Prequantum field theory

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based on joint work with
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exposition, details and references at
ncatlab.org/schreiber/show/Local+prequantum+field+theory
Let $\Sigma$ be a $(p + 1)$-dimensional smooth manifold

Given a smooth bundle $E$ over $\Sigma$, think of its sections $\phi \in \Gamma_\Sigma(E)$ as physical fields.

**Task:** Describe field theory **fully-local to fully-global**.
1. Classical field theory
2. BV-BFV field theory
3. Prequantum field theory
4. Prequantum $\infty$-CS theories
1) Classical field theory
A differential operator $D : \Gamma_{\Sigma}(E) \to \Gamma_{\Sigma}(F)$ comes from bundle map out of the jet bundle

$$\tilde{D} : J_{\Sigma}^{\infty}E \to F$$

Composition of diff ops $D_2 \circ D_1$ comes from

$$J_{\Sigma}^{\infty}E \to J_{\Sigma}^{\infty}J_{\Sigma}^{\infty}E \xrightarrow{J_{\Sigma}^{\infty}\tilde{D}_1} J_{\Sigma}^{\infty}F \xrightarrow{\tilde{D}_2} G$$

here the first map witnesses comonad structure on jets, this is composition in the coKleisli category $\text{Kl}(J_{\Sigma}^{\infty})$. 
A differential equation is an equalizer of two differential operators. Exists in the Eilenberg-Moore category $\text{EM}(J_\Sigma^\infty)$

$$
\begin{array}{c}
\mathcal{E} \\ \downarrow \\
E \xrightarrow{\tilde{D}_1} F \\
\downarrow \\
\tilde{D}_2
\end{array}
$$

Theorem [Marvan 86]:

$$
\text{EM}(J_\Sigma^\infty E) \simeq \text{PDE}_\Sigma
$$

PDE solutions are sections:

$$
\begin{array}{c}
\Sigma \\
\phi
\end{array}
\quad \xrightarrow{\phi_{\text{sol}}} \quad \begin{array}{c}
\mathcal{E} \\
\downarrow \\
E
\end{array}
$$
Def. A horizontal differential form on jet bundle $\alpha \in \Omega^k_{H}(J^\infty_{\Sigma} E)$ is diff op of the form

$$\tilde{\alpha} : E \rightarrow \wedge^k T^*\Sigma$$

This induces vertical/horizontal bigrading $\Omega^{\bullet,\bullet}(J^\infty_{\Sigma} E)$

Def: A local Lagrangian is $L \in \Omega^p_{H}(J^\infty_{\Sigma} E)$

Prop. Unique decomposition

$$d_{dR}L = EL - d_H(\Theta + d_H(\cdots))$$

with $EL \in \Omega^{p+1,1}_{S} \hookrightarrow \Omega^{p+1,1}$

depending only on vector fields along 0-jets.

This is the local incarnation of the variational principle.
**Prop.** the bigrading is preserved by pullback along diff ops

This means that

$$\Omega^{\bullet, \bullet} \in \text{Sh}(\text{DiffOp}_\Sigma) \xrightarrow{\text{Kan ext}} \text{Sh}(\text{PDE}_\Sigma)$$

is a bicomplex of sheaves on the category of PDEs.

In this sheaf topos, classical field theory looks like so:
By *transgression* of this local data we get

1) the **global action functional**

\[
[\Sigma, E]_\Sigma \xrightarrow{S = \int_\Sigma L} \mathbb{R}
\]

2) the **covariant phase space** [Zuckerman 87]:

\[
\begin{align*}
[N^\infty \Sigma p, \mathcal{E}]_\Sigma & \xrightarrow{\omega} \Omega^1 \\
\theta := \int_{\Sigma p} \Theta & \xrightarrow{d} \Omega^2
\end{align*}
\]
2) BV-BFV field theory
\( E \) need not be representable by a submanifold of \( J_\infty^\infty E \) if EL-equation is singular.

idea of BV-theory/derived geometry:

1. replace base category of smooth manifolds by smooth dg-manifolds in non-positive degree.

2. resolve singular \( E \) by realizing it as 0-cohomology of smooth dg-manifold.
If \( \{ \Phi^i \} \) are local fiber coordinates on \( E \) (field coordinates), then the derived shell \( \mathcal{E} \) has dg-algebra of functions the algebra \( C^\infty(\mathcal{E}) \) with degree-\((-1)\) generators \( \Phi^*_i \) added ("antifields") and with differential given by

\[
d_{BV} : \Phi^*_i \mapsto EL_i.
\]

Usual to write \( Q \) for \( d_{BV} \) regarded as vector field on the dg-manifold. Then this is

\[
\mathcal{L}_Q \Phi^*_i = EL_i.
\]

This gives a third grading (BV antifield grading) on differential forms on the jet bundle

\[
\Omega^{\bullet, \bullet; -\bullet}(\mathcal{E}_d)
\]
**Observation:** \( \mathcal{E}_d \) carries 2-form locally given by

\[
\Omega_{BV} = d\Phi^i \wedge d\Phi^i \in \Omega^{p+1,2;-1}(\mathcal{E}_d)
\]

which satisfies

\[
\iota_Q \Omega_{BV} = EL = dL + d_H \Theta \in \Omega^{p+1,1;0}(\mathcal{E}_d).
\]

under transgression to the space of fields this becomes

\[
\iota_Q \omega_{BV} = dS + \pi^* \theta
\]

This is the central compatibility postulate for BV-BFV field theory in [Cattaneo-Mnev-Reshetikhin 12, eq. (7)]
3) Prequantum field theory
For many field theories of interest, \( L \) is not in fact globally defined.

Simple examples:

- electron in EM-field with non-trivial magnetic charge;
- 3d \( U(1) \)-Chern-Simons theory;

Large classes of examples:

- higher \textbf{WZW-type} models
  (super \( p \)-branes, topological phases of matter)
- higher \textbf{Chern-Simons-type} models
  (AKSZ, 7dCS on String-2-connections, 11dCS,...)
Claim: There is a *systematic* solution to this problem by

1. passing to the derived category of $\text{Sh}(\text{PDE}_\Sigma)$;
2. generalizing Lagrangian forms to differential cocycles ("gerbes with connection")

If $\mathbf{V} := [\cdots \partial V \rightarrow V^2 \rightarrow V^1 \rightarrow V^0]$ is a sheaf of chain complexes, then a map in the derived category

$$E \rightarrow \mathbf{V}$$

is equivalently a cocycle in the sheaf hypercohomology of $E$ with coefficients in $\mathbf{V}$ [Brow73]. Homotopy is coboundary:

$$E \rightarrow \mathbf{V} \leftarrow \mathbf{V}$$
hence generalize sheaf of horizontal forms $\Omega^p_H$ to:

**Def.** The “variational Deligne complex”

$$B^{p+1}_H(\mathbb{R}/\mathbb{Z})_{\text{conn}} := [\mathbb{Z} \to \Omega^0_H \xrightarrow{d_H} \Omega^1_H \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{p+1}_H]$$

**Def.** A local prequantum Lagrangian is

$$L : E \longrightarrow B^{p+1}_H(\mathbb{R}/\mathbb{Z})_{\text{conn}}$$
**Consequence:** Via fiber integration in differential cohomology $L$ transgresses to globally well defined action functional

\[ S := \int_{\Sigma} [\Sigma, L] : [\Sigma, E]_{\Sigma} \to \mathbb{R}/\mathbb{Z} \]

**Theorem:** the curving of such Euler-Lagrange $p$-gerbes is given by the Euler variational operator

\[ \delta_V : B_{H}^{p+1}(\mathbb{R}/\mathbb{Z})_{\text{conn}} \to \Omega_{S}^{p+1,1} \]

principle of extremal action $\leftrightarrow$ flatness of EL-$p$-gerbes
**Def./Prop.** Prequantization of $\Theta$ is via *Lepage $p$-gerbes* $\Theta$ whose curvature in degree $(p, 2)$ is the *pre-symplectic current*.

\begin{align*}
\mathcal{E} & \quad \xrightarrow{\ker(\text{El})} \quad \Theta \\
& \quad \xrightarrow{\delta \nu} \quad \Omega_{S}^{p+1,1} \oplus \Omega^{p,2} \\
& \quad \xrightarrow{\text{fields}} \quad \mathcal{B}^{p+1}_H(\mathbb{R}/\mathbb{Z})_{\text{conn}} \\
& \quad \xrightarrow{\text{EL}} \quad \mathcal{B}^{p+1}_L(\mathbb{R}/\mathbb{Z})_{\text{conn}} \\
& \quad \xrightarrow{\delta \nu} \quad \Omega_{S}^{p+1,1} \\
\end{align*}
**Theorem:** Transgressions of single Lepage $p$-gerbe to all codimension-1 (Cauchy-)hypersurfaces $\Sigma_p \hookrightarrow \Sigma$ gives *natural* Kostant-Souriau prequantizations of all covariant phase spaces:

$$\theta := \int_{\Sigma_p} \Theta$$

Kostant-Souriau prequantum line bundle

$$\mathcal{B}(\mathbb{R}/\mathbb{Z})_{\text{conn}}$$

$$\omega := \int_{\Sigma_p} \Omega$$

canonical presymplectic form
**Theorem:** Transgression of higher prequantum gerbes $\nabla$ to fields on manifold $\Sigma$ with boundary looks like so:
**Corollary:** Transgressing Lepage $p$-gerbe to spacetime $\Sigma$ with incoming and outgoing boundary yields the prequantized Lagrangian correspondence that exhibits dynamical evolution:
4) Prequantum $\infty$-CS theories
Chern-Simons is nonabelian gauge theory

So now pass to the “nonabelian derived category” (aka $\infty$-topos) where sheaves of chain complexes are generalized to sheaves of Kan complexes [Brow73].

This serves to describe nonabelian gauge fields.

For instance there is sheaf of Kan complexes $B G_{\text{conn}}$ such that a $G$-principal connection on $X$ is a map

$$X \rightarrow B G_{\text{conn}}$$

and a gauge transformation is a homotopy

$$\begin{array}{ccc}
X & \rightarrow & B G_{\text{conn}} \\
\downarrow & & \downarrow \\
& \Rightarrow & \\
& & \\
& \Rightarrow & \\
& & \\
B G_{\text{conn}} & \leftarrow & X
\end{array}$$
One way to construct prequantum field theories is:
construct a $p$-gerbe connection on some moduli stack

$$\nabla : A_{\text{conn}} \longrightarrow B^{p+1}(\mathbb{R}/\mathbb{Z})_{\text{conn}}$$

and consider the stacky field bundle $E := \Sigma \times A_{\text{conn}}$
then pullback and project to get Euler-Lagrange and Lepage
$p$-gerbe
For instance for $G$ a simply-connected compact simple Lie group, there is a unique differential refinement

$$\nabla : B G_{\text{conn}} \longrightarrow B^3 U(1)_{\text{conn}}$$

of the canonical universal characteristic 4-class

$$c_2 : BG \longrightarrow K(\mathbb{Z}, 4)$$

This induces the standard 3d Chern-Simons Lagrangian and universally prequantizes it:

<table>
<thead>
<tr>
<th>codim 0</th>
<th>$[\Sigma_3, B G_{\text{conn}}]$ $\longrightarrow \mathbb{R}/\mathbb{Z}$</th>
<th>CS invariant</th>
</tr>
</thead>
<tbody>
<tr>
<td>codim 1</td>
<td>$[\Sigma_2, B G_{\text{conn}}]$ $\longrightarrow B(\mathbb{R}/\mathbb{Z})_{\text{conn}}$</td>
<td>CS prequantum line</td>
</tr>
<tr>
<td>codim 2</td>
<td>$[\Sigma_1, B G_{\text{conn}}]$ $\longrightarrow B^2(\mathbb{R}/\mathbb{Z})_{\text{conn}}$</td>
<td>WZW gerbe</td>
</tr>
<tr>
<td>codim 3</td>
<td>$[\Sigma_0, B G_{\text{conn}}]$ $\longrightarrow B^3(\mathbb{R}/\mathbb{Z})_{\text{conn}}$</td>
<td>Chern-Weil map</td>
</tr>
</tbody>
</table>
construct this and other examples from Lie integration of $L_\infty$-data:

**Def.** $L_\infty$-algebroid $\alpha$ is dg-manifold

Chevalley-Eilenberg algebra $\text{CE}(\alpha)$ is the dg-algebra of functions $W(\alpha)$ is dg-algebra of differential forms

cocycle is

\[
\begin{array}{c}
\alpha \xrightarrow{\mu} B^{p+2} \mathbb{R} \\
\text{CE}(\alpha) \leftarrow \text{CE}(B^{p+2} \mathbb{R})
\end{array}
\]
**Def.** universal Lie integration to the derived category over SmoothMfd is the sheaf of Kan complexes

\[ \exp(\alpha) : (U, k) \mapsto \{ \Omega^\bullet_{\text{vert}}(U \times \Delta^k) \leftarrow \text{CE}(\alpha) \} \]

**Examples:**

- for \( \mathfrak{g} \) semisimple Lie algebra, then \( \tau_1 \exp(B\mathfrak{g}) \simeq B\,G \)
- for \( \mathfrak{P} \) a Poisson Lie algebroid then \( \tau_2 \exp(\mathfrak{P}) \) is symplectic Lie groupoid.
- for string the String Lie 2-algebra, then \( \tau_2 \exp(B\text{String}) \simeq B\text{String} \)
an *invariant polynomial* \(\langle - \rangle\) on \(a\) is closed differential form on the dg-manifold.

If \(\langle - \rangle\) is binary and non-degenerate, this became also called “shifted symplectic form”

**Def.** differential Lie integration \(\exp(g)_{\text{conn}}\) is

\[
\begin{align*}
\exp(a)_{\text{conn}} : (U, k) \mapsto & \left\{ \begin{array}{c}
\Omega^\bullet(U \times \Delta^k) \leftarrow \text{CE}(a) \\
\Omega^\bullet(U \times \Delta^k) \leftarrow W(a) \\
\Omega^\bullet(U) \leftarrow \text{inv}(a)
\end{array} \right. \\
\end{align*}
\]
Def. A cocycle $\mu$ is in transgression with an invariant polynomial $\langle - \rangle$ if there is a diagram of the form

$$
\begin{align*}
\text{CE}(a) & \xleftarrow{\mu} \text{CE}(B^{p+2}\mathbb{R}) \\
W(a) & \xleftarrow{\text{cs}} W(B^{p+2}\mathbb{R}) \\
\text{inv}(a) & \xleftarrow{\langle - \rangle} \text{inv}(B^{p+2}\mathbb{R})
\end{align*}
$$

By pasting of diagrams, this data defines a map

$$
\exp(\text{cs}) : \exp(a)_{\text{conn}} \longrightarrow \exp(B^{p+2}\mathbb{R})_{\text{conn}}
$$
Theorem:
under truncation this descends to

\[
\begin{array}{cccc}
\exp(a)_{\text{conn}} & \xrightarrow{\exp(cs)} & \exp(B^{p+2}\mathbb{R})_{\text{conn}} \\
\downarrow \cosk_{p+2} & & \downarrow \\
A_{\text{conn}} & \xrightarrow{\nabla} & B^{p+2}(\mathbb{R}/\Gamma)_{\text{conn}}
\end{array}
\]

where \( \Gamma \hookrightarrow \mathbb{R} \) is the group of periods of \( \mu \).

This gives large supply of **examples** of
prequantum field theories induced from “shifted \( n \)-plectic forms”:

AKSZ including 3dCS, PSM (hence A-model/B-model), CSM, ...
7dCS on String 2-connections, 11dCS on 5brane 6-connections, ...
Outlook
Our formulation of prequantum field theory works also with \textit{generalized} differential cohomology.

For instance the “topological” Lagrangian term for super $Dp$-brane sigma models for all even or all odd $p$ at once needs to be a cocycle in differential K-theory.

And “U-duality” predicts that the topological terms for the M2 and M5 brane needs to be unified in single generalized differential cocycle with Chern character in the rational 4-sphere [Fiorenza-Sati-Schreiber 15].
References

exposition, details and full bibliography is at

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