flat V-folds

Definition 5.2.40 (Flat V-folds)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21), $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1) and $X \in V\mathrm{Folds}(\mathbf{H})$ (Def. 5.2.1). We say that X is *flat* if the classifying map (5.66) of its frame bundle (Prop. 5.2.13) factors through the $\flat$-counit (3.45), hence if it carries (G,ϕ)-structure (Def. 5.2.23) for $(G,\phi) = (\flat\mathrm{Aut}(T_eV), \varepsilon^{\flat}_{\mathrm{Aut}(T_eV)})$:



(5.116)

By the universal property of $\varepsilon^{\flat}$ and since $\flat$ commutes with $\mathbf{B}$, this means equivalently that X carries G-structure for any discrete group $G \simeq \flat\flat G$.

Proposition 5.2.41 (Flat frame bundles are V-folds)**.** *Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21), $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1) and $X \in V\mathrm{Folds}(\mathbf{H})$ (Def. 5.2.1). If X is flat (Def. 5.2.40), then*

- **(i)** *its flat frame bundle $(\flat\mathrm{Aut}(T_eV))\mathrm{Frm}(X)$ (5.82) is itself a V-fold (Def. 5.2.1),*
- **(ii)** *the bundle morphism is a local diffeomorphism (Def. 4.1.26):*

$$\big(\flat\mathrm{Aut}(T_eV)\big)\mathrm{Frm}(X) \xrightarrow{\text{ét}} X \ . \tag{5.117}$$

Proof. First consider **(ii)**: We need to show that the left square in the following pasting diagram is Cartesian:

$$
\begin{array}{ccccc}
(\flat \mathrm{Aut}(T_e V))\mathrm{Frm}(X) & \xrightarrow{\eta^{\Im}_{(\flat \mathrm{Aut}(T_e V))\mathrm{Frm}(X)}} & \Im\big((\flat \mathrm{Aut}(T_e V))\mathrm{Frm}(X)\big) & \longrightarrow & \Im * \\
{\scriptstyle p}\downarrow & & {\scriptstyle \Im p}\downarrow & \text{(pb)} & \downarrow \\
X & \xrightarrow[\eta^{\Im}_X]{} & \Im X & \xrightarrow[\Im\tau]{} & \Im \flat \mathrm{Aut}(T_e V)
\end{array}
\tag{5.118}
$$

Here the right square is Cartesian, by definition (5.82) and since $\Im$, being a right adjoint, preserves Cartesian squares (by Prop. 3.1.26). Hence, by the pasting law (Prop. 3.1.23) it is sufficient to show that the total rectangle is Cartesian. But, by the naturality of $\eta^{\Im}$, the total rectangle is equivalent to that of the following pasting diagram:

$$
\begin{array}{ccccc}
(\flat \mathrm{Aut}(T_e V))\mathrm{Frm}(X) & \longrightarrow & * & \xrightarrow{\eta^{\Im}_*} & \Im * \\
\downarrow & \text{(pb)} & \downarrow & \text{(pb)} & \downarrow \\
X & \xrightarrow[\tau]{} & \flat \mathbf{B}\mathrm{Aut}(T_e V) & \xrightarrow[\eta^{\Im}_{\flat \mathbf{B}\mathrm{Aut}(T_e V)}]{} & \Im \flat \mathbf{B}\mathrm{Aut}(T_e V)
\end{array}
\tag{5.119}
$$

Here the left square is Cartesian by the definition (5.82), while the right square is Cartesian since its two horizontal morphisms are equivalences, by elasticity. Hence the total rectangle is Cartesian by the pasting law (Prop. 3.1.23).

Regarding **(i)**: We need to exhibit a V-atlas (5.43) for the flat frame bundle. So let $V \xleftarrow{\text{ét}} U \xrightarrow{\text{ét}} X$ be a V-atlas for X, and consider the following pullback diagram:

$$
\begin{array}{ccc}
U \underset{X}{\times} (\flat \mathrm{Aut}(T_e V))\mathrm{Frm}(X) & \xrightarrow{\text{ét}} & (\flat \mathrm{Aut}(T_e V))\mathrm{Frm}(X) \\
{\scriptstyle \text{ét}}\downarrow & \text{(pb)} & \downarrow{\scriptstyle \text{ét}} \\
U & \xrightarrow[\text{ét}]{} & X \\
{\scriptstyle \text{ét}}\downarrow & & \\
V & &
\end{array}
\tag{5.120}
$$

Observe that all four morphisms in the square are effective epimorphisms (Def. 3.1.63) and local diffeomorphisms (Def. 4.1.26): The bottom one by definition, the right one by **(ii)** and hence the other two since both classes of morphisms are closed under pullback (Lemma 3.1.65 and Lemma 4.1.27). Finally, since the class of local diffeomorphisms is also closed under composition (Lemma 4.1.27), the total vertical morphisms is a local diffeomorphism, and hence the total outer diagram is a V-atlas of the flat frame bundle. □

Proposition 5.2.42 ($\flat G$-frame bundles are V-folds)**.** *Let* $\mathbf{H}$ *be an elastic* ∞*-topos (Def. 4.1.21),* $V \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1)* $X \in V\mathrm{Folds}(\mathbf{H})$ *(Def. 5.2.1),* $(G,\phi) \in \mathrm{Grp}(\mathbf{H})_{/\mathrm{Aut}(T_e V)}$ *(Prop. 3.2.1, Def. 3.2.13, Ex. 4.1.30) with* $G \simeq \flat G$ *discrete, and* $(\tau, g) \in (G,\phi)\mathrm{Structures}_X(\mathbf{H})$*. Then the corresponding G-frame bundle* (5.82) *is it-*

self a V-fold:

$$G \simeq \flat G \quad \Leftrightarrow \quad G\mathrm{Frm}(X) \in V\mathrm{Folds}(\mathbf{H})\,. \tag{5.121}$$

Proof. The proof proceeds verbatim as that for Prop. 5.2.41, just with the structure group restricted along $\flat G \to \flat \mathrm{Aut}(T_e V)$. ☐

In summary, we have found the general abstract version of the local model spaces of orbifolds:

Proposition 5.2.43 (Local orbifold model spaces)**.** *Let* $\mathbf{H}$ *be an elastic* ∞*-topos (Def. 4.1.21),* $G, V \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1), with* $G \simeq \flat G$ *discrete, and* $(V, \rho) \in G\mathrm{Actions}(\mathbf{H})$ *(Prop. 3.2.6) a restriction (Prop. 3.2.12) of the action* (V, ρ_{Aut}) *by group-automorphisms (Prop. 3.2.29). Then the homotopy quotient* (3.131)

$$V /\!\!/ G \in \mathbf{H} \tag{5.122}$$

of V regarded with its canonical framing (Prop. 5.2.19)

(i) *is a flat V-fold (Def. 5.2.40);*

(ii) *with G-structure (Def. 5.2.23)*

(iii) *whose G-frame bundle* (5.82) *is G-equivariantly (Def. 3.2.10) equivalent to V itself:*

$$G\mathrm{Frm}\big(V /\!\!/ G\big) \simeq V\,. \tag{5.123}$$

Proof. First observe that $V /\!\!/ G$ is a V-fold, by Prop. 5.2.8 applied to Ex. 5.2.3. That this is flat **(i)** is implied by **(ii)**, since G is assumed to be discrete. For **(ii)** and **(iii)** observe that the canonical framing on V is G-equivariant, by Prop. 5.2.20, so that Prop. 5.2.21 implies G-structure on $V /\!\!/ G$ classified by the action morphism ρ itself. But this means that its homotopy fiber, hence the corresponding G-frame bundle (Def. 5.82) is V itself, by (3.130) (and in accord with Prop. 5.2.42). ☐

Example 5.2.44 (Ordinary orbifold singularities)**.** Let $\mathbf{H} := \mathrm{JetSmthGrpd}_\infty$ (Ex. 4.1.24) and $V := (\mathbb{R}^n, +)$ as in Ex. 5.2.4. Then a group automorphism of V is a *linear* isomorphism, hence $\mathrm{Aut}_{\mathrm{Grp}}(\mathbb{R}^n, +) \simeq \mathrm{GL}(n)$. Therefore, in this case the assumptions of Prop. 5.2.43 hold precisely for V a *linear representation* of the discrete group G, and thus we recover the traditional local orbifold models $V /\!\!/ G$ from [Sa56] (in their incarnation as étale groupoids).

5.2.11 Orbi-V-folds

Finally, we may now easily promote V-folds to orbifolds proper, and hence promote the ∞-category of étale stacks to a ∞-category of higher proper orbifolds:

Definition 5.2.45 (Orbi-V-folds)**.** Let $\mathbf{H}$ be a singular-elastic ∞-topos (Def. 4.2.6) and $V \in \mathrm{Grp}(\mathbf{H}_{\cup})$. We say that an *orbi-V-fold* is an object $\mathscr{X} \in \mathbf{H}$ which is the orbi-singularization (Def. 4.2.7) of a V-fold (Def. 5.2.1, hence of an étale ∞-stack modeled on V, cf. Exp. 5.2.5).

(i) We write $V\mathrm{Orbfld}(\mathbf{H}) \subset \mathbf{H}$ for the full sub-∞-category on orbi-V-folds:

$$\mathscr{X} \in V\mathrm{Orbfld}(\mathbf{H}) \quad \Leftrightarrow \quad \cup \mathscr{X} \in V\mathrm{Folds}(\mathbf{H})\,. \tag{5.124}$$

This means, equivalently, that the orbi-V-folds in $\mathbf{H}$ are the orbi-singularizations (4.143) of the V-folds in $\mathbf{H}_{\cup}$:

$$V\mathrm{Folds}(\mathbf{H}_{\cup}) \underset{\text{OrbSnglr}}{\overset{\text{Smth}}{\rightleftarrows}} V\mathrm{Orbfld}(\mathbf{H}) \quad \simeq \tag{5.125}$$

$$\begin{array}{ccc} \mathrm{Smth}(\mathscr{X}) & \longleftarrow & \mathscr{X} \\ \| & & \| \\ X & \longmapsto & \mathrm{OrbSnglr}(X) \end{array}$$

(ii) Similarly, given, in addition, $(G,\phi) \in \mathrm{Grp}(\mathbf{H})_{/\mathrm{Aut}(T_eV)}$ (Def. 5.2.22), we write $(G,\phi)\mathrm{Strctrd}V\mathrm{Orbfld}(\mathbf{H}) \subset \mathbf{H}$ for the full sub-∞-category on (G,ϕ)-structured orbi-V-folds (Def. 5.2.28):

$$(G,\phi)\mathrm{Strctrd}V\mathrm{Folds}(\mathbf{H}_{\cup}) \underset{\text{OrbSnglr}}{\overset{\text{Smth}}{\rightleftarrows}} (G,\phi)\mathrm{Strctrd}V\mathrm{Orbfld}(\mathbf{H}) \quad \simeq \tag{5.126}$$

$$\begin{array}{ccc} (\mathrm{Smth}(\mathscr{X}),(\tau,g)) & \longleftarrow & (\mathscr{X},(\tau,g)) \\ \| & & \| \\ (X,(\tau,g)) & \longmapsto & (\mathrm{OrbSnglr}(X),(\tau,g)) \end{array}$$

Remark 5.2.46 (Coefficients for orbifold cohomology)**.** The point of Def. 5.2.45 is that, by regarding a V-fold in the elastic ∞-topos $\mathbf{H}_{\cup}$ equivalently as an orbi-V-fold in the larger singular-elastic ∞-topos $\mathbf{H}$, a larger class of coefficients for intrinsic cohomology theories (1.21) becomes available, notably coefficients of the form $\int \curlyvee(A /\!\!/ G)$ (see Lemma 5.1.7 below). This is what gives rise, in §6, to proper orbifold cohomology (Def. 6.2.5 below) in contrast to the coarser cohomology of underlying étale groupoids (Def. 6.2.1 below).

Remark 5.2.47 (The proper ∞-category of higher orbifolds)**.** While (5.125) is an equivalence of abstract ∞-categories,

(i) it is not an equivalence of sub-∞-categories of the ambient singular-elastic ∞-topos $\mathbf{H}$:

$$\begin{array}{ccccc} \overset{\substack{\infty\text{-category of}\\ \text{of étale groupoids}}}{V\mathrm{Folds}(\mathbf{H}_{\cup})} & \not\simeq & \overset{\substack{\text{proper}\\ \infty\text{-category}\\ \text{of orbifolds}}}{V\mathrm{Orbfld}(\mathbf{H})} & \in (\mathrm{Cat}_\infty)_{/\mathbf{H}} \\ & {}_{\text{Smth}}\searrow & & \swarrow_{\text{OrbSnglr}} & \\ & & \mathbf{H} & & \end{array} \tag{5.127}$$

(ii) To bring out this distinction, also in view of Remark 5.2.46, we call $V\mathrm{Orbfld}(\mathbf{H})$ (Def. 5.2.45) the *proper ∞-category of orbifolds*, in contrast to the ∞-category $V\mathrm{Folds}(\mathbf{H}_{\cup})$ (5.44) of étale ∞-groupoids.

(iii) It is a happy coincidence that *proper* is also the technical adjective chosen in [DHLPS19] for equivariant homotopy theories presented by ∞-presheaves over categories of orbits with compact – hence finite if discrete – isotropy groups: In this terminology the singular-cohesive ∞-topos $\mathbf{H}$ is, according to Def. 4.2.3, indeed a *proper* global equivariant homotopy theory.

Example 5.2.48 (Subcategories of smooth and of flat orbifolds)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21), $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1) and $(G, \phi) \in \mathrm{Grp}(\mathbf{H})_{/\mathrm{Aut}(T_e V)}$ (Prop. 3.2.1, Def. 3.2.13, Ex. 4.1.30). We have fully faithful inclusions into the ∞-category of (G, ϕ)-structured orbi-V-folds (Def. 5.2.45)

$$
\begin{array}{ccc}
 & (G,\phi)\mathrm{Strctrd}V\mathrm{Orbfld}(\mathbf{H}) & \\
 \overset{\text{smooth orbifolds}}{\underset{i_{\cup}}{\nearrow}} & & \overset{\text{flat orbifolds}}{\underset{i_{\flat}}{\nwarrow}} \\
(G,\phi)\mathrm{Strctrd}V\mathrm{Folds}(\mathbf{H}_0) & & (\flat G, \phi \circ \varepsilon^{\flat})\mathrm{Strctrd}V\mathrm{Orbfld}(\mathbf{H})
\end{array}
\qquad (5.128)
$$

of

(i) smooth (G, ϕ)-structured V-folds, via Lemma 4.2.20;

(ii) flat $(\flat G, \phi \circ \varepsilon^{\flat})$-structured V-folds (Def. 5.2.40).

6

Orbifold cohomology

With an internal higher topos-theoretic characterization of orbifolds in hand (from §5), we immediately obtain an induced notion of (differential, geometric, étale) *orbifold cohomology*, namely as the intrinsic cohomology (1.21) of the ambient singular-cohesive ∞-topos. Here we discuss how this new intrinsic notion of orbifold cohomology
- subsumes proper equivariant cohomology theory (§6.1)
- and unifies it with tangentially twisted cohomology (§6.2).

The main result here is a general construction of orbifold étale cohomology which we show to naturally unify

(i) tangentially twisted cohomology of smooth but curved spaces with

(ii) RO-graded proper equivariant cohomology of flat but singular spaces.

As fundamental examples, we present a new model of twisted orbifold K-theory (following [SS25d, Ex. 4.5.4]) as well as tangentially twisted orbifold Cohomotopy.

6.1 Proper equivariant cohomology

6.1.1 Proper equivariant cohomology

Definition 6.1.1 (Borel equivariant cohomology)**.** Let $\mathbf{H}_{\cup}$ be a cohesive ∞-topos (Def. 4.1.1) $G \in \mathrm{Grp}(\mathbf{H}_{\cup})$ (Prop. 3.2.1) and $(X,\tau),(A,\rho) \in G\mathrm{Actions}(\mathbf{H}_{\cup})$ (Prop. 3.2.6). Then the *Borel equivariant cohomology* of X with coefficients in A is the intrinsic cohomology (1.21) in the slice $\mathbf{H}_{/\mathbf{B}G}$ (Prop. 3.1.46) of the homotopy quotient (3.131) of X with coefficients in the shape (4.2) of the homotopy quotient of A:

$$
\begin{aligned}
\underbrace{H_{\mathrm{Borel}}(X,A)}_{\text{Borel equivariant cohomology}} \;&:=\; \pi_0\, \mathbf{H}_{/\mathbf{B}G}\big((X /\!\!/ G),\, (A /\!\!/ G)\big)\\
&=\; \left\{
\begin{array}{ccc}
(X /\!\!/ G) & \overset{\text{cocycle}}{\underset{c}{\dashrightarrow}} & (A /\!\!/ G)\\
 & \Swarrow & \\
{\scriptstyle \tau}\searrow & & \swarrow{\scriptstyle \rho}\\
 & \mathbf{B}G &
\end{array}
\right\}_{/\sim}
\end{aligned}
\tag{6.1}
$$

Definition 6.1.2 (Proper equivariant cohomology)**.** Let $\mathbf{H}$ be a singular-cohesive ∞-topos (Def. 4.2.3), $G \in \mathrm{Grp}(\mathbf{H}_\flat)$ (Prop. 3.2.1) a discrete ∞-group, and $(X,\tau),(A,\rho) \in G\mathrm{Actions}(\mathbf{H})$ (Prop. 3.2.6). Then we say that the *proper equivariant cohomology* of X with coefficients in A is the intrinsic cohomology (1.21) in the slice $\mathbf{H}_{/\curlyvee \mathbf{B}G}$ (Prop. 3.1.46) of the orbi-singularization (4.144) of the homotopy quotient (3.131) of X with coefficients in the shape (4.2) of the orbi-singularization of the homotopy quotient of A:

$$
\begin{aligned}
\underset{\text{proper equivariant cohomology}}{H_G(X,A)} \;&:=\; \pi_0\, \mathbf{H}_{/\curlyvee \mathbf{B}G}\big(\curlyvee(X /\!\!/ G), \textstyle\int \curlyvee(A /\!\!/ G)\big) \\
&= \left\{
\begin{array}{ccc}
\curlyvee(X/\!\!/G) & \overset{\text{cocycle}\; c}{\dashrightarrow} & \int \curlyvee(A/\!\!/G) \\
& \Swarrow & \\
{\scriptstyle \curlyvee(\tau)} \searrow & & \swarrow {\scriptstyle (\eta^{\int}_{\curlyvee \mathbf{B}G})^{-1} \circ \int \curlyvee(\rho)} \\
& \curlyvee \mathbf{B}G &
\end{array}
\right\}_{/\sim}
\end{aligned}
\tag{6.2}
$$

6.1.2 Recovering traditional G-equivariant cohomology

We discuss how, in the case of a finite group G, traditional G-equivariant cohomology (see §2.2) is a special case of proper equivariant cohomology (Def. 6.1.2). We take the key observation from [Re14] (Prop. 6.1.6 below).

Definition 6.1.3 (G-equivariant cohomology ∞-topos [SS25d, Def. 3.3.64])**.** Let $\mathbf{H}_\cup$ be a cohesive ∞-topos (Def. 4.1.1) and $G \in \mathrm{Grp}^{\mathrm{fin}}$ a finite group (4.154). We write

$$
G\mathbf{H}_\cup \;:=\; \mathrm{Shv}_\infty\big(G\mathrm{Orb}, \mathbf{H}_\cup\big) \;=\; \mathrm{Func}_\infty\big(G\mathrm{Orb}^{\mathrm{op}}, \mathbf{H}_\cup\big) \tag{6.3}
$$

for the ∞-topos of $\mathbf{H}_\cup$-valued ∞-sheaves on the G-orbit category (Def. 2.2.8), to be called the corresponding *G-equivariant cohesive ∞-topos*.

Remark 6.1.4 (Proper equivariant cohomology theory in singular ∞-toposes)**.** In the case $\mathbf{H}_\cup \simeq \mathrm{Grpd}_\infty$ (3.20), Def. 6.1.3 reduces to the ∞-category $G\mathrm{Grpd}_\infty$ (Def. 2.2.4) of traditional G-equivariant homotopy theory (recalled in §2.2). The intrinsic cohomology (1.21) of the ∞-topos $G\mathrm{Grpd}_\infty$ – or of its tangent ∞-topos $T\big(G\mathrm{Grpd}_\infty\big)$ (Ex. 3.1.51) in the twisted abelian case (Remark 3.2.23) – is *proper equivariant cohomology* (following terminology in [DHLPS19]), including G-Bredon cohomology [Br67a][Br67b] (review in [Blu17, §1.4][tD79, §7]), G-equivariant K-theory [Se68][AS69] (which is proper equivariant by [AS04, A3.2][FHT07, A.5][DL98]), G-equivariant Cohomotopy theory [Se71][tD79, §8][SS20][BSS19], etc.

Hence, by Remark 4.1.20, to the extent that the objects of the cohesive ∞-topos $\mathbf{H}_\cup$ in Def. 6.1.3 are ∞-groupoids equipped with further geometric or differential-geometric structure, the intrinsic cohomology theory (1.21) in $G\mathbf{H}_\cup$ (6.3) is an enhancement of plain G-equivariant cohomology to a flavor of *proper G-equivariant differential cohomology* theory (by Remark 4.1.20).

Proposition 6.1.5 (Cohesive Elmendorf theorem)**.** *Consider a cohesive ∞-topos $\mathbf{H}_\cup$ (Def. 4.1.1) with an ∞-site* Chrt *of charts (Def. 4.1.9). Then for $G \in \mathrm{Grp}^{\mathrm{fin}}$ a finite* *group, we have an equivalence of ∞-categories*

$$G\mathbf{H}_{\cup} \simeq \mathrm{Shv}_\infty\big(\mathrm{Chrt}, G\mathrm{Grpd}_\infty\big)\,, \tag{6.4}$$

where $G\mathrm{Grpd}_\infty$ *is the ∞-category of D-topological G-spaces (Def. 2.2.4).*

Proof. Consider the following sequence of ∞-functors:

$$\begin{aligned} G\mathbf{H}_{\cup} &:= \mathrm{Shv}_\infty\big(G\mathrm{Orb}, \mathbf{H}_{\cup}\big) \\ &= \mathrm{Shv}_\infty\big(G\mathrm{Orb}, \mathrm{Shv}_\infty(\mathrm{Chrt})\big) \\ &\xrightarrow{\simeq} \mathrm{Shv}_\infty\big(G\mathrm{Orb} \times \mathrm{Chrt}\big) \\ &\xrightarrow{\simeq} \mathrm{Shv}_\infty\big(\mathrm{Chrt}, \mathrm{Shv}_\infty(G\mathrm{Orb})\big) \\ &\xrightarrow{\simeq} \mathrm{Shv}_\infty\big(\mathrm{Chrt}, G\mathrm{Grpd}\big)\,. \end{aligned}$$

That the first and second of these ∞-functors are equivalences follows by the product/hom-adjunction for ∞-functors. With that, the last equivalence follows, objectwise, by Elmendorf's theorem (Prop. 2.2.10). □

Proposition 6.1.6 (G-equivariant homotopy theory embeds into G-singular cohesion)**.** *Let* $\mathbf{H}$ *be a singular-cohesive ∞-topos (Def. 4.2.3) over* Grpd_∞ (3.20) *and let* $G \in \mathrm{Grp}^{\mathrm{fin}}$ *be a finite group* (4.154).

(i) *Then there is a full sub-∞-category inclusion*

$$G\mathbf{H}_{\cup} \underset{\simeq}{\overset{\Delta_G}{\hookrightarrow}} \mathbf{H}_{/\,\mathcal{Y}^{G}} \tag{6.5}$$

of the G-equivariant non-singular cohesive ∞-topos (Def. 6.1.3) into the slice of $\mathbf{H}$ *(Prop. 3.1.46) over the generic G-orbi singularity* (4.151).

(ii) *This is such that, when pre-composed with the cohesive Elmendorf equivalence (Prop. 6.1.5), a cohesive sheaf (on* Chrt*) of* $G\mathrm{Grpd}$ (2.19) *presented* (2.22) *by D-topological G-spaces* X_U *(Def. 2.2.1) is sent to the presheaf on* Snglrt *that is given as follows:*

$$\begin{array}{ccc} \mathrm{Shv}_\infty\big(\mathrm{Chrt},\, G\mathrm{Grpd}_\infty\big) \simeq G\mathbf{H}_{\cup} & \xrightarrow{\;\sum_{\mathcal{Y}^{G}} \Delta_G\;} & \mathrm{Shv}_\infty\big(\mathrm{Chrt} \times \mathrm{Snglrt}\big) \\ \big(U \mapsto \mathrm{Shp}_{G\mathrm{Top}}(X_U)\big) & \longmapsto & \Big(\big(U, \mathcal{Y}^{K}\big) \mapsto \mathrm{Shp}_{\mathrm{Top}}\Big(\Big(\coprod_{\phi \in \mathrm{Grp}(K,G)} X_U^{\phi(K)}\Big) \underset{G}{\times} EG\Big)\Big) \end{array} \tag{6.6}$$

where on the right we have the topological shape (3.1.13) *of the Borel construction by the residual G-action on the fixed point subspaces* $X_U^{\phi(K)} \subset X_U$ (2.15).

Proof. For $\mathbf{H}_{\cup} \simeq \mathrm{Grpd}_\infty$ this is [Re14, Prop. 3.5.1]; our expression $\mathrm{Shp}_{\mathrm{Top}}\big(X_U^{\phi(K)} \underset{G}{\times} EG\big)$ is, up to convention of notation, the expression for $B\mathrm{Fun}(H, G \curvearrowright X_U)$ that is spelled out in [Re14, p. 7][Lu19, 3.2.17] (using that our G is discrete). The generalization here follows immediately by applying this equivalence objectwise in the ∞-site Chrt. □

The following is our key class of examples:

Example 6.1.7 (Cohesive shape of G-orbi-singular space is G-homotopy type)**.** In the cohesive ∞-topos $\mathbf{H}_{\cup} := \mathrm{SmthGrpd}_\infty$ (Ex. 4.1.18) consider a 0-truncated object $X \in \mathbf{H}_{\cup,0}$ equipped with a G-action (Def. 3.130) of a discrete group G, and with corresponding Cohesive G-orbispace (Prop. 5.1.6)

$$\mathscr{X} \;:=\; \mathrm{OrbSnglr}(X /\!\!/ G)$$

in $\mathbf{H} := \mathrm{SnglrSmthGrpd}_\infty$ (Ex. 4.2.11), which is either of:

(i) a smooth G-orbifold (Ex. 5.1.10):

$$X \;\in\; \mathrm{SmoothManifolds} \hookrightarrow \mathrm{DiffeologicalSpaces} \hookrightarrow \mathbf{H}_{\cup}$$

(ii) a topological G-orbi space (Ex. 5.1.11):

$$X \in \;\mathrm{TopSpc} \xrightarrow{\mathrm{Cdfflg}} \mathrm{DTopSpc} \hookrightarrow \mathbf{H}_{\cup}$$

Then the cohesive shape (4.143) of the G-orbi-singular space $\mathscr{X} \in \mathbf{H}$ is equivalent, under the identification of Prop. 6.1.6, to the G-topological shape (2.22) of the underlying topological G-space of X:

(i) By Prop. 5.1.12, comparing (5.38) with (6.6) we have:

$$\begin{array}{ccc}
G\mathrm{SmoothManifolds} & \xrightarrow[\mathrm{OrbSnglr}((-)/\!\!/G)]{\text{form Fréchet-smooth orbifold}} & \mathrm{SnglrSmthGrpd}_{\infty / \mathcal{Y}^G} \\
{\scriptstyle \text{form } G\text{-topological shape}}\; \Big\downarrow {\scriptstyle \mathrm{Shp}_{G\mathrm{Top}}(\mathrm{Dtplg}(-))} & & {\scriptstyle \mathrm{Shp}} \Big\downarrow \; {\scriptstyle \text{form cohesive shape}} \\
G\mathrm{Grpd}_\infty & \xhookrightarrow[\text{include } G\text{-equivariant homotopy theory}]{\Delta_G} & \mathrm{SingularGroupoids}_{\infty / \mathcal{Y}^G}
\end{array} \tag{6.7}$$

(ii) By Prop. 5.1.13, comparing (5.41) with (6.6) we have:

$$\begin{array}{ccc}
G\mathrm{TopSpc} & \xrightarrow[\mathrm{OrbSnglr}(\mathrm{Cdfflg}(-)/\!\!/G)]{\text{form topological } G\text{-orbi space}} & \mathrm{SnglrSmthGrpd}_{\infty / \mathcal{Y}^G} \\
{\scriptstyle \text{form } G\text{-topological shape}}\; \Big\downarrow {\scriptstyle \mathrm{Shp}_{G\mathrm{Top}}} & & {\scriptstyle \mathrm{Shp}} \Big\downarrow \; {\scriptstyle \text{form cohesive shape}} \\
G\mathrm{Grpd}_\infty & \xhookrightarrow[\text{include } G\text{-equivariant homotopy theory}]{\Delta_G} & \mathrm{SingularGroupoids}_{\infty / \mathcal{Y}^G}
\end{array} \tag{6.8}$$

Lemma 6.1.8 (Δ_G commutes with Disc)**.** *The construction Δ_G from Prop. 6.1.6 commutes with embedding of discrete cohesive structure* (4.142)*:*

$$\begin{array}{ccccc}
 & & \mathrm{Shv}_\infty\big(\mathrm{Snglrt}, \mathrm{Grpd}_\infty\big)_{/\mathcal{Y}^G} & & \\
 & \overset{\Delta_G}{\nearrow} & & \overset{\mathrm{Disc}}{\searrow} & \\
G\mathrm{Grpd} & & & & \mathrm{Shv}_\infty\big(\mathrm{Snglrt}, \mathbf{H}_{\cup}\big)_{/\mathcal{Y}^G} \\
 & \underset{\mathrm{Disc}}{\searrow} & & \underset{\Delta_G}{\nearrow} & \\
 & & G\mathbf{H}_{\cup} & &
\end{array}$$

Theorem 6.1.9 (Cohomology of good orbispaces is proper equivariant cohomology)**.** *Consider the singular-cohesive ∞-topos $\mathbf{H} := \mathrm{SnglrSmthGrpd}_\infty$ (Ex. 4.2.11) and let $G \in \mathrm{Grp}^{\mathrm{fin}}$ be a discrete group* (4.154)*. Then the intrinsic cohomology* (1.21)

(i) *of a G-orbi-singular space* $\mathscr{X} \in \mathbf{H}_{/\!\!\curlyvee^G}$ *(Def. 5.1.4) which is either*

(a) *a topological G-orbi-space (Ex. 5.1.11) with universal covering space (Def. 5.1.5)* $X_{G\mathrm{top}} \in G\mathrm{TopSpc}$ (2.12)*;*

(b) *a Fréchet-smooth G-orbifold (Ex. 5.1.10) with universal covering space (Def. 5.1.5)* $X \in$ *FréchetManifolds and underlying G-topological space* $X_{G\mathrm{top}} := \mathrm{Dtplg}(\mathrm{X})$ (3.12)*;*

(ii) *with coefficients in a cohesively discrete G-∞-groupoid A* (2.19) *(hence the G-topological shape* (2.22) *of some topological G-space* $A_{G\mathrm{top}}$*) regarded as a geometrically discrete orbi-singular ∞-groupoid* $\mathscr{A}$ *via* (6.5)*:*

$$\begin{array}{ccccccc} G\mathrm{TopSpc} & \xrightarrow{\mathrm{Shp}_{G\mathrm{Top}}} & G\mathrm{Grpd}_\infty & \xrightarrow{\mathrm{Disc}} & G\mathbf{H}_{\cup} & \xrightarrow{\Delta_G} & \mathbf{H}_{/\!\!\curlyvee^G} \\ A_{\mathrm{top}} & \longmapsto & A & \longmapsto & & & \mathscr{A} \end{array}$$

equals the proper G-equivariant cohomology (Def. 2.2.6) of $X_{G\mathrm{top}}$ *with coefficients in A:*

$$\mathbf{H}_{/\!\!\curlyvee^G}\big(\mathscr{X}, \mathscr{A}\big) \simeq G\mathrm{Grpd}_\infty\big(\mathrm{Shp}_{G\mathrm{Top}}(X_{G\mathrm{top}}), A\big)$$

$$\text{hence:} \quad \underbrace{\pi_n \mathbf{H}_{/\!\!\curlyvee^G}\big(\mathscr{X}, \mathscr{A}\big)}_{\substack{\text{intrinsic} \\ \text{equivariant differential cohomology} \\ \text{in } \infty\text{-topos of} \\ \text{singular smooth } \infty\text{-groupoids}}} \simeq \underbrace{H_G^{-n}(X_{G\mathrm{top}}, A)}_{\substack{\text{proper} \\ G\text{-equivariant cohomology}}}$$

Proof. **(i)** By Ex. 5.1.11 the topological G-orbi space $\mathscr{X}$ is given by

$$\mathscr{X} \simeq \mathrm{OrbSnglr}\big(\mathrm{Cdfflg}(X)/\!\!/G\big).$$

With this, we compute as follows:

$$\begin{aligned} \mathbf{H}_{/\!\!\curlyvee^G}\big(\mathscr{X}, \mathscr{A}\big) &= \mathbf{H}_{/\!\!\curlyvee^G}\Big(\mathrm{OrbSnglr}\big(\mathrm{Cdfflg}(X_{\mathrm{top}})/\!\!/G\big), \Delta_G \mathrm{Disc}(A)\Big) \\ &\simeq \mathbf{H}_{/\!\!\curlyvee^G}\Big(\mathrm{OrbSnglr}\big(\mathrm{Cdfflg}(X_{\mathrm{top}})/\!\!/G\big), \mathrm{Disc}(\Delta_G A)\Big) \\ &\simeq (\mathrm{Grpd}_\infty)_{/\!\!\curlyvee^G}\Big(\mathrm{Shp}\big(\mathrm{OrbSnglr}(\mathrm{Cdfflg}(X_{\mathrm{top}})/\!\!/G)\big), \Delta_G A\Big) \\ &\simeq (\mathrm{Grpd}_\infty)_{/\!\!\curlyvee^G}\big(\Delta_G X, \Delta_G A\big) \\ &\simeq G\mathrm{Grpd}\big(\mathrm{Shp}_{G\mathrm{Top}}(X_{\mathrm{top}}), A\big). \end{aligned} \tag{6.9}$$

Here the first step, after unwinding the definitions, is Lemma 6.1.8, the second step is the Shp ⊣ Disc-adjunction (4.142), the third step is Prop. 5.1.13, and the last step is Prop. 6.1.6.

(ii) By Ex. 5.1.10 the Fréchet-smooth G-orbifold $\mathscr{X}$ is given by

$$\mathscr{X} \simeq \mathrm{OrbSnglr}\big(X/\!\!/G\big).$$

With this, we compute just as in (6.9) only that now in the third step we use Prop. 5.1.12.

□

Example 6.1.10 (Orientifold cohomology)**.** Take the singular elastic ∞-topos $\mathbf{H} = \mathrm{SnglrJetSmthGrpd}_\infty$ (Ex. 4.2.11) and $V = (\mathbb{R}^n, +) \in \mathbf{H}_{\cup}$ (5.48). Then a $\mathscr{X}_{\cup} \in V\mathrm{Flds}(\mathbf{H}_{\cup})$ (Def. 5.2.1) is an ordinary n-dimensional orbifold or, more generally, an n-dimensional étale ∞-stack (by Ex. 5.2.5) with structure group (Def. 5.2.11) the ordinary general linear group $\mathbf{Aut}(T_e V) \simeq \mathrm{GL}(n)$ (by Ex. 5.2.12). Hence, the composition of the delooping (3.121) of the ordinary determinant group homomorphism $\mathrm{GL}(n) \xrightarrow{\mathrm{det}} \mathbb{Z}_2$ with the classifying map $\vdash \mathrm{Frm}(\mathscr{X}_{\cup})$ (5.66) of the frame bundle of X (Def. 5.2.13) realizes $\mathscr{X}_{\cup}$ as an object in the slice ∞-topos (Prop. 3.1.46) over $\mathbf{B}\mathbb{Z}_2$. Consequently, it realizes its orbi-singularization $\mathscr{X} := \curlyvee \mathscr{X}_{\cup} \in \mathbf{H}$ (4.2.7) as an object in the slice over $\curlyvee^{\mathbb{Z}_2}$ (4.135):

$$
\begin{array}{cc}
 & \begin{array}{c} \mathscr{X}_{\cup} \\ {\scriptstyle \mathbf{B}\mathrm{det} \circ \vdash \mathrm{Frm}(\mathscr{X}_{\cup})} \big\downarrow \\ \mathbf{B}\mathbb{Z}_2 \end{array} \in \big(\mathbf{H}_{\cup}\big)_{/\mathbf{B}\mathbb{Z}_2} \\
\Leftrightarrow & \begin{array}{c} \mathscr{X} \\ {\scriptstyle \curlyvee(\mathbf{B}\mathrm{det} \circ \vdash \mathrm{Frm}(\mathscr{X}_{\cup}))} \big\downarrow \\ \curlyvee^{\mathbb{Z}_2} \end{array} \in \big(\mathbf{H}_{\cup}\big)_{/\curlyvee^{\mathbb{Z}_2}} .
\end{array}
\tag{6.10}
$$

This is the incarnation of the orbifold as an *orbi-orientifold* [DFM11][FSS15, 4.4][SS20]. In particular, if the covering space (Def. 5.1.5)

$$X := \mathrm{fib}\big(\mathbf{B}\mathrm{det} \circ \vdash \mathrm{Frm}(\mathscr{X}_{\cup})\big)$$

happens to be an $\mathbb{R}^n$-fold (Ex. 5.2.4), we have just a plain *orientifold* (without further orbifolding) and then the intrinsic cohomology (1.21) of $\mathscr{X}$ regarded in the slice over $\curlyvee^{\mathbb{Z}_2}$(6.10) is, by Theorem 6.1.9 the proper $\mathbb{Z}_2$-equivariant cohomology of X, such as, for instance, Real K-theory [At66] (see [Mas11] for the perspective in proper equivariant cohomology) or $\mathbb{Z}_2$-Equivariant Cohomotopy [tD79, 8.4][SS20].

6.2 Proper orbifold cohomology

We introduce general *étale cohomology* of étale ∞-stacks (Def. 6.2.1), which is sensitive to geometric G-structure and to tangential structure (Def. 6.2.3). Promoting this to the *proper* incarnation of orbifolds (Remark 5.2.47), we finally obtain *tangentially twisted proper orbifold cohomology* (Def. 6.2.5) which we prove unifies tangentially twisted topological cohomology away from orbifold singularities with proper equivariant cohomology at the singularities (Thm. 6.2.6 below, cf. Fig. 1.7 on p. 21).

As a fundamental class of examples, we construct tangentially-twisted proper orbifold Cohomotopy theories (Def. 6.2.18) and observe, as an application, that these subsume the relevant cohomology theories for the physics of fractional quantum anomalous Hall systems and of M-theory, according to "Hypothesis H" (Remark 6.2.20).

6.2.1 Cohomology of V-étale ∞-stacks

Definition 6.2.1 (Étale cohomology)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21), $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1), $(G,\phi) \in \mathrm{Grp}(\mathbf{H})_{/\mathrm{Aut}(T_e V)}$ (Def. 5.2.22), and $X \in \mathrm{Intgrbl}(G,\phi)\mathrm{StrctrdVFlds}(\mathbf{H})$ (Def. 5.2.29). The *étale cohomology* of $\big(X,(\tau,g)\big)$ is its intrinsic cohomology (1.21) when regarded (via Prop. 5.2.33)

$$\begin{array}{ccc} \mathrm{Intgrbl}(G,\phi)\mathrm{StrctrdVFlds}(\mathbf{H}) & \longrightarrow & \Big(\big(\mathbf{H}_{/\mathbf{B}\mathrm{Aut}(T_eV)}\big)_{/(\mathbf{B}G,\mathbf{B}\phi)}\Big)_{/\left(\mathscr{H}\mathrm{aef}\left(V,(G,\phi)\right),(\tau_{\mathscr{H}},g_{\mathscr{H}})\right)} \\ \big(X,(\tau,g)\big) & \longmapsto & \Big(\big(X,(\tau,g)\big) \xrightarrow[\mathrm{met}]{\vdash(\tau,g)} \mathscr{H}\mathrm{aef}\big(V,(G,\phi)\big)\Big) \end{array}$$

in the iterated slice of (5.94) over the $\big(V,(G,\phi)\big)$-Haefliger stack (Def. 5.2.32) equipped with its canonical (G,ϕ)-structure $(\tau_{\mathscr{H}},g_{\mathscr{H}})$ (Prop. 5.2.33), hence is *G-structure-twisted cohomology* (Remark 3.2.21):

$$\begin{aligned} &\overset{\text{étale cohomology}}{H^{(\tau,g)}\big(X,A\big)} := \\ &\Big(\big(\mathbf{H}_{/\mathbf{B}\mathrm{Aut}(T_eV)}\big)_{/(\mathbf{B}G,\mathbf{B}\phi)}\Big)_{/\left(\mathscr{H}\mathrm{aef}\left(V,(G,\phi)\right),(\tau_{\mathscr{H}},g_{\mathscr{H}})\right)}\Big(\big(X,(\tau,g)\big)\,,\,\big(A,p\big)\Big) \\ &= \left\{ \begin{array}{ccc} X & \overset{\text{cocycle}}{\underset{c}{\dashrightarrow}} & A \\ & \Downarrow & \\ {\scriptstyle \vdash(\tau,g)} \searrow & & \swarrow {\scriptstyle p} \\ & \mathscr{H}\mathrm{aef}(V,(G,\phi)) & \end{array} \right\} \end{aligned} \tag{6.11}$$

Remark 6.2.2 (Étale cohomology is geometric)**.** As the notation in Def. 6.2.1 indicates, étale cohomology is a "geometric cohomology theory" in that it does depend (in general) on the G-structure g on the V-fold X (for instance its complex- or symplectic- or Riemannian- or Lorentzian structure structure, by Ex. 5.2.31).

Next we focus attention on the special case where the cohomology theories are not sensitive to the metric part g of a G-structure (τ,g), but just to its tangential structure τ.

Definition 6.2.3 (Tangentially twisted cohomology)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21), $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1), $(G,\phi) \in \mathrm{Grp}(\mathbf{H})_{/\mathrm{Aut}(T_eV)}$ (Def. 5.2.22), $(A,\rho) \in G\mathrm{Actions}(\mathbf{H})$ and $X \in (G,\phi)\mathrm{StrctrdVFlds}(\mathbf{H})$ (5.95). Then, for $A \in \mathbf{H}_{/\mathbf{B}\int G}$, the *tangentially twisted cohomology* of V with coefficients in A is (see Remark

3.2.21)

$$\overset{\text{tangentially twisted cohomology}}{H^{\int\tau}(X,A)} := \mathbf{H}_{/\int \mathrm{Aut}(T_eV)}\big((X,\eta^{\int}\circ\tau),(A/\!\!/G,\rho)\big) = \left\{ \begin{array}{ccc} X & \overset{\text{cocycle}}{\overset{c}{\dashrightarrow}} & (\int A)/\!\!/(\int G) \\ & {\scriptstyle \eta^{\int}\circ\tau} \searrow \; \Swarrow \; \swarrow {\scriptstyle \int\rho} & \\ & \mathbf{B}\int G & \end{array}\right\} \tag{6.12}$$

Remark 6.2.4 (Need for G-Structure vs. tangential structure)**.**

(i) The notion of tangentially twisted cohomology in Definition 6.2.3 makes sense more generally for V-folds equipped only with tangential structure (Def. 5.2.35) instead of full G-structure (Def. 5.2.23) (hence only with a reduction of the shape of their structure group, instead of the actual structure group (Def. 5.2.11)) and it only needs A to be equipped with a $\int G$-action.

(ii) We state the definition in the more restrictive form above just in order to bring out the following promotion of this notion to its proper orbifold version (Remark 5.2.47), in Def. 6.2.5 below. The process of orbi-singularization is in fact sensitive to the full G-structure, and not just to its tangential shape. More precisely, it is sensitive to the *geometric fixed point spaces* of the G-structure and not just to its homotopy fixed point spaces (as per Remark 4.2.23 Ex. 4.2.26).

6.2.2 Tangentially twisted proper orbifold cohomology

We now promote tangentially twisted cohomology of V-folds (Def. 6.2.3) to a *proper orbifold cohomology* theory in the sense of Def. 5.2.45.

Definition 6.2.5 (Tangentially twisted proper orbifold cohomology)**.** Let

- $\mathbf{H}$ be a singular-elastic ∞-topos (Def. 4.2.6).
- $V \in \mathrm{Grp}(\mathbf{H}_{\cup})$ (Prop. 3.2.1).
- $(G,\phi) \in \mathrm{Grp}(\mathbf{H}_{\cup})_{/\mathrm{Aut}(T_eV)}$ (Prop. 3.2.1, Def. 3.2.13, Ex. 4.1.30).
- $\mathscr{X}_{\cup} \in V\mathrm{Flds}(\mathbf{H}_{\cup})$ (Def. 5.2.1).
- $(\tau, g) \in (G,\phi)\mathrm{Structures}_{\mathscr{X}_{\cup}}(\mathbf{H}_{\cup})$ (Def. 5.2.23).
- $(A,\rho) \in G\mathrm{Actions}(\mathbf{H}_{\cup})$.

and set $\mathscr{A} := \curlyvee(A/\!\!/G)$ and $\mathscr{X} := \curlyvee\mathscr{X}_{\cup}$.
The *tangentially twisted proper orbifold cohomology* of $\mathscr{X}$ with coefficients in $\int\mathscr{A}$

is (see Rem. 3.2.21)

$$
\begin{aligned}
H^{\int\curlyvee\tau}\big(\mathscr{X},\mathscr{A}\big) &:= \pi_0\,\mathbf{H}_{/\int\curlyvee\mathbf{B}G}\Big(\big(\mathscr{X},\eta^{\int}\circ\curlyvee(\tau)\big),\big(\textstyle\int\mathscr{A},\int\curlyvee\rho\big)\Big)\\
&= \left\{
\begin{array}{ccc}
\mathscr{X} & \xrightarrow[\;c\;]{\text{cocycle}} & \int\curlyvee\big(A/\!\!/G\big)\\
& {\scriptstyle \eta^{\int}\circ\curlyvee(\tau)}\searrow \quad \swarrow{\scriptstyle \int\curlyvee(\rho)} & \\
& \int\curlyvee\mathbf{B}G &
\end{array}
\right\}_{/\sim}
\end{aligned}
$$

Theorem 6.2.6 (Tangentially twisted orbifold cohomology at and away from singularities)**.** *Consider the tangentially twisted orbifold cohomology of Def. 6.2.5 restricted to* **(1)** *smooth and* **(2)** *flat orbifolds, according to Ex. 5.2.48. Then* (cf. Fig. 1.7 on p. 21)*:*

(i) *The tangentially twisted orbifold cohomology of flat orbifolds for 0-truncated coefficients A is naturally equivalent to the proper equivariant cohomology (Def. 6.1.2) of the total space of their $\flat G$-frame bundle* (5.82)*:*

$$
\underset{\substack{\text{flat}\\ \text{orbifold}}}{\overset{\substack{\text{tangentially twisted}\\ \text{orbifold cohomology}}}{H^{\int\curlyvee\tau}\big(i_{\flat}\mathscr{X},\mathscr{A}\big)}} \;\simeq\; \underset{\flat G\text{-frame bundle}}{\overset{\substack{\text{proper}\\ \text{equivariant cohomology}}}{H_{\flat G}\big((\flat G)\mathrm{Frm}(\mathscr{X}_{\cup}),A\big)}}.
$$

(ii) *The tangentially twisted orbifold cohomology of smooth (non-orbi-singular) orbifolds is equivalently the tangentially twisted cohomology (Def. 6.2.1) of the underlying V-folds:*

$$
\underset{\substack{\text{smooth}\\ \text{orbifold}}}{\overset{\substack{\text{tangentially twisted}\\ \text{orbifold cohomology}}}{H^{\int\curlyvee\tau}\big(i_{\cup}\mathscr{X},\mathscr{A}\big)}} \;\simeq\; \underset{\substack{\text{0-truncated}\\ V\text{-fold}}}{\overset{\substack{\text{tangentially twisted}\\ V\text{-fold cohomology}}}{H^{\int\tau}\big(\mathscr{X}_{\cup},A\big)}}.
$$

Proof. The case **(i)** means that the classifying map of the G-structure in question factors as follows, where we use Prop. 5.2.24 to identify the leftmost morphism ρ as exhibiting the action (3.130) of $\flat G$ on $(\flat G)\mathrm{Frm}(X)$:

$$
(\flat G)\mathrm{Frm}(X)/\!\!/\flat G \xrightarrow{\;\rho\;} \mathbf{B}\flat G \xrightarrow{\;\varepsilon^{\flat}_{\mathbf{B}G}\;} \mathbf{B}G \longrightarrow \mathbf{B}\mathrm{Aut}(T_eV)\,.
$$

Now we observe:

(a) with Def. 4.2.3 that $\int$ acts objectwise over Snglrt,

(b) with Prop. 3.1.39 that the pullback of presheaves over Snglrt is computed objectwise,

(c) and with Lemma 5.1.7 that $\curlyvee(A/\!\!/G)$ is objectwise over Snglrt a homotopy quotient by G, so that Lemma 4.1.6 applies objectwise over Snglrt to give the pullback square shown on the right here:

$$
\begin{array}{ccccc}
 & & \int\prec\big(A/\!\!/\flat G\big) & \longrightarrow & \int\prec\big(A/\!\!/G\big) \\
 & \nearrow & \downarrow & \text{(pb)} & \downarrow \\
\prec\big((\flat G)\mathrm{Frm}(X)/\!\!/(\flat G)\big) & \xrightarrow{\ \prec\rho\ } & \prec\mathbf{B}\flat G & \longrightarrow & \int\prec\mathbf{B}G\,. \\
 & & \underset{\eta^{\int}\circ\prec\tau}{\longrightarrow} & &
\end{array}
$$

By the universal property of the pullback, this means that every cocycle factors naturally as shown by the dashed morphism. But by Def. 6.1.2 this dashed morphism is equivalently a cocycle in proper equivariant cohomology, as claimed.

The case **(ii)** means (using Lemma 4.2.20) that the orbi-singular space $\mathscr{X}$ is in fact smooth

$$\mathscr{X} \simeq \cup\mathscr{X} \simeq \mathrm{NnOrbSnglr}\big(\mathscr{X}_{\cup}\big)\,.$$

Therefore, we have the following natural equivalences of spaces of dashed morphisms:

$$
\begin{array}{ccccc}
\begin{array}{ccc} & \nearrow & \int\prec(A/\!\!/G) \\ & & \downarrow \\ \cup\mathscr{X}\simeq\mathscr{X} & \longrightarrow & \int\prec\mathbf{B}G \end{array}
& \Leftrightarrow &
\begin{array}{ccc} & \nearrow & \cup\int\prec(A/\!\!/G) \\ & & \downarrow \\ \cup\mathscr{X} & \longrightarrow & \cup\int\prec\mathbf{B}G \end{array}
& \Leftrightarrow &
\begin{array}{ccc} & \nearrow & \int(A/\!\!/G) \\ & & \downarrow \\ \mathscr{X}_{\cup} & \longrightarrow & \int\mathbf{B}G \end{array}
\\
\pitchfork & & & & \pitchfork \\
\mathbf{H}_{/\int\prec\mathbf{B}G}\Big(\mathscr{X},\int\prec\big(A/\!\!/G\big)\Big) & & \simeq & & \mathbf{H}_{/\int\mathbf{B}G}\Big(\mathscr{X}_{\cup},\int\big(A/\!\!/G\big)\Big)\,.
\end{array}
\tag{6.13}
$$

Here the first equivalence is by the adjunction NnOrbSnglr $\dashv$ Smth and the fully faithfulness of NnOrbSnglr (4.142). The second step uses $\cup\circ\int\simeq\int\circ\cup$ (Lemma 4.2.22) and $\cup\circ\prec\simeq\cup$ (Remark 4.2.8) But on the right of (6.13) we see the tangentially twisted cohomology of $\mathscr{X}_{\cup}$, as claimed. □

6.2.3 J-Twisted orbifold Cohomotopy theory

We discuss now the example of tangentially twisted proper orbifold cohomology (Def. 6.2.5) where the coefficients are (shapes of) spheres, specifically of *Tate V-spheres* (Def. 6.2.9), In this case the tangential twist is the *J-homomorphism* (Def. 6.2.14) whence we speak of *J-twisted Cohomotopy theory* (Def. 6.2.18).

Definition 6.2.7 (Complement of neutral element)**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30) and $V\in\mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1). Let $(V,\rho_{\mathrm{Aut}_{\mathrm{Grp}}})\in\mathrm{Aut}_{\mathrm{Grp}}(V)\mathrm{Actions}(\mathbf{H})$ denote the group-automorphism action on V (Prop. 3.2.29).

(i) Consider those subobjects (Def. 3.1.61) of the homotopy quotient $V/\!\!/\mathrm{Aut}_{\mathrm{Grp}}$ (3.175) whose pullback along the morphism

$$*/\!\!/\mathrm{Aut}_{\mathrm{Grp}}(V)\xrightarrow{e/\!\!/\mathrm{Aut}_{\mathrm{Grp}}(V)}V/\!\!/\mathrm{Aut}_{\mathrm{Grp}}(V)\ ,$$

which exhibits the neutral element as a fixed point of the group-automorphism action (Prop. 3.2.29), is empty. These are the subobjects forming the poset in the top left of the following Cartesian square (of ∞-categories):

$$\begin{array}{ccc} \mathrm{SubObjects}_{\notin}\big(V /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(V)\big) & \longrightarrow & * \\ \big\downarrow & \text{(pb)} & \big\downarrow \varnothing \\ \mathrm{SubObjects}\big(V /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(V)\big) & \xrightarrow[(e /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(V))^*]{} & \mathrm{SubObjects}(*) \end{array} \tag{6.14}$$

(ii) Consider next the union of these subobjects, hence the colimit over the left vertical functor in (6.14), which we denote as follows:

$$\big(V \setminus \{e\}\big) /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(V) \;:=\; \varinjlim \Big(\mathrm{SubObjects}_{\notin}\big(V /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(V)\big) \hookrightarrow \mathrm{SubObjects}\big(V /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(V)\big) \Big). \tag{6.15}$$

(iii) We call the homotopy fiber $V \setminus \{e\}$ of the canonical morphism from this object (6.15) to $\mathbf{B}\mathrm{Aut}_{\mathrm{Grp}}(G)$ the *complement of the neutral element of* V

$$\begin{array}{ccc} V \setminus \{e\} & \xrightarrow{\mathrm{fib}(\rho_{\mathrm{Aut}_{\mathrm{Grp}}} \setminus \{e\})} & (V \setminus \{e\}) /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(V) \\ & & \big\downarrow \\ & & V /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(V) \\ & & \big\downarrow \rho_{\mathrm{Aut}_{\mathrm{Grp}}} \\ & & \mathbf{B}\mathrm{Aut}_{\mathrm{Grp}}(V) \end{array} \quad \rho_{\mathrm{Aut}_{\mathrm{Grp}}} \setminus \{e\} \tag{6.16}$$

(iv) We regard the complement of the neutral element as equipped with the $\mathrm{Aut}_{\mathrm{Grp}}(V)$-action which is exhibited by the homotopy fiber sequence (6.16) (by Prop. 6.16):

$$\big(V \setminus \{e\}, \rho_{\mathrm{Aut}_{\mathrm{Grp}}} \setminus \{E\}\big)\big) \in \mathrm{Aut}_{\mathrm{Grp}}(V)\mathrm{Actions}(\mathbf{H}).$$

Proposition 6.2.8 (Basic properties of complement of neutral element)**.** *Let* $\mathbf{H}$ *be an* ∞*-topos (Def. 3.1.30) and* $V \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1). Then the complement* $V \setminus \{e\}$ *of the neutral element (Def. 6.2.7)*

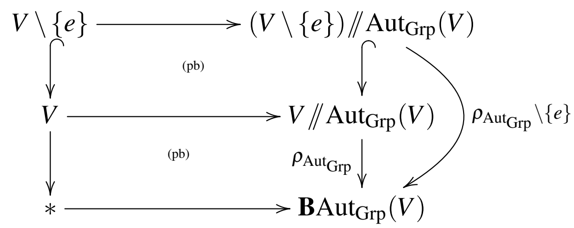
(i) *is a subobject (Def. 3.1.61) of* V

$$V \setminus \{e\} \hookrightarrow V \tag{6.17}$$

(ii) *which is disjoint from the neutral element:*

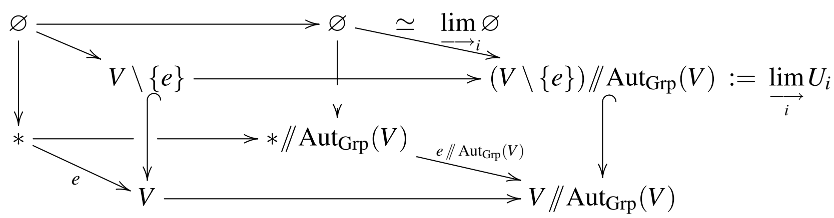
$$\begin{array}{ccc} \varnothing & \longrightarrow & V \setminus \{e\} \\ \big\downarrow & \text{(pb)} & \big\downarrow \\ * & \xrightarrow[e]{} & V \end{array}$$

Proof. For **(i)** we use the pasting law (Prop. 3.1.23) and the homotopy fiber characterization of the group-automorphism action (3.177) to decompose (6.16) as the pasting of two Cartesian squares, as follows:



Since monomorphisms are preserved by pullback (by Prop. 3.1.66), this shows the first claim from the construction (6.15).

For **(ii)** we paste to the middle horizontal morphism in this diagram the square (3.161) which exhibits the neutral element as a fixed point of the group-automorphims action (Prop. 3.2.29) and then we pull back the right vertical morphism along the boundary of that square, as shown in the following:

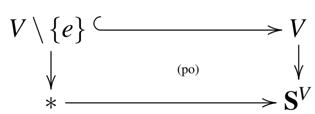


Here the right square is Cartesian since colimits in an ∞-topos are preserved by pullback (3.52) and using the definition (6.14), as indicated in the top right. Similarly the rear square is Cartesian, since pullback preserves the initial object (this being the empty colimit, Ex. 3.1.33). With this, and since the front square is Cartesian by **(i)**, the pasting law (Prop. 3.1.23) implies that also the left square is Cartesian, which was to be shown. □

Definition 6.2.9 (Tate V-sphere)**.** Let $\mathbf{H}$ be an ∞-topos (Def. 3.1.30) and $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1). Then we say that the *Tate V-sphere* is the homotopy cofiber

$$\mathbf{S}^V := V/(V \setminus \{e\})$$

of the inclusion (6.17) of the complement of the neutral element into V (Def. 6.2.7), hence the object in this homotopy pushout square:



(6.18)

Example 6.2.10 (Tate sphere in unstable motivic homotopy theory)**.** For $\mathbf{H} := \mathrm{Shv}_\infty\big(\mathrm{Schemes}_{\mathrm{Nis}}\big)$ and $V := \mathbb{A}^1$ the Tate V-sphere of Def. 6.2.9 is the Tate sphere in the traditional sense of (unstable) motivic homotopy theory, see [VRO07, 2.22].

Example 6.2.11 (Tate spheres with shape of ordinary spheres)**.** Let $\mathbf{H} = \mathrm{JetsOfSmoothGroupoids}_\infty$ (Def. 4.1.24) and $V := (\mathbb{R}^n, +)$ as in Ex. 5.2.4. Then $\mathrm{Aut}_{\mathrm{Grp}}(\mathbb{R}^n, +) = \mathrm{GL}(n)$ (as in Ex. 5.2.12) and the complement of the neutral element (Def. 6.2.7) is the ordinary complement $\mathbb{R}^n \setminus \{0\}$, whose shape is that or the

ordinary $n-1$-sphere:
$$\int(\mathbb{R}^n \setminus \{0\}) \simeq \int S^{n-1}. \tag{6.19}$$

Hence the Tate $\mathbb{R}^n$-sphere (Def. 6.2.9) is the homotopy pushout shown on the left here:

$$\begin{array}{ccc} \mathbb{R}^n \setminus \{e\} & \hookrightarrow & \mathbb{R}^n \\ \downarrow & \text{(pb)} & \downarrow \\ * & \longrightarrow & S^{(\mathbb{R}^n)} \end{array} \quad \overset{\int}{\longmapsto} \quad \begin{array}{ccc} \int S^{n-1} & \hookrightarrow & * \\ \downarrow & \text{(pb)} & \downarrow \\ * & \longrightarrow & \int \mathbf{S}^{(\mathbb{R}^n)} \end{array}$$

Since the shape modality (4.2) is left adjoint it preserves homotopy pushouts (Prop. 3.1.26), so that the shape of the Tate $\mathbb{R}^n$-sphere is that of the ordinary n-sphere:
$$\int \mathbf{S}^{\mathbb{R}^n} \simeq \int S^n. \tag{6.20}$$

In contrast, the Tate $\mathbb{R}^n$-sphere itself is the “germ of a smooth sphere”.

Proposition 6.2.12 (Canonical action on Tate V-sphere)**.** *Let* $\mathbf{H}$ *be an* ∞*-topos (Def. 3.1.30) and* $V \in \mathrm{Grp}(\mathbf{H})$ *(Prop. 3.2.1). The Tate* V*-sphere (Def. 6.2.9) inherits a canonical action (Prop. 3.2.6) of the group-automorphism group* $\mathrm{Aut}_{\mathrm{Grp}}(V)$ *(Def. 3.2.28), associated (via Prop. 3.2.14) to a group homomorphism*
$$\mathrm{Aut}_{\mathrm{Grp}}(V) \longrightarrow \mathrm{Aut}(\mathbf{S}^V) \tag{6.21}$$
whose homotopy quotient (3.131) *is given by the following homotopy pushout*
$$\begin{array}{ccc} (V \setminus \{e\}) /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(V) & \hookrightarrow & V /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(V) \\ \downarrow & \text{(po)} & \downarrow \\ * /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(V) & \longrightarrow & \mathbf{S}^V /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(V) \end{array} \tag{6.22}$$
of the defining morphisms in (6.16)*.*

Proof. Since the forgetful ∞-functor $\mathbf{H}_{/\mathbf{B}\mathrm{Aut}_{\mathrm{Grp}}(V)} \longrightarrow \mathbf{H}$ preserbes colimits (Ex. 3.1.52), the diagram (6.18) extends to a diagram over $\mathbf{B}\mathrm{Aut}_{\mathrm{Grp}}(V)$. Pulling this back along the point inclusion (3.123) and using that colimits in an ∞-topos are preserved by pullback (3.52), we find that the homotopy fiber of $\mathbf{S}^V /\!\!/ \mathrm{Aut}_{\mathrm{Grp}}(V) \to \mathbf{B}\mathrm{Aut}_{\mathrm{Grp}}(V)$ is given by the defining homotopy pushout (6.18) of the Tate V-sphere. ☐

Definition 6.2.13 (Linear group)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21) and $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1).

- **(i)** We say that V is a *linear group* if it is equipped with an equivalence
$$\mathrm{Aut}(T_e V) \underset{\simeq}{\xrightarrow{\exp}} \mathrm{Aut}_{\mathrm{Grp}}(V) \quad \in \mathrm{Grp}(\mathbf{H}) \tag{6.23}$$
between **(a)** the plain automorphism group of the local neighborhood of the neutral element (Def. 5.2.11) and **(b)** the group-automorphism group of V (Def. 3.2.28)
- **(ii)** We write
$$\mathrm{LinGrp}(\mathbf{H}) \in \mathrm{Cat}_\infty$$
for the ∞-category of linear groups in $\mathbf{H}$.

Definition 6.2.14 (Tate J-homomorphism)**.** Let $\mathbf{H}$ be an elastic ∞-topos (Def. 4.1.21) and $V \in \mathrm{LinGrp}(\mathbf{H})$ (Prop. 6.2.13).

(i) The *Tate J-homomorphism* is the composite

$$\mathbf{J}_V \;:\; \mathrm{Aut}(T_e V) \xrightarrow[\simeq]{\exp} \mathrm{Aut}_{\mathrm{Grp}}(V) \longrightarrow \mathrm{Aut}(\mathbf{S}^V) \tag{6.24}$$

of **(a)** the defining equivalence (6.23) with **(b)** the homomorphism (6.21) which reflects the canonical $\mathrm{Aut}_{\mathrm{Grp}}(V)$-action on the Tate V-sphere (Def. 6.2.12).

(ii) The corresponding $\mathrm{Aut}(T_e V)$-actions on $\mathbf{S}^V$ and on $\int(\mathbf{S}^V)$, by restriction along (6.24) and (6.26) of the canonical automorphism actions (Prop. 3.2.14), we denote, respectively, by

$$(\mathbf{S}^V, \rho_{\mathbf{J}}) \;\in\; \mathrm{Aut}(T_e V)\mathrm{Actions}(\mathbf{H})\,. \tag{6.25}$$

(iii) The actual *J-homomorphism* is the shape of the further composite with the homomorphism $\mathrm{Aut}(\eta^{\int}_{\mathbf{S}^V})$ from Prop. 4.1.7:

$$J_V \;:\; \int\mathrm{Aut}(T_e V) \xrightarrow[\simeq]{\int\exp} \int\mathrm{Aut}_{\mathrm{Grp}}(V) \longrightarrow \int\mathrm{Aut}(\mathbf{S}^V) \xrightarrow{\int\mathrm{Aut}\left(\eta^{\int}_{\mathrm{Aut}(\mathbf{S}^V)}\right)} \int\mathrm{Aut}\big(\int\mathbf{S}^V\big)\,. \qquad \left(\int\mathrm{Aut}(T_eV) \xrightarrow{\int\mathbf{J}_V} \int\mathrm{Aut}(\mathbf{S}^V)\right) \tag{6.26}$$

Example 6.2.15 (Ordinary J-homomorphism)**.** Let $\mathbf{H} = \mathrm{SnglrJetSmthGrpd}_\infty$ (Ex. 4.2.11) and $V := (\mathbb{R}^n, +)$ as in Ex. 5.2.4. This is a linear group in the sense of Def. 6.2.13, with $\mathrm{Aut}(T_0\mathbb{R}^n) \simeq \mathrm{GL}(n)$ (Ex. 5.2.12). Via Ex. 6.2.11 the induced action on the shape of the Tate $\mathbb{R}^n$-sphere (Def. 6.2.14) is the classical J-homomorphism (going back to [Wh42], reviewed in [Rav86, p. 4]):

$$J \;:\; \int\mathrm{O}(n) \;\simeq\; \int\mathrm{GL}(n) \longrightarrow \mathrm{Aut}\big(\int S^n\big) \tag{6.27}$$

being the image under topological shape (Def. 3.1.13) of the defining action of $\mathrm{GL}(n)$ on $\mathbb{R}^n$ and hence on its one-point compactification S^n.

Definition 6.2.16 (Representation spheres)**.** Let $\mathbf{H}$ be a singular-elastic ∞-topos (Def. 4.2.6), $V\mathrm{Grp}(\mathbf{H}_{\cup})$ (Prop. 3.2.1), and $(G,\phi) \in \mathrm{Grp}(\mathbf{H}_{\cup})_{/\mathrm{Aut}(T_e V)}$ (Prop. 3.2.1, Def. 3.2.13, Ex. 4.1.30). Then we say that the *representation sphere* S^{V_ϕ} of the G-action ϕ on V (via Prop. 3.2.14) is the shape (Def. 4.1.1) of the orbi-singularization (Def. 4.2.7) of the homotopy quotient (3.131) of the Tate V-sphere (Def. 6.2.9) by the restricted action (Prop. 3.2.12) along ϕ of the action $\rho_{\mathbf{J}}$ (6.25) induced by the J-homomorphism (Def. 6.2.14):

$$S^{V_\phi} \;:=\; \int \curlyvee\big(\mathbf{S}^V /\!\!/_{\phi}\, G\big) \qquad \in \mathbf{H}_{/\curlyvee G}\,.$$

Example 6.2.17 (Ordinary representation spheres)**.**
Let $\mathbf{H} = \mathrm{SnglrJetSmthGrpd}_\infty$ (Ex. 4.2.10) and $V := (\mathbb{R}^n, +)$ as in Ex. 5.2.4, whence $\mathrm{Aut}(T_e V) \simeq \mathrm{GL}(n)$ (Ex. 5.2.12). For

$$G \overset{\phi}{\hookrightarrow} \mathrm{GL}(n) \subset \mathrm{Aut}(T_e V)\,.$$

a finite subgroup, hence a linear G-representation, we have that the representation sphere $S^{\mathbb{R}^n_\phi}$ according to Def. 6.2.16 is the ordinary representation sphere, as an object in G-equivariant homotopy theory.

Definition 6.2.18 (*J*-twisted proper orbifold Cohomotopy theory)**.** Let $\mathbf{H}$ be a singular-elastic ∞-topos (Def. 4.2.6) $V \in \mathrm{Grp}(\mathbf{H})$ (Prop. 3.2.1), $W \in \mathrm{LinGrp}(\mathbf{H})$ (Def. 6.2.13) and $\phi : \mathrm{Aut}(T_e W) \longrightarrow \mathrm{Aut}(T_E V)$. Then *J-twisted proper orbifold Cohomotopy* is the tangentially twisted proper orbifold cohomotopy (Def. 6.2.5) with coefficients

$$(A, \rho) \;:=\; (\mathbf{S}^V, \rho_{\mathbf{J}})$$

the Tate W-sphere (Def. 6.2.9) with its Tate J-homomorphism action (Def. 6.2.14):

$$\underset{\text{J-twisted orbifold Cohomotopy}}{\pi^{\int \curlyvee \tau}(-)} \;:=\; \underset{\text{tangentially twisted orbifold cohomology}}{H^{\int \curlyvee \tau}}\big(-, \underset{\text{Tate } V\text{-sphere with } J\text{-homomorphism action}}{(\mathbf{S}^V, \rho_{\mathbf{J}})}\big).$$

Hence for a structured orbifold (Def. 5.2.45)

$$\big(\mathscr{X}, (\tau, g)\big) \;\in\; \big(\mathrm{Aut}(T_e W), \phi\big)\mathrm{Strctrd}V\mathrm{Orbfld}(\mathbf{H})\,,$$

we have:

$$\underset{\text{J-twisted orbifold Cohomotopy}}{\pi^{\int \curlyvee \tau}(\mathscr{X})} \;=\; \left\{ \begin{array}{ccc} \underset{\text{orbifold}}{\mathscr{X}} & \overset{\text{cocycle } c}{\dashrightarrow} & \underset{\text{orbi-singularized Tate } W\text{-sphere}}{\int \curlyvee \big(\mathbf{S}^W /\!\!/ \mathrm{Aut}(T_e W)\big)} \\ & \Swarrow & \\ {\scriptstyle \eta^{\int} \circ \curlyvee(\tau)} \searrow\ \text{tangential twist} & & \swarrow {\scriptstyle \int \curlyvee(\rho_{\mathbf{J}})}\ \text{twisting via orbi-singularized J-homomorphism} \\ & \int \curlyvee \mathbf{B}\mathrm{Aut}(T_e W) & \end{array} \right\}_{/\sim}$$

Example 6.2.19 (*J*-Twisted orbifold Cohomotopy of ordinary orbifolds)**.**
Let $\mathbf{H} = \mathrm{SnglrJetSmthGrpd}_\infty$ (Ex. 4.2.10) and $V := (\mathbb{R}^n, +)$, $W := (\mathbb{R}^p, +)$ as in Ex. 5.2.4, with $p \leq n$, and $\phi : (\mathbb{R}^p, +) \hookrightarrow (\mathbb{R}^n, +)$ be the canonical inclusion. Then the corresponding J-twisted proper orbifold Cohomotopy theory $\pi^{\int \curlyvee \tau}$ (Def. 6.2.18) is defined on ordinary n-dimensional orbifolds (by Ex. 5.2.5) with $\mathrm{GL}(p)$-structure (by Ex. 5.2.12) and it unifies the following two special cases (by Theorem 6.2.6, see the second diagram on p. 21)):

(i) On smooth orbifolds, i.e., on ordinary manifolds (Ex. 5.2.4) it reduces to non-abelian cohomology with coefficients the shape of the ordinary p-sphere (by Ex. 6.2.11) and tangentially twisted via the traditional J-homomorphism (by Ex. 6.2.15). This is the *J-twisted Cohomotopy theory* considered in [FSS20][FSS21] [BSS19].

(ii) On flat orbifolds, such as the vicinity of ordinary orbifold singularities $\mathbb{R}^p /\!\!/ G$ for finite subgroups $G \overset{\phi}{\hookrightarrow} \mathrm{GL}(p)$ (by Ex. 5.2.44), hence for linear G-representations ϕ, it reduces to proper equivariant cohomology in RO-degree ϕ and with coefficients the representation sphere $S^{\mathbb{R}^n_\phi}$ (by Ex. 6.2.17). This is the *tangentially RO-graded equivariant Cohomotopy theory* considered in [SS20][BSS19].

By way of outlook, we highlight the following:

Remark 6.2.20 (Orbifold cohomology in M-theory and *Hypothesis H*)**.** Traditional discussion of orbifold cohomology has been strongly motivated by its application to *perturbative string theory* (e.g. [AMR02][ARZ06][ALR07][BU09] [DFM11]). However, perturbative string theory is famously in need of a non-perturbative completion ("M-theory", see [HSS18, 2][FSS19] for review and pointers) whose mathematical formulation has remained an open problem. Therefore, it is to be expected that the historically rich interaction between orbifold cohomology theory and string theory is just the tip of an iceberg, whose full scope is a cohomology theory of M-theoretic orbifolds.

Elsewhere we have put forward a precise hypothesis as to the global completion of 11D supergravity towards M-theory, via *flux quantization* [SS25a] of the theory's C-field. This *Hypothesis H* says that:
(i) far from singularities, M-theory flux is quantized in twisted Cohomotopy theory [FSS20][FSS21][BSS19][FSS22];
(ii) at singularities, M-theory is quantized in RO-graded equivariant Cohomotopy theory [HSS18][SS20][BSS19].
(See these references for various consistency checks of this hypothesis.)

The impact of Theorem 6.2.6, in its specialization to Ex. 6.2.19, is to show that these two cases are indeed two aspects of a single unified cohomology theory: J-twisted proper orbifold Cohomotopy theory.

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