

Prequantum field theory

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November 10, 2015

ETH *Talks in mathematical physics*

based on joint work with
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exposition, details and references at
ncatlab.org/schreiber/show/Local+prequantum+field+theory

Let Σ be a $(p + 1)$ -dimensional smooth manifold

Given a smooth bundle E over Σ ,
think of its sections $\phi \in \Gamma_{\Sigma}(E)$ as physical fields.

Task: Describe field theory **fully-local to fully-global**.

1. Classical field theory
2. BV-BFV field theory
3. Prequantum field theory
4. Prequantum ∞ -CS theories

1)

Classical field theory

A differential operator $D : \Gamma_{\Sigma}(E) \rightarrow \Gamma_{\Sigma}(F)$ comes from bundle map out of the jet bundle

$$\tilde{D} : J_{\Sigma}^{\infty} E \longrightarrow F$$

Composition of diff ops $D_2 \circ D_1$ comes from

$$J_{\Sigma}^{\infty} E \longrightarrow J_{\Sigma}^{\infty} J_{\Sigma}^{\infty} E \xrightarrow{J_{\Sigma}^{\infty} \tilde{D}_1} J_{\Sigma}^{\infty} F \xrightarrow{\tilde{D}_2} G$$

here the first map witnesses comonad structure on jets, this is composition in the coKleisli category $\mathbf{Kl}(J_{\Sigma}^{\infty})$.

A differential equation is an equalizer of two differential operators.
Exists in the Eilenberg-Moore category $\text{EM}(J_\Sigma^\infty)$

$$\mathcal{E} \longrightarrow E \begin{array}{c} \xrightarrow{\tilde{D}_1} \\ \xrightarrow{\tilde{D}_2} \end{array} F$$

Theorem [Marvan 86]:

$$\text{EM}(J_\Sigma^\infty E) \simeq \text{PDE}_\Sigma$$

PDE solutions are sections:

A commutative triangle diagram with vertices Σ , \mathcal{E} , and E . The vertex Σ is at the bottom left, \mathcal{E} is at the top, and E is at the bottom right. An arrow labeled ϕ_{sol} points from Σ to \mathcal{E} . An arrow points from \mathcal{E} to E . An arrow labeled ϕ points from Σ to E .

Def. A horizontal differential form on jet bundle $\alpha \in \Omega_H^k(J_\Sigma^\infty E)$ is diff op of the form

$$\tilde{\alpha} : E \longrightarrow \wedge^k T^*\Sigma$$

This induces vertical/horizontal bigrading $\Omega^{\bullet,\bullet}(J_\Sigma^\infty E)$

Def: A local Lagrangian is $L \in \Omega_H^{p+1}(J_\Sigma^\infty E)$

Prop. Unique decomposition

$$d_{\text{dR}}L = \text{EL} - d_H(\Theta + d_H(\cdots))$$

with $\text{EL} \in \Omega_S^{p+1,1} \hookrightarrow \Omega^{p+1,1}$

depending only on vector fields along 0-jets.

This is the local incarnation of the variational principle.

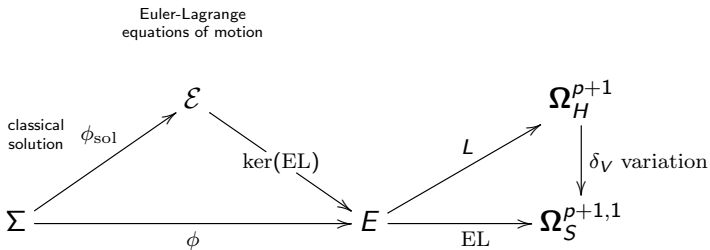
Prop. the bigrading is preserved by pullback along diff ops

This means that

$$\Omega^{\bullet,\bullet} \in \text{Sh}(\text{DiffOp}_\Sigma) \xrightarrow{\text{Kan ext}} \text{Sh}(\text{PDE}_\Sigma)$$

is a bicomplex of sheaves on the category of PDEs.

In this sheaf topos, classical field theory looks like so:



By *transgression* of this local data we get

1) the **global action functional**

$$[\Sigma, E]_{\Sigma} \xrightarrow{S = \int_{\Sigma} L} \mathbb{R}$$

2) the **covariant phase space** [Zuckerman 87]:

$$\begin{array}{ccc} & & \Omega^1 \\ & \nearrow^{\theta := \int_{\Sigma_p} \Theta} & \downarrow d \\ [N^{\infty} \Sigma_p, \mathcal{E}]_{\Sigma} & \xrightarrow{\omega} & \Omega^2 \end{array}$$

2)

BV-BFV field theory

\mathcal{E} need not be representable by a submanifold of $J_{\Sigma}^{\infty} E$
if EL-equation is singular.

idea of BV-theory/derived geometry:

1. replace base category of smooth manifolds
by smooth dg-manifolds in non-positive degree.
2. resolve singular \mathcal{E} by realizing it
as 0-cohomology of smooth dg-manifold.

If $\{\Phi^i\}$ are local fiber coordinates on E (field coordinates), then the derived shell \mathcal{E} has dg-algebra of functions the algebra $C^\infty(\mathcal{E})$ with degree-(-1) generators Φ_i^* added (“antifields”) and with differential given by

$$d_{BV} : \Phi_i^* \mapsto EL_i .$$

Usual to write Q for d_{BV} regarded as vector field on the dg-manifold. Then this is

$$\mathcal{L}_Q \Phi_i^* = EL_i .$$

This gives a third grading (BV antifield grading) on differential forms on the jet bundle

$$\Omega^{\bullet, \bullet; -\bullet}(\mathcal{E}_d)$$

Observation: \mathcal{E}_d carries 2-form locally given by

$$\Omega_{\text{BV}} = d\Phi_i^* \wedge d\Phi^i \in \Omega^{p+1,2;-1}(\mathcal{E}_d)$$

which satisfies

$$\begin{aligned} \iota_Q \Omega_{\text{BV}} &= \text{EL} \\ &= dL + d_H \Theta \in \Omega^{p+1,1;0}(\mathcal{E}_d). \end{aligned}$$

under transgression to the space of fields this becomes

$$\iota_Q \omega_{\text{BV}} = dS + \pi^* \theta$$

This is the central compatibility postulate for BV-BFV field theory in [Cattaneo-Mnev-Reshetikhin 12, eq. (7)]

3)

Prequantum field theory

For many field theories of interest, L is not in fact globally defined.

Simple examples:

- ▶ electron in EM-field with non-trivial magnetic charge;
- ▶ 3d $U(1)$ -Chern-Simons theory;

Large classes of examples:

- ▶ higher **WZW-type** models
(super p -branes, topological phases of matter)
- ▶ higher **Chern-Simons-type** models
(AKSZ, 7dCS on String-2-connections, 11dCS,...)

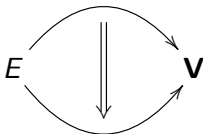
Claim: There is a *systematic* solution to this problem by

1. passing to the derived category of $\mathrm{Sh}(\mathrm{PDE}_\Sigma)$;
2. generalizing Lagrangian forms to differential cocycles (“gerbes with connection”)

If $\mathbf{V} := [\dots \xrightarrow{\partial_V} \mathbf{V}^2 \xrightarrow{\partial_V} \mathbf{V}^1 \xrightarrow{\partial_V} \mathbf{V}^0]$ is a sheaf of chain complexes, then a map in the derived category

$$E \longrightarrow \mathbf{V}$$

is equivalently a cocycle in the sheaf hypercohomology of E with coefficients in \mathbf{V} [Brow73]. Homotopy is coboundary:



hence generalize sheaf of horizontal forms Ω_H^{p+1} to:

Def. The “variational Deligne complex”

$$\mathbf{B}_H^{p+1}(\mathbb{R}/\mathbb{Z})_{\text{conn}} := [\mathbb{Z} \rightarrow \Omega_H^0 \xrightarrow{d_H} \Omega_H^1 \xrightarrow{d_H} \dots \xrightarrow{d_H} \Omega_H^{p+1}]$$

Def. A local prequantum Lagrangian is

$$\mathbf{L} : E \longrightarrow \mathbf{B}_H^{p+1}(\mathbb{R}/\mathbb{Z})_{\text{conn}}$$

Consequence: Via fiber integration in differential cohomology \mathbf{L} transgresses to globally well defined action functional

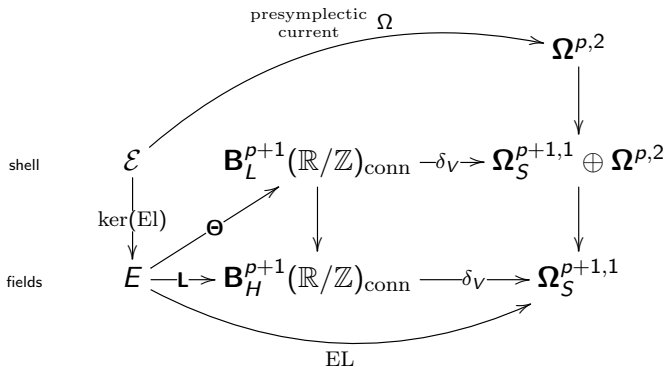
$$S := \int_{\Sigma} [\Sigma, \mathbf{L}] : [\Sigma, E]_{\Sigma} \longrightarrow \mathbb{R}/\mathbb{Z}$$

Theorem: the curving of such Euler-Lagrange p -gerbes is given by the Euler variational operator

$$\delta_V : \mathbf{B}_H^{p+1}(\mathbb{R}/\mathbb{Z})_{\text{conn}} \rightarrow \Omega_S^{p+1,1}$$

principle of extremal action \leftrightarrow flatness of EL- p -gerbes

Def./Prop. Prequantization of Θ is via *Lepage p -gerbes* Θ whose curvature in degree $(p, 2)$ is the *pre-symplectic current*.



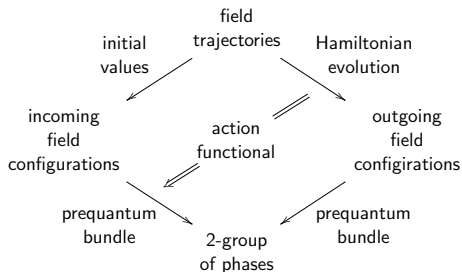
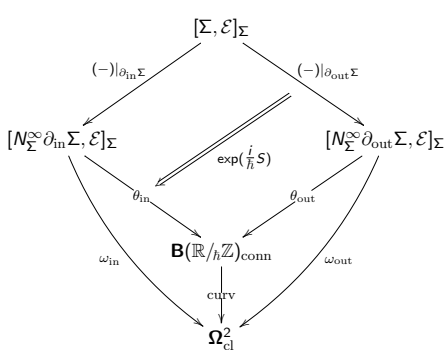
Theorem: Transgressions of single Lepage p -gerbe to all codimension-1 (Cauchy-)hypersurfaces $\Sigma_p \hookrightarrow \Sigma$ gives *natural* Kostant-Souriau prequantizations of all covariant phase spaces:

$$\begin{array}{ccc}
 & \text{Kostant-Souriau} & \mathbf{B}(\mathbb{R}/\mathbb{Z})_{\text{conn}} \\
 & \text{prequantum} & \downarrow \\
 & \text{line bundle} & \Omega^2 \\
 \text{covariant} & \nearrow \theta := \int_{\Sigma_p} \Theta & \\
 \text{phase space} & [N^\infty \Sigma_p, \mathcal{E}]_\Sigma & \xrightarrow{\omega := \int_{\Sigma_p} \Omega} \\
 \text{for } \Sigma_p \hookrightarrow \Sigma & \text{canonical presymplectic form} &
 \end{array}$$

Theorem: Transgression of higher prequantum gerbes ∇ to fields on manifold Σ with boundary looks like so:

$$\begin{array}{ccccc}
 [\Sigma, X] & \xrightarrow{[\Sigma, \nabla]} & [\Sigma, \mathbf{B}^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{\text{conn}}] & \xrightarrow{\int_{\Sigma} \text{curv}} & \Omega^{p+2-d} \\
 \downarrow (-)|_{\partial\Sigma} & & \downarrow (-)|_{\partial\Sigma} & \swarrow \int_{\Sigma} & \downarrow \\
 [\partial\Sigma, X] & \xrightarrow{[\partial\Sigma, \nabla]} & [\partial\Sigma, \mathbf{B}^{p+1}(\mathbb{R}/\hbar\mathbb{Z})_{\text{conn}}] & \xrightarrow{\int_{\partial\Sigma}} & \mathbf{B}^{p+2-d}(\mathbb{R}/\hbar\mathbb{Z})_{\text{conn}}
 \end{array}$$

Corollary: Transgressing Lepage p -gerbe to spacetime Σ with incoming and outgoing boundary yields the prequantized Lagrangian correspondence that exhibits dynamical evolution:



4)

Prequantum ∞ -CS theories

Chern-Simons is nonabelian gauge theory

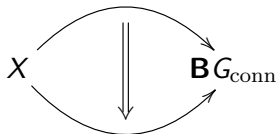
So now pass to the “nonabelian derived category” (aka ∞ -topos) where sheaves of chain complexes are generalized to sheaves of Kan complexes [Brow73].

This serves to describe nonabelian gauge fields.

For instance there is sheaf of Kan complexes $\mathbf{B}G_{\text{conn}}$ such that a G -principal connection on X is a map

$$X \longrightarrow \mathbf{B}G_{\text{conn}}$$

and a gauge transformation is a homotopy



One way to construct prequantum field theories is:
 construct a p -gerbe connection on some moduli stack

$$\nabla : \mathbf{A}_{\text{conn}} \longrightarrow \mathbf{B}^{p+1}(\mathbb{R}/\mathbb{Z})_{\text{conn}}$$

and consider the stacky field bundle $E := \Sigma \times \mathbf{A}_{\text{conn}}$
 then pullback and project to get Euler-Lagrange and Lepage
 p -gerbe

$$\begin{array}{ccccc}
 & & \Theta & \longrightarrow & \mathbf{B}_L^{p+1}(\mathbb{R}/\mathbb{Z})_{\text{conn}} \\
 & \nearrow & & \nearrow & \downarrow \\
 \Sigma \times \mathbf{A}_{\text{conn}} & \xrightarrow{\Sigma \times \nabla} & \Sigma \times \mathbf{B}^{p+1}(\mathbb{R}/\mathbb{Z})_{\text{conn}} & \longrightarrow & \mathbf{B}_H^{p+1}(\mathbb{R}/\mathbb{Z})_{\text{conn}} \\
 & \searrow & & \searrow & \\
 & & \mathbf{L} & &
 \end{array}$$

For instance for G a simply-connected compact simple Lie group, there is a unique differential refinement

$$\nabla : \mathbf{B}G_{\text{conn}} \longrightarrow \mathbf{B}^3U(1)_{\text{conn}}$$

of the canonical universal characteristic 4-class

$$c_2 : BG \longrightarrow K(\mathbb{Z}, 4)$$

This induces the standard 3d Chern-Simons Lagrangian and universally prequantizes it:

codim 0	$[\Sigma_3, \mathbf{B}G_{\text{conn}}] \longrightarrow \mathbb{R}/\mathbb{Z}$	CS invariant
codim 1	$[\Sigma_2, \mathbf{B}G_{\text{conn}}] \longrightarrow \mathbf{B}(\mathbb{R}/\mathbb{Z})_{\text{conn}}$	CS prequantum line
codim 2	$[\Sigma_1, \mathbf{B}G_{\text{conn}}] \longrightarrow \mathbf{B}^2(\mathbb{R}/\mathbb{Z})_{\text{conn}}$	WZW gerbe
codim 3	$[\Sigma_0, \mathbf{B}G_{\text{conn}}] \longrightarrow \mathbf{B}^3(\mathbb{R}/\mathbb{Z})_{\text{conn}}$	Chern-Weil map

construct this and other examples from Lie integration of L_∞ -data:

Def. L_∞ -algebroid \mathfrak{a} is dg-manifold

Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{a})$ is the dg-algebra of functions

$W(\mathfrak{a})$ is dg-algebra of differential forms

cocycle is

$$\frac{\mathfrak{a} \xrightarrow{\mu} \mathbf{B}^{p+2}\mathbb{R}}{\mathrm{CE}(\mathfrak{a}) \longleftarrow \mathrm{CE}(\mathbf{B}^{p+2}\mathbb{R})}$$

Def. universal Lie integration to the derived category over SmoothMfd is the sheaf of Kan complexes

$$\exp(\mathfrak{a}) : (U, k) \mapsto \{\Omega_{\text{vert}}^{\bullet}(U \times \Delta^k) \longleftarrow \text{CE}(\mathfrak{a})\}$$

Examples:

- ▶ for \mathfrak{g} semisimple Lie algebra, then $\tau_1 \exp(\mathbf{B}\mathfrak{g}) \simeq \mathbf{B}G$
- ▶ for \mathfrak{P} a Poisson Lie algebroid then $\tau_2 \exp(\mathfrak{P})$ is symplectic Lie groupoid.
- ▶ for \mathfrak{string} the String Lie 2-algebra, then $\tau_2 \exp(\mathbf{B}\mathfrak{string}) \simeq \mathbf{B}\text{String}$

an *invariant polynomial* $\langle - \rangle$ on \mathfrak{a} is closed differential form on the dg-manifold.

If $\langle - \rangle$ is binary and non-degenerate, this became also called “shifted symplectic form”

Def. differential Lie integration $\exp(\mathfrak{g})_{\text{conn}}$ is

$$\exp(\mathfrak{a})_{\text{conn}} : (U, k) \mapsto \left\{ \begin{array}{ccc} \Omega_{\text{vert}}^{\bullet}(U \times \Delta^k) & \longleftarrow & \text{CE}(\mathfrak{a}) \\ \uparrow & & \uparrow \\ \Omega^{\bullet}(U \times \Delta^k) & \longleftarrow & W(\mathfrak{a}) \\ \uparrow & & \uparrow \\ \Omega^{\bullet}(U) & \longleftarrow & \text{inv}(\mathfrak{a}) \end{array} \right\}$$

Def. A cocycle μ is in transgression with an invariant polynomial $\langle - \rangle$ if there is a diagram of the form

$$\begin{array}{ccc}
 \text{CE}(\mathfrak{a}) & \xleftarrow{\mu} & \text{CE}(\mathbf{B}^{p+2}\mathbb{R}) \\
 \uparrow & & \uparrow \\
 \text{W}(\mathfrak{a}) & \xleftarrow{\text{cs}} & \text{W}(\mathbf{B}^{p+2}\mathbb{R}) \\
 \uparrow & & \uparrow \\
 \text{inv}(\mathfrak{a}) & \xleftarrow{\langle - \rangle} & \text{inv}(\mathbf{B}^{p+2}\mathbb{R})
 \end{array}$$

By pasting of diagrams, this data defines a map

$$\exp(\text{cs}) : \exp(\mathfrak{a})_{\text{conn}} \longrightarrow \exp(\mathbf{B}^{p+2}\mathbb{R})_{\text{conn}}$$

Theorem:

under truncation this descends to

$$\begin{array}{ccc} \exp(\mathfrak{a})_{\text{conn}} & \xrightarrow{\exp(cs)} & \exp(\mathbf{B}^{p+2}\mathbb{R})_{\text{conn}} \\ \downarrow \text{cosk}_{p+2} & & \downarrow \\ \mathbf{A}_{\text{conn}} & \xrightarrow{\nabla} & \mathbf{B}^{p+2}(\mathbb{R}/\Gamma)_{\text{conn}} \end{array}$$

where $\Gamma \hookrightarrow \mathbb{R}$ is the group of periods of μ .

This gives large supply of **examples** of prequantum field theories induced from “shifted n -plectic forms”:

AKSZ including 3dCS, PSM (hence A-model/B-model), CSM, ...
7dCS on String 2-connections, 11dCS on 5brane 6-connections, ...

Outlook

Our formulation of prequantum field theory works also with *generalized* differential cohomology.

For instance the “topological” Lagrangian term for super Dp -brane sigma models for all even or all odd p at once needs to be a cocycle in differential K-theory.

And “U-duality” predicts that the topological terms for the M2 and M5 brane needs to be unified in single generalized differential cocycle with Chern character in the rational 4-sphere [Fiorenza-Sati-Schreiber 15].

References

exposition, details and full bibliography is at

ncatlab.org/schreiber/show/Local+prequantum+field+theory



K. Brown,

Abstract Homotopy Theory and Generalized Sheaf Cohomology, Transactions of the AMS, Vol. 186 (1973)
ncatlab.org/nlab/files/BrownAbstractHomotopyTheory.pdf



A. Cattaneo, P. Mnev, N. Reshetikhin,

Classical BV theories on manifolds with boundary,
Communications in Mathematical Physics December 2014,
Volume 332, Issue 2, pp 535-603 [arXiv:1201.0290](https://arxiv.org/abs/1201.0290)



M. Marvan,

A note on the category of partial differential equations, in
Differential geometry and its applications, Brno 1986
ncatlab.org/nlab/files/MarvanJetComonad.pdf



G. J. Zuckerman,

Action Principles and Global Geometry, in S. T. Yau (ed.),
Mathematical Aspects of String Theory, World Scientific,
Singapore, 1987, pp. 259-284
ncatlab.org/nlab/files/ZuckermanVariation.pdf