Aspects of Differential Geometry in HoTT

Felix Wellen

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$$\text{Smooth Manifolds} \hookrightarrow \text{Formal Smooth Sets} = \mathrm{Sh}(\{\mathbb{R}^n \times \mathbb{D} | n \in \mathbb{N}\})$$

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Where "Sh" means the topos of set- or ∞-groupoid-valued sheaves on:

1. $\mathbb{R}^n \times \mathbb{D}$ with smooth open good covers (ignoring the $\mathbb{D}$s).
2. Commutative, unital rings with jointly surjective inclusions of Zariski-open affine subsets.

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But $(k[X]/(X^2))_{\mathrm{red}}$ is just k, so the tangent vectors at k-points of $\Im(S)$ are just the k-points:

$$\Im(S)(\mathrm{Spec}(k[X]/(X^2))) = S(\mathrm{Spec}(k[X]/(X^2))_{\mathrm{red}}) = S(\mathrm{Spec}(k))$$

So: $\Im$ removes all differential geometric information!

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For all nilpotent $\mathbb{R}$-algebra V, formally extend the left category by

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and call this category FC. For any k, we can restrict the order:

$$\mathrm{FC}_k := \{\mathcal{C}^\infty(\mathbb{R}^n) \otimes (\mathbb{R} \oplus V) | n \in \mathbb{N}, V^{k+1} = 0\}^{\mathrm{op}} \subseteq \mathbb{R}\mathrm{-algebras}^{\mathrm{op}}$$

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Now, define $\mathfrak{I}\colon \mathrm{Sh}(\mathrm{FC}) \to \mathrm{Sh}(\mathrm{FC})$ by

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and, respectively $\mathfrak{I}_k\colon \mathrm{Sh}(\mathrm{FC}_k) \to \mathrm{Sh}(\mathrm{FC}_k)$ by the same equation

$$\mathfrak{I}_k(\mathcal{F})(\mathbb{R}^n \times \mathbb{D}_V) := \mathcal{F}(\mathbb{R}^n)$$

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The tangent bundle is also given as a pullback:

$$\begin{array}{ccc} TM & \longrightarrow & M \\ \downarrow & \text{(pb)} & \downarrow \; \iota_M \\ M & \xrightarrow[\iota_M]{} & \mathfrak{I}_1 M \end{array}$$

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2. (A is coreduced) $:\equiv$ (ι_A is an equivalence)
3. For any type A, $\Im A$ is coreduced.
4. For any $B \colon \Im A \to \mathcal{U}$, such that $\prod_{a:\Im A} B(a)$ is coreduced, a section $s \colon \prod_{a:\Im A} B(a)$ is defined by $s_0 \colon \prod_{a:A} B(\iota_A(a))$.

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5. Coreduced types have coreduced identity types.

Internal geometric notions

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Definition

For any point $x \colon A$, $\mathbb{D}_x$ is defined by

$$\begin{array}{ccc}
\mathbb{D}_x & \longrightarrow & 1 \\
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The *formal disk bundle over* A, $T_\infty A$ is defined by the pullback

$$\begin{array}{ccc} T_\infty A & \longrightarrow & A \\ \big\downarrow & \text{(pb)} & \big\downarrow \iota_A \\ A & \xrightarrow[\iota_A]{} & \Im A \end{array}$$

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2. $\mu : X \times X \to X$
3. Proof that the unit is a left and right unit, i.e. a term in each of
$$\prod_{x:X} \mu(e,x) = x \text{ and } \prod_{x:X} \mu(x,e) = x.$$
4. Proof that for any $a : X$ the right-translation $x \mapsto \mu(x,a)$ is an equivalence, i.e. there is a term of type
$$\prod_{a:X} (x \mapsto \mu(x,a)) \text{ is an equivalence.}$$

The triviality theorem

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Theorem
Let V be a left invertible H-space and $\mathbb{D}$ the formal disk at the unit in V, then:

$$\begin{array}{ccc} T_\infty V & \xrightarrow{\ \simeq\ } & \mathbb{D} \times V \\ {\scriptstyle (\pi_1, \pi_2)} \searrow & & \swarrow {\scriptstyle (d, g) \mapsto (\mu(\pi_1(d), g), g)} \\ & V \times V & \end{array}$$

This theorem has a proof similar to its topos theoretic version in [KS17].

Differential structure preserving morphisms

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A map $f\colon A \to B$ is called *formally étale* if the naturality square

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Remark

For smooth manifolds formally étale maps correspond to local diffeomorphisms.

For noetherian schemes, they correspond to étale maps.

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Theorem (needs Univalence)
Any V-Manifold has a locally trivial formal disk bundle witnessed by a classifying map

$$\chi_M \colon M \to \mathrm{BAut}(\mathbb{D}_V)$$

Cartan Geometry

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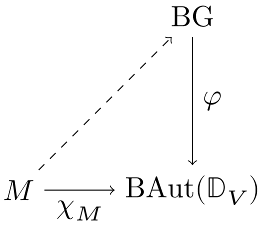
Remark

If we have a delooping BG of a group G with a map $\varphi\colon \mathrm{BG} \to \mathrm{BAut}(\mathbb{D}_V)$, we can ask if there is a lift:



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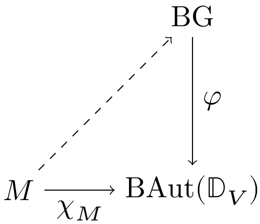

For example, such a lift for $G = O(n)$ together with another condition is a Pseudo-Riemannian structure on M.

References

A topos theoretic version of the theorem about the triviality of the formal disk bundle and more on the smooth case may be found in

I. Khavkine and U. Schreiber. "Synthetic geometry of differential equations: I. Jets and comonad structure". In: *ArXiv e-prints* (Jan. 2017). arXiv: 1701.06238 [math.DG].

A thesis titled "Formalizing Cartan Geometry in Modal Homotopy Type Theory" containing more on the topic is to appear very soon.