Prequantum field theory

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exposition, details and references at
ncatlab.org/schreiber/show/Local+prequantum+field+theory

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Let Σ be a $(p+1)$-dimensional smooth manifold

Given a smooth bundle E over Σ,
think of its sections $\phi \in \Gamma_\Sigma(E)$ as physical fields.

Task: Describe field theory **fully-local to fully-global**.

1. Classical field theory
2. BV-BFV field theory
3. Prequantum field theory
4. Prequantum ∞-CS theories

1)
Classical field theory

A *differential operator* $D : \Gamma_\Sigma(E) \to \Gamma_\Sigma(F)$ comes from bundle map out of the jet bundle

$$\tilde{D} : J^\infty_\Sigma E \longrightarrow F$$

Composition of diff ops $D_2 \circ D_1$ comes from

$$J^\infty_\Sigma E \longrightarrow J^\infty_\Sigma J^\infty_\Sigma E \xrightarrow{J^\infty_\Sigma \tilde{D}_1} J^\infty_\Sigma F \xrightarrow{\tilde{D}_2} G$$

here the first map witnesses comonad structure on jets, this is composition in the coKleisli category $\mathrm{Kl}(J^\infty_\Sigma)$.

A differential equation is an equalizer of two differential operators. Exists in the Eilenberg-Moore category $\mathrm{EM}(J^\infty_\Sigma)$

$$\begin{tikzcd} \mathcal{E} \arrow[r] & E \arrow[r, shift left, "\tilde{D}_1"] \arrow[r, shift right, "\tilde{D}_2"'] & F \end{tikzcd}$$

Theorem [Marvan 86]:

$$\mathrm{EM}(J^\infty_\Sigma E) \simeq \mathrm{PDE}_\Sigma$$

PDE solutions are sections:

$$\begin{tikzcd} & \mathcal{E} \arrow[dr] & \\ \Sigma \arrow[ur, "\phi_{\mathrm{sol}}"] \arrow[rr, "\phi"'] & & E \end{tikzcd}$$

Def. A *horizontal differential form* on jet bundle $\alpha \in \Omega^k_H(J^\infty_\Sigma E)$ is diff op of the form

$$\tilde{\alpha} : E \longrightarrow \wedge^k T^*\Sigma$$

This induces vertical/horizontal bigrading $\Omega^{\bullet,\bullet}(J^\infty_\Sigma E)$

Def: A *local Lagrangian* is $L \in \Omega^{p+1}_H(J^\infty_\Sigma E)$

Prop. Unique decomposition

$$d_{\mathrm{dR}} L = \mathrm{EL} - d_H(\Theta + d_H(\cdots))$$

with $\mathrm{EL} \in \Omega^{p+1,1}_S \hookrightarrow \Omega^{p+1,1}$
depending only on vector fields along 0-jets.

This is the local incarnation of the variational principle.

Prop. the bigrading is preserved by pullback along diff ops

This means that

$$\mathbf{\Omega}^{\bullet,\bullet} \in \mathrm{Sh}(\mathrm{DiffOp}_{\mathbf{\Sigma}}) \xrightarrow{\text{Kan ext}} \mathrm{Sh}(\mathrm{PDE}_{\mathbf{\Sigma}})$$

is a bicomplex of sheaves on the category of PDEs.

In this sheaf topos, classical field theory looks like so:

Euler-Lagrange
equations of motion

classical
solution

$$\begin{tikzcd}
 & \mathcal{E} \arrow[dr, "\ker(\mathrm{EL})"] & & \mathbf{\Omega}_H^{p+1} \arrow[d, "\delta_V \text{ variation}"] \\
\Sigma \arrow[ur, "\phi_{\mathrm{sol}}"] \arrow[rr, "\phi"'] & & E \arrow[ur, "L"] \arrow[r, "\mathrm{EL}"'] & \mathbf{\Omega}_S^{p+1,1}
\end{tikzcd}$$

By *transgression* of this local data we get

1) the **global action functional**

$$[\Sigma, E]_\Sigma \xrightarrow{S = \int_\Sigma L} \mathbb{R}$$

2) the **covariant phase space** [Zuckerman 87]:

$$\begin{tikzcd} & \mathbf{\Omega}^1 \arrow[d, "d"] \\ {[N^\infty \Sigma_p, \mathcal{E}]_\Sigma} \arrow[ur, "\theta := \int_{\Sigma_p} \Theta"] \arrow[r, "\omega"'] & \mathbf{\Omega}^2 \end{tikzcd}$$

2)
BV-BFV field theory

$\mathcal{E}$ need not be representable by a sub*manifold* of $J^\infty_\Sigma E$
if EL-equation is singular.

idea of BV-theory/derived geometry:

1. replace base category of smooth manifolds
   by smooth dg-manifolds in non-positive degree.
2. resolve singular $\mathcal{E}$ by realizing it
   as 0-cohomology of smooth dg-manifold.

If $\{\Phi^i\}$ are local fiber coordinates on E (field coordinates), then the derived shell $\mathcal{E}$ has dg-algebra of functions the algebra $C^\infty(\mathcal{E})$ with degree-(-1) generators Φ_i^* added ("antifields") and with differential given by

$$d_{BV} : \Phi_i^* \mapsto \mathrm{EL}_i \,.$$

Usual to write Q for d_{BV} regarded as vector field on the dg-manifold. Then this is

$$\mathcal{L}_Q \Phi_i^* = \mathrm{EL}_i \,.$$

This gives a third grading (BV antifield grading) on differential forms on the jet bundle

$$\Omega^{\bullet,\bullet;-\bullet}(\mathcal{E}_d)$$

Observation: $\mathcal{E}_d$ carries 2-form locally given by

$$\Omega_{\mathrm{BV}} = d\Phi_i^* \wedge d\Phi^i \quad \in \quad \Omega^{p+1,2;-1}(\mathcal{E}_d)$$

which satisfies

$$\begin{aligned} \iota_Q \Omega_{\mathrm{BV}} &= \mathrm{EL} \\ &= dL + d_H \Theta \qquad \in \Omega^{p+1,1;0}(\mathcal{E}_d)\,. \end{aligned}$$

under transgression to the space of fields this becomes

$$\iota_Q \omega_{\mathrm{BV}} = dS + \pi^* \theta$$

This is the central compatibility postulate for BV-BFV field theory in [Cattaneo-Mnev-Reshetikhin 12, eq. (7)]

3)
Prequantum field theory

For many field theories of interest, L is not in fact globally defined.

Simple examples:

- electron in EM-field with non-trivial magentic charge;
- 3d $U(1)$-Chern-Simons theory;

Large classes of examples:

- higher **WZW-type** models
  (super p-branes, topological phases of matter)
- higher **Chern-Simons-type** models
  (AKSZ, 7dCS on String-2-connections, 11dCS,...)

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Claim: There is a *systematic* solution to this problem by

1. passing to the derived category of $\mathrm{Sh}(\mathrm{PDE}_\Sigma)$;
2. generalizing Lagrangian forms to differential cocycles ("gerbes with connection")

If
$$\mathbf{V} := [\cdots \xrightarrow{\partial_V} \mathbf{V}^2 \xrightarrow{\partial_V} \mathbf{V}^1 \xrightarrow{\partial_V} \mathbf{V}^0]$$
is a sheaf of chain complexes, then a map in the derived category

$$E \longrightarrow \mathbf{V}$$

is equivalently a cocycle
in the sheaf hypercohomology of E with coefficients in $\mathbf{V}$ [Brow73].
Homotopy is coboundary:

$$\begin{tikzcd} E \arrow[r, bend left=50, ""{name=U, below}] \arrow[r, bend right=50, ""{name=D, above}] & \mathbf{V} \arrow[Rightarrow, from=U, to=D] \end{tikzcd}$$

hence generalize sheaf of horizontal forms $\mathbf{\Omega}_H^{p+1}$ to:

Def. The "variational Deligne complex"

$$\mathbf{B}_H^{p+1}(\mathbb{R}/\mathbb{Z})_{\mathrm{conn}} := [\mathbb{Z} \to \mathbf{\Omega}_H^0 \stackrel{d_H}{\to} \mathbf{\Omega}_H^1 \stackrel{d_H}{\to} \cdots \stackrel{d_H}{\to} \mathbf{\Omega}_H^{p+1}]$$

Def. A local prequantum Lagrangian is

$$\mathbf{L} : E \longrightarrow \mathbf{B}_H^{p+1}(\mathbb{R}/\mathbb{Z})_{\mathrm{conn}}$$

Consequence: Via fiber integration in differential cohomology $\mathbf{L}$ transgresses to globally well defined action functional

$$S := \int_\Sigma [\Sigma, \mathbf{L}] \;:\; [\Sigma, E]_\Sigma \longrightarrow \mathbb{R}/\mathbb{Z}$$

Theorem: the curving of such Euler-Lagrange p-gerbes is given by the Euler variational operator

$$\delta_V : \mathbf{B}_H^{p+1}(\mathbb{R}/\mathbb{Z})_{\mathrm{conn}} \to \mathbf{\Omega}_S^{p+1,1}$$

principle of extremal action $\leftrightarrow$ flatness of EL-p-gerbes

Def./Prop. Prequantization of Θ is via *Lepage p-gerbes* $\mathbf{\Theta}$ whose curvature in degree $(p, 2)$ is the *pre-symplectic current*.

$$
\begin{tikzcd}
 & & & \mathbf{\Omega}^{p,2} \arrow[d] \\
\text{shell} & \mathcal{E} \arrow[urr, bend left, "\substack{\text{presymplectic}\\ \text{current}}\ \Omega"] \arrow[d, "\ker(\mathrm{El})"] & \mathbf{B}^{p+1}_L(\mathbb{R}/\mathbb{Z})_{\mathrm{conn}} \arrow[r, "\delta_V"] \arrow[d] & \mathbf{\Omega}^{p+1,1}_S \oplus \mathbf{\Omega}^{p,2} \arrow[d] \\
\text{fields} & E \arrow[ur, "\mathbf{\Theta}"] \arrow[r, "\mathbf{L}"] \arrow[rr, bend right, "\mathrm{EL}"'] & \mathbf{B}^{p+1}_H(\mathbb{R}/\mathbb{Z})_{\mathrm{conn}} \arrow[r, "\delta_V"] & \mathbf{\Omega}^{p+1,1}_S
\end{tikzcd}
$$

Theorem: Transgressions of single Lepage p-gerbe to all codimension-1 (Cauchy-)hypersurfaces $\Sigma_p \hookrightarrow \Sigma$ gives *natural* Kostant-Souriau prequantizations of all covariant phase spaces:

Kostant-Souriau prequantum line bundle

$$
\begin{tikzcd}
& & \mathbf{B}(\mathbb{R}/\mathbb{Z})_{\mathrm{conn}} \arrow[d] \\
{[N^\infty \Sigma_p, \mathcal{E}]_\Sigma} \arrow[urr, "\theta := \int_{\Sigma_p} \mathbf{\Theta}"] \arrow[rr, "\omega := \int_{\Sigma_p} \Omega"'] & & \mathbf{\Omega}^2
\end{tikzcd}
$$

covariant phase space for $\Sigma_p \hookrightarrow \Sigma$

canonical presymplecic form

Theorem: Transgression of higher prequantum gerbes ∇ to fields on manifold Σ with boundary looks like so:

$$\begin{tikzcd}
{[\Sigma, X]} \arrow[r, "{[\Sigma,\nabla]}"] \arrow[d, "(-)|_{\partial\Sigma}"'] & {[\Sigma, \mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}}]} \arrow[r, "\int_\Sigma \mathrm{curv}"] \arrow[d, "(-)|_{\partial\Sigma}"'] & \mathbf{\Omega}^{p+2-d} \arrow[dl, Rightarrow, "\int_\Sigma"] \arrow[d] \\
{[\partial\Sigma, X]} \arrow[r, "{[\partial\Sigma,\nabla]}"'] & {[\partial\Sigma, \mathbf{B}^{p+1}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}}]} \arrow[r, "\int_{\partial\Sigma}"'] & \mathbf{B}^{p+2-d}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}}
\end{tikzcd}$$

Corollary: Transgressing Lepage p-gerbe to spacetime Σ with incoming and outgoing boundary yields the prequantized Lagrangian correspondence that exhibits dynamical evolution:

$$
\begin{tikzcd}
 & {[\Sigma, \mathcal{E}]_\Sigma} \arrow[dl, "(-)|_{\partial_{\mathrm{in}}\Sigma}"'] \arrow[dr, "(-)|_{\partial_{\mathrm{out}}\Sigma}"] \arrow[dd, phantom, "\swArrow\ \exp(\frac{i}{\hbar}S)"] & \\
{[N^\infty_\Sigma \partial_{\mathrm{in}}\Sigma, \mathcal{E}]_\Sigma} \arrow[dr, "\theta_{\mathrm{in}}"'] \arrow[ddr, bend right, "\omega_{\mathrm{in}}"'] & & {[N^\infty_\Sigma \partial_{\mathrm{out}}\Sigma, \mathcal{E}]_\Sigma} \arrow[dl, "\theta_{\mathrm{out}}"] \arrow[ddl, bend left, "\omega_{\mathrm{out}}"] \\
 & \mathbf{B}(\mathbb{R}/_{\hbar}\mathbb{Z})_{\mathrm{conn}} \arrow[d, "\mathrm{curv}"] & \\
 & \mathbf{\Omega}^2_{\mathrm{cl}} &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \text{field trajectories} \arrow[dl, "\text{initial values}"'] \arrow[dr, "\text{Hamiltonian evolution}"] \arrow[dd, phantom, "\swArrow\ \text{action functional}"] & \\
\text{incoming field configurations} \arrow[dr, "\text{prequantum bundle}"'] & & \text{outgoing field configirations} \arrow[dl, "\text{prequantum bundle}"] \\
 & \text{2-group of phases} &
\end{tikzcd}
$$

4)
Prequantum ∞-CS theories

Chern-Simons is nonabelian gauge theory

So now pass to the "nonabelian derived category" (aka ∞-topos) where sheaves of chain complexes are generalized to sheaves of Kan complexes [Brow73].

This serves to describe nonabelian gauge fields.

For instance there is sheaf of Kan complexes $\mathbf{B}G_{\mathrm{conn}}$ such that a G-principal connection on X is a map

$$X \longrightarrow \mathbf{B}G_{\mathrm{conn}}$$

and a gauge transformation is a homotopy

$$\begin{tikzcd} X \arrow[rr, bend left] \arrow[rr, bend right] \arrow[rr, phantom, "\Downarrow"] & & \mathbf{B}G_{\mathrm{conn}} \end{tikzcd}$$

One way to construct prequantum field theories is:
construct a p-gerbe connection on some moduli stack

$$\nabla : \mathbf{A}_{\mathrm{conn}} \longrightarrow \mathbf{B}^{p+1}(\mathbb{R}/\mathbb{Z})_{\mathrm{conn}}$$

and consider the stacky field bundle $E := \Sigma \times \mathbf{A}_{\mathrm{conn}}$
then pullback and project to get Euler-Lagrange and Lepage p-gerbe

$$\begin{tikzcd}
 & & \mathbf{B}^{p+1}_{L}(\mathbb{R}/\mathbb{Z})_{\mathrm{conn}} \arrow[d] \\
\Sigma \times \mathbf{A}_{\mathrm{conn}} \arrow[r, "\Sigma\times\nabla"] \arrow[urr, bend left, "\mathbf{\Theta}"] \arrow[rr, bend right, "\mathbf{L}"'] & \Sigma \times \mathbf{B}^{p+1}(\mathbb{R}/\mathbb{Z})_{\mathrm{conn}} \arrow[r] \arrow[ur] & \mathbf{B}^{p+1}_{H}(\mathbb{R}/\mathbb{Z})_{\mathrm{conn}}
\end{tikzcd}$$

For instance for G a simply-connected compact simple Lie group, there is a unique differential refinement

$$\nabla : \mathbf{B}G_{\mathrm{conn}} \longrightarrow \mathbf{B}^3 U(1)_{\mathrm{conn}}$$

of the canonical universal characteristic 4-class

$$c_2 : BG \longrightarrow K(\mathbb{Z}, 4)$$

This induces the standard 3d Chern-Simons Lagrangian and universally prequantizes it:

codim 0	$[\Sigma_3, \mathbf{B}G_{\mathrm{conn}}] \longrightarrow \mathbb{R}/\mathbb{Z}$	CS invariant
codim 1	$[\Sigma_2, \mathbf{B}G_{\mathrm{conn}}] \longrightarrow \mathbf{B}(\mathbb{R}/\mathbb{Z})_{\mathrm{conn}}$	CS prequantum line
codim 2	$[\Sigma_1, \mathbf{B}G_{\mathrm{conn}}] \longrightarrow \mathbf{B}^2(\mathbb{R}/\mathbb{Z})_{\mathrm{conn}}$	WZW gerbe
codim 3	$[\Sigma_0, \mathbf{B}G_{\mathrm{conn}}] \longrightarrow \mathbf{B}^3(\mathbb{R}/\mathbb{Z})_{\mathrm{conn}}$	Chern-Weil map

construct this and other examples from Lie integration of L_∞-data:

Def. L_∞-algebroid $\mathfrak{a}$ is dg-manifold
Chevalley-Eilenberg algebra $\mathrm{CE}(\mathfrak{a})$ is the dg-algebra of functions
$\mathrm{W}(\mathfrak{a})$ is dg-algebra of differential forms
cocycle is

$$\frac{\mathfrak{a} \xrightarrow{\mu} \mathbf{B}^{p+2}\mathbb{R}}{\mathrm{CE}(\mathfrak{a}) \longleftarrow \mathrm{CE}(\mathbf{B}^{p+2}\mathbb{R})}$$

Def. universal Lie integration to the derived category over $\mathrm{SmoothMfd}$ is the sheaf of Kan complexes

$$\exp(\mathfrak{a}) : (U, k) \mapsto \{\Omega^\bullet_{\mathrm{vert}}(U \times \Delta^k) \longleftarrow \mathrm{CE}(\mathfrak{a})\}$$

Examples:

- for $\mathfrak{g}$ semisimple Lie algebra, then $\tau_1 \exp(\mathbf{B}\mathfrak{g}) \simeq \mathbf{B}G$
- for $\mathfrak{P}$ a Poisson Lie algebroid then $\tau_2 \exp(\mathfrak{P})$ is symplectic Lie groupoid.
- for $\mathfrak{string}$ the String Lie 2-algebra, then $\tau_2 \exp(\mathbf{B}\mathfrak{string}) \simeq \mathbf{B}\mathrm{String}$

an *invariant polynomial* $\langle - \rangle$ on $\mathfrak{a}$ is closed differential form on the dg-manifold.
If $\langle - \rangle$ is binary and non-degenerate,
this became also called "shifted symplectic form"

Def. differential Lie integration $\exp(\mathfrak{g})_{\mathrm{conn}}$ is

$$\exp(\mathfrak{a})_{\mathrm{conn}} : (U, k) \mapsto \left\{ \begin{tikzcd} \Omega^\bullet_{\mathrm{vert}}(U \times \Delta^k) & \mathrm{CE}(\mathfrak{a}) \arrow[l] \\ \Omega^\bullet(U \times \Delta^k) \arrow[u] & \mathrm{W}(\mathfrak{a}) \arrow[l] \arrow[u] \\ \Omega^\bullet(U) \arrow[u] & \mathrm{inv}(\mathfrak{a}) \arrow[l] \arrow[u] \end{tikzcd} \right\}$$

Def. A cocycle μ is in transgression with an invariant polynomial $\langle - \rangle$ if there is a diagram of the form

$$\begin{tikzcd}
\mathrm{CE}(\mathfrak{a}) & \mathrm{CE}(\mathbf{B}^{p+2}\mathbb{R}) \arrow[l, "\mu"'] \\
\mathrm{W}(\mathfrak{a}) \arrow[u] & \mathrm{W}(\mathbf{B}^{p+2}\mathbb{R}) \arrow[l, "\mathrm{cs}"'] \arrow[u] \\
\mathrm{inv}(\mathfrak{a}) \arrow[u] & \mathrm{inv}(\mathbf{B}^{p+2}\mathbb{R}) \arrow[l, "\langle - \rangle"'] \arrow[u]
\end{tikzcd}$$

By pasting of diagrams, this data defines a map

$$\exp(\mathrm{cs}) \;:\; \exp(\mathfrak{a})_{\mathrm{conn}} \longrightarrow \exp(\mathbf{B}^{p+2}\mathbb{R})_{\mathrm{conn}}$$

Theorem:
under truncation this descends to

$$\begin{tikzcd}
\exp(\mathfrak{a})_{\mathrm{conn}} \arrow[r, "\exp(\mathrm{cs})"] \arrow[d, "\mathrm{cosk}_{p+2}"] & \exp(\mathbf{B}^{p+2}\mathbb{R})_{\mathrm{conn}} \arrow[d] \\
\mathbf{A}_{\mathrm{conn}} \arrow[r, "\nabla"] & \mathbf{B}^{p+2}(\mathbb{R}/\Gamma)_{\mathrm{conn}}
\end{tikzcd}$$

where $\Gamma \hookrightarrow \mathbb{R}$ is the group of periods of μ.

This gives large supply of **examples** of
prequantum field theories induced from "shifted *n*-plectic forms":

AKSZ including 3dCS, PSM (hence A-model/B-model), CSM, ...
7dCS on String 2-connections, 11dCS on 5brane 6-connections, ...

Outlook

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Our formulation of prequantum field theory
works also with *generalized* differential cohomology.

For instance the "topological" Lagrangian term
for super Dp-brane sigma models
for all even or all odd p at once
needs to be a cocycle in differential K-theory.

And "U-duality" predicts
that the topological terms for the M2 and M5 brane
needs to be unified in single generalized differential cocycle
with Chern character in the rational 4-sphere
[Fiorenza-Sati-Schreiber 15].

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References

exposition, details and full bibliography is at

ncatlab.org/schreiber/show/Local+prequantum+field+theory

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