Homotopy Quantum Field Theories meets the Crossed Menagerie: an introduction to HQFTs and their relationship with things simplicial and with lots of crossed gadgetry.

Notes prepared for the Workshop and School on Higher Gauge Theory, TQFT and Quantum Gravity

Lisbon, February, 2011

Timothy Porter
WIMCS at Bangor

Introduction

Warning:

These notes for the mini-course in Lisbon, February 2011, and have been constructed from the main body of the much larger Menagerie notes. The method used to prepare them has been to delete sections that were not more or less necessary for this course and then to add in new material. (There will be some loose ends therefore, and missing links. These will be given with a ? as the Latex refers to the original cross reference.)

If you want to follow up some of the ideas that lead out of these notes, just look at the version of the Menagerie available on the nLab, [173], and if that does not have the relevant chapter, just ask me! (Beware, the full present version is already over 800 pages in length, so, please, don't print too many copies!!!)

There are several points to make. As in the full Menagerie notes, there are no exercises as such, but at various points if a proof could be expanded, or is **left to the reader**, then, yes, bold face will be used to suggest that that is a useful place for more input from the reader. In lots of places, reading the details is not that efficient a way of getting to grips with the calculations and ideas, and there is no substitute for **doing it yourself**. That being said guidance as to how to approach the subject will often be given.

Almost needless to say, there are things that have not been discussed here (or in the Menagerie itself), and suggestions for additional material are welcome. Better still would be for the suggestions to materialise into new entries on the nLab.

Introduction to the notes

The aim of these notes is to provide some background material for discussions of Homotopy Quantum Field Theories, crossed Frobenius Algebras, simplicial group methods, algebraic models for n-types, crossed modules, etc.

Tim Porter, Anglesey, 2011.

Contents

Chapter 1

Crossed modules - definitions, examples and applications

We will give these for groups, although there are analogues for many other algebraic settings.

1.1 Crossed modules

Definition: A *crossed module*, (C, G, δ), consists of groups C and G with a left action of G on C, written $(g, c) \to {}^g c$ for $g \in G$, $c \in C$, and a group homomorphism $\delta : C \to G$ satisfying the following conditions:
CM1) for all $c \in C$ and $g \in G$,

$$\delta({}^g c) = g\delta(c)g^{-1},$$

CM2) for all $c_1, c_2 \in C$,

$${}^{\delta(c_2)}c_1 = c_2 c_1 c_2^{-1}.$$

(CM2 is called the *Peiffer identity.*)

If (C, G, δ) and (C', G', δ') are crossed modules, a *morphism*, $(\mu, \eta) : (C, G, \delta) \to (C', G', \delta')$, *of crossed modules* consists of group homomorphisms $\mu : C \to C'$ and $\eta : G \to G'$ such that

(i) $\delta'\mu = \eta\delta$ and (ii) $\mu({}^g c) = {}^{\eta(g)}\mu(c)$ for all $c \in C$, $g \in G$.

Crossed modules and their morphisms form a category, of course. It will usually be denoted $CMod$.

There is, for a fixed group G, a subcategory $CMod_G$ of $CMod$, which has, as objects, those crossed modules with G as the "base", i.e., all (C, G, δ) for this fixed G, and having as morphisms from (C, G, δ) to (C', G, δ') just those (μ, η) in $CMod$ in which $\eta : G \to G$ is the identity homomorphism on G.

Several well known situations give rise to crossed modules. The verification will be left to you.

1.1.1 Algebraic examples of crossed modules

(i) Let H be a normal subgroup of a group G with $i : H \to G$ the inclusion, then we will say (H, G, i) is a *normal subgroup pair.* In this case, of course, G acts on the left of H by conjugation and the inclusion homomorphism i makes (H, G, i) into a crossed module, an 'inclusion crossed modules'. Conversely it is an easy exercise to prove

Lemma 1 *If (C, G, ∂) is a crossed module, ∂C is a normal subgroup of G.* ∎

(ii) Suppose G is a group and M is a left G-module; let $0 : M \to G$ be the trivial map sending everything in M to the identity element of G, then $(M, G, 0)$ is a crossed module.

Again conversely:

Lemma 2 *If (C, G, ∂) is a crossed module, $K = Ker\,\partial$ is central in C and inherits a natural G-module structure from the G-action on C. Moreover, $N = \partial C$ acts trivially on K, so K has a natural G/N-module structure.* ∎

Again the proof is left as an exercise.

As these two examples suggest, general crossed modules lie between the two extremes of normal subgroups and modules, in some sense, just as groupoids lay between equivalence relations and G-sets. Their structure bears a certain resemblance to both - they are "external" normal subgroups, but also are "twisted" modules.

(iii) Let G be a group, then, as usual, let $Aut(G)$, denote the group of automorphisms of G. Conjugation gives a homomorphism

$$\iota : G \to Aut(G).$$

Of course, $Aut(G)$ acts on G in the obvious way and ι is a crossed module. We will need this later so will give it its own name, the *automorphism crossed module of the group, G* and its own notation: $\mathsf{Aut}(G)$.

More generally if L is some type of algebra then $U(L) \to Aut(L)$ will be a crossed module, where $U(L)$ denotes the units of L and the morphism send a unit to the automorphism given by conjugation by it.

This class of example has a very nice property with respect to general crossed modules. For a general crossed module, (C, P, ∂), we have an action of P on C, hence a morphism, $\alpha : P \to Aut(C)$, so that $\alpha(p)(c) = {}^p c$. There is clearly a square

$$\begin{tikzcd} C \arrow[r, "="] \arrow[d, "\partial"'] & C \arrow[d, "\iota"] \\ P \arrow[r, "\alpha"'] & Aut(C) \end{tikzcd}$$

and we can ask if this gives a morphism of crossed modules. 'Clearly' it should. The requirements are that the square commutes and that the actions are compatible in the obvious sense, (recall page 13). To see that the square commutes, we just note that, given $c \in C$, ∂c acts on an $x \in C$, by conjugation by c: ${}^{\partial c}x = c.x.c^{-1} = \iota(c)(x)$, whilst to check that the actions match correctly remember that $\alpha(p)(c) = {}^p x$ by definition, so we do have a morphism of crossed modules as expected.

(iv) We suppose given a morphism

$$\theta : M \to N$$

of left G-modules and form the semi-direct product $N \rtimes G$. This group we make act on M via the projection from $N \rtimes G$ to G.

We define a morphism

$$\partial : M \to N \rtimes G$$

by $\partial(m) = (\theta(m), 1)$, where 1 denotes the identity element of G, then $(M, N \rtimes G, \partial)$ is a crossed module. In particular, if A and B are Abelian groups, and B is considered to act trivially on A, then any homomorphism, $A \to B$ is a crossed module.

(v) Suppose that we have a crossed module, $\mathsf{C} = (C, G, \delta)$, and a group homomorphism $\varphi : H \to G$, then we can form the 'pullback group' $H \times_G C = \{(h, c) \mid \varphi(h) = \delta c\}$, which is a subgroup of the product $H \times C$. There is a group homomorphism, $\delta' : H \times_G C \to H$, namely the restriction of the first projection morphism of the product, (so $\delta'(h, c) = h$). **You are left** to construct an action of H on this group, $H \times_G C$ such that $\varphi^*(\mathsf{C}) := (H \times_G C, H, \delta')$ is a crossed module, and also such that the pair of maps φ and the second projection $H \times_G C \to C$ give a morphism of crossed modules.

Definition: The crossed module, $\varphi^*(\mathsf{C})$, thus defined, is called the *pullback crossed module of* C *along* φ

(vi) As a last algebraic example for the moment, let

$$1 \to K \xrightarrow{a} E \xrightarrow{b} G \to 1$$

be an extension of groups with K a central subgroup of E, i.e. a *central extension* of G by K. For each $g \in G$, pick an element $s(g) \in b^{-1}(g) \subseteq E$. Define an action of G on E by: if $x \in E$, $g \in G$, then

$${}^g x = s(g) x s(g)^{-1}.$$

This is well defined, since if $s(g)$, $s'(g)$ are two choices, $s(g) = ks'(g)$ for some $k \in K$, and K is central. (This also shows that this *is* an action.) The structure (E, G, b) is a crossed module.

A particular important case is: for R a ring, let $E(R)$ be, as before, the group of elementary matrices of R, $E(R) \subseteq Gl(R)$ and $St(R)$, the corresponding Steinberg group with $b : St(R) \to E(R)$, the natural morphism, (see later or [160], for the definition). Then this gives a central extension

$$1 \to K_2(R) \to St(R) \to E(R) \to 1$$

and thus a crossed module. In fact, more generally,

$$b : St(R) \to Gl(R)$$

is a crossed module. The group $Gl(R)/Im(b)$ is $K_1(R)$, the first algebraic K-group of the ring.

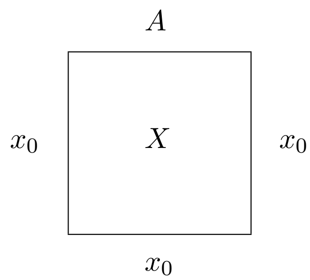
1.1.2 Topological Examples

In topology there are several examples that deserve looking at in detail as they do relate to aspects of the above algebraic cases. They require slightly more topological knowledge than has been assumed so far.

(vii) Let X be a pointed space, with $x_0 \in X$ as its base point, and A a subspace with $x_0 \in A$. Recall that the second relative homotopy group, $\pi_2(X, A, x_0)$, consists of relative homotopy classes of continuous maps

$$f : (I^2, \partial I^2, J) \to (X, A, x_0)$$

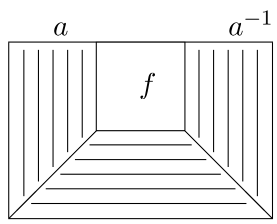
where ∂I^2 is the boundary of I^2, the square, $[0,1] \times [0,1]$, and $J = \{0,1\} \times [0,1] \cup [0,1] \times \{0\}$. Schematically f maps the square as:



so the top of the boundary goes to A, the rest to x_0 and the whole thing to X. The relative homotopies considered then deform the maps in such a way as to preserve such structure, so intermediate mappings also send J to x_0, etc. Restriction of such an f to the top of the boundary clearly gives a homomorphism

$$\partial : \pi_2(X, A, x_0) \to \pi_1(A, x_0)$$

to the fundamental group of A, based at x_0. There is also an action of $\pi_1(A, x_0)$ on $\pi_2(X, A, x_0)$ given by rescaling the 'square' given by



where f is partially 'enveloped' in a region on which the mapping is behaving like a.

Of course, this gives a crossed module

$$\pi_2(X, A, x_0) \to \pi_1(A, x_0).$$

A direct proof is quite easy to give. One can be found in Hilton's book, [119] or in Brown-Higgins-Sivera, [49]. Alternatively one can use the argument in the next example.

(viii) Suppose $F \stackrel{i}{\to} E \stackrel{p}{\to} B$ is a fibration sequence of pointed spaces. Thus p is a fibration, $F = p^{-1}(b_0)$, where b_0 is the basepoint of B. The fibre F is pointed at f_0, say, and f_0 is taken as the basepoint of E as well.

There is an induced map on fundamental groups

$$\pi_1(F) \xrightarrow{\pi_1(i)} \pi_1(E)$$

and if a is a loop in E based at f_0, and b a loop in F based at f_0, then the composite path corresponding to aba^{-1} is homotopic to one wholly within F. To see this, note that $p(aba^{-1})$ is null homotopic. Pick a homotopy in B between it and the constant map, then lift that homotopy back up to E to one starting at aba^{-1}. This homotopy is the required one and its other end gives a well defined element ${}^a b \in \pi_1(F)$ (abusing notation by confusing paths and their homotopy classes). With this action $(\pi_1(F), \pi(E), \pi_1(i))$ is a crossed module. This will not be proved here, but is not that difficult. Links with previous examples are strong.

If we are in the context of the above example, consider the inclusion map, f of a subspace A into a space X (both pointed at $x_0 \in A \subset X$). Form the corresponding fibration,

$$i^f : M^f \to X,$$

by forming the pullback

$$\begin{tikzcd} M^f \arrow[r, "\pi^f"] \arrow[d, "j^f"'] & X^I \arrow[d, "e_0"] \\ A \arrow[r, "f"'] & X \end{tikzcd}$$

so M^f consists of pairs, (a, λ), where $a \in A$ and λ is a path from $f(a)$ to some point $\lambda(1)$. Set $i^f = e_1\pi^f$, so $i^f(a, \lambda) = \lambda(1)$. It is standard that i^f is a fibration and its fibre is the subspace $F_h(f) = \{(a, \lambda) \mid \lambda(1) = x_0\}$, often called the *homotopy fibre* of f. The base point of $F_h(f)$ is taken to be the constant path at x_0, (x_0, c_{x_0}).

If we note that

$$\pi_1(F_h(f)) \cong \pi_2(X, A, x_0)$$
$$\pi_1(M^f) \cong \pi_1(A, x_0)$$

(even down to the descriptions of the actions, etc.), the link with the previous example becomes clear, and thus furnishes another proof of the statement there.

(ix) The link between fibrations and crossed modules can also be seen in the category of simplicial groups. A morphism $f : G \to H$ of simplicial groups is a fibration if and only if each f_n is an epimorphism. This means that a fibration is determined by the fibre over the identity which is, of course, the kernel of f. The $(G, \overline{W})$-links between simplicial groups and simplicial sets mean that the analogue of π_1 is π_0. Thus the fibration f corresponds to

$$Ker\, f \stackrel{\triangleleft}{\to} G$$

and each level of this is a crossed module by our earlier observations. Taking π_0, it is easy to check that

$$\pi_0(Ker\, f) \to \pi_0(G)$$

is a crossed module. In fact any crossed module is isomorphic to one of this form. (Proof left to the reader.)

If $\mathsf{M} = (C, G, \partial)$ is a crossed module, then we sometimes write $\pi_0(\mathsf{M}) := G/\partial C$, $\pi_1(\mathsf{M}) := Ker\,\partial$, and then have a 4-term exact sequence:

$$0 \to \pi_1(\mathsf{M}) \to C \xrightarrow{\partial} G \to \pi_0(\mathsf{M}) \to 1.$$

In topological situations when M provides a model for (part of) the homotopy type of a space X or a pair (X, A), then typically $\pi_1(\mathsf{M}) \cong \pi_2(X)$, $\pi_0(\mathsf{M}) \cong \pi_1(X)$.

MacLane and Whitehead, [152], showed that crossed modules give algebraic models for all homotopy 2-types of connected spaces. We will visit this result in more detail later, but loosely a 2-equivalence between spaces is a continuous map that induces isomorphisms on π_1 and π_2, the first two homotopy groups. Two spaces have the same 2-type if there is a zig-zag of 2-equivalences joining them.

1.1.3 Restriction along a homomorphism φ/ 'Change of base'

Given a crossed module (C, H, ∂) over H and a homomorphism $\varphi : G \to H$, we can form the pullback:

$$\begin{tikzcd} D \arrow[r, "\psi"] \arrow[d, "\partial'"'] & C \arrow[d, "\partial"] \\ G \arrow[r, "\varphi"'] & H \end{tikzcd}$$

in $Grps$. Clearly the universal property of pullbacks gives a good universal property for this, namely that any morphism $(\varphi', \varphi) : (C', G, \delta) \to (C, H, \partial)$ factors uniquely through (ψ, φ) and a morphism in $CMod_G$ from (C', G, δ) to (D, G, ∂'). Of course this statement depends on verification that (D, G, ∂') is a crossed module and that the resulting maps are morphisms of crossed modules, but this is routine, and will be **left as an exercise**. (You may need to recall that D can be realised, up to isomorphism, as $G \times_H C = \{(g, c) \mid \varphi(g) = \partial c\}$. It is for you to see what the action is.)

This construction also behaves nicely on morphisms of crossed modules over H and yields a functor,

$$\varphi^* : CMod_H \to CMod_G,$$

which will be called *restriction along* φ.

We next turn to the use of crossed modules in combinatorial group theory.

1.2 Group presentations, identities and 2-syzygies

1.2.1 Presentations and Identities

(cf. Brown-Huebschmann, [50]) We consider a presentation, $\mathcal{P} = (X : R)$, of a group G. The elements of X are called *generators* and those of R *relators*. We then have a short exact sequence,

$$1 \to N \to F \to G \to 1,$$

where $F = F(X)$, the free group on the set X, R is a subset of F and $N = N(R)$ is the normal closure in F of the set R. The group F acts on N by conjugation: ${}^{u}c = ucu^{-1}$, $c \in N, u \in F$ and the elements of N are words in the conjugates of the elements of R:

$$c = {}^{u_1}(r_1^{\varepsilon_1})^{u_2}(r_2^{\varepsilon_2}) \dots {}^{u_n}(r_n^{\varepsilon_n})$$

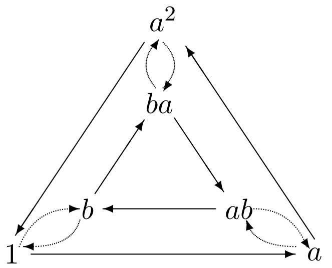
where each ε_i is $+1$ or -1. One also says such elements are *consequences* of R. Heuristically an *identity among the relations* of $\mathcal{P}$ is such an element c which equals 1. The problem of what this means is analogous to that of working with a relation in R. For example, in the presentation $(a : a^3)$ of C_3, the cyclic group of order 3, if a is thought of as being an element of C_3, then $a^3 = 1$, so why is this different from the situation with the 'presentation', $(a : a = 1)$? To get around that difficulty the free group on the generators $F(X)$ was introduced and, of course, in $F(\{a\})$, a^3 is not 1. A similar device, namely *free crossed modules* on the presentation will be introduced in a moment to handle the identities. Before that consider some examples which indicate that identities exist even in some quite common-or-garden cases.

Example 1: Suppose $r \in R$, but it is a power of some element $s \in F$, i.e. $r = s^m$. Of course, $rs = sr$ and

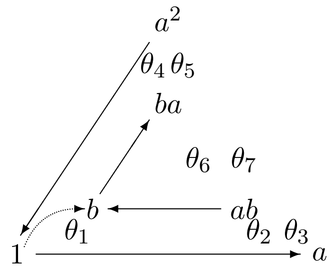
$${}^{s}rr^{-1} = 1$$

so ${}^{s}r.r^{-1}$ is an identity. In fact, there will be a unique $z \in F$ with $r = z^q$, q maximal with this property. This z is called the *root of* r and if $q > 1$, r is called a *proper power*.

Example 2: Consider one of the standard presentations of S_3, $(a, b : a^3, b^2, (ab)^2)$. Write $r = a^3$, $s = b^2$, $t = (ab)^2$. Here the presentation leads to F, free of rank 2, but $N(R) \subset F$, so it must be free as well, by the Nielsen-Schreier theorem. Its rank will be 7, given by the Schreier index formula or, geometrically, it will be the fundamental group of the Cayley graph of the presentation. This group is free on generators corresponding to edges outside a maximal tree as in the following diagram:



The Cayley graph of S_3



and a maximal tree in it.

The set of normal generators of $N(R)$ has 3 elements; $N(R)$ is free on 7 elements (corresponding to the edges not in the tree), but is specified as consisting of products of conjugates of r, s and t, and there are infinitely many of these. Clearly there must be some slight redundancy, i.e., there must be some identities among the relations!

A path around the outer triangle corresponds to the relation r; each other region corresponds to a conjugate of one of r, s or t. (It may help in what follows to think of the graph being embedded on a 2-sphere, so 'outer' and 'outside' mean 'round the back face.) Consider a loop around a region.

Pick a path to a start vertex of the loop, starting at 1. For instance the path that leaves 1 and goes along a, b and then goes around aaa before returning by $b^{-1}a^{-1}$ gives $abrb^{-1}a^{-1}$. Now the path around the outside can be written as a product of paths around the inner parts of the graph, e.g. $(abab)b^{-1}a^{-1}b^{-1}(bb)(b^{-1}a^{-1}b^{-1}a^{-1})\ldots$ and so on. Thus r can be written in a non-trivial way as a product of conjugates of r, s and t. (An explicit identity constructed like this is given in [50].)

Example 3: In a presentation of the free Abelian group on 3 generators, one would expect the commutators, $[x, y]$, $[x, z]$ and $[y, z]$. The well-known identity, usually called the Jacobi identity, expands out to give an identity among these relations (again see [50], p.154 or Loday, [144].)

1.2.2 Free crossed modules and identities

The idea that an identity is an equation in conjugates of relations leads one to consider formal conjugates of symbols that label relations. Abstracting this a bit, suppose G is a group and $f : Y \to G$, a function 'labelling' the elements of some subset of G. To form a conjugate, you need a thing being conjugated and an element 'doing' the conjugating, so form pairs (p, y), $p \in G$, $y \in Y$, to be thought of as ${}^p y$, the *formal conjugate* of y by p. Consequences are words in conjugates of relations, *formal consequences* are elements of $F(G \times Y)$. There is a function extending f from $G \times Y$ to G given by

$$\bar{f}(p, y) = pf(y)p^{-1},$$

converting a formal conjugate to an actual one and this extends further to a group homomorphism

$$\varphi : F(G \times Y) \to G$$

defined to be $\bar{f}$ on the generators. The group G acts on the left on $G \times Y$ by multiplication: $p.(p', y) = (pp', y)$. This extends to a group action of G on $F(G \times Y)$. For this action, φ is G-equivariant if G is given its usual G-group structure by conjugations / inner automorphisms. Naively identities are the elements in the kernel of this, but there are some elements in that kernel that are there regardless of the form of function f. In particular, suppose that $g_1, g_2 \in G$ and $y_1, y_2 \in Y$ and look at

$$(g_1, y_1)(g_2, y_2)(g_1, y_1)^{-1}((g_1 f(y_1)g_1^{-1})g_2, y_2)^{-1}.$$

Such an element is always annihilated by φ. The normal subgroup generated by such elements is called the Peiffer subgroup. We divide out by it to obtain a quotient group. This is the construction of the free crossed module on the function f. If f is, as in our initial motivation, the inclusion of a set of relators into the free group on the generators we call the result the *free crossed module on the presentation* $\mathcal{P}$ and denote it by $C(\mathcal{P})$.

We can now formally define the module of identities of a presentation $\mathcal{P} = (X : R)$. We form the free crossed module on $R \to F(X)$, which we will denote by $\partial : C(\mathcal{P}) \to F(X)$. The *module of identities* of $\mathcal{P}$ is $Ker\,\partial$. By construction, the group presented by $\mathcal{P}$ is $G \cong F(X)/Im\,\partial$, where $Im\,\partial$ is just the normal closure of the set, R, of relations and we know that $Ker\,\partial$ is a G-module. We will usually denote the module of identities by $\pi_{\mathcal{P}}$.

We can get to $C(\mathcal{P})$ in another way. Construct a space from the combinatorial information in $C(\mathcal{P})$ as follows. Take a bunch of circles labelled by the elements of X; call it $K(\mathcal{P})_1$, it is the 1-skeleton of the space we want. We have $\pi_1(K(\mathcal{P})_1 \cong F(X)$. Each relator $r \in R$ is a word in X so gives us a loop in $K(\mathcal{P})_1$, following around the circles labelled by the various generators making up r. This loop gives a map $S^1 \xrightarrow{f_r} K(\mathcal{P})_1$. For each such r we use f_r to glue a 2-dimensional disc e_r^2 to $K(\mathcal{P})_1$ yielding the space $K(\mathcal{P})$. The crossed module $C(\mathcal{P})$ is isomorphic to $\pi_2(K(\mathcal{P}), K(\mathcal{P})_1) \xrightarrow{\partial} \pi_1(K(\mathcal{P})_1$.

The main problem is how to calculate $\pi_{\mathcal{P}}$ or equivalently $\pi_2(K(\mathcal{P}))$. One approach is via an associated chain complex. This can be viewed as the chains on the universal cover of $K(\mathcal{P})$, but can also be defined purely algebraically, for which see Brown-Huebschmann, [50], or Loday, [144]. That algebraic - homological approach leads to 'homological syzygies'. For the moment we will concentrate on:

1.3 Cohomology, crossed extensions and algebraic 2-types

1.3.1 Cohomology and extensions, continued

Suppose we have any group extension

$$\mathcal{E}: \quad 1 \to K \to E \xrightarrow{p} G \to 1,$$

with K Abelian, but not necessarily central. We can look at various possibilities.

If we can split p, by a homomorphism $s : G \to E$, with $ps = Id_G$, then, of course, $E \cong K \rtimes G$ by the isomorphisms,

$$e \longrightarrow (esp(e)^{-1}, p(e)),$$
$$ks(g) \longleftarrow (k, g),$$

which are compatible with the projections etc., so there is an equivalence of extensions

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d, "\cong"] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & K \arrow[r] & K \rtimes G \arrow[r] & G \arrow[r] & 1.
\end{tikzcd}$$

Our convention for multiplication in $K \rtimes G$ will be

$$(k, g)(k', g') = (k^g k', gg').$$

But what if p does not split. We can build a (small) category of extensions $\mathcal{E}xt(G, K)$ with objects such as $\mathcal{E}$ above and in which a morphism from $\mathcal{E}$ to $\mathcal{E}'$ is a diagram

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d, "\alpha"] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & K \arrow[r] & E' \arrow[r] & G \arrow[r] & 1.
\end{tikzcd}$$

By the 5-lemma, α will be an isomorphism, so $\mathcal{E}xt(G, K)$ is a groupoid.

In $\mathcal{E}$, the epimorphism p is usually not splittable, but as a function between sets, it is onto so we can pick an element in each $p^{-1}(g)$ to get a transversal (or set of coset representatives), $s : G \to E$. We get a comparison pairing / obstruction map or 'factor set' :

$$f : G \times G \to E$$

$$f(g_1, g_2) = s(g_1)s(g_2)s(g_1g_2)^{-1},$$

which will be trivial, (i.e., $f(g_1, g_2) = 1$ for all $g_1, g_2 \in G$) exactly if s splits p, i.e., if s is a homomorphism. This construction assumes that we know the multiplication in E, otherwise we cannot form this product! On the other hand given this 'f', we can work out the multiplication. As a set, E will be the product $K \times G$, identified with it by the same formulae as in the split case, noting that $pf(g_1, g_2) = 1$, so 'really' we should think of f as ending up in the subgroup K, then we have

$$(k_1, g_1)(k_2, g_2) = ({k_1}^{s(g_1)}k_2 f(g_1, g_2), g_1g_2).$$

The product is *twisted* by the pairing f. Of course, we need this multiplication to be associative and, to ensure that, f must satisfy a cocycle condition:

$${}^{s(g_1)}f(g_2, g_3)f(g_1, g_2g_3) = f(g_1, g_2)f(g_1g_2, g_3).$$

This is a well known formula from group cohomology, more so if written additively:

$$s(g_1)f(g_2, g_3) - f(g_1g_2, g_3) + f(g_1, g_2g_3) - f(g_1, g_2) = 0.$$

Here we actually have various parts of the nerve of G involved in the formula. The group G 'is' a small category (groupoid with one object), which we will, for the moment, denote $\mathcal{G}$. The triple $\sigma = (g_1, g_2, g_3)$ is a 3-simplex in $Ner(\mathcal{G})$ and its faces are

$$\begin{aligned} d_0\sigma &= (g_2, g_3), \\ d_1\sigma &= (g_1g_2, g_3), \\ d_2\sigma &= (g_1, g_2g_3), \\ d_3\sigma &= (g_1, g_2). \end{aligned}$$

This is all very classical. We can use it in the usual way to link $\pi_0(\mathcal{E}xt(G, K))$ with $H^2(G, K)$ and so is the 'modern' version of Schreier's theory of group extensions, at least in the case that K is Abelian.

For a long time there was no obvious way to look at the elements of $H^3(G, K)$ in a similar way. In MacLane's homology book, [149], you can find a discussion from the classical viewpoint. In Brown's [38], the link with crossed modules is sketched although no references for the details are given, for which see MacLane's [151].

If we have a crossed module $C \xrightarrow{\partial} P$, then we saw that $Ker\,\partial$ is central in C and is a $P/\partial C$-module. We thus have a 'crossed 2-fold extension':

$$K \xrightarrow{i} C \xrightarrow{\partial} P \xrightarrow{p} G,$$

where $K = Ker\,\partial$ and $G = P/\partial C$. (We will write $N = \partial C$.)

Repeat the same process as before for the extension

$$N \to P \to G,$$

but take extra care as N is usually not Abelian. Pick a transversal $s : G \to P$ giving $f : G \times G \to N$ as before (even with the same formula). Next look at

$$K \xrightarrow{i} C \to N,$$

and lift f to C via a choice of $F(g_1, g_2) \in C$ with image $f(g_1, g_2)$ in N.

The pairing f satisfied the cocycle condition, but we have no means of ensuring that F will do so, i.e. there will be, for each triple (g_1, g_2, g_3), an element $c(g_1, g_2, g_3) \in C$ such that

$$^{s(g_1)}F(g_2, g_3)F(g_1, g_2g_3) = i(c(g_1, g_2, g_3))F(g_1, g_2)F(g_1g_2, g_3),$$

and some of these $c(g_1, g_2, g_3)$ may be non-trivial. The $c(g_1, g_2, g_3)$ will satisfy a cocycle condition correspond to a 4-simplex in $Ner(\mathcal{G})$, and one can reconstruct the crossed 2-fold extension up to equivalence from F and c. Here 'equivalence' is generated by maps of 'crossed' exact sequences:

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & C \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & K \arrow[r] & C' \arrow[r] & P' \arrow[r] & G \arrow[r] & 1, \end{tikzcd}$$

but these morphisms need not be isomorphisms. Of course, this identifies $H^3(G, K)$ with π_0 of the resulting category.

What about $H^4(G, K)$? Yes, something similar works, but we do not have the machinery to do it here, yet.

1.3.2 Not really an aside!

Suppose we start with a crossed module $\mathsf{C} = (C, P, \partial)$. We can build an internal category, $\mathcal{X}(\mathsf{C})$, in *Grps* from it. The group of objects of $\mathcal{X}(\mathsf{C})$ will be P and the group of arrows $C \rtimes P$. The source map

$$s : C \rtimes P \to P \quad \text{is} \quad s(c, p) = p,$$

the target

$$t : C \rtimes P \to P \quad \text{is} \quad t(c, p) = \partial c.p.$$

(That looks a bit strange. That sort of construction usually does not work, multiplying two homomorphisms together is a recipe for trouble! - but it does work here:

$$\begin{aligned} t((c_1, p_1).(c_2, p_2)) &= t(c_1{}^{p_1}c_2, p_1p_2) \\ &= \partial(c_1{}^{p_1}c_2).p_1p_2, \end{aligned}$$

whilst $t(c_1, p_1).t(c_2, p_2) = \partial c_1.p_1.\partial c_2.p_2$, but remember $\partial(c_1{}^{p_1}c_2) = \partial c_1.p_1.\partial c_2.p_1^{-1}$, so they are equal.)

The identity morphism is $i(p) = (1, p)$, but what about the composition. Here it helps to draw a diagram. Suppose $(c_1, p_1) \in C \rtimes P$, then it is an arrow

$$p_1 \xrightarrow{(c_1,p_1)} \partial c_1.p_1,$$

and we can only compose it with (c_2, p_2) if $p_2 = \partial c_1.p_1$. This gives

$$p_1 \xrightarrow{(c_1,p_1)} \partial c_1.p_1 \xrightarrow{(c_2,\partial c_1.p_1)} \partial c_2\partial c_1.p_1.$$

The obvious candidate for the composite arrow is (c_2c_1, p_1) and it works!

In fact, $\mathcal{X}(\mathsf{C})$ is an internal groupoid as $(c_1^{-1}, \partial c_1.p_1)$ is an inverse for (c_1, p_1).

Now if we started with an internal category

$$G_1 \overset{s}{\underset{t}{\rightrightarrows}} G_0 \;,\quad G_0 \xrightarrow{i} G_1$$

etc., then set $P = G_0$ and $C = Ker\, s$ with $\partial = t\mid_C$ to get a crossed module.

Theorem 1 *(Brown-Spencer,[55]) The category of crossed modules is equivalent to that of internal categories in Grps.* ■

You have, almost, seen the proof. As beginning students of algebra, you learnt that equivalence relations on groups need to be congruence relations for quotients to work well and that congruence relations 'are the same as' normal subgroups. That is the essence of the proof needed here, but we have groupoids rather than equivalence relations and crossed modules rather than normal subgroups.

Of course, any morphism of crossed modules has to induce an internal functor between the corresponding internal categories and *vice versa*. That is a **good exercise** for you to check that you have understood the link that the Brown-Spencer theorem gives.

This is a good place to mention 2-groups. The notion of 2-category is one that should be fairly clear even if you have not met it before. For instance, the category of small categories, functors and natural transformations is a 2-category. Between each pair of objects, we have not just a set of functors as morphisms but a small category of them with the natural transformations between them as the arrows in this second level of structure. The notion of 2-category is abstracted from this. We will not give a formal definition here (but suggest that you look one up if you have not met the idea before). A 2-category thus has objects, arrows or morphisms (or sometimes '1-cells') between them and then some 2-cells (sometimes called '2-arrows' or '2-morphisms') between them.

Definition: A *2-groupoid* is a 2-category in which all 1-cells and 2-cells are invertible.
If the 2-groupoid has just one object then we call it a *2-group*.

Of course, there are also 2-functors between 2-categories and so, in particular, between 2-groups. Again this is for **you to formulate, looking up relevant definitions, etc.**

Internal categories in *Grps* are really exactly the same as 2-groups. The Brown-Spencer theorem thus constructs the *associated 2-group of a crossed module*. The fact that the composition in the internal category must be a group homomorphism implies that the '*interchange law*' must hold. This equation is in fact equivalent via the Brown-Spencer result to the Peiffer identity. (It is **left to you** to find out about the interchange law and to check that it is the Peiffer axiom in disguise. We will see it many times later on.)

Here would be a good place to mention that an internal monoid in *Grps* is just an Abelian group. The argument is well known and is usually known by the name of the *Eckmann-Hilton argument*. This starts by looking at the interchange law, which states that the monoid multiplication must be group homomorphism. From this it derives that the monoid identity must also be the group identity and that the two compositions must coincide. It is then easy to show that the group is Abelian.

1.3.3 Perhaps a bit more of an aside ... for the moment!

This is quite a good place to mention the groupoid based theory of all this. The resulting objects look like abstract 2-categories and are 2-groupoids. We have a set of objects, K_0, a set of arrows, K_1, depicted $x \xrightarrow{p} y$, and a set of two cells

$$\begin{tikzcd} x \arrow[r, bend left=50, "p"{name=U}] \arrow[r, bend right=50, "\partial c.p"'{name=D}] & y \arrow[Rightarrow, from=U, to=D, phantom, "\Downarrow(c,p)"] \end{tikzcd} .$$

In our previous diagrams, as all the elements of P started and ended at the same single object, we could shift dimension down one step; our old objects are now arrows and our old arrows are 2-cells. We will return to this later.

The important idea to note here is that a 'higher dimensional category' has a link with an algebraic object. The 2-group(oid) provides a useful way of interpreting the structure of the crossed module *and* indicates possible ways towards similar applications and interpretations elsewhere. For instance, a presentation of a monoid leads more naturally to a 2-category than to any analogue of a crossed module, since kernels are less easy to handle than congruences in Mon.

There are other important interpretations of this. Categories such as that of vector spaces, Abelian groups or modules over a ring, have an additional structure coming from the tensor product, $A \otimes B$. They are *monoidal categories*. One can 'multiply' objects together and this is linked to a related multiplication on morphisms between the objects. In many of the important examples the multiplication is not strictly associative, so for instant, if A, B, C are objects there is an isomorphism between $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$, but this isomorphism is most definitely not the identity as the two objects are constructed in different ways. A similar effect happens in the category of sets with ordinary Cartesian product. The isomorphism is there because of universal properties, but it is again not the identity. It satisfies some coherence conditions, (a cocycle condition in disguise), relating to associativity of four fold tensors and the associahedron that we gave earlier, is a corresponding diagram for the five fold tensors. (Yes, there is a strong link, but that is not for these notes!) Our 2-group(oid) is the 'suspension' or 'categorification' of a similar structure. We can multiply objects and 'arrows' and the result is a strict 'gr-groupoid', or 'categorical group', i.e. a strict monoidal category with inverses. This is vague here, but will gradually be explored later on. If you want to explore the ideas further now, look at Baez and Dolan, [13].

(At this point, you do not need to know the definition of a monoidal category, but **remember to look it up in the not too distance future**, if you have not met it before, as later on the insights that an understanding of that notion gives you, will be very useful. It can be found in many places in the literature, and on the internet. The approach that you will get on best with depends on your background and your likes and dislikes mathematically, so we will not give one here.)

Just as associativity in a monoid is replaced by a 'lax' associativity 'up to coherent isomorphisms' in the above, gr-groupoids are 'lax' forms of internal categories in groups and thus indicate the presence of a crossed module-like structure, albeit in a weakened or 'laxified' form. Later we will see naturally occurring gr-groupoid structures associated with some constructions in non-Abelian cohomology. There is also a sense in which the link between fibrations and crossed modules given earlier here, indicates that fibrations are like a related form of lax crossed modules. In the notion of fibred category and the related Grothendieck construction, this intuition begins to be 'solidified' into a clearer strong relationship.

1.3.4 Automorphisms of a group yield a 2-group

We could also give this section a subtitle:

The automorphisms of a 1-type give a 2-type.

This is really an extended exercise in playing around with the ideas from the previous two sections. It uses a small amount of categorical language, but, hopefully, in a way that should be easy for even a categorical debutant to follow. The treatment will be quite detailed as it is that detail that provides the links between the abstract and the concrete.

We start with a look at 'functor categories', but with groupoids rather than general small categories as input. Suppose that $\mathcal{G}$ and $\mathcal{H}$ are groupoids, then we can form a new groupoid, $\mathcal{H}^{\mathcal{G}}$, whose objects are the functors, $f : \mathcal{G} \to \mathcal{H}$. Of course, functors in this context are just morphisms of groupoids, and, if $\mathcal{G}$, and $\mathcal{H}$ are $G[1]$ and $H[1]$, that is, two groups, G and H, thought of as one object groupoids, then the objects of $\mathcal{H}^{\mathcal{G}}$ are just the homomorphisms from G to H thought of in a slightly different way.

That gives the objects of $\mathcal{H}^{\mathcal{G}}$. For the morphisms from f_0 to f_1, we 'obviously' should think of natural transformations. (As usual, if you are not sufficiently conversant with elementary categorical ideas, pause and look them up in a suitable text of in Wikipedia.) Suppose $\eta : f_0 \to f_1$ is a natural transformation, then, for each x, an object of $\mathcal{G}$, we have an arrow,

$$\eta(x) : f_0(x) \to f_1(x),$$

in $\mathcal{H}$ such that, if $g : x \to y$ in $\mathcal{G}$, then the square

$$\begin{tikzcd} f_0(x) \arrow[r, "\eta(x)"] \arrow[d, "f_0(g)"'] & f_1(x) \arrow[d, "f_1(g)"] \\ f_0(y) \arrow[r, "\eta(y)"'] & f_1(y) \end{tikzcd}$$

commutes, so η 'is' the family, $\{\eta(x) \mid x \in Ob(\mathcal{G}\}$. Now assume $\mathcal{G} = G[1]$ and $\mathcal{H} = H[1]$, and that we try to interpret $\eta(x) : f_0(x) \to f_1(x)$ back down at the level of the groups, that is, a bit more 'classically' and group theoretically. There is only one object, which we denote $*$, if we need it, so we have that η corresponds to a single element, $\eta(*)$, in H, which we will write as h for simplicity, but now the condition for commutation of the square just says that, for any element $g \in G$,

$$hf_0(g) = f_1(g)h,$$

i.e., that f_0 and f_1 are *conjugate* homomorphisms, $f_1 = hf_0h^{-1}$..

It should be clear, (but **check that it is**), that this definition of morphism makes $\mathcal{H}^{\mathcal{G}}$ into a category, in fact into a groupoid, as the morphisms compose correctly and have inverses. (To get the inverse of η take the family $\{\eta(x)^{-1} \mid x \in Ob(\mathcal{G}\}$ and check the relevant squares commute.)

So far we have 'proved':

Lemma 3 *For groupoids, $\mathcal{G}$ and $\mathcal{H}$, the functor category, $\mathcal{H}^{\mathcal{G}}$, is a groupoid.* ■

We will be a bit sloppy in notation and will write H^G for what should, more precisely, be written $H[1]^{G[1]}$.

We note that it is usual to observe that, for Abelian groups, A, and B, the set of homomorphisms from A to B is itself an Abelian group, but that the set of homomorphisms from one non-Abelian group to another has no such nice structure. Although this is sort of true, the point of the above is that that set forms the set of objects for a very neat algebraic object, namely a groupoid!

If we have a third groupoid, $\mathcal{K}$, then we can also form $\mathcal{K}^{\mathcal{H}}$ and $\mathcal{K}^{\mathcal{G}}$, etc. and, as the objects of $\mathcal{K}^{\mathcal{H}}$ are homomorphisms from $\mathcal{H}$ to $\mathcal{K}$, we might expect to compose with the objects of $\mathcal{H}^{\mathcal{G}}$ to get ones of $\mathcal{K}^{\mathcal{G}}$. We might thus hope for a composition *functor*

$$\mathcal{K}^{\mathcal{H}} \times \mathcal{H}^{\mathcal{G}} \to \mathcal{K}^{\mathcal{G}}.$$

(There are **various things to check**, but we need not worry. We are really working with functors and natural transformations and with the investigation that shows that the category of small categories is 2-category. This means that if you get bogged down in the detail, you can easily find the ideas discussed in many texts on category theory.) This works, so we have that the category, *Grpds* has also a 2-category structure. (It is a '*Grpds*-enriched' category; see later for enriched categories. The formal definition is in section **??**, although the basic idea is used before that.)

We need to recall next that in any category, $\mathcal{C}$, the endomorphisms of any object, X, form a monoid, $End(X) := \mathcal{C}(X, X)$. You just use the composition and identities of $\mathcal{C}$ 'restricted to X'. If we play that game with any groupoid enriched category, C, then for any object, X, we will have a groupoid, $\mathsf{C}(X, X)$, which we might write $\mathsf{End}(X)$, (that is, using the same font to indicate 'enriched') and which also has a monoid structure,

$$\mathsf{C}(X, X) \times \mathsf{C}(X, X) \to \mathsf{C}(X, X).$$

It will be a monoid internal to *Grpds*. In particular, for any groupoid, $\mathcal{G}$, we have such an internal monoid of endomorphisms, $\mathcal{G}^{\mathcal{G}}$, and specialising down even further, for any group, G, such an internal monoid, G^G. Note that this is internal to the category of *groupoids* not of groups, as its monoid of objects is the endomorphism monoid of G, not a single element set. Within G^G, we can restrict attention to the subgroupoid on the automorphisms of G. We thus have this groupoid, $\mathsf{Aut}(G)$, which has as objects the automorphisms of G and, as typical morphism, $\eta : f_0 \to f_1$, a conjugation. It is important to note that as η is specified by an element of G and an automorphism, f_0, of G, the pair, (g, f_0), may then be a good way of thinking of it. (Two points, that may be obvious, but are important even if they are, are that the morphism η is not conjugation itself, but conjugates f_0. One has to specify where this morphism starts, its domain, as well as what it does, namely conjugate by g. Secondly, in (g, f_0), we *do* have the information on the codomain of η, as well. It is $gf_0g^{-1} = f_1$.)

Using this basic notation for the morphisms, we will look at the various bits of structure this thing has. (Remember, $\eta : f_0 \to f_1$ and $f_1 = gf_0g^{-1}$, as we will need to use that several times.) We have compositions of these pairs in two ways:

(a) as natural transformations: if

$$\begin{aligned} &\eta : f_0 \to f_1, \quad \eta = (g, f_0), \\ \text{and} \qquad &\eta' : f_1 \to f_2, \quad \eta' = (g', f_1), \end{aligned}$$

then the composite is $\eta'\sharp_1\eta = (g'g, f_0)$. (That is easy to check. As, for instance, $f_2 = g'f_1(g')^{-1} = (g'g)f_0(g'g)^{-1}$, ..., it all works beautifully). (A word of **warning** here, $(g'g)f_0(g'g)^{-1}$ is the conjugate of the automorphism f_0 by the element $(g'g)$. The bracket does not refer to f_0 applied to the 'thing in the bracket', so, for $x \in G$, $((g'g)f_0(g'g)^{-1})(x)$ is, in fact, $(g'g)f_0(x)(g'g)^{-1}$. This *is* slightly confusing so think about it, so as not to waste time later in avoidable confusion.)

b) using composition, $\sharp_0$, in the monoid structure. To understand this, it is easier to look at that composition as being specialised from the one we singled out earlier,

$$\mathcal{K}^{\mathcal{H}} \times \mathcal{H}^{\mathcal{G}} \to \mathcal{K}^{\mathcal{G}},$$

which is the composition in the 2-category of groupoids. (We really want $\mathcal{G} = \mathcal{H} = \mathcal{K}$, but, by keeping the more general notation, it becomes easier to see the roles of each $\mathcal{G}$.)

We suppose $f_0, f_1 : \mathcal{G} \to \mathcal{H}$, $f_0', f_1' : \mathcal{G} \to \mathcal{H}$, and then $\eta : f_0 \to f_1$, $\eta' : f_0' \to f_1'$. The 2-categorical picture is

$$\begin{tikzcd} \cdot \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] \arrow[r, phantom, "\Downarrow\eta"] & \cdot \arrow[r, bend left, "f_0'"] \arrow[r, bend right, "f_1'"'] \arrow[r, phantom, "\Downarrow\eta'"] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, bend left, "f_0'f_0"] \arrow[r, bend right, "f_1'f_1"'] \arrow[r, phantom, "\Downarrow\eta''"] & \cdot \end{tikzcd},$$

with η'' being the desired composite, $\eta'\sharp_0\eta$, but how is it calculated. The important point is the *interchange law*. We can 'whisker' on the left or right, or, since the 'left-right' terminology can get confusing (does 'left' mean 'diagrammatically' or 'algebraically' on the left?), we will often use 'pre-' and 'post-' as alternative prefixes. The terminology may seem slightly strange, but is quite graphic when suitable diagrams are looked at! Whiskering corresponds to an interaction between 1-cell and 2-cells in a 2-category. In 'post-whiskering', the result is the composite of a 2-cell *followed* by a 1-cell:

Post-whiskering:

$$f_0'\sharp_0\eta : f_0'\sharp_0 f_0 \to f_0'\sharp_0 f_1,$$

$$\begin{tikzcd} \cdot \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] \arrow[r, phantom, "\Downarrow\eta"] & \cdot \arrow[r, "f_0'"] & \cdot \end{tikzcd}$$

(It is convenient, here, to write the more formal $f_0'\sharp_0 f_0$, for what we would usually write as $f_0'f_0$.) The natural transformation, η is given by a family of arrows in $\mathcal{H}$, so $f_0'\sharp_0\eta$ is given by mapping that family across to $\mathcal{K}$ using f_0'. (Specialising to $\mathcal{G} = \mathcal{H} = \mathcal{K} = G[1]$, if $\eta = (g, f_0)$, then $f_0'\sharp_0\eta = (f_0'(g), f_0'f_0)$, as is easily checked; similarly for $f_1'\sharp_0\eta$.)

Pre-whiskering:

$$\eta'\sharp_0 f_0 : f_0'\sharp_0 f_0 \to f_1'\sharp_0 f_0,$$

$$\begin{tikzcd} \cdot \arrow[r, "f_0"] & \cdot \arrow[r, bend left, "f_0'"] \arrow[r, bend right, "f_1'"'] \arrow[r, phantom, "\Downarrow\eta'"] & \cdot \end{tikzcd}.$$

Here the morphism f_0 does not influence the g-part of η' at all. It just alters the domains. In the case that interests us, if $\eta' = (g', f_0')$, then $\eta'\sharp_0 f_0 = (g', f_0' f_0)$.

The way of working out $\eta'\sharp_0\eta$ is by using $\sharp_1$-composites. First,

$$\eta'\sharp_0\eta : f_0' f_0 \to f_1' f_1,$$

and we can go

$$\eta'\sharp_0 f_0 : f_0' f_0 \to f_1' f_0,$$

and then, to get to where we want to be, that is, $f_1' f_1$, we use

$$f_1'\sharp_0\eta : f_1' f_0 \to f_1' f_1.$$

This uses the $\sharp_1$-composition, so

$$\begin{aligned}
\eta'\sharp_0\eta &= (f_1'\sharp_0\eta)\sharp_1(\eta'\sharp_0 f_0)\\
&= (f_1'(g), f_1' f_0)\sharp_1(g', f_0' f_0)\\
&= (f_1'(g).g', f_0' f_0),
\end{aligned}$$

but $f_1'(g) = g' f_0(g)(g')^{-1}$, so the end results simplifies to $(g' f_0(g), f_0' f_0)$. **Hold on!** That looks nice, but we could have also calculated $\eta'\sharp_0\eta$ using the other form as the composite,

$$\begin{aligned}
\eta'\sharp_0\eta &= (\eta'\sharp_0 f_1)\sharp_1(f_0'\sharp_0\eta)\\
&= (g', f_0' f_1)\sharp_1(f_0'(g), f_0' f_0)\\
&= (g' f_0'(g), f_0' f_0),
\end{aligned}$$

so we did not have any problem. (All the properties of an internal groupoid in *Grps*, or, if you prefer that terminology, 2-group, can be derived from these two compositions. The $\sharp_1$ composition is the 'groupoid' direction, whilst the $\sharp_0$ is the 'group' one.)

We thus have a group of natural transformations made up of pairs, (g, f_0) and whose multiplication is given as above. This is just the semi-direct product group, $G \rtimes Aut(G)$, for the natural and obvious action of $Aut(G)$ on G. This group is sometimes called the *holomorph* of G.

We have two homomorphisms from $G \rtimes Aut(G)$ to $Aut(G)$. One sends (g, f_0) to f_0, so is just the projection, the other sends it to $f_1 = g f_0 g^{-1} = \iota_g \circ f_0$. We can recognise this structure as being the associated 2-group of the crossed module, $(G, Aut(G), \iota)$, as we met on page 14. We call $\mathsf{Aut}(G)$, the *automorphism 2-group* of G..

1.3.5 Back to 2-types

From our crossed module, $\mathsf{C} = (C, P, \partial)$, we can build the internal groupoid, $\mathcal{X}(\mathsf{C})$, as before, then apply the nerve construction internally to the internal groupoid structure to get a simplicial group, $K(\mathsf{C})$.

Definition: Given a crossed module, $\mathsf{C} = (C, P, \partial)$, the nerve (taken internally in *Grps*) of the internal groupoid, $\mathcal{X}(\mathsf{C})$, defined by C, will be called the nerve of C or, if more precision is needed, its *simplicial group nerve* and will be denoted $K(\mathsf{C})$.

The simplicial set, $\overline{W}(K(\mathsf{C}))$, or its geometric realisation, would be called the *classifying space* of C.

We need this in some detail in low dimensions.

$$\begin{aligned} K(\mathsf{C})_0 &= P \\ K(\mathsf{C})_1 &= C \rtimes P \qquad\qquad d_0 = t, d_1 = s \\ K(\mathsf{C})_2 &= C \rtimes (C \rtimes P), \end{aligned}$$

where $d_0(c_2, c_1, p) = (c_2, \partial c_1.p)$, $d_1(c_2, c_1, p) = (c_2.c_1, p)$ and $d_2(c_2, c_1, p) = (c_1, p)$. The pattern continues with $K(\mathsf{C})_n = C \rtimes (\ldots \rtimes (C \rtimes P)\ldots)$, having n-copies of C. The d_i, for $0 < i < n$, are given by multiplication in C, d_0 is induced from t and d_n is a projection. The s_i are insertions of identities. (We will examine this in more detail later.)

Remark: A word of caution: for G a group considered as a crossed module, this 'nerve' is not the nerve of G in the sense used earlier. It is just the constant simplicial group corresponding to G. What is often called the nerve of G is what here has been called its classifying space. One way to view this is to note that $\mathcal{X}(\mathsf{C})$ has two independent structures, one a group, the other a category, and *this* nerve is of the category structure. The group, G, considered as a crossed module is like a set considered as a (discrete) category, having only identity arrows.)

The Moore complex of $K(\mathsf{C})$ is easy to calculate and is just $NK(\mathsf{C})_i = 1$ if $i \geq 2$; $NK(\mathsf{C})_1 \cong C$; $NK(\mathsf{C})_0 \cong P$ with the $\partial : NK(\mathsf{C})_1 \to NK(\mathsf{C})_0$ being exactly the given ∂ of C. (This is left as an exercise. It is a useful one to do in detail.)

Proposition 1 *(Loday, [143]) The category CMod of crossed modules is equivalent to the subcategory of Simp.Grps, consisting of those simplicial groups, G, having Moore complexes of length 1, i.e. $NG_i = 1$ if $i \geq 2$.* ∎

This raises the interesting question as to whether it is possible to find alternative algebraic descriptions of the structures corresponding to Moore complexes of length n.

Is there any way of going directly from simplicial groups to crossed modules? Yes. The last two terms of the Moore complex will give us:

$$\partial : NG_1 \to NG_0 = G_0$$

and G_0 acts on NG_1 by conjugation via s_0, i.e. if $g \in G_0$ and $x \in NG_1$, then $s_0(g)xs_0(g)^{-1}$ is also in NG_1. (Of course, we could use multiple degeneracies to make g act on an $x \in NG_n$ just as easily.) As $\partial = d_0$, it respects the G_0 action, so CM1 is satisfied. In general, CM2 will not be satisfied. Suppose $g_1, g_2 \in NG_1$ and examine ${}^{\partial g_1}g_2 = s_0d_0g_1.g_2.s_0d_0g_1^{-1}$. This is rarely equal to $g_1g_2g_1^{-1}$. We write $\langle g_1, g_2\rangle = [g_1, g_2][g_2, s_0d_0g_1] = g_1g_2g_1^{-1}.({}^{\partial g_1}g_2)^{-1}$, so it measures the obstruction to CM2 for this pair g_1, g_2. This is often called the *Peiffer commutator* of g_1 and g_2. Noting that $s_0d_0 = d_0s_1$, we have an element

$$\{g_1, g_2\} = [s_0g_1, s_0g_2][s_0g_2, s_1g_1] \in NG_2$$

and $\partial\{g_1, g_2\} = \langle g_1, g_2\rangle$. This second pairing is called the *Peiffer lifting* (of the Peiffer commutator). Of course, if $NG_2 = 1$, then CM2 is satisfied (as for $K(\mathsf{C})$, above).

We could work with what we will call $M(G,1)$, namely

$$\overline{\partial} : \frac{NG_1}{\partial NG_2} \to NG_0,$$

with the induced morphism and action. (As $d_0d_0 = d_0d_1$, the morphism is well defined.) This is a crossed module, but we could have divided out by less if we had wanted to. We note that $\{g_1, g_2\}$ is a product of degenerate elements, so we form, in general, the subgroup $D_n \subseteq NG_n$, generated by all degenerate elements.

Lemma 4

$$\overline{\partial} : \frac{NG_1}{\partial(NG_2 \cap D_2)} \to NG_0$$

is a crossed module. ∎

This is, in fact, $M(sk_1G, 1)$, where sk_1G is the 1-skeleton of G, i.e., the subsimplicial group generated by the k-simplices for $k = 0, 1$.

The kernel of $M(G,1)$ is $\pi_1(G)$ and the cokernel $\pi_0(G)$ and

$$\pi_1(G) \to \frac{NG_1}{\partial NG_2} \to NG_0 \to \pi_0(G)$$

represents a class $k(G) \in H^3(\pi_0(G), \pi_1(G))$. Up to a notion of 2-equivalence, $M(G,1)$ represents the 2-type of G completely. This is an algebraic version of the result of MacLane and Whitehead we mentioned earlier. Once we have a bit more on cohomology, we will examine it in detail.

This use of $NG_2 \cap D_2$ and our noting that $\{g_1, g_2\}$ is a product of degenerate elements may remind you of group T-complexes and thin elements. Suppose that G is a group T-complex in the sense of our discussion at the end of the previous chapter (page **??**). In a general simplicial group, the subgroups, $NG_n \cap D_n$, will not be trivial. They give measure of the extent to which homotopical information in dimension n on G depends on 'stuff' from lower dimensions., i.e., comparing G with its $(n-1)$-skeleton. (Remember that in homotopy theory, invariants such as the homotopy groups do not necessarily vanish above the dimension of the space, just recall the sphere S^2 and the subtle structure of its higher homotopy groups.)

The construction here of $M(sk_1G, 1)$ involves 'killing' the images of our possible multiple 'D-fillers' for horns, forcing uniqueness. We will see this again later.

Chapter 2

Crossed complexes

Accurate encoding of homotopy types is tricky. Chain complexes, even of G-modules, can only record certain, more or less Abelian, information. Simplicial groups, at the opposite extreme, can encode all connected homotopy types, but at the expense of such a large repetition of the essential information that makes calculation, at best, tedious and, at worst, virtually impossible. Complete information on truncated homotopy types can be stored in the cat^n-groups of Loday, [143]. We will look at these later. An intermediate model due to Blakers and Whitehead, [221], is that of a crossed complex. The algebraic and homotopy theoretic aspects of the theory of crossed complexes have been developed by Brown and Higgins, (cf. [46, 47], etc., in the bibliography and the forthcoming monograph by Brown, Higgins and Sivera, [49]) and by Baues, [22–24]. We will use them later on in several contexts.

2.1 Crossed complexes: the Definition

We will initially look at reduced crossed complexes, i.e., the group rather than the groupoid based case.

Definition: A *crossed complex*, which will be denoted C, consists of a sequence of groups and morphisms

$$\mathsf{C} : \ldots \to C_n \stackrel{\delta_n}{\to} C_{n-1} \stackrel{\delta_{n-1}}{\to} \ldots \to C_3 \stackrel{\delta_3}{\to} C_2 \stackrel{\delta_2}{\to} C_1$$

satisfying the following:
CC1) $\delta_2 : C_2 \to C_1$ is a crossed module;
CC2) each C_n, $(n > 2)$, is a left $C_1/\delta_1 C_2$-module and each δ_n, $(n > 2)$ is a morphism of left $C_1/\delta_2 C_2$-modules, (for $n = 3$, this means that δ_3 commutes with the action of C_1 and that $\delta_3(C_3) \subset C_2$ must be a $C_1/\delta_2 C_2$-module);
CC3) $\delta\delta = 0$.

The notion of a morphism of crossed complexes is clear. It is a graded collection of morphisms preserving the various structures. We thus get a category, Crs_{red} of reduced crossed complexes.

As we have that a crossed complex is a particular type of chain complex (of non-Abelian groups near the bottom), it is natural to define its homology groups in the obvious way.

Definition: If C is a crossed complex, its n^{th} *homology group* is

$$H_n(\mathsf{C}) = \frac{Ker\,\delta_n}{Im\,\delta_{n+1}}.$$

These homology groups are, of course, functors from Crs_{red} to the category of Abelian groups.

Definition: A morphism $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$ is called a *weak equivalence* if it induces isomorphisms on all homology groups.

There are good reasons for considering the homology groups of a crossed complex as being its homotopy groups. For example, if the crossed complex comes from a simplicial group then the homotopy groups of the simplicial group are the same as the homology groups of the given crossed complex (possibly shifted in dimension, depending on the notational conventions you are using).

The non-reduced version of the concept is only a bit more difficult to write down. It has C_1 as a groupoid on a set of objects C_0 with each C_k, a family of groups indexed by the elements of C_0. The axioms are very similar; see [49] for instance or many of the papers by Brown and Higgins listed in the bibliography. This gives a category, Crs, of (unrestricted) crossed complexes and morphisms between them. This category is very rich in structure. It has a tensor product structure, denoted $\mathsf{C} \otimes \mathsf{D}$ and a corresponding mapping complex construction, $\mathsf{Crs}(\mathsf{C}, \mathsf{D})$, making it into a monoidal closed category. The details are to be found in the papers and book listed above and will be recalled later when needed.

It is worth noting that this notion restricts to give us a notion of *weak equivalence* applicable to crossed modules as well.

Definition: A morphism, $\mathsf{f} : \mathsf{C} \to \mathsf{C}'$, between two crossed modules, is called a *weak equivalence* if it induces isomorphisms on π_0 and π_1, that is, on both the kernel and cokernel of the crossed modules.

The relevant reference for π_0 and π_1 is page 18.

2.1.1 Examples: crossed resolutions

As we mentioned earlier, a resolution of a group (or other object) is a model for the homotopy type represented by the group, but which usually is required to have some nice freeness properties. With crossed complexes we have some notion of homotopy around, just as with chain complexes, so we can apply that vague notion of resolution in this context as well. This will give us some neat examples of crossed complexes that are 'tuned' for use in cohomology.

A *crossed resolution* of a group G is a crossed complex, C, such that for each $n > 1$, $Im\,\delta_n = Ker\,\delta_{n-1}$ and there is an isomorphism, $C_1/\delta_2 C_2 \cong G$.

A crossed resolution can be constructed from a presentation $\mathcal{P} = (X : R)$ as follows:

Let $C(P) \to F(X)$ be the free crossed module associated with $\mathcal{P}$. We set $C_2 = C(\mathcal{P})$, $C_1 = F(X)$, $\delta_1 = \partial$. Let $\kappa(\mathcal{P}) = Ker(\partial : C(\mathcal{P}) \to F(X))$. This is the module of identities of the presentation and is a left G-module. As the category G-Mod has enough projectives, we can form a free resolution $\mathbb{P}$ of $\kappa(\mathcal{P})$. To obtain a crossed resolution of G, we join $\mathbb{P}$ to the crossed module by setting $C_n = P_{n-2}$ for $n > 3$, $\delta_n = d_{n-2}$ for $n > 3$ and the composite from P_0 to $C(P)$ for $n = 3$.

2.1.2 The standard crossed resolution

We next look at a particular case of the above, namely the standard crossed resolution of G. In this, which we will denote by $\mathsf{C}G$, we have

(i) $C_1G =$ the free group on the underlying set of G. The element corresponding to $u \in G$ will be denoted by $[u]$.

(ii) C_2G is the free crossed module over C_0G on generators, written $[u, v]$, considered as elements of the set $G \times G$, in which the map δ_1 is defined on generators by

$$\delta[u, v] = [uv]^{-1}[u][v].$$

(iii) For $n > 3$, C_nG is the free left G-module on the set G^n, but in which one has equated to zero any generator $[u_1, \ldots, u_n]$ in which some u_i is the identity element of G.

If $n > 2$, $\delta : C_{n+1}G \to C_nG$ is given by the usual formula

$$\begin{aligned}\delta[u_1, \ldots, u_{n+1}] &= {}^{[u_1]}[u_2, \ldots, u_{n+1}] \\ &\quad + \sum_{i=1}^{n}(-1)^i[u_1, \ldots, u_iu_{i+1}, \ldots, u_{n+1}] + (-1)^{n+1}[u_1, \ldots, u_n].\end{aligned}$$

For $n = 2$, $\delta : C_3G \to C_2G$ is given by

$$\delta[u, v, w] = {}^{[u]}[v, w].[u, v]^{-1}.[uv, w]^{-1}[u, vw].$$

This is the crossed analogue of the inhomogeneous bar resolution, $\mathsf{B}G$, of the group G. A groupoid version can be found in Brown-Higgins, [45], and the abstract group version in Huebschmann, [122]. In the first of these two references, it is pointed out that $\mathsf{C}G$, as constructed, is isomorphic to the crossed complex, $\underline{\pi}(BG)$, of the classifying space of G considered with its skeletal filtration.

For any filtered space, $\underline{X} = (X_n)_{n\in\mathbb{N}}$, its *fundamental crossed complex*, $\underline{\pi}(\underline{X})$, is, in general, a non-reduced crossed complex. It is defined to have

$$\underline{\pi}(\underline{X})_n = (\pi_n(X_n, X_{n-1}, a))_{a\in X_0}$$

with $\underline{\pi}(\underline{X})_1$, the fundamental groupoid $\Pi_1X_1X_0$, and $\underline{\pi}(\underline{X})_2$, the family, $(\pi_2(X_2, X_1, a))_{a\in X_0}$. It will only be reduced if X_0 consists just of one point.

Most of the time we will only discuss the reduced case in detail, although the non-reduced case will be needed sometimes. Following that, we will often use the notation Crs for the category of *reduced* crossed complexes unless we need the more general case. This may occasionally cause a little confusion, but it is much more convenient for most of the time.

There are two useful, but conflicting, conventions as to indexation in crossed complexes. In the topologically inspired one, the bottom group is C_1, in the simplicial and algebraic one, it is C_0. Both get used and both have good motivation. The natural indexation for the standard crossed resolution would seem to be with C_n being generated by n-tuples, i.e. the topological one. (I am not sure that all instances of the other have been avoided, so please be careful!)

G-augmented crossed complexes. Crossed resolutions of G are examples of G-augmented crossed complexes. A *G-augmented crossed complex* consists of a pair (C, φ) where C is a crossed complex and where $\varphi : C_1 \to G$ is a group homomorphism satisfying
(i) $\varphi\delta_1$ is the trivial homomorphism;
(ii) $Ker\,\varphi$ acts trivially on C_i for $i \geq 3$ and also on C_2^{Ab}.

A *morphism*

$$(\alpha, Id_G) : (\mathsf{C}, \varphi) \to (\mathsf{C}', \varphi')$$

of G-augmented crossed complexes consists of a morphism

$$\alpha : \mathsf{C} \to \mathsf{C}'$$

of crossed complexes such that $\varphi'\alpha_0 = \varphi$.

This gives a category, Crs_G, which behaves nicely with respect to change of groups, i.e. if $\varphi : G \to H$, then there are induced functors between the corresponding categories.

2.2 Crossed complexes and chain complexes: I

(Some of the proofs here are given in more detail as they are less routine and are not that available elsewhere. A source for much of this material is in the work of Brown and Higgins, [47], where these ideas were explored thoroughly for the first time; see also the treatment in [49].)

We have introduced crossed complexes where normally chain complexes of modules would have been used. We have seen earlier the bar resolution and now we have the standard crossed resolution. What is the connection between them? The answer is approximately that chain complexes form a category equivalent to a reflective subcategory of Crs. In other words, there is a canonical way of building a chain complex from a crossed one akin to the process of Abelianising a group. The resulting reflection functor sends the standard crossed resolution of a group to the bar resolution. The details involve some interesting ideas.

In chapter 2, we saw that, given a morphism $\theta : M \to N$ of modules over a group G, $\partial : M \to N \rtimes G$, given by $\partial(m) = (\theta(m), 1_G)$ is a crossed module, where $N \rtimes G$ acts on M via the projection to G. That example easily extends to a functorial construction which, from a positive chain complex, D, of G-modules, gives us a crossed complex $\Delta_G(\mathsf{D})$ with $\Delta_G(\mathsf{D})_n = D_n$ if $n > 1$ and equal to $D_1 \rtimes G$ for $n = 1$.

Lemma 5 *$\Delta_G : Ch(G{-}Mod) \to Crs_G$ is an embedding.*

Proof: That Δ_G is a functor is easy to see. It is also easy to check that it is full and faithful, that is it induces bijections,

$$Ch(G{-}Mod)(\mathsf{A}, \mathsf{B}) \to Crs_G(\Delta_G(\mathsf{A}), \Delta_G(\mathsf{B})).$$

The augmentation of $\Delta_G(A)$ is given by the projection of $A_1 \rtimes G$ onto G. ■

We can thus turn a positive chain complex into a crossed complex. Does this functor have a left adjoint? i.e. is there a functor $\xi_G : Crs_G \to Ch(G{-}Mod)$ such that

$$Ch(G{-}Mod)(\xi_G(\mathsf{C}), \mathsf{D}) \to Crs_G(\mathsf{C}, \Delta_G(\mathsf{D}))?$$

If so it would suggest that chain complexes of G-modules are like G-augmented crossed complexes that satisfy some additional equational axioms. As an example of a similar situation think of 'Abelian groups' within 'groups' for which the inclusion has a left adjoint, namely Abelianisation $(G)^{Ab} = G/[G,G]$. Abelian groups are of course groups that satisfy the additional rule $[x,y] = 1$. Other examples of such situations are nilpotent groups of a given finite rank c. The subcategories of this general form are called *varieties* and, for instance, the study of varieties of groups is a very interesting area of group theory. Incidentally, it is possible to define various forms of cohomology modulo a variety in some sense. We will not explore that here.

We thus need to look at morphisms of crossed complexes from a crossed complex C to one of form $\Delta_G(\mathsf{D})$, and we need therefore to look at morphisms into a semidirect product. These are useful for other things, so are worth looking at in detail.

2.2.1 Semi-direct product and derivations.

Suppose that we have a diagram

$$\begin{tikzcd} H \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & K \rtimes G \arrow[dl, "proj"] \\ & G & \end{tikzcd}$$

where K is a G-module (written additively, so we write $g.k$ not ${}^g k$ for the action). This is like the very bottom of the situation for a morphism $f : \mathsf{C} \to \Delta_G(\mathsf{D})$.

As the codomain of f is a semidirect product, we can decompose f, as a function, in the form

$$f(h) = (f_1(h), \alpha(h)),$$

identifying its second component using the diagram. The mapping f_1 is not a homomorphism. As f is one, however, we have

$$(f_1(h_1h_2), \alpha(h_1h_2)) = f(h_1)f(h_2) = (f_1(h_1) + \alpha(h_1)f_1(h_2), \alpha(h_1h_2)),$$

i.e. f_1 satisfies

$$f_1(h_1h_2) = f_1(h_1) + \alpha(h_1)f_1(h_2)$$

for all $h_1, h_2 \in H$.

2.2.2 Derivations and derived modules.

We will use the identification of G-modules for a group G with modules over the group ring, $\mathbb{Z}[G]$, of G. Recall that this ring is obtained from the free Abelian group on the set G by defining a multiplication extending linearly that of G itself. (Formally if, for the moment, we denote by e_g, the generator corresponding to $g \in G$, then an arbitrary element of $\mathbb{Z}[G]$ can be written as $\sum_{g\in G} n_g e_g$ where the n_g are integers and only finitely many of them are non-zero. The multiplication is by 'convolution' product, that is,

$$\Big(\sum_{g\in G} n_g e_g\Big)\Big(\sum_{g\in G} m_g e_g\Big) = \sum_{g\in G}\Big(\sum_{g_1\in G} n_{g_1} m_{g_1^{-1}g} e_g\Big).$$

Sometimes, later on, we will need other coefficients that $\mathbb{Z}$ in which case it is appropriate to use the term 'group algebra' of G, over that ring of coefficients.

We will also need the augmentation, $\varepsilon : \mathbb{Z}[G] \to \mathbb{Z}$, given by $\varepsilon(\sum_{g\in G} n_g e_g) = \sum_{g\in G} n_g$ and its kernel $I(G)$, known as the *augmentation ideal.*

Definitions: Let $\varphi : G \to H$ be a homomorphism of groups. A φ-*derivation*

$$\partial : G \to M$$

from G to a left $\mathbb{Z}[H]$-module, M, is a mapping from G to M, which satisfies the equation

$$\partial(g_1g_2) = \partial(g_1) + \varphi(g_1)\partial(g_2)$$

for all g_1, $g_2 \in G$.

Such φ-derivations are really all derived from a universal one.

Definition: A *derived module* for φ consists of a left $\mathbb{Z}[H]$-module, D_φ, and a φ-derivation, $\partial_\varphi : G \to D_\varphi$ with the following universal property:

Given any left $\mathbb{Z}[H]$-module, M, and a φ-derivation $\partial : G \to M$, there is a unique morphism

$$\beta : D_\varphi \to M$$

of $\mathbb{Z}[H]$-modules such that $\beta\partial_\varphi = \partial$.

The derivation ∂_φ is called the *universal* φ *derivation.*

The set of all φ-derivations from G to M has a natural Abelian group structure. We denote this set by $Der_\varphi(G, M)$. This gives a functor from *H-Mod* to *Ab*, the category of Abelian groups. If $(D_\varphi, \partial_\varphi)$ exists, then it sets up a natural isomorphism

$$Der_\varphi(G, M) \cong H{-}Mod(D_\varphi, M),$$

i.e., $(D_\varphi, \partial_\varphi)$ represents the φ-derivation functor.

2.2.3 Existence

The treatment of derived modules that is found in Crowell's paper, [72], provides a basis for what follows. In particular it indicates how to prove the existence of $(D_\varphi, \partial_\varphi)$ for any φ.

Form a $\mathbb{Z}[H]$-module, D, by taking the free left $\mathbb{Z}[H]$-module, $\mathbb{Z}[H]^{(X)}$, on a set of generators, $X = \{\partial g : g \in G\}$. Within $\mathbb{Z}[H]^{(X)}$ form the submodule, Y, generated by the elements

$$\partial(g_1g_2) - \partial(g_1) - \varphi(g_1)\partial(g_2).$$

Let $D = \mathbb{Z}[H]^{(X)}/Y$ and define $d : G \to D$ to be the composite:

$$G \xrightarrow{\eta} \mathbb{Z}[H]^{(X)} \xrightarrow{quotient} D,$$

where η is "inclusion of the generators", $\eta(g) = \partial g$, thus d, by construction, will be a φ-derivation. The universal property is easily checked and hence $(D_\varphi, \partial_\varphi)$ exists.

We will later on construct $(D_\varphi, \partial_\varphi)$ in a different way which provides a more amenable description of D_φ, namely as a tensor product. As a first step towards this description, we shall give a simple description of D_G, that is, the derived module of the identity morphism of G. More precisely we shall identify (D_G, ∂_G) as being $(I(G), \partial)$, where, as above, $I(G)$ is the augmentation ideal of $\mathbb{Z}[G]$ and $\partial : G \to I(G)$ is the usual map, $\partial(g) = g - 1$.

Our earlier observations give us the following useful result:

Lemma 6 *If G is a group and M is a G-module, then there is an isomorphism*

$$Der_G(G, M) \to Hom/G(G, M \rtimes G)$$

where $Hom/G(G, M \rtimes G)$ is the set of homomorphisms from G to $M \rtimes G$ over G, i.e., $\theta : G \to M \rtimes G$ such that for each $g \in G$, $\theta(g) = (g, \theta'(g))$ for some $\theta'(g) \in M$. ■

2.2.4 Derivation modules and augmentation ideals

Proposition 2 *The derivation module D_G is isomorphic to $I(G) = Ker(\mathbb{Z}[G] \to \mathbb{Z})$. The universal derivation is*

$$d_G : G \to I(G)$$

given by $d_G(g) = g - 1$.

Proof:

We introduce the notation $f_\delta : I(G) \to M$ for the $\mathbb{Z}[G]$-module morphism corresponding to a derivation

$$\delta : G \to M.$$

The factorisation $f_\delta d_G = \delta$ implies that f_δ must be defined by $f_\delta(g - 1) = \delta(g)$. That this works follows from the fact that $I(G)$, as an Abelian group, is free on the set $\{g - 1 : g \in G\}$ and that the relations in $I(G)$ are generated by those of the form

$$g_1(g_2 - 1) = (g_1 g_2 - 1) - (g_1 - 1).$$

■

We note a result on the augmentation ideal construction that is not commonly found in the literature.

The proof is easy and so will be omitted.

Lemma 7 *Given groups G and H in $\mathcal{C}$ and a commutative diagram*

$$\begin{tikzcd} G \arrow[r, "\delta"] \arrow[d, "\psi"'] & M \arrow[d, "\varphi"] \\ H \arrow[r, "\delta'"'] & N \end{tikzcd} \qquad (*)$$

where δ, δ' are derivations, M is a left $\mathbb{Z}[G]$-module, N is a left $\mathbb{Z}[H]$-module and φ is a module map over ψ, i.e., $\varphi(g.m) = \psi(g)\varphi(m)$ for $g \in G$, $m \in M$. Then the corresponding diagram

$$\begin{tikzcd} I(G) \arrow[r, "f_\delta"] \arrow[d, "\psi"'] & M \arrow[d, "\varphi"] \\ I(H) \arrow[r, "f_{\delta'}"'] & N \end{tikzcd} \qquad (**)$$

is commutative. ■

The earlier proposition has the following corollaries:

Corollary 1 *The subset $Im\, d_G = \{g - 1 : g \in G\} \subset I(G)$ generates $I(G)$ as a $\mathbb{Z}[G]$-module. Moreover the relations between these generators are generated by those of the form*

$$(g_1g_2 - 1) - (g_1 - 1) - g_1(g_2 - 1).$$

■

It is useful to have also the following reformulation of the above results stated explicitly.

Corollary 2 *There is a natural isomorphism*

$$Der_G(G, M) \cong G{-}Mod(I(G), M).$$

■

2.2.5 Generation of $I(G)$.

The first of these two corollaries raises the question as to whether, if $X \subset G$ generates G, does the set $G_X = \{x - 1 : x \in X\}$ generate $I(G)$ as a $\mathbb{Z}[G]$-module.

Proposition 3 *If X generates G, then G_X generates $I(G)$.*

Proof: We know $I(G)$ is generated by the $g - 1$s for $g \in G$. If g is expressible as a word of length n in the generators X then we can write $g - 1$ as a $\mathbb{Z}[G]$-linear combination of terms of the form $x-1$ in an obvious way. (If $g = w.x$ with w of lesser length than that of g, $g-1 = w-1+w(x-1)$, so use induction on the length of the expression for g in terms of the generators.) ■

When G is free: If G is free, say, $G \cong F(X)$, i.e., is free on the set X, we can say more.

Proposition 4 *If $G \cong F(X)$ is the free group on the set X, then the set $\{x - 1 : x \in X\}$ freely generates $I(G)$ as a $\mathbb{Z}[G]$-module.*

Proof: (We will write F for $F(X)$.) The easiest proof would seem to be to check the universal property of derived modules for the function $\delta : F \to \mathbb{Z}[G]^{(X)}$, given on generators by

$$\delta(x)(y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } y \neq x; \end{cases}$$

then extended using the derivation rule to all of F using induction. This uses essentially that each element of F has a *unique* expression as a reduced word in the generators, X.

Suppose then that we have a derivation $\partial : F \to M$, define $\overline{\partial} : \mathbb{Z}[G]^{(X)} \to M$ by $\overline{\partial}(e_x) = \partial(x)$, extending linearly. Since by construction $\overline{\partial}\delta = \partial$ and is the unique such homomorphism, we are home. ■

Note: In both these proofs we are thinking of the elements of the free module on X as being functions from X to the group ring, these functions being of 'finite support', i.e. being non-zero on only a finite number of elements of X. This can cause some complications if X is infinite or has some topology as it will in some contexts. The *idea* of the proof will usually go across to that situation but details have to change. (A situation in which this happens is in profinite group theory where the derivations have to be continuous for the profinite topology on the group, see [177].)

2.2.6 (D_φ, d_φ), the general case.

We can now return to the identification of (D_φ, d_φ) in the general case.

Proposition 5 *If $\varphi : G \to H$ is a homomorphism of groups, then $D_\varphi \cong \mathbb{Z}[H] \otimes_G I(G)$, the tensor product of $\mathbb{Z}[H]$ and $I(G)$ over G.*

Proof: If M is a $\mathbb{Z}[H]$-module, we will write $\varphi^*(M)$ for the restricted $\mathbb{Z}[G]$-module, i.e. M with G-action given by $g.m := \varphi(g).m$. Recall that the functor φ^* has a left adjoint given by sending a G-module, N to $\mathbb{Z}[H] \otimes_G N$, i.e. take the tensor of Abelian groups, $\mathbb{Z}[H] \otimes N$ and divide out by $x \otimes g.n \equiv x\varphi(g) \otimes n$.

With this notation we have a chain of natural isomorphisms,

$$\begin{aligned} Der_\varphi(G, M) &\cong Der_G(G, \varphi^*(M)) \\ &\cong G{-}Mod(I(G), \varphi^*(M)) \\ &\cong H{-}Mod(\mathbb{Z}[H] \otimes_G I(G), M), \end{aligned}$$

so by universality,

$$D_\varphi \cong \mathbb{Z}[H] \otimes_G I(G),$$

as required. ■

2.2.7 D_φ for $\varphi : F(X) \to G$.

The above will be particularly useful when φ is the "co-unit" map, $F(X) \to G$, for X a set that generates G. We could, for instance, take $X = G$ as a set, and φ to be the usual natural epimorphism.

In fact we have the following:

Corollary 3 *Let $\varphi : F(X) \to G$ be an epimorphism of groups, then there is an isomorphism*

$$D_\varphi \cong \mathbb{Z}[G]^{(X)}$$

of $\mathbb{Z}[G]$-modules. In this isomorphism, the generator ∂_x, of D_φ corresponding to $x \in X$, satisfies

$$d_\varphi(x) = \partial_x$$

for all $x \in X$. ■

(You should check that you see how this follows from our earlier results.)

2.3 Associated module sequences

2.3.1 Homological background

Given an exact sequence

$$1 \to K \to L \to Q \to 1$$

of abstract groups, then it is a standard result from homological algebra that there is an associated exact sequence of modules,

$$0 \to K^{Ab} \to \mathbb{Z}[Q] \otimes_L I(L) \to I(Q) \to 0.$$

There are several different proofs of this. Homological proofs give this as a simple consequence of the Tor^L-sequence corresponding to the exact sequence

$$0 \to I(L) \to \mathbb{Z}[L] \to \mathbb{Z} \to 0$$

together with a calculation of $Tor_1^L(\mathbb{Z}[Q], \mathbb{Z})$, but we are not assuming that much knowledge of standard homological algebra. That homological proof also, to some extent, hides what is happening at the 'elementary' level, in both the sense of 'simple' and also that of'what happens to the 'elements' of the groups and modules concerned.

The second type of proof is more directly algebraic and has the advantage that it accentuates various universal properties of the sequence. The most thorough treatment of this would seem to be by Crowell, [72], for the discrete case. We outline it below.

2.3.2 The exact sequence.

Before we start on the discussion of the exact sequence, it will be useful to have at our disposal some elementary results on Abelianisation of the groups in a crossed module. Here we actually only need them for normal subgroups but we will need it shortly anyway in the more general form. Suppose that (C, P, ∂) is a crossed module, and we will set $A = Ker\partial$ with its module structure that we looked at before, and $N = \partial C$, so A is a P/N-module.

Lemma 8 *The Abelianisation of C has a natural $\mathbb{Z}[P/N]$-module structure on it.*

Proof: First we should point out that by "Abelianisation" we mean $C^{Ab} = C/[C, C]$, which is, of course, Abelian and it suffices to prove that N acts trivially on C^{Ab}, since P already acts in a natural way. However, if $n \in N$, and $\partial c = n$, then for any $c' \in C$, we have that ${}^n c' = {}^{\partial c} c' = cc'c^{-1}$, hence ${}^n c'(c')^{-1} \in [C, C]$ or equivalently

$${}^n(c'[C, C]) = c'[C, C],$$

so N does indeed act trivially on C^{Ab}. ■

Of course N^{Ab} also has the structure of a $\mathbb{Z}[P/N]$-module and thus a crossed module gives one three P/N-modules. These three are linked as shown by the following proposition.

Proposition 6 *Let (C, P, ∂) be a crossed module. Then the induced morphisms*

$$A \to C^{Ab} \to N^{Ab} \to 0$$

form an exact sequence of $\mathbb{Z}[P/N]$-modules.

Proof: It is clear that the sequence

$$1 \to A \to C \to N \to 1$$

is exact and that the induced homomorphism from C^{Ab} to N^{Ab} is an epimorphism. Since the composite homomorphism from A to N is trivial, A is mapped into $Ker(C^{Ab} \to N^{Ab})$ by the composite $A \to C \to C^{Ab}$. It is easily checked that this is onto and hence the sequence is exact as claimed. ■

Now for the main exact sequence result here:

Proposition 7 *Let*

$$1 \to K \stackrel{\varphi}{\to} L \stackrel{\psi}{\to} Q \to 1$$

be an exact sequence of groups and homomorphisms. Then there is an exact sequence

$$0 \to K^{Ab} \stackrel{\tilde{\varphi}}{\to} \mathbb{Z}[Q] \otimes_L I(L) \stackrel{\tilde{\psi}}{\to} I(Q) \to 0$$

of $\mathbb{Z}[Q]$*-modules.*

Proof: By the universal property of D_ψ, there is a unique morphism

$$\tilde{\psi} : D_\psi \to I(Q)$$

such that $\tilde{\psi}\partial_\psi = I(\psi)\partial_L$.

Let $\delta : K \to K^{Ab} = K/[K,K]$ be the canonical Abelianising morphism. We note that $\partial_\psi \varphi : K \to D_\psi$ is a homomorphism (since

$$\begin{aligned} \partial_\psi\varphi(k_1k_2) &= \partial_\psi\varphi(k_1) + \psi\varphi(k_1)\partial_\psi\varphi(k_2) \\ &= \partial_\psi\varphi(k_1) + \partial_\psi\varphi(k_2),) \end{aligned}$$

so let $\tilde{\varphi} : K^{Ab} \to D_\psi$ be the unique morphism satisfying $\tilde{\varphi}\delta = \partial_\psi\varphi$ with K^{Ab} having its natural $\mathbb{Z}[Q]$-module structure.

That the composite $\tilde{\psi}\tilde{\varphi} = 0$ follows easily from $\psi\varphi = 0$. Since D_ψ is generated by symbols $d\ell$ and $\tilde{\psi}(d\ell) = \psi(\ell) - 1$, it follows that $\tilde{\psi}$ is onto. We next turn to "$Ker\ \tilde{\psi} \subseteq Im\ \tilde{\varphi}$".

If we can prove $\alpha : D_\psi \to I(Q)$ is the cokernel of $\tilde{\varphi}$, then we will have checked this inclusion and incidentally will have reproved that $\tilde{\psi}$ is onto.

Now let $D_\psi \to C$ be any morphism such that $\alpha\tilde{\varphi} = 0$. Consider the diagram

$$\begin{tikzcd}
K \arrow[r, "\varphi"] \arrow[d, "\delta"'] & L \arrow[r, "\psi"] \arrow[d, "\partial_\psi"] & Q \arrow[d, "\partial_Q"] \\
K^{Ab} \arrow[r, "\tilde{\varphi}"] & D_\psi \arrow[r, "\tilde{\psi}"] \arrow[dr, "\alpha"'] & \mathcal{C}(Q) \\
 & & C
\end{tikzcd}$$

The composite $\alpha\partial_\psi$ vanishes on the image of φ since $\alpha\partial_\psi\varphi = \alpha\tilde{\varphi}\delta$ and $\alpha\tilde{\varphi}$ is assumed zero. Define $d : Q \to C$ by $d(q) = \alpha\partial_\psi(\ell)$ for $\ell \in L$ such that $\psi(\ell) = q$. As $\alpha\partial_\psi$ vanishes on $Im\ \varphi$, this is well defined and

$$\begin{aligned} d(q_1q_2) &= \alpha\partial_\psi(\ell_1\ell_2) \\ &= \alpha\partial_\psi(\ell_1) + \alpha(\psi(\ell_1)\partial_\psi(\ell_2)) \\ &= d(q_1) + q_1 d(q_2) \end{aligned}$$

so d factors as $\bar{\alpha}\partial_Q$ in a unique way with $\bar{\alpha} : I(Q) \to C$. It remains to prove that $\alpha = \tilde{\psi}$, but

$$\begin{aligned} \tilde{\psi}\partial_\psi &= I_C(\psi)\partial_L \\ &= \partial_Q\psi \end{aligned}$$

by the naturality of ∂. Now finally note that $\bar{\alpha}\partial_Q = d$ and $d\psi = \alpha\partial_\psi$ to conclude that $\tilde{\psi}\partial_\psi$ and $\alpha\partial_\psi$ are equal. Equality of α and $\bar{\alpha}\tilde{\psi}$ then follows by the uniqueness clause of the universal property of (D_ψ, ∂_ψ).

Next we need to check that $K^{Ab} \to D_\psi$ is a monomorphism. To do this we use the fact that there is a transversal, $s : Q \to L$, satisfying $s(1) = 1$. This means that, following Crowell, [72] p. 224, we can for each $\ell \in L$, $q \in Q$, find an element $q \times \ell$ uniquely determined by the equation

$$\varphi(q \times \ell)) = s(q)\ell s(q\psi(\ell))^{-1},$$

which, of course, defines a function from $Q \times L$ to K. Crowell's lemma 4.5 then shows

$$q \times \ell_1\ell_2 = (q \times \ell_1)(q\psi(\ell_1) \times \ell_2) \ \text{ for } \ \ell_1, \ell_2 \in L.$$

Now let $M = \mathbb{Z}[Q]^{(X)}$, with $X = \{\partial\ell : \ell \in L\}$, so that there is an exact sequence

$$M \to D_\psi \to 0.$$

The underlying group of $\mathbb{Z}[Q]$ is the free Abelian group on the underlying set of Q. Similarly M, above, has, as underlying group, the free Abelian group on the set $Q \times X$.

Define a map $\tau : M \to K^{Ab}$ of Abelian groups by

$$\tau(a, \partial\ell) = \delta(q \times \ell).$$

We check that if $p(m) = 0$, then $\tau(m) = 0$. Since $Ker\, p$ is generated as a $\mathbb{Z}[Q]$-module by elements of the form

$$\partial(\ell_1\ell_2) - \partial\ell_1 - \psi(\ell_1)\partial\ell_2,$$

it follows that as an Abelian group, $Ker\, p$ is generated by the elements

$$(q, \partial(\ell_1\ell_2)) - (q, \partial\ell_1) - (q\psi(\ell_1), \partial\ell_2).$$

We claim that τ is zero on these elements; in fact

$$\begin{aligned} \tau(q, \partial(\ell_1\ell_2)) &= \delta(q \times (\ell_1\ell_2)) \\ &= \delta(q \times \ell_1) + \delta(q\psi(\ell_1) \times \ell_2) \\ &= \tau(q, \ell_1) + \tau(q\psi(\ell_1), \ell_2). \end{aligned}$$

Thus τ induces a map $\eta : D_\psi \to K^{Ab}$ of Abelian groups.

Finally we check $\eta\tilde{\varphi} =$ identity, so that $\tilde{\varphi}$ is a monomorphism: let $b \in K^{Ab}$, $k \in K$ be such that $\delta(k) = b$, then

$$\begin{aligned} \eta\tilde{\varphi}(b) &= \eta\tilde{\varphi}\delta(k) \\ &= \eta\partial_\psi(k) \\ &= \delta(1 \times \varphi(k)), \end{aligned}$$

but $1 \times \varphi(k)$ is uniquely determined by

$$\varphi(1 \times \varphi(k)) = s(1)\varphi(k)s(1\psi\varphi(k))^{-1} = \varphi(k),$$

since $s(1) = 1$, hence $1 \times \varphi(k) = k$ and $\eta\tilde{\varphi}(b) = \delta(k) = b$ as required. ■

A discussion of the way in which this result interacts with the theory of covering spaces can be found in Crowell's paper already cited. We will very shortly see the connection of this module sequence with the Jacobian matrix of a group presentation and the Fox free differential calculus. It is this latter connection which suggests that we need more or less explicit formulae for the maps $\tilde{\varphi}$ and $\tilde{\psi}$ and hence requires that Crowell's detailed proof be used, not the slicker homological proof.

2.3.3 Reidemeister-Fox derivatives and Jacobian matrices

At various points, we will refer to Reidemeister-Fox derivatives as developed by Fox in a series of articles, see [100], and also summarised in Crowell and Fox, [73]. We will call these derivatives *Fox derivatives.*

Suppose G is a group and M a G-module and let $\delta : G \to M$ be a derivation, (so $\delta(g_1g_2) = \delta(g_1) + g_1\delta(g_2)$ for all $g_1, g_2 \in G$), then, for calculations, the following lemma is very valuable, although very simple to prove.

Lemma 9 *If $\delta : G \to M$ is a derivation, then*
(i) $\delta(1_G) = 0$;
(ii) $\delta(g^{-1}) = -g^{-1}\delta(g)$ for all $g \in G$;
(iii) for any $g \in G$ and $n \geq 1$,

$$\delta(g^n) = (\sum_{k=0}^{n-1} g^k)\delta(g).$$

Proof: As was said, these are easy to prove.
$\delta(g) = \delta(1g) + 1\delta(g)$, so $\delta(1) = 0$, and hence (i); then

$$\delta(1) = \delta(g^{-1}g) = \delta(g^{-1}) + g^{-1}\delta(g)$$

to get (ii), and finally induction to get (iii). ■

The Fox derivatives are derivations taking values in the group ring as a left module over itself. They are defined for $G = F(X)$, the free group on a set X. (We usually write F for $F(X)$ in what follows.)

Definition: For each $x \in X$, let
$$\frac{\partial}{\partial x} : F \to \mathbb{Z}F$$
be defined by
(i) for $y \in X$,
$$\frac{\partial y}{\partial x} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } y \neq x; \end{cases}$$
(ii) for any words, $w_1, w_2 \in F$,
$$\frac{\partial}{\partial x}(w_1 w_2) = \frac{\partial}{\partial x} w_1 + w_1 \frac{\partial}{\partial x} w_2.$$
Of course, a routine proof shows that the derivation property in (ii) defines $\frac{\partial w}{\partial x}$ for any $w \in F$.

This derivation, $\frac{\partial}{\partial x}$, will be called the *Fox derivative with respect to the generator x*.

Example: Let $X = \{u, v\}$, with $r \equiv uvuv^{-1}u^{-1}v^{-1} \in F = F(u, v)$, then
$$\begin{aligned} \frac{\partial r}{\partial u} &= 1 + uv - uvuv^{-1}u^{-1}, \\ \frac{\partial r}{\partial v} &= u - uvuv^{-1} - uvuv^{-1}u^{-1}v^{-1}. \end{aligned}$$
This relation is the typical braid group relation, here in Br_3, and we will come back to these simple calculations later.

It is often useful to extend a derivation $\delta : G \to M$ to a linear map from $\mathbb{Z}G$ to M by the simple rule that $\delta(g + h) = \delta(g) + \delta(h)$.

We have
$$Der(F, \mathbb{Z}F) \cong F{-}Mod(IF, \mathbb{Z}F),$$
and that
$$IF \cong \mathbb{Z}F^{(X)},$$
with the isomorphism matching each generating $x - 1$ with e_x, the basis element labelled by $x \in X$. (The universal derivation then sends x to e_x.)

For each given x, we thus obtain a morphism of F-modules:
$$d_x : \mathbb{Z}F^{(X)} \to \mathbb{Z}F$$
with
$$\begin{aligned} d_x(e_y) &= 1 \quad \text{if } y = x \\ d_x(e_y) &= 0 \quad \text{if } y \neq x, \end{aligned}$$
i.e., the 'projection onto the x^{th}-factor' or 'evaluation at $x \in X$' depending on the viewpoint taken of the elements of the free module, $\mathbb{Z}F^{(X)}$.

Suppose now that we have a group presentation, $\mathcal{P} = (X : R)$, of a group, G. Then we have a short exact sequence of groups
$$1 \to N \xrightarrow{\varphi} F \xrightarrow{\gamma} G \to 1,$$

where $N = N(R)$, $F = F(X)$, i.e., N is the normal closure of R in the free group F. We also have a free crossed module,

$$C \xrightarrow{\partial} F,$$

constructed from the presentation and hence, two short exact sequences of G-modules with $\kappa(\mathcal{P}) = Ker\,\partial$, the module of identities of $\mathcal{P}$,

$$0 \to \kappa(\mathcal{P}) \to C^{Ab} \to N^{Ab} \to 0,$$

and also

$$0 \to N^{Ab} \xrightarrow{\tilde{\varphi}} IF \otimes_F \mathbb{Z}G \to IG \to 0.$$

We note that the first of these is exact because N is a free group, (see Proposition 9, which will be proved shortly), further

$$C^{Ab} \cong \mathbb{Z}G^{(R)},$$

(the proof is left to you to manufacture from earlier results), and the map from this to N^{Ab} in the first sequence sends the generator e_r to $r[N, N]$.

We next revisit the derivation of the associated exact sequence (Proposition 7, page 43) in some detail to see what $\tilde{\varphi}$ does to $r[N, N]$. We have $\tilde{\varphi}(r[N, N]) = \partial_\gamma\varphi(r) = \partial_\gamma(r)$, considering r now as an element of F, and by Corollary 3, on identifying D_γ with $\mathbb{Z}G^{(X)}$ using the isomorphism between IF and $\mathbb{Z}F^{(X)}$, we can identify $\partial_\gamma(x) = e_x$. We are thus left to determine $\partial_\gamma(r)$ in terms of the $\partial_\gamma(x)$, i.e., the e_x. The following lemma does the job for us.

Lemma 10 *Let $\delta : F \to M$ be a derivation and $w \in F$, then*

$$\delta w = \sum_{x \in X} \frac{\partial w}{\partial x} \delta x.$$

Proof: By induction on the length of w. ∎

In particular we thus can calculate

$$\partial_\gamma(r) = \sum \frac{\partial r}{\partial x} e_x.$$

Tensoring with $\mathbb{Z}G$, we get

$$\tilde{\varphi}(r[N, N]) = \sum \frac{\partial r}{\partial x} e_x \otimes 1.$$

There is one final step to get this into a usable form:

From the quotient map $\gamma : F \to G$, we, of course, get an induced ring homomorphism, $\gamma : \mathbb{Z}F \to \mathbb{Z}G$, and hence we have elements $\gamma(\frac{\partial r}{\partial x}) \in \mathbb{Z}G$. Of course,

$$\frac{\partial r}{\partial x} e_x \otimes 1 = e_x \otimes \gamma(\frac{\partial r}{\partial x}),$$

so we have, on tidying up notation just a little:

Proposition 8 *The composite map*

$$\mathbb{Z}G^{(R)} \to N^{Ab} \to \mathbb{Z}G^{(X)}$$

sends e_r to $\sum \gamma(\frac{\partial r}{\partial x})e_x$ and so has a matrix representation given by $J_{\mathcal{P}} = \big(\gamma(\frac{\partial r_i}{\partial x_j})\big)$. ∎

Definition: The *Jacobian matrix* of a group presentation, $\mathcal{P} = (X : R)$ of a group G is

$$J_{\mathcal{P}} = \Big(\gamma(\frac{\partial r_i}{\partial x_j})\Big),$$

in the above notation.

The application of γ to the matrix of Fox derivatives simplifies expressions considerable in the matrix. The usual case of this is if a relator has the form rs^{-1}, then we get

$$\frac{\partial rs^{-1}}{\partial x} = \frac{\partial r}{\partial x} - rs^{-1}\frac{\partial s}{\partial x}$$

and if r or s is quite long this looks moderately horrible to work out! However applying γ to the answer, the term rs^{-1} in the second of the two terms becomes 1. We can actually think of this as replacing rs^{-1} by $r - s$ when working out the Jacobian matrix.

Example: Br_3 revisited. We have $r \equiv uvuv^{-1}u^{-1}v^{-1}$, which has the form $(uvu)(vuv)^{-1}$. This then gives

$$\gamma(\frac{\partial r}{\partial u}) = 1 + uv - v \quad \text{and} \quad \gamma(\frac{\partial r}{\partial v}) = u - 1 - vu,$$

abusing notation to ignore the difference between u, v in $F(u, v)$ and the generating u, v in Br_3.

Homological 2-syzygies: In general we obtain a truncated chain complex:

$$\mathbb{Z}G^{(R)} \overset{d_2}{\to} \mathbb{Z}G^{(X)} \overset{d_1}{\to} \mathbb{Z}G \overset{d_0}{\to} \mathbb{Z} \to 0,$$

with d_2 given by the Jacobian matrix of the presentation, and d_1 sending generator e^1_x to $1 - x$, so $Im\, d_1$ is the augmentation ideal of $\mathbb{Z}G$.

Definition: A *homological 2-syzygy* is an element in $Ker\, d_2$.

A homological 2-syzygy is thus an element to be killed when building the third level of a resolution of G. What are the links between homotopical and homological syzygies? Brown and Huebschmann, [50], show they are isomorphic, as $Ker\, d_2$ is isomorphic to the module of identities. We will examine this result in more detail shortly.

Extended example: Homological Syzygies for the braid group presentations: The Artin braid group, Br_{n+1}, defined using $n + 1$ strands is given by

- generators: y_i, $i = 1, \ldots, n$;
- relations: $r_{ij} \equiv y_i y_j y_i^{-1} y_j^{-1}$ for $i + 1 < j$;
  $r_{ii+1} \equiv y_i y_{i+1} y_i y_{i+1}^{-1} y_i^{-1} y_{i+1}^{-1}$ for $1 \leq i < n$.

We will look at such groups only for small values of n.

By default, Br_2 has one generator and no relations, so is infinite cyclic.

The group Br_3: (We will simplify notation writing $u = y_1$, $v = y_2$.)

This then has presentation $\mathcal{P} = (u, v : r \equiv uvuv^{-1}u^{-1}v^{-1})$. It is also the 'trefoil group', i.e., the fundamental group of the complement of a trefoil knot. If we construct $X(2) = K(\mathcal{P})$, this is already a $K(Br_3, 1)$ space, having a trivial π_2. There are no higher syzygies.

We have all the calculation for working with homological syzygies here. The key part of the complex is the Jacobian matrix as that determines d_2:

$$d_2 = \begin{pmatrix} 1 + uv - v & u - 1 - vu \end{pmatrix}.$$

This has trivial kernel, but, in fact, that comes most easily from the identification with homotopical syzygies.

The group Br_4: simplifying notation as before, we have generators u, v, w and relations

$$\begin{aligned} r_u &\equiv vwvw^{-1}v^{-1}w^{-1}, \\ r_v &\equiv uwu^{-1}w^{-1}, \\ r_w &\equiv uvuv^{-1}u^{-1}v^{-1}. \end{aligned}$$

The 1-syzygies are made up of hexagons for r_u and r_w and a square for r_v. There is a fairly obvious way of fitting together squares and hexagons, namely as a permutohedron, and there is a labelling of such that gives a homotopical 2-syzygy.

The presentation yields a truncated chain complex with d_2

$$\mathbb{Z}G^{(r_u, r_v, r_w)} \xrightarrow{d_2} \mathbb{Z}G^{(u,v,w)}$$

with

$$d_2 = \begin{pmatrix} 0 & 1 + vw - w & v - 1 - wv \\ 1 - w & 0 & u - 1 \\ 1 + uv - v & u - 1 - vu & 0 \end{pmatrix}$$

and Loday, [144], has calculated that for the permutohedral 2-syzygy, s, one gets another term of the resolution, $\mathbb{Z}G^{(s)}$, and a $d_3 : \mathbb{Z}G^{(s)} \to \mathbb{Z}G^{(r_u, r_v, r_w)}$ given by

$$d_3 = \begin{pmatrix} 1 + vu - u - wuv & v - vwu - 1 - uv - vuwv & 1 + vw - w - uvw \end{pmatrix}.$$

For more on methods of working with these syzygies, have a look at Loday's paper, [144], and some of the references that you will find there.

2.4 Crossed complexes and chain complexes: II

(The source for the material and ideas in this section is once again [47].)

2.4.1 The reflection from Crs to chain complexes

It is now time to return to the construction of a left adjoint for Δ_G.

Theorem 2 *(Brown-Higgins, [47] in a slightly more general form.) The functor, Δ_G, has a left adjoint.*

Proof: We construct the left adjoint explicitly as follows:

Let $f. : (\mathsf{C}, \varphi) \to \Delta_G(M.)$ be a morphism in Crs_G, then we have the following commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & C_2 \arrow[r, "\delta_2"] \arrow[d, "f_2"] & C_1 \arrow[r, "\delta_1"] \arrow[d, "f_1"] & C_0 \arrow[r, "\varphi"] \arrow[d, "f_0"] & G \arrow[d, "Id_G"] \\
\dots \arrow[r] & M_2 \arrow[r, "\delta_2"] & M_1 \arrow[r, "\delta_1"] & M_0 \rtimes G \arrow[r, "pr_G"] & G
\end{tikzcd}$$

Since the right hand square commutes, f_0 is given by some formula

$$f_0(c) = (\partial(c), \varphi(c)),$$

where $\partial : C_0 \to M_0$ is a φ-derivation. Thus $\partial = \tilde{f}_0 \partial_\varphi$ for a unique G-module morphism, $\tilde{f}_0 : D_\varphi \to M_0$, and f_0 factors as

$$C_0 \stackrel{\bar{\varphi}}{\to} D_\varphi \rtimes G \stackrel{\tilde{f}_0 \rtimes G}{\to} M_0 \rtimes G,$$

where $\bar{\varphi}(c) = (\partial_\varphi(c), \varphi(c))$.

The map $\partial_\varphi \delta_1 : C_1 \to D_\varphi$ is a homomorphism since

$$\begin{aligned}
\partial_\varphi \delta_1(c_1 c_2) &= \partial_\varphi \partial_1(c_1) + \varphi \partial_1(c_1) \partial_\varphi \partial_1(c_2) \\
&= \partial_\varphi \partial_1(c_1) + \partial_\varphi \partial_1(c_2),
\end{aligned}$$

$\varphi \partial_1$ being trivial (because (C, φ) is G-augmented). We thus obtain a map $d : C_1^{Ab} \to D_\varphi$ given by $d(c[C, C]) = \partial_\varphi \partial_1(c)$ for $c \in C_1$. As we observed earlier the Abelian group C_1^{Ab} has a natural $\mathbb{Z}[G]$-module structure making d a G-module morphism.

Similarly there is a unique G-module morphism,

$$\tilde{f}_1 : C_1^{Ab} \to M_1,$$

satisfying

$$\tilde{f}_1(c[C, C]) = f_1(c).$$

Since for $c \in C_1$,

$$(d_1 \tilde{f}_1(c), 1) = f_0(\delta_1 c) = (\tilde{f}_0 \partial_\varphi(\delta_1 c_1), 1),$$

we have that the diagram

$$\begin{tikzcd}
C_1^{Ab} \arrow[r, "\tilde{f}_1"] \arrow[d, "d"] & M_1 \arrow[d, "d_1"] \\
D_\varphi \arrow[r, "\tilde{f}_0"] & M_0
\end{tikzcd}$$

commutes.

We also note that since $\delta_2 : C_2 \to C_1$ maps into $Ker\ \delta_1$, the composite

$$C_2 \stackrel{\delta_2}{\to} C_1 \stackrel{\text{can}}{\to} C_1^{Ab} \stackrel{d}{\to} D_\varphi,$$

being given by $d(\delta_2(c)[C, C] = \partial_\varphi \delta_1 \delta_2(c)$, is trivial and that $\tilde{f}_1 \delta_2(c[C, C]) = f_1 \delta_2(c) = d_2 f_2(c)$, thus we can define $\xi = \xi_G(\mathsf{C}, \varphi)$ by

$$\begin{aligned}
\xi_n &= C_n \text{ if } n \geq 2 \\
\xi_1 &= C_1^{Ab}, \\
\xi_0 &= D_\varphi,
\end{aligned}$$

the differentials being as constructed. We note that as $Ker\ \varphi$ acts trivially on all C_n for $n \geq 2$, all the C_n have $\mathbb{Z}[G]$-module structures.

That ξ_G gives a functor

$$Crs \to Ch(G{-}Mod)$$

is now easy to check using the uniqueness clauses in the universal properties of D_φ and Abelianisation. Again uniqueness guarantees that the process "f goes to $\tilde{f}$" gives a natural isomorphism

$$Ch(G-Mod)(\xi_G(\mathsf{C},\varphi),\mathsf{M}) \cong Crs_G((\mathsf{C},\varphi),\Delta_G(\mathsf{M}))$$

as required. ■

It is relatively easy to extend the above natural isomorphism to handle morphisms of crossed complexes over different groups. For a detailed treatment one needs a discussion of the way that the change of groups functors work with crossed modules or crossed complexes, that is, if we have a morphism of groups $\theta : G \to H$ then we would expect to get functors between Crs_G and Crs_H induced by θ. These do exist and are very nicely behaved, but they will not be discussed here, see [177] for a full treatment in the more general context of profinite groups.

2.4.2 Crossed resolutions and chain resolutions

One of our motivations for introducing crossed complexes was that they enable us to model more of the sort of information encoded in a $K(G,1)$ than does the usual standard algebraic models, e.g. a chain complex such as the bar resolution. In particular, whilst the bar resolution is very good for cohomology with Abelian coefficients for non-Abelian cohomology the crossed version can allow us to push things further, but then comparison on the Abelian theory is very necessary! It is therefore of importance to see how this $K(G,1)$ information that we have encoded changes under the functor $\xi : Crs \to Ch(G{-}Mod)$.

We start with a crossed resolution determined in low dimensions by a presentation $\mathcal{P} = (X : R)$ of a group, G. Thus, in this case, $C_0 = F(X)$ with $\varphi : F(X) \to G$, the 'usual' epimorphism, and $C_1 \to C_0$ is $C \to F(X)$, the free crossed module on $R \to F(X)$. It is not too hard to show that $C_1^{Ab} \cong \mathbb{Z}[G]^{(R)}$, the free $\mathbb{Z}[G]$-module on R. (The proof is left as an exercise.) This maps down onto $N(R)^{Ab}$, the Abelianisation of the normal closure of R in $F(X)$ via a map

$$\partial_* : \mathbb{Z}[G]^{(R)} \to N(R)^{Ab},$$

given by $\partial_*(e_r) = r[N(R), N(R)]$, where e_r is the generator of $\mathbb{Z}[G]$ corresponding to $r \in R$.

There is also a short exact sequence

$$1 \to N(R) \xrightarrow{i} F(X) \xrightarrow{\varphi} G \to 1$$

and hence by Proposition 7, a short exact sequence

$$0 \to N(R)^{Ab} \xrightarrow{\tilde{i}} \mathbb{Z}[G] \otimes_F I(F) \xrightarrow{\tilde{\varphi}} I(G) \to 0$$

(where we have written $F = F(X)$).

By the Corollary to Proposition 5, we have

$$\mathbb{Z}[G] \otimes_F I(F) \cong \mathbb{Z}[G]^{(X)}.$$

The required map $C_1^{Ab} \to D_\varphi$ is the composite

$$\mathbb{Z}[G]^{(R)} \xrightarrow{\partial_*} N(R)^{Ab} \xrightarrow{\tilde{i}} \mathbb{Z}[G]^{(X)}.$$

We have given an explicit description of ∂_* above, so to complete the description of d, it remains to describe $\tilde{i}$, but $\tilde{i}$ satisfies $\tilde{i}\delta = \partial_\varphi i$, where $\delta : N(R) \to N(R)^{Ab}$, so $\tilde{i}(r[N(R), N(R)]) = d_\varphi(r)$. Thus if r is a relator, i.e., if it is in the image of the subgroup generated by the elements of R, then $\partial(e_r)$ can be written as a finite sum of the form $\sum_x a_x e_x$ and the elements $a_x \in \mathbb{Z}[G]$ are the images of the Fox derivatives.

This operator can best be viewed as the Alexander matrix of a presentation of a group, further study of this operator depends on studying transformations between free modules over group rings, and we will not attempt to study those here.

The rest of the crossed resolution does not change and so, on replacing $I(G)$ by $\mathbb{Z}[G] \to \mathbb{Z}$, we obtain a free pseudocompact $\mathbb{Z}[G]$-resolution of the trivial module $\mathbb{Z}$,

$$\ldots \to \mathbb{Z}[G]^{(R)} \xrightarrow{d} \mathbb{Z}[G]^{(X)} \to \mathbb{Z}[G] \to \mathbb{Z}$$

built up from the presentation. This is the complex of chains on the universal cover, $\widetilde{K(G,1)}$, where $K(G, 1)$ is constructed starting from a presentation $\mathcal{P}$.

2.4.3 Standard crossed resolutions and bar resolutions

We next turn to the special case of the standard crossed resolution of G discussed briefly earlier. Of course this is a special case of the previous one, but it pays to examine it in detail.

Clearly in $\xi = \xi(\mathsf{C}G, \varphi)$, we have:
$\xi_0 =$ the free $\mathbb{Z}[G]$-module on the underlying set of G, individual generators being written $[u]$, for $u \in G$;
$\xi_1 =$ the free $\mathbb{Z}[G]$ -module on $G \times G$, generators being written $[u, v]$;
$\xi_n = C_n G$, the free $\mathbb{Z}[G]$ -module on G^{n+1}, etc.

The map $d_2 : \xi_2 \to \xi_1$ induced from δ_2 is given by

$$d_2[u, v, w] = u[v, w] - [u, v] - [uv, w] + [u, vw],$$

and the map $d_1 : \xi_1 \to \xi_0$ by

$$\begin{aligned} d_1([u, v]) &= d_\varphi([uv]^{-1}[u][v]) \\ &= v^{-1}u^{-1}(-[uv] + [u] + u[v]), \end{aligned}$$

a unit times the usual bar resolution formula. Thus, as claimed earlier, the standard crossed resolution is the crossed analogue of the bar resolution.

2.4.4 The intersection $A \cap [C, C]$.

We next turn to a comparison of homological and homotopical syszygies. We have almost all the preliminary work already. The next ingredient is a result that will identify the intersection of the kernel of a crossed module, $A = Ker(C \xrightarrow{\partial} P)$ and the commutator subgroup of C.

The kernel of the homomorphism from A to C^{Ab} is, of course, $A \cap [C, C]$ and this need not be trivial. In fact, Brown and Huebschmann ([50], p.160) note that in examples of type $(G, Aut(G), \partial)$, the kernel of ∂ is, of course, the centre ZG of G and $ZG \cap [G, G]$ can be non-trivial, for instance, if G is dicyclic or dihedral.

We will adopt the same notation as previously with $N = \partial P$ etc.

Proposition 9 *If, in the exact sequence of groups*

$$1 \to A \to C \xrightarrow{p} N \to 1,$$

the epimorphism from C to N is split (the splitting need not respect G-action), then $A \cap [C, C]$ is trivial.

Proof: Given a splitting $s : N \to C$, (so ps is the identity on N), then the group C can be written as $A \rtimes s(N)$. The commutators in C, therefore, all lie in $s(N)$ since A is Abelian, but then, of course, $A \cap [C, C]$ cannot contain any non-trivial elements. ■

We used this proposition earlier in the case where N is free. We are thus using the fact that subgroups of free groups are free, in that case. Of course, any epimorphism with codomain a free group is split.

Brown and Huebschmann, [50], p. 168, prove that for an group G with presentation $\mathcal{P}$, the module of identities for $\mathcal{P}$ is naturally isomorphic to the second homology group, $H_2(\tilde{K}(\mathcal{P}))$, of the universal cover of $K(\mathcal{P})$, the 2-complex of the presentation. We can approach this via the algebraic constructions we have.

Given a presentation $\mathcal{P} = \langle X : R \rangle$ of a group G, the algebraic analogue of $K(\mathcal{P})$, we have argued above, is the free crossed module $C(\mathcal{P}) \xrightarrow{d} F(X)$ and the chains on the universal cover of $K(\mathcal{P})$ will be given by ξ_G of this, i.e., by the chain complex

$$\mathbb{Z}[G]^{(R)} \xrightarrow{d} \mathbb{Z}[G]^{(X)}.$$

In general there will be a short exact sequence

$$0 \to \kappa(\mathcal{P}) \cap [C(\mathcal{P}), C(\mathcal{P})] \to \kappa(\mathcal{P}) \to H_2(\xi(C(\mathcal{P})) \to 0.$$

This short exact sequence yields the Brown-Huebschmann result as $N(R)$ will a free group so the epimorphism onto $N(R)$ splits and we can use the above Proposition 9. We thus get

Proposition 10 *If $\mathcal{P} = \langle X : R \rangle$ is a free presentation of G, then there is an isomorphism*

$$\kappa \xrightarrow{\cong} H_2(\xi(C_{\mathcal{C}}(\mathcal{P})) = Ker(d : \mathbb{Z}[G]^R \to \mathbb{Z}[G]^X).$$

■

Note: Here we are using something that will not be true in all algebraic settings. A subgroup of a free group is always free, but the analogous statement for free algebras of other types is not true.

2.5 Simplicial groups and crossed complexes

2.5.1 From simplicial groups to crossed complexes

Given any simplicial group G, the formula,

$$\mathsf{C}(G)_{n+1} = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})},$$

in higher dimensions with, at its 'bottom end', the crossed module,

$$\frac{NG_1}{d_0(NG_2 \cap D_2)} \to NG_0$$

gives a crossed complex with ∂ induced from the boundary in the Moore complex. The detailed proof is too long to indicate here. It just checks the axioms, one by one.

We should have a glance at this formula from various viewpoints, some of which will be revisited later. Once again there is a clear link with the non-uniqueness of fillers for horns in a simplicial group if it is not a group T-complex. We have all those $(NG_n \cap D_n)$ terms involved!

Suppose that we had our simplicial group G and wanted to construct a quotient of it that was a group T-complex. We could do this in a silly way since the trivial simplicial group is clearly a group T-complex, but let us keep the quotient as large as possible. This problem is related to the question of whether the category of group T-complexes forms a reflexive subcategory of *Simp.Grps*. The condition $NG \cap D = 1$ looks like some sort of 'equational specification'. Our question can thus really be posed as follows: Suppose we have a simplicial group morphism $f : G \to H$ and H is a group T-complex. Remember that in group T-complexes, as against the non-algebraic ones, the thin structure is not an added bit of structure. The thin elements are determined by the degeneracies, so whether or not H is or is not a group T-complex is somehow its own affair, and nothing to do with any external factors! Does f factor universally through some 'group T-complexification' of G? Something like

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "proj"'] & & H \\ & G/T(G) \arrow[ur, "\hat{f}"'] & \end{tikzcd}$$

with $G/T(G)$ a group T-complex and $\hat{f}$ uniquely determined by the diagram.

One sensible way to look at such a question is to assume, provisionally, that such a factorisation exists and to see what $T(G)$ would have to be. In general, if $f : G \to H$ is any simplicial group morphism (with no restriction on H for the moment), then with a hopefully obvious notation,

$$f_n(NG_n \cap D(G)_n) \subseteq NH_n \cap D(H)_n,$$

since f sends degenerate elements to degenerate elements and preserves products! Back in our situation in which H is a group T-complex, then $f_n(NG_n \cap D(G)_n) = 1$, for the simple reason that the right hand side of that displayed formula is trivial by assumption. We thus have that if some such $T(G)$ exists, then we must have $NG_n \cap D(G)_n \subseteq T(G)_n$ and our first attempt might be to look at the possibility that they should be equal. This is wrong and for fairly trivial reasons. The subgroup $T(G)_n$ of G_n has to be normal if we are to form the quotient by it, and there is no reason why $NG_n \cap D(G)_n$ should be a normal subgroup in general.

We might then be tempted to take the normal subgroup generated by $NG_n \cap D(G)_n$, but that is 'defeatist' in this situation. We might hope to do detailed calculations with the subgroup and if it is specified as a normal closure, we will lose some of our ability to do that, at least without considerable more effort. (Let's be lazy and see if we can get around that difficulty.) If we look again, we find another thing that 'goes wrong' with any attempt to use $NG_n \cap D(G)_n$ as it is. This subgroup would be within NG_n, of course, and we want to induce a map from the Moore complex of G to that of $G/T(G)$. For that to work, we would need not only $NG_n \cap D(G)_n \subseteq T(G)_n$, but the image of $NG_n \cap D(G)_n$ under d_0 to be in $T(G)_{n-1}$. Going up a dimension, we thus need not only $NG_n \cap D(G)_n$, but $d_0(NG_{n+1} \cap D(G)_{n+1}) \subseteq T(G)_n$. We thus need the product subgroup $(NG_n \cap D(G)_n)d_0(NG_{n+1} \cap D(G)_{n+1})$ to be in $T(G)_n$. This looks a bit complicated. Do we need to go any further up the Moore complex? No, because d_0d_0 is trivial. We might thus try

$$T(G)_n = (NG_n \cap D(G)_n)d_0(NG_{n+1} \cap D(G)_{n+1})$$

You might now think that this is a bit silly because we would still need this product subgroup to be normal in order to form the quotient ... , but it is! The lack of normality of our earlier attempt is absorbed by the image of the next level up. (That is pretty!)

Of course, there are very good reasons why this works. These involve what are sometimes called *Peiffer pairings*. We will see some of these later.

As a consequence of the above discussion, we more or less have:

Proposition 11 *If G is a group T-complex, then NG is a crossed complex.* ■

We certainly have a sketch of

Proposition 12 *The full subcategory of Simp.Grps determined by the group T-complexes is a reflective subcategory.* ■

Of course, the details of the proofs of both of these are left for you to write down. Nearly all of the reasoning for the second result is there for you, but some of the detailed calculations for the first are quite tricky.

The close link between group T-complexes and crossed complexes is evident from these results. You might guess that they form equivalent categories. They do. We will look at the way back from crossed complexes (of groups) to simplicial groups later on, but we need to get back to cohomology.

2.5.2 Simplicial resolutions, a bit of background

We need some such means of going from simplicial groups to crossed complexes so because we can also use simplicial resolutions to 'resolve' a group (and in many other situations). We first sketch in some historical background.

In the 1960s, the connection between simplicial groups and cohomology was examined in detail. The basic idea was that given the adjoint "free-forget" pair of functors between *Groups* and *Sets*, one could generate a free resolution of a group, G, using the resulting comonad (or cotriple) (cf. MacLane, [149]). This resolution was not, however, by a chain complex but by a free simplicial group, F, say. It was then shown (Barr and Beck, [17]) that given any G-module, M, and working in the category of groups over G, one could form the cosimplicial G-module, $Hom_{Gps/G}(F, M)$, and hence, by a dual form of the Dold-Kan theorem, a cochain complex $C(G, M)$, whose homotopy type, and hence whose homology, was independent of the choice of F. This homology was the usual

Eilenberg-MacLane cohomology of G with coefficients in M, but with a shift in dimension (cf. Barr and Beck, [17]).

Other theories of cohomology were developed at about the same time by Grothendieck and Verdier, [8], André, [6, 7], and Quillen, [186, 187]. The first of these was designed for use with "sites", that is, categories together with a Grothendieck topology.

André and Quillen developed, independently, a method of defining cohomology using simplicial resolutions. Their work is best known in commutative algebra, but their methods work in greater generality. Unlike the theory of Barr and Beck (monadic cohomology), they only assume there is enough structure to construct free resolutions; a (co)monad is just one way of doing this. In particular, André, [6, 7], describes a step-by-step, almost combinatorial, process for constructing such resolutions. This ties in well with our earlier comments about using a presentation of a group to construct a crossed resolution and the important link with syzygies. André's method is the simplicial analogue of this.

We will assume for the moment that we have a simplicial resolution, F, of our group, G. Both André and Quillen then consider applying a derived module construction dimensionwise to F, obtaining a simplicial G-module. They then use "Dold-Kan" to give a chain complex of G-modules, which they call the "cotangent complex", denoted L_G or $LAb(G)$, of G (at least in the case of commutative algebras). The homotopy type of $LAb(G)$ does not depend on the choice of resolution and so is a useful invariant of G. We will need to look at this construction in more detail, but will consider a slightly more general situation to start with.

2.5.3 Free simplicial resolutions

Standard theory (cf. Duskin, [83]) shows that if F and F' are free simplicial resolutions of groups, G and H, say, and $f : G \to H$ is a morphism, then f can be lifted to $f' : F \to F'$. The method is the simplicial analogue of lifting a homomorphism of modules to a map of resolutions of those modules, which you should look at first as it is technically simpler. Any two such lifts are homotopic (by a simplicial homotopy).

Of course, f will also lift to a morphism of crossed complexes, $f : \mathsf{C}(F) \to \mathsf{C}(F')$, and any two such lifts will be *homotopic* as crossed complex morphisms. Thus whatever simplicial lift, $f' : F \to F'$, we choose, $\mathsf{C}(f')$ will be a lift in the "crossed" case, and although we do not know at this stage of our discussion of the theory if a homotopy between two simplicial lifts is transferred to a homotopy between the images under C, this does not matter as the *relation* of homotopy is preserved at least in this case of resolutions.

Any group has a free simplicial resolution. There is the obvious adjoint pair of functors

$$\begin{aligned} U &: Groups \to Sets \\ F &: Sets \to Groups \end{aligned}$$

Writing $\eta : Id \to UF$ and $\varepsilon : FU \to Id$ for the unit and counit of this adjunction (cf. MacLane, [149, 150]), we have a comonad (or cotriple) on *Groups*, the free group comonad, $(FU, \varepsilon, F\eta U)$. We write $L = FU$, $\delta = F\eta U$, so that

$$\varepsilon : L \to I$$

is the counit of the comonad whilst

$$\delta : L \to L^2$$

is the comultiplication. (For the reader who has not met monads or comonads before, (L, η, δ) behaves as if it was a monoid in the dual of the category of "endofunctors" on *Groups*, see MacLane, [150] Chapter VI. We will explore them briefly in section **??**, starting on page **??**.)

Now suppose G is a group and set $F(G)_i = L^{i+1}(G)$, so that $F(G)_0$ is the free group on the underlying set of G and so on. The counit (which is just the augmentation morphism from $FU(G)$ to G) gives, in each dimension, face morphisms

$$d_i = L^{n-i}\varepsilon L^i(G) : L^{n+1}(G) \to L^n(G),$$

for $i = 0, \ldots, n$, whilst the comultiplication gives degeneracies

$$s_i : L^n(G) \to L^{n+1}(G)$$

$$s_i = L^{n-1-i}\delta L^i,$$

for $i = 0, \ldots, n-1$, satisfying the simplicial identities.

Remark: Here we follow the conventions used by Duskin, in his Memoir, [83]. Later we will also need to look at similar resolutions where the labelling of the faces and degeneracies are reversed.

This simplicial group, $F(G)$, satisfies $\pi_0(F(G)) \cong G$ (the isomorphism being induced by $\varepsilon(G) : F_0(G) \to G$) and $\pi_n(F(G))$ is trivial if $n \geq 1$. The reason for this is simple. If we apply U once more to $F(G)$, we get a simplicial set and the unit of the adjunction

$$\eta : 1 \to UF$$

allows one to define for each n

$$\eta U(FU)^n : UL^n \to UL^{n+1},$$

which gives a natural contraction of the augmented simplicial set, $UF(G) \to U(G)$, (cf. Duskin, [83]). We will look at this in detail in our later treatment of augmentations, etc. For the moment, it suffices to accept the fact that we do get a resolution, as we do not need to know the details of why this construction works, at least not yet.

If we denote the constant simplicial group on G by $K(G, 0)$, the augmentation defines a simplical homomorphism

$$\overline{\varepsilon} : F(G) \to K(G, 0)$$

satisfying $U\overline{\varepsilon}.inc = Id$, where $inc : UK(G, 0) \to UF(G)$ is the 'inclusion' of simplicial sets given by η, and then these extra maps, $(UF)^n\eta U$, in fact, give a homotopy between $inc.U\overline{\varepsilon}$ and the identity map on $UF(G)$, i.e., $\overline{\varepsilon}$ is a weak homotopy equivalence of simplicial groups. Thus $F(G)$ is a free simplicial resolution of G. It is called the *comonadic free simplicial resolution* of G.

This simplicial resolution has the advantage of being functorial, but the disadvantage of being very big. We turn next to a 'step-by-step' method of constructing a simplicial resolution using ideas pioneered by André, [7], although most of his work was directed more towards commutative algebras, cf. [6].

2.5.4 Step-by-Step Constructions

This section is a brief résumé of how to construct simplicial resolutions by hand rather than functorially. This allows a better interpretation of the generators in each level of the resolution. These are the simplicial analogues of higher syzygies. The work depends heavily on a variety of sources, mainly [6], [137] and [164]. André only treats commutative algebras in detail, but Keune [137] does discuss the general case quite clearly. The treatment here is adapted from the paper by Mutlu and Porter, [168].

Recall of notation: We first recall some notation and terminology, which will be used in the construction of a simplicial resolution. Let $[n]$ be the ordered set, $[n] = \{0 < 1 < \cdots < n\}$. Define the following maps: the injective monotone map $\delta_i^n : [n-1] \to [n]$ is given by

$$\delta_i^n(k) = \begin{cases} k & \text{if} \quad k < i, \\ k+1 & \text{if} \quad k \geq i, \end{cases}$$

for $0 \leq i \leq n \neq 0$. The increasing surjective monotone map $\alpha_i^n : [n+1] \to [n]$ is given by

$$\alpha_i^n(k) = \begin{cases} k & \text{if } k \leq i, \\ k-1 & \text{if } k > i, \end{cases}$$

for $0 \leq i \leq n$. We denote by $\{m, n\}$ the set of increasing surjective maps $[m] \to [n]$.

2.5.5 Killing Elements in Homotopy Groups

Let G be a simplicial group and let $k \geq 1$ be fixed. Suppose we are given a set, Ω, of elements: $\Omega = \{x_\lambda : \lambda \in \Lambda\}$, $x_\lambda \in \pi_{k-1}(\mathsf{G})$, then we can choose a corresponding set of elements $\theta_\lambda \in NG_{k-1}$ so that $x_\lambda = \theta_\lambda\, \partial_k(NG_k)$. (If $k = 1$, then as $NG_0 = G_0$, the condition that $\theta_\lambda \in NG_0$ is immediate.) We want to 'kill' the elements in Ω.

We form a new simplicial group F_n where

1) F_n is the free G_n-group, (i.e., group with G_n-action)

$$F_n = \coprod_{\lambda,t} G_n\{y_{\lambda,t}\} \text{ with } \lambda \in \Lambda \text{ and } \ t \in \{n, k\},$$

where $G_n\{y\} = G_n * < y >$, the co-product of G_n and a free group generated by y.

2) For $0 \leq i \leq n$, the group homomorphism $s_i^n : F_n \to F_{n+1}$ is obtained from the homomorphism $s_i^n : G_n \to G_{n+1}$ with the relations

$$s_i^n(y_{\lambda,t}) = y_{\lambda,u} \quad \text{with} \quad u = t\alpha_i^n, \quad t : [n] \to [k].$$

3) For $0 \leq i \leq n \neq 0$, the group homomorphism $d_i^n : F_n \to F_{n-1}$ is obtained from $d_i^n : G_n \to G_{n-1}$ with the relations

$$d_i^n(y_{\lambda,t}) = \begin{cases} y_{\lambda,u} & \text{if the map} \quad u = t\delta_i^n \quad \text{is surjective,} \\ t'(\theta_\lambda) & \text{if} \quad u = \delta_k^k t', \\ 1 & \text{if} \quad u = \delta_j^k t' \quad \text{with} \ \ j \neq k, \end{cases}$$

by extending multiplicatively.

We sometimes denote the F, so constructed by $\mathsf{G}(\Omega)$.

Remark: In a 'step-by-step' construction of a simplicial resolution, (see below), there will thus be the following properties: i) $F_n = G_n$ for $n < k$, ii) F_k = a free G_k-group over a set of non-degenerate indeterminates, all of whose faces are the identity except the k^{th}, and iii) F_n is a free G_n-group on some degenerate elements for $n > k$.

We have immediately the following result, as expected.

Proposition 13 *The inclusion of simplicial groups $G \hookrightarrow F$, where $F = G(\Omega)$, induces a homomorphism*

$$\pi_n(G) \longrightarrow \pi_n(F)$$

for each n, which for $n < k-1$ is an isomorphism,

$$\pi_n(G) \cong \pi_n(F)$$

and for $n = k-1$, is an epimorphism with kernel generated by elements of the form $\bar{\theta}_\lambda = \theta_\lambda \partial_k NG_k$, where $\Omega = \{x_\lambda : \lambda \in \Lambda\}$. ■

2.5.6 Constructing Simplicial Resolutions

The following result is essentially due to André, [6].

Theorem 3 *If G is a group, then it has a free simplicial resolution $\mathbb{F}$.*

Proof: The repetition of the above construction will give us the simplicial resolution of a group. Although 'well known', we sketch the construction so as to establish some notation and terminology.

Let G be a group. The zero step of the construction consists of a choice of a free group F and a surjection $g : F \to G$ which gives an isomorphism $F/Ker\, g \cong G$ as groups. Then we form the constant simplicial group, $F^{(0)}$, for which in every degree n, $F_n = F$ and $d_i^n = \mathrm{id} = s_j^n$ for all i, j. Thus $F^{(0)} = K(F, 0)$ and $\pi_0(F^{(0)}) = F$. Now choose a set, Ω^0, of normal generators of the closed normal subgroup $N = Ker\,(F \xrightarrow{g} G)$, and obtain the simplicial group in which $F_1^{(1)} = F(\Omega^0)$ and for $n > 1$, $F_n^{(1)}$ is a free F_n-group over the degenerate elements as above. This simplicial group will be denoted by $F^{(1)}$ and will be called the *1-skeleton of a simplicial resolution of the group G.*

The subsequent steps depend on the choice of sets, Ω^0, $\Omega^1, \Omega^2, \ldots, \Omega^k, \ldots$. Let $F^{(k)}$ be the simplicial group constructed after k steps, that is, *the k-skeleton of the resolution*. The set Ω^k is formed by elements a of $F_k^{(k)}$ with $d_i^k(a) = 1$ for $0 \le i \le k$ and whose images $\bar{a}$ in $\pi_k(F^{(k)})$ generate that module over $F_k^{(k)}$ and $F^{(k+1)}$.

Finally we have inclusions of simplicial groups

$$F^{(0)} \subseteq F^{(1)} \subseteq \cdots \subseteq F^{(k-1)} \subseteq F^{(k)} \subseteq \cdots$$

and in passing to the inductive limit (colimit), we obtain an acyclic free simplicial group F with $F_n = F_n^{(k)}$ if $n \le k$. This F, or, more exactly, (F, g), is thus a simplicial resolution of the group G.

The proof of theorem is completed. ■

Remark: A variant of the 'step-by-step' construction gives: *if G is a simplicial group, then there exists a free simplicial group F and a continuous epimorphism $F \longrightarrow G$ which induces isomorphisms on all homotopy groups.* The details are omitted as they should be reasonably clear.

The key observation, which follows from the universal property of the construction, is a freeness statement:

Proposition 14 *Let $F^{(k)}$ be a k-skeleton of a simplicial resolution of G and $(\Omega^k, g^{(k)})$ k-dimension construction data for $F^{(k+1)}$. Suppose given a simplicial group morphism $\Theta : F^{(k)} \longrightarrow G$ such that $\Theta_*(g^{(k)}) = 0$, then Θ extends over $F^{(k+1)}$.*

This freeness statement does not contain a uniqueness clause. That can be achieved by choosing a lift for $\Theta_k g^{(k)}$ to NG_{k+1}, a lift that must exist since $\Theta_*(\pi_k(F^{(k)}))$ is trivial.

When handling combinatorially defined resolutions, rather than functorially defined ones, this proposition is as often as close to 'left adjointness' as is possible without entering the realm of homotopical algebra to an extent greater than is desirable for us here.

We have not talked here about the homotopy of simplicial group morphisms, and so will not discuss homotopy invariance of this construction for which one adapts the description given by André, [6], or Keune, [137]. Of course, the resolution one builds by any means would be homotopicallly equivalent to any other so, for cohomological purposes, it makes no difference how the resolution is built.

Of course, from any simplicial resolution F of G, you can get an augmented crossed complex $C(F)$ over G using the formula given earlier and this is a crossed resolution.

2.6 Cohomology and crossed extensions

2.6.1 Cochains

Consider a G-module, M, and a non-negative integer n. We can form the chain complex, $K(M, n)$, having M in dimension n and zeroes elsewhere. We can also form a crossed complex, $\mathsf{K}(M, n)$, that plays the role of the n^{th} Eilenberg-MacLane space of M in this setting. We may call it the n^{th} *Eilenberg-MacLane crossed complex* of M:

If $n = 0$, $\mathsf{K}(M, n)_0 = M \rtimes G$, $\mathsf{K}(M, n)_i = 0$, $i > 0$.

If $n \geq 1$, $\mathsf{K}(M, n)_0 = G$, $\mathsf{K}(M, n)_n = M$, $\mathsf{K}(M, n)_i = 0$, $i \neq 0$ or n.

One way to view cochains is as chain complex morphisms. Thus on looking at $Ch(\mathsf{B}G, K(M, n))$, one finds exactly $Z^{n+1}(G, M)$, the $(n + 1)$-cocycles of the cochain complex $C(G, M)$. We can also view $Z^{n+1}(G, M)$ as $Crs_G(\mathsf{C}G, \mathsf{K}(M, n))$.

In the category of chain complexes, one has that a homotopy from $\mathsf{B}G$ to $K(M, n)$ between 0 and f, say, is merely a coboundary, so that $H^{n+1}(G, M) \cong [\mathsf{B}G, K(M, n)]$, adopting the usual homotopical notation for the group of homotopy classes of maps from the bar resolution $\mathsf{B}G$ to $K(M, n)$. This description has its analogue in the crossed complex case as we shall see.

2.6.2 Homotopies

Let C, C' be two crossed complexes with Q and Q' respectively as the cokernels of their bottom morphism. Suppose λ, $\mu : \mathsf{C} \to \mathsf{C}'$ are two morphisms inducing the same map $\varphi : Q \to Q'$.

A homotopy from λ to μ is a family, $h = \{h_k : k \geq 1\}$, of maps $h_k : C_k \to C'_{k+1}$ satisfying the following conditions:

H1) $h_0 : C_1 \to C'_2$ is a derivation along μ_0 (i.e. for $x, y \in C_0$,

$$h_0(xy) = h_0(x)({}^{\mu_0}h_0(y)),)$$

such that
$$\delta_1 h_0(x) = \lambda_0(x)\mu_0(x)^{-1}, \quad x \in C_0.$$

H2) $h_1 : C_1 \to C_2'$ is a C_0-homomorphism with C_0 acting on C_2' via λ_0 (or via μ_0, it makes no difference) such that
$$\delta_2 h_1(x) = \mu_1(x)^{-1}(h_0\delta_1(x)^{-1}\lambda_1(x)) \text{ for } x \in C_1.$$

H3) for $k \geq 2$, h_k is a Q-homomorphism (with Q acting on the C_k' via the induced map $\varphi : Q \to Q'$) such that
$$\delta_{k+1}h_k + h_{k-1}\delta_k = \lambda_k - \mu_k.$$

We note that the condition that λ and μ induce the same map, $\varphi : Q \to Q'$, is, in fact, superfluous as this is implied by $H1$.

The properties of homotopies and the relation of homotopy are as one would expect. One finds $H^{n+1}(G, M) \cong [\mathsf{C}G, \mathsf{K}(M, n)]$. Given that in higher dimensions, this is the same set exactly as $[\mathsf{B}G, K(M, n)]$ means that there is not much to check and so the proof has been omitted.

2.6.3 Huebschmann's description of cohomology classes

The transition from this position to obtaining Huebschmann's descriptions of cohomology classes, [122], is now more or less formal. We will, therefore, only sketch the main points.

If G is a group, M is a G-module and $n \geq 1$, a *crossed n-fold extension* is an exact augmented crossed complex,
$$0 \to M \to C_n \to \ldots \to C_2 \to C_1 \to G \to 1.$$

The notion of similarity of such extensions is analogous to that of n-fold extensions in the Abelian Yoneda theory, (cf. MacLane, [149]), as is the definition of a Baer sum. We leave the details to you. This yields an Abelian group, $Opext^n(G, M)$, of similarity classes of crossed n-fold extensions of G by M.

Given a cohomology class in $H^{n+1}(G, M)$ realisable as a homotopy class of maps, $f : \mathsf{C}G \to \mathsf{K}(M, n)$, one uses f to form an induced crossed complex, much as in the Abelian Yoneda theory:

$$\begin{tikzcd}
& J_n(G) \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\text{pushout}"] & C_n \arrow[r] \arrow[d] & \ldots \arrow[r] & C_1 \arrow[r] \arrow[d] & G \arrow[d, equal] \\
0 \arrow[r] & M \arrow[r] & M_n \arrow[r] & \ldots \arrow[r] & M_1 \arrow[r] & G
\end{tikzcd}$$

where $J_n(G)$ is $Ker(C_nG \to C_{n-1}G)$. (Thus J_nG is also $Im(C_{n+1}G \to C_nG)$ and as the map f satisfies $f\delta = 0$, it is zero on the subgroup $\delta(C_{n+2}G)$ (i.e. is constant on the cosets) and hence passes to $Im(C_{n+1}G \to C_nG)$ in a well defined way.) Arguments using lifting of maps and homotopies show that the assignment of this element of $Opext^n(G, M)$ to $cls(f) \in H^{n+1}(G, M)$ establishes an isomorphism between these groups.

2.6.4 Abstract Kernels.

The importance of having such a description of classes in $H^n(G, M)$ probably resides in low dimensions. To describe classes in $H^3(G, M)$, one has, as before, crossed 2-fold extensions
$$0 \to M \to C_2 \stackrel{\partial}{\to} C_1 \to G \to 1,$$

where ∂ is a crossed module. One has for any group G, a crossed 2-fold extension

$$0 \to Z(G) \to G \stackrel{\partial_G}{\to} Aut(G) \to Out(G) \to 1$$

where ∂_G sends $g \in G$ to the corresponding inner automorphism of G. An *abstract kernel* (in the sense of Eilenberg-MacLane, [91]) is a homomorphism $\psi : Q \to Out(G)$ and hence provides, by pulling back, a 2-fold extension of Q by the centre, $Z(G)$, of G.

2.7 2-types and cohomology

In classifying homotopy types and in obstruction theory, one frequently has invariants that are elements in cohomology groups of the form $H^m(X, \pi)$, where typically π is the n^{th} homotopy group of some space. When dealing with homotopy types, π will be a group, usually Abelian with a π_1-action, i.e., we are exactly in the situation described earlier, except that X is a homotopy type not a group. Of course, provided that X is connected, we can replace X by a simplicial group, bringing us even nearer to the situation of this section. We shall work within the category of simplicial groups.

2.7.1 2-types

A morphism

$$f : G \to H$$

of simplicial groups is called a *2-equivalence* if it induces isomorphisms

$$\pi_0(f) : \pi_0(G) \to \pi_0(H,)$$

and

$$\pi_1(f) : \pi_1(G) \to \pi_1(H).$$

We can form a quotient category, $Ho_2(Simp.Grps)$, of $Simp.Grps$ by formally inverting the 2-equivalences, then we say two simplicial groups, G and H, have the same *2-type*, (or, more exactly, *homotopy 2-type*), if they are isomorphic in $Ho_2(Simp.Grps)$.

This is, of course, just a special case of the general notion of n-type in which "n-equivalences" are inverted, thus forming the quotient category $Ho_n(Simp.Grps)$.

We recall the following from earlier:

Definition: An *n-equivalence* is a morphism, f, of simplicial groups (or groupoids) inducing isomorphisms, $\pi_i(f)$, for $i = 0, 1, \ldots, n-1$.

Definition: Two simplicial groups, G and H, have the same *n-type* (or, more exactly, *homotopy n-type* if they are isomorphic in $Ho_n(Simp.Grps)$.

Sometimes it is convenient to say that a simplicial group, G, *is* an n-type. This is taken to mean that it represents an n-equivalence class *and* has zero homotopy groups above dimension $n-1$.

2.7.2 Example: 1-types

Before examining 2-types in detail, it will pay to think about 1-types. A morphism f as above is a 1-equivalence if it induces an isomorphism on π_0, i.e., $\pi_0(f)$ is an isomorphism. Given any group G, there is a simplicial group, $K(G, 0)$ consisting of G in each dimension with face and degeneracy maps all being identities. Given a simplicial group, H, having $G \cong \pi_0(H)$, the natural quotient map

$$H_0 \to \pi_0(H) \cong G,$$

extends to a natural 1-equivalence between H and $K(\pi_0(H), 0)$.

It is fairly routine to check that

$$\pi_0 : Simp.Grps \to Grps$$

has $K(-, 0)$ as an adjoint and that, as the unit is a natural 1-equivalence, and the counit an isomorphism, this adjoint pair induces an equivalence between the category $Ho_1(Simp.Grps)$ of 1-types and the category, *Grps*, of groups. In other words,

groups are algebraic models for 1-types.

2.7.3 Algebraic models for n-types?

So much for 1-types. Can one provide algebraic models for 2-types or, in general, n-types? We touched on this earlier. The criteria that any such “models” might satisfy are debatable. Perhaps ideally, or even unrealistically, there should be an isomorphism class of algebraic “gadgets” for each 2-type. An alternative weaker solution is to ask that a notion of equivalence between the models is possible, and that only equivalence classes, not isomorphism classes, correspond to 2-types, but, in addition, the notion of equivalence is algebraically defined. It is this weaker possibility that corresponds to our aim here.

2.7.4 Algebraic models for 2-types.

If G is a simplicial group, then we can form a crossed module

$$\partial : \frac{NG_1}{d_0(NG_2)} \to G_0,$$

where the action of G_0 is via the degeneracy, $s_0 : G_0 \to G_1$, and ∂ is induced by d_0. (As before we will denote this crossed module by $M(G, 1)$.) The kernel of ∂ is

$$\frac{Ker\, d_0 \cap Ker\, d_1}{d_0(NG_2)} \cong \pi_1(G),$$

whilst its cokernel is

$$\frac{G_0}{d_0(NG_1)} \cong \pi_0(G),$$

and so we have a crossed 2-fold extension

$$0 \to \pi_1(G) \to \frac{NG_1}{d_0(NG_2)} \to G_0 \to \pi_0(G) \to 1$$

and hence a cohomology class $k(G) \in H^3(\pi_0(G), \pi_1(G))$.

Suppose now that $f : G \to H$ is a morphism of simplicial groups, then one obtains a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \pi_1(G) \arrow[r] \arrow[d, "\pi_1(f)"'] & \frac{NG_1}{d_0(NG_2)} \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d, "f_0"] & \pi_0(G) \arrow[r] \arrow[d, "\pi_0(f)"] & 1 \\ 0 \arrow[r] & \pi_1(H) \arrow[r] & \frac{NH_1}{d_0(NH_2)} \arrow[r] & H_0 \arrow[r] & \pi_0(H) \arrow[r] & 1 \end{tikzcd}$$

If, therefore, f is a 2-equivalence, $\pi_0(f)$ and $\pi_1(f)$ will be isomorphisms and the diagram shows that, modulo these isomorphisms, $k(G)$ and $k(H)$ are the same cohomology class, i.e. the 2-type of G determines π_0, π_1 and this cohomology class, k in $H^3(\pi_0, \pi_1)$.

Conversely, suppose we are given a group π, a π-module, M, and a cohomology class $k \in H^3(\pi, M)$, then we can realise k by a 2-fold extension

$$0 \to M \to C \xrightarrow{\partial} G \to \pi \to 1$$

as above.

The crossed module, $\mathsf{C} = (C, G, \partial)$, determines a simplicial group $K(\mathsf{C})$ as follows:

Suppose $\mathsf{C} = (C, P, \partial)$ is any crossed module, we construct a simplicial group, $K(\mathsf{C})$, by

$$K(\mathsf{C})_0 = P, \qquad K(\mathsf{C})_1 = C \rtimes P,$$

$$s_0(p) = (1, p),\ d_0^1(c, p) = \partial c.p,\ d_1^1(c, p) = p.$$

Assuming $K(\mathsf{C})_n$ is defined and that it acts on C via the unique composed face map to $K(\mathsf{C})_0 = P$ followed by the given action of P on C, we set

$$\begin{aligned} K(\mathsf{C})_{n+1} &= C \rtimes K(\mathsf{C})_n; \\ d_0^{n+1}(c_{n+1}, \dots, c_1, p) &= (c_{n+1}, ..., c_2, \partial c_1.p); \\ d_i^{n+1}(c_{n+1}, \dots, c_{i+1}, c_i, \dots, c_1, p) &= (c_{n+1}, \dots, c_{i+1}c_i, \dots c_1, p) \\ &\qquad\qquad\qquad \text{for } 0 < i < n+1; \\ d_{n+1}^{n+1}(c_{n+1}, \dots, c_1, p) &= (c_n, \dots, c_1, p); \\ s_i^n(c_n, \dots, c_1, p) &= (c_n, \dots, 1, \dots, c_1, p), \end{aligned}$$

where the 1 is placed in the i^{th} position.

Clearly $Ker\, d_1^1 = \{(c, p) : p = 1\} \cong C$, whilst $Ker\, d_1^2 \cap Ker\, d_2^2 = \{(c_2, c_1, p) : (c_1, p) = (1, 1) \text{ and } (c_2c_1, p) = (1, 1)\} \cong \{1\}$, hence the "top term" of $M(K(\mathsf{C}), 1)$ is isomorphic to C itself, whilst $K(\mathsf{C})_0$ is P itself. The boundary map ∂ in this interpretation is the original ∂, since it maps $(c, 1)$ to $d_0(c)$, i.e., we have

Lemma 11 *There is a natural isomorphism*

$$\mathsf{C} \cong M(K(\mathsf{C}), 1).$$

■

This construction is the internal nerve of the corresponding internal category in *Grps*, as we noted earlier. All the ideas that go into defining the nerve of a category adapt to handling internal categories, and they produce simplicial objects in the corresponding ambient category. As we have a simplicial group $K(\mathsf{C})$, we might check if it is a group T-complex, but this is more or less immediate as $NK(\mathsf{C})_n = 1$ for $n \geq 2$, whilst $NK(\mathsf{C})_1$ is $\{(c,p) : p = 1\}$ and $s_0(K(\mathsf{C})_0 = \{(c,p) : c = 1\}$.

Suppose now that we had chosen an equivalent 2-fold extension

$$0 \to M \to C' \xrightarrow{d'} G' \to \pi \to 1$$

The equivalence guarantees that there is a zig-zag of maps of 2-fold extensions joining it to that considered earlier. We need only look at the case of a direct basic equivalence:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & C \arrow[r, "\partial"] \arrow[d] & G \arrow[r] \arrow[d] & \pi \arrow[r] \arrow[d, "="] & 1 \\ 0 \arrow[r] & M \arrow[r] & C' \arrow[r, "\partial'"] & G' \arrow[r] & \pi \arrow[r] & 1 \end{tikzcd}$$

giving a map of crossed modules, $\varphi : \mathsf{C} \to \mathsf{C}'$, where $\mathsf{C}' = (C', G', \partial')$. This induces a morphism of simplicial groups,

$$K(\varphi) : K(\mathsf{C}) \to K(\mathsf{C}'),$$

that is, of course, a 2-equivalence. If there is a longer zig-zag between C and C', then the intermediate crossed modules give intermediate simplicial groups and a zig-zag of 2-equivalences so that $K(\mathsf{C})$ and $K(\mathsf{C}')$ are isomorphic in $Ho_2(Simp.Grps)$, i.e. they have the same 2-type. This argument can, of course, be reversed.

If G and H have the same 2-type, they are isomorphic within the category $Ho_2(Simp.Grps)$, so they are linked in $Simp.Grps$ by a zig-zag of 2-equivalences, hence the corresponding cohomology classes in $H^3(\pi_0(G), \pi_1(G))$ are the same up to identification of $H^3(\pi_0(G), \pi_1(G))$ and $H^3(\pi_0(H), \pi_1(H))$. This proves the simplicial group analogue of the result of MacLane and Whitehead, [152], that we mentioned earlier, giving an algebraic model for 2-types of connected CW-complexes.

Theorem 4 *(MacLane and Whitehead, [152]) 2-types are classified by a group π_0, a π_0-module, π_1 and a class in $H^3(\pi_0, \pi_1)$.* ■

We have handled this in such a way so as to derive an equivalence of categories:

Proposition 15 *There is an equivalence of categories,*

$$Ho_2(Simp.Grps) \cong Ho(CMod),$$

where $Ho(CMod)$ is formed from $CMod$ by formally inverting those maps of crossed modules that induce isomorphisms on both the kernels and the cokernels. ■

2.8 Re-examining group cohomology with Abelian coefficients

2.8.1 Interpreting group cohomology

We have had

- A definition of group cohomology via the bar resolution: for a group G and a G-module, M:

$$H^n(G, M) = H^n(C(G, M))$$

together with an identification of $C(G, M)$ with maps from the classifying space / nerve, BG, of G to M, up to shifts in dimension;

- Interpretations

$$\begin{array}{rcl} H^0(G,M) & \cong & M^G, \text{ the module of invariants} \\ H^1(G,M) & \cong & Der(G,M)/Pder(G,M) \\ & - & \text{by inspection, where } Pder(G,M) \text{ is the submodule of} \\ & & \qquad \text{principal derivations;} \\ H^2(G,M) & \cong & Opext(G,M), \text{ i.e. classes of extensions} \\ & & \qquad 0 \to M \to H \to G \to 1 \end{array}$$

and we also have

$$\begin{array}{rcl} H^n(G,M) & \cong & Opext^n(G,M), n \geq 2, \text{ via crossed resolutions} \\ & \cong & [\mathsf{C}(G), \mathsf{K}(M,n)] \end{array}$$

Another interpretation, which will be looked at shortly is as $Ext^n(\mathbb{Z}, M)$, where $\mathbb{Z}$ is given the trivial G-module structure. This leads to

$$H^n(G, M) \cong Ext^{n-1}(I(G), M),$$

via the long exact sequence coming from

$$0 \to I(G) \to \mathbb{Z}[G] \to \mathbb{Z} \to 0.$$

2.8.2 The *Ext* long exact sequences

There are several different ways of examining the long exact sequence that we need. We will use fairly elementary methods rather than more 'homologically intensive' one. These latter ones are very elegant and very powerful, but do need a certain amount of development before being used. The more elementary ones have, though, a hidden advantage. The intuitions that they exploit are often related to ones that extend, at least partially, to the non-Abelian case and also to the geometric situations that will be studied later in the notes.

The idea is to explore what happens to an exact sequence of modules

$$\mathcal{E}: \quad 0 \to A \stackrel{\alpha}{\to} B \stackrel{\beta}{\to} C \to 0$$

over some given ring (we need it for G-modules so there the ring is $\mathbb{Z}[G]$, the group ring of G), when we apply the functor $Hom(-, M)$ for M another module. Of course one gets a sequence

$$Hom(\mathcal{E}, M) : 0 \to Hom(C, M) \stackrel{\beta^*}{\to} Hom(B, M) \stackrel{\alpha^*}{\to} Hom(A, M)$$

and it is easy to check that this is exact, but there is no reason why α^* should be onto since a morphism $f : A \to M$ may or may not extend to some g defined over the bigger module B. For instance, if $M = A$, and f is the identity morphism, then f extends if and only if the sequence splits (so $B \cong A \oplus C$). We examine this more closely.

We have

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] & C \arrow[r] & 0 \\ & M & & & \end{tikzcd}$$

and can form a new diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "f"'] & B \arrow[r, "\beta"] \arrow[d, "\overline{f}"] & C \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & M \arrow[r, "\overline{\alpha}"] & N \arrow[r, "\overline{\beta}"] & C \arrow[r] & 0 \end{tikzcd}$$

where the left hand square is a pushout. You should check that you see why there is an induced morphism $\overline{\beta} : N \to C$ 'emphusing the universal property of pushouts. (This is important as sometimes one wants this sort of construction, or argument, for sheaves of modules and there working with elements causes some slight difficulties.) The existence of this map is guaranteed by the universal property and does not depend on a particular construction of N. Of course this means that the bottom line is defined only up to isomorphism although we can give a very natural explicit model for N, namely it can be represented as the quotient of $B \oplus M$ by the submodule L of elements of the form $(\alpha(a), -f(a))$ for $a \in A$. Then we have $\overline{\beta}(b, m) = \beta(b)$. (Check it is well defined.) It is also useful to have the corresponding formulae for $\overline{\alpha}(m) = (0, m) + L$ and for $\overline{f}(b) = (b, 0) + L$. This gives an extension of modules

$$f^*(\mathcal{E}) : \quad 0 \to M \overset{\overline{\alpha}}{\to} N \overset{\overline{\beta}}{\to} C \to 0.$$

If f extends over B to give g, so $g\alpha = f$, then we have a morphism $g' : N \to M$ given by $g'((m, b) + L) = m + g(b)$. (Check that g' is well defined.)

Lemma 12 *f extends over B if and only if $f^*(\mathcal{E})$ is a split extension.*

Proof: We have done the 'only if'. If $f^*(\mathcal{E})$ is split, there is a projection $g' : N \to M$ such that $g'\overline{\alpha}(m) = m$ for all m. Define $g = g'\overline{f}$ to get the extension. ■

We thus get a map

$$Hom(A, M) \overset{\delta}{\to} Ext^1(C, M)$$

$$\delta(f) = [f^*(\mathcal{E})]$$

which extends the exact sequence one step to the right.

Here it is convenient to define $Ext^1(C, M)$ to be the set (actually Abelian group) of extensions of form

$$0 \to M \to ? \to C \to 0$$

modulo equivalence (isomorphism of middle terms with the ends fixed). The Abelian group structure is given by Baer sum (see entry in Wikipedia, or many standard texts on homological algebra).

Important aside: 'Recall' the '*snake lemma*: given a commutative diagram of modules with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\mu"'] & N \arrow[r] \arrow[d, "\nu"'] & P \arrow[r] \arrow[d, "\psi"'] & 0 \\
0 \arrow[r] & M' \arrow[r] & N' \arrow[r] & P' \arrow[r] & 0
\end{tikzcd}$$

there is an exact sequence

$$0 \to Ker\,\mu \to Ker\,\nu \to Ker\,\psi \xrightarrow{\delta} Coker\,\mu \to Coker\,\nu \to Coker\,\psi \to 0$$

This has as a corollary that if μ and ψ are isomorphisms then so is ν. (Do check that you can construct δ and prove exactness, i.e. using a simple diagram chase.)

Back to extensions: It is fairly easy to show that $Hom(\mathcal{E}, M)$ extends even further to 6 terms with

$$\ldots \xrightarrow{\beta^*} Ext^1(B, M) \xrightarrow{\alpha^*} Ext^1(A, M)$$

Here is how α^* is constructed. Suppose $\mathcal{E}_1 : 0 \to M \to N \to B \to 0$ gives an element of $Ext^1(B, M)$, then we can form a diagram

$$\begin{tikzcd}
\alpha^*(\mathcal{E}_1): & 0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & \alpha^{-1}(N) \arrow[r, "p'"] \arrow[d, "\alpha'"'] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\
\mathcal{E}_1: & 0 \arrow[r] & M \arrow[r] & N \arrow[r, "p"'] & B \arrow[r] & 0
\end{tikzcd}$$

by restricting $\mathcal{E}_1$ along α using a pull back in the right hand square. We can give $\alpha^{-1}(N)$ explicitly in the form that the usual construction of pullbacks in categories of modules gives it to us

$$\alpha^{-1}(N) \cong \{(a, n) \mid \alpha(a) = p(n)\}$$

and p' and α' are projections. The construction of β^* is done similarly using pullback along β. It is then easy to check that the obvious extension to $Hom(\mathcal{E}, M)$, mentioned above, is exact, but that there is again no reason why α^* should be onto. (Of course, knowledge of the purely homological way of getting these exact sequence will suggest that there is an $Ext^2(C, M)$ term to come.)

We examine an obstruction to it being so. Suppose given $\mathcal{E}' : 0 \to M \to N_1 \xrightarrow{p'} A \to 0$, giving us an element of $Ext'(A, M)$. If α^* were onto, we would need a $\mathcal{E}_1 : 0 \to M \to N \to B \to 0$ such that $\alpha^{-1}(N) \cong N_1$ leaving M fixed and relating to α as above by a pullback. We can splice $\mathcal{E}'$ and $\mathcal{E}_1$ together to get a suitable looking diagram

$$\begin{tikzcd}
\mathcal{E}' * \mathcal{E}_1: & 0 \arrow[r] & M \arrow[r] & N' \arrow[rr] \arrow[dr, "p'"'] & & B \arrow[r] & C \arrow[r] & 0 \\
& & & & A \arrow[ur, "\alpha"'] & & &
\end{tikzcd}$$

and the row is exact. If we change $\mathcal{E}'$ by an isomorphism than clearly this spliced sequence would react accordingly. If you check up, as suggested, on the Baer sum structure if $Ext^1(A, M)$ and $Ext^2(C, M)$ then you can again check that the above splicing construction yields a homomorphism from the first group to the second. Moreover there is no reason not to extend the splicing construction to a pairing operation on the whole graded family of Ext-groups. This is given in detail in quite a few of the standard books on Homological Algebra, so will not be gone into here.

Two facts we do need to have available are about the structure of $Ext^2(C, M)$. Let $\mathcal{E}xt^2(C, M)$ be the category of 4-term exact sequences

$$0 \to M \to N \to P \to C \to 0$$

and morphisms which are commuting diagrams

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "="] & 0 , \\ 0 \arrow[r] & M \arrow[r] & N' \arrow[r] & P' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

then $Ext^2(C, M)$ is the set of connected components of this category. The important thing to note is that the morphisms are not isomorphisms in general, so two 4-term sequences give the same element in $Ext^2(C, M)$ if they are linked by a zig-zag of intermediate terms of this form. The second fact is that the zero for the Baer sum addition is the class of the 4-term extension

$$0 \to M \to M \stackrel{0}{\to} C \to C \to 0$$

with 'equals' on the unmarked maps.

Suppose now that the top row in

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & N_1 \arrow[r, "\overline{p}"] \arrow[d, "\alpha'"'] & A \arrow[r] \arrow[d, "\alpha"] & 0 \\ 0 \arrow[r] & M \arrow[r] & N \arrow[r, "p"'] & B \arrow[r] & 0 \end{tikzcd}$$

is obtained by restriction along α from the bottom row. We now form the spliced sequence

$$0 \to M \to N_1 \stackrel{\alpha\overline{p}}{\to} B \to C \to 0.$$

We would hope that this 4-term sequence was trivial, i.e. equivalence to the zero one. We clearly must use the given element in $Ext^1(B, M)$ in a constructive way in the proof that it is trivial, so we form the pushout of $\alpha\overline{p}$ along α' getting us a diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & N_1 \arrow[r, "\alpha\overline{p}"] \arrow[d, "\alpha'"'] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "="] & 0 , \\ 0 \arrow[r] & M \arrow[r] & N \arrow[r] & B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

with the middle square a pushout. It is now almost immediate that the morphism from B to B' is split, since we can form a commutative square

$$\begin{tikzcd} N_1 \arrow[r, "\alpha\overline{p}"] \arrow[d, "\alpha'"'] & B \arrow[d, "="] \\ N \arrow[r, "p"'] & B \end{tikzcd}$$

giving us the required splitting from B' to B. It is now a simple use of the snake lemma, to show that the complementary summand of B in B' is isomorphic to C. We thus have that the bottom row of the diagram above is of the form

$$0 \to M \to N \to B \oplus C \to C.$$

This looks hopeful but to finish off the argument we just produce the morphism:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "="'] & M \arrow[r, "0"] \arrow[d] & C \arrow[r] \arrow[d, "incl_2"] & C \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & M \arrow[r] & N \arrow[r, "incl_1 p"'] & B \oplus C \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

and we have a sequence of maps joining our spliced sequence to the trivial one. (A similar argument goes through in higher dimensions.) Now you should try to prove that if a spliced sequence is linked to a trivial one then it does come from an induced one. That is quite tricky, so look it up in a standard text. An alternative approach is to use the homological algebra to get the trivialising element (coboundary or homotopy, depending on your viewpoint) and then to construct the extension from that. Another thing to do is to consider how the Ext-groups, $Ext^k(A, M)$, vary in M rather than with A. This will be left to you.

2.8.3 From Ext to group cohomology

If we look briefly at the classical homological algebraic method of defining $Ext^K(A, M)$, we would take a projective resolution $\mathsf{P}.$ of A, apply the functor $Hom(-, M)$, to get a cochain complex $Hom(\mathsf{P}., M)$, then take its (co)homology, with $H^n(Hom(\mathsf{P}., M))$ being isomorphic to $Ext^n(A, M)$, or, if you prefer, $Ext^n(A, M)$ being defined to be $H^n(Hom(\mathsf{P}., M))$. This method can be studied in most books on homological algebra (we cite for instance, MacLane, [149], Hilton and Stammbach, [118] and Weibel, [218]), so is easily accessible to the reader - and we will not devote much space to it here as a result. We will however summarise some points, notation, definitions of terms etc., some of which you probably know.

First the notion of projective module:

Definition: A module P is *projective* if, given any epimorphism, $f : B \to C$, the induced map $Hom(P, f) : Hom(P, B) \to Hom(P, C)$ is onto. In other words any map from P to C can be lifted to one from P to B.

Any free module is projective.

Of the properties of projectives that we will use, we will note that $Ext^n(P, M) = 0$ for P projective and for any M. To see this recall that any n-fold extension of P by M will end with an epimorphism to P, but such things split as their codomain is projective. It is now relatively easy to use this splitting to show the extension is equivalent to the trivial one.

A resolution of a module A is an augmented chain complex

$$\mathsf{P}. : \ldots \to P_1 \to P_0 \to M$$

which is exact, i.e. it has zero homology in all dimensions. This means that the augmentation induces an isomorphism between $P_0/\partial P_1$ and M. The resolution is projective if each P_n is a projective module.

If $\mathsf{P}.$ and $\mathsf{Q}.$ are both projective resolutions of A, then the cochain complexes $Hom(\mathsf{P}., M)$ and $Hom(\mathsf{Q}., M)$ always have the same homology. (Once again this is standard material from homological algebra so is left to the reader to find in the usual sources.)

An example of a projective resolution is given by the bar resolution, $\mathsf{B}G.$, and the construction $C^n(G, M)$ in the first chaper is exactly $Hom(\mathsf{B}G., M)$. This reolution ends with $BG_0 = \mathbb{Z}[G]$ and the resolution resolves the Abelian group $\mathbb{Z}$ with trivial G-module structure. (This can be seen from our discussion of homological syzygies where we had

$$\mathbb{Z}[G]^{(R)} \to \mathbb{Z}[G]^{(X)} \to \mathbb{Z}[G] \to \mathbb{Z}.$$

In fact we have

$$H^n(G, M) \cong Ext^n(\mathbb{Z}, M)$$

by the fact that $\mathsf{B}G.$ is a projective resolution of $\mathbb{Z}$ and then we can get more information using the short exact sequence

$$0 \to I(G) \to \mathbb{Z}[G] \to \mathbb{Z} \to 0.$$

As $\mathbb{Z}[G]$ is a free G-module, it is projective and the long exact sequence for $Ext(-, M)$ thus has every third term trivial (at least for $n > 0$), so

$$Ext^n(\mathbb{Z}, M) \cong Ext^{n-1}(I(G), M)$$

giving another useful interpretation of $H^n(G, M)$.

2.8.4 Exact sequences in cohomology

Of course, the identification of $H^n(G, M)$ as $Ext^n(\mathbb{Z}, M)$ means that, if

$$0 \to L \to M \to N \to 0$$

is an exact sequence of G-modules, we will get a long exact sequence in $H^n(G, -)$, just by looking at the long exact sequence for $Ext^n(\mathbb{Z}, -)$.

What is more interesting - but much more difficult - is to study the way that $H^n(G, M)$ varies as G changes. For a start it is not completely clear what this means! If we change the group in a short exact sequence,t

$$1 \to G \to H \to K \to 1$$

say, then what type of modules should be used fro the 'coefficients', that is to say a G-modules or one over H or K. This problem is, of course, related to the change of groups along an arbitrary homomorphism, so we will look at an group homomorphism $\varphi : G \to H$, with no assumptions as to monomorphism, or normal inclusion, at least to start with.

Suppose given such a φ, then the 'restriction functor' is

$$\varphi^* : H{-}Mod \to G{-}Mod,$$

where, if N is in $H{-}Mod$, $\varphi^*(N)$ has the same underlying Abelian group structure as N, but is a G-module via the action, $g.n := \varphi(g).n$. We have already used that φ^* has a left adjoint φ_* given by $\varphi_*(M) = \mathbb{Z}H \otimes_{\mathbb{Z}G} M$. Now we also need a right adjoint for φ^*.

To construct such an adjoint, we use the old device of assuming that it exists, studying it and then extracting a construction from that study. We have M in $G{-}Mod$ and N in $H{-}Mod$, and we assume a natural isomorphism

$$G{-}Mod(\varphi^*(N), M) \cong H{-}Mod(N, \varphi_\sharp(M)).$$

If we take $N = \mathbb{Z}H$, then, as $H{-}Mod(\mathbb{Z}H, \varphi_\sharp(M)) \cong \varphi_\sharp(M)$, we have a construction of $\varphi_\sharp(M)$, at least as an Abelian group. In fact this gives

$$\varphi_\sharp(M) \cong G{-}Mod(\varphi^*(\mathbb{Z}H), M)$$

and as $\mathbb{Z}H$ is also a right G-module, via $h.g := h.\varphi(g)$, we have a left G-module structure of $\varphi_\sharp(M)$ as expected. In fact, this is immediate from the naturality of the adjunction isomorphism using the left hand position of $G{-}Mod(\varphi^*(\mathbb{Z}H), M)$, as for fixed M, the functor converts the right G-action of $\mathbb{Z}$ to a left one on $\varphi_\sharp(M)$. This allows us to get an explicit elementwise formula for this action as follows: let $m^* : \mathbb{Z}H \to M$ be a left G-module mrphsim This can be specified by what it does to the natural basis of $\mathbb{Z}H$ (as Abelian group), and so is often written $m^* : H \to M$, where the function m^* must satisfy a G-equivariance property: $m^*(\varphi(g).h) = g.m^*(h)$. Any such function can, of course, be extended linearly to a G-module morphism of the earlier form. If $g \in G$, we get a morphism

$$-.\varphi(g) : \varphi^*(\mathbb{Z}H) \to \varphi^*(\mathbb{Z}H)$$

given by 'h goes to $h\varphi(g)$'. This *is* a G-module morphism as the G-module structure is by *left* multiplication, which is independent of this right multiplication. Applying $G{-}Mod(-, M)$, we get $g.m^*$ is given by

$$g.m^*(h) - m^*(h.\varphi(g).$$

This is a *left* G-module structure, although at first that may seem strange. That it is linear is easy to check. What take a little bit of work is to check $(g_1g_2).m^* = g_1(g_2.m^*)$: applying both sides to an element $h \in H$ gives

$$(g_1g_2).m^*(h) = m^*(h\varphi(g_1)\varphi(g_2)),$$

whilst

$$g_1(g_2.m^*)(h) = (g_2.m^*)(h.\varphi(g_1)) = m^*(h\varphi(g_1)\varphi(g_2)).$$

(The checking that $g_1.m^*$ does satisfy the G-equivariance property is left to the reader.)

Remark: There are great similarities between the above calculations and those needed later when examining bitorsors. This is certainly not coincidental!

We built $\varphi_\sharp(M)$ in such a way that it is obviously functorial in M and gives a right adjoint to φ^*. This implies that there is a natural morphism

$$i : N \to \varphi_\sharp\varphi^*(N).$$

We denote this second module by N^*, when the context removes any ambiguity, and especially when φ is the inclusion of a subgroup. The morphism sends n to $n^* : H \to N$, where $n^*(h) = h.n$. (Check that $n^*(\varphi(g).h) = g.n^*(h)$. This reminds us that the codomain of n^* is infact just the *set* N underlying both the H-module N and the G-module $\varphi^*(N)$.)

We examine the cohomology groups $H^n(H, N^*)$. These are the (co)homology groups of the cochain complex $Hom(\mathsf{P}., N^*)$, where $\mathsf{P}.$ is a projective H-module resolution of $\mathbb{Z}$. The adjunction shows that this is isomorphic to $Hom(\varphi^*(\mathsf{P}.), \varphi^*(N))$. If $\varphi^*(\mathsf{P}.)$ is a projective G-module resolution of the trivial G-module $\mathbb{Z}$ then the cohomology of this complex will be $H^n(G, N)$, where N has the structure $\varphi^*(N)$.

The condition that free or projective H modules restrict to free or projective G-modules is satisfied in one important case, namely when G is a subgroup of H, since $\mathbb{Z}H$ is a free Abelian group on the *set* H and H is a disjoint union of right G-cosets, so $\mathbb{Z}H$ splits as a G-module into a direct sum of copies of $\mathbb{Z}G$. This provides part of the proof of Shapiro's lemma

Proposition 16 *If $\varphi : G \to H$ is an inclusion, then for a H-module N, there is a natural isomorphism*

$$H^n(H, N^*) \cong H^n(G, N).$$

■

Corollary 4 *The morphism $i : N \to N^*$ and the above isomorphism yield the restriction morphism*

$$H^n(H, N) \to H^n(G, N).$$

■

This suggest other results. Suppose we have an extension

$$1 \to N \to G \to Q \to 1$$

(so here we replace H by G with N in the old role of G, but in addition, being normal in G).

If we look at $\mathsf{B}N$ and $\mathsf{B}G$ in dimension n, these are free modules over the sets N^n and G^n respectively, with the inclusion between them; G is a disjoint union of N-cosets, indexed by elements of Q, so can we use this to derive properties of the cokernel of $\mathbb{Z}G \otimes_{\mathbb{Z}N} \mathsf{B}N \to \mathsf{B}G$, and to tie them into some resolution of Q, or perhaps, of $\mathbb{Z}$ as a trivial Q-module. The answer must clearly be positive, perhaps with some restrictions such as finiteness, but there are several possible ways of getting to an answer having slightly different results. (You have in the (φ_*, φ^*) and $(\varphi^*, \varphi_\sharp)$ adjunctions, enough of the tools needed to read detailed accounts in the literature, so we will not give them here.)

This also leads to relative cohomology groups and their relationship with the cohomology of the quotient Q. We can also consider the crossed resolutions of the various groups in the extension and work, say, with the induced maps

$$\mathsf{C}(N) \to \mathsf{C}(C)$$

looking at its cokernel or better what should be called its homotopy cokernel.

Another possibility is to examine $\mathsf{C}(N)$ and $\mathsf{C}(Q)$ and the cocycle information needed to specify the extension, and to use all this to try to construct a crossed resolution of G. (We will see something related to this in our examination of non-Abelian cohomology a little later.) A simple case of this is when the extension is split, $G \cong N \rtimes Q$ and using a twisted tensor product for crossed complexes, one can produce a suitable $\mathsf{C}(N) \otimes_\tau \mathsf{C}(Q)$ resolving G, (see Tonks, [208]).

Chapter 3

Syzygies, and higher generation by subgroups

Syzygies are one of the routes to working with resolutions. They often provide insight as to how a presentation relates to geometric aspects of a group, for instance giving structured spaces such as simplicial complexes, or, better, polytopes, on which the group acts. Syzygies extend the role of 'relations' in group presentations to higher dimensions and hence are 'relations between relations ... between relations'. They thus form a very well structured (and thus simpler) case of higher dimensional rewriting. Later we will see relations between this and several important aspects of cohomology. We will also explore some links with ideas from rewriting theory.

3.1 Back to syzygies

There are both homotopical and homological syzygies. We have met homological syzygies earlier and also have:

3.1.1 Homotopical syzygies

We have built a complex, $K(\mathcal{P})$, from a presentation, $\mathcal{P}$, of a group, G. Any element in $\pi_2(K(\mathcal{P}))$ can, of course, be represented by a map from S^2 to $K(\mathcal{P})$ and, by cellular approximation, can be replaced, up to homotopy, by a cellular decomposition of S^2 and a cellular map $\phi : S^2 \to K(\mathcal{P})$. We will adopt the terminology of Kapranov and Saito, [132], and Loday, [144], and say

Definition: A *homotopical 2-syzygy* consists of a cellular subdivision of S^2 together with a map, $\phi : S^2 \to K(\mathcal{P})$, cellular for that decomposition..

Of course, such an object corresponds to an identity among the relations of $\mathcal{P}$, but is a *specific representative* of such an identity. The specification of the cellular decomposition provides valuable combinatorial and geometric information on the presentation.

Definition: A family, $\{\phi_\lambda\}_{\lambda\in\Lambda}$, of such homotopical 2-syzygies is then called *complete* when the homotopy classes $\{[\phi_\lambda]\}_{\lambda\in\Lambda}$ generate $\pi_2(K(\mathcal{P}))$.

In this case, we can use the ϕ_λ to form the next stage of the construction of an Eilenberg-Mac Lane space, $K(G,1)$, by killing this π_2. More exactly, rename $K(\mathcal{P})$ as $X(2)$ and form

$$X(3) := X(2) \cup \bigcup_{\lambda\in\Lambda} e^3_\lambda,$$

by, for each $\lambda \in \Lambda$, attaching a 3-cell, e^3_λ, to $X(2)$ using ϕ_λ. Of course, we then have

$$\pi_1(X(3)) \cong G, \quad \pi_2(X(3)) = 0.$$

Again $\pi_3(X(3))$ may be non-trivial, so we consider *homotopical 3-syzygies.* Such a thing, s, will consist of an oriented polytope decomposition of S^3 together with a continuous map, f_s from S^3 to $X(3)$, which sends the i-skeleton of that decomposition to $X(i)$, $i = 0,1,2$.

At this stage we have $X(0) = K(\mathcal{P})_0$, a point, $X(1) = K(\mathcal{P})_1$, and $X(2) = K(\mathcal{P})_2$. One wants enough such 3-syzygies, s, identified algebraically and combinatorially, so that the corresponding homotopy classes, $\{[f_s]\}$ generate $\pi_3(X(3))$.

It is clear, by induction, we get a notion of *homotopical n-syzygy.* We assume $X(n)$ has been built inductively by attaching cells of dimension $\leq n$ along homotopical k-syzygies for $k < n$, so that

$$\pi_1(X(n)) \cong G, \quad \pi_k(X(n)) = 0, \quad k = 2,\dots,n-1,$$

then a *homotopical n-syzygy*, s, is an oriented polytope decomposition of S^n and a continuous cellular map $f_s : S^n \to X(n)$.

After a choice of a set, $\mathcal{R}_n$, of n-syzygies, so that $\{[s_s] \mid s \in \mathcal{R}_n\}$ generates $\pi_n(X(n))$ as a G-module, we can form $X(n+1)$ by attaching $n+1$-dimensional cells e^{n+1}_s along these f_s for $s \in \mathcal{R}_n$.

If we can do this in a sensible way, for all n, we say the resulting system of syzygies is *complete* and the limit space $X(\infty) = \bigcup X(n)$ is then a cellular model for BG, the classifying space of the group G. We will look at classifying spaces again later.

This construction is, of course, just a homotopical version of the construction of a free resolution of the trivial G-module, $\mathbb{Z}$.

Remark: Some additional aspects of this can be found in Loday's paper [144], in particular the link with the 'pictures' of Igusa, [125, 126].

Example and construction: Given any group, G, we can find a presentation with $\{\langle g\rangle \mid g \neq 1, g \in G\}$ as set of generators and a relation, $r_{g,g'} := \langle g\rangle\langle g'\rangle\langle g'g\rangle^{-1}$, for each pair (g,g') of elements of G. (We write $\langle 1\rangle = 1$ for convenience.) We will have earlier call this the *standard presentation* of the group, G. It is closely related to the nerve of $G[1]$, and also to the various bar resolutions. (There may be a need later to consider a variant in which the identity element of G is not excluded as a generator, however that will still be loosely called the standard presentation. Note that since then $\langle 1\rangle.\langle g\rangle = \langle 1.g\rangle = \langle g\rangle$, the identification $\langle 1\rangle = 1$ is automatic.)

The relation $r_{g,g'}$ gives a triangle

$$\begin{tikzcd}
& \cdot \arrow[dr, "g'"] & \\
\cdot \arrow[ur, "g"] \arrow[rr, "g'g"'] & & \cdot
\end{tikzcd}$$

and, for each triple (g, g', g''), we get a homotopical 2-syzygy in the form of a tetrahedron.

Higher homotopical syzygies occur for any tuple, $(g_1, \ldots, g_n)$, of non-identity elements of G, by labelling a n-simplex. The limiting cellular space, $X(\infty)$, constructed from this context is just the usual model of the classifying space, BG, as geometric realisation of the *nerve* of G, or if you prefer, of the groupoid $G[1]$ with one object. The corresponding free resolution, $(C_*(G), d)$, is the classical *normalised bar resolution*. Using the bar resolution above dimension 2 together with the crossed module of the presentation at the base, one gets the standard free crossed resolution of the group, G, as we saw in section 2.1.2.

3.1.2 Syzygies for the Steinberg group

(This is adapted from Kapranov and Saito, [132].)

Let R be an associative ring with 1. Recall that the (n^{th} unstable) *Steinberg group*, $St_n(R)$, has generators, $x_{ij}(a)$, labelling the elementary matrices $\varepsilon_{ij}(a)$, having

$$\varepsilon_{ij}(a)_{k,l} = \begin{cases} 1 & \text{if } k = l \\ a & \text{if } (k,l) = (i,j), a \in R \\ 0 & \text{otherwise,} \end{cases}$$

and relations

St1 $\quad x_{i,j}(a)x_{i,j}(b) = x_{i,j}(a+b)$;

St2 $\quad [x_{i,j}(a), x_{k,\ell}(b)] = \begin{cases} 1 & \text{if } i \neq \ell, j \neq k, \\ x_{i,\ell}(ab) & i \neq \ell, j = k \end{cases}$ and in which all indices are positive integers less than or equal to n.

The terminology 'n^{th} unstable' is to make the contrast with the group $St(R)$, the stable version. The unstable version, $St_n(R)$, models 'universal' relations satisfied by the $n \times n$ elementary matrices, whilst, in $St(R)$, the indices, i, j, k etc. are not constrained to be less than or equal to n. We will look at the stable version later.

The identities / homotopical 2-syzygies are built from three types of polygon:

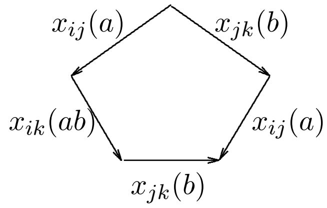
a) a triangle, $T_{ij}(a, b)$ for each i, j, $i \neq j$, coming from St1;

b) a square,

$$\begin{tikzcd} \cdot \arrow[r, "x_{ij}(a)"] \arrow[d, "x_{kl}(b)"'] & \cdot \arrow[d, "x_{kl}(b)"] \\ \cdot \arrow[r, "x_{ij}(a)"'] & \cdot \end{tikzcd}$$

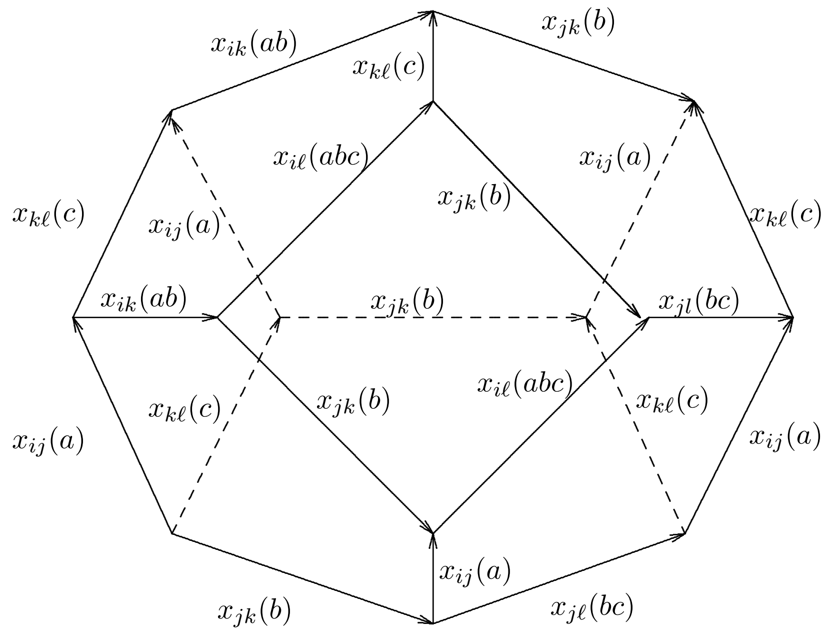
corresponding to the first case of St2 and

c) a pentagon, for the second:



Then for any pairs (i, j), (k, l), (m, p) with $x_{ij}(a)$, $x_{kl}(b)$, $x_{mp}(c)$, commuting by virtue of St2's first clause, we will have a homotopical syzygy in the form of a labelled cube.

There is also a homotopy 2-syzygy given by the associahedron labelled by generators as shown:



Remark: Kapranov and Saito, [132], have conjectured that the space $X(\infty)$ obtained by gluing labelled higher Stasheff polytopes together, is homotopically equivalent to the homotopy fibre of

$$f : BSt(R) \to BSt(A)^+,$$

where $(-)^+$ denotes Quillen's plus construction. The associahedron is a Stasheff polytope and, by encoding the data that goes to build the identities / syzygies schematically in a 'hieroglyph', Kapranov and Saito make a link between such hieroglyphs and polytopes.

3.2 A brief sideways glance: simple homotopy and algebraic K-theory

The study of the Steinberg group is closely bound up with the development of algebraic K-theory. That subject grew out of two apparently unrelated areas of algebraic geometry and algebraic topology. The second of these, historically, was the development by Grothendieck of (geometric and topological) K-theory based on projective modules over a ring, or finite dimensional vector bundles on a space. (The connection between these is that the space of global sections of a finite dimensional vector bundle on a nice enough space, X, is a finitely generated projective module over the ring of continuous real or complex functions on X. We will look at vector bundles and this link with K-theory a bit more in detail later on; see section 9.2. We will be discussing other forms of K-theory in that section as well, so will not give more detail on that more purely topological side of the subject here.)

Algebraic K-theory was initially a body of theory that attempted to generalise parts of linear algebra, notably the theory of dimension of vector spaces, and determinants to modules over arbitrary rings. It has grown into a well developed tool for studying a wide range of algebraic, geometric and even analytic situations from a variety of points of view.

For the purposes here we will give a short description of the low dimensional K-groups of a ring, R, with for initial aim to provide examples for use with the further discussion of rewriting, group presentations, syzygies, and homotopy. The discussion will, however, also look a bit more deeply at various other aspects when they seem to fit well into the overall structure of the notes.

3.2.1 Grothendieck's $K_0(R)$

For our discussion here, it will suffice to say that, given an associative ring, R, we can form the set, $[Proj_{fg}(R)]$ of isomorphism classes of finitely generated projective modules over R. Direct sum gives this a monoid structure. This is then 'completed' to get an Abelian group. We will give a more detailed discussion of this later in Proposition 102, but here we will just give the formula:

$$K_0(R) := F([Proj_{fg}(R)])/\langle [P] + [Q] - [P \oplus Q]\rangle$$

in which P and Q are finitely generated projective modules, F is the free Abelian group functor and $[P]$ indicates the isomorphism class of P. The relations force the abstract addition in the free Abelian group to mirror the direct sum induced addition on the generators.

3.2.2 Simple homotopy theory

The other area that led to algebraic K-theory was that of simple homotopy theory. J. H. C. Whitehead, following on from earlier ideas of Reidemeister, looked at possible extensions of combinatorial group theory, with its study of presentations of groups, to give a combinatorial homotopy theory; see [222]. This would take the form of an 'algebraic homotopy theory' giving good algebraic models for homotopy types, and would hopefully ease the determination of homotopy equivalences for instance of polyhedra. The 'combinatorial' part was exemplified by his two papers on 'Combinatorial Homotopy Theory' [220, 221], but raised an interesting question. In combinatorial group theory, a major role is played by Tietze's theorem:

Theorem 5 *(Tietze's theorem, 1908, [204]) Given two finite presentations of the same group, one can be obtained from the other by a finite sequence of Tietze transformations.* ∎

Proofs of this are easy to find in the literature. For instance, one based on a series of exercises is given in Gilbert and Porter, [103], p.135.

We clearly need to make precise what are the Tietze transformations.

Let $\mathcal{P} = (X : R)$ be a group presentation of a group, G and set $F(X)$ to be the free group on the set X. We consider the following transformations:

T1: Adding a superfluous relation: $(X : R)$ becomes $(X : R')$, where $R' = R \cup \{r\}$ and $r \in N(R)$, the normal closure of the relations in the free group on X, i.e., r is a consequence of R;

T2: Removing a superfluous relation: $(X : R)$ becomes $(X : R')$ where $R' = R - \{r\}$, and r is a consequence of R';

T3: Adding a superfluous generator: $(X : R)$ becomes $(X' : R')$, where $X' = X \cup \{g\}$, g being a new symbol not in X, and $R' = R \cup \{wg^{-1}\}$, where w is a word in the other generators, that is w is in the image of the inclusion of $F(X)$ into $F(X')$;

T4: Removing a superfluous generator: $(X : R)$ becomes $(X' : R')$, where $X' = X - \{g\}$, and $R' = R - \{wg^{-1}\}$ with $w \in F(X')$ and $wg^{-1} \in R$ and no other members of R' involve g.

Definition: These transformations are called *Tietze transformations.*

The question was to ask if there was a higher dimensional version of the Tietze transformations that would somehow generate all *homotopy equivalences.*

Let us imagine the transformation of the complex, $K(\mathcal{P})$, of $\mathcal{P}$ under these moves. The complex is, of course, a simple form of CW-complex, built by attaching cells in dimensions 1 and then 2. If we add a superfluous generator to $\mathcal{P}$ as above (T3), then effectively we add a 2-cell labelled by wg^{-1} and it will be glued on by an attaching map that is defined on a semi-circle in its boundary and on which the path represents the word, w. The other semi-circle yields the loop representing the new generator. This process therefore does not change the homotopy type of $K(\mathcal{P})$. On the other hand, adding a superfluous relation will change the homotopy type of the complex. The new relation corresponds to a 2-cell glued on to $K(\mathcal{P})$, but the attaching map is already null-homotopic in $K(\mathcal{P})$ as it represents a consequence of the relations. The effect is that $K(\mathcal{P}')$ has the homotopy type of $K(\mathcal{P}) \vee S^2$, and the module of identities has an extra free summand.

These thus show both types of behaviour when attaching a cell to a pre-existing complex. In the first, the relation 2-cell is attached by part of its boundary. In the second the new cell is attached by gluing along *all* of its boundary, so will change the homotopy type of $K(\mathcal{P})$. It will not change its fundamental group, just its higher homotopy groups. This raises and interesting question, and that is to mirror these Tietze transformations by higher order ones which do not change the n-type, for some n, but may change the whole homotopy type, but we need to get back towards simple homotopy theory.

Tietze transformations had given a way of manipulating presentations and thus suggested a way of manipulating complexes. The thought behind simple homotopy theory was to produce a way of constructing homotopy equivalences between complexes. This, if it worked, might simplify the task of determining whether two spaces (defined, say, as simplicial complexes) were of the same homotopy type, and if so was it possible to build up the homotopy equivalences between them in some simple way.

The resulting theory was developed initially by Reidemeister and then by Whitehead, culminating in his 1950 paper, [223]. The theory received a further important stimulus with Milnor's classic paper, [159], in which the emphasis was put on elementary expansions.

(A good source for the theory of simple homotopy is Cohen's book, [65].)

We will work here with finite CW-complexes. These are built up by induction by gluing on n-cells, that is copies of $D^n = \{x \in \mathbb{R}^n \mid \sum x_i^2 \leq 1\}$, at each stage. Each D^n has a boundary an $(n-1)$-sphere, $S^{n-1} = \{x \in \mathbb{R}^n \mid \sum x_i^2 = 1\}$. The construction of objects in the category of finite CW-complexes is by attaching cells by means of maps defined on part of all of the boundary of a cell. This will usually change the homotopy type of the space, creating or filling in a 'hole'. The homotopy type will not be changed if the attaching map has domain a hemisphere. We write $S^{n-1} = D^{n-1}_- \cup D^{n-1}_+$, with each hemisphere homeomorphic to a $(n-1)$-cell, and their intersection being the equatorial $(n-2)$-sphere, S^{n-2}, of S^{n-1}.

Given, now, a finite CW-complex, X, we can build a new complex Y, consisting of X and two new cells, e^n and e^{n-1} together with a continuous map, $\varphi : D^n \to Y$ satisfying
(i) $\varphi(D^{n-1}_+) \subseteq X_{n-1}$;
(ii) $\varphi(S^{n-2}) \subseteq X_{n-2}$;
(iii) the restriction of φ to the interior of D^n is a homeomorphism onto e^n;
and
(iv) the restriction of φ to the interior of D^{n-1}_- is a homeomorphism onto e^{n-1}.

There is an obvious inclusion map, $i : X \to Y$, which is called an *elementary expansion*. There is also a retraction map $r : Y \to X$, homotopy inverse to i, and which is called an *elementary contraction*. Both are homotopy equivalences. Can all homotopy equivalences between finite CW-complexes be built by composing such elementary ones? More precisely if we have a homotopy equivalence $f : X \to X'$, is f homotopic to a composite of a finite sequence of elementary expansions and contractions? Such a homotopy equivalence would be called *simple*. Whitehead showed that not all homotopy equivalences are simple and constructed a group of obstructions for the problem with given space X, each non-identity element of the group corresponding to a distinct homotopy class of non-simple homotopy equivalences.

3.2.3 The Whitehead group and $K_1(R)$

We will very briefly sketch how the investigation goes, skimming over the details; for them, see Milnor, [159], or Cohen's book, [65].

Starting with a homotopy equivalence, $f : X \to Y$, we can convert it to a deformation retraction using the mapping cylinder construction. (We will see this in more detail later, but do not need that detail here). This means that we have a CW-pair, (Y, X), with a deformation retraction from Y to X. Classifying the simple homotopy types of X is then transformed into a problem of classifying these. Passing first to their universal covering spaces, $\tilde{Y}$ and $\tilde{X}$, and then to the cellular chain complexes associated to both these, the problem is reduced to examining the *relative* cellular chain complex, $C(\tilde{Y}, \tilde{X})$, obtained from the exact sequence

$$0 \to C(\tilde{X}) \to C(\tilde{Y}) \to C(\tilde{Y}, \tilde{X}) \to 0$$

All of these can be considered as chain complexes of modules over the group ring of $\pi_1 X$. As there are only finitely many cells in X and Y, this chain complex has only finitely many non-zero levels in it. It is also acyclic, i.e., has zero homology because the inclusion of $C(\tilde{X})$ into $C(\tilde{Y})$ induces isomorphism on homology. The cells in $Y - X$ give a preferred basis to the modules concerned.

One further reduction takes the direct sum of the even dimensional $C(\tilde{Y}, \tilde{X})_n$, and similarly that of the odd ones, and the induced boundary from the odds to the evens. (At each stage the reduction is checked to preserve what one want, namely whether or not the inclusion of X into Y is given by some combinations of elementary expansions and contractions. (The last part of this can be examined intuitively by thinking about what happens if you add in an n-cell by a $n-1$-cell in its boundary.)

This reduces the task to one of examining an isomorphism between two based free modules over $\mathbb{Z}\pi_1 X$, and that brings us, finally, to the main point of this section namely the definition of the group $K_1(R)$. (For this original application to simple homotopy theory, one takes $R = \mathbb{Z}\pi_1 X$.)

We will not take a historical order, concentrating on K_1, which was extracted from Whitehead's work, and studied for its own sake by Bass, [20]. Other aspects relating to simple homotopy theory may be looked at later on when we have more tools available.

Let R be an associative ring with 1. As usual $G\ell_n(R)$ will denote the *general linear group* of $n \times n$ non-singular matrices over R. There is an embedding of $G\ell_n(R)$ into $G\ell_{n+1}(R)$ sending a matrix $M = (m_{i,j})$ to the matrix M' obtained from M by adding an extra row and columnof zeros except that $m'_{n+1,n+1} = 1$. This gives a nested sequence of groups

$$G\ell_1(R) \subset G\ell_2(R) \subset \ldots \subset G\ell_n(R) \subset G\ell_{n+1}(R) \subset \ldots$$

and we write $G\ell(R)$ for the colimit (union) of these. It will be called the *stable general linear group over R*

Definition: The group, $K_1(R)$, is $G\ell(R)^{Ab} = G\ell(R)/[G\ell(R), G\ell(R)]$.

This is functorial in R, so that a ring homomorphism, $\varphi : R \to S$ induces $K_1(\varphi) : K_1(R) \to K_1(S)$.

The main initial problem with the above definition of $K_1(R)$ is that of controlling the commutator subgroup of $G\ell(R)$. The key is the *stable elementary linear group*, $E(R)$.

We extend the earlier definition of elementary matrices (on page 93 from the finite dimensional case, i.e., within $G\ell_n(R)$, to being within $G\ell(R)$. Here an *elementary matrix* is of the form $e_{ij}(a) \in G\ell(R)$, for some pair (i, j) of distinct positive integers and which, thus, has an a in the (i, j) position, 1s in every diagonal position and 0 elsewhere. Although there is a small risk of confusion from notational reuse, we will, none-the-less, follow the standard notational convention and write $E_n(R)$ for the subgroup generated by the elementary matrices in $G\ell_n(R)$ and $E(R)$ for the corresponding union of the $E_n(R)$ within $G\ell(R)$. We will call $E_n(R)$ the *elementary subgroup of $G\ell_n(R)$*,

Lemma 13 *If i, j, k are distinct positive integers, then*

$$e_{ij}(a) = [e_{ik}(a), e_{kj}(1)].$$

■

This was already commented on when looking at the Steinberg group, $St_n(R)$, which abstracts the 'generic' properties of the elementary matrices. The following is now obvious.

Proposition 17 *For $n \geq 3$, $E_n(R)$ is a perfect group, i.e.,*

$$[E_n(R), E_n(R)] = E_n(R).$$

■

Now let $M = (m_{ij})$ be any $n \times n$ matrix over R. (It is not assumed to be invertible.)

We note that in $G\ell_{2n}(R)$,

$$\begin{pmatrix} I_n & M \\ 0 & I_n \end{pmatrix} = \prod_{i=1}^{n}\prod_{j=1}^{n} e_{i,j+n}(m_{ij}),$$

so this is in $E_{2n}(R)$. Similarly $\begin{pmatrix} I_n & 0 \\ M & I_n \end{pmatrix} \in E_{2n}(R)$.

Next, let $M \in G\ell_n(R)$ and note

$$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ M^{-1} - I_n & I_n \end{pmatrix}\begin{pmatrix} I_n & I_n \\ 0 & I_n \end{pmatrix}\begin{pmatrix} I_n & 0 \\ M - I_n & I_n \end{pmatrix}\begin{pmatrix} I_n & -M^{-1} \\ 0 & I_n \end{pmatrix}$$

(as is easily verified). We thus have

$$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix} \in E_{2n}(R),$$

hence it is a product of commutators.

Lemma 14 *If $M, N \in G\ell_n(R)$, then*

$$\begin{pmatrix} [M,N] & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} M & 0 \\ 0 & M^{-1} \end{pmatrix}\begin{pmatrix} N & 0 \\ 0 & N^{-1} \end{pmatrix}\begin{pmatrix} (NM)^{-1} & 0 \\ 0 & NM \end{pmatrix},$$

so is in $E_{2n}(R)$.

Proof: Just calculation. ■

Passing to the stable groups, we get the famous *Whitehead lemma*:

Proposition 18

$$[G\ell(R), G\ell(R)] = E(R).$$

■

This was, thus, very easy to prove, but it is crucial for the development of algebraic K-theory. It should be noted that it did depend on having ‘enough dimensions’, so $[G\ell_n(R), G\ell_n(R)] \subseteq E_{2n}(R)$. For our purposes here, we do not need to question whether ‘unstable’ versions of this hold, however we will mention that, if $n \geq 3$ and R is a commutative ring, then $[G\ell_n(R), G\ell_n(R)] = E_n(R)$. The proof is given in many texts on algebraic K-theory.

3.2.4 Milnor’s K_2

We have already met the definition of $K_2(R)$ (page **??**). The stable elementary linear group, $E(R)$, is a quotient of the stable Steinberg group, $St(R)$. (It will help to glance back at the presentation given on page 77 and to check that these are ‘generic’ relationships between elementary matrices.) This stable Steinberg group is obtained from the various $St_n(R)$ together with the inclusions $St_n(R) \to ST_{n+1}(R)$ obtained by including the generators of the first into the generating set of the second in the obvious way. the colimit of these 'unstable' groups yields the *stable Steinberg group*

As we mentioned early and will prove shortly, there is a central extension:

$$1 \to K_2(R) \to St(R) \xrightarrow{\varphi} E(R) \to 1$$

and thus $\varphi : St(R) \to E(R)$, a crossed module. The group, $G\ell(R)/Im(b)$, is $K_1(R)$, the first algebraic K-group of the ring.

In fact, this is a universal central extension and certain observations about such objects will help interpret what information is contained in $K_2(R)$. We will 'backtrack' a bit so as to keep things relatively self-contained.

Let, as usual, $Z(G)$ denote the centre of a group G.

Lemma 15 *(i) $Z(E(R)) = 1$;*
(ii) $Z(St(R)) = K_2(R)$.

Proof: This is elementary, but fun!

Suppose that $N \in Z(E(R))$, then $N \in E_n(R)$ for some n. Within $E_{2n}(R)$,

$$\begin{pmatrix} N & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} I & I \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}\begin{pmatrix} N & 0 \\ 0 & I \end{pmatrix},$$

since N is central in $E(R)$. This works out as

$$\begin{pmatrix} N & N \\ 0 & I \end{pmatrix} = \begin{pmatrix} N & I \\ 0 & I \end{pmatrix},$$

i.e., $N = I$.

Next suppose that $M \in Z(St(R))$, then, as φ is surjective, $\varphi(M) \in Z(E(R))$, so must be trivial, as required. ■

Proposition 19

$$1 \to K_2(R) \to St(R) \xrightarrow{\varphi} E(R) \to 1$$

is a central extension. ■

We next need to examine *universal* central extensions.

Definitions: (i) A central extension

$$1 \to K \xrightarrow{k} H \xrightarrow{\sigma} G \to 1$$

is said to be *weakly universal* if, given any other central extension of G,

$$1 \to L \xrightarrow{k'} E \xrightarrow{\sigma'} G \to 1,$$

there is a homomorphism $\psi : H \to E$ making the diagram

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r, "k"] \arrow[d, "\psi|_K"'] & H \arrow[r, "\sigma"] \arrow[d, "\varphi"'] & G \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & L \arrow[r, "k'"'] & E \arrow[r, "\sigma'"'] & G \arrow[r] & 1
\end{tikzcd}$$

commutes.

(ii) The central extension, as above, of G is *universal* if it is weakly universal and, in the previous definition, the morphism ψ is unique with that property.

Proposition 20 *Every group has a weakly universal central extension.*

Proof: Suppose that we have a presentation $(X : R)$ of G, or more usefully for us, a presentation sequence:

$$1 \to K \xrightarrow{k} F \xrightarrow{p} G \to 1,$$

(so $F = F(X)$, the free group on X, and $K = N(R)$ is the kernel of p). The subgroup, $[K, F]$. of F generated by the commutators, $[k(x), y]$, with $x \in K$, and $y \in F$, is normal , as is easily checked and is in K, so we can form an extension

$$1 \to \frac{K}{[K, F]} \to \frac{F}{[K, F]} \to G \to 1.$$

(Note that 'dividing out by this subgroup identifies all $k(x)y$ and $yk(x)$, so should make a central extension. It 'kills' the conjugation action of F on K.)

We will write $H = F/[K, F]$ with $\sigma : H \to G$ for the induced epimorphism, so we now have

$$\mathbb{E} : 1 \to Ker\,\sigma \to H \xrightarrow{\sigma} G \to 1.$$

This is a central extension, as is easily checked (**left to you**).

Now suppose

$$\mathbb{E}' : 1 \to L \xrightarrow{k} E \xrightarrow{\sigma'} G \to 1$$

is another central extension. We have to construct a morphism, $\psi : \mathbb{E} \to \mathbb{E}'$, i.e., $\varphi : H \to E$, compatibly with the projections to G, (and their kernels). As F is free and σ' is an epimorphism, we can find $\tau : F \to E$ such that $\sigma\tau = p$. Now $\sigma'\tau k = 1$, so $\tau k = k'\psi|_K : K \to L$. We examine a commutator $[k(x), y]$ with $x \in K$, $y \in F$. The image of this under τ will be $\tau[k(x), y] = [\tau k(x), \tau(y)] = [k'\tau|_K(x), \tau(y)] = 1$, since $\mathbb{E}'$ is a central extension, so τ induces a $\psi : H \to E$ compatibly with the projections to G, and hence with their kernels. ■

When will G have a *universal* central extension? The answer is: when G is perfect.

Definition: Suppose G is a group, it is *perfect* if $[G, G] = G$, i.e., it is generated by commutators.

Proposition 21 *Every perfect group, G, has a universal central extension.*

Proof: (We can pick up ideas and notation from the previous proof.) As G is perfect, we can restrict $\sigma : H \to G$ to the subgroup $[H, H]$ and still get a surjection. We thus have

$$\begin{tikzcd} 1 \arrow[r] & K \cap [H,H] \arrow[r] \arrow[d] & {[H,H]} \arrow[r, "\sigma"] \arrow[d] & G \arrow[r] \arrow[d, "="] & 1 \\ 1 \arrow[r] & K \arrow[r] & H \arrow[r, "\sigma"'] & G \arrow[r] & 1 \end{tikzcd}$$

It is clear that as the bottom is weakly universal, so is the top one.

We next need a subsidiary result.

Lemma 16 *If $1 \to Ker\,\sigma \to H \xrightarrow{\sigma} G \to 1$ is a weakly universal central extension and H is perfect, then G is perfect and the central extension is universal.*

Proof: The first conclusion should be clear, so we are left to prove 'universal'. Suppose we have $\mathbb{E}'$ as before and obtain two morphism φ and φ', from H to E such that $\sigma'\varphi = \sigma'\varphi' = \sigma$. We have, for $h_1, h_2 \in H$, $\varphi(h_1) = \varphi'(h_1)c$, and $\varphi(h_2) = \varphi'(h_2)d$ for some $c, d, \in L$. we calculate that

$$\varphi(h_1h_2h_1^{-1}h_2^{-1}) = \varphi'(h_1h_2h_1^{-1}h_2^{-1}),$$

since c and d are central in E, but as commutators generate H, $\varphi = \varphi'$ everywhere in H. ■

To complete the proof of the proposition, we show that, back in case $[[H, H]$ is itself perfect. We have

$$[H, H] = \left[\frac{F}{[K, F]}, \frac{F}{[K, F]}\right] = \frac{[F, F]}{[K, F]},$$

now as G is perfect, every element in F can be written in the form $x = ck$ with $c \in [F, F]$ and $k \in K$. (One could say 'F is perfect up to K'.)

Take, now, a $[\overline{x}, \overline{y}] \in [H, H]$, i.e., a commutator of $\overline{x}, \overline{y} \in F/[K, F]$ with $\overline{x}$ denoting the coset $x[K, F]$, etc. Set $x = ck$, $y = d\ell$, $c, d, \in [F, F]$

$$\begin{aligned} \overline{xyx^{-1}y^{-1}} &= \overline{x}.\overline{y}.\overline{x}^{-1}.\overline{y}^{-1} \\ &= \overline{c}.\overline{d}.\overline{c}^{-1}.\overline{d}^{-1} \\ &= \overline{cdc^{-1}d^{-1}} \in [[H, H], [H, H]] \end{aligned}$$

since elements of K commute with elements of F mod $[K, F]$. We thus have $[H, H] = [[H, H], [H, H]]$, as claimed. ■

To summarise, suppose we have a group presentation, $G = (X : R)$, of a perfect group, G. This gives us an exact 'presentation sequence'

$$1 \to K \to F \to G \to 1$$

where we abbreviate $N(R)$ to K. There is, then, a short exact sequence:

$$1 \to \frac{K \cap [F, F]}{[K, F]} \to \frac{[F, F]}{[K, F]} \to G \to 1$$

and this is its *universal central extension.*

Remark: The term on the left is the usual formula for the *Schur multiplier* of G and is one of the origins of group *homology*. It gives the Hopf formula for $H_2(G, \mathbb{Z})$, the second homology of G with coefficients in the trivial G-module, $\mathbb{Z}$.

To apply this theory and discussion back to the Steinberg group, $St(R)$, we need to check that $St(R)$ is a perfect group and that the central extension that we have is weakly universal. the first of these is simple.

Lemma 17 *The group $St(R)$ is perfect.*

Proof: We can write any generator $x_{ij}(a)$ as $[x_{ik}(a), x_{kj}(1)]$ for some k other than i or j, so the proof is the same as that $E_n(R)$ is perfect (for $n \geq 3$), that we gave earlier. ■

This leaves us to check that the central extension

$$1 \to K_2(R) \to St(R) \xrightarrow{\varphi} E(R) \to 1$$

that we saw earlier is weakly universal (as it will then be universal by the previous lemma).

Suppose that we have

$$1 \to L \to E \xrightarrow{\sigma} E(R) \to 1$$

is a central extension. We have to define a morphism $\psi : St(R) \to E$ projecting down to the identity morphism on $E(R)$. As we have $St(R)$ defined by a presentation, the obvious way to proceed is to find suitable images in E for the generators, $x_{ij}(a)$, and then see if the Steinberg relations are satisfied by them.

To start with, for each generator $x_{ij}(a)$ of $St(R)$, we pick an element, $y_{ij}(a)$, in E such that $\sigma(y_{ij}(a)) = e_{ij}(a)$, the corresponding elementary matrix, which is, of course, the image of $x_{ij}(a)$ in $E(R)$. (Note that any other choice of the $y_{ij}(a)$ will differ from this by a family of elements of the kernel, L, and hence by central elements of E.)

We will prove, or note, various useful identities, which will give us what we need.

- $[u, [v, w]] = [uv, w][w, u][w, v]$ for $u, v, w, \in E$;
- for convenience, for $u \in E$, write $\bar{u} = \sigma(u) \in E(R)$, and for $u, v \in E$, write $u \sim v$ if $uv^{-1} \in L$, then note that if $u \sim u'$ and $v \sim v'$, we have $[u, v] = [u', v']$;
- if $u, v, w, \in E$ with $[\bar{u}, \bar{v}] = [\bar{u}, \bar{w}] = 1$, then

  $$[u, [v, w]] = 1.$$

  To see this, put $a = [u, v]$, $b = [u, w]$, so, by assumption, $\bar{a} = \bar{b} = 1$ and $a, b \in L$. We then have $uvu^{-1} = av$, $uwu^{-1} = bw$, and $[av, bw] = [v, w]$, since $a, b \in L$. Next look at

  $$[u, [v, w]] = u[v, w]u^{-1}[v, w]^{-1} = [uvu^{-1}, uwu^{-1}][v, w]^{-1} = 1$$

  by our previous calculation.

We are now ready to look at the $y_{ij}(a)$s and see how nearly they will satisfy the Steinberg relations, (St1 and St2 of page 77). (They will not necessarily satisfy them 'on-the-nose', but we can use them to get another choice that will work.)

- If $i \neq j$, $k \neq \ell$, so the corresponding ys make sense, and further $i \neq \ell, j \neq k$ (to agree with the condition of the first part of the St2) relation), then $[y_{ij}(a), y_{k\ell}(b)] = 1$. To see this we choose n bigger than all the indices involved here, so that we can have $y_{k\ell}(b) \sim [y_{kn}(b), y_{n\ell}(1)]$, as they give the same element when mapped down to $E(R)$. We thus have

$$[y_{ij}(a), y_{k\ell}(b)] = [y_{ij}(a), [y_{kn}(b), y_{n\ell}(1)]] = 1,$$

  by the above, so the ys do go some way towards what we need, (but the other relations need not hold). We will use them, however, to make a better choice.

- Suppose i, j and n are distinct, and, as always, $a \in R$. Set

$$z_{ij}^n(a) = [y_{in}(a), y_{jn}(1)].$$

  It is easy to see that this depends on i, j and a, and, slightly less obviously, that it does not depend on the choice of the $y_{k\ell}$s. Actually it does not depend on n at all. (The details are left **for you to check**, but use the commutator rules above to show $z_{ik}^n(ab) = [y_{ij}(a), y_{jk}(b)]$. That is independent of n.) We write $z_{ij}(a)$ for $z_{ij}^n(a)$, as n is irrelevant, as long as it is sufficiently large. These $z_{ij}(a)$ will do the trick!

We define $\psi : St(R) \to E$ by defining $\psi(x_{ij}(a)) = z_{ij}(a)$ and will check that $z_{ij}(a)$ satisfies the relations of $St(R)$, (as that will mean that this assignment does define a homomorphism by what is sometimes known as von Dyck's Theorem).

Most have been done (and checking this is again **left to you**), except for

$$z_{ij}(a)z_{ij}(b) = z_{ij}(a+b).$$

Clearly their difference is central in E, but that is not enough. We calculate

$$\begin{aligned} z_{ij}(a+b) &= z_{ij}(b+a) \\ &= [z_{ik}(b+a), z_{kj}(1)] \quad \text{with} k \neq i, j \\ &= [z_{ik}(b)z_{ik}(a), z_{kj}(1)] \quad \text{as the `difference is central'} \\ &= [z_{ik}(b), z_{ij}(a)]z_{ij}(a)z_{ij}(b) \quad \text{using the first commutator identity above} \\ &= z_{ij}(a)z_{ij}(b) \end{aligned}$$

as required.

We have checked, in quite a lot of detail, that

Proposition 22

$$1 \to K_2(R) \to St(R) \xrightarrow{\varphi} E(R) \to 1$$

is a universal central extension. ■

3.2.5 Higher algebraic K-theory: some first remarks

Milnor's definition of $K_2(R)$ was initially given in a course at Princeton in 1967. The search for higher algebraic K-groups was then intense; see Weibel's excellent history of algebraic K-theory, [219]. The breakthrough was due to Quillen, who in 1969/70, gave the 'plus construction', which was a method of 'killing' the maximal perfect subgroup of a fundamental group, $\pi_1(X)$. Applying this to the classifying space, $BG\ell(R)$, of the stable general linear group, gave a space $BG\ell(R)^+$, whose homotopy groups had the right sort of properties expected of those mysterious higher groups and so were taken to be $K_n(R) := \pi_n(BG\ell(R)^+)$.

Several other constructions of $K_n(R)$ were given in 1971 and were gradually shown to be equivalent to Quillen's. One of these which was based upon the theory of 'buildings' and upper triangular subgroups was by I. Volodin, [216]. We will look at the general construction in the next few sections as it relates closely to our theme of higher syzygies.

We note that there are several other approaches that were developed at about the same time, but will not be looked at in this chapter. There are also generalisations of these ideas.

3.3 Higher generation by subgroups

We now return to more general discussions relating to presentations, syzygies and rewriting, although we will see the link with ideas and methods from K-theory coming in later on.

Often one has a group, G, and a family $\mathcal{H}$, of subgroups. For example (i) suppose G is given with a presentation, $(X : R)$, then subsets of X yield subgroups of G, and a family of subsets naturally leads to a family of subgroup, or (ii) a group may be a symmetry group of some geometric or combinatorial structure and certain substructures may be fixed by a subgroup, so families of subgroups may correspond to families of substructures. It is common, in this sort of situation, to try to see if information on G can be gleaned from information on the subgroups in $\mathcal{H}$. This will happen to some extent even if it is simply the case that the union of the elements in the subgroups generate G.

A simple example would be if G is generated by three elements a, b and c with some relations (possibly not known or not completely known), $\mathcal{H}$ consists of the subgroup generated by a, and that generated by b. There is a possibility that c is not in the subgroup generated by a and b, but how might this become apparent.

It may be that we have, instead of a presentation of G, presentations of the subgroups in $\mathcal{H}$, can we find a presentation of G, and, more generally, suppose we have knowledge of higher (homotopical or homological) syzygies of the presentations of the subgroups in $\mathcal{H}$, can we find not only a presentation of G, but build up knowledge of (at least some of) the syzygies for that presentation?

The key to attacking these problems is a knowledge of the way that the subgroups interact and by building up knowledge of the correspondence between the combinatorics of that interaction and of the induction process of building out from $\mathcal{H}$ to the whole group, G.

Various instances of this process had been studied, notably by Tits, e.g. in [205–207], since, in the situations studied in those papers, the combinatorics leads to the building of a Tits system. They also occur in the work of Behr, [25] and Soulé, [197], but, because of their general approach and the explicit link made to identities among relations, we will use the beautiful paper by Abels and Holz, [1]. This, and some subsequent developments, provides the basis for a way of calculating some syzygies in some interesting situations.

There is also a strong link with Volodin's approach to higher algebraic K-theory, but that will be slightly later in the notes. Here we sketch some of the background and intuition, giving some very elementary examples. When we have more knowledge of how to work with syzygies using both homotopical and homological methods, whether 'crossed' or not, we will return to look in more detail. We will see that this study of 'higher generation' leads in some interesting directions, towards geometric constructions and concepts of use elsewhere.

3.3.1 The nerve of a family of subgroups

We start, therefore, with a group, G, and a family, $\mathcal{H} = \{H_i \mid i \in I\}$ of subgroups of G. Each subgroup, H, determines a family of right cosets, Hg, which cover the *set*, G. Of course, these partition G, so there are no non-trivial intersections between them. If we use *all* the right cosets, $H_i g$, for *all* the H_i in $\mathcal{H}$, then, of course, we expect to get non-trivial intersections.

Remark: There is some disagreement as to which terminology for cosets is the most logical, so we should say exactly what we mean by 'right coset'. A subgroup H of G gives a left action, $H \curvearrowright G$ on the *set*, G, by multiplication on the left, and hence a groupoid whose connected components are the *right* cosets, Hg. The terminology 'right coset' corresponds to the g being on the right. If we considered the right action then we would have left cosets in the corresponding role.

Another notational point is that when writing cosets, we follow the usual rule that there is some informal set of coset representatives being used, or more exactly that the notation looks like that! This can be delicate if we step outside a set based situation, as choosing a set of coset representatives uses the axiom of choice, and in some contexts that would be 'dodgy'.

Let
$$\mathfrak{H} = \coprod_{i \in I} H_i \backslash G = \{H_i g \mid H_i \in \mathcal{H}\},$$
where the g is more as an indicator of right cosets than strictly speaking an index. This is the family of all right cosets of subgroups in $\mathcal{H}$. This covers G and we write $N(\mathfrak{H})$ for the corresponding simplicial complex, which is the *nerve* of this covering.

In many situations, 'nerves' in some form are used to help 'integrate' *local* information into *global*, since they record the way the 'localities' of the information fit together. (We will refer to this type of problem as a 'local-to-global' problem. They occur in many different contexts.) We have met nerves of categories, and will later meet nerves of open covers of topological spaces, but in that latter situation, the topological features of the construction are not central to that construction. We will consider the fairly general case of the nerve of a relation in a while, but for the moment, we will give a working definition, specific to the application that we have in mind here. We will refine and extend that definition later on.

Definition: Let G be a group and $\mathcal{H}$ a family of subgroups of G. Let $\mathfrak{H}$ denote the corresponding covering family of right cosets, $H_i g$, $H_i \in \mathcal{H}$. (We will write $\mathfrak{H} = \mathfrak{H}(G, \mathcal{H})$ or even $\mathfrak{H} = (G, \mathcal{H})$, as a shorthand as well.) The *nerve* of $\mathfrak{H}$ is the simplicial complex, $N(\mathfrak{H})$, whose vertices are the cosets, $H_i g$, $i \in I$, and where a non-empty finite family, $\{H_i g_i\}_{i \in J}$, is a simplex if it has non-empty intersection.

Examples: (i) If $\mathcal{H}$ consists just of one subgroup, H, then $\mathfrak{H}$ is just the set of cosets, $H\backslash G$ and $N(\mathfrak{H})$ is 0-dimensional, consisting just of 0-simplices / vertices.

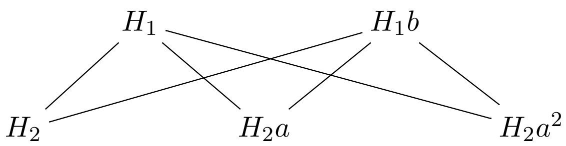
(ii) If $\mathcal{H} = \{H_1, H_2\}$, (and H_1 and H_2 are not equal!), then any right H_1 coset, $H_1 g$, will intersect some of the right H_2-cosets, for instance, $H_1 g \cap H_2 g$ always contains g. The nerve, $N(\mathfrak{H})$, is a bipartite graph, considered as a simplicial complex. (If the group G is finite, or more generally, if both subgroups have finite index, the number of edges will depend on the sizes or indcices of H_1,

H_2 and $H_1 \cap H_2$.) It is just a graphical way of illustrating the intersections of the cosets, a sort of intersection diagram. (There is an error in [16] in which it is claimed that each coset H_1 will intersect with each of those of H_2.)

As a specific very simple example, consider:

- $S_3 \equiv (a, b : a^3 = b^2 = (ab)^2 = 1)$, (so a denotes, say, the 3-cycle (1 2 3) and b, a transposition (1 2)).
- Take $H_1 = \langle a \rangle = \{1, (1\ 2\ 3), (1\ 3\ 2)\}$, yielding two cosets H_1 and $H_1 b$.
- Similarly take $H_2 = \langle b \rangle = \{1, (1\ 2)\}$ giving cosets H_2, $H_2 a$ and $H_2 a^2$.

The covering of S_3 is then $\mathfrak{H} = \{H_1, H_1 b, H_2, H_2 a, H_2 a^2\}$ and has nerve



3.3.2 n-generating families

Abels and Holz, [1], give the following definition:

Definition: A family, $\mathcal{H}$, of subgroups of G is called *n-generating* if the nerve, $N(\mathfrak{H})$, of the corresponding coset covering is $(n-1)$-connected, i.e., $\pi_i N(\mathfrak{H}) = 0$ for $i < n$.

The following results illustrate the idea and motivate the terminology. (They are to be found in [1].)

Proposition 23 *The group, G, is generated by the union of the subgroups, H, in $\mathcal{H}$ if, and only if, $N(\mathfrak{H})$ is connected.*

We will take this apart rather than use the short proof given in [1]. (Hopefully this will show how the idea works and how simple minded the proof can be!)

Proof: Suppose we have that G is generated by the various H in $\mathcal{H}$ and we are given two vertices Hg_1 and Kg_2 for $H, K \in \mathcal{H}$. (The case $H = K$ is allowed here.) Of course, $g_1 g_2^{-1} \in G$, so is a product of elements from the various H_is, say, $g_1 g_2^{-1} = h_{i_1} \ldots h_{i_n}$ with $h_{i_k} \in H_{i_k}$. (This observation suggests an induction on the length of this expression.)

To 'test the water', we assume $g_1 g_2^{-1} = h_1 \in H_1$, but then $g_1 \in Hg_1 \cap H_1 g_2$ and also $g_2 \in H_1 g_2 \cap K g_2$. (We can indicate this diagrammatically as

$$Hg_1 \overset{g_1}{\text{———}} H_1 g_2 \overset{g_2}{\text{———}} Kg_2,$$

where each edge is decorated by an element that *witnesses* that the intersection of the two cosets is non-empty.)

If we try next with $g_1 g_2^{-1} = h_1 h_2$, then $g_1 = h_1 h_2 g_2$, so we have

$$Hg_1 \overset{g_1}{\text{———}} H_1(h_2 g_2) \overset{h_2 g_2}{\text{———}} H_2 g_2 \overset{g_2}{\text{———}} Kg_2,$$

and the pattern gives the model for an induction on the length of the expression giving $g_1 g_2^{-1}$ in terms of elements of the H_is. (Note the link between the expression and the path is very simple.)

Conversely, suppose that $N(\mathfrak{H})$ is connected, then if $g \in G$, we look at Hg and H for some choice of H. There is a sequence of edges in $N(\mathfrak{H})$ joining these two vertices. We examine the length, ℓ, of such an edge path. If $\ell = 1$, there is some $h \in H \cap Hg$, so $g \in H$. If $\ell = 2$,

$$H \xrightarrow{x_1} H'g_1 \xrightarrow{x_2} Hg,$$

and we have $x_1 = h_1 = h_2 g_1$ with $h_2 \in H'$, whilst $x_2 = h_3 g_1 = h_4 g$. We thus obtain $g = h_4^{-1} h_3 g_1$ and $g_1 = h_2^{-1} h_1$, so $g = h_4^{-1} h_3 h_2^{-1} h_1$, i.e., we have an expansion of g in terms of elements of the various Hs. A proof of the general case is now easy. ■

We next form a diagram, $\mathcal{D}$, consisting of the subgroups, H_i, and all their pairwise intersections, together with the natural inclusions, and we write $H := \sqcup_{\cap} \mathcal{H}$ for $colim\, \mathcal{D}$. (Note that this colimit is within the category of groups.) More exactly, there is a poset $\{H_j, H_j \cap H_k \mid j, k \in I\}$, ordered by inclusion and $\mathcal{D}$ is the inclusion of this diagram into the category of groups. There is a presentation of H with generators x_g, $g \in \bigcup H_j$ and with relations $x_g \cdot x_h = x_{gh}$ if g and h are both in some H_i. (This group, H, is thus a 'coproduct' with amalgamated subgroups.)

There is an obvious homomorphism

$$H = \sqcup_{\cap} \mathcal{H} \to G$$

induced by the inclusions.

Proposition 24 *The family, $\mathcal{H}$, is 2-generating if, and only if, the natural homomorphism,*

$$H = \sqcup_{\cap} \mathcal{H} \to G,$$

is an isomorphism. ■

In fact,

Proposition 25 *There are isomorphisms:*
(a) $\pi_0 N(\mathfrak{H}) \cong G/\langle \bigcup H_j \rangle$;
(b) $\pi_1 N(\mathfrak{H}) \cong Ker(\sqcup_{\cap} \mathcal{H} \to G)$. ■

We almost have shown (a) in our above argument, but will postpone more detailed proofs until later. (They are, in fact, quite easy to give by direct calculation.)

Remark: It is often helpful to take the family, $\mathcal{H}$, of subgroups and to close it up under (finite) intersection and sometimes the inclusion order on the intersections comes in useful as well. This closure operation does not change the homotopy type of the nerve of the corresponding coverings by cosets, in fact, the process of taking intersections corresponds to taking the barycentric subdivision of the original nerve.

3.3.3 A more complex family of examples

An important example of the above situation is in algebraic K-theory. It occurs with the general linear group, $G\ell_n(R)$, of invertible $n \times n$ matrices together with a family of subgroups corresponding to lower triangular matrices, ... but with some subtleties involved.

Let R be an associative ring with identity and n a positive integer.

Let $\Delta = \{(i,j) \mid i \neq j, 1 \leq i,j \leq n\}$ be the set of non-diagonal positions in an $n \times n$ array. We will say that a subset, $\alpha \subseteq \Delta$, is *closed* if

$$(i,j) \in \alpha \text{ and } (j,k) \in \alpha \text{ implies } (i,k) \in \alpha.$$

Note that if $(i,j) \in \alpha$ and α is closed then $(j,i) \notin \alpha$.

Let $\Phi = \{\alpha \subseteq \Delta \mid \alpha \text{ is closed}\}$. There is a reflexive relation $\leq$ on Φ by $\alpha \leq \beta$ if $\alpha \subseteq \beta$. These αs are transitive relations on subsets of the set of integers from 1 to n, so essentially order the elements of the subset. The reason for their use is the following: suppose $(i,j) \in \Delta$ and $r \in R$. The *elementary matrix*, $\varepsilon_{ij}(r)$, is the matrix obtained from the identity $n \times n$ matrix by putting the element r in position (i,j),

$$\text{i.e.,} \qquad \varepsilon_{ij}(r)_{k,l} = \begin{cases} 1 & \text{if } k = l \\ r & \text{if } (k,l) = (i,j) \\ 0 & \text{otherwise .} \end{cases}$$

Let $G\ell_n(R)_\alpha$, for $\alpha \in \Phi$, denote the subgroup of $G\ell_n(R)$ generated by

$$\{\varepsilon_{ij}(r) \mid (i,j) \in \alpha, r \in R\}.$$

It is easy to see that $(a_{kl}) \in G\ell_n(R)_\alpha$ if and only if

$$a_{k,l} = \begin{cases} 1 & \text{if } k = l \\ \text{arbitrary} & \text{if } (i,j) \in \alpha \\ 0 & \text{if } (i,j) \in \Delta \backslash \alpha. \end{cases}$$

If $\alpha \leq \beta$, then there is an inclusion, $G\ell_n(R)_{\alpha \leq \beta}$ of $G\ell_n(R)_\alpha$ into $G\ell_n(R)_\beta$.

We will consider the $G\ell_n(R)_\alpha$ as forming a family, $\mathcal{G}\ell_n(R)$, of subgroups of $G\ell_n(R)$.

Remark: Although a similar idea is found in Wagoner's paper [217], I actually learnt the idea for this approach to these subgroups from papers by A. K. Bak, [14, 15], and, with others, in [16], and from talks he gave in Bangor and Bielefeld. In these sources, this construction leads on to a discussion of his notion of a global action, and, in the third paper cited, the variant known as a *groupoid atlas*. The motivation, there, is to study the *unstable* algebraic K-theory groups, whilst Volodin's original and Wagoner's approach are more centred on the stable version.

There is a lot more that could be said about these groupoid atlasses, which were introduced to handle the intrinsic homotopy involved in Volodin's definition of a form of algebraic K-theory, [216]. We will not use them explicitly here, but will attempt to show the link between the above and the question of syzygies, higher generation by subgroups, etc.

The nerve of this family would consist of the cosets of these subgroups, linked via their intersections. We need to extract another description of the homotopy type of this simplicial complex and for that will examine the intersections of cosets, and of the subgroups. We will do this in a slightly strange way in as much as we will turn first, or rather after some preparation, to descriptions related to Volodin's version of the higher K-theory of an associative ring. Our approach will be via *Volodin spaces* as used, for instance, in a paper by Suslin and Wodzicki, [202] and then an examination of the various nerves of a relation, before returning to this setting.

3.3.4 Volodin spaces

Let X be a non-empty set, and denote by $E(X)$, the simplicial set having $E(X)_p = X^{p+1}$, so a p-simplex is a $p+1$ tuple, $\underline{x} = (x_0, \ldots, x_p)$, each $x_i \in X$, and in which

$$d_i(\underline{x}) = (x_0, \ldots, \hat{x_i}, \ldots x_p),$$

and

$$s_j(\underline{x}) = (x_0, \ldots, x_j, x_j, \ldots x_p),$$

so d_i omits x_i, whilst s_j repeats x_j.

Lemma 18 *The simplicial set, $E(X)$, is contractible.*

Proof: We thus have to prove that the unique map $E(X) \to \Delta[0]$ is a homotopy equivalence. (That this is the case is well known, but we will none the less give a sketch proof of it as firstly we have not assumed that much knowledge of simplicial homotopy and also as it gives some interesting insights into that subject in a very easy situation.) We pick some $a_0 \in X$ and obtain a map $\Delta[0] \xrightarrow{a_0} E(X)$ by mapping the single 0-simplex of $\Delta[0]$ to the 0-simplex, (a_0) in $E(X)$. We now show that the identity map on $E(X)$ is homotopic to the composite map, $E(X) \to \Delta[0] \xrightarrow{a_0} E(X)$, that 'sends all simplices to a_0'.

We will look at simplicial homotopies in more detail later, (in particular around page 292), but clearly, a homotopy $h : f \simeq g : K \to L$, between two simplicial mapsa $f, g : K \to L$, should be a simplicial map $h : K \times \Delta[1] \to L$, restricting to f and g on the two ends of $K \times \Delta[1]$.. Here we need a homotopy $h : E(X) \times \Delta[1] \to E(X)$ and we look at what this must be on a cylinder over a simplex, $(x_0, \ldots, x_p)$. To see what to do, look at almost the simplest case, $p = 1$, then a schematic representation of h on $(x_0, x_1) \times \Delta[1]$ must look like:

$$\begin{tikzcd}
a_0 \arrow[r] & a_0 \\
x_0 \arrow[u] \arrow[r] \arrow[ur] & x_1 \arrow[u]
\end{tikzcd}$$

More precisely, the two simplices of $E(X) \times \Delta[1]$ that we need have two forms

$$\sigma_1 = ((x_0, 0), (x_1, 0), (x_1, 1))$$

and

$$\sigma_2 = (x_0, 0), (x_0, 1), x_1, 1))$$

being, respectively the bottom right and the top left hand ones. We need $h(\sigma_1) = (x_0 x_1, a_0)$ and $h(\sigma_2) = (x_0, a_0, a_0)$. Now it is easy to see how to set up h, in general, giving the required contracting homotopy. ■

Remark: Any homotopy can be specified by a family of maps, $h_i^n : K_n \to L_{n+1}$, satisfying some rules that will be given later (page 294). It is then easy to specify the $h_i^n : E(X)_n \to E(X)_{n+1}$ generalising the formula we have given above. (We **leave this to you** if you have not seen it before, as it is easy, but also instructive.)

The case we are really interested in is when we replace the general set, X, by the underlying set of a group, G. (As usual, we will not introduce a special notation for the underlying set of G, just writing G for it.) In this case we have the simplicial set $E(G)$ and the group, G, acts freely on $E(G)$ by

$$g \cdot (g_0, \ldots, g_p) = (gg_0, \ldots, gg_p).$$

(Here we have used a left action of G, and **leave you to check** that the evident right action could equally well be used and we will use it later on.)

The quotient simplicial set of orbits, will be denoted $G\backslash E(G)$. It is often useful to write $[g_1, \ldots, g_p]$ for the orbit of the p-simplex $(1, g_1, g_1g_2, \ldots, g_1g_2 \ldots g_p) \in E(G)_p$.

It is 'instructive' to calculate the faces and degeneracy maps in this notation. We will only look at $[g_1, g_2]$ in detail. This element has representative $(1, g_1, g_1g_2)$. We thus have:

- $d_0(1, g_1, g_1g_2) = (g_1, g_1g_2) \equiv (1, g_2)$, so $d_0[g_1, g_2] = [g_2]$;
- $d_1(1, g_1, g_1g_2) = (1, g_1g_2)$, so $d_1[g_1, g_2] = [g_1g_2]$;
- $d_2(1, g_1, g_1g_2) = (1, g_1)$, so $d_2[g_1, g_2] = [g_1]$.

(That looks familiar!)

For the degeneracies,

- $s_0(1, g_1, g_1g_2) = (1, 1, g_1, g_1g_2)$, so $s_0[g_1, g_2] = [1, g_1, g_2]$;
- $s_1(1, g_1, g_1g_2) = (1, g_1, g_1, g_1g_2)$, so $s_1[g_1, g_2] = [g_1, 1, g_2]$;

and similarly $s_2[g_1, g_2] = [g_1, g_2, 1]$.

The general formulae are now easy to guess and to prove - so they will be **left to you**, and then the following should be obvious.

Lemma 19 *There is a natural simplicial isomorphism,*

$$G\backslash E(G) \xrightarrow{\cong} Ner(G[1]) = BG.$$

■

We thus have that $G\backslash E(G)$ is a 'classifying space' for G.

We note that this shows that $G\backslash E(G)$ is a Kan complex, since we already have that $Ner(G[1])$ is one. It is easy enough to check it directly. Of course, $E(G)$ is Kan as well. Jumping ahead of ourselves, we will sketch that the fundamental group of $G\backslash E(G)$ is $\pi_1(G\backslash E(G)) \cong G$, whilst for $k > 1$, $\pi_k(G\backslash E(G))$ is trivial. (We will have to 'fudge' the details as they either need material that will not be directly handled in these notes (and hence, for which the reader is referred to standard texts on simplicial homotopy theory), or they may depend on ideas that will be only explored later on in the notes, so we will sketch enough to whet the appetite!)

First we take on trust that if K is a connected Kan complex, then the k^{th} homotopy group of K can be 'calculated' by looking at homotopy classes of mappings from the boundary of a $k+1$-simplex into K, based at a base point. If you have a map, $\partial\Delta[k+1] \to Ner(G[1])$, then you have *all* the information needed to extend it to a map defined on $\Delta[k+1]$, i.e., the map you started with is null homotopic. (If you want more intuition on this, try looking at the case $k = 2$ and writing down what the various faces in $\partial\Delta[3]$ will give and then see how they determine a 3-simplex in $Ner(G[1])$.)

For dimension 1, the construction of π_1 is, of course, that of the fundamental group(oid), so gives a presentation with set of generators $\{[g] \mid g \in G\}$ and, for each pair (g_1, g_2), a relation r_{g_1,g_2} corresponding to $[g_1, g_2] \in G\backslash E(G)_2$, and which gives $[g_1][g_2][g_1g_2]^{-1}$, but this was our prime example of a presentation of G, so $\pi_1(G\backslash E(G)) \cong G$.

There is, here, another **useful fact for the reader to check**. The quotient map from $E(G)$ to $G\backslash E(G)$ is a Kan fibration (and this is a **useful example to do in detail** if you are not that conversant with Kan fibrations). The fibre of this quotient map is a constant (or 'discrete') simplicial set with *value* G, so is a $K(G, 0)$. As is well known, and as we will introduce and use later, there is a long exact sequence of homotopy groups for any pointed fibration sequence, $F \to E \to B$, so we can apply this to

$$K(G, 0) \to E(G) \to G\backslash E(G)$$

to get $\pi_i(G\backslash E(G) \cong \pi_{i-1}(K(G, 0))$ and another proof that $G\backslash E(G)$ is an 'Eilenberg Mac Lane space' for G, i.e., a $K(G, 1)$ in the usual notation, (... and yes, this is related to covering spaces ...).

Returning to the construction of what are called 'Volodin spaces' (cf. [202]), we put ourselves back in the context of a group, G, and a family, $\mathcal{H}$, of subgroups of G. We suppose that $\mathcal{H} = \{H_i \mid i \in I\}$ for some indexing set, I. (We may assume extra structure on I, as before, when we get further into the construction.)

Definition: (Suslin-Wodzicki, [202], p. 65.) We denote by $V(G, \mathcal{H})$, or $V(\mathfrak{H})$, the simplicial subset of $E(G)$ formed by simplices, $(g_0, \ldots, g_p)$, that satisfy the condition that there is some $i \in I$ such that, for all $0 \leq j, k \leq p$, $g_j g_k^{-1} \in H_i$.

The simplicial set, $V(G, \mathcal{H})$, will be called the *Volodin space* of $(G, \mathcal{H})$.

Remark: The actual definition given in [202] uses $g_j^{-1} g_k \in H_i$, as there the convention on cosets is gH rather than our Hg.

The subobject, $V(G, \mathcal{H})$, of $E(G)$ is a G-subobject, i.e., it is invariant under the action of G. The corresponding quotient simplicial set $G\backslash V(G, \mathcal{H})$ coincides with the union of the BH_i within the classifying space, BG.

Remark: The construction of $V(G, \mathcal{H})$ is usually ascribed to Volodin in his approach to the higher K-theory groups of a ring, but in fact, the basic construction is essentially much older, being due to Vietoris in the 1920s, but in a different setting, namely that of a simplicial complex associated to an open covering of a space. This was further studied by Dowker, [82], in 1952, where he abstracted the situation to construct two simplicial complexes from a relation between two sets.

3.3.5 The two nerves of a relation: Dowker's construction

The results of the next few sections are of much more general use than just for a group and a family of its subgroups. We therefore present things in an abstract version.

Let X, Y be sets and R a relation between X and Y, so $R \subseteq X \times Y$. We write xRy for $(x, y) \in R$.

Fairly generic example: Let X be a set (often a topological space) and Y be a collection of (usually open) subsets of X covering X, i.e., $\bigcup Y = X$. The classical case is when Y is an index set for an open cover of X. The relation is xRy if and only if $x \in y$, or, more exactly, x is in the subset indexed by y.

Returning to the abstract setting, we define two simplicial complexes associated to R, as follows:

(i) $K = K_R$:

  (a) the set of vertices is the set X;

  (b) a p-simplex of K is a set $\{x_0, \cdots, x_p\} \subseteq X$ such that there is some $y \in Y$ with $x_i R y$ for $i = 0, 1, \ \cdots, \ p$.

(ii) $L = L_R$:

  (a) the set of vertices is the set, Y;

  (b) p-simplex of K is a set $\{y_0, \cdots, y_p\} \subseteq Y$ such that there is some $x \in X$ with xRy_j for $j = 0, 1, \ \cdots, \ p$.

Clearly the two constructions are in some sense dual to each other. The original motivating example was as above. It had X, a space, and $Y = \mathcal{U} = \{U_\alpha : \alpha \in A\}$, an open cover of X, and, in that case, K_R is the *Vietoris complex* of $\mathcal{U}$, $V(\mathcal{U})$ or $V(X,\mathcal{U})$, of the cover. The 'dual' construction has the open cover, $\mathcal{U}$, or better, the indexing set, A, as its set of vertices, and $\sigma = \langle \alpha_0, \alpha_1, ..., \alpha_p \rangle$, belongs to L_R if and only if the open sets, U_{α_j}, $j = 0, 1, \ldots, p$, have non-empty common intersection. This is the simplicial complex known as the *Čech complex*, *Čech nerve* or simply, *nerve*, of the open covering, $\mathcal{U}$, and it will be denoted $N(X,\mathcal{U})$, or $N(\mathcal{U})$. We will have occasion to repeat this definition later, both when considering Čech non-Abelian cohomology, (starting on page 254), and also when looking at triangulations when examining methods of constructing some simple topological quantum field theories, page 349.

We will extend the terminology so that for a given relation, R, K_R will be called the *Vietoris nerve* of R, whilst L_R is its *Čech nerve*. (This is rather arbitrary as the Vietoris nerve of R is the Čech nerve of the opposite relation, R^{op}, from Y to X.)

In the situation in this chapter, we have a pair, $(G, \mathcal{H})$, and X is G, whilst Y is the family, $\mathfrak{H}$, of right cosets of subgroups from the family $\mathcal{H}$. The relation is 'xRy if and only if $x \in y$'.

The simplicial complex, K_R, thus has G as its set of vertices and $(g_0, \ldots, g_p)$ is a p-simplex of K_R if, and only if, all the g_ks are in some common right coset, $H_i x$, in the family $\mathfrak{H}$. It is then just a routine calculation to check that this is the same as saying that the simplex is in $V(\mathfrak{H})$. In other words, the Volodin complex of $(G, \mathcal{H})$ is the same as the Vietoris complex of $\mathfrak{H}$, and it is convenient that both names begin with the letter 'V'! The one difference is that the Vietoris complex is a simplicial complex, whilst the Volodin space is a simplicial set. For each p-simplex $\{g_0, \ldots, g_p\}$, of $V(\mathfrak{H})$, there are $p!$ simplices in the Volodin space.

The corresponding Čech nerve, L_R, is $N(\mathfrak{H})$ as introduced earlier, so, if $\sigma \in N(\mathfrak{H})_p$, $\sigma = \{H_0g_0, \cdots, H_pg_p\}$ with the requirement that $\cap\sigma = \bigcap_{i=0}^{p} H_ig_i \neq \emptyset$.

Before turning to Dowker's result, we will examine barycentric subdivisions as these play a neat role in his proof.

3.3.6 Barycentric subdivisions

Combinatorially, if K is a simplicial complex with vertex set, V_K, then one associates to K the partially ordered set of its simplices. (We avoid our earlier notation of $V(K)$ for the vertex set as being too ambiguous here.) Explicitly we write $S(K)$ (or sometimes S_K), for the set of simplices of K and $(S(K), \subseteq)$ for the partially ordered set with $\subseteq$ being the obvious inclusion. The *barycentric subdivision*, K', of K has $S(K)$ as its set of vertices and a finite set of vertices of K' (i.e., simplices of K) is a simplex of K' if it can be totally ordered by inclusion.) We may sometimes write $Sd(K)$ instead of K'.)

Remark: It is important to note that there is, in general, no natural simplicial map from K' to K. If, however, V_K is given an order in such a way that the vertices of any simplex in K are totally ordered (for instance by picking a total order on V_K), then one can easily specify a map,

$$\varphi : K' \to K,$$

by:
if $\sigma' = \{x_0, \cdots, x_p\}$ is a vertex of K' (so $\sigma' \in S(K)$), let $\varphi\sigma'$ be the least vertex of σ' in the given fixed order.

This preserves simplices, but reverses order so if $\sigma'_1 \subset \sigma'_2$ then $\varphi(\sigma'_1) \geq \varphi(\sigma'_2)$.

If one changes the order, then the resulting map is *contiguous*:

Definition: Let $\varphi, \psi : K \to L$ be two simplicial maps between simplicial complexes. They are said to be *contiguous* if for any simplex σ of K, $\varphi(\sigma) \cup \psi(\sigma)$ forms a simplex in L.

Contiguity gives a constructive form of homotopy applicable to simplicial maps between simplicial complexes.

If $\psi : K \to L$ is a simplicial map, then it induces $\psi' : K' \to L'$ after subdivision. As there is no way of knowing/picking compatible orders on V_K and V_L in advance, we get that on constructing

$$\varphi_K : K' \to K$$

and

$$\varphi_L : L' \to L$$

that $\varphi_L\psi'$ and $\psi\varphi$ will be contiguous to each other, but rarely equal.

3.3.7 Dowker's lemma

Returning to K_R and L_R, we order the elements of X and Y, then suppose y' is a vertex of L'_R, so $y' = \{y_0, \cdots, y_p\}$, a simplex of L_R and there is an element $x \in X$ with $xRy_i, i = 0, 1, \cdots, p$. Set $\psi y' = x$ for one such x.

If $\sigma = \{y'_0, \cdots, y'_q\}$ is a q-simplex of L'_R, assume y'_0 is its least vertex (in the inclusion ordering)

$$\varphi_L(y'_0) \in y'_0 \subset y' \text{ for each } y_i \in \sigma,$$

hence $\psi y'_i R\varphi_L(y'_0)$ and the elements $\psi y'_0, \cdots, \psi y'_q$ form a simplex in K_R, so $\psi : L'_R \to K_R$ is a simplicial map. It, of course, depends on the ordering used and on the choice of x, but any other choice $\bar{x}$ for $\psi y'$ gives a contiguous map.

Reversing the rôles of X and Y in the above, we get a simplicial map,

$$\bar{\psi} : K'_R \to L_R.$$

Applying barycentric subdivisions again gives

$$\bar{\psi}' : K''_R \to L'_R,$$

and composing with $\psi : L'_R \to K_R$ gives a map

$$\psi\bar{\psi}' : K''_R \to K_R.$$

Of course, there is also a map

$$\varphi_K\varphi'_K : K''_R \to K_R.$$

Proposition 26 *(Dowker, [82] p.88). The two maps $\varphi_K\varphi'_K$ and $\psi\bar{\psi}'$ are contiguous.*

Before proving this, note that contiguity implies homotopy and that $\varphi\varphi'$ is homotopic to the identity map on K_R after realisation, i.e., this shows that

Corollary 5

$$|K_R| \simeq |L_R|.$$

■

The actual homotopy depends on the ordering of the vertices and so is not natural.

Proof of the Proposition:

Let $\sigma''' = \{x''_0, x''_1, \cdots, x''_q\}$ be a simplex of K''_R and as usual assume x''_0 is its least vertex, then for all $i > 0$

$$x''_0 \subset x''_i.$$

We have that φ'_K is clearly order reversing, so $\varphi'_K x''_i \subseteq \varphi'_K x''_0$. Let $y = \bar{\varphi}\varphi'_K x''_0$, then for each $x \in \varphi'_K x''_0$, xRy. Since $\varphi_K\varphi'_K x''_i \in \varphi'_K x''_i \subseteq \varphi'_K x''_0$, we have $\varphi_K\varphi'_K x''_i Ry$.

For each vertex x' of $x''_i, \bar{\psi}x' \in \bar{\psi}'x''_i$, hence as $\varphi'_K x''_0 \in x''_0 \subset x''_i, y = \bar{\psi}\varphi'_K x x''_0 \in \bar{\psi}' x''_i$ for each x''_i, so for each $x''_i, \psi\bar{\psi}' x''_i Ry$, however we therefore have

$$\varphi_k\varphi'_K(\sigma'') \cup \psi\bar{\psi}(\sigma''') = \bigcup \varphi_k\varphi'_K(x''_i) \cup \psi\bar{\psi}; x''_i$$

forms a simplex in K_R, i.e., $\varphi_K\varphi'_K$ and $\psi\bar{\psi}'$ are contiguous. ■

To prove this we had to choose orders on the two sets, and thus we were working with the non-degenerate simplices of the corresponding simplicial sets. (Abels and Holz, [1], use the neat notation of writing $N^{simp}(R)$, etc. for the corresponding simplicial set, either dependent on order or taking all possible orders, i.e., a p-tuple is a simplex in the simplicial set if its underlying set of elements is a simplex in the simplicial complex. Which method is used make essentially no difference much of the time. Their notation can be useful, but we will sometimes tend to ignore the difference as the homotopy groups and homotopy types are independent of which approach one takes. We have briefly discussed this on page **??** and we will revisit it in more detail later in this chapter, in section 3.4.3.)

3.3.8 Flag complexes

The construction of the barycentric subdivision is closely related to that of a flag complex of a poset.

Suppose that $\mathcal{P} = (P, \leq)$ is a partially ordered set (poset), then we can consider is as a category and hence look at its nerve. This is the associated simplicial set of the flag complex of $\mathcal{P}$, which is a simplicial complex, whose construction uses some ideas that can be of use later on, so we will briefly discuss how it relates to our situation.

Definition: A subset, σ,of $\mathcal{P} = (P, \leq)$ is said to be a *flag* if it satisfies, for all $x.y \in P$, either $x \leq y$ or $y \leq x$.

A finite non-empty flag, thus, is a linearly ordered subset of P, i.e., is of the form $\{x_0, \ldots x_p\}$, where $x_0 < \ldots x_n$ are elements of the set P.

Definition: Let $\mathcal{P} = (P, \leq)$ be a poset. The *flag complex*, *Flag*$(\mathcal{P})$ of $\mathcal{P}$ is the simplicial complex having the elements of P as its vertices and in which a p-simplex will be a non-empty flag, $x_0 < \ldots x_n$. in $\mathcal{P}$.

This is often also called the *order complex* of the poset.

Lemma 20 *The flag complex construction gives a functor*

$$Flag : Posets \to SimComp,$$

from the category of partially ordered sets and order preserving maps, to the category of simplicial complexes and simplicial morphisms between them. ■

As a simplicial complex, K, consists of a set, $V(K)$ of vertices and a set $S(K) \subseteq P(V(K)) - \{\emptyset\}$, $S(K)$ can naturally be ordered by inclusion to get a partially ordered set $U(K) = (S(K), \subseteq)$. This gives a functor,

$$U : SimpComp \to Posets.$$

The composite functor,

$$Flag \circ U : SimpComp \to SimpComp$$

is the *barycentric subdivision* functor, Sd.

If X is a set and $\mathcal{U} = \{U_i \mid i \in I\}$ is a family of subsets of X, we may think of $\mathcal{U}$ as being ordered by inclusion and thus get a poset. (Of course, this will only be significant if there are some inclusions between the U_is, for instance if $\mathcal{U}$ is closed under finite intersection.) This gives a poset, $(\mathcal{U}, \subseteq)$ and we will abbreviate $Flag(\mathcal{U}, \subseteq)$ to $F(\mathcal{U})$.

The links between nerves and flag complexes are strong.

Proposition 27 *(Abels and Holz, [1], p. 312) Suppose given $(X, \mathcal{U})$ as above, and that $\mathcal{U}$ is such that, if U and V are in $\mathcal{U}$ and $U \cap V$ is not empty, then $U \cap V \in \mathcal{U}$, then there is a natural homotopy equivalence,*

$$|N(\mathcal{U})| \simeq |F(\mathcal{U})|.$$

We cannot give a full proof here as it involves a result, namely Quillen's Theorem A, [185], that will not be discussed in these notes. We can however give a sketch (based on the treatment in [1]).

Sketch proof: Abusing notation so as to consider the simplicial complex, $N(\mathcal{U})$, as being the same as the poset of its simplices, we define a mapping:

$$f : N(\mathcal{U}) \to \mathcal{U}$$

sending $\sigma = \{U_0, \ldots, U_p\}$ to $U_\sigma = \cap_{i=0}^{p} U_i$. This is order reversing. (Note that it, of course, needs $\mathcal{U}$ to be closed under pairwise non-empty intersections.) Writing $\mathcal{U}^{op}$ for the poset, $(\mathcal{U}, \supseteq)$, that is with the opposite order, the poset $U \downarrow f$ of objects under some $U \in \mathcal{U}^{op}$ is just $\{\tau \in N(\mathcal{U}) \mid U_\tau \supseteq U\}$, so is a directed poset, and hence is contractible. By Quillen's theorem A, f induces a homotopy equivalence as claimed. ■

Remark: An interesting variant of these nerve and flag complex constructions combines some aspects of the Vietoris complex construction with the idea of flags to construct a bisimplicial set. A (p, q)-simplex will be pair consisting of a subset $\{x_0, \ldots, x_p\}$ of X together with a flag $U_0 \subset U_1 \subset \ldots \subset U_q$, such that all the x_i are in U_0. We will not explore this idea here as we have not discussed bisimplicial sets in any detail yet.

Within geometric group theory, the term 'flag complex' is also applied to a closely related, but distinct, concept. These 'flag complexes' are abstract simplicial complexes that satisfy a particular defining property, rather than being defined by how they are constructed. We will see other similar ideas later on in less geometric contexts, but for the moment will give a brief discussion based on the treatment of Bridson and Haefliger, [36], p. 210.

Definition: Let L be a simplicial complex with set of vertices V_L. It satisfies the *no triangles condition* if every finite subset of V_L that is pairwise joined by edges, is a simplex. More precisely, if $\{v_0, \ldots, v_n\}$ is such that for each $i, j \in \{1, \ldots, n\}$, $\{v_i, v_j\}$ is a 1-simplex of L, then $\{v_0, \ldots, v_n\}$ is a simplex of L.

An alternative name for the condition are the '*no empty simplices*' condition. It is also said that in this case L is *determined by its 1-skeleton*. The point is

Proposition 28 *If simplicial complex, L, is an order complex of some partially ordered set then it is determined by its 1-skeleton.* ■

The proof should be evident.

Geometric group theory contains many other examples of this sort of construction, especially with relation to Coxeter groups. (Perhaps we will return to this later one)

3.3.9 The homotopy type of Vietoris-Volodin complexes

Returning to $V(\mathfrak{H})$, the second complex associated to a pair $(G, \mathcal{H})$, it is possible to extract some homotopy information from it using fairly elementary methods. To go into its structure more deeply we will need to bring more explicitly in the group action of G as well, but that is for later.

The great advantage now is that as we know $N(\mathfrak{H})$ and $V(\mathfrak{H})$ have the same homotopy type (after realisation) so we can use either when working out homotopy invariants. We can also use $N^{simp}(\mathfrak{H})$, or $V^{simp}(\mathfrak{H})$ the corresponding simplicial sets, although, in fact, the Volodin *space* was actually defined as a simplicial set. We will usually leave out the difference between the simplicial complex and the simplicial set as that distinction is largely unnecessary.

If we look at any $gH_i \in \mathcal{H}$, then we have a subcomplex of $V(\mathfrak{H})$ consisting of those $(g_0, \ldots, g_p)$ all of which are in gH_i. In the simplest case, where $g = 1$, this is a copy of $E(H_i)$, and, in general, it is a translated copy of $E(H_i)$, so each forms a contractible subcomplex.

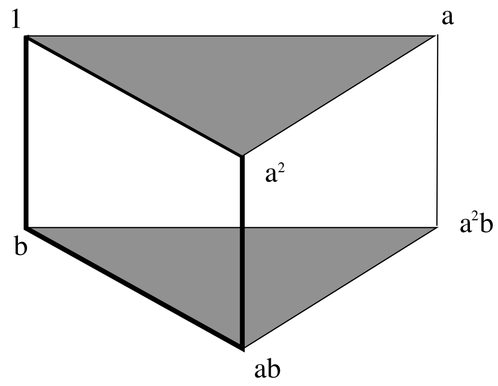
Example: (already considered in section 3.3.1)

$$\begin{aligned}
G &= S_3 = (a, b \mid a^3 = b^2 = (ab)^2 = 1), \text{ with } a = (1,2,3),\ b = (1,2);\\
H_1 &= \langle a \rangle = \{1,\ (1,2,3),\ (1,3,2)\},\\
H_2 &= \langle b \rangle = \{1,\ (1,2)\};\\
\mathcal{H} &= \{H_1, H_2\}
\end{aligned}$$

The intersection diagram given in our earlier look at this example, on page 91, is just the nerve, $N(\mathfrak{H})$, having 5 vertices and 6 edges. The other complex, $V(\mathfrak{H})$, is almost as simple. It has 6 vertices corresponding to the 6 elements of S_3, and each orbit yields a simplex

- $H_1 = \{1, a, a^2\}$ gives a 2-simplex (and three 1-simplices),
- $H_1 b = \{b, ab, a^2 b\}$ also gives a 2-simplex;
- $H_2 = \{1, b\}$ yields a 1-simplex, as do its cosets $H_2 a$ and $H_2 a^2$.

We can clearly see here the contractible subcomplexes mentioned earlier. We have that $V(\mathfrak{H})$ looks like two 2-simplices joined by 1-simplices at the vertices, (see below).



$V(S_3, \{\langle a\rangle, \langle b\rangle\})$

As $N(\mathfrak{H})$ is a connected with 5 vertices and 6 edges, we know $\pi_1 N(\mathfrak{H})$ is free on 2 generators. (The number of generators is the number of edges outside a maximal tree.) This same rank can be read of equally easily from $V(\mathfrak{H})$ as that complex is homotopically equivalent to a bouquet of 2 circles, (i.e., a figure eight). The generators of $\pi_1 V(\mathfrak{H})$ can be identified with words in the free product $H_1 * H_2$ (one such being shown in the picture) and relate to the kernel of the natural homomorphism from $H_1 * H_2$ to S_3. The heavy line in the figure corresponds to a loop at 1 given by

$$1 \xrightarrow{(1,b)} b \xrightarrow{(b,ab)} ab \xrightarrow{(ab,a^2)} a^2 \xrightarrow{(a^2,1)} 1$$

We write $g_0 \xrightarrow{(g_0,g_1)} g_1$ as there is an edge, (g_0, g_1) joining g_0 to g_1 in $V(\mathfrak{H})$. We, thus, have that there is a g and an index i such that $\{g_0, g_1\} \in H_i g$, but the index and the elements are not necessarily uniquely determined. We saw that this means that $g_1 g_0^{-1} \in H_i$, so $g_1 = hg_0$ for some $h \in H_i$, and we could equally well abbreviate the notation to $g_0 \xrightarrow{h} g_1$. Note that the only condition required is that h is in some H_i, so the lack of uniqueness mention above is without importance. In our example, we can redraw the diagram corresponding to the heavier loop and we get

$$1 \xrightarrow{b} b \xrightarrow{a} ab \xrightarrow{b} a^2 \xrightarrow{a} 1$$

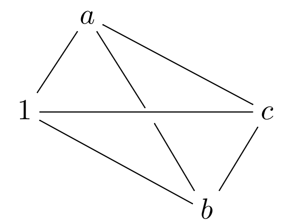
so the loop, representing an element in $\pi_1 N(\mathfrak{H})$, is given by the word $baba \in C_2 * C_3$, which, of course, is in the kernel of the homomorphism from $C_2 * C_3$ to S_3. The reason that this works is clear. Starting at 1, each part of the loop corresponds to a left multiplication either by an element of $H_1 \cong C_3$ or of $H_2 \cong C_2$. We thus get a word in $H_1 * H_2 \cong C_2 * C_3$. As the loop also finishes at 1, we must have that the corresponding word must evaluate to 1 when projected down into S_3.

Note that the two subgroups had simple presentations that combine to give a partial presentation of S_3. The knowledge of the fundamental group, $\pi_1 N(\mathfrak{H})$, then provides information on the 'missing' relations.

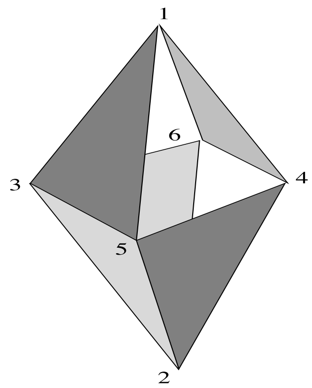
In more complex examples, the interpretation of $\pi_1(V(\mathfrak{H}), 1)$ will be the similar, but sometimes when G has more elements, $N(\mathfrak{H})$ may be easier to analyse than $V(\mathfrak{H})$, but the second may give links with other structure and be more transparent for interpretation. The important idea to retain is that the two complexes give the same information, so either can be used or both together.

Example: $G = K_4$, the Klein 4 group, $\{1, a, b, c\} \cong C_2 \times C_2$, so $a^2 = b^2 = c^2 = 1$ and $ab = c$; $\mathcal{H} = \{H_a, H_b, H_c\}$ where $H_a = \{1, a\}$, etc. Set $\mathfrak{H}_{K4} = (K_4, \mathcal{H})$.

The cosets are $H_a, H_a b, H_b, H_b a, H_c, H_c a$, each with two elements, so $V(\mathfrak{H}_{K4}) \cong$ the 1-skeleton of $\Delta[3]$:



$N(\mathfrak{H}_{K4})$ is "prettier" and a bit more 'interesting': Labelling the cosets from 1 to 6 in the order given above, we have 6 vertices, 12 1-simplices and 4 2-simplices. For instance, $\{1, 3, 5\}$ has the identity in the intersection, $\{1, 4, 6\}$ gives $H_a \cap H_b a \cap H_c a$, so contains a and so on. The picture is of the shell of an octahedron with 4 of the faces removed.



$N(\mathfrak{H}_{K4})$

From either diagram it is clear that $\pi_1 \mathfrak{H}_{K4}$ is free of rank 3. Again explicit representations for elements are easy to give. Using $V(\mathfrak{H})$ and the maximal tree given by the edges 1a, 1b and 1c, a typical generating loop would be

$$1 \to a \to b \to 1,$$

i.e., $(1, a, b, 1)$ as the sequence of points. There is an obvious representative word for this, namely

$$1 \xrightarrow{a} a \xrightarrow{c} b \xrightarrow{b} 1 .$$

In general, any based path at 1 in an $V(G, \mathcal{H})$ will yield a word in $\sqcup\mathcal{H}$, the free product of the family $\mathcal{H}$. We will think of the path as being represented by a (finite) sequence $(f(n))$ of elements in G, linked by transitions, h_i in the various subgroups. Whether or not that representative is unique depends on whether or not there are non-trivial intersections and "nestings" between the subgroups in the family $\mathcal{H}$, since, for instance, if H_i is a subgroup of H_j, then if $f(n) \to f(n+1)$ using $g \in H_i$, it could equally well be taken to be $g \in H_j$. As we have mentioned before, the characteristic of the Vietoris-Volodin spaces, $V(G, \mathcal{H})$, is that there is only one possible *element* of G linking $f(n)$ to the next $f(n+1)$ namely $f(n+1)f(n)^{-1}$, but this may be in several of the H_i. We thus have a strong link between $\pi_1(V(G, \mathcal{H}))$ and $\underset{\cap}{\sqcup}\mathcal{H}$, the 'amalgamated product' of $\mathcal{H}$ over its intersections, and an analysis of homotopy classes will prove (later) that

$$\pi_1(V(G, \mathcal{H}), 1) \cong \operatorname{Ker}(\underset{\cap}{\sqcup}\mathcal{H} \to G),$$

since a based path $(g_1, g_2, \cdots, g_n)$ ends at 1 if and only if the product $g_1 \cdots g_n = 1$. These identifications will be investigated more fully shortly.

We note that composites of such 'paths' may involve two adjacent transitions between elements being in the same H_i in which case we can use the rewriting system determined by the contractible $E(H_i)$ to simplify the representatives.

Example: The number of subgroups in $\mathcal{H}$ clearly determines the dimension of $N(\mathfrak{H})$, when $\mathfrak{H} = \mathfrak{H}(G, \mathcal{H})$. Here is another 3 subgroup example.

Take $q8 = \{1, i, j, k, -1, -i, -j, -k\}$ to be the quaternion group, so $i^4 = j^4 = k^4 = 1$, and $ij = k$. Set $H_i = \{1, -1, i, -i\}$ etc., so $H_i \cap H_j = H_i \cap H_k = H_j \cap H_k = \{1, -1\}$ and let $\mathcal{H} = \{H_i, H_j, H_k\}$, and $\mathfrak{H}_{q8} = \mathfrak{H}(q8, \mathcal{H})$.

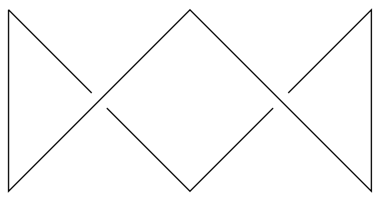
Then $N(\mathfrak{H}_{q8})$ is, as above in Example 3.3.9, a shell of an octahedron with 4 faces missing. Note however that $V(\mathfrak{H}_{q8})$ has 8 vertices and, comparing with $V(\mathfrak{H}_{K4})$, each edge of that diagram has become enlarged to a 3-simplex. It is still feasible to work with $V(\mathfrak{H}_{q8})$ directly, but $N(\mathfrak{H}_{q8})$ gives a clearer indication that

$$\pi_1(\mathfrak{H}_{q8}, 1) \text{ is free of rank 3.}$$

Example: Consider next the symmetric group, S_3, given by the presentation

$$S_3 := (x_1, x_2 \mid x_1^2 = x_2^2 = 1, (x_1x_2)^3 = 1)$$

Take $H_1 = \langle x_1 \rangle$, $H_2 = \langle x_2 \rangle$, so both are of index 3. Each coset intersects two cosets in the other list giving a nerve of form (see below):



so $\pi_1 N(\mathfrak{H}(S_3, \mathcal{H}))$ is infinite cyclic.

Example: The next symmetric group, S_4, has presentation

$$S_4 := (x_1, x_2, x_3 \mid x_1^2 = x_2^2 = x_3^2 = 1, (x_1x_2)^3 = (x_2x_3)^3 = 1, (x_1x_3)^2 = 1).$$

Take $H_1 = \langle x_1, x_2 \rangle$, $H_2 = \langle x_2, x_3 \rangle$, $H_3 = \langle x_1, x_3 \rangle$. H_1 and H_2 are copies of S_3, but H_3 is isomorphic to the Klein 4 group, K_4. Thus there are $4 + 4 + 6$ cosets in all. There are 36 pairwise intersections and each edge is in two 2-simplices. Each vertex is either at the centre of a hexagon or a square, depending on whether it corresponds to a coset of H_1, H_2 or of H_3. There are 24 triangles, and $N(S_4, \mathcal{H})$ is a surface. Calculation of the Euler characteristic gives 2, so this is a triangulation of S^2, the two sphere. (Thanks to Chris Wensley for help with the calculation using GAP.)

The fundamental group of $N(S_4, \mathcal{H})$ is thus trivial and, using the result mentioned above,

$$S_4 \cong \bigsqcup_{\cap} H_i,$$

the coproduct of the subgroups amalgamated over the intersection.

Accepting Proposition 25 for the moment, we can examine an important class of examples.

Example: Some graphs of groups. Let us suppose that $\mathcal{H} = \{H_1, H_2\}$, so just two subgroups of G, then we have

$$H_1 \underset{H_1\cap H_2}{\sqcup} H_2 \to G.$$

This is an isomorphism if and only if $N(\mathfrak{H})$ is a connnected graph which has trivial fundamental group, thus exactly when $N(\mathfrak{H})$ is a *tree*. The vertices of $N(\mathfrak{H})$ are the cosets in $(H_1\backslash G) \sqcup (H_2\backslash G)$ and H_1g_1 and H_2g_2 are connected by an edge if they intersect. This gives us one of the two basic types of a *graph of groups* as defined by Serre, [194, 195],

$$H_1 \underset{H_1\cap H_2}{\rule{4em}{0.4pt}} H_2$$

corresponding to a free product with amalgamation. Note this does not give us the other basic type of graph of groups, which corresponds to an HNN-extension. We will explore the connection with this theory in more detail a bit later or, more exactly, we will see a connection with the generalisation *complexes of groups* due to Corson, [35, 68–71] and Haefliger, [110, 111], and developed extensively in the book by Bridson and Haefliger, [36].

We have now seen, somewhat informally, discussions of the low dimensional homotopy invariants of these two nerves, both in examples and, to some extent, in general. We turn now to more formal calculations of those, and in the process will prove Proposition 25.

We will approach the determination of the invariants in an 'elementary' but reasonably formal way. We will repeat some arguments that we have already seen partially to get everything in the same place, but also to impose some more consistent notation.

The set, $\pi_0(V(G,\mathcal{H}))$, of connected components: The vertex set of $V(G,\mathcal{H})$ is the set of elements of G, so we have to work out when two vertices, g and g', are in the same connected component.

Suppose they are connected by a path, that is a sequence of edges, $(\langle g_0, g_1\rangle, \langle g_1, g_2\rangle, \ldots \langle g_{n-1}, g_n\rangle)$, in $V(G,\mathcal{H})$ and for some n. We have that an edge such as $\langle g_0, g_1\rangle$ has $d_0\langle g_0, g_1\rangle = g_1$ and $d_1\langle g_0, g_1\rangle = g_0$ and it is an edge because there is some $H_{\alpha_1} \in \mathcal{H}$ and some $x_1 \in G$ such that g_0 and g_1 are in the coset $H_{\alpha_1}x_1$. Of course, this means that there are $h_0, h_1 \in H_{\alpha_1}$ with $g_0 = h_0x_1$ and $g_1 = h_1x_1$, hence that $g_0g_1^{-1} \in H_{\alpha_1}$. (Conversely if $g_0g_1^{-1} \in H_{\alpha_1}$, then both g_0 and g_1 are in $H_{\alpha_1}g_1$, so $\langle g_0, g_1\rangle$ is an edge.)

We thus have from our path that there are indices $\alpha_1, \ldots, \alpha_n$ such that $g_{i-1}g_i^{-1} \in H_{\alpha_i}$ for each i, whilst $g = g_0$ and $g' = g_n$. We then note that gg'^{-1} is in $\langle\bigcup\mathcal{H}\rangle$, the subgroup generated by the union of the subgroups in the family $\mathcal{H}$, so, if g and g' are in the same component, then $gg'^{-1} \in \langle\bigcup\mathcal{H}\rangle$.

Conversely, suppose $gg'^{-1} \in \langle\bigcup\mathcal{H}\rangle$, then there is a finite sequence of indices, $\alpha_1, \ldots, \alpha_n$ for some n and elements $h_i \in H_{\alpha_i}$ such that $gg'^{-1} = h_1h_2\ldots h_n$. We define $g_0 = g$, $g_i = h_i^{-1}g_{i-1}$ and note that $g_{i-1}, g_i \in H_{\alpha_i}g_i$, thus giving us a path from g to $g_n = h_n^{-1}g_{n-1} = h_n^{-1}\ldots h_1^{-1}g_0 = g'$.

We thus have proved that $\pi_0(V(G,\mathcal{H}))$ is in bijection with $G/\langle\bigcup\mathcal{H}\rangle$, that is the first part of Proposition 25.

The fundamental group, $\pi_1(V(G,\mathcal{H}),1)$, and groupoid, $\Pi_1(V(G,\mathcal{H}))$: Although $V(G,\mathcal{H})$ comes with a natural choice of basepoint, namely 1, and we will eventually be looking at loops at 1, it is more in tune with our just previous discussion to look at the fundamental groupoid $\Pi_1(V(G,\mathcal{H}))$ rather than the fundamental group $\pi_1(V(G,\mathcal{H}),1)$ of $V(G,\mathcal{H})$ based at 1. We will sometimes abbreviate $\Pi_1(V(G,\mathcal{H}))$ to $\Pi_1\mathfrak{H}$.

The set of objects of this groupoid will be the vertices of $V(G,\mathcal{H})$ and so are the elements of G, and the set of arrows $\Pi_1\mathfrak{H}(g,g')$ will be the set of homotopy classes of paths from g to g'. We saw that a path from, g to g' corresponds to a finite sequence, $\underline{h} = (h_1,h_2,\ldots,h_n)$, of elements from the various subgroups H_{α_i} in $\mathcal{H}$. It is convenient to write

$$g \xrightarrow{(h_1,h_2,\ldots,h_n)} g' = g \xrightarrow{\underline{h}} g',$$

where $h_n^{-1}\ldots h_1^{-1}g = g'$. We can see that given two composable paths

$$g \xrightarrow{\underline{h}} g' \xrightarrow{\underline{h}'} g'',$$

the defining sequence of the composite is given by the concatenation of the two sequences,

$$\underline{hh'} = (h_1,h_2,\ldots,h_n,h'_1,h'_2,\ldots,h'_m).$$

Remark: This notation is not quite accurate. The $\underline{h}$ does not indicate from where the arrow, so labelled, starts. Of course, it is visually clear, but 'really' we should denote the arrows by $(g,\underline{h})$, so then

$$(g,\underline{h})\cdot(\underline{h}^{-1}g,\underline{h}') = (g,\underline{hh'}),$$

or similar. This is clearly a form related to, but not identical, to some sort of 'action groupoid', but that does not quite fit. For a start, it does not give a groupoid as where are the inverses? It does give a category, however. (It is **left for you to check** that $\langle g_0,g_0\rangle$ is the identity at the 'object' g_0.)

The paths between the vertices are not the actual arrows in the fundamental groupoid $\Pi_1\mathfrak{H}$. For that we need to divide out by relations coming from 2-simplices.

For any simplicial complex or simplicial set, K, one can form the fundamental groupoid, (also called in this context the *edge path groupoid*), by taking the free groupoid on the directed graph given by the 1-skeleton and then dividing out by the 2-simplices. (We will see this several times later; see pages 214, and **??**. It is the classical edge-path groupoid to be found, for instance, in Spanier's book, [198].) The arrows are sequences of concatenated edges and then, if $\langle v_0,v_1,v_2\rangle$ is a 2-simplex, we add a 'relation'

$$\langle v_0,v_1\rangle\langle v_1,v_2\rangle = \langle v_0,v_2\rangle,$$

or if you prefer, rewrite rules:

$$\langle v_0,v_1\rangle\langle v_1,v_2\rangle \Leftrightarrow \langle v_0,v_2\rangle.$$

For $\Pi_1\mathfrak{H}$, a 2-simplex in $V(G,\mathcal{H})$ will, of course, be a triple, (g_0,g_1,g_2), of elements of G contained in some $H_\alpha x$. We explore this in detail as before. There will be three elements, h_0,h_1,h_2 in H_α with $g_i = h_ix$ for $i = 0,1,2$ and thus $g_ig_j^{-1} \in H_\alpha$, for each i and j.

Dividing out by these relations has several neat consequences which 'control' the paths and their compositions. For instance, working in the simplicial set version of $V(G,\mathcal{H})$, if we have $\langle g_0, g_1\rangle$ in $V(G,\mathcal{H})$, then $\langle g_1, g_0\rangle$ is there as well, and so is $\langle g_0, g_0\rangle$ and as $\langle g_0, g_1, g_0\rangle$ is in $V(G,\mathcal{H})_2$, we have that

$$\langle g_0, g_1\rangle\langle g_1, g_0\rangle = \langle g_0, g_0\rangle,$$

so $\langle g_0, g_1\rangle$ has $\langle g_1, g_0\rangle$ as its inverse. Another important result of these relations is that it allows simplification of the path labelling sequences. Suppose we have a composite path

$$g_0 \xrightarrow{h_1} g_1 \xrightarrow{h_2} g_2$$

which stays more than one step in a given coset, i.e., both h_1 and h_2 are in some H_α. In this case we can clearly replace that path, up to homotopy, that is, modulo the relations, by

$$g_0 \xrightarrow{h_1h_2} g_2$$

as $\langle g_0, g_1, g_2\rangle$ is a 2-simplex. This means that every arrow in $\Pi_1\mathfrak{H}$ has a representative whose corresponding sequence $\underline{h}$ corresponds to an element of the coproduct (aka free product), $\sqcup H_i$, of the groups in $\mathcal{H}$. This is still not a unique representative however. We may have a situation

$$g_0 \xrightarrow{h_1} g_1 \xrightarrow{h_2} g_2 \xrightarrow{h_3} g_3$$

where $h_1, h_2 \in H_i$ and $h_2, h_3 \in H_j$, so we will have an overlap with $\langle g_0, g_1\rangle\langle g_1, g_2\rangle\langle g_2, g_3\rangle$ rewriting both to $\langle g_0, g_2\rangle\langle g_2, g_3\rangle$ and to $\langle g_0, g_1\rangle\langle g_1, g_3\rangle$, and so we have to *amalgamate* the coproduct over intersections.

Let us be a bit more precise about this. We form up a diagram of the subgroups H_i in $\mathcal{H}$, together with their pairwise intersections, $H_i \cap H_j$. We write $H = \underset{\cap}{\sqcup}\mathcal{H}$ for its colimit.

Definition: Given a family, $\mathcal{H}$, of subgroups of G, its *free product* or *coproduct amalgamated along the intersections* is the colimit, H, specified above.

This group, H, can be given as simple presentation. Take as set of generators a set, $X = \{x_g \mid g \in \bigcup H_j\}$, in bijection with the elements of the union of the underlying sets of subgroups in $\mathcal{H}$, and for relations all $x_{h_1}x_{h_2} = x_{h_1h_2}$ where h_1 and h_2 are both in some group, H_i, of the family.

The inclusion of each H_j into G gives a cocone on the diagram of groups, so induces a homomorphism, $p : \underset{\cap}{\sqcup}\mathcal{H} \to G$, which will be essential in our description. This homomorphism, p, thus takes a sequence $\underline{h} = (h_1, \ldots, h_n)$ representing some element of H and evaluates it within G mapping it to the product $h_1 \ldots h_n \in G$.

Clearly we have

Proposition 29 *The fundamental groupoid, $\Pi_1\mathfrak{H}$, has for objects the elements of G and an arrow from g to g' is representable, uniquely, by an element h in $\underset{\cap}{\sqcup}\mathcal{H}$ such that $g = p(h)g'$.* ∎

The proof is by comparison of the two presentations.

Corollary 6 *There is an isomorphism*

$$\pi_1\mathfrak{H} \cong Ker(p : \underset{\cap}{\sqcup}\mathcal{H} \to G)$$

Proof: The group $\pi_1(V(G,\mathcal{H}),1)$ is the vertex group at 1 of the edge path groupoid, so consists of the $h$in H, which evaluate to 1, since here $g = g' = 1$, i.e. the vertex group is just $Ker\, p$. ■

This means that we have $p : H \to G$, whose 'cokernel', $G/p(H)$, 'is' $\pi_0(V(G,\mathcal{H}))$ and whose kernel is $\pi_1(V(G,\mathcal{H}),1)$.

What about $\pi_2 V(\mathfrak{H})$? We will limit ourselves, here, to a special case, and will merely quote a result from the paper of Abels and Holz, [1]. We suppose as always that we are given $(G,\mathcal{H})$ and now assume that we use the standard presentation $\mathcal{P}_j := (X_j : R_j)$ of each H_j. Combining these we get $X = \bigcup X_j$, $R = \bigcup R_j$. We have $\mathcal{H}$ is 2-generating for G if and only if $\mathcal{P} = (X,R)$ is a presentation of G. (That is nice, since it says that there are no hidden extra relations needed, and that corresponds to the intuitions that we were mentioning earlier. There is better to come!) Assuming that $\mathcal{P}$ is a presentation of G, we have a module of identities, $\pi_{\mathcal{P}}$. We also have all the $\pi_{\mathcal{P}_j}$, the identity modules for each of the presentations, $\mathcal{P}_j$. The inclusions of generators and relations induce morphisms of the crossed modules, $C(\mathcal{P}_j) \to C(\mathcal{P})$, and hence of the modules $\pi_{\mathcal{P}_j} \to \pi_{\mathcal{P}}$, although here there is the slight complication that this is a morphism of modules over the inclusion of H_j into G, which we will not look further into here. We let $\pi_{\mathcal{H}}$ be the sub G-module of $\pi_{\mathcal{P}}$ generated by the images of these $\pi_{\mathcal{P}_j}$. We can think of $\pi_{\mathcal{H}}$ as the sub-module of $\pi_{\mathcal{P}}$ consisting of those identities that come from the presentations of the subgroups.

In the above situation, i.e., with standard presentations for the subgroups, we have ([1] Cor. 2.9.)

Proposition 30 *If $\mathcal{H}$ is 2-generating, then there is an isomorphism:*

$$\pi_2(N(\mathfrak{H}) \cong \pi_{\mathcal{P}}/\pi_{\mathcal{H}}.$$

■

We should therefore, and in this case at least, interpret $\pi_2(N(\mathfrak{H})$ as telling us about the 2-syzygies that are not due to the presentations of the subgroups. We will give shortly a neat example of this but first would note that this does not interpret the 2-type of $V(\mathfrak{H})$ in general, and that somehow is a lack in the theory as developed so far. Abels and Holz do extend thie away from the standard presentations of the subgroups, but this requires a bit more than we have available at this stage in the notes so will be 'put on hold' until later.

This gives all the easily available data on these Vietoris-Volodin complexes as far as their elementary homotopy information is concerned. We can, and will, extract more later on, but now want to look at the main example for their original introduction.

3.3.10 Back to the Volodin model ...

Our 'more complex family' of section 3.3.3 leads to a link with higher algebraic K-theory in the version developed initially by Volodin. The usual approach, however, uses a slightly different notation and for some of its details ends up looking different, so here we will give the version of that example nearer to that given by, for instance, Suslin and Wodzicki, [202], or Song, [196]. Let, as before, R be an associative ring, and now let σ be a partial order on $\{1,\ldots,n\}$. If i is less that j in the partial order σ, it is convenient to write $i \overset{\sigma}{<} j$. (Note that this means that some of the elements may only be related to themselves and hence are really not playing a role in such a σ.) We will write $PO(n)$ for the set of partial orders of $\{1, \ldots, n\}$.

Definition: We say an $n \times n$ matrix, $A = (a_{ij})$ is *σ-triangular* if, when $i \overset{\sigma}{\nleq} j$, $a_{ij} = 0$, and all diagonal entries, a_{ii} are 1.

We let $T_n^\sigma(R)$ be the subgroup of $G\ell_n(R)$ formed by the σ-triangular matrices.

Lemma 21 *If $n \geq 3$, $T_n^\sigma(R)$ has a presentation with generators $x_{ij}(a)$, where $i \overset{\sigma}{<} j$ and $a \in R$, and with relations:*

$$x_{ij}(a)x_{ij}(b) = x_{ij}(a+b) \qquad i \overset{\sigma}{<} j, \quad a, b \in R$$

and

$$[x_{ij}(a), x_{jk}(b)] = x_{ik}(ab) \qquad i \overset{\sigma}{<} j \overset{\sigma}{<} k, \quad a, b \in R,$$

$$x_{ij}(a)x_{k\ell}(b) = x_{k\ell}(b)x_{ij}(a), \qquad i \neq \ell, j \neq k, \quad a, b \in R.$$

■

Remark: In fact, Kapranov and Saito, [132], mention that, not only is this a presentation of $T_n^\sigma(R)$, but with the addition of the syzygies that they describe (and which up to dimension 2 are those given in our section 3.1.2) gives a complete set of syzygies, of dimension 3.

We can 'stablise' the above, since it σ is a partial order on $\{1, \ldots, n\}$, then it extends uniquely to one on $\{1, \ldots, n+1\}$ by specifying that $n+1$ is related to itself in the extended version, but to no other. (The notation and treatment for this is not itself that 'stable' and some sources do not go into a detailed handling of this point, presumably because it is clear what is going on.) We will write $\mathfrak{T}_n = (G\ell_n(R), \mathcal{T}_n)$, where $\mathcal{T}_n = \{T_n^\sigma(R) \mid \sigma \in PO(n)\}$, and then, letting n 'go to infinity' write $\mathfrak{T}$ for the corresponding system based on $G\ell(R)$ with all σ-triangular subgroups for all partial orders having finite 'support', i.e., in which outside some finite set, (its support), the partial order is trivial.

Proposition 31 *For $n \geq 3$, the subgroup of $G\ell_n(R)$ generated by the union of the $T_n^\sigma(R)$ is $E_n(R)$, the elementary subgroup of $G\ell_n(R)$.*

Proof: This should be more or less clear as, by definition, any elementary matrix is σ-triangular for many σ', and conversely, any $T_n^\sigma(R)$ is given as a subgroup of $E_n(R)$. ■

Corollary 7 *The Volodin nerve, $V(\mathfrak{T})$, has*

$$\pi_0 V(\mathfrak{T}) \cong K_1(R).$$

■

The obvious next question to pose is what $\pi_1(V(\mathfrak{T}), 1)$ will be. We know it to be the kernel of $\underset{\cap}{\sqcup} T_n^\sigma(R) \to E(R)$, and the obvious guess would be that it was Milnor's $K_2(R)$. That's right. Proofs are given in several places in the literature, but usually they require a bit more machinery than we have been assuming up to this point in these notes, so we will not give one of those proofs here. The most usual proofs use the natural action of G on $N(\mathfrak{H})$ and a covering space argument.

We will mention this in a bit more detail after we have looked at a sketch proof and will explore aspects of this sort of approach more in a later chapter, but here will attempt to give that sketch proof which, it is hoped, seems more direct and which starts from the descriptions of $\pi_0 V(\mathfrak{T})$ that are consequences of what we have already done above. (We will still need a covering space-type argument, which, since central extensions behave like covering spaces from many points of view, is suggestive of a general approach that is, it seems, nowhere given in the literature with the conceptual simplicity it seems to deserve. Kervaire's treatment of universal central extensions, [136], perhaps goes some way towards what is needed.) We start by looking at paths in $V(\mathfrak{T})$, especially, but not only, those which start at 1. We will be, in part, following Volodin's original treatment in [216] as this is very elementary and 'constructive' in nature. As we said above, he uses covering space intuitions as well, as this seems almost optimal for the identification we need. (Remember that one classical construction of universal covering spaces is from the space of paths that start at the base point, followed by quotienting by fixed end point homotopy as a relation.)

A path in $V(\mathfrak{T})$ as it is of finite length, must live in some $V(\mathfrak{T}_n)$. We thus can represent it by a pair, $(g, \underline{t})$, with $\underline{t} = (t_1, \ldots, t_k)$ for some k, a word with each t_i in some $T_n^{\sigma_i}(R)$, and g in $E_n(R)$ which will be the starting element of the path. (Of course, this representation is not unique, because of the amalgamated subgroups, and we will need to break each t_i up as a product of elementary matrices shortly. The non-uniqueness will be taken account of later on.)

We say that t_i is a *segment* of the path, and that the paths is *elementary* if all the t_is used are elementary matrices.

We now need some 'elementary' linear algebra. We will look at it with respect to the standard maximal linear order on $\{1, \ldots, n\}$ and hence for upper triangular matrices.

Lemma 22 *Let $B = (b_{ij})$ be an upper triangular matrix (with 1s on its diagonal), so b_{ij} is zero if $j < i$. There is a factorisation*

$$B = \prod_{(i,j)} e_{ij}(b_{ij}),$$

with the order of multiplication given by increasing lexicographic order, so $(i, j) > (i_1, j_1)$ if either a) $j > j_1$ of b) $j = j_1$ and $i > i_1$. ■

The proof should be obvious.

We can replace t_k by a path consisting only of elementary matrices (for the ordering σ_i) and with the order of terms given by a lexicographic order in the (i, j)s relative to $\overset{\sigma_i}{<}$. The resulting $t_k = \prod_{(i,j)} e_{ij}(b_{ij})$ and can be 'lifted' to an element

$$\bar{t}_k = \prod_{(i,j)} e_{ij}(x_{ij}) \in St_n(R).$$

This element maps down to the element t_k in $E_n(R)$.

Suppose s is a loop, based at 1, in $V(\mathfrak{T})$, but consisting just of elementary matrices in some $T_n^{\sigma_k}(R)$. (We will say s is an elementary loop. We will work with the standard linear order.) As s is a loop at 1, it has a representation as $(1, \underline{s})$, where $\underline{s} = (s_1, \ldots, s_N)$ and the s_ks are in lexicographic order, each s_k is some $e_{ij}(a_ij)$ and, as the path s is a loop, $\prod_{(i,j)} e_{ij}(a_{ij}) = 1$.

Lemma 23 *If s is an elementary loop at 1 in $T_n(R)$, then its lift $\overline{s}$ is $1 \in St_n(R)$.*

Before giving a proof, remember the intuition that seems to be built in Volodin's approach. The $T_n^\sigma(R)$ are seen as patches over which there is a way of lifting paths, so you decompose a long path into bits in the various patches, and then lift them successively. The lifted bits give elements in $St_n(R)$, and 'up there' we have divided out by the homotopy that comes from the relations / rewriting 2-cells. In each patch we expect to get that the lift of s that we are using gives a trivial element (i.e. something like a null-homotopic loop. We thus expect to have to use the presentation of $St(R)$ and, in particular, the embryonic homotopies given by the rewriting 2-cells / relations. As we will see that is exactly what happens.

Proof: We let m be larger than all the i, j involved in the expression for s. (We will generally write $x_{ij}(a)$ etc where a is variable and is really just a 'place marker'.) As $x_{im}(a)x_{kj}(b) = x_{kj}(b)x_{im}(a)$ for $i \neq j$, $k \neq m$, and

$$x_{im}(a)x_{ki}(b) = x_{km}(-ab)x_{ki}(b)x_{im}(a) = x_{ki}(b)x_{km}(-ab)x_{im}(a),$$

we can move all terms of form $x_{im}(a)$ to the right of the product expression for $\overline{s}$. In $St_m(R)$, we thus have

$$\prod_{i<j\leq m} x_{ij}(a) = \prod_{i<j\leq m-1} x_{ij}(a) \cdot \prod_{i<m} x_{im}(a),$$

where, as we said, the a is just a place marker. We thus have that $\overline{s}$ in $St(R)$ can be decomposed as the product of two parts corresponding to loops (down in $E(R)$). These are $\prod_{i<j\leq m-1} x_{ij}(a)$ and $\prod_{i<m} x_{im}(a)$. (As this latter is in the subgroup of $St_m(R)$ generated by the $x_{im}(a)$, this must itself evaluate to 1 as the product does, hence also the other factor must.) Working on the product $\prod_{i<m} x_{im}(a)$ and using the facts firstly that the terms commute with each other by the first rule we recalled above, and then using the first Steinberg relation: $St1 : x_{im}(a)x_{im}(b) = x_{im}(a+b)$, we can now check that this word must itself be trivial as it evaluates to 1.

We now can use backwards induction on m to gradually you get back to the minimal value possible and get the result. ■

Corollary 8 *If s is an elementary loop in some $T_n^\sigma(R)$, then the corresponding lifted word in $St(R)$ is trivial.*

Proof: We have done most of this, except it was in the case of the standard linear order. One can either adapt the above to the general case, or more neatly note that s conjugates, using permutation matrices, to give an element in that linear case. The lifting goes across to $St(R)$ and so the result follows after a bit of checking. ■

Now look at any path in $V(\mathfrak{T})$, starting at 1. Take an elementary representative and examine the initial segment, $1 \xrightarrow{t_1} t_1^{-1}$, so $t_1 \in T_n^{\sigma_1}(R)$. We can lift t_1 to give an element $\bar{t}_1 \in St_n(R)$. This will, in general, depend on the choice of σ_1, but if σ_1' is another possible partial order (i.e., $t_1 \in T_n^{\sigma_1}(R) \cap T_n^{\sigma_1'}(R)$, then the resulting two lifts of t_1 will form a 'loop' $\bar{t}_1 \cdot \overline{t'}_1^{-1}$ in $St_n(R)$, but then this loop must be trivial by the lemma and its corollary. We pass to the next 'node' in the path and continue. The next segment does not start at 1, but the argument adapts easily as the corresponding labelling element in the coproduct with amalgamation is all that is used.

This gives that each path s in $V(\mathfrak{T})$ uniquely determines an element $\overline{s}$ in $St(R)$. It is now fairly clear where the argument has to go. The standard classical construction of a universal covering space is via paths starting at some base point 'modulo' fixed endpoint homotopy, so one checks that homotopic paths lift to the same element of $St(R)$. (This is Volodin's Lemma 3.4 of [216], but it is easy to see how it is to go.) Volodin is using the 'patches' given by the $T_n^\sigma(R)$ to lift a path in $E_n(R)$. (This mix of topological intuition with combinatorics and algebra is the starting point of Bak's theory of global actions, [14, 15], that was mentioned earlier.)

It is now feasible to complete the proof à la Volodin, that the universal cover of $V(E_n(R), \{T_n^\sigma(R)\})$ is 'related to' $St_n(R)$, but that is not really satisfactory as it mixes the categories in which we are working. (A simplicial complex is not a group!) We have a more limited aim, namely to note that if we have an element in $\pi_1(V(\mathfrak{T}), 1)$, then we can pick a loop, s, representing it. We can lift s uniquely by lifting over each 'patch' $T_n^\sigma(R)$ that it uses, to obtain an element in $St(R)$, but as it is a loop its evaluation, back down in $G\ell(R)$ will be trivial. (Topologically its endpoint is over the basepoint!) It is in the kernel of the homomorphism from $St(R)$ to $G\ell(R)$, so determines an element of $K_2(R)$. Finally one reverses the argument to say that if $\overline{s} \in K_2(R)$, then it is in the image of this morphism. We have thus given an idea of how Volodin's theorem, below, can be proved, using fairly elementary ideas.

Theorem 6 *(Volodin, [216], Theorem 2)*

$$\pi_1(V(\mathfrak{T}), 1) \cong K_2(R).$$

$\blacksquare$

Remark: The usual proofs of this result given in more recent sources tend to use the classifying spaces, $BT_n^\sigma(R)$ together with the induced mappings to $BG\ell(R)$ to obtain

$$\bigcup BT_n^\sigma(R) \to BG\ell(R),$$

which is then shown to give the 'homotopy fibre' of the map to $BG\ell(R)^+$. This does seem slightly too reliant on spatially based methods from homotopy theory and a more purely combinatorial group theoretic or 'rewriting' analysis of the constructions, related to Volodin's original proof, should be possible.

We hope to return to the study of the Volodin model for higher algebraic K-theory later on, but are near to the limit of what can be done with the limited tools at our disposal here, so will put it aside for the moment.

3.3.11 The case of van Kampen's theorem and presentations of pushouts

The above example / case study coming from algebraic K-theory is very rich in its structure and its applications, but *is* complex, so we will return to a simpler situation to indicate the direction that this theory of 'higher generation by subgroups' can lead us to. To motivate this recall the formulation of the classical form of van Kampen's theorem.

Theorem 7 *(van Kampen) Let $X = U \cup V$, where U, V and $U \cap V$ are non-empty, open and arc-wise connected. Let $x_0 \in U \cap V$ be chosen as a base point, then the diagram*

$$\begin{tikzcd}
\pi_1(U \cap V) \arrow[r, "j_{V*}"] \arrow[d, "j_{U*}"'] & \pi_1(V) \arrow[d, "i_{V*}"] \\
\pi_1(U) \arrow[r, "i_{U*}"'] & \pi_1(X)
\end{tikzcd}$$

is a pushout square of groups, where each fundamental group is based at x_0. ■

Proofs can be found in many places in 'the literature', for instance, in Massey's introduction, [154], or in Crowell and Fox, [73]. A proof of a neat more general form of the result is given in Brown's book, [42]. There the result is given in terms of fundamental groupoids, which is very useful for many applications and several variants are also given there. We may have need for some of these later on, but for the moment what we want is the version in terms of group presentations, cf. [73], page 71, for example. This just translates the above pushout result into one about presentations.

Theorem 8 *(van Kampen: alternative form) Let* $X = U \cup V$*, etc., be as above. Suppose*

- *that* $\pi_1(U, x_0)$ *has a presentation,* $(\mathbf{X} : \mathbf{R})$*,*
- *that* $\pi_1(V, x_0)$ *has a presentation,* $(\mathbf{Y} : \mathbf{S})$*,*

and

- *that* $\pi_1(U \cap V, x_0)$ *has one,* $(\mathbf{Z} : \mathbf{T})$*,*

then $\pi_1(X, x_0)$ *has a presentation,*

$$(\mathbf{X} \cup \mathbf{Y} : \mathbf{R} \cup \mathbf{S} \cup \{(\overline{j_{U*}(z)})(\overline{j_{V*}(z)})^{-1} \mid z \in \mathbf{Z}\}),$$

where $\overline{j_{U*}(z)}$ *is a word in the free group,* $F(\mathbf{X})$ *representing* $j_{U*}(z)$*, and similarly for* $\overline{j_{V*}(z)}$. ■

This form gives a way of calculating a presentation, $\mathcal{P}$, of $\pi_1(X, x_0)$ given presentations of the parts. If we see a presentation as the first part of a recipe to construct a resolution of a group, or alternatively to construct an Eilenberg-Mac Lane space for the group, then this is useful and, of course, is used in courses on elementary algebraic topology to calculate the fundamental groups of surfaces. The obvious points to note are that we take the union of the two generating sets, $\mathbf{X}$ and $\mathbf{Y}$, to be the generating set of $\pi_1(X, x_0)$, but use the generators in $\mathbf{Z}$ to help form *relations* in the pushout presentation, then we use the union of the two sets of relations to give the other relations (which seems sort of natural). This leaves a query. Whatever happened to the relations in the presentation of $\pi_1(U \cap V, x_0)$? To get some idea of what they do, think along the following somewhat vague lines. As those relations correspond to maps of 2-discs into the complex, $K(\mathcal{P})$, of the presentation, $\mathcal{P}$, used to 'kill' the corresponding words, we have two 2-discs with 'the same' boundary and hence map of a 2-sphere into $K(\mathcal{P})$ with no reason for it being homotopically trivial. This suggests that the relations in $\mathbf{T}$ are going to give homotopical 2-syzygies, and this is the case. It also suggests that to build an Eilenberg-MacLane / classifying space from the presentation, $\mathcal{P}$, we could do worse than take the pushout of the complexes of the various other presentations involved.

It is a good idea to abstract this out a bit away from the van Kampen situation for the moment. We suppose that $G = A *_C B$ is a 'free product with amalgamation', so we can describe G by means of a pushout of groups:

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[d, "i"'] & B \arrow[d, "i'"] \\ ar[r]_{j'} & G \end{tikzcd}$$

It is a standard result that if i and j are injective, then so are i' and j'.

The van Kampen examples can be a bit complex to work through, but we can, in fact, gain some intuition about them from one of the simplest examples of such situations. Consider the trefoil knot group, $G(T_{2,3})$. This has a presentation $(a, b : a^3b^{-2} = 1)$. It is therefore an amalgamated coproduct / pushout of three infinite cyclic groups:

$$\begin{tikzcd}
(z:\emptyset) \arrow[r, "j"] \arrow[d, "i"'] & (b:\emptyset) \arrow[d] \\
(a:\emptyset) \arrow[r] & G(T_{2,3})
\end{tikzcd}$$

where $i(z) = a^3$ and $j(z) = b^2$. We note that all the input presentations are with empty sets of relations, yet $G(T_{2,3})$ has a single non-trivial relation. If we took the complexes of each presentation, we would merely have a circle for each, and that of the presentation of $G(T_{2,3})$ has to have a 2-cell in it, hence we can see that the construction of the presentation of $G(T_{2,3})$ does not just result from a 'pushout of presentations'! (In fact, what is needed is a homotopy pushout, or, in more general situations than the pushout of a diagram of group, a homotopy colimit. We will say a bit more on this shortly.) We now return to our general situation.

Our abstracted situation is that we have presentations, $\mathcal{P}_Q = (X_Q : R_Q)$ for $Q = A, B$ and C, and get the corresponding presentation for G, given by the analogue of that in the above discussion. We take complexes $K(\mathcal{P}_Q$ modelling each of the presentations in turn. The morphisms between the groups in the diagram lift give a diagram

$$\begin{tikzcd}
K(\mathcal{P}_C) \arrow[r, "j_*"] \arrow[d, "i_*"'] & K(\mathcal{P}_B) \arrow[d, "i'_*"] \\
K(\mathcal{P}_A) \arrow[r, "j'_*"'] & K(\mathcal{P}_G)
\end{tikzcd}$$

but as the lifts have to be *chosen*, they are only determined up to homotopy, and this will in general only be a square that is homotopy coherent, i.e., commutative up to a specified homotopy, (see the later discussion in Chapter **??**). In fact, as we do not know that i_* and j_* are injective, the result need not be a pushout, so does not tell us much. An alternative is to see what we can construct from the 'corner':

$$\begin{tikzcd}
K(\mathcal{P}_C) \arrow[r, "j_*"] \arrow[d, "i_*"'] & K(\mathcal{P}_B) \\
K(\mathcal{P}_A) &
\end{tikzcd}$$

from this we can take its 'homotopy pushout' which begins to be more like the square we had. We have not met this construction yet; it is a double mapping cylinder. This would form a cylinder on $K(\mathcal{P}_C)$ and use the maps to glue copies of the other spaces to its two ends. In here, we will be getting a cylinder with the discs corresponding to the relations in $\mathcal{P}_C$ and these will to cylindrical 2-cells in that double mapping cylinder and hence to a potential homotopical 2-syzygy. This will be picked up by the crossed module of that space or better still the crossed complex. An analysis of this can be found in Brown-Higgins-Sivera, [49], starting on page 338. This is based on an earlier paper by Brown, Moore, Porter and Wensley, [53]. (As an exercise, it is **worth looking at the**

trefoil group from this viewpoint and to draw what intuitively the mapping cylinder must look like ... as much as this is feasible.)

We have used this discussion above for two main reasons, first to suggest that the situation *naturally* leads to having to take the homotopies seriously and that implies a study of (at least some) homotopy coherence theory, and homotopy colimits in particular. The other reason is that it suggests that it provides a key set of concepts, as yet at a vague intuitive level, to understand more fully the theory of 'higher generation by subgroups' of Abels and Holz, [1]. If we get our group G, and a 1-generating family of subgroups, $\mathcal{H}$, and want to work out the 'syzygies of G', i.e., some combinatorial information to enable a (crossed) resolution or a small model of a $K(G,1)$ to be formed, then the idea is that by calculating the syzygies of each of the input groups, the n-syzygies of G should involve those of the H_is, but also the $(n-1)$-syzygies of the pairwise intersections, $H_i \cap H_j$, and then, why not, the $(n-2)$-syzygies of the triple intersections, and so on. We certainly do not have the machinery to pursue this here, and so will leave it vague.

(In addition to the above references on the pushout, which use homotopy colimits of crossed complexes over groupoids, the original paper of Abels and Holz, [1], also uses homotopy colimit techniques, but this time with chain complexes. It uses these to prove results on the homological finiteness properties of certain groups. That paper is well worth reading. This use of homotopy colimits is also explored in Stephan Holz's thesis, [121].)

3.4 Group actions and the nerves

Although we will be continuing the theme of the previous section, we will be expanding out our view of the area slightly so as to gain additional tools for the study of the nerve and those Vietoris-Volodin complexes, but also some more interesting potential applications.

Further information on the nerve, $N(\mathfrak{H})$, and the Vietoris-Volodin complex, $V(\mathfrak{H})$, of the coset covering corresponding to a family, $\mathcal{H}$, of subgroups of G, can be obtained by exploiting the natural action of G on these simplicial complexes. This leads to a further connection of these objects to simple examples of key ideas from geometric group theory via the notion of a complex of groups due to Haefliger, [110, 111] and Corson, [68–70].

3.4.1 The G-action on $N(\mathfrak{H})$

Let G be a group, and, as before, $\mathcal{H} = \{H_i \mid i \in I\}$, a family of subgroups of G. Usuall we will assume that $\mathfrak{H} = (G, \mathcal{H})$ is 1-connected, so that G is generated by the union of the subgroups in $\mathcal{H}$. We, again as before, write $N(\mathfrak{H})$ for the nerve of the covering of G formed by the right cosets of the H_is and $V(\mathfrak{H})$ for the corresponding Vietoris complex / Volodin space.

There is an obvious right G-action on $N(\mathfrak{H})$ given as follows:

An n-simplex of $N(\mathfrak{H})$ has the form

$$\sigma = \{H_{\alpha_0}x_0, \ldots, H_{\alpha_n}x_n\},$$

where

$$\bigcap \sigma = \bigcap_{k=0}^{n} H_{\alpha_k}x_k$$

is non-empty. If we have $g \in G$, we define $\sigma \cdot g = \{H_{\alpha_0}x_0g, \ldots, H_{\alpha_n}x_ng\}$ multiplying each coset representative on the right by g.

It is easy to see that if $y \in \bigcap \sigma$, then $yg \in \bigcap(\sigma \cdot g)$, so $\sigma \cdot g$ is an n-simplex of $N(\mathfrak{H})$.

This action has various very nice features. The terminology on these is a bit of a minefield. The action is 'without inversion' in the sense of Haefliger, [110], (but, in his later paper, he changed terminology and redefined that term). That type of action is also called 'regular' by Abels and Holz, [1], but 'regular' is used for another type of condition in several other articles. We will define both these slightly later. (This confusing situation is not that serious for us as the action we have satisfies all the different variants, so for what we need we do not have to worry about which form is being used or if they are equivalent conditions. When we need to consider a more general case, we will take the approach used in Bridson and Haefliger, [36], and so will avoid the potential confusion.)

Proposition 32 *If $\sigma = \{H_{\alpha_0}x_0, \ldots, H_{\alpha_n}x_n\}$ is a simplex in $N(\mathfrak{H})$, there is an element g of G such that $\sigma \cdot g = \sigma_0 := \{H_{\alpha_0}, \ldots, H_{\alpha_n}\}$.*

Proof: As σ is a simplex of $N(\mathfrak{H})$, $\bigcap \sigma$ is non-empty . Let $h \in \bigcap \sigma$, then $h \in H_{\alpha_0}x_0$, so has the form $h = h_0x_0$ for some $h_0 \in H_{\alpha_0}$. It equally well has the form $h = h_ix_i$ for some $h_i \in H_{\alpha_i}$. Now take $g = h^{-1}$, then we have $H_{\alpha_i}x_ig = H_{\alpha_i}$ for all i, that is, $\sigma \cdot g = \sigma_0$. ■

We will sometimes refer to the σ_0 determined as here, as the basic 'supporting' simplex of σ. We will be using this slightly later on.

One of the definitions of 'regular action' is as follows (cf. Corson, [71]):

Definition: Suppose the G act simplicially on a simplicial complex, K, then the action is *regular* and we say that K is a *regular G-complex* if given elements $g_0, \ldots, g_n \in G$ and a simplex, $\sigma = \{v_0, \ldots, v_n\}$ of K such that $\tau = \{v_0g_0, \ldots, v_ng_n\}$ is also a simplex of K, then there is a single element, $g \in G$ such that $\sigma \cdot g = \tau$.

We will see, a bit later on, that this definition as stated hides a difficulty in its interpretation. That difficulty does not, however, occur in this case, so we will ignore it for the moment.

Corollary 9 *The action of G on $N(\mathfrak{H})$ is regular.*

Proof: Suppose $\sigma = \{H_{\alpha_0}x_0, \ldots, H_{\alpha_n}x_n\}$, and $g_0, \ldots, g_n$ are in G, then the τ that we get, according to the definition, will be $\{H_{\alpha_0}x_0g_0, \ldots, H_{\alpha_n}x_ng_n\}$.

Let us write $\sigma_0 = \{H_{\alpha_0}, \ldots, H_{\alpha_n}\}$ for the basic 'supporting' simplex of σ

Using the proposition, we have there is a $h_1 \in G$ such that $\sigma \cdot h_1 = \sigma_0$. Similarly we have $h_2 \in G$ such that $\tau \cdot h - 1 = \sigma_0$, as the basic supporting simplex of τ is once again σ_0 as it involves exactly the same subgroups from $\mathcal{H}$. Now it is easy to take $g = h_1h_2^{-1}$ to get $\sigma \cdot g = \tau$, as required. ■

We note that if we needed to write down such a g explicitly, we need only pick h_1 and h_2 according to the recipe in the proof of the proposition.

As we said before, Abels and Holz, [1], use 'regular' in a seemingly different way, one that is the same as 'action without inversion' of Haefliger, [110].

Definition: An *action without inversion of a group, G,* on a simplicial complex, K, is an action by simplicial automorphisms such that if $\sigma = \{v_0, \ldots, v_n\}$ and $\sigma \cdot g = \sigma$, then $v_i g = v_i$ for all $i \in \{0, \ldots, n\}$.

Proposition 33 *The action of G on $N(\mathfrak{H})$ is without inversion. In more detail, if*

$$\sigma = \{H_{\alpha_0}x_0, \ldots, H_{\alpha_n}x_n\}$$

is a simplex of $N(\mathfrak{H})$, as before, and $g \in G$ is such that $\sigma \cdot g = \sigma$, then g fixes all the cosets, $H_{\alpha_i}x_i$, that is to say, $H_{\alpha_i}x_i g = H_{\alpha_i}x_i$ for all $i \in I$.

Proof: As we are working with the *simplicial complex*, $N(\mathfrak{H})$, (and not with a corresponding simplicial set), all the H_{α_i} must be distinct (since otherwise the simplex would have empty intersection). Now suppose $\sigma \cdot g = \sigma$, then $H_{\alpha_i}x_i g = H_{\alpha_k}x_k$ for some k, and it is then immediate that H_{α_i} and H_{α_k} must be equal. ■

Stabilisers: One of the ways to understand an action is via its stabiliser subgroups. If we have a simplex, $\sigma = \{H_{\alpha_0}x_0, \ldots, H_{\alpha_n}x_n\}$, it is now quite easy to work out the stabiliser of σ. First, for convenience, recall that

$$Stab_G(\sigma) = \{g \mid \sigma \cdot g = \sigma\}.$$

If $\sigma \cdot g = \sigma$, then $H_{\alpha_i}x_i g = H_{\alpha_i}x_i$, so $g \in x_i^{-1}H_{\alpha_i}x_i$. This does not seem to be that useful, *but* we can take all the x_is to be the sme, and that then gives a complete answer. Let us backtrack a bit. We have that if $a \in \bigcap \sigma$, then $\sigma = \sigma_0 \cdot a$, so if $\sigma \cdot g = \sigma$, $aga^{-1} \in Stab_G(\sigma_0)$. It therefore remains to ask when $H_{\alpha_i}g' = H_{\alpha_i}$ for all i and clearly this is when $g' \in \bigcap \sigma$. We thus have

Proposition 34 *For $\sigma \in N(\mathfrak{H})$ and $a \in \bigcap \sigma$,*

$$Stab_G(\sigma) = a^{-1} \bigcap \sigma a.$$

■

Note that this is independent of the choice of a in $\bigcap \sigma$.

The 'space' of orbits, $N(\mathfrak{H})/G$: Suppose $\mathcal{H}$ is a *finite* family of subgroups of G, then we have a special maximal dimensional simplex in $N(\mathfrak{H})$, namely the family $\mathcal{H}$ itself. If $\mathcal{H}$ has n members then the dimension of this simplex will be $n - 1$. This acts like a fundamental domain of a group action, say on the plane, so we will call it the *fundamental domain simplex* of $N(\mathfrak{H})$. It will usually be denoted σ_0. We have:

Proposition 35 *If $\mathcal{H}$ has n elements, then $N(\mathfrak{H})/G$ is an $(n-1)$-simplex.* ■

Examples: We will look back at some of the examples of nerves that we have given before. In many cases, it is possible to see the group action on the nerve quite clearly and to illustrate the way in which each maximal dimensional simplex is a 'translate' of the fundamental domain simplex.

1. $G = S_3, H_1 = \{1, (1\ 2\ 3), (1\ 3\ 2)\}, H_2 = \{1, (1\ 2)\}$, as on page 91. The nerve $N(\mathfrak{H})$ in this case is the graph given in example 3.3.1 with vertices

$$\begin{array}{ccccc} & H_1 & & H_1b & \\ H_2 & & H_2a & & H_2a^2 \end{array}$$

(where, as there, $a = (1\ 2\ 3)$, $b = (1\ 2)$). The action is given by: a fixes H_1 and H_1b and permutes the cosets of H_2 in the obvious way; b permutes H_1 and H_1b *and* H_2a and H_2a^2, but fixes H_2 (of course). On 1-simplices, the action follows on an edge is determined by what it does to the two ends. so, for instance,

$$a \in H_1 \cap H_2a \quad \text{so} \quad H_1a^{-1} \cap H_2aa^{-1} = H_1 \cap H_2 \neq \emptyset$$

and so on. It is thus easy to see directly that $N(\mathfrak{H})/S_3 \cong \Delta[1]$.

As to the stabilisers: on vertices,

$$Stab_{S_3}(H_1) = H_1;$$

$$Stab_{S_3}(H_1b) = b^{-1}H_1b = H_1,$$

and note that, in this case, $\sigma = \{H_1b\}$, so $b \in \bigcap \sigma$, and the result is as predicted by the proposition on stabilisers.

On 1-simplices, as $\sigma_0 = \{H_1, H_2\}$, this has trivial stabiliser, hence so do all 1-simplices.

2. $G = K_4 = \{1, a, b, c\}$ with $\mathcal{H} = \{H_a, H_b, H_c\}$, where $H_x = \langle x \rangle$, (cf. page 104). Here $N(\mathfrak{H}_{K4})$ is the octahedral shell with 4 faces removed.

Using the same notation as before: a fixes 1 and 2, permutes 3 and 4, and also 5 and 6, so in the diagram in example 3.3.9, a corresponds to a rotation through $180°$ about the vertical axis. Similarly for b and c, but about the two horizontal axes. The orbit space is $\Delta[2]$ as this example has 3 subgroups.

We will leave the determination of the stabilisers **to the reader**.

3. $G = q8$, (cf. page 105): $N(\mathfrak{H}_{q8})$ is as in the previous example and has the action of $q8$ given via the quotient homomorphism to K_4 and the action outlined before in 2. Of course, $N(\mathfrak{H}_{q8})/q8$ is again a 2-simplex. (Again the stabilisers are **left to you**. It is interesting to reflect on the relationship between the stabilisers here and in that previous example.)

4. S_4 with three subgroups, $H_1 = \langle(1, 2), (2, 3)\rangle$, $H_2 = \langle(2, 3), (3, 4)\rangle$ and $H_3 = \langle(1, 2), (3, 4)\rangle$, as on page 105. Of course, $N(\mathfrak{H})/S_4$ is a 2-simplex. The stabilisers are not too difficult to calculate.

5. For our last example, for the moment, we will consider a 'generic' one and take up the example of an amalgamated 'free product' / coproduct from page 106. We will write $G = A \sqcup_C B$, where $C = A \cap B$. (The change in notation is for convenience of typing and has no significance.) The family of subgroups is $\mathcal{H} = \{A, B\}$ or more exactly the images of A and B in G, so the cosets have form Aw_A and Bw_b. If we think about the coset representatives w_A may be assumed to start with a b that is not in C, and similarly in the w_B, we may assume that it starts with an a

not in C. The nerve, $N(\mathfrak{H})$, is a bipartite graph and is, in fact, a tree as each w_A or w_B provides a unique direct path back to A or B. For instance,

$$A \text{ ------ } Ba_2 \text{ ------ } Ab_2a_2 \text{ ------ } Ba_1b_2a_2 \text{ ------ } Ab_1a_1b_2a_2$$

whilst A is linked to B.

The action is fairly easy to visualise and the orbit space is a 1-simplex, of course.

What about stabilisers? The result we showed earlier reduces the problem to looking at the stabilisers of the vertices of the 'fundamental domain' simplex. These we will denote simply by $v_A = \{A\}$, $v_B = \{B\}$ and, the edge joining them, $\sigma_0 = \{v_a, v_B\}$.

$$Stab_G(v_A) = \{g \mid Ag = A\},$$

so is simply A, and similarly $Stab_G(v_B)$ is B, whilst

$$Stab_G(\sigma_0) = \{g \mid \sigma_0 \cdot g\sigma_0\} = C.$$

If we draw the fundamental domain simply labelled by the corresponding stabilisers, we get (surprise, surprise!)

$$A \xrightarrow[A\cap B]{\qquad} B\ ,$$

which is the usual picture for a graph of groups, (cf. Serre, [194, 195]) and which we saw earlier on page 106.

This last example suggests the generic case for more subgroups should be related to some 'simplex of groups' or more generally to 'complexes of groups', and, of course, that is what we will be discussing shortly.

3.4.2 The G-action on $V(\mathfrak{H})$

There is, of course, an equally natural group action of G on $V(\mathfrak{H})$, but its properties are not so 'combinatorial'.

If $\sigma = \{g_0, \ldots, g_n\}$ is an n-simplex of $V(\mathfrak{H})$ and $g \in G$, then the 'obvious' guess for $\sigma \cdot g$ would be $\{g_0g, \ldots, g_ng\}$. (We saw and used the corresponding left G-action earlier when we first introduced Volodin spaces on page 95.) It is immediate that this is a simplex in $V(\mathfrak{H})$, since there is some coset $H_\alpha x$ containing σ as a subset, and $\sigma \cdot g \subseteq H_\alpha xg$.

Where this action is different from that on $N(\mathfrak{H})$ is that it is not '*without inversion*' in general. For instance, if one of the subgroups of $\mathcal{H}$, say $H_1 \subset G$ is a subgroup of order 2, $H_1 = \{1, a\}$ with $a^2 = 1$, then $\sigma = \{1, a\}$ is a 1-simplex of $V(\mathfrak{H})$, but $\sigma \cdot a = \sigma$, whilst a moves both vertices of σ.

This 'irregular' behaviour is not a worry as any simplicial action can be made regular by passing to a barycentric subdivision.

We saw that $V(\mathfrak{H})$ was related to the simplicial set, $E(G)$, which had a G-action whose quotient gave $Ner(G[1])$, the nerve, or simplicial classifying space, BG, of G. We have, for any coset $H_\alpha x$, a sub-simplicial set of $V(\mathfrak{H})$ consisting of those simplices that are within $H_\alpha x$. This is really just a copy of $E(H_\alpha x)$ and is isomorphic to $E(H_\alpha)x$, which, in turn, is isomorphic to $E(H_\alpha)$. What about the action? How does it operate on these parts? For ease of analysis let us suppose we make a choice of a set of coset representatives for H_α in G, then given $g \in G$, we can write $g = hx$

for $h \in H_\alpha$ and for some (chosen) coset representative, x. Now if we have $\sigma \in E(H_\alpha) \subseteq V(\mathfrak{H})$, $\sigma \cdot g = (\sigma \cdot h)cdotx$, so we can think of the action of g as consisting of a part that shifts σ around within $E(H_\alpha)$, followed by a part that 'translates' $E(H_\alpha)$ to $E(H_\alpha)x$. If $\sigma \in E(H_\alpha)y$ to start with then $\sigma \cdot y^{-1} \in \sigma \in E(H_\alpha)$ so we write $yg = hx$ and use the simpler analysis above.

In other words, the action can be thought of as being partially within each $E(H_\alpha)x$ and partially as permuting the different $E(H_\alpha)x$s amongst themselves. This seems, of course, very 'H_α-centric', i.e., seen from the viewpoint of H_α and its cosets, but, as the action is defined the same way irrespective of where one is in $V(\mathfrak{H})$, the different viewpoints are compatible. (Here a detailed treatment would be simpler if $\mathcal{H}$ is closed under intersection, but is not to difficult **to write down** in any case.)

If we now look at $V(\mathfrak{H})/G$, we can easily check that:

Proposition 36 *The inclusion of $V(\mathfrak{H})$ into $E(G)$ induces an isomorphism*

$$V(\mathfrak{H})/G \xrightarrow{\cong} \bigcup E(H_\alpha)/H_\alpha = \bigcup BH_\alpha,$$

between $V(\mathfrak{H})/G$ and the union of the classifying spaces, $BH_\alpha = NerH_\alpha[1]$, within BG. ■

Remarks: (i) This result is to be found in Suslin and Wodzicki's treatment, [202], of Volodin spaces.

(ii) Just as $\bigcup H_\alpha$ will not usually be a subgroup of G, in general, $\bigcup BH_\alpha$ will not be a Kan complex (within BG which *is* a Kan complex). For instance, in a 2-horn the two given edges may be in different groups of the family, $\mathcal{H}$, so the filler (within BG) will not necessarily be in $\bigcup BH_\alpha$.

3.4.3 Group actions on simplicial complexes

We have been using some ideas on actions of groups on simplicial complexes. For future use, we need this in a bit more depth and generality than merely on the nerves of coverings by cosets. We will not give full details, but need to discuss the regularity conditions that we have already met.

Suppose, as ever, that G is a group and K is a simplicial complex with vertex set V_K and with S_K as its poset of simplices. We have group of simplicial automorphisms, $Aut(K)$ (not to be confused with the *simplicial group* of automorphisms of the simplicial set K^{simp}). An action of G on K is, by definition, a homomorphism from G to $Aut(K)$, so it is a *simplicial action* or an *action by simplicial automorphisms*. The regularity conditions are needed to help ensure that 'obvious' quotienting operations are well behaved.

Remark: From some points of view, some of the problems that we will be examining are partially obscured by our use of group actions rather than converting those actions into some sort of 'action groupoid' as we introduced in section **??** for the simpler case of a group acting on a set. That viewpoint is highly relevant and will be taken on board in more detail later on, however the links with more 'traditional' viewpoints are also very important not only as the allow transfer of results and ideas between the differently focussed ways of seeing a particular area but also as a source of examples, interpretation and intuition.

Example: The most obvious simple example is the 1-simplex, Δ^1, with the C_2 action that flips the interval about, so let us set this up a bit formally. Let $G = C_2$, the cyclic group of order 2, which we will write as $\{1, a\}$ where, of course, $a^2 = 1$. We take $K = \Delta^1$ so with vertex set, $\{0, 1\}$ and in which the action is given by $0a = 1$, which immediately implies $1a = 0$. The 1-simplex, $\sigma = \{0, 1\}$ is fixed by a, but clearly the individual vertices are not, so this is an action that is not 'without inversion' (see page 118 for the definition). The action is not 'regular' either, and here we meet the slight problem that we mentioned back on page 117 when we introduced the term 'regular'. The problem is one of interpretation. That definition uses the phrasing '*if given elements* $g_0, \ldots, g_n \in G$ *and a simplex,* $\sigma = \{v_0, \ldots, v_n\}$ *of* K *such that* $\tau = \{v_0 g_0, \ldots, v_n g_n\}$ *is also a simplex of* K'. In our case, taking $g_0 = 1, g_1 = a$, then are we to take $\{0, 1a\}$ to be a simplex or not? Is it the 0-simplex $\{0\}$? The answer is 'yes', so as to be consistent with the definition of simplicial map. (You may recall or check that we left this to you to workout or look up. The complication is that if $f : K \to L$ is a simplicial map, then it is a map on the vertices, which preserves 'simplexness'. This means that if $\sigma \in S_K$ then $f(\sigma) \in S_L$, but note $f(\sigma)$ is the subset of V_L given by the images of the vertices in σ, hence it may, and usually will, have smaller dimension. (We will look at this point in quite a lot of detail very shortly.) This means that saying $\tau = \{v_0 g_0, \ldots, v_n g_n\}$ is a simplex, is not quite accurate as it does not mean that the elements in the listing are distinct. (We will introduce some additional terminology to keep track of this shortly, but for the moment please excuse the slightly sloppy notation.) In our case, this is $\{0\}$. Now it is clear that the action is not regular, as there is no $g \in C_2$ sending $\{0, 1\}$ to $\{0\}$!

This example is simple but quite important as it highlights some weaknesses in both terminology and notation. It also suggests that we should ask the question: 'what are these conditions 'about'? To answer this, we need to look at 'quotienting' and its relationship with the passage from simplicial complexes to simplicial sets and to clarify several issues in the process.

Quotienting operations on simplicial complexes

We will give this in more generality then we actually need. Suppose that K is a simplicial complex with vertex set, V_K, and V_L is a set (not yet of 'vertices' of anything). Suppose we have a surjection $f : V_K \to V_L$. We want to construct a simplicial complex, L from the set V_L and the function, f. There is an obvious way to do it.

Definition: Define a subset $\{w_0, \ldots, w_n\} \subseteq V_L$ to be a simplex if there is a simplex $\{v_0, \ldots, v_n\} \in S_K$ such that $f(v_i) = w_i$ for $i = 0, 1, \ldots, n$. We will call $\{v_0, \ldots, v_n\}$ a *witness* for $\{w_0, \ldots, w_n\}$ in this case.

It should be clear that this *is* a simplicial complex structure on V_L and we will call it the induced simplicial complex along f, or similar terminology. If f is obtained explicitly by some equivalence relation (and, of course, it can always be considered to be given in such a way), then L would be called the *quotient* of K by that equivalence relation.

Lemma 24 *With this structure,* f *induces a morphism of simplicial complexes,* $f : K \to L$.

Proof: This is more or less obvious, at least at first sight. There is a detail, however, that is worth pointing out. If $\sigma = \{v_0, \ldots, v_n\}$ is a simplex in K, $f(\sigma)$ is a subset of V_L, but it may have fewer than the $n + 1$ elements that σ had, so we cannot, necessarily, use σ as a witness for $f(\sigma)$ being a simplex of L. There will, however, be a subset / face of σ that maps bijectively onto $f(\sigma)$, so $f(\sigma)$ *is* a simplex of L. ■

As an example of this construction, we can return to the 1-simplex, Δ^1, with the C_2-action. To recap, we have $K = \Delta^1$, $V_K = \{0, 1\}$, $0a = 1$, $1a = 0$ and we take $V_L = V_K/C_2 = \{0G\}$, so it is a 1 element set. (In general, here we will write vG for the orbit $\{vg : g \in G\}$ of v under the G-action, as being a fairly self evident notation.) The only simplex in L is thus the 0-simplex, $\{0G\}$. This again illustrates the point in the proof, as $f : K \to L$ sends $\{0, 1\}$ to $\{0G\}$, which is a 0-simplex of L, since, for instance, it is witnessed by $\{0\}$ in K.

We note that this, of course, works well with G-complexes in general, and not just in this simple example. If G acts on the right of K, we can take V_L to be the set of orbits, V_K/G, with $f : V_K \to V_L$, the obvious function assigning the orbit vG to a vertex v. The resulting simplicial complex will then be what we have been calling the *quotient of K by the G-action*. The usual notation will then be K/G for this.

Quotienting operations on simplicial sets

A (right) G-action on a simplicial set, K can be most elegantly defined as a functor $K : G[1]^{op} \to \mathcal{S}$, from the opposite of the category $G[1]$ (with a single object, conveniently denoted by $*$ if needed and with $G[1](*, *) = G$ with composition given by the multiplication) to the category of simplicial sets. equivalently, we can write the action as a group homomorphism $G^{op} \to Aut(K)$, where $Aut(K)$ is the group of automorphisms of K, or, again equivalently, as a simplicial map

$$K \times K(G, 0) \to K$$

satisfying certain fairly obvious properties, where $K(G, 0)$ is the constant simplicial group with value G in each dimension.

A simplicial set with G action also corresponds to a simplicial G-set, that is, as simplicial object in the category of G-sets.

The quotient simplicial set of orbits is obtained as $(K/G)_n = K_n/G$, the set of G-orbits of n-simplices, with $d_i(\sigma G) = (d_i\sigma)G$, etc. (This works because G is acting via simplicial automorphisms, but that is **left to you to check and to think of other ways of putting it)**. If we think of the G-action as a functor (as above) then K/G is the colimit of $K : G[1]^{op} \to \mathcal{S}$; again left to you to check, but this is important when considering the homotopical aspect, such as homotopical syzygies of presentations, as it is then natural that, in such a context, one should replace these colimits by homotopy colimits, ..., but that comes quite a lot later!

From simplicial complexes to simplicial sets

Earlier (page **??**) we briefly discussed ways of converting a simplicial complex to a simplicial set. We there concentrated on a construction that was going to give a fairly small simplicial set. This picked a total order, $\leq$, on the vertices of the simplicial complex, K, and then used each $\sigma = \{v_0, \ldots, v_n\}$ by saying that we will put the v_i in order according to the given order. (For ease of notation we will assume that this is the given order we wrote them in, in the 'set' σ, so the corresponding simplex will be $\sigma = \{v_0 \leq \ldots \leq v_n\}$. We thus have, for example, that the simplicial complex, Δ^1, converts to the simplicial set, $\Delta[1]$. (Of course, this is almost cheating as we might have chosen a total order on $\{0, 1\}$ in which $1 \leq 0$, or in the general case have $v_3 \leq v_2 \leq v_0 \leq v_1 \leq v_n \leq \ldots\}$ within σ. In the case of Δ^1, the example is so simple that we get isomorphic simplicial sets whichever way we choose the order.) We will write $\langle v_0, \ldots, v_n\rangle$ for the simplex corresponding to $\{v_0 \leq \ldots \leq v_n\}$. The resulting structure will have nicely behaved face maps but not yet degeneracies. For those we have to add in more simplices and it helps to introduce some terminology, that will be useful in other places as well.

Definition: Let X be a set and $\underline{x} = \langle x_0, \ldots, x_n \rangle$, an $(n+1)$-tuple of elements of X. The *support* or *range* of $\underline{x}$ is the set of components of $\underline{x}$.

As an example, the support of the 5-tuple, $\langle 5, 1, 3, 2, 1 \rangle$ is the set $\{1, 2, 3, 5\}$.

Given a total order on V_K, an n-simplex in the associated simplicial set (N.B., 'associated' to the pair $(K, \leq)$, not just to K) will be an $(n+1)$-tuple, $\langle v_0, \ldots, v_n \rangle$, of elements of V_K such that (i) $\{v_0 \leq \ldots \leq v_n\}$ in the order $\leq$ on V_K and (ii) the support of $\langle v_0, \ldots, v_n \rangle$ forms a simplex of K. Faces and degeneracies are defined in a fairly obvious way by deletion of a position and its entry and repetition of a component, so $s_1\langle v_0, v_1, v_2 \rangle = \langle v_0, v_1, v_1, v_2 \rangle$, for instance.

This construction is very useful as it gives a fairly small simplicial set that models the homotopy type of the simplicial complex, however it is not functorial on simplicial complexes as such because of the choice of order. If one works with simplicial complexes with an order on vertices together with order preserving simplicial maps between them then it will be functorial, but this is not feasible for most situations in which there is a G-action involved. The G-action is very unlikely to preserve the order and so the associated simplicial set will not inherit a G-action. We thus discard this method here, although it can be useful elsewhere.

The second construction does not need a choice of an ordering of the vertex set and so gives us a functor.

Definition: If K is a simplicial complex with vertex set V_K, the *associated simplicial set*, K^{simp}, of K is the simplicial set having $\sigma \in (K^{simp})_n$ if and only if $\sigma \in (V_K)^{n+1}$ and the support of σ is a simplex of K. The face and degeneracy mappings are defined in the usual way via deletions and repetitions.

This is much bigger than the ordered version, for instance if $\{v_0, v_1, v_2\} \in S_K$ is a simplex of K, then not only do we have $\langle v_0, v_1, v_2 \rangle \in (K^{simp})_2$, but also $\langle v_1, v_0, v_2 \rangle, \langle v_2, v_1, v_0 \rangle$, etc. as they all have support $\{v_0, v_1, v_2\}$. Also in K_2 there will be degenerate 2-simplices such as $\langle v_0, v_0, v_1 \rangle$, and $\langle v_0, v_2, v_2 \rangle$, but that is not all as $\langle v_2, v_1, v_2 \rangle$ is there, but is not a degenerate simplex as in degenerate 2-simplices either the first two or the last two vertices will be the same. These all have support that is a non-empty subset of $\{v_0, v_1, v_2\}$, so are simplices in K^{simp}.

With regard to our example, we note that $(\Delta^1)^{simp}$ is *not* $\Delta[1]$, as in addition to $\langle 0, 1 \rangle$, we also have $\langle 1, 0 \rangle$ in $(\Delta^1)^{simp}_1$.

We record for future use the lemma:

Lemma 25 $(-)^{simp}$ *gives a functor from the category, Simp.Comp, of simplicial complexes, to that of simplicial sets.*

Proof: Although this is really **left to you** to check as being more-or-less 'obvious', it is worth commenting that if $f : K \to L$ is in *Simp.Comp*, we can have $\sigma = \{v_0 \leq \ldots \leq v_n\}$ in K, so there are $(n+1)!$-different simplices in K^{simp} corresponding to σ; now $f(\sigma)$ may contain fewer elements than σ if f is not 1-1 on vertices. Each σ^{simp} that corresponds to σ will be of the form $\sigma^{simp} = \langle v_0, \ldots, v_n \rangle$ for some ordering of the elements of σ, and $f^{simp}(\sigma^{simp})$ will then be $\langle f(v_0), \ldots, f(v_n) \rangle$, which will be an n-simplex of L^{simp} for the 'obvious' reason. This assignment will, of course, respect the faces and degeneracies, as is easily checked. ■

Corollary 10 *If K ia a simplicial G-complex, i.e. a simplicial complex with simplicial G-action, then K^{simp} is a simplicial G-set, i.e. the simplcial set, K^{simp}, inherits a simplicial G-action.*

Proof: A (right) G-action on K 'is' a functor $K : G[1]^{op} \to Simp.Comp$, now compose that with $(-)^{simp}$ to get K^{simp}.

If you prefer a more 'hands-on' viewpoint, if $\sigma = \{v_0 \leq \ldots \leq v_n\}$, then $\sigma \cdot g = \{v_0 g \leq \ldots \leq v_n g\}$, and any simplex, $\sigma^{simp} = \langle v_0, \ldots, v_n \rangle$ with support σ gets sent by the action of g to $\langle v_0 g, \ldots, v_n g \rangle$ which has support $\sigma \cdot g$. it then just remains to **check that it all fits together well**. ■

This corollary means that we seem to have *two* ways of getting a simplicial set of orbits from a simplicial G-complex K. We can either form $(K/G)^{simp}$ or K^{simp}/G. They need not be the same.

Example revisited: Again $K = \Delta^1$, $G = C_2$, with $0a = 1$, $1a = 0$.

- K/G is Δ^0, so $(K/G)^{simp}$ is $\Delta[0]$;
- K^{simp} has $K_0^{simp} = \{\langle 0 \rangle, \langle 1 \rangle$, and the two 0-simplices are swapped by the G-action, $K_1^{simp} = \{\langle 0, 0 \rangle, \langle 1, 1 \rangle, \langle 0, 1 \rangle, \langle 1, 0 \rangle\}$, and a exchanges the first two of these and also the last two, so $\langle 0, 1 \rangle a = \langle 1, 0 \rangle$, etc. and thus

$$(K^{simp}/G)_1 = \{\langle 0, 0 \rangle G, \langle 0, 1 \rangle G\},$$

so two orbits. (**You are left** to identify what happens in higher dimensions, and to check that $(K^{simp}/G) \cong S^1 = \Delta[1]/\partial\Delta[1]$, one version of the simplicial circle. Note that this is *not* the simplicial set associated to any simplicial complex.

Another example: There is a similar example with a bit more subtlety. This time take $K = \partial\Delta^2$, the 1-skeleton of the 2-simplex, hence an empty triangle. The vertex set of this is $\{0, 1, 2\}$, whilst the simplices are $\{0, 1\}, \{1, 2\}$, and $\{0, 2\}$, together with the obvious singletons coming from the vertices. Take $G = C_3 = \{1, a, a^2\}$ (and, of course, $a^3 = 1$), with action on K given by $0 \cdot a = 1$, $1 \cdot a = 2$ and, of course, $2 \cdot a = 0$, so a rotation.

- Again K/G has just one vertex, so $(K/G)^{simp}$ is $\Delta[0]$.
- The corresponding K^{simp}/G, has just one vertex, but has two non-degenerate 1-simplices, $\langle 0, 1 \rangle G$, and $\langle 1, 0 \rangle G$.
- In fact, it also has some non-degenerate 2-simplices, $\langle 0, 1, 0 \rangle G$ and $\langle 1, 0, 1 \rangle G$
- . . . and so on.

This is quite neat. The two non-degenerate 1-simplices are, in some sense, homotopy inverses to each other with the homotopies encoded by the 2-simplices that we have given. Those two homotopies are themselves homotopic by the next level and so on. This is very much typical of a homotopy coherent situation, for the meaning of which see later.

This is an example of an action which is 'without inversion' but is not regular.

Our examples have had $(K/G)^{simp}$ a 0-simplex, and so there is a unique map from K^{simp}/G to $(K/G)^{simp}$ by virtue of that, but such a map always exists.

Proposition 37 *For any simplicial complex, K, with G-action, there is a natural degreewise surjective simplicial morphism,*

$$\varphi : K^{simp}/G \to (K/G)^{simp}.$$

Proof: Suppose that $\langle v_0, \ldots, v_n\rangle \in K^{simp}_n$, take $\varphi(\langle v_0, \ldots, v_n\rangle G) = \langle v_0G, \ldots, v_nG\rangle$, and **check this is a simplicial morphism**. We are left to check it is a degreewise surjective morphism. (Naturality is easy to verify, so is **left to you**. In fact, as K^{simp}/G is the colimit of the functor giving the action, naturality is more-or-less forced to be the case.)

Suppose we have $\sigma = \langle v_0G, \ldots, v_nG\rangle$ is an n-simplex in $(K/G)^{simp}$, then its support is a simplex of K/G. Deleting repeats if any from the list, $(v_0G, \ldots, v_nG)$, we get a sublist, $(v_{i_0}G, \ldots, v_{i_k}G)$, whose elements give the support of σ, so $\{v_{i_0}G, \ldots, v_{i_k}G\} \in S_{K/G}$. By the definition of the simplicial complex structure on K/G, this means that there are elements g_{i_j} of G, for $j = 0, \ldots, k$, such that $\{v_{i_0}g_{i_0}, \ldots, v_{i_k}g_{i_k}\}$ is in S_K. We will write $I = \{i_0, \ldots, i_k\}$.

We can now use this to build a simplex of K^{simp}, as follows:

- examine v_0G; there is some index, which we suppose to be $j_0 \in I$ such that $v_0G = v_{j_0}G$, hence we have some $g_0 \in G$ such that $v_0g_0 = v_{j_0}g_{j_0}$;
- we repeat for v_1; there is some index $j_1 \in I$ such that $v_1G = v_{j_1}G$, hence we have some $g_1 \in G$ such that $v_1g_1 = v_{j_1}g_{j_1}$
- and so on.

This gives us a potential simplex $\langle v_0g_0, \ldots v_ng_n\rangle$. Its support is $\{v_{i_0}g_{i_0}, \ldots, v_{i_k}g_{i_k}\}$, which is in S_K, so $\tau = \langle v_0g_0, \ldots v_ng_n\rangle \in K^{simp}$ and $\varphi(\tau G) = \sigma$, so φ is degreewise surjective. ■

It is a natural question to ask when φ has additional properties and, for us most importantly, when φ is an isomorphism.

Theorem 9 *The natural morphism,*

$$\varphi : K^{simp}/G \to (K/G)^{simp},$$

is an isomorphism if, and only if, K is a regular G-complex.

Proof: Suppose that φ is one-to-one, (and hence is an isomorphism by the above proposition), and now suppose $\{v_0, \ldots, v_n\} \in S_K$ and $g_0, \ldots, g_n \in G$ are such that $\{v_0g_0, \ldots, v_ng_n\}$ is also a simplex of K. We look at $\langle v_0, \ldots, v_n\rangle G$ and $\langle v_0g_0, \ldots, v_ng_n\rangle G$, and note that φ maps then both to $\langle v_0G, \ldots, v_nG\rangle$, hence, as φ is one-to-one, those two simplices must, in fact, be equal. The second of these contains $\langle v_0g_0, \ldots, v_ng_n\rangle$, so there must be a $g \in G$ such that $\langle v_0, \ldots, v_ng\rangle = \langle v_0g_0, \ldots, v_ng_n\rangle$, so the action makes K into a regular G-complex.

Conversely, suppose that K is a regular G-complex and that $\langle v_0, \ldots, v_n\rangle G$ and $\langle v'_0, \ldots, v'_n\rangle G$ have the same image under G. This translates to there being an equality

$$\langle v_0G, \ldots, v_nG\rangle = \langle v'_0G, \ldots, v'_nG\rangle,$$

so there are elements $g_0, \ldots, g_n \in G$ such that, for each i, $v_ig_i = v'_i$, We can thus use the condition of regularity to find a single $g \in G$ such that $\langle v_0, \ldots, v_n\rangle g = \langle v'_0, \ldots, v'_n\rangle$, but that implies that in fact

$$\langle v_0, \ldots, v_n\rangle G = \langle v'_0, \ldots, v'_n\rangle G,$$

so φ is one-to-one, hence an isomorphism. ■

We could continue looking at the various conditions of G-complexes for instance, the 'without inversion' one that we mentioned earlier, but will rather leave that to the reader to follow up, for instance, in Prasolov's book, [184], or in Bredon's notes, [30], and will start on some of the simpler ideas of the theory of Complexes of Groups.

3.5 Complexes of groups

This situation that we saw with the group actions on the nerve, $N(\mathfrak{H})$, is a simple form of a general one considered by Haefliger (cf. [36, 110, 111]) and Corson (cf. [68–70]). They consider a simplicial complex (or more generally a simplicial cell complex, cf. Haefliger, [110] or a scwol (small category without loops), cf. Bridson and Haefliger, [36]) on which a group G acts 'without inversion' or, in the variant used by Corson, with a regular G-complex. Their work introduced complexes of groups, a notion generalising that of graphs of groups as in Bass-Serre theory, [194, 195] and also, [21], but developed into a central part of geometric group theory later on. We will give definitions shortly, but first need to revise some of the more detailed notation and terminology relating to barycentres, barycentric subdivisions, etc. extending our discussion in section 3.3.6. Here we will be limiting ourselves initially to the simpler form of the ideas, but will generalise later.

These complexes of groups are important not only for discussion of properties relating to syzygies, but because they provide fairly simple examples of orbifolds, and topological stacks, both of which are ideas that we will encounter (much) later on in these notes.

3.5.1 Simplicial complexes, barycentres and scwols

If K is a simplicial complex, we can encode the information in K in a simply way by considering K as a partially ordered set. The elements of this partially ordered set are the elements of S_K, the set of simplices of K, ordered by inclusion. As we mentioned earlier, the barycentric subdivision of K is then just the (categorical) nerve of the poset $(S_K, \subseteq)$. We will follow Haefliger [110] in orienting the edges of K' in the following way:

The vertices of $K'(= Sd(K))$ are the simplices of K. An (unoriented) edge of K' consists of a pair (σ, τ) with either $\sigma \subset \tau$ or $\tau \subset \sigma$. If a is an edge of K' contained in a simplex, σ, of K, then the *initial point* $i(a)$ of a will be the barycentre of σ, (i.e. σ as a vertex of K') and its *terminal point*, $t(a)$, will the barycentre of some smaller simplex, τ. We write $i(a) = \sigma$, $t(a) = \tau$ and so have $a = (\tau, \sigma)$, with $\tau \subset \sigma$. (This is perhaps the opposite order from that which seems natural, but it avoids considering dual posets later.)

Examples:

(i) The simplest case is for the 1-simplex, Δ^1, which has, as we have said earlier, vertex set $\{0, 1\}$ or if you prefer, $\{1, 2\}$, (to keep the same notation as Haefliger), and all non-empty subsets are simplices. This gives, a slightly abbreviated notation,

$$\{1\} \longleftarrow \{1,2\} \longrightarrow \{2\}$$

(ii) For the 2-simplex, considered as the simplicial complex of non-empty subsets of $\{1, 2, 3\}$, this gives

$$\begin{tikzcd}
& & \{2\} & & \\
& \{1,2\} \arrow[ur] \arrow[ddl] & & \{2,3\} \arrow[ul] \arrow[ddr] & \\
& & \{1,2,3\} \arrow[uu] \arrow[ul] \arrow[ur] \arrow[dll] \arrow[d] \arrow[drr] & & \\
\{1\} & & \{1,3\} \arrow[ll] \arrow[rr] & & \{3\}
\end{tikzcd}$$

It is quite usual to consider partially ordered sets as categories, so we could just leave things like this and use this partially ordered set of simplices in K as the categorical model of subdivision. Haefliger, however, wanted a more general type of complex than merely simplicial complexes, so introduces a specific construction of a small category associated to K, (cf. [110]), extracts an abstraction of the key properties that he needs from this subdivision category and the uses that abstraction (a *scwol*) as a means to build the generalisation he wanted. We will follow his construction explicitly, including the notation, as his conventions are not always identical to those we have used earlier (e.g., because of the graph-theoretic link with graphs of groups, the terminology *initial* and *terminal* instead of *source* and *target* for the two ends of an arrow in a small category, are used. This results in a use of i and t, instead of s and t as notation. This should not be too confusing, but otherwise the comparison and cross referencing to the original sources would be difficult.) Before giving the precise definition of a small category without loops, or *scwol*, we will given the example of the scwol associated to a simplicial complex.

Example: Given a simplicial complex, define a category, $C(K)$, with set of objects, S_K, the set of vertices of the barycentric subdivision, K', of K and with arrows, $Arr(C(K)) = E_{K'} \sqcup S_K$, the set of edges of K' together with S_K. (Of course, the vertices are being considered as identity arrows at themselves.) Two edges a and b are considered composable if $i(a) = t(b)$ and the composite is $c = ba$ such that a, b, c form the boundary of a 2-simplex in K':

$$\begin{tikzcd}
& \cdot \\
\cdot \arrow[ur, "c"] \arrow[r, "b"'] & \cdot \arrow[u, "a"']
\end{tikzcd}.$$

This category, $C(K)$, is an example of a *small category without loops* as introduced by Haefliger [36, 110]. In general, in this section, we shall consider a small category, C, to consist of a set, $V(C)$, of vertices or objects (denoted here by Greek letters, τ, σ, etc.) and a set, $E(C)$, of edges (denoted by Latin letters, a, b, $\dots$), together with maps:

- identity, $id : V(C) \to E(C)$;
- $i : E(C) \to V(C)$, the *initial vertex* or map,

- $t : E(C) \to V(C)$, the *terminal vertex* or *target* map,
  and a composition:

- $E^{(2)}(C) \to E(C)$,
  where $E^{(2)}(C) = \{(a, b) \in E(C) \times E(C) : i(a) = t(b)\}$,

together with rules requiring associativity of composition, correct behaviour of the identities, so $i(id(v) = t(id(v)) = v$, etc., and the rules $i(ba) = i(b)$, $t(ba) = t(a)$ for ba, the composite of a and b.

Definition: A small category, C, is a *small category without loops*, or *scwol*, if for all a in $E(C)$, $i(a) \neq t(a)$.

Remark: Haefliger's definition of a small category without loops in [36] (p.521) is optimised for the statement of the no loops condition, but actually omits to define composition of an arbitrary arrow with an identity at a vertex. This is handled correctly (p.573) in an appendix. This does not influence the later development.

In general, it is clear that scwols need not be posets, and so are not restricted to come from simplicial complexes. Scwols do have associated simplicial cell complexes, essentially obtained by taking their nerve, (cf. page **??**), and then taking the geometric realisation so this is the *classifying space of the scwol*. This is sometimes considered, however, together with explicit orderings on the cells that result, retaining in this way the important amount of 'directionality' that is within the scwol, but not in the classifying space as such. In this case, the term *ordered simplicial cell complex* is used by Haefliger, [110, 111]. Some neat examples of ordered simplicial cell complexes are given by Bridson and Haefliger, starting on page 524 of [36].

For the moment, we will move attention back to the 'geometric' situation and the definition of a complex of groups and will pretend that we have a simplicial (cell) complex, K. We will later make the necessary changes to get a complex of groups defined directly on a scwol and will give, somewhat later in the noted, categorical interpretations of what the 'geometry' is handling here.

3.5.2 Complexes of groups: introduction

As we said above, we will start by giving a 'geometric' form of the notion of a complex of groups, here. Our aim is not to explore all the resulting theory, so we will restrict attention to those aspects that seem to have evident uses as examples, etc., later on.

Definition: A *complex of groups*, $G(K)$, on K is specified by the data, $(\{G_\sigma\}, \{\psi_a\}, \{g_{a,b}\})$ given by

1) a group, G_σ, for each simplex, σ, of K;

2) an injective homomorphism,
$$\psi_a : G_{i(a)} \to G_{t(a)},$$
for each edge, $a \in E_K$, of the barycentric subdivision of K;

3) for each pair of composable edges, a and b, in E_K, an element $g_{a,b} \in G_{t(a)}$ is given such that

$$g_{a,b}^{-1}\psi_{ba}(-)g_{a,b} = \psi_a\psi_b$$

and such that the "cocycle condition"

$$g_{a,cb}\psi_a(g_{b,c}) = g_{ab,c}g_{a,b}$$

holds.

(If the dimension of K is less than 3, this last condition is trivially satisfied, since the existence of a triple of composable (non-identity) edges implies that there would be some 3-simplices in K.)

The groups, G_σ, are sometimes called the local groups of teh complex of groups.

This definition is quite 'bare hands' and so we will, of course, need some examples. Later we will generalise and through that generalisation obtain a neater more elegant formulation as well. We will give some simple examples shortly, but before that there is a construction giving a 'generic' example, or almost.

Almost generic example: Developable complexes of groups.

Suppose we have a simplicial complex, $\tilde{K}$, with a right G-action, which is "without inversion" or is regular, both are used. Write $K = \tilde{K}/G$ for the quotient complex. We will specify a complex of groups, $G(K)$, on K:
Set $p : \tilde{K} \to K$ to be the quotient mapping.

For a simplex, σ, of K, pick a $\tilde{\sigma} \in K$ with $p(\tilde{\sigma}) = \sigma$. We say $\tilde{\sigma}$ is the *chosen lift* of σ. Set

$$\begin{aligned} G_\sigma =& G_{\tilde{\sigma}}, \text{ the stabiliser subgroup of } \tilde{\sigma}, \\ =& \{g : \tilde{\sigma}g = \tilde{\sigma}\}. \end{aligned}$$

For each $a \in E_K$ with $i(a) = \sigma$, let $\tilde{a}$ be the edge in $\tilde{\sigma}$, whose projection is a, i.e., $p(\tilde{a}) = a$ and $i(\tilde{a}) = \tilde{\sigma}$. There is then some $h_a \in G$ with $t(\tilde{a}.h_a) = \tilde{\tau}$, where $\tilde{\tau}$ is the chosen lift of $\tau = t(a)$. (If $t(\tilde{a}) = \tilde{\tau}$ already, we agree to take H_a to be the identity of G.)

Define

$$\psi_a : G_{i(a)} \to G_{t(a)}$$

by

$$\psi_a(g) = h_a^{-1}gh_a \quad \text{for } g \in G_{i(a)}.$$

Given two composable edges a and b, we have a configuration such as

$$\begin{tikzcd} & \cdot \\ \cdot \arrow[ur, "ba"] \arrow[r, "b"'] & \cdot \arrow[u, "a"'] \end{tikzcd}$$

and hence a diagram

$$\begin{tikzcd} & G_{t(a)} \\ G_{i(b)} \arrow[ur, "\psi_{ba}"] \arrow[r, "\psi_b"'] & G_{i(a)} \arrow[u, "\psi_a"'] \end{tikzcd}$$

but there is no reason why it should be commutative, in fact:

$$\psi_a\psi_b(g) = h_a^{-1}h_b^{-1}gh_bh_a,$$

whilst

$$\psi_{ba} = h_{ba}^{-1}gh_{ba},$$

so we take

$$g_{a,b} = h_{ba}^{-1}h_bh_a.$$

Verification of conditions (Although easy to do, this helps the intuition.):

(i) Suppose $g \in G_{i(a)}$, then $\tilde{a} = (\tilde{\tau}h_a^{-1}, \tilde{\sigma})$ or $\tilde{a}.h_a = (\tilde{\tau}, \tilde{\sigma}.h_a)$. As $\tilde{\sigma}g = \tilde{\sigma}$, and $\tilde{\tau}h_a^{-1} \subset \tilde{\sigma}$, we have

$$\tilde{\tau}h_a^{-1}g = \tilde{\tau}h_a^{-1}$$

and $h_a^{-1}gh_a \in G_{t(a)}$, i.e., $\psi_a(g) \in G_{t(a)}$.

(ii) It is clear that $g_{a,b}$ as defined above does the job as it was chosen to do so!

(iii) There remains the cocycle condition:

$$g_{a,cb}\psi_a(g_{b,c}) = h_{cba}^{-1}h_{cb}h_a.h_a^{-1}h_{cb}^{-1}h_ch_bh_a$$

$$g_{a,cb}.g_{a,b} = h_{cba}^{-1}h_ch_{ba}.h_{ba}^{-1}h_bh_a,$$

so it all does check out correctly.

In the case of a group, G acting on the nerve of a family of subgroups, $\mathcal{H}$, where $\mathcal{H} = \{H_1, \cdots, H_n\}$ with $H_i < G$, then $N(\mathfrak{H})/G \cong \Delta^{n-1}$. Suppose $\sigma \in S_{\Delta^{n-2}}$ then if $\sigma = \{\alpha_1, \cdots, \alpha_r\}$, we can always choose $\tilde{\sigma} = \{H_{\alpha_1}, \cdots, H_{\alpha_r}\}$. If a is an edge of $Sd(\Delta^{n-1})$ then, for $i(a) = \sigma$ and $t(a) = \tau, \tilde{\tau} \subset \tilde{\sigma}$, hence

$$G_\tau = G_{\tilde{\tau}} = \bigcap\{H_i \mid i \in \tilde{\tau}\},$$
$$G_\sigma = G_{\tilde{\sigma}} = \bigcap\{H_i \mid i \in \tilde{\sigma}\},$$

so there is no need to have $h_a \neq 1$. Because of this, ψ_a is simply an inclusion of a subgroup and $g_{a,b}$ can be chosen to be 1. The Abels-Holz situation, thus, leads to simplices of groups of a particularly simple kind.

In general, not all complexes of groups are *developable*. Shortly we will give Haefliger's characterisation of the developable ones. All graphs of groups are developable and we turn to them next.

3.5.3 Graphs of groups

The notion of a complex of groups was a natural development of that of a graph of groups due to Bass and Serre, [194, 195] and also, [21]. It seems a good idea to give some definitions of some main elementary ideas from that theory as they provide some insight into the generalised form. (We will adapt the definition as given by Corson in [68].)

We first introduce some notation. Let Γ be a graph and e an edge of Γ. We will choose an orientation for each edge, and e together with that chosen orientation will be denote either by e, itself, or if more precision is needed by e^+. That edge with the opposite orientation will be denoted e^-. As we will be using both, this does not mean that we have a directed graph, merely that, for convenience we need to be able to talk of each edge together with both possible orientations and this is one way of handling that need. We will adopt the notation $i(e)$ for the initial vertex of $e = e^+$, and $t(e)$ for the 'other end'. Of course, $i(e^-) = t(e^+)$, etc. Note that we can also think of this as being a directed graph,

$$E_\Gamma \underset{t}{\overset{i}{\rightrightarrows}} V_\Gamma \ ,$$

together with an involution on the edges,

$$^- : E_\Gamma \to E_\Gamma$$

called *edge reversal*, satisfying some obvious properties. This is particularly useful for various definitions slightly later on.

Definition: A *graph of groups*, $\mathcal{G}$, is a pair, $\mathcal{G} = (\Gamma, G)$, consisting of an (abstract) connected graph, Γ and an assignment, G, which assigns to each vertex, v of Γ, a group G_v, and to each *oriented edge* (i.e., an edge of Γ together with a direction on it), a group G_e, such that $G_e = G_{e^-}$ and an *monomorphism*, $\mu_e : G_e \to G_{i(e)}$.

If we consider Γ as a 1-dimensional simplicial complex, and work with its associated poset, S_Γ, then the above gives a functor from the opposite category, S_Γ^{op} to the category of groups and monomorphisms between them. This category, S_Γ^{op}, has objects the vertices and the edges and for each edge there is one morphism $e \to i(e)$, (and, of course, another $e \to i(e^-)$).

It should be fairly obvious that a graph of groups is a simple example of a complex of groups. We leave the detailed checking to the diligent reader. (Note the exposition here is adapted from various sources on graphs of groups, so there will be some minor things to check, in particular that the extra structure given in the case of complexes of groups has no content in this simple case.)

Examples: Suppose that T is a tree, and therefore, in particular, a graph, and there is a group π acting on the right on T (an action, which is assumed to be without inversions). The orbit graph $\Gamma = T/\pi$ supports a natural structure of a graph of groups. In this, the vertex groups, G_v, are the vertex stabilisers of the actions, so

$$G_v = \{g \in \pi \mid v.g = v\},$$

and the edge group of an edge, e, is

$$G_e = \{g \in \pi \mid e.g = v\}.$$

As any automorphism of T that fixes e must fix both $i(e)$ and $t(e)$, the group G_e is a subgroup of both $G(i(e))$ and $G(t(e))$. This gives the information necessary for a graph of groups based on Γ.

In fact, given any graph of groups, one can find a tree, T, and a group acting on it, but for this we need the idea of the fundamental group of a graph of groups.

3.5.4 The fundamental group(oid) of a graph of groups

The fundamental group of a graph of groups, (Γ, G), can be defined in several equivalent ways. There are basically two approaches one topological and the other algebraic. In the algebraic one the usual starting point is to choose a maximal tree in Γ. This seems a bit counter to our approach so we will, instead, first define the fundamental *groupoid* of (Γ, G), a definition first given Higgins in [116] and an equivalent one is given in [162], where the formulation is optimised for computational uses. We will explicitly use the description of Γ as having an involution, $^{-}$, which 'reverses' arrows.

Definition: Given a graph of groups, $\mathcal{G} = (\Gamma, G)$, its *fundamental groupoid*, Π_1 , $\Pi_1(\mathcal{G})$ or, if more detail is needed, $\Pi_1(\Gamma, G)$, is the groupoid specified as follows:

- the objects of Π_1 are the vertices of Γ;
- a generating graph of arrows for Π_1 is given by Γ, together with the elements of all the groups, G_v, for $v \in V_\Gamma$, where
- if $v \in V_\Gamma$ and $g \in G_v$, both the source and target of g are equal to v, i.e., for each $g \in G_v$ we have a loop labelled g at the vertex v in this generating graph.

The defining relations are

(i) if $v \in V_\Gamma$, and $a, b, c \in G_v$ satisfy $ab = c$, then $ab = c \in \Pi_1$;

(ii) if $e \in E_\Gamma$ and $a \in G_e$, then
$$\mu_e(a) = e\mu_{\bar{e}}(a)\bar{e}.$$

Remarks: (i) As a consequence of the last relation, we get $\bar{e} = e^{-1}$, so the algebraic inverse within the groupoid coincides with the more 'geometric' inverse obtained by the edge reversal involution.

(ii) Again in this last relation, it is worth taking this apart a bit as it is here that the edge groups interact with the vertex groups, whilst it is only the vertex groups that give generators. (It is worth comparing this situation with our earlier discussion on the van Kampen theorem and presentations of pushouts of groups in section 3.3.11, as that was a closely related situation, although, as there, we still do not have enough machinery to do it justice, and to explain 'what is going on'.)

We have $e \in E_\Gamma$ is an edge, going from $i(e)$ to $t(e)$, there is an edge, $\bar{e}$, going in the reverse direction. We therefore have two injections, $\mu_e : G_e \to G_{i(e)}$ and $\mu_{\bar{e}} : G_e \to G_{t(e)}$. We also have two generators in Π_1, $e : i(e) \to t(e)$ and $\bar{e} : t(e) \to i(e)$. If $a \in G_e$, we have $\mu_e(a) \in G_{i(e)}$ and the composite $e\mu_{\bar{e}}(a)\bar{e}$ is also in this vertex group, $G_{i(e)}$. (We are reading off the composite as
$$i(a) \xrightarrow{e} t(e) \xrightarrow{\mu_{\bar{e}}(a)} t(a) \xrightarrow{\bar{e}} i(e),$$
so in 'concatenation order'.)

Definition: The *fundamental group* of (Γ, G) at a vertex v, is the vertex group, $\Pi_1(\mathcal{G})(v)$, of $\Pi_1(\mathcal{G})$ at v.

As Γ is connected, the fundamental groups at any two vertices are isomorphic. This is slightly deceptive, however, as they may be isomorphic by many different isomorphisms corresponding to different paths between those vertices. This is essentially the same point as saying that a presentation of a group has, *really*, to be given together with an explicit isomorphism to the group, although for many (most?) purposes this is not useful information.

If a presentation of $\Pi_1(\mathcal{G})(v)$ is desired, it can be obtained by choosing a maximal tree, T, in the graph Γ.

Proposition 38 *Given a maximal tree, T, in Γ, the fundamental group, $\Pi_1(\mathcal{G})(v)$, has a presentation, $(X : R)$, where X is the disjoint union of the G_vs and the set E_Γ, of edges of Γ, and the relations, R, are the relations are*

(i) if $v \in V_\Gamma$, and $a, b, c \in G_v$ satisfy $ab = c$, then $ab = c \in \Pi_1(\mathcal{G})(v)$;

(ii) if $e \in E_\Gamma$ and $a \in G_e$, then

$$\mu_e(a) = e\mu_{\overline{e}}(a)\overline{e},$$

and

(iii) $e = 1$ if $e \in T$. ■

3.5.5 A graph of 2-complexes

If we have a graph of groups, $\mathcal{G} = (\Gamma, G)$, then one way to obtain a 'presentation' of $\mathcal{G}$ is via a graph of 2-complexes. The ideas is easily accessible in Corson, [68], but is also discussed in the lecture notes of Scott and Wall, [193], where graphs of spaces, in more generality, are introduced. We will return to this later.

Clearly the graph of groups $\mathcal{G}$ can be considered as a functor $G : S_\Gamma^{op} \to Grp$, with the proviso that each morphism of S_Γ is sent to a monomorphism of groups. This viewpoint will be useful very shortly.

Definition: (i) A graph of 2-complexes, (Γ, X) is a functor, $X : S_\Gamma^{op} \to CW$, from S_Γ^{op} to the category of CW-complexes such that, for each vertex, v, (resp. each edge, e) of Γ, the space X_v, (resp. X_e) is a (pointed connected) 2-complex, and, for each edge, e, the maps $X_e \to X_{i(e)}$ (and $X_e \to X_{t(e)}$) are cellular and preserve base points.

(ii) The graph of 2-complexes, (Γ, X) is a *presentation* of a graph of groups, $\mathcal{G} = (\Gamma, G)$, if there are (given) isomorphisms

- $\pi_1(X_v, *_v) \cong G_v$;
- $\pi_1(X_e, *_e) \cong G_e$,

which are compatible with the edge monomorphisms, so

$$\begin{tikzcd} \pi_1(X_e, *_e) \arrow[r, "\cong"] \arrow[d] & G_e \arrow[d, "\mu_e"] \\ \pi_1(X_{i(e)}, *_{i(e)}) \arrow[r, "\cong"'] & G_{i(e)} \end{tikzcd}$$

commutes; similarly for $\mu_{\bar{e}}$.

Put more succinctly, applying the fundamental group functor, π_1, to (Γ, X) gives (Γ, G) up to natural isomorphism.

Of course, the various X_v and X_e are essentially given by a presentation of the corresponding groups (except that we do not state that the 2-complexes will be reduced), so it is natural to extend our prebvious discussion of higher syzygies to this case. We mention that in the case of a pushout of groups, a double mapping cylinder allowed one to write out the 2-syzygies in at least a simple case. In this more general case, we have an analogous construction, which generalises that and which we introduce next. (We will later on see this as a simple example of a homotopy colimit.)

Definition: Given a graph of 2-complexes, (Γ, X), its *total space* is the space constructed as follows:

- take the coproduct (so disjoint union) of the spaces X_v for $v \in V_\Gamma$ together with the spaces $X_e \times [0,1]$ for $e \in E_\Gamma$;
- identify along the following maps:
  - $X_e \times [0,1] \to X_{\bar{e}} \times [0,1]$ sending (x,t) to $(x, 1-t)$;
  - on the subspace, $X_e \times \{0\}$, of $X_e \times [0,1]$, use $X_e \times \{0\} \cong X_e \to X_{i(e)}$, given by the structure map of $X : S_\Gamma^{op} \to CW$, (and similarly for $\bar{e}$).

We will denote the resulting space by $Tot(\Gamma, X)$, or sometimes simply X_Γ.

Example: Take Γ to be the graph with two vertices, 0 and 1, and one edge, $(0,1)$, joining them, then a graph of 2-complexes is given by a 'span' diagram

$$X_0 \longleftarrow X_{0,1} \longrightarrow X_1 \; .$$

The resulting total space is given by the colimit of the diagram:

$$\begin{tikzcd} & X_{0,1} \arrow[r] \arrow[d, "e_1"] & X_1 \\ X_{0,1} \arrow[r, "e_0"] \arrow[d] & X_{0,1} \times [0,1] & \\ X_0 & & \end{tikzcd}$$

where $e_i : X_{0,1} \to X_{0,1} \times [0,1]$ sends x to (x, i) for $i = 0, 1$, so is the appropriate double mapping cylinder.

Proposition 39 *If (Γ, X) is a graph of 2-complexes that presents (Γ, G), then the fundamental group of $Tot(\Gamma, X)$ is isomorphic to the fundamental group of (Γ, G) (based at any vertex).* ■

We will not give a proof. One is given in [193] and we will later see a generalisation of it, so including one here seems inessential. A proof can be given using the van Kampen theorem in a more general form than we have quoted above.

The above strays out of our usual, more algebraic territory as it uses topological methods. An intermediate approach which uses the algebraic ideas combined with some combinatorial constructs can be usefully obtained by looking at the corresponding construction within the category of groupoids. This is taken, here, from Emma Moore's thesis, [162]. As usual, $\mathcal{I}$ denotes the interval groupoid, which has two objects 0 and 1 and morphisms $\iota : 0 \to 1$ and its inverse, together with, of course, the identity arrows at each object, also, if H is a group, $H[1]$ denotes the one object groupoid to which it corresponds. (Reminder we will be working within the category of groupoids in the following definition, so, in particular, coproduct, $\sqcup$, has to be interpreted accordingly.)

Definition: Given a graph of groups, $\mathcal{G} = (\Gamma, G)$, the *total groupoid*, $Tot(\mathcal{G})$ is defined as the quotient of $(\bigsqcup\{G_v[1] \mid v \in V_\Gamma\}) \sqcup (\bigsqcup(\{G_e[1] \times \mathcal{I} \mid e \in E_\Gamma\})$ by the relations corresponding to

- $G_e[1] \times \mathcal{I} \overset{\cong}{\leftrightarrow} G_{\bar{e}}[1] \times \mathcal{I}$ by $(g, \iota) \leftrightarrow (g, \iota^{-1})$,
  and
- $G_e[1] \times \{0\} \to G_{i(e)}[1]$ given by $(g, 0) \leftrightarrow \mu_e(g)$.

Example: For a graph of groups, $\mathcal{G} = (\Gamma, G)$, where Γ has just two vertices and one edge, $Tot(\mathcal{G})$, is the groupoid double mapping cylinder.

Theorem 10 *The total groupoid of a graph of groups, $\mathcal{G}$ is isomorphic to the fundamental groupoid of $\mathcal{G}$.*

Sketch proof: First of all we note that the set of objects of $\Pi_1\mathcal{G}$ is V_Γ, whilst that of $Tot(\mathcal{G})$ is obtained by quotienting from a set made up as the disjoint union of V_Γ with two copies of E_Γ, one labelled $E_\Gamma \times \{0\}$, the other $E_\Gamma \times \{1\}$. The first relation identifies each $(e, 0)$ with the corresponding $(\bar{e}, 1)$, so after that there is but one copy of each 'edge vertex', then that edge vertex, $(e, 0)$, is identified with the vertex of $G_{i(e)}$. We thus have $Ob(Tot(\mathcal{G}))$ is bijective with V_Γ, and as each equivalence class of objects contains exactly one element naturally identify the two sets without risk of losing naturality.

We next construct a morphism from $Tot(\mathcal{G})$ to $\Pi_1\mathcal{G}$. To do this, we use the natural morphisms from the coproduct used to construct $Tot(\mathcal{G})$ to $\Pi_1\mathcal{G}$, and then check that the relations / identifications are consistent with the relations in the presentation of $\Pi_1\mathcal{G}$. This is easy to check at an intuitive level, but needs a bit of care for the detail. (We will see generalisations of such results later, so **leave these details for your consideration**.) ∎

This does not *directly* help in our search for machinery to calculate syzygies, but note that for calculations with graphs of 2-complexes should be mirrored by calculations with some sort of graph of crossed modules, and this is in part suggested by the result of Abels and Holz that we mentioned earlier, (Proposition 30, page 109), and also our comments on the 2-syzygies of a pushout presentation, discussed in section 3.3.11, however these idea would seem to be better explored in the more general context of complexes of groups, so we will put them aside for the moment.

3.5.6 A brief glance at the Bass-Serre theory

We will, very briefly, now turn to some aspects relating to the formulation and (sketch) proof of one of the main theorems of Serre's theory of graphs of groups. (We will, in part, use ideas from the discussion in K. Brown's book, [39], on cohomology of groups, as it links the theory into standard material on equivariant homology, and for that is thoroughly to be recommended. The relevant sections are at the end of Chapter II, starting on page 52, and then section 9 of Chapter VII, page 178.)

Suppose G acts on a *tree*, Γ, and let $e \in E_\Gamma$ be an edge of Γ, having vertices v and w.

Definition: The edge, e, is called a *fundamental domain* for the G-action if (i) given any edge, e' of Γ, there is a $g \in G$ such that $e \cdot g = e'$, and (ii) every vertex of Γ is equivalent, modulo the action, to eith v or w, but not both.

This second condition, of course, implies that, if we have some $v' \in V_\Gamma$ and $g \in G$ such that $v' = v \cdot g$, then $\{h \mid h \in G, v' = w \cdot h\}$ is empty. We thus have that the subgraph given just by e, v and w maps isomophically to Γ/G under the projection from Γ to Γ/G.

Lemma 26 *Suppose e is a fundamental domain for the action of G on Γ, then*

$$G_e = G_v \cap G_w.$$

Proof: First note that as Γ is a tree, $G_e \supseteq G_v \cap G_w$, since there is only the edge, e between v and w, so, if $v \cdot g = v$ and $w \cdot g = w$, then the only possibility is that $e \cdot g = e$.

On the other hand, v and w are not in the same G-orbit, by the fact that e is a fundamental domain, hence, if $e \cdot g = e$, then g must fix both v and w, which proves the opposite inclusion. ■

Theorem 11 *(Serre) Let G be a group acting on a tree Γ in such a way that there is a fundamental domain, (which is an edge, e, as above), then G is a 'free product with amalgamation' $G \cong G_v \sqcup_{G_e} G_w$, i.e., there is a pushout square,*

$$\begin{tikzcd} G_e \arrow[r] \arrow[d] & G_v \arrow[d] \\ G_w \arrow[r] & G \end{tikzcd}$$

in which the top horizontal and left vertical arrows are inclusions of subgroups, (and hence the other two arrows are monomorphisms).

Conversely given a pushout,

$$\begin{tikzcd} A \arrow[r] \arrow[d] & G_1 \arrow[d] \\ G_2 \arrow[r] & G \end{tikzcd}$$

in which $A \to G_1$ and $A \to G_2$ are monomorphisms, there is a tree on which G acts, as above, with G_1, G_2 and A being G_v, G_w and G_e for the obvious notational choice, and in which e is a fundamental domain.

Sketch proof: There is clearly a morphism from $G_v \sqcup_{G_e} G_w$ to G since the square given in the statement of the result clearly commutes. We have to show this is an isomorphism.

We form the graph of groups corresponding to the G-action on Γ, which will be

$$G_v \underset{G_e}{\text{———}} G_w \ ,$$

or, in the notation we have used earlier,

$$G_v \longleftarrow G_e \longrightarrow G_w \ ,$$

and which has $G_v \sqcup_{G_e} G_w$ as its fundamental group. If, now, $g \in G$, we can look at the unique path in the *tree*, Γ, from v to $v \cdot g$. This must start something like the following

$$v \overset{e}{\text{———}} w \overset{eg_1}{\text{———}} vg_1 \overset{eg_1g_2}{\text{———}} wg_1g_2 \text{———} \dots ,$$

but then, as $wG_1 = w$, $g_1 \in G_w$, and as $g_2 \in G_{vg_1}$, $g_1g_2g_1^{-1} \in G_v$, and you continue on in this form until eventually you get to $v \cdot g$. Hence g can be written as a word in the elements of G_v and G_w. This shows that the natural map in onto. (There is ambiguity in the word one gets but the amalgamation handles that. That also handles the injectivity of the natural map. We **leave the details for you to complete**. They are very similar to the arguments we looked at when examining $\pi_1(N(\mathfrak{H}))$. The similarity is not by accident as is fairly clear it is hoped.)

We now turn to the converse. Given that $G = G_1 \sqcup_A G_2$, there is virtually no choice as to how to construct a graph, Γ which will 'undo' the construction above. We must take the set of vertices to be $(G_1\backslash G) \sqcup (G_2\backslash G)$, and the set of edges to be $A\backslash G$, in each case the set of right cosets of the respective subgroup. As $A \subseteq G_1$, there is a natural map $A\backslash G \to G_1\backslash G$, and similarly $A\backslash G \to G_2\backslash G$, giving the source and target maps of the graph, Γ, that we are constructing, so $Ag \in A\backslash G$ joins G_1g to G_2g. The group, G, acts on the right on Γ with any edge a fundamental domain and with G_1, G_2 and A as the corresponding stabilisers. The graph Γ is connected by an argument similar to that above, whilst it can have no non-trivial reduced loops, so is a tree. ■

Remark: The observant reader will, of course, have noticed that, as G_1 and G_2 are subgroups of G, we can form $N(\mathfrak{H})$, for $\mathcal{H} = (G_1, G_2)$. This means, as we have stated before when discussing families of groups and their nerves, that $N(\mathfrak{H})$ has as vertices the elements of $(G_1\backslash G) \sqcup (G_2\backslash G)$. What about the 1-simplices?

We have $\sigma = \langle G_1g, G_2h\rangle$ is a 1-simplex of $N(\mathfrak{H})$ if there is an $x \in G_1g \cap G_2h$, and, as we saw earlier, then $\sigma x^{-1} = \langle G_1, G_2\rangle$. This edge of $N(\mathfrak{H})$ is $A1$, the coset of A with representative 1, and hence the basic edge of the graph, Γ, going between the cosets / vertices G_11 and G_21. We can thus define

$$N(\mathfrak{H}) \to \Gamma$$

by $\langle G_ig\rangle$ goes to G_ig, of course, and $\langle G_1g, G_2h\rangle = \langle G_1, G_2\rangle x$ goes to Ax in Γ. This is a simplicial isomorphism (or as both are 1-dimensional simplicial complexes, i.e. graphs, an isomorphism of graphs). There is thus a considerable overlap between the Abels-Holz theory and (part of) the theory of graphs and complexes of groups. It is only a part of that theory, however, since the above result, and these remarks, only handle the case of actions having a fundamental domain.

If the action of G has no fundamental domain then some edges would have both ends within the same G-orbit. This means that the action would not be regular. Examples of this occur with

HNN-extensions. We have some group, H, with a subgroup, A, together with a monomorphism, $\theta : A \to H$. The HNN-extension, $H*_{(A,\theta)}$, is obtained by adjoining an element t to H, subject to the relations:

$$t^{-1}at = \theta(a)$$

for all $a \in A$. To see the relationship with graphs of groups examine the fundamental group of the graph of groups, $\mathcal{G}$, with underlying graph the graph with one vertex, v, and one edge, e, and nothing else. Take the vertex group, G_v, to be H, the edge group, G_e, to be A, and the two morphisms from G_e to Gv are the inclusion of A into H and the given monomorphism, θ. Clearly $\Pi_1(\mathcal{G})(v) \cong H*_{(A,\theta)}$.

The interesting question is: if we have (higher) presentation data on A and H, can we get similar information on $H_{(A,\theta)}$?

Remark: There is a neat way of considering HNN-extensions via pushouts of groupoids, (rather than of groups) that ties in nicely with our glance at total groupoids above. As before, we let H be a group and A a subgroup together with a monomorphism, $\theta : A \to H$. We form a pushout of groupoids:

$$\begin{tikzcd} \{0,1\} \times A[1] \arrow[r, "k"] \arrow[d, "i"'] & H[1] \arrow[d] \\ \mathcal{I} \times A[1] \arrow[r] & \mathcal{G} \end{tikzcd}$$

where, as ever, $\mathcal{I}$ is the interval groupoid, $k(0,1) = a$, $k(1,a) = \theta(a)$, and i is the inclusion of the two ends of the cylinder. As $H[1]$ is a 1-object group so is $\mathcal{G}$, so it is $G[1]$ for some group, G. This group, G can be written as a factor group

$$C_\infty * H/\{(t^{-1}a^{-1}t)\theta(a) \mid a \in A\},$$

where $C_\infty = \langle t \mid \emptyset\rangle$, is the infinite cyclic group generated by an element t, so G is $H*_{(A,\theta)}$ in our earlier notation.

For HNN-extensions as here, there is also a tree, Γ, with an action of the group $H*_{(A,\theta)}$ in this case, but it is not as easy to describe. In general, given any graph of groups, $\mathcal{G}$ with fundamental group $G = \Pi_1(\mathcal{G})(v)$, there is a tree, T, with an action of G on it, such that the graph of groups that results from that action is isomorphic to $\mathcal{G}$. The idea of the construction is an analogue of the construction of a universal covering space using homotopy classes of paths based at a base point. We will not give it here, as we will see generalisations later on.

3.5.7 Fundamental group(oid) of a complex of groups

We now will go back to the higher dimensional situation, since, apart from any other reason, the Abels-Holz context with a family of n subgroups naturally leads to a $(n-1)$-simplex of groups, so graphs of groups are not general enough for their study. In fact, as we mentioned earlier, general complexes of groups are closely related to orbifolds and more generally to topological stacks, so the families of groups case is just one example of a situation leading to their study.

We now need to extend the definition of a fundamental group from applying to graphs of groups to the more general case.

Let $G(K) = (K, G_\sigma, \psi_a, g_{a,b})$ be a complex of groups as before, and, for convenience, let $E_K^\pm$ denote the set of edges of the barycentric subdivision, K', with an orientation, $a^+ = a$, and a^- to be a with the opposite orientation, so $i(a^-) = t(a^+)$, etc.

First define $FG(K)$ to be the group generated by

$$\bigsqcup\{G_\sigma : \sigma \in V_K\} \cup E_K^\pm$$

subject to the relations

- the relations of each G_σ,
- $(a^+)^{-1} = a^-$ and $(a^-)^{-1} = a^+$,
- $\psi_a(g) = a^- g a^+$ for $g \in G_{i(a)}$,
- $(ba)^+ g_{a,b} = b^+ a^+$ for composable a, b.

The image of G_σ in $FG(K)$ will be denoted $\bar{G}_\sigma$.

Haefliger defines $\pi_1(G(K), \sigma_0)$ in two equivalent ways:

Definition: Version 1: If $\sigma_0, \sigma_1 \in V_K$, the vertices of K, a $G(K)$-*path*, c, from σ_0 to σ_1 is a sequence, $(g_0, e_1, g_1, \cdots, e_n, g_n)$, where $(e_1, \cdots, e_n)$ is an edge path in K' from $i(e_1) = \sigma_0$ to $t(e_n) = \sigma_1$, $e_i \in E_K^\pm$, for $i = 1, \cdots, n$, and where $g_k \in G_{t(e_k)} = G_{i(e_{k+1})}$.

Such a $G(K)$-path, c, represents $g_0 e_1 \cdots e_n g_n \in FG(K)$. Two such paths from σ_0 to σ_1 are said to be *homotopic* if they represent the same element of $FG(K)$. We set $\Pi_1(G(K), \sigma_0, \sigma_1)$ equal to the subset of $FG(K)$ represented by $G(K)$-paths from σ_0 to σ_1. These can be used to form a *fundamental groupoid* of $G(K)$. When $\sigma_0 = \sigma_1$, we write

$$\pi_1(G(K), \sigma_0) = \Pi_1(G(K), \sigma_0, \sigma_0).$$

This is a subgroup of $F(G)$ and is called the *fundamental group* of $G(K)$.

Definition: Version 2: Assume K is connected and pick a maximal tree, T, in the 1-skeleton of $Sd(K) = K'$. Let $N(T)$ be the normal subgroup of $FG(K)$ generated by $\{a^+ : a \in T\}$, then

$$\pi_1(G(K), T) \cong FG(K)/N(T),$$

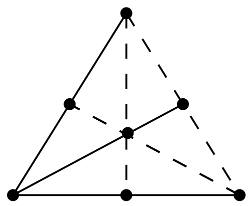
and hence has a presentation:

- generators	$\sqcup G_\sigma \sqcup E_K$	
- relations :	- $g_1 \cdot g_2 = g_1 g_2$	within any particular G_σ
	- $\psi_a(g) = a^{-1} g a$	$g \in G_{i(a)}$,
	- $(ba) g_{a,b} = b.a$	if $a, b \in E_K$ are composable
	- $a = 1$	if $a \in T$.

Clearly if we just have a complex of groups which is a graph of groups then the above gives the fundamental group of that in the previous sense.

If we restrict back to the Abels-Holz situation of group with a family of subgroups we get the following:

Example: Suppose $\mathfrak{H} = (G, \mathcal{H})$, $\tilde{K} = N(\mathfrak{H})$, $\mathcal{H} = \{H_1, \cdots, H_n\}$, so $K = \Delta^{n-1}$. Pick the maximal tree with edges radiating out from the vertex $\{H_1\}$, e.g. if $n = 3$, we get figure 3.5.7



Barycentric Subdivision of Δ^2 with the chosen maximal tree shown.

There is an obvious collapse of Δ^{n-1} to T. We have already noted that all the $g_{a,b}$ are trivial in these examples so we can prove (inductively via the collapsing order) that if a is any edge in $Sd(\Delta^{n-1})$, the fact that $\alpha = 1$ for $\alpha \in T$ implies that $a = 1$ in $\pi_1(G(K), 1)$. Thus $\pi_1(G(K), 1)$ has a presentation with

- generators $\sqcup G_\sigma$
- relations :
  - $g_1 \cdot g_2 = g_1 g_2$ within any particular G_σ
  - $\psi_\alpha(g) = g$ for $g \in G_{i(\alpha)}$

As $G_\sigma = \bigcap\{H_i \mid i \in \sigma\}$, we have

$$\pi_1(G(K), 1) \cong \underset{\cap}{\sqcup} H_i,$$

the coproduct of the H_i amalgamated over the intersection.

It is noticeable that there is, as before, a homomorphism

$$\pi_1(G(K), 1) \to G$$

with kernel $\pi_1 N(\mathfrak{H})$.

3.5.8 Haefliger's theorem on developable complexes of groups

Theorem 12 *Let K be a connected simplicial (cell) complex and $G(K)$ be a complex of groups on K. Assume that for each σ. the natural homomorphism from G_σ to $FG(K)$ is injective, then $G(K)$ is developable, (i.e., one can construct a $\tilde{K}$ with a group G acting on it, with $\tilde{K}/G$ isomorphic to K and $G(K)$ isomorphic to the associated complex of groups of the G-action).* ■

Instead of a detailed proof we will just give the construction, **leaving you to check through the details** or to **look them up** for instance in [36].

Construction: For simplicity, we will assume that K is a simplicial complex. The more general case of it being a simplicial cell complex and thus determined by a scwol is handled in [36]. Choose a maximal tree, T in the 1-skeleton of $K' = Sd(K)$, the barycentric subdivision of K,, then there is a simply connected $\tilde{K}$ and an action of $G = \pi_1 G(K)$ (without inversion) on $\tilde{K}$ and a projection $p : \tilde{K} \to K$ inducing $\tilde{K}/G \cong K$. The complex $\tilde{K}$ is specified via its associated category, $C(\tilde{K})$, as follows:

- the set, $V(\tilde{K})$, of simplices of $\tilde{K}$ is the set,

$$\{(G_\sigma g, \sigma) \mid \sigma \in S_K = V_{Sd(K)}, g \in G = \pi_1 G(K)\};$$

  these form the objects of $C(\tilde{K})$, then $p((G_\sigma g, \sigma) = \sigma$;

- the set, $E(\tilde{K})$, of 'edges' will be $\{(G_\sigma g, a) \mid a \in E(X), i(a) = \sigma\}$, then $p(G_\sigma g, a) = a$, whilst $i(G_\sigma g, a) = (G_\sigma g, i(a))$ and $t(G_\sigma g, a) = (G_{t(a)} a g, t(a))$.

This is well defined for $h \in G_{i(a)}$, since $ahg = \psi_a(h) a g$ in $FG(K)$.

The action of G is obvious and there is an obvious lifting of T to a maximal tree, $\tilde{T}$ in $\tilde{K}$ via a 'choice', $\tilde{\sigma} = (G_\sigma, \sigma)$, $\tilde{a} = (G_{i(a)}, a)$, which yields an associated complex of groups isomorphic to $G(K)$.

The following is a continuation of the same theme:

Theorem 13 *If $G(K)$ is obtained from a group, $\bar{G}$, acting one some complex, $\bar{K}$, with $K = \bar{K}/\bar{G}$, and there is given a lifting, $\bar{T}$, of the maximal tree, T, then there is a surjective homomorphism,$\varphi : G \to \bar{G}$, and a φ-equivariant covering map $f : \tilde{K} \to \bar{K}$ mapping $\tilde{T}$ to T over K. The kernel of φ is the fundamental group of $\bar{K}$, viewed as the 'algebraic' Poincaré-Galois group of $\bar{K}$.* ■

By the 'algebraic' Poincaré-Galois group, here we are meaning the analogue here of the group of deck transformations in covering space theory, as in Grothendieck's theory of the fundamental group. This result should be compared with classical results on covering spaces.

3.5.9 Over to scwols

As we said earlier, the use of simplicial complexes is unduly restrictive in the above theory and in the development of that theory, both Haefliger and Corson used various types of cell complex which were more general than merely simplicial complexes. Eventually it has become clear that what is essential in all these was the *combinatorial data* relating to the face inclusions of the cell complexes. as an example, we have used the partially ordered set, $(S_K, \supseteq)$, of simplices of a simplicial complex, K, where $\sigma \leq \tau$ if $\sigma \supseteq \tau$. (This was used to form the barycentric subdivision of the complex, K.) We note that this means that in the small category, $C(K)$, associated to this poset if there is an arrow from σ to τ, then τ has to be a subset of σ so has at most the same number of elements as does σ, so morphisms in $C(K)$ 'drop dimension' (except for the identities of course). It is thus clear for several reasons that $C(K)$ is 'without loops'. This is obvious because it is a partially ordered set, but also since, if a non-identity arrow drops dimension, it cannot give one a loop. This gives an argument that is slightly more general and has some intuitive additional geometric content.

We saw (page 139) that HNN-extensions corresponded, in Bass-Serre theory to a graph of groups whose underlying graph was not a poset. The basic graph had one vertex and one edge,

$$\begin{tikzcd} v \cdot \arrow[loop right, "e"] \end{tikzcd}$$

and hence $C(K)$ was

$$\begin{tikzcd} e \arrow[r, bend left] \arrow[r, bend right] & v \end{tikzcd}$$

so is most definitely *not* a poset! It is, however, 'without loops'. 'Small categories without loops' (i.e., scwols), therefore seem a good generalisation of both the simplicial complex based theory and one based on graphs. We gave the definition of a scwol earlier (on page 129) and so will not repeat it here. We did mention that most of the theory that has been developed above generalises without difficulty from the partially order set (and thus simplicial complex) case to that using scwols. This latter theory is described in detail in Bridson and Haefliger, [36], so we will not develop it here, except in so much as it overlaps with the themes of these notes. We will however give a few of the definitions to show how, in going from the simplicial complex / poset case to that involving scwols, one can adapt ideas quite easily. (It may help to think of simplicial cell complexes as they are geometric realisations of scwols, just as simplicial complexes are for posets. Again these definitions and properties are discussed in Chapter III of [36].)

First we look at encoding the 'action without inversion' and 'regular actions' type of condition in terms of scwols. We suppose given a scwol, $\mathcal{X}$, with vertex set, $V(\mathcal{X})$, and edge / arrow set, $E(\mathcal{X})$. As usual, a group G acts on $\mathcal{X}$ if there is a given homomorphism from G to the automorphism group of $\mathcal{X}$.

When G acts on $\mathcal{X}$, it will be assumed that this is 'without inversion':

Definition: The action of G on $\mathcal{X}$ is *without inversion* if

(i) for all $a \in E(\mathcal{X})$ and $g \in G$, $i(a) \cdot g \neq t(a)$ (so no g 'flips' an edge);

(ii) for all $g \in G$ and $a \in E(\mathcal{X})$, if $i(a) \cdot g = i(a)$ then $a \cdot g = a$.

Remarks: (a) The second condition means that the stabiliser of $i(a)$ contains that of a.

(b) If $\mathcal{X}$ is *finite dimensional* in the sense that it contains no infinite sequence of composable arrows, then condition (i) will automatically be satisfied since, if $i(a) \cdot g = t(a)$, we could form $a, a \cdot g, a \cdot g^2, \ldots)$ an infinite sequence since it cannot loop as $\mathcal{X}$ has no loops.

(c) Suppose that condition (ii) is satisfied and that v is a vertex / object of $\mathcal{X}$, such that $v \cdot g = v$, then g fixes any simplex, $\sigma = (a_1, a_2, \ldots, a_n) \in Ner(\mathcal{X})$ for which $i(a_1) = v$, by induction on n.

3.5.10 Quotient of a scwol by an action

We suppose that G acts on $\mathcal{X}$, and try to organise the data in $V(\mathcal{X})/G$ and $E(\mathcal{X})/G$ to form a category $\mathcal{Y}$, which, hopefully will be a scwol. We take $V(\mathcal{Y}) = V(\mathcal{X})/G$, and $E(\mathcal{Y}) = E(\mathcal{X})/G$. If a is an edge of $\mathcal{X}$, then the orbit aG is an edge from $i(a)G$ to $t(a)G$. We thus have at least a graph. If aG and bG are composable edges in that graph, then $t(a)G = i(b)G$, so $t(a) = i(b) \cdot g$ for some $g \in G$. There is thus an edge $b' = b \cdot g$ such that ab' is defined in $\mathcal{X}$. We define $aG.bG = (ab')G$, then **check this is independent of the choices made**. Associativity is easy to check as also are the identities, as $1_{vG} = 1_v G$, etc. we thus have a category (and did not use either condition for that). We write $\mathcal{Y} = \mathcal{X}/G$.

Lemma 27 *If the group action satisfies condition (i), then $\mathcal{Y}$ is a scwol.*

Proof: Suppose $a \in E(\mathcal{X})$, and let us explore the consequences of the equation $i(a)G = t(a)G$, i.e. that aG is a loop in $\mathcal{Y}$, then $t(a) = i(a) \cdot g$ for some $g \in G$. If condition (i) is satisfied, this does not happen, which proves the lemma. ■

It is clear that the right notion of a morphism of scwols should be just that of a functor. We next need some ideas from the algebraic approach to category theory given by Higgins, [117].

Definition: Given a small category, $\mathcal{X}$, and an object v of $\mathcal{X}$, the *star* of v is the set, $Star_n(\mathcal{X}) = \{a \in E(\mathcal{X}) \mid i(a) = v\}$. The *costar*, $Costar_v(\mathcal{X})$, is similarly defined with the condition $t(a) = v$.

The costar of v in $\mathcal{X}$ is, of course, the same as the star of v in $\mathcal{X}^{op}$.

Definition: if $f : \mathcal{X} \to \mathcal{X}'$ is a morphism of scwols, (with $\mathcal{X}$ non-empty), we say f is *star bijective* if for all $v \in V(\mathcal{X})$, f induces, by restriction, a bijection

$$Star_v(\mathcal{X}) \to Star_{f(v)}(\mathcal{X}').$$

We may also use related terminology: *star injective*, *star surjective*, *costar injective*, *costar surjective* and *costar bijective*.

Remark: Bridson and Haefliger, [36], p. 526, use the term 'non-degenerate' for what we have called 'star bijective'. We have chosen this latter term since 'non-degenerate' is a much overused term, but also because the terminology of 'star bijective' etc. was already used by Higgins in 1971, as was mentioned, but also is current in groupoid theory, cf. Brown, [42], which is a rewrite of his earlier books, (cf. [41]). There 'star bijective' and 'costar bijective' coincide as, in a groupoid, all arrows are invertible. The connection with covering spaces, covering groupoids, etc., is examined in these latter references and provides some insight as to the usefulness of the idea here.

If we have a group, G, acting on a scwol $\mathcal{X}$ and $\mathcal{Y} = \mathcal{X}/G$, the obvious assignment of vG to v and aG to a defines a morphism,

$$p : \mathcal{X} \to \mathcal{Y},$$

called the *projection morphism.*

Lemma 28 *If action satisfies condition (ii), above, the projection morphism is star bijective.*

Proof: We suppose that v is an object of $\mathcal{X}$ and $aG \in Star_{vG}(\mathcal{Y})$, then $i(a) \in vG$ so there is some g such that $i(a) = v \cdot g$. We thus have $i(a \cdot g^{-1}) = v$ and $a \cdot g^{-1} \in Star_v(\mathcal{X})$ has image aG, i.e., p is start surjective.

We now suppose $a, a' \in Star_v(\mathcal{X})$ are such that $p(a) = p(a')$. We thus have $a = a' \cdot g$ for some $g \in G$. We knew $i(a) = i(a') = v$ and now have $i(a) = i(a') \cdot g$ as well. If condition (ii) holds, then we can conclude that $a = a'$, so p is star injective as well. ■

The above favours the star rather than the costar, but, of course, there is a dual condition to (ii) that can be put on the action namely : if $t(a) \cdot g = t(a)$, then $a \cdot g = a$. We could call this 'condition (ii)*' as it is the dual of condition (ii) above, and is true of p if condition (ii) is true of p^{op}. It is then clear that the dual of the lemma applies and if condition (ii)* holds then p is costar bijective. If both (ii) and (ii)* are satisfied then, of course, p will be both star and costar bijective. This leads to the following:

3.5.11 Coverings of small categories

Definition: A functor, $f : \mathcal{X} \to \mathcal{X}'$, between small categories is a *covering* if it is surjective on objects, star and costar bijective.

Note on terminology: Bridson and Haefliger, [36], make the definition for connected categories and with $\mathcal{X}$ non-empty, otherwise one can get to check the star and costar bijectivity conditions at no vertices, and, of course, they are then both satisfied at all the vertices that one checks! For a similar reason, for the non-connected case, they require surjectivity as otherwise there might be a component of $\mathcal{X}'$ which has no pre-image in $\mathcal{X}$ which would not seem to correspond to some of the intuitions on coverings. The terminology in Higgins, [117], is different and he uses covering to mean star bijective, with costar bijective being refereed to as co-covering. He also does not insist that coverings be surjective.

Corollary 11 *If $\mathcal{X}$ is a connected scwol, and G acts on $\mathcal{X}$ without inversion, and satisfies (ii)*, then $p : \mathcal{X} \to \mathcal{X}/G$ is a covering. In particular, any free action of G on $\mathcal{X}$ yields a covering $p : \mathcal{X} \to \mathcal{X}/G$.*

Proof: This is just a question of collecting up and repackaging some of the earlier stuff, except for the last statement, but, if G acts freely, then the condition $i(a) \cdot g = i(a)$ implies $g = 1_G$, and similarly for that involving $t(a)$, thus both (i) and (ii)* are both automatically satisfied. ■

Definition: A *Galois covering* of a scwol, $\mathcal{Y}$, with Galois group, G, is a covering, $p : \mathcal{X} \to \mathcal{Y}$, together with a *free* action of G on $\mathcal{X}$ such that p induces an isomorphism between $\mathcal{X}/G$ and $\mathcal{Y}$.

In particular, if $p : \mathcal{X} \to \mathcal{Y}$ is a Galois covering, as above, then G must act in a simply transitive way, i.e., given $x, x' \in V(\mathcal{X})$ with $p(x) = p(x')$, there is a *unique* $g \in G$ such that $x \cdot g = x'$, and similarly on edges. This is a particular case of a *principal G-bundle*, or *G-torsor*, over $\mathcal{Y}$ in this context of small categories. We will be seeing various variants of this idea of a G-torsor later on in the notes.

3.5.12 Fundamental group(oid) of a scwol:

All this mention of coverings is reminiscent of the corresponding theory in the topological context (and even more for the 'groupoidal' context), and so there should be some analogue, one might guess, to some fundamental group or groupoid of the scwols involved. The fundamental group or groupoid of a scwol is fairly simple to define as a scwol is a small category. It therefore has a nerve (or if you prefer 'classifying space'), as defined in section **??**. This will be a simplicial set, but usually is not a Kan complex, so that in general its homotopy groups are slightly less easy to define, or rather to write down, than otherwise. The fundamental groupoid of any simplicial set is very easy to define, however, as we can mimic the 'classical' edge-path definition for simplicial complexes. We saw this earlier on page 107, so just recall it below.

If K is a simplicial set, its vertices and 1-simplices form a directed graph, the 1-skeleton of K. We form the fundamental groupoid, $\Pi_1 K$, by taking the free groupoid on this directed graph and then dividing out by the 2-simplices, explicitly for each $\sigma \in K_2$, we have a relation

$$d_2(\sigma)d_0(\sigma) \Leftrightarrow d_1(\sigma).$$

We have also seen this recently in the context of a complex of groups, since a complex of groups with all its local groups trivial is nothing more than the 'underlying' complex or scwol. We have a definition of the fundamental group(oid) of a complex of groups, so have one of that underlying scwol. This gives us a presentation. (In fact, we seem to be a bit lax here because the definitions we gave were officially for the case of a simplicial complex and its poset (scwol) of simplices, however a glance at that definition will show that it does not in anyway depend on that fact, so does fit the bill here as well.) We thus get a presentation of the fundamental groupoid, $\Pi_1\mathcal{X}$, of a scwol, $\mathcal{X}$, by defining it to be $\Pi_1 Ner(\mathcal{X})$ and then by means of this recipe. We will shortly see how to define a 'classifying space' for a complex of groups, $\mathcal{G}$, which will be the nerve of a small category associated with $\mathcal{G}$ and whose fundamental groupoid, according to this recipe again, the fundamental groupoid of the complex of groups.

If we pick a basepoint, v_0 in $\mathcal{X}$, or assume that it is connected, or restrict attention to a single chosen component, then we can define $\pi_1(\mathcal{X}, v_0)$, the fundamental group of $\mathcal{X}$ based at v_0, to be the vertex group, $\Pi_1\mathcal{X}(v_0)$. (The usual results as to this not depending on the choice of v_0 within its connected component, of course, hold true. It is worth noting, however, that this always hides a quite important fact. If v_0 and v_1 are two different choices of base point within the same component of $\mathcal{X}$, then yes, there is an isomorphism between $\pi_1(\mathcal{X}, v_0)$ and $\pi_1(\mathcal{X}, v_1)$. The isomorphism will depend on the path chosen between v_0 and v_1, so it is not always sufficient just to say the two groups are isomorphic, and that is why, quite often, it is good to use $\Pi_1\mathcal{X}$ instead of $\pi_1(\mathcal{X}, v_0)$, even when $\mathcal{X}$ is connected.)

We can also give a presentation of $\pi_1(\mathcal{X}, v_0)$ by choosing a maximal tree, T, in $\mathcal{X}$ and then setting any generating edge that is in T to be equal to 1. We can also use the terminology of edge paths and homotopies between them to build a useful equivalent description of $\pi_1(\mathcal{X}, v_0)$ and this can, also, be a useful intuition to have to hand, so we will next describe that briefly.

As usual, given $\mathcal{X}$, an *edge path* joining objects / vertices v_0 and v_1 , is a sequence, $\mathsf{e} = (e_1, \ldots, e_k)$, of elements of $E^{\pm}(\mathcal{X})$, i.e. e_i is either in $E(\mathcal{X})$ or its reversal $e^- \in E(\mathcal{X})$. These elements satisfy $i(e_1) = v_0$, $i(e_j) = t(e_{j-1})$, $j = 1, \ldots, k$, $t(e_k) = v_1$, (and where $i(e^-) = t(e)$, etc., as always).

Remark: The empty sequence causes a bit of a nuisance here. We 'really' need to specify an edge path to be a sequence $(x_0, e_1, x_1, \ldots, e_k, x_k)$ with the $x_i \in V(\mathcal{X})$ and $e_i \in E^{\pm}(\mathcal{X})$ having $i(e_j) = x_{j-1}$, $t(e_j) = x_j$ for $j = 1, \ldots, k$, then the identity at x_0 is the sequence (x_0). We will usually abuse notation however and omit the intermediate objects.

The description of $\Pi_1\mathcal{X}$ or of $\pi_1(\mathcal{X}, v_0)$, then proceeds via a notion of homotopy of edge paths, where homotopy is determined by reduction rules: $e \cdot e^- \Leftrightarrow 1_{i(e)}$ together the composition rules corresponding to composition within $\mathcal{X}$, (since up to this point we have not used that $\mathcal{X}$ is a small *category*, so has a composition). We will look at this in more detail shortly.

As we might expect, coverings have nice lifting properties with respect to paths. Suppose that we have a covering, $f : \mathcal{X} \to \mathcal{Y}$, a path, $\mathsf{e} = (e_1, \ldots, e_k)$, starting at some $y_0 \in \mathcal{Y}$, and an object, x_0, of $\mathcal{X}$ such that $f(x_0) = y_0$. For ease of exposition, let us first assume $e_1 \in E(\mathcal{X})$ with $i(e_1) = y_0$ As f is star bijective, it induces a bijection $Star_{x_0}(\mathcal{X}) \to Star_{y_0}(\mathcal{Y})$, so there is a unique edge $\bar{e}_1$ in $Star_{x_0}(\mathcal{X})$ such that $f(\bar{e}_1) = e_1$. If e_1 was the reverse of an edge of $\mathcal{Y}$, then we would use the bijective correspondence between $Costar_{x_0}(\mathcal{X})$ and $Costar_{y_0}(\mathcal{Y})$ instead. A simple proof by induction on the length of e then gives:

Proposition 40 *Any covering map, $f : \mathcal{X} \to \mathcal{Y}$, has unique path lifting.* ■

There is also a result about lifting homotopies, but this will be **left to you**. (This is similar to the above, but it would probably be helpful if we gave some more details on homotopies of edge paths, packaged in a way that was more conducive for that result!)

Homotopies of edge paths: It will help to be a bit more explicit about homotopy of edge paths in $\mathcal{X}$, not only for the above, but so as to able to mimic the topological construction of simply connected / universal coverings.

Let $\mathsf{e} = (e_1, \ldots, e_n)$ be an edge path in $\mathcal{X}$, joining v_0 to v_1. We can clearly perform the following operations on e (corresponding to the rewrites / relations that we used when presenting the fundamental groupoid of $\mathcal{X}$).

(A) Assume that, for some j, $e_j = e$ and $e_{j+1} = e^-$, for some $e \in E(\mathcal{X}$, (or more generally for $e \in E^{\pm}(\mathcal{X})$), then we can rewrite e to e', where e' is obtained from e by deleting the two entries, e_j and e_{j+1} and 'closing up'. (Note this also includes the case $e_j = e^-$ and $e_{j+1} = e$, since $(e^-)^- = e$.) We can also use the revers operation, inserting an edge and its revers in any suitable place.

(B) If we have $e_j = e$ $e_{j+1} = e'$ with both of $e, e' \in E(\mathcal{X})$, (i.e., they are both 'original edges'), then we can replace e by $\mathsf{e}' = (e_1, \ldots, e_j e_{j+1}, \ldots, e_n)$, where $e_j e_{j+1}$ is the composite in $\mathcal{X}$ of the two arrows. We also have the reverse of this operation, factorising an arrow within $\mathcal{X}$ and inserting the two factors in place of their composite in the string of edges.

(C) If an edge, e_j, is an identity arrow at some $v = i(e_j)$, then we can replace e by the contracted path leaving out e_j. We can also insert appropriate identities.

These moves or rewrites may sometimes be called *elementary homotopies*. Note they do not change the start or end vertex of a path. A homotopy between two edge paths from v_0 to v_1 will be a sequence of such elementary homotopies (and their inverses). The usual terminology and related notation will be used such as 'homotopy classes', $[\mathsf{e}]$, denoting the homotopy class of e, and so on.

3.5.13 Constructing a simply connected covering of a connected category:

Continuing that use of the usual terminology, we will say that a connected small category $\mathcal{X}$ is *simply connected* if $\pi_1(\mathcal{X}, v)$ is the trivial group.

The obvious way to construct a covering of a small category, $\mathcal{X}$, is by mimicking the standard construction of a universal covering from topology. This construction for categories is given in detail in [36], p. 580, so we will just sketch what is needed here.

We pick a vertex / object, v_0, in $\mathcal{X}$, to act as a base-point and form a category $\mathcal{X}'$ in which the objects are the homotopy classes, $[\mathsf{e}]$, of paths starting at v_0. The functor $p : \mathcal{X}' \to \mathcal{X}$ sends and object, $[\mathsf{e}]$, to the 'other end', $t(e_n)$, of $[\mathsf{e}]$. As before let $\mathsf{e} = (e_1, \ldots, e_n)$ be an edge path. An arrow starting at $[\mathsf{e}]$ is a pair, $([\mathsf{e}], e')$, where $e' \in E(\mathcal{X})$ with $i(e') = t(e_n) = p([\mathsf{e}])$; the start vertex of $([\mathsf{e}], e')$ will be $[\mathsf{e}]$, as we said, whilst the target vertex will be $[\mathsf{e} \cdot e']$, the homotopy class of the path obtained by concatenating e with the edge e', so if $\mathsf{e} = (e_1, \ldots, e_n)$, $\mathsf{e} \cdot e' = (e_1, \ldots, e_n, e')$.

Composition is the final structure to define on $\mathcal{X}'$, and it is obtained in the obvious way so as to make p a functor. Associativity etc. are then **easy to check**.

Lemma 29 *The functor, $p : \mathcal{X}' \to \mathcal{X}$, is a covering.* ∎

This is just a question of **checking** star and costar bijectivity. The first is more or less immediate; the second only slightly less so.

As in the topological situation (or the simplicial one, or the groupoid one, etc., cf. Gabriel and Zisman, [102], or Brown, [42]), this covering is a universal covering, is a simply connected covering, and has lots of nice properties. We will start making this more precise and giving (sketch) proofs.

Definition: A covering, $p : \mathcal{X}' \to \mathcal{X}$, of a connected $\mathcal{X}$ is said to be *universal* if $\mathcal{X}'$ is connected and, for every covering $q : \mathcal{X}'' \to \mathcal{X}$ of $\mathcal{X}$ together with objects v' of $\mathcal{X}'$ and v'' of $\mathcal{X}''$ satisfying $p(v') = q(v'')$, there is a functor $f : \mathcal{X}' \to \mathcal{X}''$ over $\mathcal{X}$ (so $qf = p$) such that $f(v') = v''$.

It is usual to expect uniqueness on such a universal property, but here it is a consequence of the other properties.

Lemma 30 *(i) Any functor $f : (\mathcal{X}', p) \to (\mathcal{X}'', q)$ over $\mathcal{X}$ between (connected) coverings is itself a covering.*

(ii) If f and f' both satisfy the properties in the definition, then they are equal.

Proof: (i) First we note that the result does not depend on $(\mathcal{X}', p)$ being constructed as above, and that connectedness is only needed to avoid some obvious silly counterexamples. We thus start with arbitrary connected $(\mathcal{X}', p)$ and $(\mathcal{X}'', q)$ and a morphism f between them (so a functor with $qf = p$). Let v' be an object of $\mathcal{X}'$. We look at f restricted to $Star_{v'}(\mathcal{X}')$. We have to show that this is a bijection. We know $p : Star_{v'}(\mathcal{X}') \to Star_{p(v')}(\mathcal{X})$ is a bijection as is $q : Star_{f(v')}(\mathcal{X}'') \to Star_{q(f(v'))}(\mathcal{X}) = Star_{p(v')}(\mathcal{X})$, (where we have omitted using a different notation for the restrictions to the stars in each case). We therefore have that f restricted to $Star_{v'}(\mathcal{X}')$ is a bijection (with inverse $p^{-1}q$ after suitable restrictions). Of course, the same argument applies to costars with minor changes. We thus have that f is a covering.

(ii) We have $qf = qf' = p$ and $f(v') = f'(v')$, so we look at f and f' restricted to $Sta_{v'}(\mathcal{X}')$. Here they coincide as both are $q^{-1}p$; similarly on the costars. If we want to see what they do to some object or arrow of $\mathcal{X}'$, then we can find a 'zig-zag' of arrows joining v' to that object or arrow, since $\mathcal{X}'$ is connected. we use 'continuation' along this zig-zag to show that f and f' must agree on the given object or arrow. The details are **left to you** to write down. ∎

Proposition 41 *The covering, $(\mathcal{X}', p)$, constructed above is a universal covering.*

Sketch proof: Suppose we have some connected covering, $q : \mathcal{X}'' \to \mathcal{X}$, together with an object $v' = [\mathsf{e}]$ in $\mathcal{X}'$ and another, v'', in $\mathcal{X}''$ satisfying $q(v'') = p(v')$.

We first look at the simple case, where $v' = [(v_0)]$, i.e., the homotopy class of the constant path at v_0. (We write v'_0 for this and will use it later.) If $w' = [\mathsf{e}']$ is another object of $\mathcal{X}'$, then e' is a path from v_0 to $p(w')$, which we lift to a path, $\tilde{\mathsf{e}'}$ in $\mathcal{X}''$ starting at v'', our given object in $\mathcal{X}''$. (Remember, as we are in the case $v' = v'_0$, $q(v'') = v_0$, so this does work.) We define $f(w') = t(\tilde{\mathsf{e}'})$. This *does* give $qf(w') = p(w')$. We can see this works as a functor, since, if $([\mathsf{e}'], e')$ is an arrow starting at w', then the end of $([\mathsf{e}'], e')$ is $t(\mathsf{e}' \cdot e') = t(e')$, and unique path lifting give a path from $f(w')$ to $f(t(e'))$ in $\mathcal{X}''$. This will be $f([\mathsf{e}'], e')$. This not only gives a functor, but shows that f *must* be constructed like this as the value of $f(w')$ is determined bit-by-bit by the path, e', to which it corresponds.

Next we go to the general case with $v' = [\mathsf{e}]$, v'' in $\mathcal{X}''$, etc. If $w' = [\mathsf{e}']$ is in $\mathcal{X}'$, then $\mathsf{e}^{-1} \cdot \mathsf{e}'$ is a path from $p(v')$ to $p(w')$, which can be uniquely lifted to $\mathcal{X}''$ to a path, $\tilde{\mathsf{e}}^{-1} \cdot \widetilde{\mathsf{e}'}$ starting at v', and we try $f(w') = t(\tilde{\mathsf{e}}^{-1} \cdot \widetilde{\mathsf{e}'})$. It remains only to check that everything works, but **that is left to you**. (We note the extent to which unique path and homotopy lifting is used in this.) ■

Other facets of the construction and the properties of universal coverings go across from the classical topological case to this categorical one in a fairly easy manner. We will give a few of these, chosen, in the main, for links with ideas that will be developed later on in the notes. First some more general information about coverings and path lifting.

Proposition 42 *Let $q : \mathcal{Y} \to \mathcal{X}$ be a covering of a connected category, $\mathcal{X}$. Let x_0 in $\mathcal{X}$ and y_0 in $\mathcal{Y}$ be objects such that $q(y_0) = x - 0$, then the induced homomorphism,*

$$q_* : \pi_1(\mathcal{Y}, y_0) \to \pi_1(\mathcal{X}, x_0),$$

is an injection. ■

This is a consequence of the fact that unique path lifting holds and that homotopies lift.

Proposition 43 *(i) For any covering, $q : \mathcal{Y} \to \mathcal{X}$, there is a functor $\mathsf{L}(q) : \Pi_1\mathcal{X}^{op} \to Sets$ defined by $\mathsf{L}(q) : (x) = q^{-1}(x)$, the 'fibre' over x, and, if $[\mathsf{e}] : x \to x'$ in $\Pi_1\mathcal{X}$, then, for $y \in q^{-1}(x)$, $\mathsf{L}(q) : ([\mathsf{e}])(y) = t(\tilde{\mathsf{e}})$, where $\tilde{\mathsf{e}}$ is the unique path in $\mathcal{Y}$, starting at y and covering e.*

(ii) Let $\mathsf{R}/\mathcal{X}$ denote the category of coverings of $\mathcal{X}$, then there is a functor,

$$\mathsf{L} : \mathsf{R}/\mathcal{X} \to Sets^{\Pi_1\mathcal{X}^{op}},$$

given by assigning $\mathsf{L}(q) :$ to q.

Proof: (i) This is mostly just checking that things are functorial, so that part of the proof will be **left to you, the reader**. For part (ii), it is worth noting that $\mathsf{R}/\mathcal{X}$ is a subcategory of the category of small categories over $\mathcal{X}$, and that $Sets^{\Pi_1\mathcal{X}^{op}}$ is the category of functors from $\Pi_1\mathcal{X}^{op}$ to the category of sets. (These are sometimes called *local systems*, hence the letter L.) If $f : (\mathcal{Y}_1, p_1) \to (\mathcal{Y}_2, p_2)$ is a morphism of coverings, we have $\mathsf{L}(p_i) : \Pi_1\mathcal{X}^{op} \to Sets$, for $i = 1, 2$ and need a natural transformation, $\mathsf{L}(f) : \mathsf{L}(p_1) \to \mathsf{L}(p_2)$,between them, and that means, for each x in $\mathcal{X}$, we need $\mathsf{L}(f)(x) : p_1^{-1}(x) \to p_2^{-1}(x)$. The obvious simplest thing to try is simply f restricted to the fibre over x, and then compatibility with paths, etc., follows again through unique path lifting and lifting of homotopies. The remaining checking can again be **left to you**. ■

3.5.14 The groupoid case

Although it has special features which mean that the picture there simplifies, these also give an indication of what a general theory of coverings in these categorical contexts may involve. The theory in some generality is handled in Gabriel and Zisman, [102], p. 140, and various other sources. A slightly different approach has been given in [177], as well, adapting the Gabriel-Zisman treatment to fit in with the general theory of coverings put forward by Grothendieck in SGA1, [108], which in part explains the use of R for covering as it reflects the initial letter of 'revêtement', which is the French for covering in this sense.

To any covering $p : \mathcal{R} \to \mathcal{G}$ of a (connected) groupoid $\mathcal{G}$, we can assign the local system, $L(p) : \mathcal{G}^{op} \to Sets$, which sends an object x in $\mathcal{G}$ to the fibre over x, that is, $p^{-1}(x)$. Conversely given any $L : \mathcal{G}^{op} \to Sets$, we can assign a covering, $\mathsf{R}(L) = (p_L : \mathcal{R}(L) \to \mathcal{G})$, and we turn to this next.

The above description of the construction of a 'universal' covering category was 'bare hands', i.e., the building of $\mathcal{X}'$ was given without much machinery and with minimal motivation that it would work. The description of $p_L : \mathcal{R}(L) \to \mathcal{G}$, given in [102] is more 'elegant' and starts towards general categorical considerations that will be examined later on, in particular, the Grothendieck construction (section **??**) and the homotopy colimit, section **??**. This construction is therefore very important as part of the 'categorifcation' process, although it is still fairly non-technical. It is clearly a categorical version of a semi-direct product construction.

Given a local system, $L : \mathcal{G}^{op} \to Sets$, on $\mathcal{G}$, the set of objects of $\mathcal{R}(L)$ is the disjoint union, $\bigsqcup\{L(x) \mid x \in Ob(\mathcal{G})\}$. (We actually need L to be non-empty if what follows is to work without glitches.) As usual, it is useful to write elements of a disjoint union as pairs, one part indicating an index the other an element in the set corresponding to that index. The order does not matter, but here we will take (y, x), where x is an index, in this case an object of $\mathcal{G}$, and $y \in L(x)$ is simply an element. (It is worth recalling that this care is important, for instance, if L was a constant functor, since the same y might occur in many different sets of the family, $\{L(x) \mid x \in Ob(\mathcal{G})\}$.) It is sometimes useful to write this ordered pair in the form, $y \otimes x$, or $x \otimes y$, depending on the conventions, as if we had a tensor product, even if we are in a non-additive situation. The functor, p_L, on objects, will map (y, x) to x.

A morphism in $\mathcal{R}(L)$ from (y, x) to (y', x') will consist of a morphism $f : x \to x'$. This will induce $L(f) : L(x') \to L(x)$, since L is contravariant. We need, then to specify that $y = L(f)(y')$. Again it is often useful to write the morphism as (y', f) as otherwise the same f will occur as the name of many morphisms. (This is the same point we made back on page **??** when talking of action groupoids. In fact, if a group G acts *on the right* on a set X, then this corresponds to a local system $L : G[1]^{op} \to Sets$, and our $\mathcal{R}(L)$ in this case is just the action groupoid of that action, adjusted for the right actions rather than the left action as given back there.) In this notation, $(y', f) : (L(f)(y'), x) \to (y', x')$. The 'functor', $p_L : \mathcal{R}(L) \to \mathcal{G}$, then sends (y', f) to f.We say 'functor' since we have not yet got a composition in the structure that we have denoted by $\mathcal{R}(L)$. (Beware, this is where the details get a bit confusing, because we are using both the function composition and the algebraic/concatenation conventions in the same structure, so we advise **take your time and check the details!**) Composition in $\mathcal{R}(L)$ is defined as follows: if $f : x \to x'$ and $g : x' \to x''$, then $fg : x \to x''$. That looks simple, but we need to see this in the more detailed notation. We have $(y', f) : (L(f)(y'), x) \to (y', x')$ and, say, $y'' : (L(g)(y''), x') \to (y'', x'')$, so the composite actually requires that $y' = L(g)(y'')$. we then have

$$L(f)L(g)(y'') \to (L(g)(y''), x') \to (y'', x'').$$

If things fit together as they 'should', this would correspond to a morphism from $(L(fg)(y''), x)$ to (y'', x''), but L is 'contravariant', so does this really work? Yes, it does. Look again at $L(fg)$ as a composite

$$L(x'') \xrightarrow{L(g)} L(x') \xrightarrow{L(f)} L(x),$$

so, using ordinary function notation we can apply this to $y'' \in L(x'')$. This will give $L(f)L(g)(y'')$, so it did work: $(L(fg)(y'') = L(f)L(g)(y'')$. Taking a bit of care, it is then easy to show that $\mathcal{R}(L)$ is a groupoid, and that p_L is a functor.

Remark: In fact we did nothing here that required $\mathcal{G}$ to be a groupoid and we could have started with $\mathcal{X}$ a small category with $L : \mathcal{X}^{op} \to Sets$ and would have obtained $\mathcal{R}(L)$ with $p_L : \mathcal{R}(L) \to \mathcal{X}$.

Returning to the case of a connected groupoid, $\mathcal{G}$, we note that L and R define an equivalence of categories between $\mathsf{R}/\mathcal{G}$ and $Sets^{\mathcal{G}^{op}}$. (This is quite **routine to check**.) This means that coverings in the groupoid case are more or less the same as local systems of sets, i.e. functors from $\mathcal{G}^{op}$ to $Sets$. We would expect, therefore, that properties of such functors should correspond to properties of coverings. They do and, in at least one case, give useful new insights on both sides of the equivalence.

Recall from elementary category theory, that a functor $F : \mathcal{C} \to Sets$ is said to be *representable* if there is an object, c of $\mathcal{C}$ and a natural isomorphism of functors between F and the functor $\mathcal{C}(c, -) : \mathcal{C} \to Sets$. (We, in fact need the case $\mathcal{C} = \mathcal{G}^{op}$, so have a functor of the form, $\mathcal{G}(-, x)$, here. It is important to think 'geometrically' here as this functor is an analogue of the costar of x in $\mathcal{G}$, recording the arrows that *end* at x, as a family indexed by their domains.)

Suppose we start with a representable local system, $L : \mathcal{G}^{op} \to Sets$ (which will eventually can be identified with $L(p)$ for some groupoid covering, $p : \mathcal{R} \to \mathcal{G}$.) There is, then by representability, some object x_0 in $\mathcal{G}$ such that $L(x) \cong \S, \S_{\prime}$, naturally in x. we look at $\mathcal{R}(L)$, or more exactly $\mathcal{R}(\mathcal{G}(-, x_0)$. This groupoid will have as its objects pairs (g, x) with $x \in Ob(\mathcal{G})$ and $g \in \mathcal{G}(x, x_0)$, so it seems that $\mathcal{R}(L) \cong \mathcal{G}/x_0$, the slice groupoid of 'elements over x_0'. We have to check the morphisms. A morphism from (g, x) to (g', x') will be a morphism $h : x \to x'$ such that $g = L(h)(g')$, but $L(h) : L(x') \to L(x)$ is just pre-composition with h, i.e., $L(h)(g') = hg'$. We thus do have $\mathcal{R}(L) \cong \mathcal{G}/x_0$, but better than that we can see that $\mathcal{R}(L)$ must be a simply connected groupoid, i.e., $\mathcal{R}(L)$ has only trivial vertex groups, since, if we have any (g, x) and look for an $h : x \to x$ such that h gives a loop at (g, x), we have to have $hg = g$, so h will be the identity. (Note that this argument *does* use that $\mathcal{G}$ is a groupoid as we have used the g was invertible.)

Suppose now we start with a covering, $p : \mathcal{R} \to \mathcal{G}$, in which $\mathcal{R}$ is a simply connected and connected groupoid. In fact, (R, p) must be a universal covering of $\mathcal{G}$. To see this suppose $(\mathcal{S}, q)$ is another covering and that we have x an object of $\mathcal{R}$, y an object of $\mathcal{S}$ such that $p(x) = q(y)$. As $\mathcal{R}$ is simply connected, it is *codiscrete*, i.e., there is exactly one arrow between any to objects of $\mathcal{R}$. Suppose that x' and $r : x' \to x$ are respectively an object of $\mathcal{R}$ and the unique morphism from it to x, then $p(r) : p(x') \to p(x)$ is an arrow ending at $p(x) = q(y)$, so we use costar bijectivity to lift $p(r)$ to a uniquely defined arrow, which we will denote $f(r) : f(x') \to y$. Uniqueness ensures that this is a functor.

Finally any simply connected covering is representable. (This is now not too difficult to see, so is **left to the reader**.)

In the wider context of small categories rather than groupoids, it is easy to check that simply connected coverings are universal and conversely, but the link with representability does not go through so easily. (The point is that representability favours the costar and ignores the star. For groupoids, invertibility of arrows eliminates the problem. There are interesting problems in handling the various analogues of covering spaces in these situations as it depends on the intended use of the theory and hence on the exact definition of covering that is used. The one we have given is not the only possibility as we have already mentioned as the question of the directed nature of paths etc. can be useful to add in as in discussions of directed homotopy. For the moment we will leave the general discussion at that.) In both the small category and the groupoid case, there *is* a Galois theory of coverings, yielding equivalences of categories between $\mathsf{R}/\mathcal{X}$ and $Sets^{\Pi_1 \mathcal{X}^{op}}$. As $\mathcal{G}$

is assumed to be connected,, this latter category is equivalent to $Sets^{\pi_1\mathcal{X}^{op}}$, which is a category of right $\pi_1\mathcal{X}$-sets. We will not follow this up for the moment as it would lead us a bit too far away from complexes of groups, at least as we have been exploring them here. One final observation is that, if $\mathcal{X}$ is a connected scwol, with base point v_0, and $p : \tilde{\mathcal{X}} \to \mathcal{X}$ is a covering, (so $\tilde{\mathcal{X}}$ is also a scwol), then $\pi_1(\mathcal{X}, v_0)$ acts on $\tilde{\mathcal{X}}$, so that $p : \tilde{\mathcal{X}} \to \mathcal{X}$ is a Galois covering with $\pi_1(\mathcal{X}, v_0)$ as its Galois group. This corresponds to the local system given by the action of $\pi_1(\mathcal{X}, v_0)$ on itself by multiplication on the right.

3.6 Complexes of groups on a scwol

Earlier we pretended to use a simplicial complex as the main 'base' on which to build complexes of groups, and said that it was easy to make the transition to the more general one defined on a simplicial cell complex, and thus on a scwol. It *is* easy and could be 'left to you', but it is also convenient to now have the definition given *explicitly* before going further, so here it is. (The main reference for this is, as usual, Bridson and Haefliger, [36], p. 535.)

3.6.1 General complexes of groups revisited

Definition: Let $\mathcal{Y}$ be a scwol. A *complex of groups*, $G(\mathcal{Y}) = \{\{G_v\}, \{\psi_a\}.\{g_{a,b}\}\}$, over $\mathcal{Y}$ is given by the following data:

(1) a family, $\{G_v \mid v \in V(\mathcal{Y})$, of groups, indexed by the objects of $\mathcal{Y}$, the group G_v being called the *local group at* v;

(2) for each edge $a \in E(\mathcal{Y})$, a monomorphism,

$$\psi_a : G_{i(a)} \to G_{t(a)};$$

if a is an identity arrow in $\mathcal{Y}$, ψ_a is to be the identity homomorphism;

(3) for each pair of composable arrows, $(a, b) \in E^{(2)}(\mathcal{Y})$, a *twisting element*, $g_{a,b} \in G_{t(a)}$,

$$g_{a,b}^{-1}\psi_{ba}(\_)g_{a,b} = \psi_a\psi_b$$

and such that the "cocycle condition" holds, i.e., that for each triple in $E^{(3)}(\mathcal{Y})$ of composable elements,

$$g_{a,cb}\psi_a(g_{b,c}) = g_{ab,c}g_{a,b}.$$

(This definition does need checking for the composition conventions it uses.)

If the twisting elements of $G(\mathcal{Y})$ are trivial, then we say the complex of groups is *simple*.

It is sometimes useful to write $Ad(g)$ for the (right action by) conjugation by g, and then the first condition for the ψ_as becomes

$$Ad(g_{a,b})\psi_{ba} = \psi_a\psi b.$$

Remark: As usual in this chapter, $E^{(k)}(\mathcal{Y})$ denoted the set of composable k-tuples of elements of edges / arrows of $\mathcal{Y}$, and so is just another notation for $Ner(\mathcal{Y})_{k-1}$, the set of $(k-1)$-simplices of the nerve of $\mathcal{Y}$. This simple observation will allow us, later on, to give other descriptions of complexes of groups and to put them into a much wider context, which can be useful both for their area of geometric group theory, but for suggesting analogous higher dimensional categorified analogues of complexes of groups.

The cocycle condition, of course, reminds one of 'cohomology' and the cocycle condition for the factor set of an extension (cf. page **??**), and so one should expect there to be some form of coboundary around, and there is. Suppose that we have, for each edge $a \in E(\mathcal{Y})$, an element $g_a \in G_{t(a)}$, then we can form a new complex of groups, $G'(\mathcal{Y})$ over $\mathcal{Y}$ with $G'_v = G_v$, $\psi'_a = Ad(g_a)\psi_a$ and $g'_{a,b} = g_a\psi_a(g_b)g_{a,b}g_{ba}^{-1}$(needs checking that it works) and we say that $G'(\mathcal{Y})$ is obtained from $G(\mathcal{Y})$ by *deformation* using the coboundary, $\{g_a\}$, or that it is *deduced* from $G(\mathcal{Y})$ by that coboundary.

Morphisms of complexes of groups

Chapter 4

Beyond 2-types

The title of this chapter promises to go beyond 2-types and in particular, we want to model what is there algebraically. We have so far only done this with the crossed complexes. These do give all the homotopy groups of a simplicial group, but the homotopy types they represent are of a fairly simple type, as they have vanishing Whitehead products.

We will return to crossed complexes later on, but will first look at the general idea of n-types, going into what was said earlier in more detail.

4.1 n-types and decompositions of homotopy types

We will start with a fairly classical treatment of the ideas behind the idea of n-types of topological spaces.

4.1.1 n-types of spaces

We earlier (starting in section 2.7.1) discussed 'n-equivalences' and 'n-types'. As homotopy types are enormously complex in structure, we can try to study them by 'filtering' that information in various ways, thus attempting to see how the information at the n^{th}-level depends on that at lower levels. The informational filtration by n-type is very algebraic and very natural. It has two very satisfying interacting aspects. It gives complete models for a subclass of homotopy types, namely those whose homotopy groups vanish for all high enough n, but, at the same time, gives a set of approximating notions of equivalence that, on all 'spaces', give useful information on weak equivalences.

We start with one version of the topological notion:

Definition: Given a cellular mapping, $f : (X, x_0) \to (Y, y_0)$, between connected pointed spaces, f is said to be an *n-equivalence* if the induced homomorphisms, $\pi_k(f) : \pi_k(X, x_0) \to \pi_k(Y, y_0)$, for $1 \leq k \leq n$, are all isomorphisms. More generally, on relaxing the connectedness requirements on the spaces, a cellular mapping, $f : X \to Y$, is an *n-equivalence* if it induces a bijection on π_0, that is, $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection, and for each $x_0 \in X$ and $1 \leq k \leq n$, $\pi_k(f) : \pi_k(X, x_0) \to \pi_k(Y, f(x_0))$ is an isomorphism.

Remark: It is important to note that here the mappings are cellular, not just continuous. We will see consequences of this later.

There are alternative descriptions and these can be useful. We recall them next, emphasising certain facts and viewpoints that perhaps have not yet been stressed enough in our earlier treatments, but can be useful for our use of these ideas here.

We start by recalling some standard notions of classical homotopy theory. We let CW be the category of all CW-complexes and *cellular* maps, and CW_{c*} be the corresponding category of pointed connected complexes, again with cellular maps. (The notions below generalise easily to the non-connected multi-pointed case.) If X is such a CW-complex, then we will write X^n for its n-*skeleton*, that is, the union of all the cells in X of dimension at most n. We say that X has *dimension* n if $X = X^n$.

It is important to remember that the homotopy type of X^n is *not* an invariant of the homotopy type of X. (Just think about subdivision if you are in doubt about this.) It was partially to handle this that Whitehead introduced the notion of N-type, as this does give such invariants. The two ways of viewing n-types, which we have already mentioned, are both important. We recall that in one, they are certain equivalence classes of CW-complexes, whilst in the other, they are homotopy types of certain spaces with special characteristics. (Useful sources for this topic include Baues' Handbook article on 'Homotopy Types', [24].)

Let CW_{c*}^{n+1} be the full subcategory of CW_{c*} consisting of complexes of dimension $\leq n+1$. (To emphasise where we are working, we will sometimes write X^{n+1}, Y^{n+1}, etc. for objects here.) Let $f, g : X^{n+1} \to Y^{n+1}$ be two maps in CW_{c*}^{n+1} and $f|_{X^n}, g|_{X^n} : X^n \to Y^{n+1}$ their restrictions to the n-skeleton of X. (Note that the codomain is still the $n+1$-skeleton of Y.)

Definition: We say f, and g, as above, are *n-homotopic* if $f|_{X^n} \simeq g|_{X^n}$ (that is, within Y^{n+1}). We write $f \simeq_n g$ in this case.

It can be useful to remember that f and g, in this, need only be defined on the $(n+1)$-skeleton of X. (This statement is true, but is deliberately silly. We, in fact, assumed that X had dimension $\leq n+1$, but what we said is still useful, since if we have any complex, X, we can restrict to its $(n+1)$-skeleton, X^{n+1}, yet do not need f or g to be defined on *all* of X, merely on X^{n+1}.)

Our first version of (connected) n-types, in this approach, is obtained by taking $CW_{c*}^{n+1}/\simeq_n$, that is, taking the complexes of dimension $\leq n+1$ and the cellular maps between them, and then dividing out the hom-sets by the equivalence relation, $\simeq_n$. From this perspective, we have:

Definition: (à là Whitehead.) A connected *n-type* is an isomorphism class in the category, $CW_{c*}^{n+1}/\simeq_n$.

That sets up, a bit more formally, the first type of description of n-types. If we have a connected CW-complex, X, then we assign to it the isomorphism class of X^{n+1} in $CW_{c*}^{n+1}/\simeq_n$ (for any choice of base point) to get its n-type. From this viewpoint, we get a notion of n-equivalence from the notion of n-homotopy:

Definition: A cellular map, $f : X \to Y$, between CW-complexes is an *n-equivalence* if $f^{n+1} : X^{n+1} \to Y^{n+1}$ gives an isomorphism in $CW_{c*}^{n+1}/\simeq_n$.

This is also called *n-homotopy equivalence*, with the earlier version, that based on the homotopy groups, then called *n-weak equivalence*. It amounts to f^{n+1} having a *n-homotopy inverse*, $g^{n+1} : Y^{n+1} \to X^{n+1}$, so $f^{n+1}g^{n+1} \simeq_n 1_{Y^{n+1}}$ $g^{n+1}f^{n+1} \simeq_n 1_{X^{n+1}}$. Here it is *not* claimed that there is some $g : Y \to X$ that extends g^{n+1} to the whole of Y, merely there is a map, g, defined on the $(n+1)$-skeleton.

(These are stated for connected spaces, but as usual the extension to non-connected complexes is easy to do.)

Let us take these ideas apart one stage more. Suppose that P is a CW-complex of dimension $\leq n$, and $f : X \to Y$ is a n-equivalence in the above sense. We note that, as we are looking at cellular maps and cellular homotopies, the inclusion $i^{n+1} : X^{n+1} \to X$ induces a bijection

$$[P, i^{n+1}] : [P, X^{n+1}] \to [P, X],$$

but then it is clear that

$$[P, f] : [P, X] \to [P, Y]$$

is also a bijection. (Note that if we had required P to have dimension $n+1$, then $[P, i^{n+1}] : [P, X^{n+1}] \to [P, X]$ might not be *injective* as two non-homotopic maps with image in X^{n+1} may be homotopic within the whole of X. That being so $[P, i^{n+1}]$ will be surjective, but just not a bijection. The same would be true for $[P, f]$.)

So much for the first viewpoint, i.e., as equivalence classes of objects in CW_{c*}. For the second approach, that is, n-types as homotopy types of certain spaces delineated by conditions, we work in the bigger category of (pointed connected) CW-complexes and *all continuous maps*, i.e., not just the cellular ones (although, remember, the classical cellular approximation theorem tells us that any (general continuous) map is homotopic to a cellular one). We will temporarily call this category 'spaces', (following the treatment in Baues' Handbook article, [24]). We form $\mathsf{spaces}/\simeq$, the quotient category of 'spaces' and homotopy classes of maps.

Definition: The subcategory, $\mathsf{n\!-\!types}$, of $\mathsf{spaces}/\simeq$, is the full subcategory consisting of spaces, X, with $\pi_i(X) = 0$ for $i > n$. Such spaces, or their homotopy types, may also be called *n-types*. The generalisation to the non-connected case should be clear.

We now have two different definitions of n-type of CW-complexes (and that is without mentioning n-types of simplicial sets, simplicial groups $\mathcal{S}$-groupoids, etc.). We need to check on the relationship between them. For this, we introduce *Postnikov functors* and in a later section will study the related *Postnikov tower* that decomposes a homotopy type. Note the Postnikov functors are usually defined so as to be functorial at the level of the homotopy categories, not at the level of the spaces and maps, although this is possible. We will comment on this a bit more later on, but let us describe the main ideas first as these directly relate to the comparison of the two ways of approaching n-types.

Definition: The n^{th} *Postnikov functor*,

$$P_n : CW_{c*}/\simeq \to \mathsf{n\!-\!types}$$

is defined by killing homotopy groups above dimension n, that is, we *choose* a CW-complex, P_nX, with

$$(P_nX)^{n+1} = X^{n+1},$$

and, by attaching cells to X in dimensions $> n$, with $\pi_i(P_nX) = 0$ for $i > n$. If $f : X \to Y$ is a cellular map, we choose a map $P_nf : P_nX \to P_nY$, so that $(P_nf)^{n+1} = f^{n+1}$. The functor P_n takes the homotopy class, $[f]$, to $[P_nf]$.

The first point to note is that the choices are absorbed by the homotopy. To examine this more deeply we make several:

Remarks: (i) First a word about 'killing homotopy groups'. (This is very like the construction of resolutions of a group.)

Suppose that we have a space, X, and a set of representatives, $\varphi_g : S^{n+1} \to X$, of generators, g, of the homotopy group, $\pi_{n+1}(X)$, then we form

$$X(1) := X \sqcup_{\{\varphi_g\}} \bigsqcup_g D^{n+2},$$

i.e., we glue $(n+2)$-dimensional discs to X, along their boundaries, using the representing maps. We now take $\pi_{n+2}(X(1))$ and a generating set for that, form $X(2)$ by the same sort of construction, and continue to higher dimensions.

If $f : X \to Y$, then each $f(\varphi_g) : S^{n+1} \to Y$ defines an element of $\pi_{n+1}(Y)$, and this will be 'killed' within $\pi_{n+1}(Y(1))$. There is thus a null homotopy for that map within $Y(1)$. We choose one such and use it to extend f over the disc attached by φ_g. Doing this for each generator, we extend f to $f(1) : X(1) \to Y(1)$, and so on.

This is unbelievably non-canonical and non-functorial at the level of spaces, but the different choices can fairly easily be shown to yield homotopy equivalent spaces and homotopic maps. This is discussed in many of the standard algebraic topology textbooks, see, for instance, Hatcher, [115].

The basis of these constructions is a simple extension lemma, (cf. Hatcher, [115], lemma 4.7, p.350, for instance).

Lemma 31 *Given a CW pair, (X, A), and a map, $f : A \to Y$, with Y path connected, then f can be extended to a map $X \to Y$ if $\pi_{n-1}(Y) = 0$ for all n such that $X - A$ has cells in dimension n.* ■

(ii) Things are clearer when working with simplicial sets as we will see shortly. In that case, there is a good functorial 'Postnikov tower' of Postnikov functors, defined at the level of simplicial sets, and morphisms and not merely at the homotopy level. That works beautifully for what we need, but at the slight cost of moving from 'spaces' to simplicial sets, there using Kan complexes (which is no real bother, as singular complexes are Kan), and finally taking geometric realisations to get back to the spaces. As we said, we will look at this shortly.

There are inclusion maps, $P_n(X) : X \to P_nX$, whose homotopy classes give a natural transformation from the identity to P_n. (This is defined on the homotopy categories of course.) For

$f : X \to Y$ in CW_{c*}, then $P_n f$ can be chosen to make the square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p_n(X)"'] & Y \arrow[d, "p_n(Y)"] \\
P_n X \arrow[r, "P_n f"'] & P_n Y
\end{tikzcd}$$

commutative 'on the nose'. We note that these maps make each $(P_{n+1}X, X)$ into a CW-pair and, as $P_{n+1}X - X$ has only cells of dimension $n+3$ or greater, and $\pi_i(P_nX) = 0$ in those dimensions, we can apply the extension lemma to the map, $p_n(X) : X \to P_nX$ and thus extend it to $P_{n+1}X$, giving $p_n^{n+1}(X) : P_{n+1}X \to P_nX$, and this satisfies $p_n^{n+1}(X) \cdot p_{n+1}(X) = p_n(X)$. These map, $p_n^{n+1}(X)$ fit into a tower diagram with a 'cone' of maps from X:

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P_3X \arrow[d, "p_2^3(X)"] \\
 & P_2X \arrow[d, "p_1^2(X)"] \\
X \arrow[r, "p_1(X)"'] \arrow[ur, "p_2(X)"] \arrow[uur, "p_3(X)"] \arrow[uuur, dashed] & P_1(X)
\end{tikzcd}$$

The limit of the tower is isomorphic to X itself. This is known as a *Postnikov tower* for X. We will return to such towers in section 4.1.3.

It is useful to refer to $X \to P_nX$, or more loosely to P_nX as a *Postnikov section* of X, or as the n^{th}-*Postnikov section* of X, even though it is only determined up to homotopy equivalence.

We return to the n^{th} Postnikov functor, P_n, and can use it to define n-equivalences in a different way.

Definition: A map, $f : X \to Y$, is called a P_n-equivalence if the induced morphism, $[P_nf]$, in $\mathsf{n-types}$ is an isomorphism.

Of course, we expect these P_n-equivalences to just be n-equivalences under another name. To examine this, we look again at P_n.

We had the Postnikov functor:

$$P_n : CW_{c*}/\simeq \to \mathsf{n-types}.$$

If we look at $CW_{c*}^{n+1}/\simeq_n$, we need to see that a P_n construction adapts to give a functor

$$P_n : CW_{c*}^{n+1}/\simeq_n \to \mathsf{n-types},$$

as this does not follow trivially from the previous case. Suppose X and Y are $(n+1)$-dimensional connected pointed CW complexes and $f \simeq_n g : X \to Y$, then $f|_{X^n} \simeq g|_{X^n}$. We have to check that $P_n f \simeq P_n g$.

We have some $h : f|_{X^n} \simeq g|_{X^n} : X_n \times I \to Y^{n+1} \hookrightarrow P_n Y$, and also have the map from $P_n X \times \{0,1\}$ to $P_n Y$ given by $P_n f$ and $P_n g$. These are compatible so define a map from the subcomplex, $X_n \times I \cup P_n X \times \{0,1\}$ of $P_n X \times I$, to $P_n Y$. The cells in $P_n X \times I$ that are not in that subcomplex, all have dimension $n+3$ or greater, since P_n is obtained from X^{n+1} by adding cells. We have $\pi_i(P_n Y) = 0$ for $i > n$, so an application of the extension lemma gives us an extension ver $P_n X \times I$ giving a homotopy between $P_n f$ and $P_n g$, as required. This proves

Lemma 32 *P_n give a functor from $CW^{n+1}_{c*}/\simeq_n$ to* n−types. ∎

We claim that this functor is an equivalence of categories, which will show, after a bit more checking, that the two notions of n-equivalence coincide and will relate the main notions of (topological) homotopy n-types.

To prove that P_n is an equivalence of categories, it is, perhaps, easiest to look for a functor in the opposite sense that might serve as a 'quasi-inverse'. If we have that X is a (connected, pointed) CW-complex with $\pi_i(X) = 0$ for $i > n$, then we can take its $(n+1)$-skeleton, X^{n+1} to get something in CW^{n+1}_{c*}. This is not quite a functor, since not all the morphisms in spaces are cellular. Each continuous map between such complexes is *homotopic* to a cellular map, but, whilst taking the $(n+1)$-skeleton *is* a functor with respect to cellular maps, we have to verify that if we choose two cellular approximations for some $f : X \to Y$, then their $(n+1)$-skeletons are, at least, n-homotopic.

Suppose that $f_0, f_1 : X \to Y$ are two cellular maps between n-types (to be thought of, in the first instance, as two 'rival' cellular approximations to some $f : X \to Y$). We assume they are homotopic by a homotopy $h : f_0 \simeq f_1$, which again using cellular approximation, can be assumed to be a cellular homotopy. We take f_0^{n+1} and f_1^{n+1} and see if they are n-homotopic.- Yes they are. They may not be homotopic, since h may use $n+2$-cells in the process of 'homotoping' between f_0^{n+1} and f_1^{n+1} within Y, but $F_0|_{X^n}$ and $f_1|_{X^n}$ are homotopic via h restricted to $X_n \times I$, i.e., exactly what is needed.

We have checked not only that our idea of taking $(n+1)$-skeletons is compatible with the cellular approximations, but also that that assignment induces a functor from n−types to $CW^{n+1}_{c*}/\simeq_n$. (Of course, in fact, this is the restriction of a functor from spaces to $CW_{c*}/\simeq_n$, as we nowhere use that X and Y were n-types.)

Theorem 14 *The n^{th} Postnikov functor, P_n, gives an equivalence of categories between $CW_{c*}/\simeq_n$ and* n−types*. A quasi inverse is given by the $(n+1)$-skeleton functor.*

Proof: We examine the two composite functors.

If X is in CW_{c*}, then $(P_n X)^{n+1} = X^{n+1}$, by definition. The inclusion of X^{n+1} into X gives an isomorphism in $CW_{c*}/\simeq_n$, since $\simeq_n$ uses nothing in X above dimension $n+1$.

The other composite starts with an n-type, Y, say, takes Y^{n+1}, then forms $P_n(Y^{n+1})$. The inclusion of Y^{n+1} into Y extends by the extension lemma, to a map $P_n(Y^{n+1}) \to Y$, which induces isomorphisms on all homotopy groups, so is a weak homotopy equivalence, and thus, as we are handling CW-complexes, is a homotopy equivalence, i.e., an isomorphism in $\mathsf{n-types}$, which completes the proof. ■

Remark: It is worth noting that, in the above, we have 'naturally' defined maps from X *to* $(P_nX)^{n+1}$ and from $P_n(Y^{n+1})$ *to* Y, which suggests an adjointness behind the equivalence. In fact, we actually did *not* assume that X was in $CW_{c*}^{n+1}/\simeq_n$, so, in some sense, proved that $\mathsf{n-types}$ was equivalent to a homotopically reflective subcategory of CW_{c*}. (Of course, connectedness has nothing to do with the picture and was for convenience only.)

We thus have a fairly complete picture of homotopy n-types and n-equivalence in the topological case. If $f : X \to Y$ is such that $[P_nf]$ is an isomorphism in $\mathsf{n-types}$, then $[f^{n+1}]$ is an isomorphism in $CW_{c*}^{n+1}/\simeq_n$, hence an n-equivalence á lá Whitehead.

If X and Y are (connected, pointed) $(n+1)$-dimensional CW-complexes, and $f : X \to Y$ is cellular, then f is an n-equivalence if, and only if, it induces isomorphisms on all π_i for $i \leq n$. In general, i.e., with no dimensional constraint, as we have defined it, f is an n-equivalence if, and only if f^{n+1} is an n-equivalence in this more restricted sense.

We write $Ho_n(Top)$ for the category of CW-complexes (or more generally, topological spaces, after inverting the n-equivalences. If we are just considering the CW-complexes, this is just the same as $\mathsf{n-types}$ up to equivalence and n-types are just isomorphism classes of objects in this category. (If considering spaces other than those having the homotopy types of CW-complexes, then this is better thought of as the singular n-types, but we will not usually need this level of generality in our development.) It seems that, in his original thoughts on algebraic homotopy theory, Whitehead hoped to find algebraic models for n-types, that is, to find algebraic descriptions of isomorphism classes of spaces within $Ho_n(Top)$. Classifying 1-types is 'easy' as they have models that are just groups, so classification reduces to classifying groups up to isomorphism. This is still not an easy task, but there are a wide range of tools available for it. As was previously mentioned, Mac Lane and Whitehead, [152], gave a complete algebraic model for 2-types. (Note: their 3-types are modern terminology's 2-types.) The model they proposed was the crossed module and we have seen the extension of their result to n-types given by Loday.

It should be pointed out that, although n-equivalence is defined in terms of the π_k, $0 \leq k \leq n$, the interactions between the various π_ks mean that not every sequence $\{\varphi_k : \pi_k(X) \to \pi_k(Y)\}_{0\leq k\leq n}$ can be realised as the induced morphisms coming from some $f : X \to Y$, even if the φ_k are all isomorphisms.

One approach that we will be looking at in our exploration of the basics of Whitehead's idea of Algebraic Homotopy and its implications and developments, is to convert the problems to ones in the study simplicial groups or, more generally, in $\mathcal{S}$-groupoids. For this we will need a knowledge of the corresponding theory for n-types of simplicial sets. This is very elegant, so would, in any case, be worth looking at in some detail.

4.1.2 n-types of simplicial sets and the coskeleton functors

(Sources for this section include, at a fairly introductory level, the description of the coskeleton functors in Duskin's Memoir, [83], his paper, [86], and Beke's paper, [26]. There is also a description of the skeleton and coskeleton constructions in the nLab, [173], (search on 'simplicial skeleton'). The original introduction of this construction would seem to be by Verdier in SGA4, [8], with an early use being in Artin and Mazur's *Étale homotopy*, Lecture Notes, [10].)

First let us summarise some basic ideas. For simplicial sets and simplicially enriched group(oid)s, the definitions of n-equivalence are analogous, and we give them now for convenience:

Definition: For $f : G \to H$ a morphism of $\mathcal{S}$-groupoids, f is an *n-equivalence* if $\pi_0 f : \pi_0 G \to \pi_0 H$ is an equivalence of the fundamental groupoids of G and H and for each object $x \in Ob(G)$ and each k, $1 \leq k \leq n$,

$$\pi_k f : \pi_k(G\{x\}) \to \pi_k(H\{f(x)\})$$

is an isomorphism.

We write $Ho_n(\mathcal{S}-Grpd)$ for the corresponding category of n-types, i.e., $\mathcal{S}-Grpd(\Sigma_n^{-1})$, where Σ_n is the class of all n-equivalences of $\mathcal{S}$-groupoids. An *n-type of $\mathcal{S}$-groupoids* is an isomorphism class within $Ho_n(\mathcal{S}-Grpd)$.

Cautionary note: If K is a simplicial set, then as $\pi_k(K) \cong \pi_{k-1}(GK)$, the n-type of K corresponds to the $(n-1)$-type of GK.

We need to look at simplicial n-types, in general, and in some more detail, and will start by the theory for simplicial sets. On a first reading the above summary may suffice.

The theory sketched out in the previous section uses the $(n+1)$- and n-skeletons of a CW-complex in a neat way. If we go over to simplicial sets as models for homotopy types then skeletons are easy to define, but some points do need making about them.

The n-skeleton of a CW-complex is the union of all cells of dimension less than or equal to n, so the set of higher dimensional cells in an n-skeleton is, clearly, empty. On the other hand, a simplicial set, K, has in addition to the simplices in each dimension, the face and degeneracy operators, i.e., the various $d_i : K_n \to K_{n-1}$ and $s_j : K_n \to K_{n+1}$, so to get the n-skeleton of K, we cannot *just* take the k-simplices for $k \leq n$, throwing away everything in higher dimensions, and hope to get a simplicial set. If $\sigma \in K_n$, then the $s_j\sigma$ are in K_{n+1}, so K_{n+1} cannot be empty. The point is rather that, in the n-skeleton, *all* simplices in dimensions greater than n will be degenerate.

Our first task, therefore, is to set this up more abstractly and categorically. A simplicial set, K is a functor, $K : \mathbf{\Delta}^{op} \to Sets$ and we want to restrict attention to those parts of K in dimensions less than or equal to n, discarding, initially, all higher dimensional simplices, before reinstating those that we need.

(We will introduce the ideas for simplicial sets, but we can, and will later, extend the discussion to simplicial groups, and, in general, to simplicial objects in a category, $\mathcal{A}$. The latter situation will require some conditions on the existence of various limits and colimits in $\mathcal{A}$, but we will introduce these as we go along. The ability to use more general categories is a great simplification for later developments.)

Recall that the category, $\mathbf{\Delta}$, consists of all finite ordinals and all order preserving maps between them. Given any natural number n, we can form a full subcategory, $\mathbf{\Delta}[0,n]$, with objects the ordinals $[0],\ldots,[n]$, and all order preserving maps between *them*. As the category of simplicial sets is $\mathcal{S} = Sets^{\mathbf{\Delta}^{op}}$, there is a restriction functor, call *n-truncation* or, more fully, *simplicial n-truncation*,

$$tr^n : \mathcal{S} \to Sets^{\mathbf{\Delta}[0,n]^{op}},$$

which, to a simplicial set, K, assigns the n-truncated simplicial set, $tr^n(K)$, with the same data in dimensions less than $n+1$, but which forgets all information on higher dimensions. A functor, $K : \mathbf{\Delta}[0,n]^{op} \to Sets$ is equivalent to a system, $K = \{(K_k)_{0\leq k\leq n}, d_i, s_j\}$, of sets and functions, (or more generally of objects and arrows of $\mathcal{A}$). These are to be such that the d_i and s_j verify the simplicial identities wherever they make sense.

Remark: Setting up notation and terminology for the more general case, we have a category $Tr^nSimp.\mathcal{A} = \mathcal{A}^{\mathbf{\Delta}[0,n]^{op}}$ of n-truncated simplicial objects in $\mathcal{A}$. The category of n-truncated simplicial sets is then $Tr^nSimp.Sets = Tr^n\mathcal{S} = Sets^{\mathbf{\Delta}[0,n]^{op}}$. Back in the general case, the analogue of the above restriction functor gives us a restriction functor:

$$tr^n : Simp.\mathcal{A} \to Tr^nSimp.\mathcal{A}.$$

If the category $\mathcal{A}$ has finite colimits, then this functor, tr^n has a left adjoint, which we will denote sk^n, and which is called the *n-skeleton* of the truncated simplicial object. The proof that this left adjoint exists is most neatly seen by using the theory of Kan extensions, for which see Mac Lane, [150], here with a discussion starting in section **??**, or the nLab, [173], (search on 'Kan extension'.)

The *idea* of the construction of that left adjoint is, however, quite simple and is just an encoding of the intuitive idea that we sketched out above. We first look at it in the case of a simplicial set. We have K in $Tr^n\mathcal{S}$, and want $(sk^nK)_{n+1}$, that is the first missing level, (after that we can presumably repeat the idea to get the higher levels of sk^nK). We clearly need degenerate copies of all simplices in K_n and that suggests, (slightly incorrectly), that we take this $(sk^nK)_{n+1}$ to be the disjoint union of sets, $s_i(K_n) = \{s_i(x) \mid x \in K_n\}$. (The elements $s_i(x)$ are just copies of x indexed by the degeneracy mapping. If you prefer another notation, use pairs (x, s_i) as this corresponds more to one of the usual models of disjoint unions.) This is not right, since these $s_i(x)$ are not independent of each other. If x is already a degenerate element, say $x = s_jy$ then $s_ix = s_is_jy$ and, as we will need the simplicial identities to hold in the end result, this must be *the same element* as $s_{j+1}s_iy$, (this is if $i \leq j$). In other words, we should not use a disjoint union of these sets, $s_i(K_n)$, but will have to identify elements according to the simplicial identities, that is, we must form some sort of colimit. In fact, one forms a diagram consisting of copies of K_n and K_{n-1}, and then forms its colimit to get $(sk^nK)_{n+1}$. The next task is to define the face and degeneracy maps linking the new level with the old ones, so as to get an $(n+1)$-truncated simplicial sets. (It is a **good idea** to try this out in some simple cases such as for $n = 1$ and 2 and then to **look up** a 'slick' version, as then you will, more easily, see what makes the slick version work.)

Of course, the use of simplicial *sets* here is not crucial, but if working with simplicial objects in some $\mathcal{A}$, then we will need, as we mentioned earlier, that $\mathcal{A}$ has finite colimits so as to be able to form $(sk^nK)_{n+1}$. The process is then repeated as we now have a $(n+1)$-truncated object.

Remark: Shortly we will be using skeletons (and coskeletons) of simplicial *groups*. In such a context, it should be noted that not all elements in $(sk^nG)_m$, for $m > n$, need be, themselves, degenerate. For instance, we might have g, and g', in G_n, so have for two different indices, i, j, elements s_ig and s_jg' in $(sk^nG)_{n+1}$, but, more often than not, their product $s_ig.s_jg'$, will not be a degenerate element. This fact is crucial and is one reason why, in homotopy theory, it is possible to have non-trivial homotopy groups above the dimension of a space.

If we are considering simplicial sets, or, more generally, simplicial objects in $\mathcal{A}$, where $\mathcal{A}$ has finite *limits*, the truncation functor, tr^n, has a *right* adjoint, which will be denoted $cosk^n$. This is called the *n-coskeleton functor*. (WARNING: this term will also be used for the *composite* $cosk^n \circ tr^n$, from $Simp.\mathcal{A}$ to itself as it is too useful to 'waste' on the more restrictive situation! Usually no confusion will arise, especially as we will use a slightly different notation.)

The fact that $cosk^n$ is right adjoint to tr^n means that, at least in the case of simplicial sets, $cosk^n$ has a very simple description. If K is a simplicial set and L is an n-truncated simplicial set, then we have

$$Tr^n\mathcal{S}(tr^n(K), L) \cong \mathcal{S}(K, cosk^n L).$$

Taking $K = \Delta[m]$, the simplicial m-dimensional simplex, we get

$$(cosk^n L)_m = \mathcal{S}(\Delta[m], cosk^n L) \cong Tr^n\mathcal{S}(tr^n(\Delta[m]), L),$$

giving us a recipe for the simplices of $cosk^n L$ in all dimensions. As $tr^n\Delta[m]$ is an n-dimensional shell of a m-dimensional simplex, we can think of it intuitively as being a family of n-simplices stuck together along lower dimensional bits in some neat way (governed by the simplicial identities). We thus would expect $cosk^L_m$ to be made up of compatible families of n-simplices of L, and this suggests a 'limit' - which makes sense as sk^nL was thought of as a colimit.

As with the left adjoint of tr^n, the right adjoint can be described as a Kan extension, which would give an explicit 'end' formula and also a limit formula that we could take apart. At this stage in the notes, it is not being assumed that those parts of categorical toolbag are available to us. (They *are* discussed later with Kan extension starting on page **??** and with ends (and coends) discussed in section **??**.) Because of this it seems better to adopt a fairly 'barehands' approach, which is more elementary and nearer the initial intuition of what is needed, but the way to go beyond the limitations of this approach is to understand Kan extensions fully. (The approach that we will use will be adapted from Duskin's memoir, [83].)

For a category, $\mathcal{A}$, with finite limits, we suppose given an n-truncated simplicial object, $L \in Tr^nSimp.\mathcal{A}$ and we consider all the face maps at level n

$$d_0, \ldots, d_n : L_n \to L_{n-1}.$$

Definition: An object, K_{n+1}, together with morphisms $p_0, \ldots, p_{n+1} : K_{n+1} \to L_n$ is said to be the *simplicial kernel* of $(d_0, \ldots, d_n)$ if the family $(p_0, \ldots, p_{n+1})$ satisfies the simplicial identities with respect to the d_is and, moreover, has the following universal property: given any family, $x_0, \ldots, x_{n+1}$ of morphisms from some object, T, to L_n, which satisfy the simplicial identities with respect to the face morphisms, $d_0, \ldots, d_n$ (so that for $0 \leq i < j \leq n+1$, $d_i x_j = d_{j+1} x_i$), there is a unique morphism $x = \langle x_0, \ldots, x_{n+1} \rangle : T \to K_{n+1}$ such that for each i, $p_i x = x_i$.

This is clearly just a special type of limit. We would expect to get this K_{n+1}, together with the projections, p_i, as some sort of multiple pullback, corresponding to the 'naive' description we gave above. (To gain intuition on this point, **look at** the case $n = 1$, so we have $d_0, d_1 : L_1 \to L_0$ and want K_2 with maps $p_0, p_1, p_2 : K_2 \to L_1$, and these must satisfy the simplicial identities. It is **worth your while**, if you have not seen this before, to draw a diagram, consisting of some copies of L_1 and L_0, and the face maps built from $d_0, d_1 : L_1 \to L_0$, so that the limit of the diagram is K_2.)

If the simplicial kernel is to do the job, we should be able to use it to take $(cosk^n L)_{n+1} = K_{n+1}$, that is to form a $(n+1)$-truncated simplicial objects from it having the right properties. We, first, need face and degeneracy morphism defined in a natural way. As the p_i were to satisfy the face simplicial identities, they are the obvious candidates for the face morphisms. We will, then, need to define for each j between 0 and n, a morphism $s_j : L_n \to K_{k+1}$. The universal property of K_{n+1} gives that such a morphism will be of the form

$$s_j = \langle s_{j,0}, \ldots, s_{j,n+1} \rangle,$$

for $s_{j,k} : L_n \to L_n$, and, of course, in this notation $d_i : K_{n+1} \to L_n$ will be the i^{th} projection, p_i. This gives us the recipe for determining the $s_{j,k}$ as we must have, for instance, if $k < j$,

$$s_{j,k} = d_k s_j = s_{j-1} d_k,$$

so as to make sure that the s_j satisfy the simplicial identities. (**It is useful** to list the various cases yourself.) It is now clear that the following holds:

Lemma 33 *The data $((cosk^n L)_k, (d_i), (s_j))$, where*

(i) $(cosk^n L)_k$ is equal to L_k for $k \leq n$ and $(cosk^n L)_{n+1} = K_{n+1}$, the simplicial kernel (as above),

(ii) the d_i are the structural limit cone projections,
and

(iii) the s_j are defined by the universal property and the simplicial identities,

defines an $(n+1)$-truncated simplicial object. ∎

We denote this by $tr^{n+1} cosk^n L$, as it is the next step in the construction of $cosk^n L$.

We have as a consequence the following:

Proposition 44 *Suppose given a simplicial object, T, and a morphism, $f : tr^n T \to L$, then there is a unique morphism,*

$$\tilde{f} : tr^{n+1} T \to tr^{n+1} cosk^n L,$$

that extends f in the obvious sense. ∎

We may now construct $cosk^n L$ by successive simplicial kernels in the obvious way, and, generalising the above proposition to each successive dimension, prove that the result gives a right adjoint to tr^n.

Remarks: (i) The n-skeleton functor, that we saw earlier, can be given by an analogous simplicial cokernel construction using the degeneracy operators instead of the faces to give a universal object, and then applying the universal property to obtain the face morphisms. The object $sk^n(L)$ is then obtained by iterating that construction. (This is a **good exercise to follow up on** as it sheds useful light on what the skeleton will be in other situations where our intuitions are less strong than for simplicial sets.)

(ii) We are often, in fact, usually, interested more bby the composites

$$sk_n := sk^n \circ tr^n,$$

and

$$cosk_n := cosk^n \circ tr^n,$$

which will be called the n-skeleton and n-coskeleton functors on $Simp.\mathcal{A}$. (The superfix / suffix notation is just to distinguish them and no special significance should be read into it.)

Proposition 45 *(i) If $p \geq q$, then $cosk_p cosk_q = cosk_q$.*
(ii) If $p \leq q$, then $cosk_p cosk_q = cosk_p$.

Proof: This is a simple **exercise** in the definition, or, alternatively, in the constructions, so is **left to the reader** to work out or check up on in the literature. ■

A similar result holds for skeletons, and this is, again, **left to you** to investigate.

So far in this section we have just looked at the skeleton and coskeleton functors, but we are wanting these for a discussion of simplicial n-types. If we adopt the view that an n-type is a homotopy type with vanishing homotopy groups above dimension n, this goes across without pain to the context of simplicial sets, and, in fact, to many other situations such as simplicial sheaves on a space or simplicial objects in a (Grothendieck) topos, $\mathcal{E}$.

Aside: A good reason for briefly looking at this is that it introduces several useful concepts and the linked terminology. These in the main are due to Jack Duskin, who developed them for the study of simplicial objects in a topos. We will give the definitions and subsequent discussion within the classical setting of *Sets*, but this is really only because we have not given a thorough and detailed treatment of toposes earlier. The basic point is that if the arguments used in the development are 'constructive' then, usually with some minor changes, the theory will generalise from a category of sets, to one of sheaves, and eventually to any Grothendieck topos. To make that statement more precise would require quite a lot more discussion, and would take us away from our main themes, so investigation is **left to you**.

We start with a slight variant of the Kan fibration definition that we met earlier, (see page **??**). We recall that $\Lambda^i[n]$ is the (n,i)-*horn* or (n,i)-*box*, obtained by discarding the top dimensional n-simplex and its i^{th} face and all the degeneracies of those simplices.

Definition: A simplicial map $p : E \to B$ is a *Kan fibration*, or *satisfies the Kan lifting condition*, *in dimension* n if, in every commutative square (of solid arrows) of form

$$\begin{tikzcd} \Lambda^i[n] \arrow[r, "f_1"] \arrow[d, "inc"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "f_0"'] \arrow[ur, dashed, "f"] & B \end{tikzcd}$$

a diagonal map (indicated by the dashed arrow) exists, i.e., there is an $f : \Delta[n] \to E$ such that $pf = f_0$, $f.inc = f_1$, so f lifts f_0 and extends f_1.

We thus have that p is a Kan fibration if it is one in *all dimensions*. We can refine the above (following Duskin, [84]).

Definition: A simplicial map $p : E \to B$ *satisfies the exact Kan lifting condition in dimension* n if, in every commutative square (as above), precisely one diagonal map f exists.

Starting with the Kan fibration condition, we singled out the Kan complexes as being those simplicial sets for which the unique map $K \to \Delta[0]$ was a Kan fibration. We clearly can do a similar thing here.

Definition: A simplicial set K is an *exact* n*-type*, or n*-hypergroupoid*, if $K \to \Delta[0]$ is a Kan fibration that is exact in dimensions greater than n.

The definition of n-hypergroupoid used by Glenn, [104], is slightly different from this as it only requires the (exact) Kan condition in dimensions greater than n, so not requiring K to 'be' a Kan complex in lower dimensions. The n-hypergroupoid terminology is due to Duskin, [84], whilst 'exact n-type' is Beke's, [26].

If we need a version of these ideas in $Simp(\mathcal{E})$ or $Simp.\mathcal{A}$, then we can easily adapt our earlier discussion of horns and Kan objects in that context. For instance:

Proposition 46 *If $\mathcal{A}$ is a finite limit category, a morphism, $p : E \to B$, in $Simp.\mathcal{A}$ is an exact Kan fibration in dimension n if, and only if, the natural maps $E_n \to \Lambda^k[n](E) \times_{\Lambda^k[n](B)} B_n$ are all isomorphisms in $\mathcal{A}$.* ∎

Corollary 12 *In $Simp.\mathcal{A}$, an object, K, is an exact n-type (or n-hypergroupoid) if, and only if, the natural map, $K_k \to \Lambda^j[k](K)$, is an epimorphism for $k \leq n$ and an isomorphism for $k > n$.* ∎

To begin to take 'exact n-types' apart, we will need to look again at look at the coskeleton functors. It is very useful for our purposes to have a description of when a simplicial set, K, is isomorphic to its own n-coskeleton. The following summary is actually adapted from Beke's paper, [26], but is quite well known and moderately easy to prove, so the proof will be **left as an exercise**.

Proposition 47 *For a simplicial set, K, the following are equivalent:*

1) K is isomorphic to an object in the image of $cosk_n$.

2) The natural morphism $K \to cosk_n(K)$ is an isomorphism.

3) Writing $\partial\Delta_k(K)$ for the set

$$\partial\Delta_k(K) = \{(x_0, \ldots, x_k) \mid x_i \in K_{k-1} \text{ and, whenever } i < j,\ d_i x_j = d_{j-1} x_i\},$$

(so $\partial\Delta_k(K) \cong \mathcal{S}(\partial\Delta[k], K)$), the natural 'boundary' map $b_k(x) = (d_0 x, \ldots, d_k x)$, from K_k to $\partial\Delta_k(K)$ is a bijection for all $k > n$.

4) The natural map, $K_k \to Sets^{\mathbf{\Delta}[0,n]^{op}}(tr^n\Delta[k], tr^n(K))$, which sends a k-simplex x of K, considered as its 'name', $\ulcorner x \urcorner : \Delta[k] \to K$, to the n-truncation, of $\ulcorner x \urcorner$, is a bijection for all $k > n$.

5) For any $k > n$, and any pair of (solid) arrows

$$\begin{tikzcd}
\partial\Delta[k] \arrow[r] \arrow[d] & K \\
\Delta[k] \arrow[ur, dashed] &
\end{tikzcd}$$

there is precisely one *(dotted) arrow making the diagram commute.* ■

As we said, the proof is **left to you**, as it is just a question of translating between different viewpoints.

Definition: If K satisfies any, and hence all, of the above conditions, it is called *n-coskeletal.*

The first two conditions can be transferred verbatim for simplicial objects in any category with finite limits, and thus for simplicial objects in a topos. Condition 3 can also be handled in those contexts, using iterated pullbacks to construct $\partial\Delta_k(K)$. Condition 4) can also be used if the category of simplicial objects has finite cotensors (see the discussion of tensors and cotensors in simplicially enriched categories in section **??**, page **??**). A similar comment may be made about 5), since using cotensors allows one to 'internalise' the condition - but it ends up then being 3) in an enriched form. The details will not be needed in our later discussion, so are **left to you if you need them**.

We use this notion of n-coskeletal object in the following way

Proposition 48 *(cf. Beke, [26], proposition 1.3) (i) If K satisfies the exact Kan condition above dimension n, then K must be $(n+1)$-coskeletal.*
(ii) If K is n-coskeletal, then it satisfies the exact Kan condition above dimension $n+1$.
(iii) If K is an n-coskeletal Kan complex, then it has vanishing homotopy groups in dimensions n and above.
(iv) An exact n-type has vanishing homotopy groups above dimension n.

Before we prove this, it needs noting that there is an internal version in $Simp(\mathcal{E})$ for $\mathcal{E}$ a topos, see [26]. We have refrained from giving it only to avoid the need to define the homotopy groups of such an object internally.

Proof: (i) Suppose we are given a map $b : \partial\Delta[k] \to K$ for $k > n+1$, then we can omit $d_0 b$ to get a $(k,0)$-horn in K. By assumption, this horn has a filler, $f : \Delta[k] \to K$, so we consider both $d_0 f$ and $d_0 b$. As they have the same boundary and since K satisfies the exact Kan condition above dimension n, they must coincide. We have thus that f is a filler for b. By exactness, we have that it is unique.

(ii) If $m > n+1$, $tr_n(\Lambda^k[m]) \to tr_n(\Delta[m])$ is fairly obviously an isomorphism. Now $cosk_n(K)$ satisfies the exact Kan condition in dimension m if, and only if, for any horn, $\underline{x} : \Lambda^k[m] \to K$, there is a diagram

$$\begin{tikzcd} \Lambda^k[m] \arrow[r, "\underline{x}"] \arrow[d, "inc"'] & cosk_n K \arrow[d] \\ \Delta[k] \arrow[r] \arrow[ur, dashed, "\exists!"] & 1 \end{tikzcd}$$

with unique diagonal. Using the adjunction, this gives a diagram

$$\begin{tikzcd} tr_n\Lambda^k[m] \arrow[r, "\overline{x}"] \arrow[d, "inc"'] & K \arrow[d] \\ tr_n\Delta[k] \arrow[r] \arrow[ur, dashed, "?"] & 1 \end{tikzcd}$$

and we have noted that the left hand side is an isomorphism if $m > n+1$.

(iii) If K is Kan, the topological description of homotopy groups goes over to K, i.e., as the group of homotopy classes of maps from $\partial\Delta[n]$ to K mapping a vertex to chosen basepoint. Such a map will fill in dimensions $k \geq n$, so all the $\pi_k(K)$ will be trivial for any base point. (**You should fill in the details of this argument.**)

(iv) This just combines (i) and (iii). ∎

We note that (iv) above says that exact n-types are n-types!

4.1.3 Postnikov towers

In the topological case, we saw above that given any (connected) CW-complex, X, we could construct a sequence of Postnikov sections, $P_n X$, and maps between them, $P_{n+1}X \to P_n X$. We referred to this as a *Postnikov tower* for X. In the simplicial case, we found that the coskeletons gave us a corresponding construction, (and we will shortly see an alternative, if related, one). It is often useful to demand a bit more structure in the tower, structure that is always potentially there but which is usually not in its 'optimal form'. To make them more 'useful', we first review the definition of Postnikov towers and some of their properties. (We refer the reader, who wants a slightly more detailed introduction, to Hatcher's book, [115], p. 410.) First a redefinition,(adapted to our needs from [115])

Definition: A *Postnikov tower* for a (connected) space X is a commutative diagram:

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_3 \arrow[d, "p^3_2(X)"] \\
 & X_2 \arrow[d, "p^2_1(X)"] \\
X \arrow[r, "p_1(X)"'] \arrow[ur, "p_2(X)"] \arrow[uur, "p_3(X)"] \arrow[uuur, dashed] & X_1
\end{tikzcd}$$

such that

(i) the map $X \to X_n$ induces an isomorphism on π_i for $i \leq n$;

(ii) $\pi_i(X_i) = 0$ for $i > n$.

Remark: A Postnikov tower for X always exists by our discussion in section 4.1.1 and, hidden in that discussion is the information that shows that the tower is unique up to a form of homotopy equivalence for towers.

If we convert each maps $X_n \to X_{n-1}$ into a fibration (in the usual way be pulling back the pathspace fibration on X_{n-1} along this map, see the discussion of the corresponding construction for chain complexes, in section **??**, where the term *mapping cocone* is used), then its fibre (which is, then, the *homotopy fibre* of the original map), will be an Eilenberg-Mac Lane space, $K(\pi_n X, n)$, as the difference between the homotopy groups of X_n and X_{n-1} is exactly $\pi_n(X)$ in dimension n. (More exactly, we should look at the long exact homotopy sequence for this fibration, but we do not have this available within the notes so far so if you need more precision on this refer to Hatcher, [115], or other texts on homotopy theory.)

Definition: A *fibrant Postnikov tower* for X is a Postnikov tower (as above) in which each $X_n \to X_{n-1}$ is a fibration.

The discussion above shows that any Postnikov tower can be replaced, up to homotopy equivalence, by a fibrant one. There is here a technical remark that is worth making, but requires that the reader has met the theory of model categories. (It can safely be ignored if you have not yet met this.) On the category of towers of spaces (or or simplicial sets, etc.) there is a model category structure in which these fibrant towers are exactly the fibrant objects.

Moving over to the simplicial case, we restrict attention to Kan complexes, as they are much better behaved, homotopically, than arbitrary ones. We have the n^{th} coskeleton, $cosk_n K$ of a Kan complex, K, and the first query is whether it is a Kan complex itself. Certainly in dimensions lower than n, as it agrees with K there, any k-horn will have a filler. We thus look at an $(n+1)$-horn, $x_0, \ldots, \hat{x}_i, \ldots, x_{n+1}$, corresponding to the map, $\underline{x} : \Lambda^i[n+1] \to cosk_n K$, (using the usual convention with a 'hat' indicating the missing face). All the faces, x_k, are in $(cosk_n K)_n = K_n$, so all toegther they form a $(n+1)$-horn in K, which, of course, can be filled by some $y \in K_{n+1}$ We have its naming map $\ulcorner y \urcorner : \Delta[n+1] \to K$, which we restrict to $sk_n\Delta[n+1]$ to get a filler for our original $\underline{x}$. We thus do have that $cosk_nK$ satisfies the Kan filler condition in dimension $n+1$.

We look, next, at dimension $n = 2$ (expecting, of course, that the situation there will tell us how to handle the general case in higher dimensions). In fact, we have already seen the argument that we will use above.

Suppose $\underline{x} : \Lambda^i[n+2] \to cosk_nK$, then $\underline{x}$ corresponds, under the adjunction to a map, $\overline{x} : sk_n\Lambda^i[n+2] \to K$, but, and this is the neat argument we saw before, $sk_n\Lambda^i[n+2] = sk_n\Delta[n+2]$ (or, if you want to be precise, the inclusion of $\Lambda^i[n+2]$ into $\Delta[n+2]$ restricts to the 'identity' isomorphism on the n-skeletons). This means that $\overline{x}$ is already in $(cosk_nK)_{n+2}$. (Of course, dotting i's and crossing t's, that statement is also not true, but means $\Lambda^i[\ell] \to \Delta[\ell]$ induces a bijection

$$\mathcal{S}(\Delta[\ell], cosk_nK) \xrightarrow{\cong} \mathcal{S}(\Lambda^i[\ell], cosk_nK)$$

for $\ell = n+2$, and, in fact, for all $\ell \geq n+2$, so $sk_n\Lambda^i[n+2] \xrightarrow{\cong} sk_n\Delta[n+2]$ for all $\ell \geq n+2$.) We summarise this in a proposition for possible later use.

Proposition 49 *If K is a Kan complex, then so is $cosk_nK$.* ■

We next glance at the canonical map

$$p_n^{n+1} : cosk_{n+1}K \to cosk_nK.$$

This does not seem to be a fibration, but that is not too worrying since (i) we can replace is by a fibration as in the topological case, and (ii) we will see there is a subtower of this *cosk* tower which is fibrant and very neat and we turn to it next. Its beauty is that it adapts well to many other simplicial settings, such as that of simplicial groups, without much adjustment, and it is functorial.

The canonical map, $p_n = \eta(K) : K \to cosk_nK$, which is the unit of the adjunction, can be very easily described in combinatorial terms, since $(cosk_nK)_m = \mathcal{S}(sk_n\delta[m], K)$. If x is a m-simplex in K, then its 'name' $\ulcorner x \urcorner : \Delta[m] \to K$ determines it precisely and conversely, (by the Yoneda lemma and the equation $\ulcorner x \urcorner \iota_n = x$). There is an inclusion, $i_m : sk_n\Delta[m] \to \Delta[m]$, and $\ulcorner x \urcorner \circ i_m$ is an m-simplex in $cosk_nK$. This is $eta(x)$.

In $(cosk_nK)_m$, there can be simplices that are not restrictions of m-simplices in K and these are, for instance, simplices that, together, 'kill' the homotopy groups (above dimension n, that is.) As K is Kan, $\pi_m(K) \cong [S^m, K]$, the set of pointed homotopy classes of pointed maps from $S^m = \partial\Delta[m+1]$ or alternatively, $S^m = \Delta[m]/\partial\Delta[m]$. (Both identifications are useful and we can go from one to the other since they are weakly homotopy equivalent.) We note that, for instance, $sk_{m-1}S^m = sk_{m-1}\Delta[m]$, so any m-sphere in K has a canonical filler in $cosk_{m-1}K$. Other cases are slightly more tricky, but can be **left to you**, as, in any case, when we consider these more formally slightly later on we will use a slightly different argument.

The image of $\eta(K)$ is, in each dimension m, obtained by dividing K_m by the equivalence relation determined by $\eta(K)_m$, i.e., define $\sim_n$ on K_m by $x \sim_n y$ if, and only if, the representing maps, $x, y : \Delta[m] \to K$ agree on $sk_n\Delta[m]$. (We will dispense with the 'name' notation, $\ulcorner x \urcorner$, here, as it tends to clutter the notation and is not needed, if no confusion is likely to occur. We are thus pretending that $K_m = \mathcal{S}(\Delta[m], K)$, rather than merely being naturally isomorphic.)

We write $[x]_n$ for the $\sim_n$-equivalence class of x. We note that if $m \leq n$ then $\sim_n$ is simply equality as the n-skeleton of $\Delta[m]$ is all of $\Delta[m]$.

Definition: The simplicial set, $K(n) := K/\sim_n$ is called the n^{th} *Postnikov section* of K.

That $\sim_m$ is compatible with the face and degeneracy maps is **easy to check**, so $K(n)$ is a simplicial set and , equally simply, the natural quotient, $q_n : K \to K(n)$, so $q_n(x) = [x]_n$, is simplicial. (It is the codomain restriction of $p_n = \eta(K)$.) This is best seen using the fact that is is induced from the *cosimplicial* inclusions $sk_n\Delta[m] \to \Delta[m]$. The cosimplicial viewpoint also gives that the inclusions $sk_n\Delta[m] \to sk_{n+1}\Delta[m]$ induce the quotient maps, $q_n^{n+1} : K(n+1) \to K(n)$, (which are the restrictions of the p_n^{n+1}), and that $q_n^{n+1}q_{n+1} = q_n$.

Lemma 34 *For a (connected) Kan complex, K, and for each n:*
(i) The map $q_n : K \to K(n)$ is a Kan fibration, and $K(n)$ is a Kan complex.
(ii) The map, $q_n^{n+1} : K(n+1) \to K(n)$, is a Kan fibration.
(iii) The map, q_n, induces an isomorphism on π_i for $0 \leq i \leq n$.
(iv) The homotopy groups of $K(n)$ are trivial above dimension n, $K(n)$ is an n-type.

Proof: (i) Suppose we have a commutative diagram

$$\begin{tikzcd} \Lambda^i[m] \arrow[r, "{(x_0,\ldots,\hat{x_i},\ldots,x_m)}"] \arrow[d] & K \arrow[d, "q_n"] \\ \Delta[m] \arrow[r, "{[y]_n}"'] & K(n) \end{tikzcd}$$

where we have written the i-horn as an $(m+1)$-tuple of $(m-1)$-simplices, with a gap at the 'hat'. We need to lift $[y]_n$ to some y agreeing with the x_ks, i.e., $d_ky = x_k$.

If $m \leq n$, there is no problem as q_n the identity in those dimensions.

For $m = n+1$, we have if y is a representative of $[y]_n$, then as $\sim_n$ is the identity relation in dimension n, $d_ky = x_k$ for $k \neq i$, so y is a suitable lift.

For $m > n+1$, we use that K is Kan to find a filler $x \in K_nm+1$ for the (m,i)-horn, so $d_kx = x-k$ for $k \neq i$. Now $sk_n\Lambda^i[m] = sk_n\Delta[m]$, as we have used before, and so $q_n(x) = [x]_n = [y]_n$.

In general, if $p : K \to L$ is a surjective Kan fibration and K is a Kan complex, then L is Kan, so the last part of (i) follows.

(ii) Look at $K(n+1)$ and form $K(n+1)(n)$, i.e. divide it out by $\sim_n$. This gives $K(n)$ with the quotient being just q_n^{n+1}. By (i), this will be a fibration.

We next pick a base vertex, $v \in K_0$ and look at the various $\pi_m(K,v)$ and $\pi_m(K(n),[v]_n)$. Clearly, as q_n 'is the identity' in dimensions $m \leq n$, the induced morphisms $\pi_m(q_n)$ 'is the identity' in dimensions $m < n$. For (iii), we have, thus, only to examine $\pi_n(q_n)$. Suppose $f : \Delta[n] \to K$ sends $\partial\Delta[n]$ to $\{v\}$, i.e., represents an element of $\pi_n(K)$, and that q_nf is null-homotopic, then q_nf extends to a map,$\overline{F} : \Delta[n+1] \to K(n)$ such that $q_nf = d_0\overline{F}$, and $d_i\overline{F} = v$ for $i \neq 0$. We can lift $\overline{F}$ to a map $F : \Delta[n+1] \to K$, since q_n is surjective *and* the n-dimensional faces are mapped by the identity. We thus have that f itself was null-homotopic, so $\pi_n(q_n)$ is a monomorphism. As $\pi_n(q_n)$ is cearly an epimorphism, this handles (iii).

(iv) Any map $f : \Delta[m] \to K(n)$ is determined by its restriction,$f| : sk_n\Delta[m] \to K$, but

$$sk_n\partial\Delta[m] \to sk_n\Delta[m]$$

is the identity if $m > n$, and $f|_{\partial\Delta[m]}$ is constant with value v, so $\pi_m(K(n)) = 0$ if $m > n$. ■

We thus have proved the connected case of the following:

Theorem 15 *If K is a Kan complex, $(K(n), q_n^{n+1}, q_n)$, forms a (functorial) fibrant Postnikov tower for K.* ■

The non-connected case is a simple extension of this connected one involving disjoint unions, so ....

Of course, the inclusion of $K(n)$ into $cosk_nK$ is a weak equivalence.

Remarks: (i) A note of caution seems in order. Some sources tend to confuse $K(n)$ and $cosk_nK$, and whilst, for many homotopical purposes, this is not critical, for certain purposes the use of one is prefereable to that of the other, so it seems better to keep the restriction.

(ii) The study of Postnikov complexes, which abstract the properties of the $K(n)$, is important in the study of coskeletal simplicial sets and nerves of higher categories, for which see the important paper of Duskin, [86].

(iii) Putting a naturally defined model category structure on the category of n-types (and on the corresponding simplicial presheaves and sheaves) has been done using these Postnikov sections, see Biedermann, [27]. He notes that his construction depends on using the Postnikov section approach that we have just outlined, rather than the coskeleton, as that latter one disturbs some of the necessary structure.

(iv) If you need more on Postnikov towers in simplicial sets, a good source is Goerss and Jardine, [105], Chapter 6, whilst Duskin's paper, [86], mentioned above, gives some powerful tools for manipulating them and also coskeletons.

4.1.4 Whitehead towers

Postnikov towers approximate a homotopy type by its tower of n-types, that is, by 'n-co-connected' spaces. The Whitehead tower of a homotopy type produces a sequence of n-connected approximations to it. Before we look at this in detail, let us consider what this should mean. (As sources, we will initially use Hatcher, [115], p. 356 in the topological case, before looking at the simplicial case. Another useful source is the nLab page on 'Whitehead towers', ([173], and search on 'Whitehead tower').)

What we would expect from a naive dualisation of Postnikov tower for a pointed space, X, would be a diagram,

$$\begin{tikzcd} \cdots \arrow[r] \arrow[drrr, dashed] & Z_2 \arrow[r] \arrow[drr] & Z_1 \arrow[r] \arrow[dr] & Z_0 \arrow[d] \\ & & & X \end{tikzcd}$$

with Z_n an n connected space, (so $\pi_i(Z_n) = 0$ for $i \leq n$), and the composite map $Z_n \to X$ inducing an isomorphism on all homotopy groups, π_i for $i > n$. The space Z_0 would be path connected and homotopy equivalent to the component of X containing the base point. The next space, Z_1 would be simply connected and would have the homotopy properties of the universal cover of Z_0. We would then think of $Z_n \to X$ as an 'n-connected cover' of the (pointed connected component, Z_0, of the)space, X.

Definition: The *Whitehead tower* of a pointed space, (X, x) is a sequence of fibrations

$$\ldots \to X\langle n\rangle \to \ldots \to X\langle 1\rangle \to X\langle 0\rangle \to X$$

where each $X\langle n\rangle \to X\langle n-1\rangle$ induces isomorphisms on the homotopy groups, π_i, for $i > n$ and such that $X\langle n\rangle$ is n-connected, so $\pi_k(X\langle n\rangle)$ is trivial for all $k \leq n$.

The problem of constructing such a tower was posed by Hurewicz and solved by George Whitehead in 1952. We will assume that we have chosen a Postnikov tower for a CW-complex, X, so giving a map $p_n : X \to P_nX$.

We next to form the *homotopy fibre* or *mapping cocone* of this map, over the basepoint, x_0, of P_nX. We have already seen this idea, page 17, so will just briefly review how it is constructed. We first form the pullback

$$\begin{tikzcd} M^{p_n} \arrow[r, "\pi^{p_n}"] \arrow[d, "j^{p_n}"'] & (P_nX)^I \arrow[d, "e_0"] \\ X \arrow[r, "p_n"'] & P_nX \end{tikzcd}$$

so M^{p_n} consists of pairs, (x, λ), where $x \in X$ and $\lambda : I \to P_nX$ is a path with $\lambda(0) = p_n(x)$. We set $i^{p_n} = e_1 \circ \pi^{p_n}$, so that $i^{p_n}(x, \lambda) = \lambda(1)$. The fact that $i^{p_n} : M^{p_n} \to P_nX$ is a fibration is standard, as is that $j^{p_n} : M^{p_n} \to X$ is a homotopy equivalence. (If you want a proof of these, **after trying to give one yourself**, there are proofs in many standard textbooks, such as that of Hatcher, and the abstract setting of such results is discussed in Kamps and Porter, [131]. This all fits well into a 'homotopical' context, and that is explored more on the nLab, [173], search under 'mapping cocone' and follow the links.) For brevity, we will write $\overline{X}$ for M^{p_n}, $\overline{p_n} : \overline{X} \to P_nX$ for i^{p_n}. The homotopy fibre of p_n is then the fibre of $\overline{p_n}$ over the base point of P_nX. It is $F^h(p_n) = \{(x, \lambda) \mid \lambda(1) = x_0\}$.

We thus have a fibration sequence,

$$F^h(p_n) \to \overline{X} \to P_nX,$$

and, hence, by standard homotopy theory, a long exact sequence of homotopy groups,

$$\ldots \to \pi_k(F^h(p_n)) \to \pi_k(\overline{X}) \to \pi_k(P_nX) \to \pi_{k-1}(F^h(p_n)) \to \ldots$$

Note that $\pi_k(\overline{X}) \cong \pi_k(X)$, since j^{p_n} is a homotopy equivalence. (If you have not met this long fibration exact sequence before, **check it up, briefly** in any standard book on homotopy theory. We will look at it, and also the dual situation in cohomology, in more detail later on, starting in section **??**.)

If we look at this long exact sequence, below the value $k = n$, the homomorphism $\pi_k(\overline{X}) \to \pi_k(P_nX)$ is an isomorphism, so $\pi_k(F^h(p_n)) = 0$ in that range, whilst as $\pi_k(P_nX) = 0$ if $k > n$, there $\pi_k(F^h(p_n)) \to \pi_k(\overline{X})$ is an isomorphism. Thus the homotopy fibre, $F^h(p_n)$ is n-connected.

This looks good, as this is a functorial construction (or, more exactly, any lack of functoriality is due to a lack of functoriality of the Postnikov tower). We have a composite map $F^h(p_n) \to \overline{X} \to X$. This sends (x, λ) to x, of course. We will write $X\langle n\rangle := F^h(p_n)$, in the expectation that it will form part of a 'Whitehead tower'.

The next ingredient that we need will be a map

$$X\langle n+1\rangle \to X\langle n\rangle.$$

We do have a (chosen) map $p_n^{n+1} : P_{n+1}X \to P_nX$, which is compatible with the 'projections' $p_n : X \to P_nX$, so $p_n^{n+1}p_{n+1} = p_n$. This induces a map from the homotopy fibre of p_{n+1} to that of p_n. (This is **left to you to check**. The usual proof uses the functoriality of $(-)^I$ and the naturality of the various mappings, and then the universal property of pullbacks. Everything is being 'chosen up to homotopy' so there are subtleties that **do** need thinking about, and it is a good idea to try to get a reasonably homotopy 'coherent' argument going on behind the proof. The construction is a 'homotopy pullback' and the property you a looking for is the analogue of the universal property of pullbacks to this more structured setting. It is, in the long term, important to get used to this sort of situation as well as to the sort of geometric / higher categorical picture that it corresponds to, as this is needed for generalisations.)

We note that the fibre of $X\langle n+1\rangle \to X\langle n\rangle$ is a $K(\pi_n(X), n)$.

Remarks: (i) The above hides slightly the fact that the construction of a Whitehead tower is only really 'natural' up to homotopy as that was already the case for the Postnikov tower in the topological case.

(ii) For the simplicial case, we can use either the coskeleton based tower or, better, the Postnikov section one, as that is already fibrant as we saw. As the p_n and p_n^{n+1} are fibrations in that case, we can replace the homotopy pullbacks by pullbacks, and the homotopy fibres by fibres, thus gaining more insight into the relationship of the objects in the corresponding Whitehead tower to the Kan complex being 'resolved'. (The detailed description is **left to you**.)

(iii) The theory and constructions adapt well to other simplicial contexts such as that of simplicial groups, where, as fibrations are simply degreewise epimorphisms, many of the constructions take on a much simpler algebraic aspect.

The case of a topological group, G: In this case, one can find a topological model for each $G\langle n\rangle$ which is a topological group, and, as there is a topological Abelian group model for the $K(\pi, n)$s occurring as the fibres in the tower, there is a short exact sequence

$$1 \to K(\pi_n(G), n) \to G\langle n+1\rangle \to G\langle n\rangle \to 1.$$

Example: The Whitehead tower of the orthogonal group, $O(n)$.

For large n, the orthogonal group, $O(n)$, has the following homotopy groups:

i	0	1	2	3	4	5	6	7
$\pi_i(O(n))$	C_2	C_2	0	C_∞	0	0	0	C_∞

There are then periodicity results for higher dimensions giving $\pi_{k+8}(O(n)) \cong \pi_k(O(n))$. The first space of the Whitehead tower of $O(n)$ is, of course, $O(n)\langle 0\rangle = SO(n)$, as it is the (0-)connected component of the identity element.

The next space is the group, $O(n)\langle 1\rangle = Spin(n)$, (which we will look at in more detail later; see section 9.1.3). There is a short exact sequence:

$$1 \to C_2 \to Spin(n) \to SO(n) \to 1.$$

The next homotopy group is trivial and $O(n)\langle 2\rangle = O(n)\langle 3\rangle = String(n)$. This is a very interesting group, but we have not yet the machinery to do it justice. (For more on it in our sort of setting, see, for instance, Jurco, [130], Schommer-Pries, [192]. We will return to it later.)

4.2 Crossed squares

We next turn back to algebraic models of these n-types that we have now introduced more formally. We have already seen models for 2-types, namely the crossed modules that we looked at earlier, now we turn to 3-types. There are several different types of model here. We start with one that is relatively simple in its apparent structure.

4.2.1 An introduction to crossed squares

We saw earlier that crossed modules were like normal subgroups except that the inclusion map is replaced by a homomorphism that need not be a monomorphism. We even noted that all crossed modules are, up to isomorphism, obtainable by applying π_0 to a simplicial "inclusion crossed module".

Given a pair of normal subgroups M, N of a group G, we can form a square

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & N \arrow[d] \\ M \arrow[r] & G \end{tikzcd}$$

in which each morphism is an inclusion crossed module and there is a commutator map

$$h : M \times N \to M \cap N$$

$$h(m, n) = [m, n].$$

This forms a *crossed square* of groups, in fact, it is a special type of such that we will call an *inclusion crossed square*. Later we will be dealing with crossed squares as crossed n-cubes, for $n = 2$. Here we will give an interim definition of crossed squares. The notion is due to Guin-Walery and Loday, [109], and this slightly shortened form of the definition is adapted from Brown-Loday, [52].

4.2.2 Crossed squares, definition and examples

Definition: (First version) A *crossed square* (more correctly *crossed square of groups*) is a commutative square of groups and homomorphisms

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"] & P \end{tikzcd}$$

together with actions of the group P on L, M and N (and hence actions of M on L and N via μ and of N on L and M via ν) and a function $h : M \times N \to L$. This structure is to satisfy the following axioms:

(i) the maps λ , λ' preserve the actions of P, furthermore with the given actions, the maps μ, ν and $\kappa = \mu\lambda = \mu'\lambda'$ are crossed modules;

(ii) $\lambda h(m, n) = m{}^n m^{-1}$, $\lambda' h(m, n) = {}^m n n^{-1}$;

(iii) $h(\lambda\ell, n) = \ell{}^n\ell^{-1}$, $h(m, \lambda'\ell) = {}^m\ell\ell^{-1}$;

(iv) $h(mm',n) = {}^m h(m',n)h(m,n)$, $h(m,nn') = h(m,n)^n h(m,n')$;
(v) $h({}^p m, {}^p n) = {}^p h(m,n)$;
for all $\ell \in L$, $m, m' \in M$, $n, n' \in N$ and $p \in P$.

There is an evident notion of morphism of crossed squares, just preserve all the structure, and we obtain a category Crs^2, the *category of crossed squares.*

Examples

In addition to the above class of examples, we have the following:
(a) Given any simplicial group, G, and two simplicial normal subgroups, M and N, the square

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & N \arrow[d] \\ M \arrow[r] & G \end{tikzcd}$$

with inclusions and with $h = [\ ,\ ] : M \times N \to G$ is a simplicial "inclusion crossed square" of simplicial groups. Applying π_0 to the diagram gives a crossed square and, in fact, all crossed squares arise in this way (up to isomorphism).

b) Any simplicial group, G, yields a crossed square, $M(G,2)$, defined by

$$\begin{tikzcd} \dfrac{NG_2}{d_0(NG_3)} \arrow[r] \arrow[d] & Ker\, d_1 \arrow[d] \\ Kerd_2 \arrow[r] & G_1 \end{tikzcd}$$

for suitable maps. This is, in fact, part of the construction that shows that all connected 3-types are modelled by crossed squares.

Another way of encoding 3-types is using the truncated simplicial group and Conduché's notion of 2-crossed module.

4.3 2-crossed modules and related ideas

4.3.1 Truncations.

Definition: Given a chain complex, (X, ∂), and an integer n, the *truncation of* X *at level* n is the complex $t_{n]}X$ defined by

$$(t_{n]}X)_i = \begin{cases} 0 & \text{for } i > n \\ X_n/Im\,\partial_n & \\ X_i & \text{for } i < n. \end{cases}$$

For $i < n$, the differential of $t_{n]}X$ is the same as that of X, whilst the n^{th}-differential is induced by ∂.

(For more on truncations see Illusie [127, 128]). Truncation is, of course, functorial.

Remark on terminology: There are several schools of thought on the terminology here. The problem is whether this should be 'truncation' or 'co-truncation'. To some extent both are 'wrong' as n-truncated chain complexes should not have any information available in dimensions greater than n, if the model of simplicial sets was to be followed. This would then be expected to have right and left adjoints, which would correspond, approximately to the coskeleton and skeleton functors of simplicial set theory that we have already seen. At the moment the 'jury' seems to be out and the terminological conventions fairly lax. (We may thus decide to change this later on if convincing arguments are presented.)

This construction will work for chain complexes of groups provided each $Im\,\partial$ is a *normal* subgroup of the corresponding X, i.e., provided X is a *normal chain complex of groups.*

Proposition 50 *There is a truncation functor $t_{n]} : Simp.Grps \to Simp.Grps$ such that there is a natural isomorphism*

$$t_{n]}NG \cong Nt_{n]}G,$$

where N is the Moore complex functor from Simp.Grps to the category of normal chain complexes of groups.

Proof: We first note that $d_0(NG_{n+1})$ is contained in G_n as a normal subgroup and that all face maps of G vanish on it. We can thus take

$$\begin{aligned} (t_{n]}G)_i &= G_i \text{ for all } i < n \\ (t_{n]}G)_n &= G_n/d_0(NG_{n+1}) \end{aligned}$$

and for $i > n$, we take the semidirect decomposition of G_i, which we will see shortly, given by Proposition 63, delete all occurrences of NG_k for $k > n$ and replace any NG_n by $NG_n/d_0(NG_{n+1})$. The definition of face and degeneracy is easy as is the verification that $t_{n]}N$ and $Nt_{n]}$ are the same and that the various actions are compatible. ■

This truncation functor has nice properties. (In the chain complex case, these are discussed in Illusie, [127].)

Proposition 51 *Let $T_{n]}$ be the full subcategory of Simp.Grps defined by the simplicial groups whose Moore complex is trivial in dimensions greater than n and let $i_n : T_{n]} \to Simp.Grps$ be the inclusion functor.*

a) The functor $t_{n]}$ is left adjoint to i_n. (We will usually drop the i_n and so also write $t_{n]}$ for the composite functor.)

b) The natural transformation, η, co-unit of the adjunction, is a natural epimorphism which induces an isomorphism on π_i for $i \leq n$. The unit of the adjunction is isomorphic to the identity transformation, so $T_{n]}$ is a reflective subcategory of Simp.Grps.

c) For any simplicial group G, $\pi_i(t_{n]}G) = 0$ if $i > n$.

d) To the inclusion, $T_{n]} \to T_{n+1]}$, there corresponds a natural epimorphism η_n from $t_{n+1]}$ to $t_{n]}$. If G is a simplicial group, the kernel of $\eta_n(G)$ is a $K(\pi_{n+1}(G), n+1)$, i.e., has a single non-zero homotopy group in dimension $n+1$, that being $\pi_{n+1}(G)$, i.e., is an 'Eilenberg-MacLane space' of type $(\pi_{n+1}(G), n+1)$. ■

As each statement is readily verified using the Moore complex and the semidirect product decomposition, the proof of the above will be left to you, however you will need Proposition 63, page 215.

Definition: We will say that a simplicial group, G, is *n-truncated* if $NG_k = 1$ for all $k > n$.

Of course, $T_{n]}$ is the category of n-truncated simplicial groups.

A comparison of these properties with those of the *coskeleton functors* (cf., above, section 4.1.2, page 162, or for an 'original' source, Artin and Mazur, [10]) is worth making. We will not look at this in detail here, but will just summarise the results. We will meet them again later on from time to time.

Given any integer $k \geq 0$, there is a functor, $cosk_k$, defined on the category of simplicial sets, which is the composite of a truncation functor (differently defined) and its right adjoint. The n-simplices of $cosk_k\, X$ are given by $Hom(sk_k\Delta[n], X)$, the set of simplicial maps from the k-skeleton of the n-simplex, $\Delta[n]$, to the simplicial set, X. There is a canonical map from X to $cosk_k\, X$, whose homotopy fibre is $(k-1)$-connected. The canonical map from $cosk_k\, X$ to $cosk_{k-1}\, X$ thus has homotopy fibre an Eilenberg-MacLane 'space' of type $(\pi_k(X), k)$.

This k-coskeleton is constructed using finite limits and there is an analogue in any category of simplicial objects in a category, $\mathcal{D}$, provided only that $\mathcal{D}$ has finite limits, thus in particular in $Simp.Grps$. Conduché, [66], has calculated the Moore complex of $cosk_{k+1}\, G$ for a simplicial group, G, using a construction described in Duskin's Memoir, [83]. His result gives

$$\begin{aligned} N(cosk_{k+1}G)_r &= 0 \qquad \text{if } r > k+2 \\ N(cosk_{k+1}G)_{k+2} &= Ker(\partial_{k+1} : NG_{k+1} \to NG_k), \end{aligned}$$

and

$$N(cosk_{k+1}G)_r = NG_r \qquad \text{if } r \leq k+1.$$

There is an epimorphism from $cosk_{n+1}G$ to $t_{n]}G$, which, on passing to Moore complexes, gives

$$\begin{tikzcd} 0 \arrow[r] & Ker\,\partial_{k+1} \arrow[r] \arrow[d] & NG_{k+1} \arrow[r] \arrow[d] & NG_k \arrow[r, "\partial_{k+1}"] \arrow[d] & NG_{k-1} \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & NG_k/Im/\,\partial_{m+1} \arrow[r] & NG_{k-1} \end{tikzcd}$$

This epimorphism of chain complexes thus has a kernel with trivial homology. The epimorphism therefore induces an isomorphism on all homotopy groups and hence is a weak homotopy equivalence. We may thus use either $t_{n]}G$ or $cosk_{n+1}G$ as a model of the n-type of G.

4.3.2 Truncated simplicial groups and the Brown-Loday lemma

The theory of crossed n-cubes that we have hinted at above is not the only way of encoding higher n-types. Another method would be to use these truncated simplicial groups as suggested above. A detailed study of this is complicated in high dimension, but feasible for 3-types and, in fact, reveals some interesting insights into crossed squares in the process.

As a first step to understanding truncated simplicial groups a bit more, we will give a variant of an argument that we have already seen. We will look at a 1-truncated simplicial group. The analysis is really a simple use of the sort of insights given by the Brown-Loday lemma.

Proposition 52 *(The Brown-Loday lemma) Let N_2 be the (closed) normal subgroup of G_2 generated by elements of the form*

$$F_{(1),(0)}(x,y) = [s_1x, s_0y][s_0y, s_0x]$$

for $x, y \in NG_1 = Ker\, d_1$. Then $NG_2 \cap D_2 = N_2$ and consequently

$$\partial(NG_2 \cap D_2) = [Ker\, d_0, Ker\, d_1].$$

■

Note the link with group T-complex type conditions through the intersection, $NG_2 \cap D_2$.

The form of this element, $F_{(1),(0)}(x,y)$, is obtained by taking the two elements, x and y, of degree 1 in the Moore complex of a simplicial group, G, mapping them up to degree 2 by complementary degeneracies, and then looking at the component of the result that is in the Moore complex term, NG_2. (It is easy to show that G_2 is a semidirect product of NG_2 and degenerate copies of lower degree Moore complex terms.) The idea behind this pairing can be extended to higher dimensions. It gives the *Peiffer pairings*

$$F_{\alpha,\beta} : NG_p \times NG_q \to NG_{p+q}.$$

In general, these take $x \in NG_p$ and $y \in NG_q$ and (α, β) a complimentary pair of index strings (of suitable lengths), and sends (x, y) to the component in NG_{p+q} of $[s_\alpha x, s_\beta y]$; see the series of papers [166–170]. This again uses the Conduché decomposition lemma, [66], that we will see later on, cf. page 215. It is also worth noting that the Peiffer pairing ends up in $NG_{p+q} \cap D_{p+q}$, so would all be zero in a group T-complex.

A very closely related notion is that of hypercrossed complex as in Carrasco and Cegarra, [62, 63]. There one uses the component of $s_\alpha x.s_\beta y$ in NG_{p+q} to give a pairing and adds cohomological information to the result to get a reconstruction technique for G from NG, i.e., an *ultimate Dold-Kan theorem*, thus hypercrossed complexes generalise 2-crossed modules and 2-crossed complexes to all dimensions.

4.3.3 1- and 2-truncated simplicial groups

Suppose that G is a simplicial group and that $NG_i = 1$ for $i \geq 2$. This leaves us just with

$$\partial : NG_1 \to NG_0.$$

We make $NG_0 = G_0$ act on NG_1 by conjugation as before

$${}^g c = s_0(g)cs_0(g)^{-1} \text{ for } g \in G_0, c \in NG_1,$$

and, of course, $\partial({}^g c) = g.\partial c.g^{-1}$. Thus the first crossed module axiom is satisfied. For the other one, we note that $F_{(1),(0)}(c_1, c_2) \in NG_2$, which is trivial, so

$$\begin{aligned} 1 &= d_0([s_1c_1, s_0c_2][s_0c_2, s_0c_1]) \\ &= [s_0d_0c_1, c_2][c_2, c_1] = ({}^{\partial c_1}c_2)(c_1c_2c_1^{-1})^{-1}, \end{aligned}$$

so the Peiffer identity holds as well. Thus $\partial : NG_1 \to NG_0$ is a crossed module. As we have already seen that the functor $\mathcal{G}$ provides a way to construct a simplicial group from a crossed module and that the result has Moore complex of length 1, we have the following slight reformulation of earlier results:

Proposition 53 *The category of crossed modules is equivalent to the subcategory $T_{1]}$ of 1-truncated simplicial groups.* ∎

The main reason for restating and proving this result in this form is that we can glean more information from the proof for examining the next level, 2-truncated simplicial groups.

If we replace our 1-truncated simplicial group by an arbitrary one, then we have already introduced the idea of a Peiffer commutator of two elements, and there we used the term 'Peiffer lifting' without specifying what particular interest the construction had. We recall that here: Given a simplicial group, G, and two elements $c_1, c_2 \in NG_1$ as above, then the *Peiffer commutator* of c_1 and c_2 is defined by

$$\langle c_1, c_2 \rangle = ({}^{\partial c_1}c_2)(c_1 c_2 c_1^{-1})^{-1}.$$

We met earlier, $F_{(1),(0)}$, which gives the *Peiffer lifting* denoted

$$\{-,-\} : NG_1 \times NG_1 \to NG_2,$$

where

$$\{c_1, c_2\} = [s_1c_1, s_0c_2][s_0c_2, s_0c_1]$$

and we noted

$$\partial\{c_1, c_2\} = \langle c_1, c_2 \rangle.$$

These structures come into their own for a 2-truncated simplicial group. Suppose that G is now a simplicial group, which is 2-truncated, so its Moore complex looks like:

$$\ldots 1 \to NG_2 \xrightarrow{\partial_2} NG_1 \xrightarrow{\partial_1} NG_0.$$

For the moment, we will concentrate our attention on the morphism ∂_2.

The group NG_1 acts on NG_2 via conjugation using s_0 or s_1. We will use s_0 for the moment, so that if $g \in NG_1$ and $c \in NG_2$,

$${}^{g}c = s_0(g)cs_0(g)^{-1}.$$

It is once again clear that $\partial_2({}^{g}c) = g.\partial_2(c).g^{-1}$ and, as before, we consider, for $c_1, c_2 \in NG_2$ this time, the Peiffer pairing given by

$$[s_1c_1, s_0c_2][s_0c_2, s_0c_1],$$

which is, this time, the component of $[s_1c_1, s_0c_2]$ in NG_3. However that latter group is trivial, so this element is trivial, and hence, so is its image in NG_2. The same calculation as before shows that, with this s_0-based action of NG_1 on NG_2, (NG_2, NG_1, ∂_2) is a crossed module.

We also know that there is a Peiffer lifting

$$\{-,-\} : NG_1 \times NG_1 \to NG_2,$$

which measures the obstruction to $NG_1 \to NG_0$ being a crossed module, since $\partial\{-,-\}$ is the Peiffer commutator, whose vanishing is equivalent to $NG_1 \to NG_0$ being a crossed module. We do not have yet in our investigation a detailed knowledge of how the two structures interact, nor any other distinguishing properties of $\{-,-\}$. We will not give such a detailed derivation here, but from it we can obtain the following:

Proposition 54 *Let G be a 2-truncated simplicial group. The Peiffer lifting*

$$\{-,-\}: NG_1 \times NG_1 \to NG_2,$$

has the following properties:
(i) it is a map such that if $m_0, m_1 \in NG_1$,

$$\partial\{m_0, m_1\} = {}^{\partial m_0}m_1.(m_0 m_1 m_0^{-1})^{-1};$$

(ii) if $\ell_0, \ell_1 \in NG_2$,

$$\{\partial\ell_0, \partial\ell_1\} = [\ell_0, \ell_1];$$

(iii) if $\ell \in NG_2$ and $m \in NG_1$, then

$$\{m, \partial\ell\}\{\partial\ell, m\} = {}^{\partial m}\ell.\ell^{-1};$$

(iv) if $m_0, m_1, m_2 \in NG_1$, then
a) $\{m_0, m_1 m_2\} = \{m_0, m_1\}^{(m_0 m_1 m_0^{-1})}\{m_0, m_2\}$,
b) $\{m_0 m_1, m_2\} = {}^{\partial m_0}\{m_1, m_2\}\{m_0, m_1 m_2 m_1^{-1}\}$;
(v) if $n \in NG_0$ and $m_0, m_1 \in NG_1$, then

$$^{n}\{m_0, m_1\} = \{^{n}m_0, {}^{n}m_1\}.$$

■

The above can be encoded in the definition of a 2-crossed module.

4.3.4 2-crossed modules, the definition

Definition: A *2-crossed module* is a normal complex of groups

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N,$$

together with an action of N on all three groups and a mapping

$$\{-,-\}: M \times M \to L$$

such that

(i) the action of N on itself is by conjugation, and ∂_2 and ∂_1 are N-equivariant;

(ii) for all $m_0, m_1 \in M$,
$$\partial_2\{m_0, m_1\} = {}^{\partial_1 m_0}m_1.m_0 m_1^{-1} m_0^{-1};$$

(iii) if $\ell_0, \ell_0 \in L$, then
$$\{\partial_2\ell_0, \partial_2\ell\} = [\ell_1, \ell_0];$$

(iv) if $\ell \in L$ and $m \in M$, then
$$\{m, \partial\ell\}\{\partial\ell, m\} = {}^{\partial m}\ell.\ell^{-1};$$

(v) for all $m_0, m_1, m_2 \in M$,

(a) $\{m_0, m_1m_2\} = \{m_0, m_1\}\{\partial\{m_0, m_2\}, (m_0m_1m_0^{-1})\}\{m_0, m_2\}$;

(b) $\{m_0m_1, m_2\} = {}^{\partial m_0}\{m_1, m_2\}\{m_0, m_1m_2m_1^{-1}\}$;

(vi) if $n \in N$ and $m_0, m_1 \in M$, then

$$^{n}\{m_0, m_1\} = \{{}^{n}m_0, {}^{n}m_1\}.$$

The pairing $\{-,-\} : M \times M \to L$ is often called the *Peiffer lifting* of the 2-crossed module.

The only one of these axioms that looks 'daunting' is (v)a). Note that we have not specified that M acts on L. We could have done that as follows: if $m \in M$ and $\ell \in L$, define

$$^{m}\ell = \{\partial\ell, m\}\ell.$$

Now (v)a) simplifies to the expression

$$\{m_0, m_1m_2\} = \{m_0, m_1\}\,{}^{(m_0m_1m_0^{-1})}\{m_0, m_2\}.$$

We denote such a 2-crossed module by $\{L, M, N, \partial_2, \partial_1\}$, or similar, only adding in notation for the actions and the pairing if explicitly needed for the context. A *morphism of 2-crossed modules* is, fairly obviously, given by a diagram

$$\begin{tikzcd} L \arrow[r, "\partial_2"] \arrow[d, "f_2"'] & M \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & N \arrow[d, "f_0"'] \\ L' \arrow[r, "\partial_2'"'] & M' \arrow[r, "\partial_1'"'] & N' \end{tikzcd},$$

where $f_0\partial_1 = \partial_1' f_1, f_1\partial_2 = \partial_2' f_2$,

$$f_1({}^{n}m) = {}^{f_0(n)}f_1(m), \quad f_2({}^{n}\ell) = {}^{f_0(n)}f_2(\ell),$$

and

$$\{-,-\}(f_1 \times f_1) = f_2\{-,-\},$$

for all $\ell \in L$, $m \in M$, $n \in N$.

These compose in an obvious way giving a category which we will denote by $2{-}CMod$.

The following should be clear.

Theorem 16 *The Moore complex of a 2-truncated simplicial group is a 2-crossed module. The assignment is functorial.* ■

We will denote this functor by $\mathsf{C}^{(2)} : T_{2]} \to 2{-}CMod$. It is an equivalence of categories.

4.3.5 Examples of 2-crossed modules

Of course, the construction of 2-crossed modules from simplicial groups gives a generic family of examples, but we can do better than that and show how these new crossed gadgets link in with others that we have met earlier.

Example 1: Any crossed module gives a 2-crossed module, since if (M, N, ∂) is a crossed module, we need only add a trivial $L = 1$, and the resulting sequence

$$L \to M \to N$$

with the 'obvious actions' is a 2-crossed module! This is, of course, functorial and $CMod$ can be considered to be a full subcategory of $2-CMod$ in this way. It is a reflective subcategory since there is a reflection functor obtained as follows:

If

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

is a 2-crossed module, then $Im\,\partial_2$ is a normal subgroup of M and we have (with a small abuse of notation):

Proposition 55 *If $L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$ is a 2-crossed module then there is an induced crossed module structure on*

$$\partial_1 : \frac{M}{Im\,\partial_2} \to N.$$

■

But we can do better than this:

Example 2: Any crossed complex of length 2, that is one of form

$$\ldots \to 1 \to 1 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0,$$

gives us a 2-crossed complex on taking $L = C_2$, $M = C_1$ and $N = C_0$, with $\{m, m'\} = 1$ for all $m, m' \in M$. We will check this in a moment, but note that this gives a functor from $Crs_{2]}$ to $2-CMod$ extending the one we gave in Example 1.

Of course, (i) crossed complexes of length 2 are the same as 2-truncated crossed complexes.

4.3.6 Exploration of trivial Peiffer lifting

Suppose we have a 2-crossed module

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N,$$

with the extra condition that $\{m_0, m_1\} = 1$ for all $m_0, m_1 \in M$. The obvious thing to do is to see what each of the defining properties of a 2-crossed module give in this case.

(i) There is an action of N on L and M and the ∂s are N-equivariant. (This gives nothing new in our special case.)

(ii) $\{-,-\}$ is a lifting of the Peiffer commutator - so if $\{m_0, m_1\} = 1$, the Peiffer identity holds for (M, N, ∂_1), i.e. that is a crossed module;

(iii) if $\ell_0, \ell_1 \in L$, then $1 = \{\partial_2\ell_0, \partial_2\ell_1\} = [\ell_1, \ell_0]$, so L is Abelian
and,

(iv) as $\{-,-\}$ is trivial ${}^{\partial m}\ell = \ell$, so ∂M has trivial action on L.

Axioms (v) and (vi) vanish.

We leave the reader, if they so wish, to structure this into a formal proof that the 2-crossed module is precisely a 2-truncated crossed complex.

Our earlier discussion should suggest:

Proposition 56 *The category $Crs_{2]}$ of crossed complexes of length 2 is equivalent to the full subcategory of $2-CMod$ given by those 2-crossed complexes with trivial Peiffer lifting.* ■

We leave the proof of this to the reader.

A final comment is that in a 2-truncated simplicial group, G, one obviously has that it satisfies the thin filler condition (cf. page **??**) in dimensions greater than 2, since $NG_k = 1$ for all $k > 2$ and if the Peiffer lifting is trivial in the corresponding 2-crossed module, G satisfies it in dimensions 2 as well. (As D_1 is $s_0(G_0)$, any simplicial group satisfies the thin filler condition in dimension 1.)

In the next section we will give other examples of 2-crossed modules, those coming from crossed squares.

4.3.7 2-crossed modules and crossed squares

We now have several 'competing' models for homotopy 3-types. Since we can go from simplicial groups to both crossed square and 2-crossed modules, there should be some link between the latter two situations. In his work on homotopy n-types, Loday gave a construction of what he called a 'mapping cone' for a crossed square. Conduché later noticed that this naturally had the structure of a 2-crossed module. This is looked at in detail in a paper by Conduché, [67].

Suppose that

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & P \end{tikzcd}$$

is a crossed square, then its mapping cone complex is

$$L \overset{\partial_2}{\to} M \rtimes N \overset{\partial_1}{\to} P,$$

where $\partial_2 \ell = (\lambda\ell^{-1}, \lambda'\ell)$ and $\partial_1(m,n) = \mu(m)\nu(n)$.

We first note that the semi-direct product $M \rtimes N$ is formed by making N act on M *via* P, i.e.

$${}^{n}m = {}^{\nu(n)}m,$$

where the P-action is the given one. The fact that (λ^{-1}, λ') and $\mu\nu$ are homomorphisms is an interesting and instructive, but easy, exercise:

i) $(m,n)(m',n') = (m^{\nu(n)}m', nn')$, so

$$\begin{aligned} \partial_1((m,n)(m',n')) &= \mu(m^{\nu(n)}m').\nu(nn') \\ &= \mu(m)\nu(n)\mu(m')\nu(n)^{-1}\nu(n)\nu(n') \\ &= (\mu(m)\nu(n))(\mu(m')\nu(n')); \end{aligned}$$

(ii) if $\ell, \ell' \in L$, then, of course,

$$\begin{aligned} \partial_1(\ell\ell') &= (\lambda(\ell\ell')^{-1}, \lambda'(\ell\ell')) \\ &= (\lambda(\ell')^{-1}\lambda(\ell)^{-1}, \lambda'(\ell)\lambda'(\ell')). \end{aligned}$$

whilst

$$\begin{aligned}\partial_1(\ell)\partial_1(\ell') &= (\lambda(\ell)^{-1},\lambda'(\ell))(\lambda(\ell')^{-1},\lambda'(\ell'))\\ &= (\lambda(\ell)^{-1}.{}^{\nu\lambda'(\ell^{-1})}\lambda(\ell')^{-1},\lambda'(\ell\ell')),\end{aligned}$$

thus the second coordinates are the same, but, as $\nu\lambda' = \mu\lambda$, the first coordinates are also equal.

These elementary calculations are useful as they pave the way for the calculation of the Peiffer commutator of $x = (m, n)$ and $y = (c, a)$ in the above complex:

$$\begin{aligned}\langle x,y\rangle &= {}^{\partial x}y.xy^{-1}x^{-1}\\ &= {}^{\mu m.\nu n}(c,a).(m,n)({}^{a^{-1}}c^{-1},a^{-1})({}^{n^{-1}}m^{-1},n^{-1})\\ &= ({}^{\mu m\nu n}c,{}^{\mu m\nu n}a)(m^{\nu(na^{-1})}c^{-1}.{}^{\nu(na^{-1}n^{-1})}m^{-1},na^{-1}n^{-1}),\end{aligned}$$

which on multiplying out and simplifying is

$$({}^{\nu(na^{-1}n^{-1})}m.m^{-1},{}^{\mu m}(nan^{-1}).(na^{-1}n^{-1})).$$

(Note that any dependence on c vanishes!)

Conduché defined the Peiffer lifting in this situation by

$$\{x,y\} = h(m,nan^{-1}).$$

It is immediate to check that this works

$$\begin{aligned}\partial_2\{x,y\} &= (\lambda h(m,nan^{-1}),\lambda' h(m,nan^{-1}))\\ &= ({}^{\nu(na^{-1}n^{-1})}m.m^{-1},{}^{\mu m}(nan^{-1}).(na^{-1}n^{-1}),\end{aligned}$$

by the axioms of a crossed square.

We will not check all the axioms for a 2-crossed module for this structure, but will note the proofs for one or two of them as they illustrate the connection between the properties of the h-map and those of the Peiffer lifting.

2CM(iii) : $\{\partial\ell_0,\partial\ell_1\} = [\ell_1,\ell_0]$. As $\partial\ell = (\lambda\ell^{-1},\lambda'\ell)$, this needs the calculation of

$$h(\lambda\ell_0^{-1},\lambda'(\ell_0\ell_1\ell_0^{-1})),$$

but the crossed square axiom :

$$h(\lambda\ell,n) = \ell.{}^{n}\ell^{-1},\text{ and } h(m,\lambda'\ell) = {}^{m}\ell.\ell^{-1},$$

together with the fact that the map $\lambda : L \to M$ is a crossed module, give

$$\begin{aligned}h(\lambda\ell_0^{-1},\lambda'(\ell_0\ell_1\ell_0^{-1})) &= {}^{\mu\lambda(\ell_0^{-1}}(\ell_0\ell_1\ell_0^{-1}).\ell_0\ell_1^{-1}\ell_0^{-1})\\ &= [\ell_1,\ell_0].\end{aligned}$$

We need $\{(m,n),(\lambda\ell^{-1},\lambda'\ell)\}\{(\lambda\ell^{-1},\lambda'\ell),(m,n)\}$ to equal ${}^{\mu(m)\nu(n)}\ell.\ell^{-1}$, but evaluating the initial expression gives

$$\begin{aligned}h(m,n.\lambda'\ell.n^{-1})h(\lambda\ell^{-1},\lambda'\ell.n.\lambda'\ell^{-1}) &= h(m,\lambda'({}^{n}\ell))h(\lambda\ell^{-1},\lambda'\ell.n.\lambda'\ell^{-1})\\ &= {}^{\mu(m)\nu(n)}\ell.{}^{\nu(n)}\ell^{-1}.\ell^{-1}.{}^{\nu\lambda'(\ell).\nu(n).\nu\lambda'\ell^{-1}}\ell,\end{aligned}$$

and this does simplify as expected to give the correct results.

We thus have two ways of going from a simplicial group, G, to a 2-crossed module:
(a) directly to get

$$\frac{NG_2}{\partial NG_3} \to NG_1 \to NG_0;$$

(b) indirectly via $M(G,2)$ and then by the above construction to get

$$\frac{NG_2}{\partial NG_3} \to Ker\, d_0 \rtimes Ker\, d_1 \to G_1$$

and they clearly give the same homotopy type. More precisely G_1 decomposes as $Ker\, d_0 \rtimes s_0G_0$ and the $Ker\, d_0$ factor in the middle term of (b) maps down to that in this decomposition by the identity map, thus d_0 induces a quotient map from (b) to (a) with kernel isomorphic to

$$1 \to Ker\, d_0 \overset{=}{\to} Ker\, d_0,$$

which is acyclic/contractible.

4.3.8 2-crossed complexes

(These were not discussed in the lectures in Buenos Aires due to lack of time.) Crossed complexes are a useful extension of crossed modules allowing not only the encoding of an algebraic model for the 2-type, but also information on the 'chains on the universal cover', e.g. if G is a simplicial group, earlier, in section 2.5.1, we had $\mathsf{C}(G)$, the crossed complex constructed from the Moore complex of G, given by

$$C(G)_n = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})},$$

in higher dimensions and having at its 'bottom end' the crossed module,

$$\frac{NG_1}{d_0(NG_2 \cap D_2)} \to NG_0.$$

For a crossed complex, $\pi(X)$, coming from a CW-complex (as a filtered space, filtered by its skeleta), these groups in dimensions ≥ 3 coincide with the corresponding groups of the complex of chains on the universal cover of X. In general, the analogue of that chain complex can be extracted functorially from a general crossed complex; see [49] or [177]. The tail on a crossed complex allows extra dimensions, not available just with crossed modules, in which homotopies can be constructed. The category Crs is very much better structured than is $CMod$ itself and so 'adding a tail' would seem to be a 'good thing to do', so with 2-crossed modules, we can try and do something similar, adding a similar 'tail'.

We have an obvious normal chain complex of groups that ends

$$\ldots \to C(G)_3 \to \frac{NG_2}{d_0(NG_3 \cap D_3)} \to NG_1 \to NG_0.$$

Here there are more of the structural Peiffer pairings of the Moore complex NG that survive to the quotient, but it should be clear that, as they take values in the $NG_n \cap D_n$, in general these will again be almost all trivial if the receiving dimension, n, is greater than 2. For $n \leq 2$, these pairings are those that we have been using earlier in this chapter. The one exceptional case that is important here, as in the crossed complex case, is that which gives the action of NG_0 on $C_n(G)$ for $n \geq 3$, which, just as before, gives $C_n(G)$ the structure of a $\pi_0 G$-module. Abstracting from this gives the definition of a 2-crossed complex.

Definition: A *2-crossed complex* is a normal complex of groups

$$\ldots \to C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots \longrightarrow C_0,$$

together with a 2-crossed module structure given on $C_2 \to C_1 \to C_0$ by a Peiffer lifting function $\{-,-\} : C_1 \times C_1 \to C_2$, such that, on writing $\pi = Coker(C_1 \to C_0)$,

(i) each C_n, $n \geq 3$ and $Ker\,\partial_2$ are π-modules and the ∂_n for $n \geq 4$, together with the codomain restriction of ∂_3, are π-module homomorphisms;

(ii) the π-module structure on $Ker\,\partial_2$ is the action induced from the C_0-action on C_2 for which the action of $\partial_1 C_1$ is trivial.

A 2-crossed complex morphism is defined in the obvious way, being compatible with all the actions, the pairings and Peiffer liftings. We will denote by $2 - Crs$, the corresponding category.

There are reduced and unreduced versions of this definition. In the discussion and in the notation we use, we will quietly ignore the groupoid based non-reduced version, but it is easy to give simply by replacing simplicial groups by simplicially enriched groupoids, and making fairly obvious changes to the definitions.

Proposition 57 *The construction above defines a functor, $\mathsf{C}^{(2)}$, from Simp.Grps to $2 - Crs$.* ■

There are no prizes for guessing that the simplicial groups whose homotopy types are accurately encoded in $2 - Crs$ by this functor are those that satisfy the thin condition in dimensions greater than 3. In fact, the construction of the functor $\mathsf{C}^{(2)}$ explicitly kills off the intersection $NG_k \cap D_k$ for $k \geq 3$.

We have noted above that any 2-crossed module,

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N,$$

gives us a short crossed complex by dividing L by the subgroup $\{M, M\}$, the image of the Peiffer lifting. (We do not need this, but $\{M, M\}$ is easily checked to be a normal subgroup of L.) We also discussed those 2-crossed complexes that had trivial Peiffer lifting. They were just the length 2 crossed complexes. This allows one to show that crossed complexes form a reflexive subcategory of $2 - Crs$ and to give a simple description of the reflector:

Proposition 58 *There is an embedding*

$$Crs \to 2{-}Crs,$$

which has a left adjoint, L say, compatible with the functors defined from Simp.Grps to $2{-}Crs$ and to Crs, i.e. $\mathsf{C}(G) \cong \mathsf{L}\mathsf{C}^{(2)}(G)$. ■

4.4 Cat^n-groups and crossed n-cubes

4.4.1 Cat^2-groups and crossed squares

In the simplest examples of crossed squares, μ and μ' are normal subgroup inclusions and $L = M \cap N$, with h being the conjugation map. Moreover this type of example is almost 'generic' since, if

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & M \arrow[d] \\ N \arrow[r] & G \end{tikzcd}$$

is a simplicial crossed square constructed from a simplicial group, G, and two simplicial normal subgroups, M and N, then applying π_0, the square gives a crossed square and, up to isomorphism, all crossed squares arise in this way.

Although when first defined by D. Guin-Walery and J.-L. Loday, [109], the notion of crossed squares was not linked to that of cat^2-groups, it was in this form that Loday gave their generalisation to an n-fold structure, cat^n-groups (see [143] and below).

Definition: A *cat^1-group* is a triple, (G, s, t), where G is a group and s, t are endomorphisms of G satisfying conditions
(i) $st = t$ and $ts = s$.
(ii) $[Ker\, s,\ Ker\, t] = 1$.

A cat^1-group is a reformulation of an internal groupoid in *Grps*. (The interchange law is given by the $[Ker, Ker]$ condition; left for you to check) As these latter objects are equivalent to crossed modules, we expect to be able to go between cat^1-groups and crossed modules without hindrance, and we can:

Setting $M = Ker\, s$, $N = Im\, s$ and $\partial = t|M$, then the action of N on M by conjugation within G makes $\partial : M \to N$ into a crossed module. Conversely if $\partial : M \to N$ is a crossed module, then setting $G = M \rtimes N$ and letting s, t be defined by

$$s(m, n) = (1, n)$$

and

$$t(m, n) = (1, \partial(m)n)$$

for $m \in M$, $n \in N$, we have that (G, s, t) is a cat^1-group. Again this is one of those simple, but key calculations that are well worth doing yourself.

For a cat^2-group, we again have a group, G, but this time with two independent cat^1-group structures on it. Explicitly:

Definition: A *cat^2-group* is a 5-tuple (G, s_1, t_1, s_2, t_2), where (G, s_i, t_i), $i = 1, 2$, are cat^1-groups and

$$s_i s_j = s_j s_i, \quad t_i t_j = t_j t_i, \quad s_i t_j = t_j s_i$$

for $i, j = 1, 2, \quad i \neq j$.

There is an obvious notion of morphism between cat^2-groups and with this we obtain a category, $Cat^2(Grps)$.

Theorem 17 *[143] There is an equivalence of categories between the category of cat^2-groups and that of crossed squares.*

Proof: The cat^1-group (G, s_1, t_1) will give us a crossed module with $M = Ker\, s_1$, $N = Im\, s_1$, and $\partial = t|M$, but, as the two cat^1-group structures are independent, (G, s_2, t_2) restricts to give cat^1-group structures on both M and N and makes ∂ a morphism of cat^1-groups as is easily checked. We thus get a morphism of crossed modules

$$\begin{tikzcd}
Ker\, s_1 \cap Ker\, s_2 \arrow[r] \arrow[d] & Im\, s_1 \cap Ker\, s_2 \arrow[d] \\
Ker\, s_2 \cap Im\, s_1 \arrow[r] & Im\, s_1 \cap Im\, s_2,
\end{tikzcd}$$

where each morphism is a crossed module for the natural action, i.e., conjugation in G. It remains to produce an h-map, but this is given by the commutator within G, since, if $x \in Ker\, s_2 \cap Im\, s_1$ and $y \in Im\, s_2 \cap Ker\, s_1$, then $[x, y] \in Ker\, s_1 \cap Ker\, s_2$. It is easy to check the axioms for a crossed square. The converse is left as an exercise. ■

4.4.2 Interpretation of crossed squares and cat^2-groups

We have said that crossed squares and cat^2-groups give equivalent categories and we will see that, similarly, for the crossed n-cubes and cat^n-groups, which will be introduced shortly. The simplest case of that general situation is one that we have already already met namely that of crossed modules and cat^1-groups, and there we earlier saw how to interpret a crossed modules as being the essential data for a 2-group(oid).

We thus have, you may recall (combining ideas from pages 23 and 189), that a crossed module, (C, P, ∂), gives us a cat^1-group / 2-group, $(C \rtimes P, s, t)$, with $s(c, p) = p$ being the source of an element (c, p) and $t(c, p) = \partial c.p$ being its target. The definition of cat^2-group does not explicitly use the language of 'internal categories', we mentioned that the $[Ker\, s, Ker\, t] = 1$ condition is a version of the interchange law, and that a cat^{1}group can be interpreted as an internal category in *Grps*. This leads to pictures such as

$$p_1 \xrightarrow{(c_1,p_1)} \partial c_1.p_1,$$

(cf. section 1.3.2, page 23) indicating that (c, p) interprets as an arrow having source and target as indicated. We could equally well use the 2-category or 2-group(oid) style diagram:

$$\begin{tikzcd}
\cdot \arrow[r, bend left=50, "p"{name=U}] \arrow[r, bend right=50, "\partial c.p"'{name=D}] & \cdot
\arrow[from=U, to=D, phantom, "\Downarrow (c,p)"]
\end{tikzcd}$$

as we discussed earlier in section 1.3.3.

If we start with a cat^1-group, (G, s, t), then the picture is

$$s(g) \xrightarrow{g} t(g).$$

It thus looks that the source and target are 'objects' of the category structure that we know to be there. Where do they live? Clearly in $Im\, s$ or $Im\, t$, or both. Life is easy on us however. We note that $Im\, s = Im\, t$, since $st = t$ implies that $Im\, t \subseteq Im\, s$, whilst we also have $ts = s$, giving the other inclusion. The subgroup $Im\, s$, corresponds to the group P of the crossed module, considered as a subgroup of the 'big group' $C \rtimes P$.

It is sometimes more convenient to write an internal category in the form

$$\begin{tikzcd} G_1 \arrow[r, shift left=2, "\sigma"] \arrow[r, "\tau"'] & G_0 \arrow[l, shift left=2, "\iota"] \end{tikzcd},$$

so that G_1 is an object of arrows and G_0 the object of objects, in our case, the 'group of objects'. The cat^1-group notation replaces the source, target and identity maps by the composites $s = \iota\sigma$ and $t = \iota\tau$. This, of course, gives endomorphisms of G_1, which are simpler to handle than having a 'many sorted' picture with two separate groups. The downside of that simplicity is that the object of objects is slightly hidden. Of course, it is this subgroup, $Im\, s$, and the inclusion of that subgroup into $G = G_1$ is the morphism denoted ι. It is therefore reasonable to draw the 'objects' as blobs or points rather than as elements of G, e.g., as loops on the single real object of the group thought of as a single object groupoid. The resulting pictures *are* easier to draw! and to interprete.

A cat^2-group is similarly a category-like structure, internal to cat^1-groups, so is a double category internal to the category of groups, as the two category structures are independent of each other. This is emphasised if we look at the elements of a cat^2-group in an analogous way to the above. First suppose that (G, s_1, t_1, s_2, t_2) is a cat^2-group, then we might draw, for each $g \in G$, a square diagram:

$$\begin{tikzcd} \cdot \arrow[r, "t_2 g"] & \cdot \\ \cdot \arrow[u, "s_1 g"] \arrow[r, "s_2 g"'] \arrow[ur, phantom, "g"] & \cdot \arrow[u, "t_1 g"'] \end{tikzcd}$$

Now the left vertical arrow is in the subgroup, $Im\, s_1 = Im\, t_1$. (We can refer to $s_1 g$ as the 1-source, and $t_1 g$ as the 1-target, of g, and similarly for 2-source, and so on.) The square is a schema consistent the the equations: $s_1 t_2 = t_2 s_1$, and the three other similar ones. The element $s_1 t_2 g$ is the 1-source of the 2-target of g, so is the vertex at the top left of the square. It is also the 2-target of the 1-source of g, of course.

Such squares compose horizontally and vertically, provided the relevant sources and targets match, but how does this relate to the group structure on G?

Looking back, once more, to a cat^1-group, (G, s, t) and a resulting composition

$$s(g) \xrightarrow{g} t(g) = s(g') \xrightarrow{g'} t(g'),$$

it is not immediately clear how the composite is to be studied, but look back to the corresponding crossed module based description and it becomes clearer. We had in section 1.3.2,

$$p \xrightarrow{(c,p)} \partial c.p \xrightarrow{(c\prime,\partial c.p)} \partial c'\partial c.p,$$

and the composition was given as $(c', \partial c.p) \star (c, p) = (c'c, p)$. Back in cat^1-group language, this corresponds to $g' \star g = g's(g')^{-1}g$. (We can check that $s(g's(g')^{-1}g) = s(g)$ and that $t(g's(g')^{-1}g) = t(g')$, as we would expect.)

We can extend this to cat^2-groups giving a way of composing the squares that we have in this context. For instance, for horizontal composition, we have

$$\begin{tikzcd} \cdot \arrow[r] & \cdot \arrow[r] & \cdot \\ \cdot \arrow[r] \arrow[u] \arrow[ur, phantom, "g"] & \cdot \arrow[r] \arrow[u] \arrow[ur, phantom, "g'"] & \cdot \arrow[u] \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r] & \cdot \\ \cdot \arrow[r] \arrow[u] \arrow[ur, phantom, "g' s_1(g')^{-1} g"] & \cdot \arrow[u] \end{tikzcd}$$

and similarly for vertical composition, replacing s_1 by s_2.

That gives a double category interpretation for a cat^2-group, but how does this relate to a crossed square,

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & P \end{tikzcd}$$

with h-map $h : M \times N \to L$. The construction hinted at earlier is first to form the cat^1-groups of the two vertical crossed modules, giving

$$\partial : L \rtimes N \to M \rtimes P, \text{ with } \partial(\ell, n) = (\lambda(\ell), \nu(n)),$$

with ∂ the induced map. There is an action of $M \rtimes P$ on $L \rtimes N$ (which will be examined shortly) giving a crossed module structure to the result. This action is non-trivial to define (or discover), so here is a way of thinking of it that may help.

We 'know' that a crossed square is meant to be a crossed module of crossed modules, so, if the above ∂ and action does give a crossed module, we will then be able to form a 'big group', $(L \rtimes N) \rtimes (M \rtimes P)$, with a cat^2-group structure on it. The action of $M \rtimes P$ on $L \rtimes N$ will need to correspond to conjugation within this 'big group' as the idea of semi-direct products is, amongst other things, to realise an action: if G acts on H, $H \rtimes G$ has multiplication given by $(h_1, g_1)(h_2, g_2) = (h_1 {}^{g_1}h_2, g_1g_2)$. In particular, it is easy to **work out**

$$(h, g)^{-1} = ({}^{g^{-1}}h^{-1}, g^{-1}),$$

so

$$(1, g)(h, 1)(1, g)^{-1} = ({}^g h, 1).$$

In our situation, we thus can work out the conjugation,

$$((1,1),(m,p))((\ell,n),(1,1))((1,1),({}^{p^{-1}}m^{-1}, p^{-1})) = ({}^{(m,p)}(\ell, n), (1,1)).$$

Now this looks as if we are getting nowhere, but let us remember that any crossed square is isomorphic to the π_0 of an 'inclusion crossed square' of simplicial groups, (this was mentioned on page 177). This suggests that we first look at a group G, and a pair of normal subgroups M, N, and the inclusion crossed square

$$\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & N \arrow[d] \\ M \arrow[r] & G \end{tikzcd}$$

with $h(m,n) = [m,n]$. If we track the above discussion of the action and the definition of ∂ in this example, we get the induced map, ∂, is the inclusion of $(M \cap N) \rtimes N$ into $M \rtimes G$. Here, therefore, there is, '*gratis*', an action of $M \rtimes G$ on $(M \cap N) \rtimes N$, namely by inner automorphisms / conjugation:

$$\begin{aligned}(m,g)(\ell,n)({}^{g^{-1}}m^{-1}, g^{-1})) &= (m,g)(\ell.n.{}^{g^{-1}}m.n^{-1}, ng)\\ &= (m.{}^{g}\ell.{}^{g}n.m.{}^{g}n^{-1}, gmg^{-1}),\end{aligned}$$

which can conveniently be written

$$({}^{mg}\ell.[m, {}^{g}n], {}^{g}n).$$

This suggests a formula for an action in the general case

$$\begin{aligned}{}^{(m,p)}(\ell,n) &= {}^{m}({}^{p}\ell, {}^{p}n)\\ &= ({}^{\mu(m)p}\ell.h(m, {}^{p}n), {}^{p}n).\end{aligned}$$

If we start with a simplicial inclusion crossed square, and form its 'big simplicial group' simplicially using the previous formula, then this *will* give the action of $M \rtimes P$ on $L \rtimes N$ in the general case, so our guess looks as if it is correct. Note that in both the particular case of the inclusion crossed square and this general case, we can derive $h(m,n)$ as a commutator within the 'big group'. (Of course, for the first of these, the h-map was defined as a commutator within G.)

We could go on to play around with other facets of this construction. This would be **well worthwhile** - but is better **left to the reader**. For instance, one obvious query is that $(L \rtimes N) \rtimes (M \rtimes P)$ should not be dependent on thinking of a crossed square as a morphism of (vertical) crossed modules. It is also a morphism of horizontal crossed modules, so this 'big group', if it is to give a useful object, should be isomorphic to $(L \rtimes M) \rtimes (N \rtimes P)$. It is, but what is a specific natural isomorphism doing the job. As somehow M has to 'pass through' N, we should expect to have to use the h-map.

There are other 'games to play'. Central extensions gave an instance of crossed modules, so what is their analogue for crossed squares. Double central extensions have been introduced by Janelidze in [129] and have been further studied by others, [95, 106, 190]. They provide a related idea. It is **left to you** to explore any connections that there are.

If we start with a crossed square, as above, what is the analogue of the picture

$$p_1 \xrightarrow{(c_1,p_1)} \partial c_1.p_1,$$

representing an element of the 'big group' of a crossed module. Suppose (ℓ, n, m, p) is such an element, then it is easy to see the 2-cell that corresponds to it must be:

$$\begin{tikzcd}
\nu(n)p \arrow[r, "\lambda(\ell).{}^{\nu(n)}m{,}\nu(n)p)"] & \mu(\lambda(\ell))\nu(n)\mu(m)p \\
p \arrow[u, "(n{,}p)"] \arrow[r, "(m{,}p)"'] \arrow[ur, phantom, "(\ell{,}n{,}m{,}p)"] & \mu(m)p \arrow[u, "\lambda'(\ell)n{,}\mu(m)p)"']
\end{tikzcd}$$

The details of how to compose, etc. are again **left to you**. It is, however, worth just checking the way in which the two edges on the top and on the right do match up. The right hand edge will clearly end at $\nu(\lambda'(\ell))\nu(n)\mu(m)p$, which, as $\nu\lambda' = \mu\lambda$, gives the expression on the top right vertex. Of more fun is the top edge. This ends at

$$\mu(\lambda(\ell)).\mu({}^{\nu(n)}m).\nu(n).p = \mu(\lambda(\ell)).\nu(n)\mu(m)\nu(n)^{-1}\nu(n)p,$$

so is as required, using the fact that μ is a crossed module.

In such a square 2-cell, the square itself is in the 'big group', the edges are in the cat^1-groups corresponding to vertical and horizontal crossed modules of the crossed square, and the vertices are in P.

Particularly interesting is the case of two crossed modules, $\mu : M \to P$ and $\nu : N \to P$, together with the corresponding $L = M \otimes N$, the Brown-Loday tensor product of the two, (cf. [51, 52]). Approximately, $M \otimes N$ is the universal codomain for an h-map based on the two given sides of the resulting crossed square. (A treatment of this construction has been included in the notes, [177], - please ignore the profinite conditions if using it 'discretely'.)

4.4.3 Cat^n-groups and crossed n-cubes, the general case

Of the two notions named in the title of this section, the first is easier to define.

Definition: A *cat^n-group* is a group G together with $2n$ endomorphisms $s_i, t_i, (1 \leq i \leq n)$ such that

$$s_i t_i = t_i, \text{ and } t_i s_i = s_i \text{ for all } i,$$

$$s_i s_j = s_j s_i, \quad t_i t_j = t_j t_i, \quad s_i t_j = t_j s_i \text{ for } i \neq j$$

and, for all i,

$$[Ker\, s_i, Ker\, t_i] = 1.$$

A cat^n-group is thus a group with n independent cat^1-group structures on it.

As a cat^1-group can also be reformulated as an internal groupoid in the category of groups, a cat^n-group, not surprisingly, leads to an internal n-fold groupoid in the same setting.

The definition of crossed n-cube as an n-fold crossed module was initially suggested by Ellis in his thesis. The only problem was to determine the sense in which one crossed module should act on another. Since the number of axioms controlling the structure increased from crossed modules to crossed squares, one might fear that the number and complexity of the axioms would increase drastically in passing to higher 'dimensions'. The formulation that resulted from the joint work, [94], of Ellis and Steiner showed how that could be avoided by encoding the actions and the h-maps in the same structure.

We write $\langle n \rangle$ for the set $\{1, \ldots, n\}$.

Definition: A *crossed n-cube*, M, is a family of groups, $\{M_A : A \subseteq \langle n \rangle\}$, together with homomorphisms, $\mu_i : M_A \to M_{A-\{i\}}$, for $i \in \langle n \rangle, A \subseteq \langle n \rangle$, and functions, $h : M_A \times M_B \to M_{A\cup B}$, for $A, B \subseteq \langle n \rangle$, such that if ${}^a b$ denotes $h(a, b)b$ for $a \in M_A$ and $b \in M_B$ with $A \subseteq B$, then for $a, a' \in M_A$, $b, b' \in M_B$, $c \in M_C$ and $i, j \in \langle n \rangle$, the following axioms hold:

(1) $\mu_i a = a$ if $a \notin A$
(2) $\mu_i \mu_j a = \mu_j \mu_i a$
(3) $\mu_i h(a,b) = h(\mu_i a, \mu_i b)$
(4) $h(a,b) = h(\mu_i a, b) = h(a, \mu_i b)$ if $i \in A \cap B$
(5) $h(a,a') = [a,a']$
(6) $h(a,b) = h(b,a)^{-1}$
(7) $h(a,b) = 1$ if $a = 1$ or $b = 1$
(8) $h(aa',b) = {}^a h(a',b) h(a,b)$
(9) $h(a,bb') = h(a,b)^b h(a,b')$
(10) ${}^a h(h(a^{-1},b),c)^c h(h(c^{-1},a),b)^b h(h(b^{-1},c),a) = 1$
(11) ${}^a h(b,c) = h({}^a b, {}^a c)$ if $A \subseteq B \cap C$.

A morphism of crossed n-cubes

$$\{M_A\} \to \{M'_A\}$$

is a family of homomorphisms, $\{f_A : M_A \to M'_A \mid A \subseteq \langle n \rangle\}$, which commute with the maps, μ_i, and the functions, h. This gives us a category, Crs^n, equivalent to that of cat^n-groups.

Remarks: 1. In the correspondence between cat^n-groups and crossed n-cubes (see Ellis and Steiner, [94]), the cat^n-group corresponding to a crossed n-cube, (M_A), is constructed as a repeated semidirect product of the various M_A. Within the resulting "big group", the h-functions interpret as being commutators. This partially explains the structure of the h-function axioms.

2. For $n = 1$, these eleven axioms reduce to the usual crossed module axioms. For $n = 2$, they give a crossed square:

$$\begin{tikzcd} M_{\langle 2 \rangle} \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] & M_{\{1\}} \arrow[d, "\mu_1"] \\ M_{\{2\}} \arrow[r, "\mu_2"'] & M_{\emptyset} \end{tikzcd} ,$$

with the h-map, that was previously specified, being $h : M_{\{1\}} \times M_{\{2\}} \to M_{\langle 2 \rangle}$. The other h-maps in the above definition correspond to the various actions as explained in the definition itself.

Theorem 18 *[94] There are equivalences of categories*

$$Crs^n \simeq Cat^n(Grps),$$

■

4.5 Loday's Theorem and its extensions

In 1982, Loday proved a generalisation of the MacLane-Whitehead result that stated that connected homotopy 2-types (they called them 3-types) were modelled by crossed modules. The extension used cat^n-groups, and, as cat^1-groups 'are' crossed modules, we should expect cat^n-groups to model connected $(n+1)$-types (if the MacLane-Whitehead result is to be the $n = 1$ case, see page 177).

We have mentioned that 'simplicial groupoids' model all homotopy types and had a construction of both a crossed module $M(G,1)$ and a crossed square, $M(G,2)$ from a simplicial group, G. These are the $n = 1$ and $n = 2$ cases of a general construction of a crossed n-cube from G that we will give in a moment First we note a rather neat result.

We saw early on in these notes, (Lemma 1, page 14), that if $\partial : C \to P$ was a crossed module, then $\partial C \lhd P$, i.e. is a normal subgroup of P. A crossed square

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\mu'"'] & P \end{tikzcd}$$

can be thought of as a (horizontal or vertical,) crossed module of crossed modules:

$$\begin{array}{c} L \\ \downarrow \\ N \end{array} \longrightarrow \begin{array}{c} M \\ \downarrow \\ P \end{array}$$

(λ, ν) gives such a crossed module with domain (L, N, λ') and codomain (M, P, μ) and so on. (Working out the precise meaning of 'crossed module of crossed modules' and, in particular, what it should mean to have an action of one crossed module on another, is a very useful exercise; try it!) The image of (λ, ν) is a normal sub-crossed module of (M, P, μ), so we can form a quotient

$$\overline{\mu} : M/\lambda L \to P/\nu N,$$

and this is a crossed module. (This is not hard to check. There are lots of different ways of checking it, but perhaps the best way is just to show how $P/\nu N$ acts on $M/\lambda L$, in an obvious way, and then to check the induced map, $\overline{\mu}$, has the right properties - just by checking them. This gives one a feeling for how the various parts of the definition of a crossed square are used here.)

Another result from near the start of these notes, (Lemma 2), is that $Ker\,\partial$ is a central subgroup of C and ∂C acts trivially on it, so $Ker\,\partial$ has a natural $P/\partial C$-module structure. Is there an analogue of this for a crossed square? Of course, referring again to our crossed square, above, the kernel of (λ, ν) would be $\lambda' : Ker\,\lambda \to Ker\,\nu$ (omitting any indication of restriction of λ' for convenience). Both $Ker\,\lambda$ and $Ker\,\nu$ are Abelian, as they themselves are kernels of crossed modules, so $Ker\,\lambda$ is a $M/\lambda L$-module and $Ker\,\nu$ is a $P/\nu N$-module. (It is left to the diligent reader to work out the detailed structure here and to explore crossed modules that are modules over other ones.)

We had, for a given simplicial group, G, the crossed square

$$\begin{tikzcd} \dfrac{NG_2}{d_0(NG_3)} \arrow[r] \arrow[d] & Ker\,d_1 \arrow[d] \\ Kerd_2 \arrow[r] & G_1 \end{tikzcd}$$

which was denoted $M(G, 2)$. (The top horizontal and left vertical maps are induced by d_0.) Let us examine the horizontal quotient and kernel.

First the quotient, this has NG_1/d_0NG_2 as its 'top' group and $G_1/Ker\,d_0 \cong G_0$, as its bottom one. Checking all the induced maps shows quite quickly that the quotient crossed module is $M(G, 1)$, up to isomorphism.

What about the kernel? Well, the bottom horizontal map is an inclusion, so has trivial kernel, whilst the top is induced by d_0, and so the kernel here can be calculated to be $Ker\, d_0 \cap NG_2$, divided by $d_0(NG_3)$, but that is $Ker\,\partial / Im\,\partial$ in the Moore complex, so is $H_2(NG)$ and thus is $\pi_2(G)$. We thus have, from previous calculations, that for $M(G,1)$, there is a crossed 2-fold extension

$$\pi_1(G) \to \frac{NG_1}{\partial NG_2} \to NG_0 \to \pi_0(G)$$

and for $M(G,2)$, a similar object, a crossed 2-fold extension of crossed modules:

$$\begin{tikzcd}
1 \arrow[r] & \pi_2(G) \arrow[r] \arrow[d] & Ker\, d_1 \arrow[r] \arrow[d] & NG_2/d_0(NG_3) \arrow[r] \arrow[d] & Ker\, d_1 NG_1/d_0(NG_2) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & 1 \arrow[r] & Ker\, d_0 \arrow[r] & G_1 \arrow[r] & G_0 \arrow[r] & 1
\end{tikzcd}$$

'Obviously' this should give an element of '$H^3(M(G,2),(\pi_2(G)\to 1))$', but we have not given any description of what that cohomology group should be. It can be done, but we will not go in that direction for the moment. Rather we will use the route via simplicial groups.

4.5.1 Simplicial groups and crossed n-cubes, the main ideas

We have that simplicial groups yield crossed squares by the $M(G,2)$ construction, and that, from $M(G,2)$, we can calculate $\pi_0(G)$, $\pi_1(G)$, and $\pi_2(G)$. If G represents a 3-type of a space (or the 2-type of a simplicial group), then we would expect these homotopy groups to be the only non-trivial ones. (Any simplicial group can be truncated to give one with these π_i as the only non-trivial ones.) This suggests that going from 3-types to crossed squares in a nice way should be just a question of combining the functorial constructions

$$\mathsf{Spaces} \xrightarrow{Sing} \mathsf{Simplicial\ Sets}$$

$$\mathsf{Simplicial\ Sets} \xrightarrow{G(\ )} \mathcal{S}\!-\!\mathsf{Groupoids}$$

$$\mathcal{S}\!-\!\mathsf{Groupoids} \xrightarrow{M(\ ,2)} \mathsf{Crossed\ squares}.$$

Of course, we would need to see if, for $f : X \to Y$ a 3-equivalence (so f induces isomorphisms on π_i for $i = 0,1,2,3$), what would be the relationship between the corresponding crossed squares. We would also need to know that each crossed square was in sense 'equivalent' to one of the form $M(G,2)$ for some G constructed from it, in other words to reverse, in part, the last construction. (The other constructions have well known inverses at the homotopy level.)

We will use a 'multinerve' construction, generalising the nerve that we have already met. We will denote this by $E^{(n)}(\mathsf{M})$ for M a crossed n-cube.

For $n = 1$, $E^{(1)}$ is just the nerve of the crossed module, so if $\mathsf{M} = (C,P,\partial)$, we have $E^{(1)}(\mathsf{M}) = K(\mathsf{M})$ as given already on page 30.

For $n = 2$, i.e., for a crossed square, M, we form the 'double nerve ' of the associated cat^2-group of M. From M, we first form the 'crossed module of cat^1-groups'

$$L \rtimes N \xrightarrow{(\lambda,\nu)} M \rtimes P,$$

where, for instance, in $M \rtimes P$ the source endomorphism is $s(m,p) = (1,p)$ and the target is $t(m,p) = (1, \partial m.p)$. (We could repeat in the horizontal direction to form $(L \rtimes N) \rtimes (M \rtimes P)$, which is the 'big group' of the cat^2-group associated to M, but, in fact, will not do this except implicitly, as it is easier to form a simplicial crossed module in this situation. This,

$$E^{(1)}(L \xrightarrow{\lambda'} N) \longrightarrow E^{(1)}(M \xrightarrow{\mu} P),$$

is obtained by applying the $E^{(1)}$ construction to the vertical crossed modules. The two parts are linked by a morphism of simplicial groups induced from (λ, ν) and which is compatible with the action of the right hand simplicial group on the left hand one. (This action is not that obvious to write down - unless you have already done the previously suggested 'exercises'. It uses the h-maps from $M \times N$ to L, etc. in an essential way, and is, in some ways, best viewed within $(L \rtimes N) \rtimes (M \rtimes P)$ as being derived from conjugation. Details are, for instance, in Porter, [177] or [178] as well as in the discussion of the equivalence between cat^n-groups and crossed n-cubes in the original, [94].)

With this simplicial crossed module, we apply the nerve in the second horizontal direction to get a bisimplicial group, $\mathcal{E}^{(2)}(\mathsf{M})$. (Of course, if we started with a crossed n-cube, we could repeat the application of the nerve functor n-times, one in each direction to get an n-simplicial group $\mathcal{E}^{(n)}(\mathsf{M})$.)

There are two ways of getting from a bisimplicial set or group to a simplicial one. One is the diagonal, so if $\{G_{p,q}\}$ is a bisimplicial group, $\mathsf{diag}(G_{\bullet,\bullet})_n = G_{n,n}$ with fairly obvious face and degeneracy maps. The other is the *codiagonal* (also sometimes called the 'bar construction'). This was introduced by Artin and Mazur, [9]. It picks up related terms in the various $G_{p,q}$ for $p+q=n$. (An example is for any simplicial group, G, on taking the nerve in each dimension. You get a bisimplicial set whose codiagonal is $\overline{W}(G)$, with the formula given later in these notes.) We will consider the codiagonal in some detail later on, (starting on page **??**). The two constructions give homotopically equivalent simplicial groups. Proofs of this can be found in several places in the literature, for instance, in the paper by Cegarra and Remedios, [64]. Here we will set $E^{(n)}(\mathsf{M}) = \mathsf{diag}\mathcal{E}^{(n)}(\mathsf{M})$.

At this stage, for the reader trying to understand what is going on here, it is worth calculating the Moore complex of these simplicial groups. This is technically quite tricky as it is easy to make a slip, but it is not hard to see that they are 'closely related' to the 2-crossed module / mapping cone complex:

$$L \to M \rtimes N \to P$$

that we met earlier, (page 185), that is due to Loday and Conduché, see [67]. Of course, such detailed calculations are much harder to generalise to crossed n-cubes and other techniques are used, see [178] or the alternative version based on the technology of cat^n-groups due to Bullejos, Cegarra and Duskin, [57].

In any of these approaches from a crossed n-cube or cat^n-group, you either extract a n-simplicial group and then a simplicial group, by diagonal or codiagonal, or going one stage further, applying the nerve functor to the n-simplicial group to get a $(n+1)$-simplicial set, which is then 'attacked' using the diagonal or codiagonal functors to get out a simplicial set. This end result is the simplicial model for the crossed n-cube and has the same homotopy groups as M. It is known as the *classifying space of the crossed n-cube or cat^n-group.* (That term is usual, but it actually gives rise to an interesting obvious question, which has a simple answer in some ways but not if one looks at it thoroughly. That question is : *what does this classifying space classify?* That question will to some extent return to haunt us later one. The simple answer would be certain types of simplicial fibre bundles with fibre a $n+1$-type, but that throws away all the hard work to get the crossed n-cube itself, so ... .

Returning to the simplicial group approach, one applies the $M(-,n)$-functor, that we have so far seen only for $n = 1$ and 2, to get back a new crossed n-cube. This is not M itself in general, but is 'quasi-isomorphic' to it.

Definition: A morphism, $f : \mathsf{M} \to \mathsf{N}$, of crossed n-cubes will be called a *trivial epimorphism* if $\mathcal{E}^{(n)}(f) : \mathcal{E}^{(n)}(\mathsf{M}) \to \mathcal{E}^{(n)}(\mathsf{N})$ is an epimorphism (and thus a fibration) of simplicial groups having contractible kernel.

Starting with the category, Crs^n, of crossed n-cubes, inverting the trivial epimorphisms gives a category, $Ho(Crs^n)$, and f will be called a *quasi-isomorphism* if it gives an isomorphism in this category.

Remark: Any trivial epimorphism of crossed modules is a *weak equivalence* in the sense of section 2.1, page 34. This follows from the long exact fibration sequence. Conversely any such weak equivalence is a quasi-isomorphism.

We can now state Loday's result in the form given in [178]:

Theorem 19 *The functor*

$$M(-,n) : Simp.Grps \to Crs^n$$

induces an equivalence of categories

$$Ho_n(Simp.Grps) \xrightarrow{\simeq} Ho(Crs^n).$$

■

As yet we have not actually given the definition of $M(G,n)$ for $n > 2$ so here it is:

Definition Given a simplicial group, G, the crossed n-cube, $M(G,n)$, is given by:
(a) for $A \subseteq \langle n \rangle$,

$$M(G,n)_A = \frac{\bigcap\{Ker\, d_j^n \,:\, j \in A\}}{d_0(Ker\, d_1^{n+1} \cap \bigcap\{Ker\, d_{j+1}^{n+1} \,:\, j \in A\})};$$

(b) if $i \in \langle n \rangle$, the homomorphism $\mu_i : M(G,n)_A \to M(G,n)_{A\setminus\{i\}}$ is induced from the inclusion of $\bigcap\{Ker\, d_j^n \,:\, j \in A\}$ into $\bigcap\{Ker\, d_j^n \,:\, j \in A \setminus \{i\}\}$;
(c) representing an element in $M(G,n)_A$ by $\overline{x}$, where $x \in \bigcap\{Ker\, d_j^n \,:\, j \in A\}$, (so the overbar denotes a coset), and, for $A, B \subseteq \langle n \rangle$, $\overline{x} \in M(G,n)_A$, $\overline{y} \in M(G,n)_B$,

$$h(\overline{x},\overline{y}) = \overline{[x,y]} \in M(G,n)_{A\cup B}.$$

Where this definition 'comes from' and why it works is a bit to lengthy to include here, so we refer the interested reader to [177]. From its many properties, we will mention just the following one, linking $M(G,n)$ with $M(G,n-1)$ in a similar way to that we have examined for $n=2$.

We will use the following notation: $M(G,n)_1$ will denote the crossed $(n-1)$-cube obtained by restricting to those $A \subseteq \langle n\rangle$ with $1 \in A$ and $M(G,n)_0$ that obtained from the terms with $A \subseteq \langle n\rangle$ with $1 \notin A$.

Proposition 59 *Given a simplicial group G and $n \geq 1$, there is an exact sequence of crossed $(n-1)$-cubes:*

$$1 \to K \to M(G,n)_1 \xrightarrow{\mu_1} M(G,n)_0 \to M(G,n-1) \to 1,$$

where, if $B \subseteq \langle n-1\rangle$ and $B \neq \langle n-1\rangle$, then $K_B = \{1\}$, whilst $K_{\langle n-1\rangle} \cong \pi_n(G)$. ■

There are some special cases of crossed n-cubes, or the associated cat^n-groups that are worth looking at. For instance in [176], Paoli gives a new perspective on cat^n groups. It identifies a full subcategory of them (which are called *weakly globular*) which is sufficient to model connected $n+1$-types, but which has much better homotopical properties than the general ones. This, in fact, gives a more transparent algebraic description of the Postnikov decomposition and of the homotopy groups of the classifying space, and it also gives a kind of minimality property. Using weakly globular cat^n groups one can also describe a comparison functor to the Tamsamani model of $n+1$-types (cf. Tamsamani, [203]) which preserves the homotopy type.

4.5.2 Squared complexes

We have met crossed squares and 2-crossed modules and the different ways they encode the homotopy 3-type. We have extended 2-crossed modules to 2-crossed complexes, so it is natural curiosity to try to extend crossed squares to a 'cube' formulation. We will see this is just the start of another hierarchy which is in some ways simpler than that suggested by the hypercrossed complexes, and their variants, etc. The first step is the following which was introduced by Ellis, [92].

Definition: A *squared complex* consists of a diagram of group homomorphisms

$$\begin{tikzcd} & & & & N \arrow[dr, "\mu"] & \\ \ldots \arrow[r] & C_4 \arrow[r, "\partial_4"] & C_3 \arrow[r, "\partial_3"] & L \arrow[ur, "\lambda'"] \arrow[dr, "\lambda"'] & & P \\ & & & & M \arrow[ur, "\mu'"'] & \end{tikzcd}$$

together with actions of P on L, N, M and C_i for $i \geq 3$, and a function $h : M \times N \longrightarrow L$. The following axioms need to be satisfied.

(i) The square $\left(\begin{array}{ccc} L & \xrightarrow{\lambda} & N \\ {\scriptstyle\lambda'}\downarrow & & \downarrow{\scriptstyle\mu} \\ M & \xrightarrow[\mu']{} & P \end{array}\right)$ is a crossed square;

(ii) The group C_n is Abelian for $n \geq 3$

(iii) The boundary homomorphisms satisfy $\partial_n\partial_{n+1} = 1$ for $n \geq 3$, and $\partial_3(C_3)$ lies in the intersection $Ker\,\lambda \cap Ker\,\lambda'$;

(iv) The action of P on C_n for $n \geq 3$ is such that μM and $\mu' N$ act trivially. Thus each C_n is a π_0-module with $\pi_0 = P/\mu M \mu' N$.
(v) The homomorphisms ∂_n are π_0-module homomorphisms for $n \geq 3$.

This last condition does make sense since the axioms for crossed squares imply that $Ker\,\mu' \cap Ker\mu$ is a π_0-module.

Definition: A *morphism of squared complexes*,

$$\Phi : \left(C_*, \left(\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & N \arrow[d, "\mu"] \\ M \arrow[r, "\mu'"'] & P \end{tikzcd}\right)\right) \longrightarrow \left(C'_*, \left(\begin{tikzcd} L' \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & N' \arrow[d, "\mu"] \\ M' \arrow[r, "\mu'"'] & P' \end{tikzcd}\right)\right)$$

consists of a morphism of crossed squares $(\Phi_L, \Phi_N, \Phi_M, \Phi_P)$, together with a family of equivariant homomorphisms Φ_n for $n \geq 3$ satisfying $\Phi_L \partial_3 = \partial'_3 \Phi_L$ and $\Phi_{n-1}\partial_n = \partial'_n \Phi_n$ for $n \geq 4$. There is clearly a category $SqComp$ of squared complexes.

A squared complex is thus a crossed square with a 'tail' attached.

Any simplicial group will give us such a gadget by taking the crossed square to be $M(sk_2 G, 2)$, that is,

$$\begin{tikzcd} \dfrac{NG_2}{d_0(NG_3 \cap D_3)} \arrow[r] \arrow[d] & Ker\, d_1 \arrow[d] \\ Kerd_2 \arrow[r] & G_1 \end{tikzcd}$$

and then, for $n \geq 3$,

$$C_n(G) = \frac{NG_n}{(NG_n \cap D_n) d_0(NG_{n+1} \cap D_{n+1})}.$$

The above complex contains not only the information for the crossed square $M(G, 2)$ that represents the 3-type, but also the whole of $\mathsf{C}^{(2)}(G)$, the 2-crossed complex of G and thus the crossed complex and the 'chains on the universal cover' of G.

The advantage of working with crossed squares or squared complexes rather than the more linearly displayed models is that they can more easily encode 'non-symmetric' information. We will show this in low dimensions here but will later indicate how to extend it to higher ones. For instance, one gets a building process for homotopy types that reflects more the algebra. In examples, given two crossed modules, $\mu : M \to P$ and $\nu : N \to P$, there is a universal crossed square defining a 'tensor product' of the two crossed modules. We have

$$\begin{tikzcd} M \otimes N \arrow[r, "\lambda"] \arrow[d, "\lambda'"'] & M \arrow[d, "\mu"] \\ N \arrow[r, "\nu"'] & P \end{tikzcd}$$

is a crossed square and hence represents a 3-type. It is universal with regard to crossed squares having the same right-hand and bottom crossed modules, (see [51, 52] for the original theory and [177] for its connections with other material).

Equivalently we could represent its 3-type as a 2-crossed module

$$M \otimes N \longrightarrow M \rtimes N \xrightarrow{\mu\nu} P$$

or

$$M \otimes N \longrightarrow \frac{(M \rtimes N)}{\sim} \longrightarrow \frac{P}{\mu M},$$

where $\sim$ corresponds to dividing out by the μM action. However, of these, the crossed square lays out the information in a clearer format and so can often have some advantages.

4.6 Crossed $\mathbb{N}$-cubes

4.6.1 Just replace n by $\mathbb{N}$?

We have already suggested (page 180) how one might model all homotopy types using hypercrossed complexes, i.e. by adding more of the potential structure to the Moore complex of a simplicial group. We also saw how crossed modules (which are, from this viewpoint, 1-truncated hypercrossed complexes) generalised to crossed complexes, which have a better structured homotopical and homological algebra. We have seen earlier the transition from 2-crossed modules (= 2-truncated hypercrossed complexes) to 2-crossed complexes and briefly in the previous section, how crossed squares generalised to give squared complexes.

We will end this progression by looking at an elegant theoretical treatment of a generalisation of both crossed complexes and squared complexes. These gadgets are related to the "Moore chain complexes of order $(n+1)$ of a simplicial group", as briefly studied by Baues in [23], but have some of the advantages of crossed squares over 2-crossed modules, namely they can be 'non-symmetric', and hence are easily specified by, say, an 'inclusion crossed n-cube' consisting of a simplicial group and n simplicial normal subgroups. This allows for extra freedom in constructions. Also the axioms are very much simpler!

The definition of a crossed n-cube involves the set $\langle n\rangle = \{1, 2, \ldots, n\}$. One obvious way to extend this, eliminating dependence on n, is to try replacing $\langle n\rangle$ by $\mathbb{N} = \{1, 2, \ldots\}$ and taking the subsets A, B, C, in that definition to be finite, a condition previously automatic. This gives the notion of a crossed $\mathbb{N}$-cube:

Definition: A *crossed* $\mathbb{N}$-*cube*, M, is a family of groups,

$$\{M_A \mid A \subset \mathbb{N}, A \text{ finite}\},$$

together with homomorphisms, $\mu_i : M_A \to M_{A-\{i\}}$, $(i \in \mathbb{N},\ A \subset_{fin} \mathbb{N})$, and functions, $h : M_A \times M_B \to M_{A\cup B}, (A, B \subset_{fin} \mathbb{N})$, such that if ${}^a b$ denotes $h(a, b)b$ for $a \in M_A$ and $b \in M_B$ with $A \subseteq B$, then for $a, a' \in M_A$, b, $b' \in M_B$, $c \in M_C$ and $i, j \in \mathbb{N}$, the following axioms hold:

(1) $\mu_i a = a$ if $a \notin A$
(2) $\mu_i\mu_j a = \mu_j\mu_i a$
(3) $\mu_i h(a, b) = h(\mu_i a, \mu_i b)$
(4) $h(a, b) = h(\mu_i a, b) = h(a, \mu_i b)$ if $i \in A \cap B$
(5) $h(a, a') = [a, a']$
(6) $h(a, b) = h(b, a)^{-1}$
(7) $h(a, b) = 1$ if $a = 1$ or $b = 1$

(8) $h(aa', b) = {}^a h(a', b)h(a, b)$
(9) $h(a, bb') = h(a, b)^b h(a, b')$
(10) ${}^a h(h(a^{-1}, b), c)^c h(h(c^{-1}, a), b)^b h(h(b^{-1}, c), a) = 1$
(11) ${}^a h(b, c) = h({}^a b, {}^a c)$ if $A \subseteq B \cap C$.

(We have written $A \subset_{fin} \mathbb{N}$ as a shorthand for $A \subset \mathbb{N}$ with A finite.) Of course, these are formally identical to those given previously except in as much as there is no bound on the size of the finite sets A, B, C involved.

Examples: The first example is somewhat obvious, the second slightly surprising.

(i) As, for any n, $\langle n\rangle \subset \mathbb{N}$, if M is a crossed n-cube, then we can extend it trivially to an crossed $\mathbb{N}$-cube by defining $M_A = M_A$ if $A \subseteq \langle n\rangle$, and $M_A = 1$ otherwise. The h-maps $M_A \times M_B \to M_{A\cup B}$ are then clearly determined by those of the original crossed n-cube.

(ii) Suppose $\mathsf{M} = \{M_A, \mu_i, h\}$ is a crossed $\mathbb{N}$-cube, which is such that M_A is trivial unless A is of form $\langle n\rangle$ for some n, (where we interpret $\emptyset$ as being $\langle 0\rangle$, and so $M_\emptyset$ is not required to be trivial). We will write $C_n = M_{\langle n\rangle}$ and $\partial_n : C_n \to C_{n-1}$ for the morphism $\mu_n : M_{\langle n\rangle} \to M_{\langle n-1\rangle}$.

We note that $\partial_{n-1}\partial_n$ is trivial as it factorises via the trivial group:

$$\begin{tikzcd} M_{\langle n\rangle} \arrow[r] \arrow[d] & M_{\langle n-1\rangle} \arrow[d] \\ M_A \arrow[r] & M_{\langle n-2\rangle} \end{tikzcd}$$

where $A = \langle n\rangle - \{n-1\}$, so $M_A = 1$. We thus have that (C_n, ∂_n) is a complex of groups.

There is a pairing

$$C_0 \times C_n \to C_n$$

given by $h : M_\emptyset \times M_{\langle n\rangle} \to M_{\langle n\rangle}$, and thus an action

$${}^a b = h(a, b)b,$$

whilst $\partial({}^a b) = {}^a\partial b$, since $\mu_n h(a, b) = h(\mu_n a, \mu_n b)$, which is $h(a, \mu_n b)$, since $n \notin \emptyset$!

The map $\partial_1 : C_1 \to C_0$ is a crossed module by exactly the proof that a crossed 1-cube is a crossed module.

If $a = \partial_1 b$, then for $c \in C_n$, $n \geq 2$,

$$\begin{aligned} ac &= h(\partial_1 b, c)c \\ &= h(b, \mu_1 c)c, \end{aligned}$$

since $1 \in \langle 1\rangle \cap \langle n\rangle$, but $\mu_1 c \in M_{\langle n\rangle - \{1\}}$, the trivial group so

$${}^a c = c.$$

We will not systematically check *all* the axioms, but clearly (C_n, ∂) is a crossed complex. (The detailed checking *is* best left to the reader.) Conversely any crossed complex gives a crossed $\mathbb{N}$-cube.

These examples show that both crossed n-cubes, for all n, and crossed complexes are examples of crossed $\mathbb{N}$-cubes. The obvious question, given our previous discussion, is to try to put Ellis' squared complex in the same framework. There is an obvious method to try out, and it works! One takes $M_A = 1$ unless $A = \langle n\rangle$ for some $n \in \mathbb{N}$ or if $A \subseteq \langle 2\rangle$. This does it, but it also indicates an effective way of encoding higher dimensional analogues of these squared complexes.

To do this, given $n \geq 1$, we have a subcategory of the category of crossed $\mathbb{N}$-cubes specified by the crossed n-cube complexes, that is, by $M_A = 1$ unless $A = \langle m\rangle$ for some $m \in \mathbb{N}$ or if $A \subseteq \langle n\rangle$ for the given n.

As we are going to explore these gadgets in a bit of detail, we introduce some notation.

$Crs^{\mathbb{N}}$ will denote the category of crossed $\mathbb{N}$-cubes of groups; $Crs^n.Comp$ will denote the subcategory of $Crs^{\mathbb{N}}$ determined by the crossed n-cube complexes. Thus, for instance, $Crs^1.Comp$ becomes an alternative notation for the category of crossed complexes.

4.6.2 From simplicial groups to crossed n-cube complexes

To show how these gadgets relate to ordinary 'bog-standard' models of homotopy types, we will show how to obtain a crossed n-cube complex from a simplicial group G.

To obtain a crossed n-cube complex from a simplicial group G, one analyses the constructions giving crossed complexes and crossed square complexes. For crossed complexes, one used the relative homotopy groups of G, so that the base crossed module is

$$\frac{NG_1}{(NG_1 \cap D_1)d_0(NG_2 \cap D_2)} \to G_0,$$

but $NG_1 \cap D_1 = 1$ since D_1 is generated by the $s_0(g)$ with $g \in G_0$.

For an arbitrary simplicial group, H, the crossed module $M(H, 1)$ was given by

$$\frac{NH_1}{d_0(NH_2)} \to H_0,$$

so the earlier crossed module was $M(sk_1G, 1)$, as $N(sk_1G)_2 = NG_2 \cap D_2$.

Similarly for the crossed square complex associated to G, we explicitly took the 'base' crossed square to be $M(sk_2G, 2)$.

Proposition 60 *Let G be a simplicial group and $n \in \mathbb{N}$. Define a family M_A, $A \subset \mathbb{N}$, A finite, by*
(i) if $A = \langle m\rangle$ and $m > n$, then

$$M_A = \frac{NG_m}{(NG_m \cap D_m)d_0(NG_{m+1} \cap D_{m+1})};$$

(ii) if $A \subseteq \langle n\rangle$,

$$\begin{aligned} M_A &= M(sk_nG, n)_A \\ &= \frac{\bigcap\{Ker\, d^n_j : j \in A\}}{d_0(Ker\, d^{n+1}_1 \cap \bigcap\{Ker\, d^{n+1}_{j+1} : j \in A\} \cap D_{n+1})} : \end{aligned}$$

(iii) if A is otherwise, then M_A is trivial.
Further define $\mu_i : M_A \to M_{A-\{i\}}$ by
(iv) if $i \in A$, then μ_i is the identity morphism;
(v) if $A = \langle m\rangle$, with $m > n$ and $i = m$, then μ_m is induced by d_0, and is trivial if $i \neq m$;

(vi) if $A \subseteq \langle n\rangle$, then μ_i is induced by the inclusions of intersections (i.e. as in $M(sk_nG, n)$);
(vii) otherwise μ_i is trivial.

Finally define $h : M_A \times M_B \to M_{A\cup B}$ by
(viii) if $A = \emptyset$ and $B = \langle m\rangle$ with $m > n$ then as $M_\emptyset = G_{n-1}$ and $M_B = C(G)_m$, if $a \in M_\emptyset$ and $b \in M_B$,

$$h(a, b) = [s_0^{m-n+1}(a), b] \in M_B;$$

similarly if $A = \langle m\rangle$ and $B = \emptyset$;
(ix) if $A, B \subseteq \langle n\rangle$, h is defined as in $M(sk_nG, n)$;
(x) otherwise h is trivial.

This data defines a crossed $\mathbb{N}$-cube which is, in fact, a crossed n-cube complex.

Proof: Much of this can be safely 'left to the reader'. It uses results from earlier parts of the notes. Note, however, that (viii) and (x) effectively say that it is only the $s_0^{n-1}G_0$ part of G_{n-1} that acts on any $M_{\langle m\rangle}$ and even then the image of $d_0 : NG_1 \to G_0$ acts trivially. To see this note that any $a \in G_{n-1}$ that is in some $Ker\, d_i$ is in the image of some μ_i, hence $a = \mu_i x$ say, but then

$$\begin{aligned} h(a, b) &= h(\mu_i x, b) \\ &= h(x, \mu_i b) \\ &= 1, \end{aligned}$$

by necessity if the structure is to be crossed $\mathbb{N}$-cube. Thus to check that the h-maps, and, in particular, those involved with part (viii) of the definition, satisfy the axioms, it suffices to use the methods mentioned earlier for checking that $\mathsf{C}(G)$ was a crossed complex, see [177]. ■

We might denote this crossed n-cube complex by $\mathsf{C}(G, n)$, as it combines both the technology of the $M(G, n)$ and the $\mathsf{C}(G)$. These models have yet to be explored in any depth, but see [177] and below for some preliminary results.

4.6.3 From n to $n - 1$: collecting up ideas and evidence

We noted earlier that given $M(G, n)$, the quotient crossed $(n - 1)$-cube was $M(G, n - 1)$. Is a similar result true here? Is there an epimorphism from $\mathsf{C}(G, n)$ to $\mathsf{C}(G, n-1)$? In fact this is linked with another problem. We have a nested sequence of full categories of $Crs^{\mathbb{N}}$,

$$Crs^1.Comp \subset Crs^2.Comp \subset \ldots \subset Crs^n.Comp \subset \ldots \subset Crs^{\mathbb{N}}.$$

Does the inclusion of $Crs^{n-1}.Comp$ into $Crs^n.Comp$ have a left adjoint, in other words, is $Crs^{n-1}.Comp$ a reflexive subcategory of $Crs^n.Comp$? We investigate this question here only for $n = 2$ as this is at the same time easiest to see and also one of the most useful cases.

In this case, the crossed square complexes can be neatly represented as

$$\mathsf{C} := \quad \begin{tikzcd} \ldots \arrow[r] & C_3 \arrow[r, "\mu_3"] & C_{\langle 2\rangle} \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] & C_{\langle 1\rangle} \arrow[d, "\mu_1"] \\ & & C_{\{2\}} \arrow[r, "\mu_2"'] & C_\emptyset \end{tikzcd} ,$$

whilst those corresponding to crossed complexes look like

$$\mathsf{D} := \quad \begin{tikzcd} \ldots \arrow[r] & D_3 \arrow[r, "\mu_3"] & D_{\langle 2\rangle} \arrow[r, "\mu_2"] \arrow[d, "\mu_1"'] & D_{\langle 1\rangle} \arrow[d, "\mu_1"] \\ & & 1 \arrow[r, "\mu_2"'] & D_\emptyset \end{tikzcd} .$$

A map φ in $Crs^2.Comp$ from C to D, clearly, must kill off $C_{\{2\}}$ and hence must also kill off $\mu_2(C_{\{2\}})$, which is normal in $C_\emptyset$. That is not all. If $a \in C_{\{2\}}$, $b \in C_{\{1\}}$ or $C_{\langle 2\rangle}$, then

$$\varphi(h(a,b)) = h(\varphi a, \varphi b) = 1,$$

and $\varphi a = 1$, thus φ must kill off the action of $C_{\{2\}}$ on $C_{\langle 2\rangle}$, and all elements of this form, $h(a,b)$ with $a \in C_{\{2\}}$, $b \in C_{\{1\}}$ or $C_{\langle 2\rangle}$.

Example: To illustrate what is happening let us examine the case of an inclusion crossed square. Suppose G is a group and M, N normal subgroups, then

$$\mathsf{C} = \left(\begin{tikzcd} M \cap N \arrow[r] \arrow[d] & M \arrow[d] \\ N \arrow[r] & G \end{tikzcd} \right)$$

is a crossed square. Any 2-truncated crossed complex also gives a crossed square

$$\mathsf{D} = \left(\begin{tikzcd} D_2 \arrow[r] \arrow[d] & D_1 \arrow[d] \\ 1 \arrow[r] & D_0 \end{tikzcd} \right),$$

and any map from C to D factors through

$$\begin{tikzcd} \frac{M\cap N}{[M,N]} \arrow[r] \arrow[d] & \frac{M}{[M,N]} \arrow[d] \\ 1 \arrow[r] & G/N \end{tikzcd}$$

Proposition 61 *The inclusion of $Crs^1.Comp$ into $Crs^2.Comp$ has a left adjoint, denoted L. This left adjoint is a reflection, fixing the objects of the subcategory.* ■

The proof should be fairly obvious so we will leave it as an exercise.

From $\mathsf{C}(G,2)$ to $\mathsf{C}(G,1)$: What happens if we apply this L to $\mathsf{C}(G,2)$? The answer is not that much of a surprise!

Proposition 62 *If G is a simplicial group, then there is a natural isomorphism*

$$\mathsf{L}(\mathsf{C}(G,2)) \cong \mathsf{C}(G,1).$$

■

(Of course, the 'crossed 1-cube complex', $\mathsf{C}(G,1)$, is just the crossed complex $\mathsf{C}(G)$ under another name.)

This does generalise to higher dimensions. We thus have a series of crossed approximations to homotopy types, each one reflecting nicely down to the previous one, but what do these crossed gadgets tell us about the spaces being modelled? To explore that we must go back to crossed modules and their classifying spaces. There is a two way process here, algebraic gadgets tell us information about spaces, but conversely spaces can inform us about algebra.

Chapter 5

Classifying spaces, and extensions

We will first look in detail at the construction of classifying spaces and their applications for the non-Abelian cohomology of *groups*. This will use things we have already met. Later on we will need to transfer some of this to a sheaf theoretic context to handle 'gerbes' and to look at other forms of non-Abelian cohomology.

5.1 Non-Abelian extensions revisited

We again start with an extension of groups:

$$\mathcal{E}: \quad 1 \to K \to E \xrightarrow{p} G \to 1.$$

From a section, s, we constructed a factor set, f, but this *is* a bit messy. What do we mean by that? We are working in the category of groups, but neither s nor f are group morphisms. For s, there is an obvious thing to do. The function s induces a homomorphism, k_1, from $C_1(G)$, the free group on the set, G, to E and

$$\begin{tikzcd} C_1(G) \arrow[r] \arrow[d, "k_1"] & G \arrow[d, "="] \\ E \arrow[r, "p"] & G \end{tikzcd}$$

commutes. One might be tempted to do the same for f, but f is partially controlled by s, so we try something else. When we were discussing identities among relations (page **??**), we looked at the example of taking $X = \{\langle g \rangle \mid g \neq 1, g \in G\}$ and a relation $r_{g,g'} := \langle g \rangle \langle g' \rangle \langle gg' \rangle^{-1}$ for each pair (g, g') of elements of G. (Here we will write $\langle g_1, g_2 \rangle$ for r_{g_1,g_2}.)

We can use this presentation $\mathcal{P}$ to build a free crossed module

$$C(\mathcal{P}) := C_2(G) \to C_1(G).$$

We noted earlier that the identities were going to correspond to tetrahedra, and that, in fact, we could continue the construction by taking $C_n(G) =$ the free G-module on $\langle g_1, \ldots, g_n \rangle$, $g_i \neq 1$, i.e. the normalised bar resolution. This is very nearly the usual bar resolution coming from the nerve of G, but we have a crossed module at the base, not just some more modules.

We met this structure earlier when we were looking at syzygies, and later on with crossed n-fold extensions, but is it of any use to us here?

We know $pf(g_1, g_2) = 1$, so $f(g_1, g_2) \in K$, and $C_2(G)$ is a free crossed module ... . Also, $K \to E$ is a normal inclusion, so is a crossed module ... . Thinking along these lines, we try

$$k_2 : C_2(G) \to K$$

defined on generators by f, i.e., $i(k_2(\langle g_1, g_2\rangle) = f(g_1, g_2)$. It is fairly easy to check this works, that

$$\partial k_2(\langle g_1, g_2\rangle) = k_1\partial(\langle g_1, g_2\rangle),$$

and that the actions are compatible, i.e., $\mathbf{k} : C(\mathcal{P}) \to \mathcal{E}$, where will write $\mathcal{E}$ also for the crossed module (K, E, i).

In other words, it seems that the section and the resulting factor set give us a morphism of crossed modules, $\mathbf{k}$. We note however that f satisfies a cocycle condition, so what does that look like here? To answer this we make the boundary, $\partial_3 : C_3(G) \to C_2(G)$, precise.

$$\partial_3\langle g_1, g_2, g_3\rangle = {}^{\langle g_1\rangle}\langle g_2, g_3\rangle\langle g_1, g_2g_3\rangle\langle g_1g_2, g_3\rangle^{-1}\langle g_1, g_2\rangle^{-1}$$

and, of course, the cocycle condition just says that $k_2\partial_3$ is trivial.

We can use the idea of a crossed complex as being a crossed module with a tail which is a chain complex, to point out that $\mathbf{k}$ gives a morphism of crossed complexes:

$$\begin{tikzcd}
\mathsf{C}(G) : ... \arrow[r] \arrow[d] & C_3(G) \arrow[r] \arrow[d] & C_2(G) \arrow[r] \arrow[d, "k_2"] & C_1(G) \arrow[r] \arrow[d, "k_1"] & G \arrow[d] \\
\mathcal{E} : ... \arrow[r] & 1 \arrow[r] & K \arrow[r] & E \arrow[r] & G
\end{tikzcd}$$

where the crossed module $\mathcal{E}$ is thought of as a crossed complex with trivial tail.

Back to our general extension,

$$\mathcal{E} : \quad 1 \to K \to E \overset{p}{\to} G \to 1,$$

we note that the choice of a section, s, does not allow the use of an action of G on K. Of course, there is an action of E on K by conjugation and hence s does give us an action of $C_1(G)$ on K. If we translate 'action of G on a group, K', to being a functor from the *groupoid*, $G[1]$, to *Grps* sending the single object of $G[1]$ to the object K, then we can consider the 2-category structure of *Grps* with 2-cells given by conjugation, (so that if K and L are groups, and $f_1, f_2 : K \to L$ homomorphisms, a 2-cell $\alpha : f_1 \Longrightarrow f_2$ will be given by an element $\ell \in L$ such that

$$f_2(x) = \ell f_1(x)\ell^{-1}$$

for all $x \in K$). With this categorical perspective, s *does* give a lax functor from $G[1]$ to *Grps*. We essentially replace the action $G \to Aut(K)$, when s is a splitting, by a lax action (see Blanco, Bullejos and Faro, [28]);

$$\begin{tikzcd}
{} \arrow[r] & C_2(G) \arrow[r] \arrow[d, "k_2"] & C_1(G) \arrow[d, "k_1"] \\
 & K \arrow[r] \arrow[d, "="] & E \arrow[d] \\
 & K \arrow[r] & Aut(K).
\end{tikzcd}$$

Using this lax action and $\mathbf{k}$, we can reinterpret the classical reconstruction method of Schreier as forming the semidirect product $K \rtimes C_1(G)$, then dividing out by all pairs,

$$(k_2(\langle g_1, g_2\rangle), \partial_2(\langle g_1, g_2\rangle)^{-1}).$$

(We give Brown and Porter's article, [54], as a reference for a discussion of this construction.)

By itself this reinterpretation does not give us much. It just gives a slightly different viewpoint, however two points need making. This formulation is nearer the sort of approach that we will need to handle the classification of gerbes and the use of $K \to Aut(K)$ to handle the lax action of G reveals a problem and also a power in this formulation.

Dedecker, [80], noted that any theory of non-Abelian cohomology of groups must take account of the variation with K. Suppose we have two groups, K and L, and lax actions of G on them. What should it mean to say that some homomorphism $\alpha : K \to L$ is compatible with the lax actions?

A lax action of G on K can be given by a morphism of crossed modules / complexes, $Act_{G,K} : \mathsf{C}(G) \to \mathsf{Aut}(K)$, but $\mathsf{Aut}(K)$ is not functorial in K, so we do not automatically get a morphism of crossed modules, $\mathsf{Aut}(\alpha) : \mathsf{Aut}(K) \to \mathsf{Aut}(L)$. Perhaps the problem is slightly wrongly stated. One might say α is compatible with the lax G-actions if such a morphism of crossed modules existed and such that $Act_{G,L} = \mathsf{Aut}(\alpha)Act_{G,K}$. It is then just one final step to try to classify extensions with a finer notion of equivalence.

Definition: Suppose we have a crossed module, $\mathsf{Q} = (K, Q, q)$. An extension of K by G of the type of Q is a diagram:

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d, "="'] & E \arrow[r] \arrow[d, "\omega"] & G \arrow[r] & 1 \\ & K \arrow[r, "q"'] & Q & & \end{tikzcd}$$

where ω gives a morphism of crossed modules.

There is an obvious notion of equivalence of two such extensions, where the isomorphism on the middle terms must commute with the structural maps ω and ω'. The special case when $Q = Aut(K)$ gives one the standard notion. In general, one gets a set of equivalence classes of such extensions $Ext_{K\to Q}(G, K)$ and this can be related to the cohomology set $H^2(G, K \to Q)$. This can also be stated in terms of a category $\mathcal{E}xt_{\mathsf{Q}}(G)$ of extensions of type Q, then the cohomology set is the set of components of this category.

This latter object can be defined using any free crossed resolution of G as there is a notion of homotopy for morphisms of crossed complexes such that this set is $[\mathsf{C}(G), \mathsf{Q}]$. Any other free crossed resolution of G has the same homotopy as $\mathsf{C}(G)$ and so will do just as well. Finding a complete set of syzygies for a presentation of G will do.

Example:

$$G = (x, y \mid x^2 = y^3)$$

This is the trefoil group. It is a one relator presentation and has no identities, so $C(\mathcal{P})$ is already a crossed resolution. A morphism of crossed modules, $\mathbf{k} : C(\mathcal{P}) \to \mathsf{Q}$, is specified by elements

$q_x, q_y \in Q$, and $a_r \in K$ such that $\mathbf{k}(a_r) = (q_x)^2(q_y)^{-3}$. Using this one can give a presentation of the E that results.

Remark: Extensions correspond to 'bitorsors' as we will see. These in higher dimensions then yields gerbes with action of a gr-stack and a corresponding cohomology. In the case of gerbes, as against extensions, a related notion was introduced by Debremaeker, [76–79]. This has recently been revisited by Milne, [156], and Aldrovandi, [3], who consider the special case where both K and Q are Abelian and the action of Q is trivial. This links with various important structures on gerbes and also with Abelian motives and hypercohomology. In all these cases, Q is being viewed as the coefficients of the cohomology and the gerbes / extensions have interpretations accordingly. Another very closely related approach is given in Breen, [31, 33]. We explore these ideas later in these notes.

We can think of the canonical case $K \to Aut(K)$ as being a 'natural' choice for extensions by K of a group, G. It is the structural crossed module of the 'fibre'. The crossed modules case says we can restrict or, alternatively, lift this structural crossed module to Q. This may, perhaps, be thought of as analogous to the situation that we will examine shortly where geometric structure corresponds to the restriction or the lifting of the natural structural group of a bundle. Both restricting to a subgroup and lifting to a covering group are useful and perhaps the same is true here.

5.2 Classifying spaces

The classifying spaces of crossed modules are never far from the surface in this approach to cohomology and related areas. They will play a very important role in the discussion of gerbes, as, for instance, in Larry Breen's work, [31–33] and later on here.

Classifying spaces of (discrete) groups are well known. One method of construction is to form the nerve, $Ner(G)$, of the group, G, (considered as a small groupoid, $\mathcal{G}$ or $G[1]$, as usual). The classifying space is obtained by taking the geometric realisation, $BG = |Ner(G)|$.

To explore this notion, and how it relates to crossed modules, we need to take a short excursion into some simplicially based notions.

A classifying space of a group classifies principal G-bundles (G-torsors) over a space, X, in terms of homotopy classes of maps from X to BG, using a universal principal G-bundle $EG \to BG$.

This is very topological! If possible, it is useful to avoid the use of geometric realisations, since (i) this restricts one to groups and groupoids and makes handling more general 'algebras' difficult and (ii) for algebraic geometry, the topology involved is not the right kind as a sheaf-theoretic, topos based construction would be more appropriate. Thus the classifying space is often replaced by the nerve, as in Breen, [33].

How about classifying spaces for crossed modules? Given a crossed module, $\mathsf{M} = (C, G, \theta)$, say, we can form the associated 2-group, $\mathcal{X}(\mathsf{M})$. This gives a simplicial group by taking the nerve of the groupoid structure, then we can form $\overline{W}$ of that to get a simplicial set, $Ner(\mathsf{M})$. To reassure ourselves that this *is* a good generalisation of $Ner(G)$, we observe that if C is the trivial group, then $Ner(\mathsf{M}) = Ner(G)$. But this raises the question:

What does this 'classifying space' classify?

To answer that we must digress to provide more details on the functors G and $\overline{W}$, we mentioned earlier.

5.2.1 Simplicially enriched groupoids

We denote the category of simplicial sets by $\mathcal{S}$ and that of simplicially enriched groupoids by $\mathcal{S}-Grpds$. This latter category includes that of simplicial groups, but it must be remembered that a simplicial object in the category of groupoids will, in general, have a non-trivial simplicial set as its 'object of objects', whilst in $\mathcal{S} - Grpds$, the corresponding simplicial object of objects will be constant. This corresponds to a groupoid in which each collection of 'arrows' between objects is a simplicial set, not just a set, and composition is a simplicial morphism, hence the term 'simplicially enriched'. We will often abbreviate the term 'simplicially enriched groupoid' to '$\mathcal{S}$-groupoid', but the reader should note that in some of the sources on this material the looser term 'simplicial groupoid' is used to describe these objects, usually with a note to the effect that this is not a completely accurate term to use.

Remark: Later, in section **??**, we will need to work with $\mathcal{S}$-categories, i.e., simplicially enriched categories. Some brief introduction can be found in [131], in the notes, [181] and the references cited there. We *will* give a fairly detailed discussion of the main parts of the elementary theory of $\mathcal{S}$-categories later.

The loop groupoid functor of Dwyer and Kan, [87], is a functor

$$G : \mathcal{S} \longrightarrow \mathcal{S}-Grpds,$$

which takes the simplicial set K to the simplicially enriched groupoid GK, where $(GK)_n$ is the free groupoid on the directed graph

$$K_{n+1} \overset{s}{\underset{t}{\rightrightarrows}} K_0 \ ,$$

where the two functions, s, source, and t, target, are $s = (d_1)^{n+1}$ and $t = d_0(d_2)^n$ with relations $s_0x = id$ for $x \in K_n$. The face and degeneracy maps are given on generators by

$$\begin{aligned} s_i^{GK}(x) &= s_{i+1}^K(x), \\ d_i^{GK}(x) &= d_{i+1}^K(x), \text{for} \ \ x \in K_{n+1}, 1 < i \leq n \end{aligned}$$

and

$$d_0^{GK}(x) = (d_0^K(x))^{-1}(d_1^K(x)).$$

This loop groupoid functor has a right adjoint, $\overline{W}$, called the *classifying space* functor. The details as to its construction will be given shortly. It is important to note that if K is reduced, i.e. has just one vertex, then GK will be a simplicial group, so is a well known type of object. This helps when studying these gadgets as we can often use simplicial group constructions, suitable adapted, in the $\mathcal{S}$-groupoid context. The first we will see is the Moore complex.

Definition: Given any $\mathcal{S}$-groupoid, G, its Moore complex, NG, is given by

$$NG_n = \bigcap_{i=1}^{n} Ker(d_i : G_n \longrightarrow G_{n-1})$$

with differential $\partial : NG_n \longrightarrow NG_{n-1}$ being the restriction of d_0. If $n \geq 1$, this is just a disjoint union of groups, one for each object in the object set, O, of G. If we write $G\{x\}$ for the simplicial group of elements that start and end at $x \in O$, then at object x, one has

$$NG\{x\}_n = (NG_n)\{x\}.$$

In dimension 0, one has $NG_0 = G_0$, so the $NG_n\{x\}$, for different objects x, are linked by the actions of the 0-simplices, acting by conjugation via repeated degeneracies.

The quotient $NG_0/\partial(NG_1)$ is a groupoid, which is the *fundamental groupoid* of the simplicially enriched groupoid, G. We can also view this quotient as being obtained from the $\mathcal{S}$-enriched category G by applying the 'connected components' functor π_0 to each simplicial hom-set $G(x, y)$. If $G = G(K)$, the loop groupoid of a simplicial set K, then this fundamental groupoid is exactly the fundamental groupoid, ΠK, of K and we can take this as defining that groupoid if we need to be more precise later. This means that ΠK is obtained by taking the free groupoid on the 1-skeleton of K and then dividing out by relations corresponding to the 2-simplices: if $\sigma \in K_2$, we have a relation

$$d_2(\sigma).d_0(\sigma) \equiv d_1(\sigma).$$

(**You are left** to explore this a bit more, justifying the claims we have made. You may also like to review the treatment in the book by Gabriel and Zisman, [102].)

For simplicity in the description below, we will often assume that the $\mathcal{S}$-groupoid is *reduced*, that is, its set O, of objects is just a singleton set $\{*\}$, so G is just a simplicial group.

Suppose that NG_m is trivial for $m > n$.

If $n = 0$, then NG_0 is just the group G_0 and the simplicial group (or groupoid) represents an Eilenberg-MacLane space, $K(G_0, 1)$.

If $n = 1$, then $\partial : NG_1 \longrightarrow NG_0$ has a natural crossed module structure.

Returning to the discussion of the Moore complex, if $n = 2$, then

$$NG_2 \xrightarrow{\partial} NG_1 \xrightarrow{\partial} NG_0$$

has a 2-crossed module structure in the sense of Conduché, [66] and above section 4.3. (These statements are for groups and hence for connected homotopy types. The non-connected case, handled by working with simplicially enriched groupoids, is an easy extension.)

In all cases, the simplicial group will have non-trivial homotopy groups only in the range covered by the non-trivial part of the Moore complex.

Now relaxing the restriction on G, for each $n > 1$, let D_n denote the subgroupoid of G_n generated by the degenerate elements. Instead of asking that NG_n be trivial, we can ask that $NG_n \cap D_n$ be. The importance of this is that the structural information on the homotopy type represented by G includes structure such as the Whitehead products and these all lie in the subgroupoids $NG_n \cap D_n$. If these are all trivial then the algebraic structure of the Moore complex is simpler, being that of a crossed complex, and $\overline{W}G$ is a simplicial set whose realisation is the *classifying space of that crossed complex*, cf. [48]. The simplicial set, $\overline{W}G$, is isomorphic to the *nerve* of the crossed complex.

Notational warning. As was mentioned before, the indexing of levels in constructions with crossed complexes may cause some confusion. The Dwyer-Kan construction is essentially a 'loop' construction, whilst $\overline{W}$ is a 'suspension'. They are like 'shift' operators for chain complexes. For example G decreases dimension, as an old 1-simplex x yields a generator in dimension 0, and so on. Our usual notation for crossed complexes has C_0 as the set of objects, C_1 corresponding to a relative fundamental groupoid, and C_n abstracting its properties from $\pi_n(X_n, X_{n-1}, p)$, hence the natural topological indexing *has* been used. For the $\mathcal{S}$-groupoid $G(K)$, the set of objects is separated out and $G(K)_0$ is a groupoid on the 1-simplices of K, a dimension shift. Because of this, in the notation being used here, the crossed complex $\mathsf{C}(G)$ associated to an $\mathcal{S}$-groupoid, G, will have a dimension shift as well: explicitly

$$C(G)_n = \frac{NG_{n-1}}{(NG_{n-1} \cap D_{n-1})d_0(NG_n \cap D_n)} \quad \text{for } n \geq 2,$$

$C(G)_1 = NG_0$, and, of course, C_0 is the common set of objects of G. In some papers where only the algebraic constructions are being treated, this convention is not used and C is given without this dimension shift relative to the Moore complex. Because of this, care is sometimes needed when comparing formulae from different sources.

5.2.2 Conduché's decomposition and the Dold-Kan Theorem

The category of crossed complexes (of groupoids) is equivalent to a reflexive subcategory of the category $\mathcal{S}-Grpds$ and the reflection is defined by the obvious functor : take the Moore complex of the $\mathcal{S}$-groupoid and divide out by the $NG_n \cap D_n$, see [88, 89]. We will denote by $C : \mathcal{S}-Grpds \longrightarrow Crs$ the resulting composite functor, Moore complex followed by reflection. Of course, we have the formula, more or less as before, (cf. page 54),

$$C(G)_{n+1} = \frac{NG_n}{(NG_n \cap D_n)\ d_0(NG_{n+1} \cap D_{n+1})}.$$

The Moore complex functor itself is part of an adjoint (Dold-Kan) equivalence between the category $\mathcal{S} - Grpds$ and the category of hypercrossed complexes, [63], and this restricts to the Ashley-Conduché version of the Dold-Kan theorem of [11].

In order to justify the description of the nerve, and thus the related classifying space, of a crossed complex C, we will specify the functors involved, namely the Dold-Kan inverse construction and the $\overline{W}$. (We will leave **the reader** to chase up the detailed proof of this crossed complex form of the Dold-Kan theorem. The functors will be here, but the detailed proofs that they do give an equivalence will be left to you to give or find in the literature.)This will also give us extra tools for later use. We will first need the Conduché decomposition lemma, [66].

Proposition 63 *If G is a simplicial group(oid), then G_n decomposes as a multiple semidirect product:*

$$G_n \cong NG_n \rtimes s_0NG_{n-1} \rtimes s_1NG_{n-1} \rtimes s_1s_0NG_{n-2} \rtimes s_2NG_{n-1} \rtimes \ldots s_{n-1}s_{n-2}\ldots s_0NG_0$$

■

The order of the terms corresponds to a lexicographic ordering of the indices $\emptyset$; 0; 1; 1,0; 2; 2,0; 2,1; 2,1,0; 3; 3,0; ... and so on, the term corresponding to $i_1 > \ldots > i_p$ being $s_{i_1} \ldots s_{i_p} NG_{n-p}$.

The proof of this result is based on a simple lemma, which is easy to prove.

Lemma 35 *If G is a simplicial group(oid), then G_n decomposes as a semidirect product:*

$$G_n \cong Ker\ d^n_n \rtimes s^{n-1}_{n-1}(G_{n-1}).$$

■

We next note that in the classical (Abelian) Dold-Kan theorem, (cf. [74]), the equivalence of categories is constructed using the Moore complex and a functor K constructed via the original direct sum / Abelian version of Conduché's decomposition, cf. for instance, [74].

For each non-negatively graded chain complex, $\mathsf{D} = (D_n, \partial)$. in Ab, $K\mathsf{D}$ is the simplicial Abelian group with

$$(K\mathsf{D})_n = \oplus_a (D_{n-\sharp(a)}, s_a),$$

the sum being indexed by all descending sequences, $a = \{n > i_p \geq ... \geq i_1 \geq 0\}$, where $s_a = s_{i_p}...s_{i_1}$, and where $\sharp(a) = p$, the summand D_n corresponding to the empty sequence.

The face and degeneracy operators in $K\mathsf{D}$ are given by the rules:
(1) if $d_i s_a = s_b$, then d_i will map (D_{n-p}, s_a) to $(D_{(n-1)-(p-1)}, s_b)$ by the identity on D_{n-p}; its components into other direct summands will be zero;
(2) if $d_i s_a = s_b d_0$, then d_i will map (D_{n-p}, s_a) to (D_{n-p-1}, s_b) as the homomorphism $\partial_{n-p} : D_{n-p} \to D_{n-p-1}$; its components into other direct summands will be zero;
(3) if $d_i s_a = s_b d_j$, $j > 0$, then $d_i(D_{n-p}, s_a) = 0$;
(4) if $s_i s_a = s_b$, then s_i maps (D_{n-p}, s_a) to $(D_{(n+1)-(p+1)}, s_b)$ by the identity on D_{n-p}; its components into other direct summands will be zero.

This suggests that we form a functor

$$K : Crs \to \mathcal{S} - Grpds$$

using a semidirect product, but we have to take care as there will be a dimension shift, our lowest dimension being C_1:
if C is in Crs, set

$$K(\mathsf{C})_n = C_{n+1} \rtimes s_0 C_n \rtimes s_1 C_n \rtimes s_1 s_0 C_{n-1} \rtimes \cdots \rtimes s_{n-1} s_{n-2} \ldots s_0 C_1.$$

The order of terms is to be that of the proposition given above. The formation of the semidirect product is as in the proof we hinted at of that proposition, that is the bracketing is inductively given by

$$(C_{n+1} \ldots \rtimes s_{n-2} \ldots s_0 C_2) \rtimes (s_{n-1} C_n \rtimes \ldots \rtimes s_{n-1} \ldots s_0 C_1);$$

each $s_\alpha(C_{n+1-\sharp(\alpha)})$ is an indexed copy of $C_{n+1-\sharp(\alpha)}$; the action of

$$s_{n-1} C_{n-1} \rtimes \ldots \rtimes s_{n-1} \ldots s_0 C_0 \quad (\cong s_{n-1} K(\mathsf{C})_{n-1})$$

on $C_{n+1} \rtimes \ldots s_{n-2} \ldots s_0 C_1$, is given componentwise by the actions of each C_i and as C is a crossed complex, these are all via C_0. This implies, of course, that the majority of the components of these actions are trivial.

To see how this looks in low dimensions, it is simple to give the first few terms of the simplicial group(oid). As we are taking a reduced crossed complex as illustration, the result is a simplicial group, $K(\mathsf{C})$, having

- $K(\mathsf{C})_0 = C_1$
- $K(\mathsf{C})_1 = C_2 \rtimes s_0(C_1)$
- $K(\mathsf{C})_2 = (C_3 \rtimes s_0 C_2) \rtimes (s_1 C_2 \rtimes s_1 s_0 C_1)$
- $K(\mathsf{C})_3 = (C_4 \rtimes s_0 C_3 \rtimes s_1 C_3 \rtimes s_1 s_0 C_2) \rtimes (s_2 C_3 \rtimes s_2 s_0 C_2 \rtimes s_2 s_1 C_2 \rtimes s_2 s_1 s_0 C_1)$.

and so on.

The face and degeneracy maps are determined by the obvious rules adapting those in the Abelian case, so that if $c \in C_k$, the corresponding copy of c in $s_\alpha C_k$ will be denoted $s_\alpha c$ and a face or degeneracy operator will usually act just on the index. The exception to this is if, when renormalised to the form $s_\beta d_\gamma$ using the simplicial identities, γ is non-empty. If $d_\gamma = d_0$ then $d_\gamma c$ becomes $\delta_k c \in C_{k-1}$, otherwise $d_\gamma c$ will be trivial.

Lemma 36 *The above defines a functor*

$$K : Crs \to \mathcal{S} - Grpds$$

such that $\mathsf{C}K \cong Id$. ■

This extends the functor $K : CMod \to Simp.Grps$, given earlier, to crossed complexes as there $C_k = 1$ for $k > 2$.

One obvious question, given our earlier discussion of group T complexes, and its fairly obvious adaptation to groupoid T-complexes, is if we start with a crossed complex C and construct this simplicially enriched groupoid $K(\mathsf{C})$, is this a groupoid T-complex? As the thin filler condition for groupoid T-complexes involves the Moore complex, it is enough to look at the single object simplicial group case. We have the following:

Proposition 64 *If* C *is a crossed complex, then* $K\mathsf{C}$ *is a group* T*-complex.*

Proof: We have to check that $NK(\mathsf{C})_n \cap D_n = 1$. We suppose $g \in NK(\mathsf{C})_n$ is a product of degenerate elements, then, using the semidirect decomposition, we can write g in the form

$$g = s_1(g_1)\dots s_{n-1}(g_{n-1}). \qquad (*)$$

The only problem in doing this is handling any element that comes from C_0, but this can be done via the action of C_0 on the C_i.

As $g \in Ker\, d_n$, we have

$$1 = d_n g = s_1 d_{n-1}(g_1)\dots s_{n-2}d_{n-1}(g_{n-2}).g_{n-1},$$

so we can replace g_{n-1} by a product of degenerate elements and use $s_{n-1}s_i = s_i s_{n-2}$ and rewriting to obtain a new expression for g in the form (*), but with no s_{n-1} term. Repeating using d_{n-1} on this new expression yields that the new g_{n-2} is also in D_{n-1} and so on until we obtain

$$g = s_0(g^{(1)})$$

where $g^{(1)} \in D_{n-1}$, writing $g^{(1)}$ in the form (*) gives

$$g = s_0 s_0(g_1{}^{(1)}\dots s_0 s_{n-2}(g_{n-2}{}^{(1)}),$$

but $d_1 d_n g = 1$, so $g_{n-2}{}^{(1)} \in D_{n-2}$. Repeating we eventually get $g = s_0 s_0(g^{(n)})$ with $g^{(2)} \in D_{n-2}$. This process continues until we get $g = s_0{}^{(n)}(g^{(n)})$ with $g^{(n)} \in K(\mathsf{C})_0$, but $d_1 \dots d_n g = g^{(n)}$ and $d_1 \dots d_n g = 1$, so $g = 1$ as required. ■

Note that this proof, which is based on Ashley's proof that simplicial Abelian groups are group T-complexes (cf., [11]), depends in a strong way on being able to write g in the form $(*)$, i.e., on the triviality of almost all the actions together with the explicit nature of the action of C_0.

Collecting up the pieces we have all the main points in the proof of the following Dold-Kan theorem for crossed complexes.

Theorem 20 *There is an equivalence of categories*

$$Grpd.T-comp. \overset{\simeq}{\longleftrightarrow} Crs.$$

■

Checking that we do have all the parts necessary and providing any missing pieces is a good exercise, so will be **left to you**. A treatment more or less consistent with the conventions here can be found in [177].

5.2.3 $\overline{W}$ and the nerve of a crossed complex

We next need to make explicit the $\overline{W}$ construction. The simplicial / algebraic description of the nerve of a crossed complex, C, is then as $\overline{W}(K(\mathsf{C}))$. We first give this description for a general simplicially enriched groupoid.

Let H be an $\mathcal{S}$-groupoid, then $\overline{W}H$ is the simplicial set described by

- $(\overline{W}H)_0 = ob(H_0)$, the set of objects of the groupoid of 0-simplices (and hence of the groupoid at each level);

- $(\overline{W}H)_1 = arr(H_0)$, the set of arrows of the groupoid H_0:

and for $n \geq 2$,

- $(\overline{W}H)_n = \{(h_{n-1}, \ldots, h_0) \mid h_i \in arr(H_i) \text{ and } s(h_{i-1}) = t(h_i), 0 < i < n\}$.

Here s and t are generic symbols for the domain and codomain mappings of all the groupoids involved. The face and degeneracy mappings between $\overline{W}(H)_1$ and $\overline{W}(H)_0$ are the source and target maps and the identity maps of H_0, respectively; whilst the face and degeneracy maps at higher levels are given as follows:

The face and degeneracy maps are given by

- $d_0(h_{n-1}, \ldots, h_0) = (h_{n-2}, \ldots, h_0)$;

- for $0 < i < n$, $d_i(h_{n-1}, \ldots, h_0) = (d_{i-1}h_{n-1}, d_{i-2}h_{n-2}, \ldots, d_0h_{n-i}h_{n-i-1}, h_{n-i-2}, \ldots, h_0)$;

and

- $d_n(h_{n-1}, \ldots, h_0) = (d_{n-1}h_{n-1}, d_{n-2}h_{n-2}, \ldots, d_1h_1)$;

whilst

- $s_0(h_{n-1}, \ldots, h_0) = (id_{dom(h_{n-1})}, h_{n-1}, \ldots, h_0)$;

and,

- for $0 < i \leq n$, $s_i(h_{n-1}, \ldots, h_0) = (s_{i-1}h_{n-1}, \ldots, s_0h_{n-i}, id_{cod(h_{n-i})}, h_{n-i-1}, \ldots, h_0)$.

Remark: We note that if H is a constant simplicial groupoid, $\overline{W}(H)$ is the same as the nerve of that groupoid for the algebraic composition order. Later on, when re-examining the classifying space construction, we may need to rework the above definition in a form using the functional composition order.

To help understand the structure of the nerve of a (reduced) crossed complex, C, we will calculate $Ner(\mathsf{C}) = \overline{W}(K(\mathsf{C}))$ in low dimensions. This will enable comparison with formulae given earlier. The calculations are just the result of careful application of the formulae for $\overline{W}$ to $H = K(\mathsf{C})$:

- $Ner(\mathsf{C})_0 = *$, as we are considering a *reduced* crossed complex - in the general case, this is C_0;
- $Ner(\mathsf{C})_1 = C_1$, as a *set* of 'directed edges' or arrows - we will avoid using a special notation for 'underlying set of a group(oid)';
- $Ner(\mathsf{C})_2 = \{(h_0, h_1) \mid h_1 = (c_2, s_0(c_1)), h_0 = c'_1, \text{ with } c_2 \in C_2, c_1, c'_1 \in C_1\}$, and such a 2-simplex has faces given as in the diagram

$$\begin{tikzcd}
& {} \arrow[dr, "c'_1"] \arrow[d, phantom, "(h_1{,}h_0)"] & \\
{} \arrow[ur, "c_1"] \arrow[rr, "\delta c_2.c_1.c'_1"'] & {} & {}
\end{tikzcd}$$

  Note that $h_1 : c_1 \longrightarrow \delta c_2.c_1$ in the internal category corresponding to the crossed module, (C_2, C_1, δ), so the formation of this 2-simplex corresponds to a right whiskering of that 2-cell (in the corresponding 2-groupoid) by the arrow c'_1;
- $Ner(\mathsf{C})_3 = \{(h_2, h_1, h_0) \mid h_1 = (c_3, s_0c_2^0, s_1c_2^1, s_1s_0c_1), h_1 = (c'_2, s_0(c'_1)), h_0 = c''_1\}$ in the evident notation. Here the faces of the 3-simplex (h_2, h_1, h_0) are as in the diagrams, (in each of which the label for the 2-simplex itself has been abbreviated):

d_3 :
$$\begin{tikzcd}
& {} \arrow[dr, "c'_1"] \arrow[d, phantom, "c_2^1"] & \\
{} \arrow[ur, "c_1"] \arrow[rr, "\delta c_2^1.c_1.c'_1"'] & {} & {}
\end{tikzcd}$$
; d_2 :
$$\begin{tikzcd}
& {} \arrow[dr, "\delta c'_2.c'_1.c''_1"] \arrow[d, phantom, "c_2^0c_2^1"] & \\
{} \arrow[ur, "c_1"] \arrow[rr, "\delta(c_2^0c_2^1).c_1.\delta c'_2.c'_1.c''_1"'] & {} & {}
\end{tikzcd}$$

d_1 :
$$\begin{tikzcd}
& {} \arrow[dr, "c''_1"] \arrow[d, phantom, "\delta c_3.c_2^0.{}^{\delta c_2^1.c_1}c'_2"] & \\
{} \arrow[ur, "\delta c_2^1.c_1.c'_1"] \arrow[rr, "\delta(c_2^0c_2^1).c_1.\delta c'_2.c'_1.c''_1"'] & {} & {}
\end{tikzcd}$$
; d_0 :
$$\begin{tikzcd}
& {} \arrow[dr, "c''_1"] \arrow[d, phantom, "c'_2"] & \\
{} \arrow[ur, "c'_1"] \arrow[rr, "\delta c'_2.c'_1.c''_1"'] & {} & {}
\end{tikzcd}$$

The only face where any real thought has to be used is d_1. In this the d_1 face has to be checked to be consistent with the others. The calculation goes like this:

$$\begin{aligned}\delta(\delta c_3.c_2^0.{}^{\delta c_2^1.c_1}c_2').(\delta c_2^1.c_1.c_1').c_1'' &= \delta c_2^0.(\delta c_2^1.c_1.\delta c_2'.c_1^{-1}.(\delta c_2^1)^{-1}).\delta c_2^1.c_1.c_1'.c_1'' \\ &= \delta(c_2^0 c_2^1).c_1.\delta c_2'.c_1'.c_1''\end{aligned}$$

This uses (i) $\delta\delta c_3$ is trivial, being a boundary of a boundary, and (ii) the second crossed module rule for expanding $\delta({}^{\delta c_2^1.c_1}c_2')$ as $\delta c_2^1.c_1.\delta c_2'.c_1^{-1}.(\delta c_2^1)^{-1}$.

This diagrammatic representation, although useful, is limited. A recursive approach can be used as well as the simplicial / algebraic one given above. In this, $Ner(\mathsf{C})$ is built up via its skeletons, specifying a simplex in $Ner(\mathsf{C})_n$ as an element of C_n, together with the empty simplex that it 'fills', i.e. the set of compatible $(n-1)$-simplices. This description is used by Ashley, ([11], p.37). More on nerves of crossed complexes can be found in Nan Tie, [171, 172]. There is also a very neat 'singular complex' description, $Ner(\mathsf{C})_n = Crs(\pi(n), \mathsf{C})$, where $\pi(n)$ is the free crossed complex on the n-simplex, $\Delta[n]$. We will have occasion to see this in more detail later.

This singular complex description shows another important feature. If we have an n-simplex $f : \pi(n) \to \mathsf{C}$, we will say it is *thin* if the image $f(\iota_n)$ of the top dimensional generator in $\pi(n)$ is trivial. The nerve together with the filtered set of thin elements forms a T-complex in the sense of section **??**. This is discussed in Ashley, [11], and Brown-Higgins, [48].

5.3 Simplicial Automorphisms and Regular Representations

The usual enrichment of the category of simplicial sets is given by :
for each $n \geq 0$, the set of n-simplices is

$$\underline{\mathcal{S}}(K, L)_n = \mathcal{S}(K \times \Delta[n], L),$$

together with obvious face and degeneracy maps.
Composition : for $f \in \underline{\mathcal{S}}(K, L)_n$, $g \in \underline{\mathcal{S}}(L, M)_n$, so $f : \Delta[n] \times K \to L$, $g : \Delta[n] \times L \to M$,

$$g \circ f := (\Delta[n] \times K \xrightarrow{diag \times K} \Delta[n] \times \Delta[n] \times K \xrightarrow{\Delta[n] \times f} \Delta[n] \times L \xrightarrow{g} M);$$

Identity : $id_K : \Delta[0] \times K \xrightarrow{\cong} K$.

Definition: The simplicial set, $\underline{\mathcal{S}}(K, L)$, defined above, is called the *simplicial mapping space of maps from K to L.*

This **clearly** is functorial in both K and L. (Of course, with differing 'variance'. It is 'contravariant' in K, so that $\underline{\mathcal{S}}(-, L)$ is a functor from $\mathcal{S}^{op}$ to $\mathcal{S}$, but $\underline{\mathcal{S}}(K, -) : \mathcal{S} \to \mathcal{S}$. In the category, $\mathcal{S}$, each of the functors 'product with K' for K a simplicial set, has a right adjoint, namely this $\underline{\mathcal{S}}(K, -)$. Technically $\mathcal{S}$ is a *Cartesian closed category*, a notion we will explore briefly in the next section. In any such setting we can restrict to looking at endomorphisms of an object, and, here we can go further and get a simplicial group of automorphisms of a simplicial set, K, analogously to our construction of the automorphism 2-group of a group (recall from section 1.3.4).

Explicitly, for fixed K, $\underline{\mathcal{S}}(K, K)$ is a simplicial monoid, called the *simplicial endomorphism monoid of* K and $\mathsf{aut}(K)$ will be the corresponding simplicial group of invertible elements, that is the *simplicial automorphism group of* K.

If $f : K \times \Delta[n] \longrightarrow L$ is an n-simplex, then we can form a diagram

$$\begin{tikzcd} K \times \Delta[n] \arrow[rr, "(f,p)"] \arrow[dr] & & L \times \Delta[n] \arrow[dl] \\ & \Delta[n] & \end{tikzcd}$$

in which the two slanting arrow are the obvious projections, (so $(f,p)(k,\sigma) = (f(k,\sigma),\sigma)$). Taking $K = L$, $f \in \mathsf{aut}(K)$ if and only if (f,p) is an isomorphism of simplicial sets.

Given a simplicial set K, and an n-simplex, x, in K, there is a representing map,

$$\mathbf{x} : \Delta[n] \longrightarrow K,$$

that sends the top dimensional generating simplex of $\Delta[n]$ to x.

As was just said, the mapping space construction, above, is part of an adjunction,

$$\mathcal{S}(K \times L, M) \cong \mathcal{S}(L, \underline{\mathcal{S}}(K,M)),$$

in which, given $\theta : K \times L \longrightarrow M$ and $y \in L_n$, the corresponding simplicial map

$$\bar{\theta} : L \longrightarrow \underline{\mathcal{S}}(K,M)$$

sends y to the composite

$$K \times \Delta[n] \xrightarrow{K \times \mathbf{y}} K \times L \xrightarrow{\theta} M\ .$$

In a simplicial group G, the multiplication is a simplicial map, $\#_0 : G \times G \longrightarrow G$, and so, by the adjunction, we get a simplicial map

$$G \longrightarrow \underline{\mathcal{S}}(G,G)$$

and this is a simplicial monoid morphism. This gives the *right regular representation* of G,

$$\rho = \rho_G : G \longrightarrow \mathsf{aut}(G).$$

We will look at this idea of *representations* in more detail later.

This morphism, ρ, needs careful interpretation. In dimension n, an element $g \in G_n$ acts by multiplication on the right on G, but even in dimension 0, this action is not as simple as one might think. (NB. Here $\mathsf{aut}(G)$ is the simplicial group of 'simplicial automorphisms of the underlying simplicial set of G' as, of course, multiplication by an element does not give a mapping that respects the group structure.) Simple examples are called for:

In general, 0-simplices give simplicial maps corresponding to multiplication by that element, so that for $g \in G_0$, and $x \in G_n$,

$$\rho(g)(x) = x \#_0 s_0^{(n)}(g).$$

Suppose, now, $g \in G_1$, then $\rho(g) \in \mathsf{aut}(G)_1 \subset \underline{\mathcal{S}}(G,G)_1 = \mathcal{S}(G \times \Delta[1], G)$. In other words, $\rho(g)$ is a homotopy between $\rho(d_1 g)$ and $\rho(d_0 g)$. Of course, it is an invertible element of $\underline{\mathcal{S}}(G,G)_1$ and this will have implications for its properties as a homotopy, and, to use a geometric term, we will loosely refer to it as an *isotopy*.

In dimension 1, we, thus, have that elements give isotopies, and in higher dimensions, we have 'isotopies of isotopies', and so on.

Of course, the existence of these automorphism simplicial groups, $\mathsf{aut}(K)$, leads to a notion of a *(permutation) representation for a simplicial group*, G, as being a simplicial group morphism from G to $aut(K)$ for some simplicial set K. Likewise, if we have a simplical vector space, V, then we can construct a group of its automorphisms and thus consider linear representations as well. We will return to this later so give no details here.

5.4 Simplicial actions and principal fibrations

We saw, back in the first chapter, (page **??**), the idea of a group, G, acting on a set, X. This is clearly linked to what was discussed in the previous section. A group action was given by a map,

$$a : G \times X \to X,$$

(and we may write $g.x$, or simply gx, for the image $a(g, x)$), satisfying obvious conditions such as an 'associativity' rule $g_2.g_1).x = g_2.(g_1.x)$ and an 'identity' rule $1_G.x = x$, both for all possible gs and xs. Of course, this 'action by g' gives a permutation of X, that is, a bijection form X to itself.

5.4.1 More on 'actions' and Cartesian closed categories

We know that the behaviour we have just been using for simplicial sets is also 'there' in the much simpler case of *Sets*, i.e., given sets X, Y and Z, there is a natural isomorphism

$$Sets(X \times Y, Z) \cong Sets(X, Sets(Y, Z)),$$

given by sending a 'function of two variables', $f : X \times Y \to Z$, to $\tilde{f} : X \to Sets(Y, Z)$, where $\tilde{f}(x) : Y \to Z$ sends y to $f(x, y)$. (We often write Z^Y for $Sets(Y, Z)$, since, for instance, if $Y = \{1, 2\}$, a two element set, $Sets(Y, Z) \cong Z \times Z = Z^2$, in the usual sense.) Technically, this is saying that $- \times Y$ has an adjoint given by $Sets(Y, -)$.

Definition: A category, $\mathcal{C}$, is *Cartesian closed* or a *ccc*, if it has all finite products and for any two objects, Y and Z, there is an *exponential*, Z^Y, in $\mathcal{C}$, so that $(-)^Y$ is right adjoint to $- \times Y$.

Recall or note: To say that $\mathcal{C}$ has all products says that, for any two objects X and Y in $\mathcal{C}$, their product $X \times Y$ is also there, and that there is a *terminal object*, and conversely. If you have not really met 'terminal objects' explicitly before an object T is *terminal* if, for any X in $\mathcal{C}$, there is a *unique* morphism from X to T. The simplest examples to think about are (i) any one element (singleton) set is terminal in *Sets*, (ii) the trivial group is terminal in *Groups*, and so on. The dual notion is *initial object*. An object, I, is *initial* if there is a unique morphism from I to X, again for all X in $\mathcal{C}$. The empty set is initial in *Sets*; the trivial group is initial in *Groups*.

If you have not formally met these, now is a good time to check up in texts that give an introduction to category theory and categorical ideas. In particular, it is worth thinking about why the terminal object in a category, if it exists, is the 'empty product', i.e., the product of an empty family of objects. This can initially seem strange, but is a *very useful insight* that will come in later, when we discuss sheaves.

We can use this property of $Sets$, and $\mathcal{S}$, or more generally for any ccc, to give a second description of a group action. The function $a : G \times X \to X$ gives, by the adjunction, a function

$$\tilde{a} : G \to Sets(G, G).$$

This set, $Sets(G, G)$, is a monoid under composition, and we can pick out $Perm(X)$ or if you prefer the notations, $Symm(X)$ or $Aut(X)$, the subgroup of self bijections or permutations of G. In this guise, an action of G on X is a group homomorphism from G to $Perm(X)$. (You might like to **consider how this selection of the invertibles** in the 'internal' monoid, $\mathcal{C}(X, X)$, could be done in a general ccc.)

As we mentioned, the category, $\mathcal{S}$, is also Cartesian closed, and we can use the above observation, together with our identification of the simplicial group of automorphisms, $\mathsf{aut}(Y)$, of a simplicial set Y from our earlier discussion, to describe the action of a simplicial group, G, on a simplicial set, Y. A *simplicial action* would thus be, equivalently, a simplicial map,

$$a : G \times Y \to Y,$$

satisfying associativity and identity rules, or a morphism of simplicial groups,

$$\tilde{a} : G \to \mathsf{aut}(Y).$$

We thus have the well known equivalence of 'actions' and 'representations'. This will be another recurring theme throughout these notes with embellishments, variations, etc. in different contexts. it is sometimes the 'aut'-object version that is easiest to give, sometimes not, and for some contexts, although $\mathcal{C}(X, X)$ will always be a monoid internal to some base category, the automorphisms may be hard to 'carve' out of it. (The structure may only be 'monoidal' not 'Cartesian' closed, for instance.) For this reason it pays to have both approaches.

We can identify various properties of group actions for a special mention. Here G may be a group or a simplicial group (or often more generally, but we do not need that yet) and X will be a set respectively a simplicial set, etc. (We choose a slightly different form of condition, than we will be using later on. The links between them can be **left to you**.)

Definition: (i) A left group action

$$a : G \times X \to X,$$

is said to be *effective* (or *faithful*) if $gx = x$ for all $x \in X$ implies that $g = 1_G$.

(ii) The G-action is said to be *free* (or sometimes, *principal*, cf. May, [155]) if $gx = x$ for some $x \in X$ implies $g = 1_G$.

(iii) If $x \in X$, the *orbit* of x is the set $\{g.x \mid g \in G\}$.

Clearly (i) can be, more or less equivalently, stated as, if $g \neq 1_G$, then there is an $x \in X$ such that $gx \neq x$. This is a form sometimes given in the literature. Whether or not you consider it equivalent depends on your logic. The use of negation means that in some context this formulation of the condition is less easy to use than the former.

For future use, it will be convenient to also have slightly different, but equivalent, ways of viewing these simplicial actions. For these we need to go back again to the simplicial mapping space, $\underline{\mathcal{S}}(K,L)$ and the composition, (see page 220). Suppose we have, as there, three simplicial sets, K, L and M, and the composition:

$$\underline{\mathcal{S}}(K,L)\times\underline{\mathcal{S}}(L,M)\to\underline{\mathcal{S}}(K,M).$$

(The product is symmetric so this is equivalent to

$$\underline{\mathcal{S}}(L,M)\times\underline{\mathcal{S}}(K,L)\to\underline{\mathcal{S}}(K,M).$$

The former is the viewpoint of the 'algebraic' concatentation composition order, the latter is the 'analytic' and 'topological' one. Of course, which you choose is up to you. We will tend to use the second, but sometimes .... .)

We want to look at the situation where $K=\Delta[0]$. As $\Delta[0]$ is the terminal object in $\mathcal{S}$, $\Delta[0]\times\Delta[n]\cong\Delta[n]$, so $\underline{\mathcal{S}}(\Delta[0],L)\cong L$. If we substitute from this back into the previous composition, we get

$$eval: L\times\underline{\mathcal{S}}(L,M)\to M.$$

(It is equally valid, to write the product around the other way, giving

$$eval:\underline{\mathcal{S}}(L,M)\times L\to M,$$

which correspond better to the 'analytic' Leibniz composition order. We will often use this form as well.) In either notational form, this is the simplicially enriched evaluation map, the analogue of $eval(x,f)=f(x)$ in the set theoretic case. (We will usually write *eval* for this sort of map.) Of course, if $L=M$, this situation is exactly that of the simplicial action of the simplicial monoid of self maps of L on L itself.

We can take the simplicial version apart quite easily, to see what makes it work.

Going back one stage, if $g\in\underline{\mathcal{S}}(K,L)_n$ and $f\in\underline{\mathcal{S}}(L,M)_n$, we can form their composite using the trick we saw earlier, in the discussion in section 5.3, page 220. We can replace $g:K\times\Delta[n]\to L$, by a map over $\Delta[n]$, given by $\overline{g}=(g,p_2):K\times\Delta[n]\to L\times\Delta[n]$, and then compose with $f:L\times\Delta[n]\to M$ to get the composite $f\circ g\in\underline{\mathcal{S}}(K,M)_n$, or use the 'over $\Delta[n]$ version to get $\overline{f\circ g}=\overline{f}\overline{g}:K\times\Delta[n]\to M\times\Delta[n]$. We note

$$\overline{f\circ g}(k,\sigma)=(f(g(k,\sigma),\sigma),\sigma),$$

(yes, we do need all those σs!).

Next we try the formulae with $K=\Delta[0]$ and '$g=\ulcorner x\urcorner$', the 'naming' map for an n-simplex, x, in L. That is not quite right, and to make things 'crystal clear', we had better be precise. The naming map for x has domain $\Delta[n]$ and we need the corresponding map, g, defined on $\Delta[0]\times\Delta[n]$. (Here the notation is getting almost 'silly', but to track things through it is probably necessary to do this, at least once! It shows how the details are there and can be taken out from the abstract packaging if and when we need them.) This map g is defined by $g(s_0^{m)}\iota_0,\sigma)=\ulcorner x\urcorner(\sigma)$, and this is 'really' given by $g(s_0^{(n)}(\iota_0),\iota_n)$ as that special case determines the others by the simplicial identities, so that, for $\sigma\in\Delta[n]_m$, so $\sigma:[m]\to[n]$, $g(s_0^{m)}\iota_0,\sigma)=L_\sigma g(s_0^{(n)}(\iota_0),\iota_n)$. (It may help here to think of σ as one of the usual face inclusions or degeneracies, at least to start with.) We have not yet used what g is, but $g(s_0^{(n)}(\iota_0), \iota_n) = x$, that is all! We can now work out (with all the identifications taken into account),

$$eval(x, f) = \overline{f \circ g}(s_0^{(n)}\iota_0, \iota_n) = f(x, \iota_n).$$

We might have guessed that this was the formula, ... what else could it be? This derivation, however, obtains it consistently with the natural 'action' formula, without having to check any complicated simplicial identities.

We will use this formula in the next chapter when discussing the structure of fibre bundles in the simplicial context.

5.4.2 G-principal fibrations

Specialising down to the simplicial case for now, suppose that G is a simplicial group acting on a simplicial set, E, then we can form a quotient complex, B, by identifying x with $g.x$, $x \in E_q$, $g \in G_q$. In other words the q-simplices of B are the orbits of the q-simplices of E, under the action of G_q. We note that this works (for **you to check**).

Lemma 37 *(i) The graded set, $\{B_q\}_{q\geq 0}$ forms a simplicial set with induced face and degeneracy maps, so that, if $[x]_G$ denotes the orbit of x under the action of G_q, then $d_i^B[x]_G = [d_i^E x]_G$, and similarly $s_i^B[x]_G = [s_i^E x]_G$.*

(ii) The graded function, $p : E \to B$, $p(x) = [x]_G$, is a simplicial map. ■

Definition: A map of the form $p : E \to B$, as above, is called a *principal fibration*, or, more exactly, *G-principal fibration* if we need to emphasise the simplicial group being used.

A morphism between two such objects will be a simplicial map over B, which is G-equivariant for the given G-actions.

(Any such morphism will be an isomorphism; **for you to check**.)

We will denote the set of isomorphism classes of G-principal fibrations on B by $Princ_G(B)$.

This definition really only makes sense if such a p is a fibration. Luckily we have:

Proposition 65 *Any map $p : E \to B$, as above, is a Kan fibration.*

Proof: Suppose $p : E \to B$ is a principal fibration. We assume that we have (cf. page **??**) a commutative diagram

$$\begin{tikzcd} \Lambda^i[n] \arrow[r, "f_1"] \arrow[d, "inc"'] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r, "f_0"'] & B \end{tikzcd}$$

and will write $b = f_0(\iota_n)$ for the corresponding n-simplex in B, and $(x_0, \ldots, x_{i-1}, -, x_{i+1}, \ldots, x_n)$ a compatible set of $(n-1)$-simplices up in E, in other words, a (n, i)-horn in E and a filler, b, for its image down in B.

Pick a $x \in E_n$ such that $p(x) = b$, then as $d_j p(x) = p(x_j)$, we have there are unique elements $g_j \in G_{n-1}$ such that $d_j x = g_j x_j$. ('Uniqueness' comes from the assumed properties of the action.)

It is easy to **check** (again using 'uniqueness') that the g_js give a (n,i)-horn in G, which, since G is a 'Kan complex', has a filler (use the algorithm in section **??**). Let g be the filler and set $y = g^{-1}x$. It is now **easy to check** that $d_k y = x_k$ for all $k \neq i$, i.e., that y is a suitable filler. ■

We need to investigate the class of these principal fibrations (for some fixed G). (We will tend to omit specific mention of the simplicial group G being used if, within a context, it is 'fixed', so, for instance, if we are not concerned with a 'change of groups' context.)

Let us suppose that $p : E \to B$ is a principal fibration and that $f : X \to B$ is any simplicial map. We can form a pullback fibration

$$\begin{tikzcd} E_f \arrow[r, "f'"] \arrow[d, "f^*(p)"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B. \end{tikzcd}$$

Is this pullback a G-principal fibration? Or to use terminology that we introduced earlier (section **??**), is the class of principal fibrations pullbacks stable?

There are several proofs of the result that it is, some of which are very neat, but here we will use the trusted method of 'brute force and ignorance', using as little extra machinery as possible. We have a reasonable model for E_f, so we should expect to be able to give it an explicit G-action in a fairly obvious natural way. We then can see what the orbits look like. That sounds a simple plan and it in fact works nicely.

We will model E_f as $E \times_B X$. (Previously, we had it around the other way as $X \times_B E$, but the two are isomorphic and this way is marginally easier notationally.) Recall the n-simplices in $E \times_B X$ are pairs (e, x) with $e \in E_n$, $x \in X_n$ and $p(e) = f(x)$. The G-action is staring at us. It surely must be

$$g \cdot (e, x) = (g \cdot e, x),$$

but does this work? We note $p(e) = [e]_G$, the G-orbit of e, so $p(g \cdot e) = p(e) = f(x)$, so we end up in the correct object. (**You are left** to check that this *is* a G-action and that it is free and effective.) What are the orbits?

We have (e, x) and (e', y) will be in the same orbit provided that there is a g such that $(g \cdot e, x) = (e', y)$, but that means that $x = y$ and that e and e' are in the same G-orbit within E. This has various consequences, which you are **left to explore**, but it is clear that, up to isomorphism, the map $f^*(p)$, which is projection onto the x component, *is* the quotient by the action. We have verified (except for the bits **left to you**:

Proposition 66 *If $p : E \to B$ is a G-principal fibration, and $f : X \to B$ is a simplicial map, then $(E_f, X, f^*(p))$ is a G-principal fibration.* ■

Of particular interest is the case when $X = \Delta[n]$, so that f is a 'naming' map, (cf. page **??**), $\ulcorner b \urcorner$, for some n-simplex, $b \in B_n$. We can, in this case, think of E_f as being the 'fibre' over b, although b is in dimension n.

This is very useful because of the following:

Lemma 38 *If $p : E \to \Delta[n]$ is a G-principal fibration, then $E \cong \Delta[n] \times G$, with p corresponding to the first projection.*

Before launching into the proof, it should be pointed out that here $\Delta[n] \times G$, should really be written $\Delta[n] \times U(G)$, where $U(G)$ is the underlying simplicial set of G. Of course there is a natural free and effective G-action on $U(G)$, with exactly one orbit. We have suppressed the U as this is a common 'abuse' of notation.

Proof: We have a single non-degenerate n-simplex in $\Delta[n]$, namely ι_n, which corresponds to the identity map in $\Delta[n]_n = \mathbf{\Delta}([n],[n])$. We pick any $e_n \in p^{-1}(\iota_n)$ and get a map, $\ulcorner e_n \urcorner : \Delta[n] \to E$, naming e_n. Of course, the composite, $p \circ \ulcorner e_n \urcorner$, is the identity on $\Delta[n]$. (This means that the fibration is 'split', in a sense we will see several times later on.)

Suppose $e \in E_m$, then $p(e) = \mu \in \Delta[n]_m = \mathbf{\Delta}([m],[n])$. We have another possibly different element in $p^{-1}(\mu)$, since $\mu : [m] \to [n]$ induces $E(\mu) : E_n \to E_m$, and so we have an element $E(\mu)(e_n)$. (**You can easily check** that, as p is a simplicial map, $p(E(\mu)(e_n)) = \mu$, i.e. $E(\mu)(e_n) \in p^{-1}(\mu)$, but therefore there is a unique element $g_m \in G_m$ such that $g_m \cdot E(\mu)(e_n) = e$. Starting with e, we got a unique pair $(\mu, g_m) \in (\Delta[n] \times G)_m$ and, from that pair, we can retrieve e by the formula. (You are **left to check** that this yields a simplicial isomorphism over $\Delta[n]$.) ■

We will see this sort of argument several times later. We have a 'global section,' here $\ulcorner e_n \urcorner$, of some G-principal 'thing' (fibration, bundle, torsor, whatever) and the conclusion is that the 'thing' is trivial' that is, a product thing.

5.4.3 Homotopy and induced fibrations

A key result that we will see later is that, if you use homotopic maps to pullback something like a fibration, or its more structured version, a fibre bundle, then you get 'related' pullbacks. Here we will look at the simplest, least structured, case, where we are forming pullbacks of *fibrations*. As this is a very important result, we will include quite a lot of detail.

As $\Delta[1]_0 = \Delta([0],[1])$, it has two elements, which we will write as e_0 and e_1, where $e_i(0) = i$, for $i = 0, 1$. (We will use this simplified notation several times later in the notes and should point out that e_0 corresponds to δ_1, and so induces d_1 if passing to simplicial notation, whilst $e_1 is\ \delta_0$, corresponding to d_1, which is the 'face opposite 1', hence is 0. This is slightly confusing, but the added intuition of $K \times \Delta[1]$ being a cylinder with $K \times \ulcorner e_0 \urcorner : K \cong K \times \Delta[0] \to K \times \Delta[1]$ being inclusion at the bottom end is too good to pass by!)

In what follows, we will quietly write e_i instead of $\ulcorner e_i \urcorner$, as it is a lot more convenient.

Proposition 67 *Let $p : E \to B$ be a Kan fibration and let $f, g : A \to B$ be homotopic simplicial maps, with $F : f \simeq g$, a specific homotopy, then there is a homotopy equivalence* over A *between $f^*(p) : E_f \to A$ and $g^*(p) : E_g \to A$.*

Proof: We first write $f = F \circ (A \times e_0)$, then we form E_f in two stages, by forming two pullbacks:

$$\begin{tikzcd}
E_f \arrow[r, "i_f"] \arrow[d, "f^*(p)"'] & E_F \arrow[r] \arrow[d, "F^*(p)"] & E \arrow[d, "p"] \\
A \arrow[r, "A\times e_0"'] & A \times \Delta[1] \arrow[r, "F"'] & B
\end{tikzcd}$$

A similar construction works, of course, for E_g using $A \times e_1$.

We have, from Lemma **??**, that, as $F^*(p)$ is a Kan fibration, so is $q_f := \underline{S}(E_f, F^*(p))$, and so also is $q_g := \underline{S}(E_g, F^*(p))$. These maps just compose with $F^*(p)$, so

$$q_f(i_f) = f^*(p) \times e_0.$$

Next we note that $f^*(p) \times \Delta[1] : E_f \times \Delta[1] \to A \times \Delta[1]$, so is in $\underline{S}(E_f, A \times \Delta[1])_1$ and $f^*(p) \times e_0 = d_1(f^*(p) \times \Delta[1])$. We now have a (1,1)0-horn, $(-, i_f)$ in $\underline{S}(E_f, E_F)$, whose image $(-, q - f(i_f))$ in $\underline{S}(E_f, A \times \Delta[1])$ has a filler, namely $f^*(p) \times \Delta[1]$. We can thus lift that filler to one y_f, say, in $\underline{S}(E_f, E_F)_1$, with $d_1(y_f) = i_f$, and, of course, $q_f(y_f) = f^*(p) \times \Delta[1]$. What is the other end, $d_0(y_f)$?

This is also in $\underline{S}(E_f, E_F)_0$, so is a simplicial map from E_f to E_F. This suggests it might be a map of fibrations. Does

$$\begin{tikzcd} E_f \arrow[r, "d_0(y_f)"] \arrow[d, "f^*(p)"'] & E_F^{F^*(p)} \\ A \arrow[r, "A\times e_1"'] & A \times \Delta[1] \end{tikzcd}$$

commute? We calculate,

$$\begin{aligned} F^*(p)d_0(y_f) &= q_F(d_0(y_f)) \\ &= d_0(q_f(y_f)) \\ &= d_0(f^*(p) \times \Delta[1]) \\ &= (A \times e_1) \circ f^*(p), \end{aligned}$$

so it is, but this means that, as bottom 'right-hand corner' of the square, had E_g as its pullback, we get a map, $\alpha : E_f \to E_g$, over A, so that $f^*(p) = g^*(p)\alpha$, and $d_0(y_f) = i_g\alpha$. This gives us the first part of our homotopy equivalence.

Reversing the roles of f and g, we get a y_g in $\underline{S}(E_g, E_F)_1$ with $d_0(y_g) = i_g$, then $q_g(y_g) = g^*(p) \times \Delta[1]$, and we get a $\beta : E_g \to E_f$ such that $f^*(p)\beta = g^*(p)$ and $i_f\beta = d_1(y_g)$.

We now have to look at the composites $\alpha\beta$ and $\beta\alpha$, and to show they are homotopic (over A) to the identities. Of course, we need only produce one of these as the other will follow 'similarly', on reversing the roles of f and g.

Considering $s_0(\alpha) \in \underline{S}(E_f, E_g)_1$ and $y_g \in \underline{S}(E_g, E_F)_1$, we have a composite (really a composite homotopy), that we will denote by $\xi \in \underline{S}(E_f, E_F)_1$. We can check (**for you to do**) that $d_0(\xi) = d_0(y_f)$ and $d_1(\xi) = d_i(y_g)\alpha = i_f\beta\alpha$. We thus have a horn

$$h = \begin{tikzcd} & i_f\beta\alpha \arrow[dr, "\xi"] & \\ i_f \arrow[rr, "y_f"'] & & d_0(y_f) \end{tikzcd}$$

in $\underline{S}(E_f, E_F)$. We look at its image in $\underline{S}(E_f, A\times\Delta[1])$, and **check** it can be filled by $s_0(f^*(p)\times\Delta[1])$, that means that, as $F^*(p)$ is a Kan fibration, we can find a filler, z, for h, so set $w := d_2(z)$. (This is a composite homotopy, as if it was topologically 'y_f followed by the reverse of ξ.') this homotopy, w, is in $\underline{S}(E_f, E_F)$, not in $\underline{S}(E_f, E_f)$, but otherwise does the right sort of thing.

To get a homotopy with E_f as codomain, we use the left hand pullback square of the above double pullback diagram, so have to work out $F^*(p)(w)$. This is just our $q_f(w)$ and that, by the description of z as a filler is $d_2s_0(f^*(p) \times \Delta[1]) = s_0d_1(f^*(p) \times \Delta[1]) = f^*(p).pr_{E_f}.(A \times e_0)$, so we have a map $w' : E_f \times \Delta[1] \to E_f$, as in the diagram

$$\begin{tikzcd}
E_f \times \Delta[1] \arrow[dr, "w'"] \arrow[drr, bend left, "w"] \arrow[ddr, bend right, "f^*(p).pr_{E_f}"'] & & \\
& E_f \arrow[r, "i_f"'] \arrow[d, "f^*(p)"] & E_F \arrow[d, "F^*(p)"] \\
& A \arrow[r, "A\times e_0"'] & A \times \Delta[1],
\end{tikzcd}$$

where $pr_{E_f} : E_f \times \Delta[1] \to E_f$ is the projection. Note that w' is a homotopy *over* A, so is 'in the fibres'.

This w' certainly goes between the right objects, but is it the required homotopy. We check

$$i_f.w'.e_1 = w.e_1 = i_f\beta\alpha,$$

but i_f is the induced map from $A \times e_0$, which is a (split) monomorphism, so i_f is itself a monomorphism, and so $w'.e_1 = \beta\alpha$. Similarly $w'.e_0 = id_{E_f}$, so w' does what was hoped for.

We reverse the roles of α and β, and of f and g, to get the last part of the proof. ∎

5.5 $\overline{W}$, W and twisted Cartesian products

Suppose we have simplicial sets, Y, a potential 'fibre' and B, a potential 'base', which will be assumed to be pointed by a vertex, $*$. Inspired by the sort of construction that works for the construction of group extensions, we are going to try to construct a fibration sequence,

$$Y \longrightarrow E \longrightarrow B.$$

Clearly the product $E = B \times Y$ will give such a sequence, but can we somehow *twist* this Cartesian product to get a more general construction? We will try setting $E_n = B_n \times Y_n$ and will change as little as possible in the data specifying faces and degeneracies. In fact we will take all the degeneracy maps to be exactly those of the Cartesian product, and all but d_0 of the face maps likewise. This leaves just the zeroth face map.

In, say, a covering space considered as a fibration with discrete fibre, the fundamental group(oid) of the base acts by automorphisms / permutations on the fibre, and the fundamental group(oid) is generated by the edges, hence by elements of dimension one greater than that of the fibre, so we try a formula for d_0 of form

$$d_0(b, y) = (d_0b, t(b)(d_0y)),$$

where $t(b)$ is an automorphism of Y, determined by b in some way, hence giving a function $t : B_n \longrightarrow \mathsf{aut}(Y)_{n-1}$. Note here Y is an arbitrary simplicial set, not the underlying simplicial set of a simplicial group as was previously the case when we considered aut, but this makes no difference to the definition.

Of course, with these tentative definitions, we must still have that the simplicial identities hold, but it is easy to check that these will hold exactly if t satisfies the following equations

$$\begin{aligned} d_i t(b) &= t(d_{i-1}b) \quad \text{for} \quad i > 0, \\ d_0 t(b) &= t(d_1 b)\#_0 t(d_0 b)^{-1}, \\ s_i t(b) &= t(s_{i+1}b) \quad \text{for} \quad i \geq 0, \\ t(s_0 b) &= *. \end{aligned}$$

A function, t, satisfying these equations will be called a *twisting function*, and the simplicial set E, thus constructed, will be called a *regular twisted Cartesian product* or $T.C.P.$ We write $E = B \times_t Y$.

It is often useful to assume that the twisting function is 'normalised' so that $t(*)$ is the identity automorphism. We usually will tacitly make this assumption if the base is pointed.

If this construction is to make sense, then we really need also a 'projection' from E to B and Y should be isomorphic to its fibre over the base point, $*$. The obvious simplicial map works, sending (b, y) to b. It *is* simplicial and clearly has a copy of Y as its fibre.

Of course, a twisting function is not a simplicial map, but the formulae it satisfies look closely linked to those of the Dwyer-Kan loop group(oid) construction, given earlier, page 213. In fact:

Proposition 68 *A twisting function, $t : B \longrightarrow \mathsf{aut}(Y)$, determines a unique homomorphism of simplicial groupoids $t : GB \to \mathsf{aut}(Y)$, and conversely.* ■

Of course, since G is left adjoint to $\overline{W}$, we could equally well note that t gave a simplicial morphism $t : B \longrightarrow \overline{W}(\mathsf{aut}(Y))$, and conversely.

Of course, we could restrict attention to a particular class of simplicially enriched groupoids such as those coming from groups (constant simplicial groups), or nerves of crossed modules, or of crossed complexes, etc. We will see some aspects of this in the following chapter, but we will be generalising it as well.

This adjointness gives us a 'universal' twisting function for any simplicial group, H. We have the general natural isomorphism,

$$\mathcal{S}(B, \overline{W}H) \cong Simp.Grpds(G(B), H),$$

so, as usual in these situations, it is very tempting to look at the special case where $B = \overline{W}H$ itself and hence to get the counit of the adjunction from $G\overline{W}(H)$ to H corresponding to the identity simplicial map from $\overline{W}H$ to itself. By the general properties of adjointness, this map 'generates' the natural isomorphism in the general case.

From our point of view, the two natural isomorphic sets are much better viewed as being $\mathsf{Tw}(B, H)$, the set of twisting functions $\tau : B \to H$, so the key case will be a 'universal' twisting function, $\tau_H : \overline{W}H \to H$ and hence a universal twisted Cartesian product $\overline{W}H \times_{\tau_H} H$. (Notational point: the context tells us that the fibre H is the underlying simplicial set of the simplicial group, H, but no special notation will be used for this here.) This universal twisted Cartesian product is called the *classifying bundle for* H and is denoted WH. We can unpack its definition from its construction, but will not give the detailed derivation (which is suggested as a **useful exercise**). Clearly

$$(WH)_n = H_n \times_t \overline{W}(H)_n,$$

so from our earlier description of $\overline{W}(H)$, we have

$$WH_n = H_n \times H_{n-1} \times \ldots \times H_0.$$

The face maps are given by

$$d_i(h_n, \ldots, h_0) = (d_i h_n, \ldots, d_0 h_{n-i}.h_{n-i-1}, h_{n-i-2}, \ldots, h_0)$$

for all i, $0 \leq i \leq n$, whilst

$$s_i(h_n, \ldots, h_0) = (s_i h_n, \ldots s_0 h_{n-i}, 1, h_{n-i-1}, \ldots, h_0).$$

(It is noteworthy that $d_0(h_n, \ldots, h_0) = (d_0 h_n.h_{n-1}, h_{n-2}, \ldots, h_0)$ so the universal twist, τ_H, must somehow be built in to this. In fact τ_H is an 'obvious' map as one would hope. We have $\overline{W}(H)_n = H_{n-1} \times \ldots \times H_0$ and we need $(\tau_H)_n : \overline{W}(H)_n \to H_{n-1}$, since it is to be a twisting map and so has degree -1. The obvious formula to try is that τ_H is the projection map - and it works. The details are left to you. A glance back at the formula for the general d_0 in a twisted Cartesian product will help.)

We start by showing that $p : W(H) \to \overline{W}(H)$ is a principal fibration. This simplicial map just is the projection onto the second factor in the T.C.P. To prove this is such a principal fibration, we first examine $W(H)$ more closely and then at an obvious action. The simplicial set, $W(H)$, contains a copy of (the underlying simplicial set of) H as the fibre over the element $(1, 1, \ldots, 1) \in \overline{W}(H)$. There is then a fairly obvious action of H on $W(H)$, given by, in dimensions n,

$$h'.(h_n, \ldots, h_0) = (h' h_n, \ldots, h_0).$$

In other words, just using multiplication on the first factor. As multiplication is a simplicial map, $H \times H \to H$, or simply glancing at the formulae, we have that this *is* a simplicial action.

That action is *free*, since the regular representation is free as an action. (After all, this is just saying that, if $gx = x$ for some $x \in H$, then $g = 1$, so is obvious!) The action is also faithful / effective, for similar reasons. What are the orbits? As the action only changes the first coordinate, and does that freely and faithfully, the orbits coincide with the fibres of the projection map from $W(H)$ to $\overline{W}(H)$, so that p is also the quotient map coming from the action. It follows that

Lemma 39 *The simplicial map*

$$W(H) \to \overline{W}(H),$$

is a principal fibration. ■

The following observations now are either corollaries of this, simple to check or should be looked up in 'the literature'.

1). The simplicial set, $W(H)$, is a Kan complex.

2). $W(H)$ is contractible, i.e., is homotopy equivalent to $\Delta[0]$.

3). The simplicial map,

$$W(H) \to \overline{W}(H),$$

is a Kan fibration with fibre the underlying simplicial set of H, (so the long exact sequence of homotopy groups together with point 2) shows that $\pi_n(\overline{W}H) \cong \pi_{n-1}(H)$).

4). If $p : E \to B$ is a principal H-bundle, that is, E is $H \times_t B$ for some twisting function, $t : B \to H$, then we have a simplicial map
$$f_t : B \to \overline{W}(H)$$
given by $f_t(b) = (t(b), t(d_0 b), \ldots, t(d_0^{n-1} b))$, and we can pull back $(W(H) \to \overline{W}(H))$ along f_t to get a principal H-bundle over B
$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"'] & W(H) \arrow[d] \\ B \arrow[r, "f_t"'] & \overline{W}(H). \end{tikzcd}$$
We can, of course, calculate E' and p' precisely:
$$\begin{aligned} E' &\cong \{((h_n, h_{n-1}, \ldots, h_0), b) \mid h_{n-1} = t(b), \ldots h_0 = t(d_0^{n-1} b)\} \\ &\cong \{(h_n, b) \mid h_n \in H_n, b \in B_n\} \\ &= H_n \times B_n. \end{aligned}$$
It should come as no surprise to find that $E' \cong H \times_t B$, so is E itself up to isomorphism, and that p' is p in disguise.

The assignment of f_t to t gives a one-one correspondence between the set, $Princ_H(B)$, of H-equivalence classes of principal H-bundles with base B, and the set, $[B, \overline{W}(H)]$, of homotopy classes of simplicial maps from B to $\overline{W}(H)$.

An important thing to remember is that not all T.C.Ps are principal fibrations. To get a T.C.P., we just need a fibre Y, a base, B, and a simplicial group, G, acting on Y, together with our twisting function, $t : B \to \overline{W}(G)$. From B and t, we can build a principal fibration which is, of course, a T.C.P. but has fibre the underlying simplicial set of G. To build the T.C.P., $B \times_t Y$, we need the *additional* information about the *representation* $G \to \mathsf{aut}(Y)$, that is, the action of G on the fibre, and, of course, that representation need not be an isomorphism. In general, we have: 'fibre bundle = principal fibration plus representation', as a rule of thumb. This is not just in the simplicial case. (We will consider fibre bundles and similar other structures in a lot more detail in the next chapter.)

A good introduction to simplicial bundle theory can be found in Curtis' classical survey article, [74] section 6, or, for a thorough treatment, May's book, [155]. For full details, you are invited to look there, at least to know what is there. We have not gone into all the detail here. We will revisit the overall theory several times later on, drawing parallels and comparisons that will, it is hoped, shed light both on it and on geometrically related theories elsewhere in the area.

5.6 More examples of Simplicial Groups

We have already seen several general constructions of simplicial groups, for instance, the simplicial resolutions of a group, the loop group on a reduced simplicial set, the internal nerve of a crossed module / cat^1-group, and so on. The previous few sections give some ideas for other construction leading to simplicial groups. We will concentrate on two such.

Let G be a topological (or Lie) group (so a group internal to 'the' category of topological spaces - whichever one is most appropriate for the situation). The singular complex functor, $Sing : Top \to \mathcal{S}$, preserves products,

$$Sing(X \times Y) \cong Sing(X) \times Sing(Y),$$

so it follows that, as the multiplication on G is continuous, there is an induced simplicial map,

$$Sing(G) \times Sing(G) \to Sing(G).$$

With the map induced from the maps that picks out the identity element and that give the inverse, this makes $Sing(G)$ into a simplicial group. This gives a large number of interesting simplicial groups, corresponding to general linear, orthogonal, and other topological (or Lie) groups of various dimension. Of course, the homotopy groups of these simplicial groups correspond to those of the groups themselves.

A closely related construction involves a similar idea to the $\mathsf{aut}(K)$ simplicial group, that we used when discussing simplicial bundles, twisted Cartesian products, etc., a few sections ago. We had a simplicial set, K, and hence a simplicial monoid, $\underline{\mathcal{S}}(K,K)$, of endomorphisms of K. The simplicial group, $\mathsf{aut}(K)$, was the corresponding simplicial group of simplicial automorphisms of K. We had a representation of such an $f : K \times \Delta[k] \to K$ as $(f,p) : K \times \Delta[k] \to K \times \Delta[k]$ and this was an automorphism *over* $\Delta[k]$, (look back to page 221).

This sort of construction will work in any situations where the basic category being studied is 'simplicially enriched', i.e. the usual hom-sets of the category form the vertices of simplicial hom-sets and the composition maps between these are simplicial. We will formally introduce this idea later, (see Chapter **??**, and in particular section **??**, page **??**). Here we will give some examples of this type of idea in situations that are useful in geometric and topological contexts.

We will assume that X is a (locally finite) simplicial complex. In applications X is often $\mathbb{R}^n$, or S^n or similar. We think of the product, $\Delta^k \times X$, as a 'bundle over the k-simplex, Δ^k, or, if working in the piecewise linear (PL) setting, a PL bundle over Δ^k. The simplicial group, $\mathcal{H}(X)$, is then the simplicial group having $\mathcal{H}(X)_k$ being the set of homeomorphisms of $\Delta^k \times X$ over Δ^k, or, alternatively, the (PL) bundle isomorphisms of $\Delta^k \times X$. As a variant, if $A \subset X$ is a subcomplex, one can restrict to those bundle isomorphisms that fix $\Delta^k \times A$ pointwise.

Various examples of this were used to study the problem of the existence and classification of triangulations and smoothings for manifolds. The construction occurs, for instance, in Kuiper and Lashof, [141, 142]. Later on starting in section 9.1.4, we will look at another variant of these examples concerning microbundle theory, (see Buoncristiano, [59, 60]), as it gives a nice interpretation of some simplicial bundles in a geometric setting.

Chapter 6

Non-Abelian Cohomology: Torsors, and Bitorsors

One of the problems to be faced when presenting the applications of crossed modules, etc., is that such is the breadth of these applications that they may safely be assumed to be potentially of interest to mathematicians of very differing backgrounds, algebraists of many different hues, geometers both algebraic and differential, theoretical physicists and, of course, algebraic topologists. To make these notes as useful as possible, some part of the more basic 'intuitions' from the background material from some of these areas has been included at various points. This cannot be 'all inclusive' nor 'universal' as different groups of potential readers have different needs. The real problems are those of transfer of 'technology' between the areas and of explanation of the differing terminology used for the same concept in different contexts. Often, essentially the same idea or result will appear in several places. This repetition is not just laziness on the authors behalf. The introduction of a concept bit-by-bit from various angles almost necessitates such a treatment.

For the background on bundle-like constructions (sheaves, torsors, stacks, gerbes, 2-stacks, etc.), the geometric intuition of 'things over X' or X-parametrised 'things' of various forms, does permeate much of the theory, so we will start with some fairly basic ideas, and so will, no doubt, for some of the time, be 'preaching to the converted', however that 'bundle' intuition is so important for this and later sections that something more than a superficial treatment is required.

(In the original lectures at Buenos Aires, I did assume that that intuition was understood, but in any case concentrated on the 'group extension' case rather than on 'gerbes' and their kin. By this means I avoided the need to rely too heavily on material that could not be treated to the required depth in the time available. However I cannot escape the need to cover some of that material here!)

Initially crossed modules, etc., will not be that much in evidence, *but* it is important to see how they do enter in 'geometrically' or their later introduction can seem rather artificial.

We start by looking at descent, i.e., the problem of putting 'local' bits of structure into a global whole.

6.1 Descent: Bundles, and Covering Spaces

(Remember, if you have met 'descent' or 'bundles', then you should 'skim' this section only / anyway.)

We will look at these structures via some 'case studies' to start with.

6.1.1 Case study 1: Topological Interpretations of Descent.

Suppose A and B are topological spaces and $\alpha : A \to B$ a continuous map (sometimes called a 'space over B' or loosely speaking a 'bundle over B', although that can also have a more specialised meaning later). The space, B, will usually be called the *base*, whilst A is the *total space* of the bundle, α.

An obvious and important example is a product, $A = B \times F$, with α being the projection. We call this a *trivial bundle* on B.

If $U \subset B$ is an open set, then we get a restriction $\alpha_U : \alpha^{-1}(U) \to U$. If $V \subset B$ is another open set, we, of course, have $\alpha_V : \alpha^{-1}(V) \to V$ and over $U \cap V$ the two restrictions 'coincide', i.e., if we form the pullbacks

$$\begin{tikzcd} ? \arrow[r] \arrow[d] & \alpha^{-1}(U) \arrow[d] \\ U \cap V \arrow[r] & U \end{tikzcd} \qquad \begin{tikzcd} ? \arrow[r] \arrow[d] & \alpha^{-1}(V) \arrow[d] \\ U \cap V \arrow[r] & V \end{tikzcd}$$

the resulting spaces over $U \cap V$ are 'the same'. (We have to be a bit careful since we formed them by pullbacks so they are determined only 'up to isomorphism' and we should take care to interpret 'the same' as meaning 'being isomorphic' as spaces over $U \cap V$. This care will be important later.) Now assume that for each $b \in B$, we choose an open neighbourhood $U_b \subset B$ of b. We then have a family

$$\alpha_b : A_b \to U_b \qquad\qquad b \in B,$$

where we have written A_b for $\alpha^{-1}(U_b)$, and we know information about the behaviour over intersections.

Can we reverse this process? More precisely, can we start with a family $\{\alpha_b : A_b \to U_b : b \in B\}$ of maps (with A_b now standing for an arbitrary space) and add in, say, information on the 'compatibility' over the intersections of the cover $\{U_b : b \in B\}$ so as to rebuild a space over B, $\alpha : A \to B$, which will restrict to the given family.

We will need to be more precise about that 'compatibility', but will leave it aside until a bit later. Clearly, indexing the cover by the elements of B is a bit impractical as usually we just need, or are given, some (open) cover, $\mathcal{U}$, of B, and then can choose, for each $b \in B$, some set of the cover containing b. This way we do not repeat sets unless we expressly need to. Thinking like this we have a cover $\mathcal{U}$ and for each U in $\mathcal{U}$, a space over U, $\alpha_U : A_U \to U$. To encode the condition on compatibility on intersections, we need some (temporary) notation: If $U, U' \in \mathcal{U}$, write $(A_U)_{U'}$ for the restriction of A_U over the intersection $U \cap U'$, similarly $(\alpha_U)_{U'}$ for the restriction of α_U to $U \cap U'$. This is given by the further pullback of α_U along the inclusion of $U \cap U'$ into U, so we also get a map

$$(\alpha_U)_{U'} : (A_U)_{U'} \to U \cap U'.$$

We noted that if the family $\{\alpha_U \mid U \in \mathcal{U}\}$ did come from a single $\alpha : A \to B$, then the α_Us agreed up to isomorphism on the intersections, i.e., we needed homeomorphisms

$$\xi_{U,U'} : (A_U)_{U'} \stackrel{\cong}{\to} (A_{U'})_U$$

over $U \cap U'$ if we were going to give an adequate description. (These are sometimes called the *transition functions* or gluing *cocycles*.) This, of course, means that

$$(\alpha_{U'})_U \circ \xi_{U,U'} = (\alpha_U)_{U'}.$$

Clearly we should require
1. $\xi_{U,U}$ = identity,
but also if U'' is another set in the cover, we would need
2. $\xi_{U',U''} \circ \xi_{U,U'} = \xi_{U,U''}$
over the triple intersection $U \cap U' \cap U''$.

(This condition 2. is a *cocycle condition*, similar in many ways to ones we have met earlier in apparently very different contexts.)

These two conditions are inspired by observation on decomposing an original bundle. They give us 'descent data', but are our 'descent data' enough to construct and, in general, to classify such spaces over B? The obvious way to attempt construction of an α from the data $\{\alpha_U; \xi_{U,U'}\}$ is to 'glue' the spaces A_U together using the $\xi_{U,U'}$. 'Gluing' is almost always a colimiting process, but as that can be realised using coproducts (disjoint union) and coequalisers (quotients by an equivalence relation), we will follow a two step construction

Step 1: Let $C = \sqcup_{U\in\mathcal{U}} A_U$ and $\gamma : C \to \sqcup_{U\in\mathcal{U}} U$, the induced map. Thus if we consider a specific U in $\mathcal{U}$, we will have inclusions of A_U into C and U into $\sqcup U$ and a diagram

$$\begin{tikzcd} A_U \arrow[r, hook] \arrow[d, "\alpha_U"'] & C = \sqcup A_U \arrow[d, "\gamma"] \\ U \arrow[r, hook] & \sqcup U \end{tikzcd} .$$

Remember that a useful notation for elements in a disjoint union is a pair, (element, index), where the index is the index of the set in which the element is. We write (a, U) for an element of C, then $\gamma(a, U) = (\alpha_U(a), U)$, since $a \in A_U$.

Step 2: We relate elements of C to each other by the rule:

$$(a, U) \sim (a', U')$$

if and only if
(i) $\alpha_U(a) = \alpha_{U'}(a')$,
and
(ii) we want to glue corresponding elements in fibres over the same point of B so need something like $\xi_{U,U'}(a) = a'$. Although intuitively correct, as it says that if a and a' are over the same point of $U \cap U'$ then they are to be 'related' or 'linked' by the homeomorphism, $\xi_{U,U'}$, a close look at the formula shows it does not quite make sense. Before we can apply $\xi_{U,U'}$ to a, we have to restrict a to be in $(A_U)_{U'}$ and the result will be in $(A_{U'})_U$. Perhaps the neatest way to present this is to look at another disjoint union, this time $\sqcup_{U,U'}(A_U)_{U'}$, and to map this to $C = \sqcup_{U\in\mathcal{U}} A_U$ in two ways. The first of these, $\mathbf{a}$, say, takes the component $(A_U)_{U'}$ and injects it into C via the injection of A_U. The second map, $\mathbf{b}$, first sends $(A_U)_{U'}$ to $(A_{U'})_U)$ using $\xi_{U,U'}$ then sends that second component to $(A_{U'})$ and thus into C. We thus get the correct version of the formula for (ii) to be:

there is an $x \in \sqcup_{U,U'}(A_U)_{U'}$ *such that* $\mathbf{a}(x) = a$ *and* $\mathbf{b}(x) = a'$.

The two conditions on the homeomorphisms ξ readily imply that this is an equivalence relation and that the α_U together define a map

$$\alpha : A = C/_\sim \ \to B$$

given by

$$\alpha[(a, U)] = \alpha_U(a),$$

on the equivalence class, $[(a, U)]$ of (a, U). For this to be the case, we only needed $\alpha_U(a) = \alpha_{U'}(a')$ to hold. Why did we impose the second condition, i.e., the cocycle condition? Simply, if we had not, we would risked having an equivalence relation that crushed C down to B. Each fibre $\alpha^{-1}(b)$ might have been a single point since each $\alpha_U^{-1}(a)$ could have been in a single equivalence class.

We now have a space over B, $\alpha : A \to B$ (with A having the quotient topology, which ensures that α will be continuous).

If we had started with such a space, decomposed over $\mathcal{U}$, then had constructed a 'new space' from that data, would we have got back where we started? Yes, up to isomorphism (i.e., homeomorphism over B). To discuss this, it helps to introduce the category, Top/B, of spaces over B. This has continuous maps $\alpha : A \to B$ (often written (A, α)) as its objects, whilst a map from (A, α) to $\alpha' : A' \to B$ will be a continuous map $f : A \to A'$ making the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\alpha"'] & & A' \arrow[dl, "\alpha'"] \\ & B & \end{tikzcd}$$

commutative. This, however, raises another question.

If we have such an f and an (open) cover $\mathcal{U}$ of B, we restrict f to $\alpha^{-1}(U)$ to get

$$f_U : A_U \to A'_U$$

which, of course, is in Top/U. If we have data,

$$\{\alpha_U : A_U \to U, \{\xi_{U,U'}\}\}$$

for (A, α) and similarly for (A', α'), and morphisms

$$\{f_U : A_U \to A'_U\},$$

when can we 'rebuild' $f : A \to A'$? We would expect that we would need a compatibility between the various f_U and the $\xi_{U,U'}$ and $\xi'_{U,U'}$. The obvious condition would be that whenever we had U, U' in $\mathcal{U}$, the diagram

$$\begin{tikzcd} (A_U)_{U'} \arrow[r, "(f_U)_{U'}"] \arrow[d, "\xi_{U,U'}"'] & (A'_U)_{U'} \arrow[d, "\xi'_{U,U'}"'] \\ (A_{U'})_U \arrow[r, "(f_{U'})_U"] & (A'_{U'})_U \end{tikzcd}$$

should commute, where we have extended our notation to use $(f_U)_{U'}$ for the restriction of f_U to $\alpha^{-1}(U \cap U')$. To codify this neatly we can form each category, Top/U, for $U \in \mathcal{U}$, then form the category, D, consisting of families of objects, $\{\alpha_U : U \in \mathcal{U}\}$, of $\prod Top/U$ together with the extra structure of the $\xi_{U,U'}$. Morphisms in D are families $\{f_U\}$ as above, compatible with the structural isomorphisms $\xi_{U,U'}$.

Remark: For any specific pair consisting of a family, $\mathcal{A} = \{(A_U, \alpha_U) : U \in \mathcal{U}\}$ and the extra $\xi_{U,U'}$s is a *set of descent data for* $\mathcal{A}$. We will look at both this construction and its higher dimensional relatives in quite a lot of detail and generality later on. The category of these things and the corresponding morphisms can be called the *category of descent data relative to the cover*, $\mathcal{U}$.

The reason for the use of the word 'descent' is that, in many geometric situations, structure is easily encoded on some basic 'patches'. This structure, that is locally defined, 'descends' to the space giving it a similar structure. In many cases, the A_U have the fairly trivial form $U \times F$ for some fibre F. This fibre often has extra structure and the $\xi_{U,U'}$ have then to be structure preserving automorphisms of the space, F. The term 'bundle' is often used in general, but some authors restrict its use to this *locally trivial* case. The classic case of a locally trivial bundle is a Möbius band as a bundle over the circle. Locally, on the circle, the band is of form $U \times [-1, 1]$, but globally one has a twist. A bit more formally, and for use later, we will define:

Definition: A bundle $\alpha : A \to B$ is said to be *locally trivial* if there is an open cover $\mathcal{U}$ of B, such that, for each U in $\mathcal{U}$, A_U is homeomorphic to $U \times F$, for some fibre F, compatibly with the projections, α_U and $p_U : U \times F \to U$.

We will gradually build up more precise intuitions about what 'compatibly' means, and as we do so, the above definition will gain in precision and strength.

6.1.2 Case Study 2: Covering Spaces

This is a classic case of a class of 'spaces over' another space. It is also of central importance for the development of possible generalisations to higher 'dimensions', (cf. Grothendieck's *Pursuit of Stacks*, [107].) We have a continuous map

$$\alpha : A \to B$$

and for any point $b \in B$, there is an open neighbourhood U of b such that $\alpha^{-1}(U)$ is the disjoint union of open subsets of A, each of which is mapped homeomorphically onto U by α. The map α is then called a *covering projection*. On such a U, $\alpha^{-1}(U)$ is $\sqcup U_i$ over some index set which can be taken to be $\alpha^{-1}(b) = F_b$, the fibre over b. Then we may identify $\alpha^{-1}(U)$ with $U \times F_b$ for any $b \in U$. This F_b is 'the same' up to isomorphism for all $b \in U$. If B is connected then for any b, $b' \in B$, we can link them by a chain of pairwise intersecting open sets of the above form and hence show that $F_b \cong F_{b'}$. We can thus take each $\alpha^{-1}(U) \cong U \times F$ and F will be a discrete space provided B is nice enough. The descent data in this situation will be the local covering projections

$$\alpha_U : U \times F \to U$$

together with the homeomorphisms

$$\xi_{U,U'} : (U \cap U') \times F \to (U \cap U') \times F$$

over $(U \cap U')$. Provided that $(U \cap U')$ is connected, this $\xi_{U,U'}$ will be determined by a permutation of F.

We often, however, want to allow for non-connected $(U \cap U')$. For instance, take B to be the unit circle S^1, $F = \{-1, 1\}$,

$$U_1 = \{\underline{x} \in S^1 \mid \underline{x} = (x, y), x > -0.1\}$$

$$U_2 = \{\underline{x} \in S^1 \mid \underline{x} = (x, y), x < 0.1\}.$$

The intersection, $U_1 \cap U_2$, is not connected, so we specify ξ_{U_1,U_2} separately on the two connected components of $U_1 \cap U_2$. We have

$$U_1 \cap U_2 = \{(x, y) \in S^1 \mid |x| < 0.1, y > 0\} \cup \{(x, y) \mid |x| < 0.1, y < 0\}.$$

Let $\xi_{U_1,U_2}((x,y),t) = \begin{cases} ((x,y),t) & \text{if } y > 0 \\ ((x,y),-t) & \text{if } y < 0, \end{cases}$
so on the part with negative y, ξ exchanges the two leaves. The resulting glued space is either viewed as the edge of the Möbius band or as the map,

$$S^1 \to S^1$$

$$e^{i\theta} \mapsto e^{i2\theta}.$$

Remark: The well known link between covering spaces and actions of the fundamental group $\pi_1(B)$ on Sets is at the heart of this example.

A neat way to picture the n-fold covering spaces of S^1 for $n \geq 2$ is to consider a knot on the surface of a torus, $S^1 \times S^1$, for instance the trefoil. The projection to the first factor of $S^1 \times S^1$ gives a covering, as does the second projection. It is **also instructive** to consider the covering space $\mathbb{R}^2 \to S^1 \times S^1$, working out what the various transitions are for a cover. We note the way that quotients of $\mathbb{R}^n$ by certain geometrically defined group actions, yields neat examples of coverings (although some may be 'ramified', an area we will not stray into here.)

In general, when we have a local product structure, so $\alpha^{-1}(U) \cong U \times F$, the homeomorphisms $\xi_{U,U'}$ have a nicer description than the general one, since being 'over' the intersection, they have to have the form that interprets at the product levels as being $\xi_{U,U'}(x, y) = (x, \xi'_{U,U'}(x)(y))$ where $\xi'_{U,U'} : U \cap U' \to Aut(F)$. In the case of covering spaces F is discrete, so $\xi'_{U,U'}(x)$ will give a permutation of F.

6.1.3 Case Study 3: Fibre bundles

The examples here are to introduce / recall how torsors / principal fibre bundles are defined topologically and also to give some explicit instances of how fibre bundles arise in geometry.

(Often in this context, the terminology 'total space' is used for the source of the bundle projection.)

First some naturally occurring examples.

(i) Let S^n denote the usual n-sphere represented as a subspace of $\mathbb{R}^{n+1}$,

$$S^n = \{\underline{x} \in \mathbb{R}^{n+1} \big| \; \|\underline{x}\| = 1\},$$

where $\|\underline{x}\| = \sqrt{\langle \underline{x} \mid \underline{x} \rangle}$ for $\langle \underline{x} \mid \underline{y} \rangle$, the usual Euclidean inner product on $\mathbb{R}^{n+1}$. The *tangent bundle* of S^n, τS^n is the 'bundle' with total space,

$$TS^n = \{(\underline{b}, \underline{x}) \bigm| \langle \underline{b} \mid \underline{x} \rangle = 0\} \subset S^n \times \mathbb{R}^{n+1}.$$

We thus have a projection

$$p : TS^n \to S^n$$

given by $p(\underline{b}, \underline{x}) = \underline{b}$, as a space over S^n.

Similarly the *normal bundle*, νS^n, of S^n is given with total space,

$$NS^n = \{(\underline{b}, \underline{x}) \mid \underline{x} = k\underline{b} \text{ for some } k \in \mathbb{R}\} \subset S^n \times \mathbb{R}^{n+1}.$$

The projection map $q : NS^n \to S^n$ gives, as before, a space over S^n, $\nu S^n = (NS^n, q, S^n)$.

Another example extends this to a geometric context of great richness.

(ii) First we need to introduce generalisations, the Grassmann varieties, of projective spaces and in order to see what topology it is to have, we look at a related space first. The *Stiefel variety* of k-frames in $\mathbb{R}^n$, denoted $V_k(\mathbb{R}^n)$, is the subspace of $(S^{n-1})^k$ such that $(v_1, \ldots, v_k) \in V_k(\mathbb{R}^n)$ if and only if each $\langle v_i \mid v_j \rangle = \delta_{i,j}$, so that it is 1 if $i = j$ and is zero otherwise. Note $V_1(\mathbb{R}^n) = S^{n-1}$.

The *Grassmann variety* of k-dimensional subspaces of $\mathbb{R}^n$, denoted $G_k(\mathbb{R}^n)$, is the set of k-dimensional subspaces of $\mathbb{R}^n$. There is an obvious function,

$$\alpha : V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n),$$

mapping $(v_1, \ldots, v_k)$ to $span_{\mathbb{R}}\langle v_1, \ldots, v_k \rangle \subseteq \mathbb{R}^n$, that is, the subspace with $(v_1, \ldots, v_k)$ as basis. We give $G_k(\mathbb{R}^n)$ the quotient topology defined by α. (For $k = 1$, we have $G_1(\mathbb{R}^n)$ is the real projective space of dimension $n - 1$.)

This geometric setting also produces further important examples of 'bundles', this time on these Grassmann varieties.

Consider the subspace of $G_k(\mathbb{R}^n) \times \mathbb{R}^n$ given by those (V, x) with $x \in V$. Using the projection $p(V, x) = V$ gives the bundle,

$$\gamma_k^n = (\gamma_k^n, p, G_k(\mathbb{R}^n)).$$

This is *canonical k-dimensional vector bundle on $G_k(\mathbb{R}^n)$.*

Similarly the *orthogonal complement bundle*, ${}^*\gamma_k^n$, has total space consisting of those (V, x) with $\langle V \mid x \rangle = 0$, i.e., x is orthogonal to V.

All of these 'bundles' have vector space structures on their fibres. They are all *locally trivial* (so in each case $\alpha^{-1}(U) \cong U \times F$ for suitable open subsets U of the base), and the resulting $\xi_{U,U'}$ have form

$$\xi_{U,U'}(x, t) = (x, \xi'_{U,U'}(x))(t)$$

where $\xi'_{U,U'} : U \cap U' \to G\ell_M(\mathbb{R})$ for suitable M. (As usual, $G\ell_M(\mathbb{R})$, which may sometimes also be denoted $G\ell(M, \mathbb{R})$, is the *general linear group of non-singular $M \times M$ matrices over* $\mathbb{R}$. Here it is considered as a topological group. It also has a smooth structure and is an important example of a *Lie group*.) Such *vector bundles* are prime examples of the situation in which the fibres have extra structure.

We will see, use and study vector bundles in more detail later on, for the moment, we introduce the example of a *trivial vector bundle* in addition to those geometrically occurring ones above. We will work over the real numbers as our basic field, but could equally well use $\mathbb{C}$ or more generally.

Definition: A *trivial (real) vector bundle* of dimension m, on a space B is one of the form $\mathbb{R}^m \times B \to B$, the mapping being, naturally, the projection. We will denote this by ε_B^m.

Even more structure can be encoded, for instance, by giving each fibre an inner product structure with the requirement that the $\xi'_{U,U'}$ take values in $O_M(\mathbb{R})$, or $O(M,\mathbb{R})$, the *orthogonal group*, hence that they preserve that extra structure. Abstracting from this we have a group, G, which acts by automorphisms on the space, F, and have our descent data isomorphisms $\xi_{U,U'}$ of the form $\xi_{U,U'}(x,t) = (x, \xi'_{U,U'}(x))(t)$ for some continuous $\xi'_{U,U'} : U \cap U' \to G$.

As usual, if G is a (topological) group, by a *G-space*, we mean a space X with an action (left action):

$$G \times X \to X,$$

$$(g,x) \to g.x.$$

The action is *free* if $g.x = x$ implies $g = 1$. The action is *transitive* if given any x and y in X there is a $g \in G$ with $g.x = y$. Let X^* be the subspace

$$X^* = \{(x, g.x) : x \in X, g \in G\} \subseteq X \times X,$$

(cf. our earlier discussion of action groupoids on page **??**).

There is a function (called the *translation function*)

$$\tau : X^* \to G$$

such that $\tau(x,x')x = x'$ for all $(x,x') \in X^*$. We note

(i) $\tau(x,x) = 1$,

(ii) $\tau(x',x'')\tau(x,x') = \tau(x,x'')$,

(iii) $\tau(x',x) = \tau(x,x')^{-1}$

for all $x, x', x'' \in X$.

A G-space, X, is called *principal* provided X is a free, transitive G-space with *continuous* translation function $\tau : X^* \to G$.

Proposition 69 *Suppose X is a principal G-space, then the mapping*

$$G \times X \to X \times X$$

$$(g,x) \to (x, g.x)$$

is a homeomorphism.

Proof: The mapping is continuous by its construction. Its inverse is (τ, pr_1), which is also continuous. ■

This is often taken as the definition of a principal G-space, so you could try to prove the converse. We, in fact, need a fibrewise version of this.

Given any G-space, X, we can form a quotient X/G with a continuous map $\alpha : X \to X/G$. A bundle $\mathsf{X} = (X, \alpha, B)$ is called a *G-bundle* if X has a G-action, so that B is homeomorphic to X/G compatibly with the projections from X. The bundle is a *principal G-bundle* if X is a principal G-space *over* B. What does this mean? In a G-bundle, as above, the fibres of α are orbits of the G-action, so the action is 'fibrewise'. We can replace G by $\underline{G} = G \times B$ and, thinking of it as a space over B, perhaps rather oddly, write the action within the category Top/B. We replace the product in Top by that in Top/B, which is just the pullback along projections in Top. The action is thus

$$\underline{G} \times_B X \to X$$

over B, or just $\underline{G} \times \mathsf{X} \to \mathsf{X}$ in the notation valid in Top/B. Now 'principalness' will say that the action is free and transitive, and that the translation function is a continuous map *over* B. A neater way to handle this is to use the above proposition and to define X to be a principal G-bundle if the corresponding morphism over B,

$$\underline{G} \times \mathsf{X} \to \mathsf{X} \times \mathsf{X}$$

is an isomorphism in Top/B. We will not explore this more here as that *is*, more or less, the way we will define G-torsors later on, except that we will be using a bundle or sheaf of groups rather than simply $\underline{G}$.

We note that if $\xi = (X, p, B)$ is a principal G-bundle then the fibre $p^{-1}(b)$ is homeomorphic to G for any point $b \in B$. It is usual in topological situations to require that the bundle be locally trivial. For the moment, we can summarise the idea of principal G-bundle as follows:

A principal G-bundle is a fibre bundle $p : X \to B$ together with a continuous left action $G \times X \to X$ by a topological group G such that G preserves the fibers of p and acts freely and transitively on them.

Later we will see other more categorical views of principal G-bundles. As we have mentioned, they will reappear as 'G-torsors' in various settings. For the moment we need them to provide the link to the general notion of fibre bundle.

For F, a (right) G-space with action $G \times F \to F$, we can form a quotient, X_F, of $F \times X$ by identifying (f, gx) with (fg, x). The composite

$$F \times X \overset{pr_2}{\to} X \to X/G$$

factors via X_F to give $\beta : X_F \to X/G$, where $\beta(f, x)$ is the orbit of x, i.e., the image of x in X/G. The earlier examples of 'bundles' were all examples of this construction. The resulting (X_F, β, B) is called a *fibre bundle* over B $(= X/G)$.

Note: The theory of fibre bundles was developed by Cartan and later by Ehresmann and others from the 1930s onwards. Their study arose out of questions on the topology and geometry of manifolds. In 1950, Steenrod's book, [199], gave what was to become the first reasonably full treatment of the theory. Atiyah, Hirzebruch and then, in book form, Husemoller, [123] in 1966 linked this theory up with K-theory, which had come from algebraic geometry. The books contain much of the basic theory including the local coordinate description of fibre bundles which is most relevant for the understanding of the descent theory aspects of this area (cf. Chapter 5 of Husemoller, [123]). The restriction of looking at the local properties relative to an open cover makes this treatment slightly too restrictive for our purposes. It *is* sufficient, it seems, for many of the applications in algebraic topology, differential geometry and topology and related areas of mathematical physics, however as Grothendieck points out (SGA1, [108], p.146), in algebraic geometry *localisation of properties*, although still linked to certain types of "base change" (as here with base change along the map

$$\sqcup\, \mathcal{U} \to B$$

for $\mathcal{U}$ an open cover of B), needs to consider other families of base change. These are linked with some problems of commutative algebra that are interesting in their own right and reveal other aspects of the descent problem, see [29]. For these geometric applications, we need to replace a purely topological viewpoint by one in which *sheaves* take a front seat role.

(The Wikipedia entries for principal G-space, principal bundle and 'fiber' bundle are good places to start seeing how these concepts get applied to problems in geometry. For a picture of how to build a fibre bundle out of wood, see http://www.popmath.org.uk/sculpmath/pagesm/fibundle.html.)

6.1.4 Change of Base

This is a theme that we will revisit several times. Suppose that we have a good knowledge of 'bundles' over some space, B', but want bundles over another space, B. We have a continuous map, $f : B \to B'$, and hope to glean information on bundles on B by comparing them with those on B', using f in some way. (We could be looking to transfer the information the other way as well, but this way will suffice for the moment!)

What we have used when restricting to open subsets of a base space was pullback and that works here as well. Suppose $p' : A' \to B'$ is a principal G-bundle over B', then we form the pullback

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d, "p'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

Categorically the pullback, as it is characterised by a universal property, is only determined up to isomorphism, but we can pick a definite model for A in the form

$$A' \times_{B'} B = \{(a, b) \mid p'(a) = f(b)\},$$

with $a \in A'$ and $b \in B$. The projection of A onto B is given by sending (a, b) to b and the map from A to A' by the obvious other projection. As we have an action of G on the left of A' it is tempting to see if there is one on A and the obvious thing to attempt is $g.(a, b) = (g.a, b)$. Does this make sense? Yes, because $p'(g.a) = p'(a)$, since B' is the space of orbits of the action of G on A'. Is $A \to B$ then a principal G-bundle? Again the answer is yes. To gain some idea why look at the fibres. We know the fibres of a principal G bundle are copies of the *space* G, and fibres of the pullback are the same as fibres of the original. The action is concentrated in the fibres as the orbit space of the action *is* the base.

The one question is whether the map

$$\underline{G} \times_B A \to A \times_B A$$

is an isomorphism. You can see that it is in two ways. The elements of A are pairs (a, b), as above. The map is $((g, b), (a, b)) \mapsto ((a, b), (g.a, b)$ and this is clearly in the fibres as the second component in each pair is the same. It has an inverse surely, (since an element in $A \times BA$, has the form $((a_1, b), (a_2, b))$ and since A' is a principal bundle we can continuously find g such that $a_2 = g.a_1$). The alternative approach is to note that the map fits into a diagram with lots of pull back squares and to note that is is induced from the corresponding map for (A', B', p').

We thus have, it would seem, that $f : B \to B'$ induces a 'functor' from the category of principal G-bundles over B' to the corresponding one over B. (The word 'functor' is given between inverted commas since we have not discussed morphisms between bundles of this form. That is left to you both to formulate the notion and to check that the inverted commas can be removed. In any case we will be considering this in the more general setting of G-torsors slightly later in this chapter.)

We thus have induced bundles, $f^*(\mathsf{A}')$, but different maps, f, can lead to isomorphic bundles. More precisely, suppose f and g are two maps from B to B', then if f and g are homotopic (under mild compactness conditions on the spaces) it is fairly easy to prove that for any (principal) bundle A' on B', the two bundles $f^*(\mathsf{A}')$, and $g^*(\mathsf{A}')$, are isomorphic. We will not give the details here as they are in most text books on the area, (see, for instance, [123], or [134]), but the idea is that if $H : B \times I \to B'$ is a homotopy between f and g, we get a bundle $H^*(\mathsf{A})$ with base $B \times I$. You now use local triviality of the bundle to cover $B \times I$ by open sets over which this bundle trivialises. Using compactness of B, we get a sequence of points t_i in I and an open cover of $B \times I$ made up of open sets of the form $U \times (t_i, t_{i+2})$. Now we work our way up the cylinder showing that the bundle over each slice $B \times \{t_i\}$ is isomorphic to that on the previous slice. (There are lots of details left vague here and you should look them up if you have not seen the result before.)

This result shows that categories of principal bundles over homotopically equivalent spaces will be equivalent, and, in particular, that over any contractible space, all principal bundles are isomorphic to each other and hence are all isomorphic to the product principal bundle. It also shows that if we can cover B with an open cover made up of contractible open sets that all bundles trivialise over that cover.

Remarks: In many different theories of bundle-like objects there is an *induced bundle* construction given by pullback along a continuous map on the 'bases'. In *most* of those cases, it seems, homotopic maps induce isomorphic 'bundles', again with possibly a compactness requirement of some sort on the bases.. This happens with vector bundles, (as follows from the result on principal bundles mentioned above.) In these cases, the only bundles of that type on a contractible space will be product bundles. (We will keep this vague directing the reader to the literature as before.)

6.2 Descent: simplicial fibre bundles

To understand topological descent, as in the theory of fibre bundles as sketched out above, it is useful to see the somewhat simpler simplicial theory. This has aspects that are not so immediately obvious as in the topological case, yet some of these will be very useful when we get further in our study handling sheaves and later on stacks.

The basics of simplicial fibre bundle theory were developed in the 1950s and early 1960s, the start being in a paper by Barratt, Gugenheim and Moore, [18]. We have already discussed several of the features of this theory. A useful survey is given by Curtis, [74], and a full description of the theory are available in May's book, [155], with many aspects also treated in Goerss and Jardine, [105].

6.2.1 Fibre bundles, the simplicial viewpoint

We earlier saw how, in the simplicial setting, the G-principal fibrations, when pulled back over any simplex of their base, gave a trivial product fibration. It is this feature that we abstract to get a working notion of simplicial fibre bundle.

Definition: A *(simplicial) fibre bundle* with fibre, Y, over a simplicial set, B, is a simplicial map, $f : E \to B$ such that for any n-simplex, $b \in B_n$, (for any n), the pullback over the representing ('naming') map, $\ulcorner b \urcorner : \Delta[n] \to B$, is a trivial bundle, that is, isomorphic to a product of Y with $\Delta[n]$ together with its projection onto $\Delta[n]$.

We thus have a diagram

$$\begin{tikzcd} Y \times \Delta[n] \arrow[r] \arrow[d, "p_2"'] & E \arrow[d, "f"] \\ \Delta[n] \arrow[r, "\ulcorner b \urcorner"'] & B, \end{tikzcd}$$

which is a pullback.

It is worthwhile just thinking about the comparison between this and what we have been looking at for topological bundles. The role played there by the open covering is taken by the family of *all* simplices of the base. (From this one can build a neat category, and in a very similar way from a plain classical open cover you can form all finite (non-empty) intersections, add them into the cover and build a category from these and the inclusions between them. It will pay to retain that thought for when we launch into discussion of sheaves, and, in particular, stacks, etc.)

It is, thus, important to note that in any simplicial fibre bundle, we have fibres over all simplices, not just the 'vertices'. The 'fibre' over an n-simplex, b, of the base, is given by the pullback

$$\begin{tikzcd} E \times_B \Delta[n] \arrow[r] \arrow[d, "p_2"'] & E \arrow[d, "f"] \\ \Delta[n] \arrow[r, "\ulcorner b \urcorner"'] & B, \end{tikzcd}$$

The usual notion of 'fibre' then corresponds to the case where $n = 0$. We will sometimes write $E(b) = E \times_B \Delta[n]$, since $E \times_B \Delta[n]$ as a notation, does not actually record the b being considered. For instance, given $e \in E_n$, we have the fibre through e will be $E(p(e))$.

Examples of fibre bundles: (i) **Trivial product bundles**:

Lemma 40 *The trivial product bundle, $p_B : Y \times B \to B$, is a fibre bundle in this sense.*

Proof: To see this, we pick an arbitrary, $\ulcorner b \urcorner : \Delta[n] \to B$, and embed it in the commutative diagram:

$$\begin{tikzcd} Y \times \Delta[n] \arrow[r, "Y \times \ulcorner b \urcorner"] \arrow[d, "p_2"'] & Y \times B \arrow[r] \arrow[d, "p_B"] & Y \arrow[d] \\ \Delta[n] \arrow[r, "\ulcorner b \urcorner"'] & B \arrow[r] & \Delta[0], \end{tikzcd}$$

where the two arrows with codomain $\Delta[0]$ are the unique such maps, (since $\Delta[0]$ is terminal in $\mathcal{S}$). This means that both the right-hand square and the outer rectangle are pullbacks, and then it is an elementary (standard) exercise of category theory to show that the left hand square is also a pullback, which completes the proof. ∎

(ii) **Any G-principal fibration** is a fibre bundle, since we saw, Lemma 38, that the fibre bundle condition was satisfied. The fibre in this case is the underlying simplicial set of the simplicial group, G.

6.2.2 Atlases of a simplicial fibre bundle

The idea of atlases originally emerged in the theory of manifolds. manifolds are specified by local 'charts' and, of course, a collection of charts makes, yes you guessed, ... Here we will see how that idea can be adapted to a simplicial setting.

Let (E, B, p) be a fibre bundle with fibre, Y, then we see that, for any $b \in B_n$, there is an isomorphism

$$\alpha(b) : Y \times \Delta[n] \to E \times_B \Delta[n],$$

given by the diagram:

$$\begin{tikzcd}
Y \times \Delta[n] \arrow[r, "\alpha(b)", "\cong"'] \arrow[dr, "p_2"'] & E \times_B \Delta[n] \arrow[r, "p_1"] \arrow[d, "p_2"] & E \arrow[d, "p"] \\
 & \Delta[n] \arrow[r, "\ulcorner b \urcorner"'] & B
\end{tikzcd}$$

using the universal property of pullbacks. Set $a(b) : Y \times \Delta[n] \to E$ to be the composite $p_1\alpha(b)$.

Remark: If we think of b as a 'patch' over which (E, B, p) trivialises, then $\alpha(b)$ is the trivialising isomorphism identifying E 'restricted to the patch b' with a product. A face of b may be shared with another n-simplex, so we can expect interactions / transitions between the different descriptions / trivialisations.

Definition: The family $\alpha = \{\alpha(b) \mid b \in B\}$ (or, equivalently, $\mathbf{a} = \{a(b) \mid b \in B\}$) will be called an *atlas* for (E, B, p).

That α determines $\mathbf{a}$ is obvious, but we have also $\alpha(b)(y, \sigma) = (a(b)(y, \sigma), \sigma)$, so $\mathbf{a}$ also determines α. We should also point out that in the definition, we are using $b \in B$ as a convenient shorthand for $b \in \bigsqcup_n B_n$.

It is often useful to think of $\alpha(b)$ as an element of $\underline{S}(Y, E \times_B \Delta[n])_n$ and $a(b) \in \underline{S}(Y, E)_n$, since this makes the following idea very clear.

Suppose we consider the automorphism simplicial group, $\mathsf{aut}(Y)$, (cf. page 220) and a subsimplicial group, G, of it. Pick a family $\mathbf{g} = \{g(b) \mid b \in B\}$, of elements of G, where, if $b \in B_n$, $g(b) \in G_n$. There is a new atlas $\alpha \cdot \mathbf{g} = \{\alpha(b)g(b) \mid b \in B\}$ obtained by 'precomposing' with $\mathbf{g}$. (We can also use $\mathbf{a} \cdot \mathbf{g}$ with the obvious definition.)

Definition: Two atlases, α and α', are said to be *G-equivalent* is $\alpha' = \alpha \cdot \mathbf{g}$ for some family, $\mathbf{g}$, of elements from G.

So far, there has been no requirement on the atlas α to respect faces and degeneracies in any way. In fact, we do not really *want* to match faces, since, even in such a simple case as the Möbius band, strict preservation of faces (something like $a(d_ib) = d_i(a(b))$, perhaps) would not allow the 'twisting' that we would need.) On the other hand, if we have $a(b)$ defined for a non-degenerate simplex, b, then we already have a suitable $a(s_ib)$ around, namely $s_ia(b)$, so why not take that! (You may like to **investigate** this with regard to theuniversal property that we used to define the $\alpha(b)$s.)

Definition: An atlas, $\mathbf{a}$, is *normalised* if, for each $b \in B$, $a(s_ib) = s_ia(b)$ in $\underline{S}(Y, E)$.

Lemma 41 *Given any atlas,* $\mathbf{a}$*, there is a normalised atlas,* $\mathbf{a}'$*, that agrees with* $\mathbf{a}$ *on the non-degenerate simplices of* B. ■

The proof, which is simply a question of making a definition, then verifying that it works is **left to you**.

Turning to the face maps, as we said, we do not necessarily have $a(d_ib) = d_ia(b)$, but we might expect the two sided to be linked by an automorphism of the fibre, of some type. We know

$$d_i(\alpha(b)) = (Y \times \Delta[n-1] \stackrel{Y\times\delta_i}{\to} Y \times \Delta[n] \stackrel{\alpha(b)}{\to} E \times_B \Delta[n],$$

is an isomorphism onto its image. The i^{th} face inclusion $\delta : \Delta[n-1] \to \Delta[n]$ also induces

$$E \times \delta_i : E \times_B \Delta[n-1] \to E \times_B \Delta[n],$$

which we will call θ, and which element-wise is given by $\theta(e, \sigma) = (e, \delta_i \circ \sigma)$, and the image of $\theta \circ \alpha(d_ib)$ is the same as that of $d_i(\alpha(b))$, namely elements of the form $(e, \delta_i \circ \sigma)$. We thus obtain an automorphism, $t_i(b)$, of $Y \times \Delta[n-1]$ with

$$\alpha(d_ib) \circ t_i(b) = d_i(\alpha(b)).$$

('Corestricting' $\alpha(d_ib)$ and $d_i(\alpha(b))$ to that image, we have $t_i(b) = \alpha(d_ib)^{-1} \circ d_i(\alpha(b))$, so $t_i(b)$ is uniquely determined.)

This 'corestriction' argument is reasonably clear as an element based level, but it leaves a lot to check. It is useful to give an equivalent more categorical construction of t, which gets around the verification, for instance, that $t_i(b)$ is a simplicial map - which was 'swept under the carpet' in the above - and is more 'universally valid' as it shows what categorical and simplicial properties are being used.

Let us go back a stage, therefore, and take things apart as 'pullbacks' and in quite some detail. This is initially a bit tedious perhaps, but it is worth doing.

- $\ulcorner d_ib \urcorner$ is the composite
$$\Delta[n-1] \stackrel{\delta_i}{\to} \Delta[n] \stackrel{\ulcorner b\urcorner}{\to} B,$$
and so $\alpha(d_ib)$ fits in a diagram:

$$\begin{tikzcd}
Y \times \Delta[n-1] \arrow[r, "\alpha(d_ib)", "\cong"'] \arrow[dr, "p_2"'] & E \times_B \Delta[n-1] \arrow[r, "E\times_B\delta_i"] \arrow[d, "p_2"] & E \times_B \Delta[n] \arrow[r, "p_1"] \arrow[d, "p_2"] & E \arrow[d, "p"] \\
& \Delta[n-1] \arrow[r, "\delta_i"'] & \Delta[n] \arrow[r, "\ulcorner b\urcorner"'] & B
\end{tikzcd}$$

- We have $\alpha(b) : Y \times \Delta[n] \to E \times_B \Delta[n]$ and want to obtain a restriction of it to the i^{th} face, i.e., to $Y \times \Delta[n-1]$ along $Y \times \delta_i$, *and*, at the same time, that 'corestriction' to $E \times_B \Delta[n-1]$. We want to form the square diagram

$$\begin{tikzcd}
Y \times \Delta[n-1] \arrow[r, "\widetilde{d_i(\alpha(b))}"] \arrow[d, "Y\times\delta_i"'] & E \times_B \Delta[n-1] \arrow[d, "E\times_B\delta_i"] \\
Y \times \Delta[n] \arrow[r, "\alpha(b)", "\cong"'] & E \times_B \Delta[n],
\end{tikzcd}$$

  where the top horizontal arrow, $\widetilde{d_i(\alpha(b))}$, is 'induced from' $\alpha(b)$. We should check how exactly it is built. As it is goinginto an object specified by a pullback, we need only specify its two components, that is, the projections onto E and $\Delta[n-1]$. (Of course, this is exactly what we did in in the element-wise description.) The component going to E is just found by going the other way around the square and folowing that composite by p_1 down to E. The component to $\Delta[n-1]$ is just the projection, p_2. (To see what is going on **draw a diagram yourself**.) We have to verify that the square commutes. This uses the pullback 'uniqueness' clause for $E \times_B \Delta[n]$.

- We note that the corestriction, $\widetilde{d_i(\alpha(b))}$, is a monomorphism, as its composite with $E \times_B \delta_i$ is one. We claim it is an isomorphism. It remains to show, for instance, that it is a split epimorphism. (That is relative easy to try, so is a good place to attack what is needed.)

  First note that

$$\begin{tikzcd}
Y \times \Delta[n-1] \arrow[r, "Y\times\delta_i"] \arrow[d] & Y \times \Delta[n] \arrow[d] \\
\Delta[n-1] \arrow[r, "\delta_i"'] & \Delta[n]
\end{tikzcd}$$

  is a pullback, as is also

$$\begin{tikzcd}
E \times_B \Delta[n-1] \arrow[r, "E\times_B\delta_i"] \arrow[d] & E \times_B \Delta[n] \arrow[d] \\
\Delta[n-1] \arrow[r, "\delta_i"'] & \Delta[n].
\end{tikzcd}$$

  (In each case, you can put an obvious pullback square to the right, so that the composite 'rectangle' is again a pullback - that same argument again.) We build the inverse to $\tilde{d} := \widetilde{d_i(\alpha(b))}$, using the first of these two squares. The component of that inverse going to $\Delta[n-1]$ is the obvious one, whilst to $Y \times \Delta[n]$, we use $\alpha(b)$. (You are **left to check commutativity**.) To check then that this map we have constructed, does split $\tilde{d}$, we use the uniqueness clause for the second of these pullbacks.

  The final step in proving that $\tilde{d}$ is an isomorphism is the 'usual' proof that if a morphism is both a monomorphism and a split epimorphism then the splitting is, in fact, the inverse for the original monomorphism (which is thus an isomorphism). (**If you have not seen this before**, first **check** the categorical meaning of monomorphism, then **work out a proof** of the fact.)

We, therefore, have

$$Y \times \Delta[n-1] \xrightarrow[\cong]{\alpha(d_i b)} E \times_B \Delta[n-1]$$

and

$$Y \times \Delta[n-1] \xrightarrow[\cong]{\tilde{d}} E \times_B \Delta[n-1] ,$$

both over $\Delta[n-1]$, as you **easily check** from the above. We thus get

$$t_i(b) = \alpha(d_i b)^{-1}.\tilde{d},$$

and this is in $\mathsf{aut}(Y)_{n-1}$. We note that these elements are completely determined by the normalised atlas.

Definition: The automorphisms, $t_i(b)$, for $b \in B$ are called the *transition elements* of the atlas, α.

If the transition elements all lie in a subgroup, G, of $\mathsf{aut}(Y)$, then we say α, (or, equivalently, a), is a *G-atlas.*

An atlas, α, is *regular* if, for $i > 0$, its transition elements, $t_i(b)$, are all identities.

We thus have that, in a regular normalised atlas, we just need to specify the $t_0(b)$, as these may be non-trivial. (To see where this theory is going at this point, you may find it helps to think $t =$ 'twisting', as well as, $t =$ 'transition', and to look back at our discussion of T.C.P.s (section 5.5, page 229).)

Lemma 42 *Every (normalised) G-atlas is G-equivalent to a (normalised) regular G-atlas.*

Proof: We start with a G-atlas, which we will assume normalised. (The unnormalised case is more or less identical.) We will use it in the form $\mathbf{a}$, rather than α, but, of course, this really makes no difference. We will build, by induction, a G-equivalent regular one, $\mathbf{a}'$.

On vertices, we take $a'(b) = a(b)$. That gets us going, so we now assume $a'(b)$ is defined for all simplices of dimension less than n, and that $\mathbf{a}'$ is regular and G-equivalent to $\mathbf{a}$, to the extent that this makes sense. We next want to define $a'(b)$ for b, a (non-degenerate) n-simplex. (The degenerate ones are handled by the normalisation condition.)

We look at the $(n, 0)$-horn in B corresponding to b, i.e., made up of all the $d_i b$ for $i \neq 0$. We have elements $g_i(b)$ such that

$$a'(d_i b) = a(d_i b) g_i(b),$$

since $\mathbf{a}'$ is G-equivalent to $\mathbf{a}$ in this dimension, then, using

$$a(d_i b) t_i(b) = d_i(a(b)),$$

we get $a'(d_i b) = d_i(a(b)).t_i(b)^{-1}.g_i(b) = d_i(a(b)).h_i$, where we have set $h_i = t_i(b)^{-1}.g_i(b)$. Since $\mathbf{a}'$, so far defined is regular, we have, for $0 < i \leq j$, after a bit of simplicial identity work (**for you**), that

$$d_i d_j(a(b)) d_i h_j = d_i d_j(a(b)) d_{j-1} h_i,$$

which implies that $d_ih_j = d_{j-1}h_i$, the the hs form a $(n,0)$-horn in G. we now wheel out our method for filling horns in G to get a $h \in G_n$ with $d_ih = h_i$, for $i > 0$, and we set $a'(b) = a(b)h$. we heck

$$\begin{aligned} d_ia'(b) &= d_ia(b))d_ih \\ &= d_ia(b)h_i \\ &= a'(d_ib). \end{aligned}$$

The resulting $\mathbf{a}'$, is now defined up to and including dimension n, is normalised and regular, and G-equivalent to $\mathbf{a}$. We get this in all dimensions by induction. ■

6.2.3 Fibre bundles are T.C.P.s

We saw earlier that G-principal fibrations were locally trivial and hence are fibre bundles, and that twisted Cartesian products (T.C.Ps) are principal fibrations. We now have regular atlases, yielding structures that look like twisting functions. This suggests that the various ideas are really 'the same'. We will not comlete all the details that show that they are, since that theory is in various texts (for instance, May's book, [155]), but will more-or-less complete our *sketch* of the interrelationships.

There remains, for our sketch, an investigation of the transition elements for simplicial fibre bundles and a 'sketch proof' that fibre bundles are just T.C.Ps.

Suppose we have some simplicial fibre bundle and a normalised regular G-atlas, $\mathbf{a} = \{a(b) \mid b \in B\}$, giving as the only possibly non-trivia transition elements, the $t(b) := t_0(b)$. We thus have

$$d_0a(b) = a(d_0b).t(b).$$

(To avoid looking back all the time to the definition of twisting function, we repeat it here for convenience and also to adjust conventions. We had:

a function, t, satisfying the following equations will be called a *twisting function*:

$$\begin{aligned} d_it(b) &= t(d_{i-1}b) \quad \text{for} \quad i > 0, \\ d_0t(b) &= t(d_0b)^{-1}t(d_1b), \\ s_it(b) &= t(s_{i+1}b) \quad \text{for} \quad i \geq 0, \\ t(s_0b) &= *. \end{aligned}$$

(**Warning:** The version on page 230 corresponded to the 'algebraic' diagrammatic composition order, and here we have used the 'Leibniz' composition order so we have adjusted the second equation accordingly.)

Lemma 43 *The transition elements, $t(b)$, above, define a twisting function.*

Proof: We use the defining equation (above) for the $t(b)$ and, in particular, the uniqueness of these elements with this property, (together with the 'regular' and 'normalised' conditions for $\mathbf{a}$). We leave the majority of the cases **to you**, since conce you have seen one or two of these, the others are easy.

(We wil do a very easy one as a 'warm up', then the important, and more tricky, one relating toe d_0 and d_1, i.e., the twist.)

Applying the equation above to s_0b, we get

$$d_0a(s_0b) = a(d_0s_0b).t(s_0b) = a(b).t(s_0b),$$

but **a** is normalised, so $a(s_0b) = s_0(b)$ and the left hand side is thus just $a(b)$. we can thus conclude that $t(s_0b)$ is the identity. (That was easy!)

We now turn to the relation involving $t(d_0b)$ and $t(d_1b)$, etc.:

$$d_0a(d_1b) = a(d_0d_1b).t(d_1b),$$

but we also have

$$d_0a(d_0b) = a(d_0d_0b).t(d_0b),$$

and, of course, $d_0d_1b) = d_0d_0b$.

We next apply d_0 to the 'master equation', simply giving

$$d_0d_0a(b) = d_0a(d_0b).d_0t(b),$$

and to $d_1a(b) = a(d_1b)$ to get

$$d_0d_1a(b) = d_1a(d_1b).$$

Again using the simplicial identity $d_0d_1 = d_0d_0$, we rearrange terms algebraically to get

$$d_0t(b) = t(d_0b)^{-1}t(d_1b),$$

as expected.

The other equations are **left to you**. (You just mix applying a d_i or s_i to the 'master equation' inside (i.e, on b) and outside, then use normalisation, regularity and the simplicial identities.) ■

It is thus possible to use E to find **a** and thus t, and thence to form $B \times_t Y$. We need now to compare $B \times_t Y$ with E.

To start with we will do something that looks as if it is 'cheating'. We have, for $b \in B_n$ that $a(b) \in \underline{S}(Y, E)$, so *do* have a *graded* map

$$\mathbf{a} : B \to \underline{S}(Y, E).$$

Our assumptions about **a** being regular, normalised, etc., imply that this is very nearly a simplicial map. (The only thing that goes wrong is the d_0-face compatibility.)

If **a** was simplicial, we could 'fli[it' through the adjunction to get $\xi : B \times Y \to E$. We know how to do this. We form the composite

$$B \times Y \xrightarrow{\mathbf{a}\times Y} \underline{S}(Y, E) \times Y \xrightarrow{eval} E,$$

where *eval* is the map we met earlier (page 225), and which, as you will recall, we worked hard to get a complete description of. For $y \in Y_n$, and $f : Y \times \Delta[n] \to E \in \underline{S}(Y, E)_n$, we had that

$$eval(f, y) = f(y, \iota_n),$$

where, as always, ι_n is the unique non-degenerate n-simplex in $\Delta[n]$, corresponding to the identity map on $[n]$ in the description $\Delta[n] = \mathbf{\Delta}(-, [n])$. We can pretend that **a** is simplicial, see what ξ is given by and then see how much it is or is not simplicial. We can read off, if $y \in Y_n$ and $b \in B_n$,

$$\xi(b, y) = a(b)(y, \iota_n).$$

This map ξ is 'as simplicial as is **a**'. We will check this, or part of it, by hand, but although it follows from generalities on the adjunction process, verifying the conditions needs care.

First we note that if $f : Y \times \Delta[n] \to E$, then $d_i f = f \circ (Y \times \Delta[\delta - i])$, where $\delta_i : [n-1] \to [n]$ is the i^{th} face inclusion (so we get $\Delta[\delta_i] : \Delta[n-1] \to \Delta[n]$). We examine the evaluation map in detail as it is the key to the calculation. By its construction, it is bound to be simplicial, but we need also to see what that means at this 'elementary' level. We have

$$\begin{tikzcd}
\underline{S}(Y,E)_n \times Y_n \arrow[r, "eval"] \arrow[d, "d_i"'] & E_n \arrow[d, "d_i"] \\
\underline{S}(Y,E)_{n-1} \times Y_{n-1} \arrow[r, "eval"] & E_{n-1}
\end{tikzcd}$$

and, for $i > 0$,

$$\begin{aligned}
d_i\xi(b,y) &= d_i(a(b)(y,\iota_n) = eval(d_i a(b), d_i y) \\
&= eval(a(d_i b), d_i y) = \xi(d_i b, d_i y) = \xi d_i(b,y).
\end{aligned}$$

Similarly, we have, for s_i that $s_i\xi = \xi s_i$. That just leaves $d_0\xi$ and, of course

$$d_0\xi(b,y) = eval(a(d_0 b).t(b), d_0 y),$$

by the same sort of argument, and then this is $a(d_0(b))(t(b)d_0 y, \iota_{n-1}) = \xi(d_0 b, t(b)d_0 y)$. (You may want to check this last bit for yourself. You need to translate to-and-fro between a G-actions on Y as being $a : G \times Y \to Y$ and the adjoint $a : G \to \mathsf{aut}(Y)$, again using *eval*.)

This gives us that, if we *define* a new d_0 on this product by *twisting* it using t (and, of course, this is just giving us $B \times_t Y$ as we have already seen it, on page 229) with, explicitly,

$$d_0(b,y) = (d_0 b, t(b)(d_0 y)),$$

then we actually obtain

$$\xi : B \times_t Y \to E$$

as a simplicial map. We note that $p\xi = p_B$, the projection onto B of the T.C.P., so ξ is 'over B'.

Proposition 70 *This map ξ is an isomorphism (over B).*

Proof: We start by constructing, for each $b \in B_n$, a map $\nu(b) : E(b) \to Y$, where, as before, $E(b) = E \times_B \Delta[n]$, the pullback of E along $\ulcorner b \urcorner$, so is the 'fibre over b'. We have $\alpha(b) : Y \times \Delta[n] \to E(b)$ is an isomorphism, and so we can form $\nu(b) := pr_Y \alpha(b)^{-1} : E(b) \to Y$. Using this we send and n-simplex e to $(p(e), \nu(p(e))(e, \iota_n))$, where $(e, \iota_n) \in E(p(e))$ This gives us something in $B \times_t Y$ and ξ is then easily seen to send that n-simplex back to e. That the other composite is the identity ia also easy (for **you to check**). ∎

We thus have a pretty full picture of how principal fibrations are principal fibre bundles, given by twisted Cartesian products of a particular type, that principal H-fibre bundles are classified by $\overline{W}(H)$, since $Princ_H(B) \cong [B, \overline{W}(H)]$, that general fibre bundles in the simplicial context are T.C.P.s and so correspond to a principal bundle and a representation of the corresponding group, and probably some other things as well. As these have been spread over different chapters, since we wanted to make use of the ideas as we went along, **you may find it helpful** to now read one of the texts, such as [155] or the survey, [74], that give the whole theory in one go. We will periodically be recalling part of this, making comparisons with other ideas and methods, and possibly pushing this theory on new directions (as this is 'classical').

6.2.4 ... and descent in all that?

In earlier sections, we looked at descent in a topological context. There we used an open cover, $\mathcal{U}$, of the base space and had transitions, $\xi_{U,U'}$, on intersections of these open patches, with a condition on triple intersections. The idea was to take the A_U for the various open sets, U, of the cover $\mathcal{U}$, and to glue them together, using the $\xi_{U,U'}$ to get the right amount of 'twisting' from patch to patch, with the cocycle condition to ensure the different gluings are compatible.

That somehow looks initially very different from what we have been doing in our discussion of simplicial fibre bundles. We would not expect to have 'open sets', but what takes their place in the simplicial context. We will look at this only briefly, but from several directions. The ideas that we would use for a full treatment will be studied in more depth in the following chapters. This therefore is a 'once over lightly' treatment of just a few of the ideas and insights. The ideas will be recalled, and treated in some depth in later chapters, but not always from the same perspective.

We start by looking at the open cover from a simplicial viewpoint.

Definition: The *Čech complex*, *Čech nerve* or simply, *nerve*, of the open covering, $\mathcal{U}$, is the simplicial complex, $N(\mathcal{U})$, specified by:

- Vertex set : the collection of open sets in $\mathcal{U} = \{U_a \mid a \in A\}$ (alternatively, the set, A, of labels or indices of $\mathcal{U}$);
- Simplices : the set of vertices, $\sigma = \langle \alpha_0, \alpha_1, ..., \alpha_p\rangle$, belongs to $N(\mathcal{U})$ if and only if the open sets, U_{α_j}, $j = 0, 1, \ldots, p$, have non-empty common intersection.

As usual, if we choose an order on the indexing set, i.e., the set of vertices of $N(\mathcal{U})$, then we can construct a neat simplicial set out of this, so the $\langle U_0, U_1\rangle \in N(\mathcal{U})_1$ means $U_0 \cap U_1 \neq \emptyset$ and U_0 is listed before U_1 in the chosen order. (We could, of course, not bother about the order and just consider all possible simplices. For instance, $\langle U_0, U_0, U_1\rangle$ woud be $s_0\langle U_0, U_1\rangle$, but apparently the same simplex, $\langle U_1, U_0, U_0\rangle = s_1\langle U_1, U_0\rangle$, will also be there. This gives a larger simplicial set, but does have the advantage of being constructed without involving an order. **You are left to investigate** if this second construction gives something really different from the other. It is larger, but does it retract to the other form, for instance.)

(For simplicity of exposition, we will assume local triviality, so $A_U = U \times F$, for some 'fibre' F.) Looking at our transition functions, $\xi_{U,U'}$, they assign elements of the group, G, which acts on F, to these 1-simplices, $\langle U, U'\rangle$. (We assume G is a discrete group, not one of the more complex topological groups that also occur in this context.) Taking the group, G, we can form the constant simplicial group $K(G, 0)$, which has G in all dimensions and identity maps for all face and degeneracy morphism. This, then, gives a simplicial map from $N(\mathcal{U})$ to $\overline{W}K(G, 0)$. (You can check this if you wish, *but* we will be looking at it in great detail later on anyway.) We thus get a twisted Cartesian product $N(\mathcal{U}) \times_t K(G, 0)$. That gives us one way of seeing simplicial fibre bundles as being generalisations of the topological ones. They replace a very simple constant simplicial group by an arbitrary one, so have 'higher order transitions' acting as well. Untangling the complex intuitions and interpretations of this simple idea will be one of the themes from now on, not constantly 'up front', but quietly increasing in importance as we go further.

Another way of thinking of descent data is as 'building plans' for the fibre bundle given the bits, $A_U \cong U \times F$. We took the disjoint union, $\sqcup_U A_U$, then 'quotiented' by the gluing instructions encoded in the descent data, (see section 6.1.1). This is a fairly typical simple example of a colimit construction. We will study the categorical notion of colimit (and limit) later in some detail and will use it, and generalisations, many times. (These notes are intended to be reasonably accessible to people who have not had much formal contact with the theory of categories, although some basic knowledge of terminology is assumed as has been mentioned several times already. If you have not met 'colimits' formally, then **do** look up the definition. It may initially not 'mean' much to you, but it will help if you have some intuition. Something like: colimits are 'gluing' processes. You form a 'disjoint union' (coproduct), putting pieces out ready for use in the construction, then 'divide out' by an equivalence relation given, or at least, generated, by some maps between the different pieces.) We will see, more formally, the way that topological descent fits into this colimit / gluing intuition later on, but it is clearly also here in this simplicial context.

We have our basic pieces, $Y \times \Delta[n]$, and we glue them together using the 'combinatorial' information encoded in the simplicial set B. One way to view that is by using a neat construction of a category from a simplicial set.

Suppose we have a simplicial set, B. then we can form a small category $Cat(B)$ (also denoted (Yon, B), as it is an example of a *comma category*). This has as its set of objects the simplices, b, of B, or, more usefully, their representing maps, such as $\ulcorner b \urcorner : \Delta[n] \to B$. If $\ulcorner b \urcorner$ and $\ulcorner c \urcorner : \Delta[m] \to B$ are two such, not necessarily of the same dimension, then a morphism in $Cat(B)$ from $\ulcorner b \urcorner$ to $\ulcorner c \urcorner$ 'is' a diagram:

$$\begin{tikzcd} \Delta[n] \arrow[rr, "\Delta[\mu]"] \arrow[dr, "\ulcorner b \urcorner"'] & & \Delta[m] \arrow[dl, "\ulcorner c \urcorner"] \\ & B & \end{tikzcd}$$

i.e., $\mu : [n] \to [m]$ is a morphism in $\mathbf{\Delta}$, so is a 'monotone map' which induces $\Delta[\mu]$ as shown. Saying that the diagram commutes says, of course, that $\ulcorner b \urcorner = \ulcorner c \urcorner \circ \Delta[\mu]$. Again, of course, $b \in B_n$ and $c \in B_m$ and μ induces a map $B_\mu : B_m \to B_n$. The obvious relationship corresponding to 'commutative' is that $B_\mu(c) = b$ and this holds. (You can take this, in the definition of morphism, to replace commutativity of the triangle as it is equivalent, then it comes out as saying 'a morphism $\mu : \ulcorner b \urcorner \to \ulcorner c \urcorner$ is a $\mu : [n] \to [m]$ such that $B_\mu(c) = b$, but it is very worth while checking through the above at a categorical level as well.)

If now you look back at our discussion of the reconstruction of (E, B, p) from the various patches, $Y \times \Delta[n]$, which corresponded to an n-simplex b in B, the process of gluing these together is completely analogous to our earlier discussion. It is again a 'colimit'. (You may, quite rightly ask, 'how come we get a twisted Cartesian *product* from a disjoint union type construction?' This is neat - and, of course, you may have seen it before. Looking just at sets A and B, if we form $A \times B$, then $A \times B = \coprod\{\{a\} \times B \mid a \in A\}$, so we can write a product as a disjoint union of (identical) labelled copies of the second set, each indexed by an element of the first one. (First and second here are really interchangeable of course.) We will see this type of construction several times later on. For instance if G is a simplicial groupoid and K is a simplicial set, we can form a new simplicial groupoid $K \otimes G$ with $(K \otimes G)_n$ being a disjoint union (coproduct) of copies of G_n indexed by the n-simplices of K. We will see this in detail later on, so this mention is 'in passing', but it is hopefully suggestive as to the sort of viewpoint we can use and adapt later.

The structure of simplicial fibre bundles is thus closely linked to the same intuitions and techniques used in the topological case. We now turn to sheaves, and will see those same ideas coming out again, with of course, their own flavour in the new context.

6.3 Descent: Sheaves

(As with previous sections, this should be 'skimmed' if you have met the subject matter, here sheaves, before. A good accessible account and brief introduction to this is Ieke Moerdijk's Lisbon notes, [161]. These also are useful for alternative developments of later material and are thoroughly to be recommended.)

6.3.1 Introduction and definition

Sheaves provide a useful alternative to bundles when handling 'local-to-global' constructions. The intuition is, in many ways, the same as that of bundles. We have a space B and for each $b \in B$, a 'fibre' over b, i.e., a set F_b, and we want to have F_b varying in some continuous way as we vary b continuously. In other words, naively a sheaf is a continuously varying family of 'sets'.

That is much too informal to use as a definition as it has employed several terms that have not been defined. Before seeing how that intuition might be encoded more exactly, we will return to the 'spaces over B'. Let $\alpha : A \to B$ be a space over B as before, and, once again, let $U \subset B$ be an open set. This time we will not consider $\alpha^{-1}(U)$, but will look at *local sections of α over U*. A *(local) section* of α, over U is a continuous map $s : U \to A$ such that, for all $x \in U$, $\alpha s(x) = x$, that is, $s(x)$ is always in the fibre over x. We write $\Gamma_A(U)$ for the set of such local sections, although this notation *does not* record the all important map, α, in it.

If $V \subset U$ is another open set of B and $s : U \to A$ is a local section of α over U, then the restriction, $s|_V$, of s to V is a local section of α over V. We thus get, from $V \subset U$, an induced 'restriction' map

$$\mathsf{res}^U_V : \Gamma_A(U) \to \Gamma_A(V).$$

Of course, if $W \subset V$ is another such

$$\mathsf{res}^U_V \circ \mathsf{res}^V_W = \mathsf{res}^U_W.$$

There is a little teasing problem here. Suppose V is empty. Of course, the empty set is a subset of all the other open sets, so what should $\Gamma_A(\emptyset)$ be? The empty space is the initial object in the category of spaces so there is a unique map from it to A and, of course, this is a local section! (You can either check the condition at all points of the domain or argue that composition of this empty local section with the projection p yields the unique map from $\emptyset$ into B, as required.)

Back to the generalities, there is, again of course, a neat, and well known, categorical description of this setting.

Let $Open(B)$ denote the partially ordered set of open sets of B with the usual order coming from inclusion, and consider it as a category in the usual way. The above construction just gave a functor

$$\Gamma_A : Open(B)^{op} \to Sets,$$

a *presheaf* on B. Any functor $F : Open(B)^{op} \to Sets$ is called a *presheaf*, but not all presheaves come from 'spaces over B' by the local sections construction, as it is fairly clear that Γ_A has some special properties, for instance, we saw that such a presheaf must send $\emptyset$ to the singleton set, but we also have the gluing property:

Suppose $s_1 \in \Gamma_A(U_1)$ and $s_2 \in \Gamma_A(U_2)$ are two local sections and

$$\mathsf{res}^{U_1}_{U_1 \cap U_2}(s_1) = \mathsf{res}^{U_2}_{U_1 \cap U_2}(s_2),$$

so these local sections agree on the intersection of their domains, then define

$$s : U_1 \cup U_2 \to A$$

by

$$s(x) = \begin{cases} s_1(x) & \text{if } x \in U_1 \\ s_2(x) & \text{if } x \in U_2. \end{cases}$$

It is easy to prove that s is continuous and so gives a local section over $U_1 \cup U_2$. We need not stop with just two local sections. If we have any family of local sections, over a family of open sets, that coincide on pairwise intersections, then they can be glued together, just as above, to give a unique local section on the union of those open sets, restricting to the given ones with which we started on their original domains. This gluing property is the defining property of the sheaves amongst the presheaves on B:

Definition: A presheaf $F : Open(B)^{op} \to Sets$ is a *sheaf* if given any family $\mathcal{U}$ of open sets of B, say $\mathcal{U} = \{U_i\}_{i \in I}$, and elements $s_i \in F(U_i)$ for $i \in I$, such that for $i, j \in I$ $\mathsf{res}^{U_i}_{U_i \cap U_j}(s_i) = \mathsf{res}^{U_j}_{U_i \cap U_j}(s_j)$, there is a *unique* $s \in F(U)$, for $U = \bigcup U_j$, such that $\mathsf{res}^{U}_{U_i}(s) = s_i$ for all i.

Query: Does this gluing property imply the normalisation condition that $F(\emptyset)$ is a singleton? **For you to investigate!**

Example and Definition: Let $\alpha : A \to B$ be a 'bundle', then, for U open in B, take $\Gamma_\alpha(U) = \{s : U \to A \mid \alpha s(x) = x \text{ for all } x \in U\}$, defines a presheaf on B. It is a sheaf. The functions, s, are called *local sections*, as before, and Γ_α is called the *sheaf of local sections of* α. (We will sometimes, as above, slightly abuse notation and write Γ_A instead of Γ_α, if the map α is unambiguous in the context.)

For later purposes and comparisons, we will note that a *compatible family* s_i *of local elements*, as above, gives an element $\underline{s}$ in the product set $\prod\{F(U_i) : i \in I\}$. Not just any family of elements however. We also have a product of the parts over the intersections. We write $U_{ij} = U_i \cap U_j$ and get a product $\prod\{F(U_{i,j}) : i, j \in I\}$. There are two functions, which we will call a and b for convenience only, defined from $\prod\{F(U_i) : i \in I\}$ to $\prod\{F(U_{ij}) : i, j \in I\}$. To specify these we see how they project onto the factors $F(U_{ij})$. (Technically, we have maps $\prod F(U_{ij}) \stackrel{p_{ij}}{\to} F(U_{ij})$, being the $\{ij\}^{th}$ projection of the product.) The specifications are

$$p_{ij} a(\underline{s}) = res^{U_i}_{U_{ij}}(s_i),$$

whilst

$$p_{ij} b(\underline{s}) = res^{U_j}_{U_{ij}}(s_i).$$

We can now give the compatibility condition as $\underline{s}$ is a compatible family of local elements exactly if $a(\underline{s}) = b(\underline{s})$:

$$Eq(a,b) \longrightarrow \prod F(U_j) \overset{a}{\underset{b}{\rightrightarrows}} \prod F(U_{ij}) \ ,$$

i.e., $\underline{s}$ is in the *equaliser* $Eq(a, b)$ of a and b. This equaliser is sometimes called the *set of descent data for the presheaf relative to the cover.* It may be denoted $Des(\mathcal{U}, F)$.

From this perspective, we note that the restriction maps give a map

$$c : F(U) \to \prod F(U_i),$$

with $p_i des(s) = res^U_{U_i}(s)$ and we know $a.c = b.c$. We thus get a function, des, from $F(U)$ to $Eq(a, b)$ assigning $des(s) := c(s)$ to s. We have F is a sheaf exactly when this map, des, is a bijection; it is a *separated presheaf* when this map is one-one, see below.

This scenario is quite useful for sheaves, but it really comes into its own when we look at higher dimensional analogues such as stacks.

We will note quite a lot of facts about sheaves and presheaves, but will not give a detailed development, since here is not a suitable place to give a lengthy treatment of sheaf theory.

6.3.2 Presheaves and sheaves

The category, $Sh(B)$, of sheaves on a space, B, is a reflective subcategory of the category, $Presh(B) = [Open(B)^{op}, Sets]$, of presheaves on B.

We first note a half-way house between general presheaves and sheaves.

The presheaf F is *separated* if there is at most one $s \in F(U)$ such that $res^U_{U_i}(s) = s_i$ for all i. ('Sheafness' would also require this, but, in addition, asks for the existence of such an s, not just uniqueness if it exists.) In fact:

The functors

$$Sh(B) \to Sep.Presh(B) \to Presh(B)$$

have left adjoints.

If F is a presheaf, we will write $s(F)$ for the corresponding separated presheaf and $a(F)$ for the associated sheaf. We can give explicit constructions of $s(F)$ and $a(F)$.

- Define an equivalence relation $\sim_U$ on $F(U)$, where, if $a, b \in F(U)$, then $a \sim b$ if and only if $res^U_{U_i}(a) = res^U_{U_i}(b)$ for all i, then $s(F)$ given by $s(F)(U) = F(U)/\sim_U$ is a separated presheaf. (For you to check the presheaf structure.)

- Suppose F is separated (if not replace it by $s(F)$ and rename!) Form $F_\mathcal{U}$, the set of compatible families (relative to $\mathcal{U}$) of elements in the $F(U_i)$. If $\mathcal{V} < \mathcal{U}$ is a finer cover of U, (so for each $V \in \mathcal{V}$, there is a $U \in \mathcal{U}$ with $V \subseteq U$), then there is a function $res^\mathcal{U}_\mathcal{V} : F_\mathcal{U} \to F_\mathcal{V}$ where $res^\mathcal{U}_\mathcal{V}(\underline{s})_j = res^{U_i}_{V_j}(s_i)$ if $V_j \subseteq U_i$. (**Check** it is well defined.)

  Varying $\mathcal{U}$, we get a diagram of sets and form

  $$a(F)(U) = colim_\mathcal{U} F_\mathcal{U}.$$

  Explicitly we generate an equivalence relation on the union of the $F_\mathcal{U}$s by

  $$\underline{s}_\mathcal{U} \sim \underline{s}_\mathcal{V}$$

  if $\mathcal{V} < \mathcal{U}$ and $res^\mathcal{U}_\mathcal{V}(\underline{s}_\mathcal{U}) = \underline{s}_\mathcal{V}$, and then form the quotient.

(The details are well known and, if you have not met them before should be checked or looked up, e.g. in a related context, [29], p.268. The sort of constructions used will be useful throughout this chapter. It is a good idea to try to rewrite this in terms of the equaliser description given earlier, to see what is happening there.)

6.3.3 Sheaves and étale spaces

The category, $Sh(B)$, is equivalent to the category of étale spaces over B.

A continuous map, $f : X \to Y$, between topological spaces is *étale* if, for every $x \in X$, there is an open neighbourhood U of x in X and an open neighbourhood, V, of $f(x)$ in Y such that f restricts to a homeomorphism $f : U \to V$. We also say that X is an *étale space over* Y.

Given a presheaf, F on B and $b \in B$, let

$$F_b = colim_{b\in U} F(U).$$

and $\mathsf{germ}_b : F(U) \to F_b$, be the natural map. The colimit is constructed using a disjoint union followed by using an equivalence relation. This germ map just send an element to its equivalence calss. More precisely: the set, F_b is the 'stalk' of F at b. It is made up of equivalence classes of 'germs' of *locally defined elements*, i.e., (U, b, x), where b is the point at which we are looking, U is an open set with $b \in U$ and $x \in F(U)$. If (U, b, x_U) and (V, b, x_V) are two such germs, they are equivalent if there is a $W \subset U \cap V$, again open in B, such that

$$res^U_W(x_U) = res^V_W(x_V),$$

i.e.,x_U and x_V agree 'near to b'. Now let $E(F) = \bigsqcup_{b\in B} F_b$ be the disjoint union with $\pi : E(F) \to B$, the obvious projection.

The topology on $E(F)$ is given by basic open sets: if $x \in F(U)$, $B(x) = \{\mathsf{germ}_b(x) \mid b \in U\}$ is to be open. (The idea is that we make x into a continuous local section of $E(F)$ over U by this means.) This makes $(E(F), \pi)$ an étale space over B.

We could construct $a(F)$ in (i) as $\Gamma_{E(F)}$, i.e., the sheaf of local sections of $E(F)$.

6.3.4 Covering spaces and locally constant sheaves

A covering space is an étale space, which is locally trivial, and it then corresponds to a locally constant sheaf on B.

For any set S, there is a constant sheaf, defined by the presheaf $F(U) = S$ for all $U \in Open(B)$. The corresponding étale space is $B \times S$ with its projection onto B and where S is given the discrete topology. A sheaf is *locally constant* if for each $b \in B$, there is an open set U_b containing b such that the restriction of F to U_b is a constant sheaf or, more strictly speaking, is isomorphic to a constant sheaf.

We can rephrase this in a neat way that introduces viewpoints that will be useful later on. The open sets U_b give us an open cover of B, so we could pick a subcover with the same trivialising property. We thus assume that we have a cover $\mathcal{U}$ and form a space $\bigsqcup \mathcal{U}$ by taking the disjoint union of the open sets in $\mathcal{U}$. (Recall that a convenient way of working with $\bigsqcup \mathcal{U}$ is to denote its elements by pairs (b, U) with $b \in U$ and $U \in \mathcal{U}$. We then have a copy of each b for each open set from the cover of which it is an element.) There is an obvious projection map

$$p : \bigsqcup \mathcal{U} \to B,$$

which is $p(b, U) = b$, and this is, fairly obviously, an étale map. We pull back F along p to get a sheaf on $\bigsqcup \mathcal{U}$ and, of course, this pulled back sheaf is constant.

This trick of turning a (topological) open cover into a map is very important. It forms the basis of the theory of Grothendieck topologies. In that theory, one replaces $Open(B)$ by a category $\mathcal{C}$, so a presheaf on $\mathcal{C}$ is just a functor $F : \mathcal{C}^{op} \to Sets$. The sheaf condition is adapted to this setting by specifying what (families of) morphisms in $\mathcal{C}$ are to be considered 'coverings' with an axiomatisation of their desired properties. For instance, for an open covering, $\mathcal{U}$ of B, if for each $U \in \mathcal{U}$, we pick an open covering of it and then combine these open coverings together we get an open covering of B. That is mirrored by a condition on the covering families in the Grothendieck topology.

We will not treat Grothendieck topologies in great detail here as, once again, that might take us too far away from the 'crossed menagerie' and the related issues of cohomology. We will give a definition shortly. It *will* be necessary, however, to have such a definition of a Grothendieck topos, i.e., the category of sheaves for such a Grothendieck topology and we will attempt to show how it relates to some of the topics we are considering. For greater detail from a very approachable viewpoint, the approach from Borceux and Janelidze's book, [29], is suggested, but we warn the reader that they also avoid very lengthy discussions of the topic, as their aim is not topos theory *per se*, but generalised Galois theory.

6.3.5 A siting of Grothendieck toposes

Definition: A *Grothendieck topos* is a category, $\mathcal{E}$, which is equivalent to a full reflective subcategory

$$\mathcal{E} \underset{}{\overset{a}{\leftrightarrows}} [\mathcal{C}^{op}, Sets]$$

of a presheaf category, $Presh(\mathcal{C}) = [\mathcal{C}^{op}, Sets]$, where the left adjoint, a, preserves finite limits.

The reflective nature of this category means that when considering morphisms *from* a (pre)sheaf to a sheaf, it is enough to give them at the presheaf level, since they will automatically be sheafified.

We had early on in our discussion of sheaves, the statement: *The category, $Sh(B)$, of sheaves on a space, B, is a reflective subcategory of the category, $Presh(B) = [Open(B)^{op}, Sets]$, of presheaves on B.* We can now rephrase this as a proposition:

Proposition 71 *The category, $Sh(B)$, of sheaves on a space, B, is a Grothendieck topos.* ■

In addition to the category of sheaves on a space, B, we also have several other important examples of the notion.

Example: (i) For any $\mathcal{C}$, the presheaf category, $Presh(\mathcal{C})$, is itself a full reflective subcategory of itself! It thus is a Grothendieck topos.

In particular, the category, $\mathcal{S}$, of simplicial sets is a Grothendieck topos (by taking $\mathcal{C} = \mathbf{\Delta}$). Later we will consider sheaves and bundles of groups, i.e., group objects in the topos of sheaves on a (base) space B. Equally well, we could look at group objects in presheaf toposes such as $[\mathcal{C}^{op}, Sets]$, and these are the group valued presheaves, and thus, in particular, $Simp.Grps$ is just the category of presheaves of groups on $\mathbf{\Delta}$.

We can take this 'analogy' further. If we have an étale space, $\alpha : A \to B$, over B, then a local section is a map $s : U \to A$ for $U \in Open(B)$, such that $\alpha s(x) = x$ for all $x \in U$. A presheaf, $F : Open(B)^{op} \to Sets$, is thought of as having $F(U)$ as being the local sections over

U of 'something' over B. That does not quite give an idea which is wholly expressed within the category of (pre)sheaves itself, as we needed to talk about U itself as well, but, from U, we can get a presheaf, much as above, namely the representable presheaf

$$\hat{U} = Open(B)(-, U).$$

This presheaf takes value a singleton on V if $V \subseteq U$ and is empty otherwise. The inclusion of U into B is the étale map that corresponds to this, so our local section $s : U \to A$ is the analogue of, (in fact, corresponds exactly to), a map of presheaves

$$s : \hat{U} \to \Gamma_A$$

and if $F : Open(B)^{op} \to Sets$ is arbitrary, $F(U) = Presh(B)(\hat{U}, F)$ by the Yoneda lemma, with each presheaf morphism φ from $\hat{U}$ to F yielding an element $\varphi_U(id_U) \in F(U)$. (Remember presheaf morphisms are merely natural transformations between the corresponding functors.)

Example: (ii) Another very important example of a presheaf topos, as above, comes from any group, G. We can, as we have done several times already, consider G as a one object groupoid, $G[1]$. It is then a suitable instance of a small category, which can be fed into the machine of the previous example. The category, $Presh(G[1])$, will be a Grothendieck topos, but what is the interpretation of these objects? From a straightforward perspective, they are set valued functors on $G[1]^{op}$. Suppose that $X : G[1]^{op} \to Sets$ is one such, then, abusing notation like mad, write $X = X(*)$ for the image of the single object $*$ of $G[1]^{op}$, and if $g \in G$, and $x \in X$, write $X(g)(x) = x.g$, then (and this is *left to you*) we can easily check that X is a *right G-set*. Conversely any right G-set, gives a presheaf on $G[1]$ and this sets up an equivalence of categories. (You should also check on morphisms.) If you prefer left G-sets, replace G by the opposite group, G^{op}.

This example is important as it provides the bridge between the cohomology of groups and the cohomology of spaces via a cohomology of toposes. We will see the above argument several times in what follows. (Following the idea that the reader should be able to 'dip' into these notes, we may repeat the point again and again!)

Example: (iii) Any category with a Grothendieck topology on it leads to a Grothendieck topos. We need a definition.

Definition: A *Grothendieck topology* on a category $\mathcal{C}$ is an assignment of families of 'coverings', $\{U_\alpha \to U\}_\alpha$ for each object U in $\mathcal{C}$ such that

- If $\{U_\alpha \to U\}_\alpha$ and $\{U_{\alpha\beta} \to U_\alpha\}_\beta$ are coverings, so is $\{U_{\alpha\beta} \to U\}_{\alpha\beta}$, i.e., 'coverings of coverings are coverings';
- If $\{U_\alpha \to U\}_\alpha$ is a covering family and $V \to U$ is a morphism in $\mathcal{C}$, then the pullback family $\{U_\alpha \times_U \to V\}_\alpha$ is a covering family for V, i.e., 'coverings are pullback stable';
- If $\{V \xrightarrow{\cong} U\}$ is an isomorphism, then this singleton family is a covering family.

A category together with a Grothendieck topology is called a *site*.

Given a site based on $\mathcal{C}$, a presheaf $F : \mathcal{C}^{op} \to Sets$ is called a *sheaf* on the site if for any object U and covering family $\{U_\alpha \to U\}_\alpha$, the sequence

$$F(U) \longrightarrow \prod F(U_\alpha) \rightrightarrows \prod F(U_\alpha \times_U U_\beta) \ ,$$

is an equaliser. (If the left hand morphism is merely injective then F will be a 'separated presheaf' in this context'.) The category of sheaves for a given site gives a Grothendieck topos.

Returning to the general case of $[\mathcal{C}^{op}, Sets]$, the Yoneda lemma shows the importance of the representable presheaves. In our key example with $\mathcal{C} = \mathbf{\Delta}$, these representable presheaves are just the simplices $\Delta[n] = \mathbf{\Delta}(-, [n])$. Our observations above point out that if K is a simplicial set, $K_n = K[n] \cong \mathcal{S}(\Delta[n], K)$ and this is the analogue of $F(U)$, i.e., the analogue of the set of local sections of F. Of course, there is no notion of topological continuity in the classical sense here, and as, in the 'presheaf topos' $\mathcal{S}$, all presheaves are sheaves, we have that in some sense 'all sections are as if they were continuous'. (The topological language is being pushed to breaking point here, so the corresponding intuitions would need refining if we were to follow them up properly. One *can* do this with the language of Grothendieck topologies, but we will not explore that further here. To some extent this is done in [29] with a different end point in mind. Here our purpose is to explain loosely why $\mathcal{S}$ is a topos, and why that may be useful and, reciprocally, what do the simplicial ideas, seen from that presheaf / sheaf viewpoint, suggest about general toposes.)

One further fact worth noting is that if $\mathcal{E}$ is a topos and B is an object in $\mathcal{E}$, then the 'slice category', $\mathcal{E}/B$, is also a topos. It thus is Cartesian closed, i.e., not only does it have finite limits, but the functor $- \times A : \mathcal{E} \to \mathcal{E}$, which sends an object X to $X \times A$ for some fixed object A, has a right adjoint $(-)^A$ thought of as being the object of maps from A to whatever. General results can be found in the various books on topos theory, which give very general constructions of these mapping space objects in settings such as the slice toposes. We will need some elementary ideas about Cartesian closed categories later.

6.3.6 Hypercoverings and coverings

It is sometimes necessary to mention 'hypercoverings', instead of 'coverings' when looking at generalisations of sheaves.

In any topos $\mathcal{E}$, there is a precise sense in which $\mathcal{E}$ behaves like a generalisation of the category of sets, but with a logic that replaces the two truth values $\{0, 1\}$ of ordinary Boolean logic by a more general object of truth values. In the topos $Sh(B)$ of sheaves on a space B, this truth value object is the lattice of open sets, $Open(B)$. This may seem a bit weird, but in fact works beautifully. (The logic is non-Boolean in general, so occasionally you need to take care with classical arguments.) This allows one to do things like simplicial homotopy theory *within* $\mathcal{E}$. This replaces the category, $\mathcal{S}$, of simplicial sets by $Simp(\mathcal{E})$ and if $\mathcal{E} = Sh(B)$, then the objects are just simplicial sheaves on B, i.e., sheaves of simplicial sets on B.

Any open cover $\mathcal{U}$ of a space B yields $\bigsqcup \mathcal{U}$, as before, and one can take repeated pullbacks to construct a simplicial sheaf on B from that cover. It is fun to view this in another way as it illustrates some of the ideas working within the topos $\mathcal{E}$ and, in particular, within $Sh(B)$.

Firstly, in $Sets$, there is a terminal object, 1, 'the one point set'. In a topos $\mathcal{E}$, there is a terminal object, $1_\mathcal{E}$, and, for $\mathcal{E} = Sh(B)$, this is the constant sheaf with value the one point set.

Viewed as an étale space, it is just the identity map, $B \stackrel{id}{\to} B$. (This multitude of viewpoints may initially seem to lead to confusion, but it does give a beautifully rich context in which to work, with different intuitions and analogies interacting and combining.)

Within $\mathcal{E}$, we have a product, so if $A_1, A_2 \in \mathcal{E}$, we can form $A_1 \times A_2$. What does this looks like for $\mathcal{E} = Sh(B)$? The A_i gives étale spaces $\alpha_i : A_i \to B$, $i = 1, 2$ and $A_1 \times A_2$ corresponds to the pullback

$$A_1 \times_B A_2 \to B.$$

In particular, if $\mathcal{U}$ is an open covering of B, write $U \to 1$ for $\mathcal{U}$ viewed as a sheaf / étale space, $\bigsqcup \mathcal{U} \to B$, within $Sh(B)$, then the product

$$\begin{tikzcd} U \times U \arrow[r, shift left=2] \arrow[r] & U \arrow[l, shift left=2] \end{tikzcd}$$

makes U into a groupoid / equivalence relation within $\mathcal{E} = Sh(B)$. The simplicial object defined by multiple pullbacks is just the nerve of this groupoid, which will be denoted $N(U)$, or more often $N(\mathcal{U})$. In low dimensions, this looks like

$$N(U): \qquad \ldots \underset{\longrightarrow}{\overset{\longrightarrow}{\vdots}} U \times \cdots \times U \underset{\longrightarrow}{\overset{\longrightarrow}{\vdots}} \cdots \overset{d_0}{\underset{d_2}{\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}}} U \times U \overset{d_0}{\underset{d_1}{\substack{\longrightarrow \\ \longrightarrow}}} U \xrightarrow{p} 1.$$

(In the case when B is a manifold and $\mathcal{U}$ is an open covering by contractible open sets such that all the finite intersections of sets from $\mathcal{U}$ are also contractible (sometimes called a 'Leray cover', cf. [146]), the groupoid above is called a 'Leray groupoid', see the same cited paper.)

(In terms of étale spaces over B, you just replace $\times$ by $\times_B$ and 1 by B.) In cases where B is not a 'locally nice space', or if we replace $Sh(B)$ by a more general topos, the simplicial sheaf given by $\mathcal{U}$ is too far away from being an internal Kan complex and so we have to replace the nerve of a cover by a 'hypercovering', which is a 'Kan' simplicial sheaf, K, with an 'augmentation map' $K \to 1$, which is a 'weak homotopy equivalence'. (Look up papers on hypercoverings for a much more accurate treatment of them than we have given here.) Of course, this is very like the situation in group cohomology, where one starts with a 'resolution' of G. This is a resolution of B or better of 1 by a simplicial object.

It will be useful later on to give a 'down-to-earth' description of the various levels of $N(\mathcal{U})$. The zeroth level $N(\mathcal{U})_0$ is just the sheaf $\mathcal{U} = \sqcup\{U : U \in \mathcal{U}\}$, or rather the local sections of this over B. A point in this étale space can be represented by a pair (b, U) where $b \in U$, i.e., the point b of B indexed by U. The projection to B, of course, sends (b, U) to b. This notation is one way of labelling points in a disjoint union, namely the point and an index labelling in which of the sets of the collection is it being considered to be for that part of the disjoint union. Now a point of the pullback over B will be a pair of such points with the same b, so is easily represented as (b, U_0, U_1) where (b, U_0) and (b, U_1) are both points in the above sense. This however implies that $b \in U_0 \cap U_1$, and here, and in higher levels, this idea works: a point in the multiple pullback occurring at level n is of the form $(b, U_0, \ldots, U_n)$, where $b \in \bigcap_{i=0}^n U_i$.

There is yet another useful point to make about this multiple way of considering an open covering as a sheaf (or a family or a simplicial sheaf or groupoid or étale space). It tells us what a morphism between open coverings might be and hence what the category of open coverings of a space B 'is'.

We will take a naive viewpoint (as that is often a good place to start), and then may refine it slightly if we hit problems. An open covering of a space B is a *family*, $\mathcal{U} = \{U_i \mid i \in I(\mathcal{U})\}$, of open sets of B, where we refer to $I(\mathcal{U})\}$ as the index set of the family. Of course, we need $\bigcup \mathcal{U} = B$ as well.

If $\mathcal{V}$ is another such covering family, then we would expect a map of coverings $\alpha : \mathcal{V} \to \mathcal{U}$ to be a map of families. Here it will help to have a formal definition of the category of families in an abstract category, $\mathbb{A}$. (A good reference for this notion is chapter 6 of the book by Borceux and Janelidze, [29], that we have mentioned several times before.)

Definition: Let $\mathbb{A}$ be a category. A *family*, $\mathcal{A}$ of objects of $\mathbb{A}$ is a function $\mathcal{A} : I(\mathcal{A}) \to Ob(\mathbb{A})$, from the index set $I(\mathcal{A})$ of the family to the collection of objects of the category, $\mathbb{A}$. For a set, I, we say that $\mathcal{A}$ is an *I-indexed family* if $I(\mathcal{A}) = I$.

A morphism $\alpha : \mathcal{A} \to \mathcal{B}$ of families consists of a map $|(\alpha) : I(\mathcal{A}) \to I(\mathcal{B})$ and an $I(\mathcal{A})$-indexed family of morphism $\{\alpha_i : A_i \to B_{I(\alpha)(i)}$. The category $\mathsf{Fam}(\mathbb{A})$ is the category of such families and the morphisms between them.

An open covering $\mathcal{U}$, of a space B is then a family in the category $Open(B)$ of open sets of B and inclusions between them satisfying the condition $\bigcup \mathcal{U} = B$. This leads to a category, $Cov(B)$, of open coverings of B.

Remark: The above definition is very closely related to the idea of refinement of open coverings that one finds in classical treatments of Čech homology and cohomology, for instance, see Spanier, [198]. It is notable that to handle the constructions of these well one has to take the relation of 'finer than' and chose a 'refinement map' which realises the relation in a more 'constructive' way. (The relations says that there is a function 'doing the job', the refinement map picks out one of the possible ones.) This is very like a situation we will meet many times later on. The classical approach asks for the *existence* of something, the more modern approach needs that something to be specified.

We have each open covering, $\mathcal{U}$, of our space B gives a sheaf, namely the *sheaf of local sections* of the étale space, $\bigsqcup \mathcal{U} \to B$. We note the following:

Lemma 44 *If $\mathcal{V}$ and $\mathcal{U}$ are open coverings of a space B, then a morphism, α, from $\mathcal{V}$ to $\mathcal{U}$, induces a map of the corresponding étale spaces over the base B:*

$$\begin{tikzcd}
\bigsqcup \mathcal{V} \arrow[rr, "\alpha"] \arrow[dr] & & \bigsqcup \mathcal{U} \arrow[dl] \\
& B &
\end{tikzcd}$$

■

Of course, as you would expect, any such morphism will induce a morphism of the corresponding groupoids or simplicial sheaves.

We have to be a bit careful here, since if the sets in the coverings are not connected, we could get maps between these étale spaces that did not correspond to morphisms of the coverings. We will **leave you to explore this**, but also suggest looking at [29].

6.3.7 Base change at the sheaf level

Changing the base induces a pair of adjoint functors.

It is often necessary to examine what happens when we 'change the base space' for our sheaves. Suppose X is a space and $Sh(X)$ the corresponding category of sheaves on X. We might have a subspace A of X, and ask for the relationship between $Sh(X)$ and $Sh(A)$, for instance: Is there an induced functor? In which direction? If so, when does it have nice properties? and so on. More generally, if $f : X \to Y$ is a continuous map, then we expect to have some 'induced functors' between $Sh(X)$ and $Sh(Y)$.

First take a look at presheaves, and so naturally we need to look at the behaviour of f on open sets. The partially ordered sets $Open(X)$ and $Open(Y)$ can be thought of as categories as we already have done, and since continuity of f is just : if V is open in Y, then $f^{-1}(V)$ is open in X, f corresponds to a functor

$$f^{-1} : Open(Y) \to Open(X).$$

(You should **check functoriality**. It is routine.)

As a presheaf F on X is just a functor $F : Open(X)^{op} \to Sets$, we can precompose with $(f^{-1})^{op}$ to get a presheaf on Y, i.e., we have a presheaf, $f_*(F)$. This is then given by $f_*(F)(V) = F(f^{-1}(V))$. If $\mathcal{V} = \{V_i\}$ is an open cover of V, then $f^{-1}(\mathcal{V}) = \{f^{-1}(V_i)\}$ is an open cover of $f^{-1}(V)$, so it is easy to check that, if F is a sheaf on X, $f_*(F)$ is a sheaf on Y. (An interesting exercise is to consider the inclusion, f, of a subspace, A, into Y and a sheaf F on A. What is the value of $f_*(F)(V)$ if $A \cap V = \emptyset$ and why?) The sheaf $f_*(F)$ is often called the *direct image* of F under f, but this is not always a good name as it is not really an 'image'.

The construction gives a functor

$$f_* : Sh(X) \to Sh(Y),$$

and, clearly, if $g : Y \to Z$ as well, then $(gf)_* = g_* f_*$, whilst $(Id_X)_* = Id_{Sh(X)}$. (Note we are saying that f_* is a functor, but also that writing $Sh(f)$ for f_* would give us a 'sheaf category functor'. That is more or less true, but things are, in fact, richer and more complex than just this.) The richness of the situation is that f also induces a functor going in the other direction, that is from $Sh(Y)$ to $Sh(X)$. This is easier to see if we change our view of sheaves back from special presheaves to étale spaces over the base.

Suppose we have a space over Y, $p : A \to Y$, then we can form the pullback $X \times_Y A$. This is, in fact, 'only specified 'up to isomorphism' as it is defined by a universal property. (You should check up on this point if you are unsure, although we will discuss it in some more detail as we go along.) There is a 'usual construction' of it namely as a subspace of the product $X \times A$:

$$X \times_Y A = \{(x, a) \mid f(x) = p(a)\},$$

but this is not 'the' pullback, just a choice of representing object within the class of isomorphic objects satisfying the specifying universal pullback property - and we also need the structural maps $p_X : X \times_Y A \to X$ and $X \times_Y A \to A$ in order to complete the picture. Of course, for instance, $p_X(x, a) = x$. There is no canonical choice of pullback possible and the resulting coherence situation is the source of much of the higher dimensional structure that we will be meeting later.

We will find it useful to use the universal property more or less explicitly, so it may be good to recall it here:

We have a square

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "p_X"'] & A \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

such that (i) it commutes: $pf' = fp_X$, and (ii) given any object B and maps $q : B \to A$ such that $pg = qf$, then there is a *unique* morphism $\alpha : B \to P$ such that $p_X\alpha = q$ and $f'\alpha = g$.

We repeat that this property determines P, p_X and f' up to isomorphism only. Our construction of P as $X \times_Y A$ for the situation in the category of spaces shows that such a P exists, but does not impose any odour of 'canonisation' on the object constructed.

We next look at local sections of (P, p_X). We have $s : U \to P$ such that $p_X s(x) = x$ for all $x \in U$. This means that s determines, and is determined by, a map from U to A, namely $f's$, such that $f(x) = pf's(x)$ for all $x \in U$. This looks a bit like a local section of $A \xrightarrow{p} Y$ over $f(U)$, but we do not know if $f(U)$ is open in Y. To make things work, we can take $f^*(F)(U) = colim\{F(V) : V \text{ open in } Y, f(U) \subseteq V\}$, so we have the elements of $f^*(F)(U)$ are germs of local sections of F, whose domain contains $f(U)$. (You should check this works in giving us a sheaf on X, and that it is functorial, giving us a functor

$$f^* : Sh(Y) \to Sh(X).$$

See why it works yourself, but looks up the details in a sheaf theory textbook.) Of course, warned by previous comments, you will want to check that if $g : Y \to Z$, $(gf)^*$ and f^*g^* will be naturally isomorphic, (but usually not 'equal'). This will be very important later on.

If $F \in Sh(X)$, the sheaf we have just constructed is variously called the *pullback of F along f*, the *inverse image sheaf* or if f is the inclusion of a subspace into Y, the *restriction of F to X*. This construction is also said to lead to *induced sheaves* or sometimes *co-induced sheaves* depending on the style of terminology being used.

Now suppose $f : X \to Y$ and so we have

$$f_* : Sh(X) \to Sh(Y),$$

and

$$f^* : Sh(Y) \to Sh(X).$$

These functors must be related somehow! In fact if $F \in Sh(Y)$ and $G \in Sh(X)$, then

$$Sh(X)(f^*(F), G) \cong Sh(Y)(F, f_*(G)).$$

We sketch a bit of this, leaving the details to be looked for. Suppose $\varphi : F \to f_*(G)$ in $Sh(Y)$, then for an open set V in Y, we have

$$\varphi_V : F(V) \to G(f^{-1}(V)).$$

Now suppose U is open in X and $V \supseteq f(U)$, then $f^{-1}(V) \supseteq U$, so we have

$$F(V) \xrightarrow{\varphi} G(f^{-1}(V)) \to G(U),$$

and passing to the colimit we get a map from $f^*(F)(U)$ to $G(U)$. The other way around is similar, so is left for you to worry out for yourselves.

Of course, the above natural isomorphism says f^* is left adjoint to f_*, and this implies a lot of nice properties that are often used.

This makes for quite a lot of 'facts' about sheaves and their uses, but we need one more observation before passing to other things. Often geometric information is encoded by a sheaf, sometimes 'of rings', sometimes 'of modules' or 'of chain complexes'. For instance, on a differential manifold, one has a sheaf of differential functions and also the de Rham complex which is a sheaf of differential graded algebras. In algebraic geometry, the usual basic object is a scheme, which is a space together with a sheaf of commutative rings on it that is 'locally' like the prime spectrum of a commutative ring. There are many other examples. We will also be looking at sheaves of groups and sheaves of crossed modules.

It would have been nice to show how a sheaf theoretic viewpoint provides the link between covering space theory and Galois theory, but again this would take us too far afield so we refer to Borceux and Janelidze, [29], and the references therein.

6.4 Descent: Torsors

(Some of the best sources for the material in this section are in the various notes and papers of Breen, [31, 32] and, in particular, his Astérisque monograph, [33] and his Minneapolis notes, [34].)

The demands of algebraic geometry mean that principal G-bundles for G a (topological) group are not sufficient to handle all that one would like to do with such things. One generalisation is to vary G over a base. This may be to replace G by a sheaf of groups or by a group object in Top/B, i.e., a group bundle. (This is the topological analogue of a group scheme.) The situation that we considered earlier then corresponds to a constant sheaf of groups or the group bundle $G_B := (B \times G \to B)$ given by projection from the product. It also includes the vector bundles that we briefly saw earlier. The more general case, however, does not change things much. We have a parametrised family of groups G_b, $b \in B$, acting on a parametrised family of spaces, X_b, $b \in B$. The sheaf of groups viewpoint corresponds to an étale space on B and thus to a group bundle on B with each G_b discrete as a topological group. We will let, in the following, G be a bundle of groups on a space B. (We will on occasion abuse notation and write G instead of G_B for the 'constant G' example.)

Technically we will need to be working in a setting where we can talk of a bundle of locally defined maps from one bundle to another. This is fine in the sheaf theoretic setting, and will be assumed to be the case in the general case of a suitable category of bundles within the ambient category, Top/B. It corresponds to the functor $- \times A$ always having a right adjoint $(-)^A$, the function bundle of locally defined maps from A to whatever. Technically we are assuming that our category of bundles on B, Bun/B is a Cartesian closed category.

6.4.1 Torsors: definition and elementary properties

Definition: A left G-*torsor* on B is a space $P \xrightarrow{\pi} B$ over B together with a left group action

$$G \times_B P \to P$$

$$(g, p) \longmapsto g.p$$

such that the induced morphism

$$\phi : G \times_B P \to P \times_B P$$

$$(g, p) \longmapsto (g.p, p)$$

is an isomorphism. In addition we require that there exists a family of local sections, $s_i : U_i \to P$, for some open cover, $\mathcal{U} = (U_i)_{i \in I}$, of B.

A right G-torsor is defined similarly with a right G-action. If P is a left G-torsor, there is an associated right G-torsor, P^o, with action $p.g = g^{-1}.p$.

When we refer to a G-torsor, without mentioning 'left' or 'right', we will mean a left G-torsor.

The connection with our earlier definition of principal G-bundle can be made more evident if we note that, on writing $\theta = \phi^{-1} : P \times_B P \to G \times_B P$, then the analogue of the *translation function* of page 242, is the *translation morphism*, $\tau : P \times_B P \to G$, given by $pr_1 \circ \theta$. The morphism θ then equals (τ, pr_2).

The effect of the requirement that local sections exist is to ensure that the bundle $P \xrightarrow{\pi} B$ is locally trivial, i.e., locally like $G \to B$. This is a consequence of the following lemma.

Lemma 45 *Suppose $P \xrightarrow{\pi} B$ is a G-torsor for which there is a global section*

$$s : B \to P$$

of π, then there is an isomorphism

$$G \xrightarrow{f} P$$

of spaces over B.

Proof: Define a function $f : G \to P$ by $f(g) = (g.s(b))$, where $g \in G_b$. As the projection of the group bundle G is continuous, f is continuous. To get an inverse for f, consider the map

$$P \xrightarrow{\pi} B \xrightarrow{s} P.$$

For any $p \in P$, $s\pi(p)$ is in the same fibre as p itself, so we get a continuous map

$$P \xrightarrow{(id, s\pi)} P \times_B P \xrightarrow{\cong} G \times_B P$$

on composing with the inverse of the torsor's structural isomorphism. Finally projecting on to G gives a map $h : P \to G$. This is continuous and checking what it does on fibres shows it to be the required inverse for f. ■

This does not, of course, transfer a group structure to P, but says that P is like G with 'an identity crisis'. It no longer knows what its identity is!

The group bundle, $G \to B$, considered as a space over B is naturally a G-torsor with multiplication on the left giving the G-action. Check the conditions. It has a global section, since we required it to be a group object in Top/B, so there is a continuous map, e, over B from the terminal object of Top/B to G, which plays the role of the identity. As that terminal object is (isomorphic to) the identity on B, $B \to B$, this splits $G \to B$,

$$\begin{tikzcd} B \arrow[rr, "e"] \arrow[dr, "="'] & & G \arrow[dl] \\ & B & \end{tikzcd}$$

This trivial G-torsor will be denoted T_G.

Applying this to a general G-torsor, the local section $s_i : U_i \to P$ makes $P_{U_i} = \pi^{-1}(U_i)$, the restricted torsor over the open set U_i, into the trivial G_{U_i}-torsor over U_i, so P is *locally trivial.* It is important to note again that this means that P looks locally like G, (but if G is not a product bundle, P will not be locally a product, so need not be locally trivial in the stronger sense used in topological situations). The way that P differs globally from G is measured by cohomology. (An important visual example is, once again, the boundary circle of the Möbius band, i.e., the double cover of the circle, S^1, that twists as you go around that base circle. It is locally a product $U \times \{-1, 1\}$, but not globally so.)

The next observation is very important for us as it shows how the language of G-torsors starts to interact with that of groupoids. First an obvious definition.

Definition: If P and Q are two left G-torsors, then a *morphism, $f : P \to Q$, of G-torsors* (over B) is a continuous map over B such that $f(g.p) = g.f(p)$ for all $g \in G$, $p \in P$.

Here and elsewhere, it is to be understood that we only write $g.p$ if $g \in G_b$ and $p \in P_b$ for the same b. This avoids our constantly repeating mention of the base space and its points. If working with sheaves on a site, i.e., a category $\mathcal{C}$, with a Grothendieck topology, the g and p correspond to locally defined 'elements' in some $G(C)$ and $P(C)$ respectively, so the same (abusive) notation suffices.

Lemma 46 *Any morphism $f : P \to Q$ is an isomorphism.*

Proof: We have trivialising covers, $\mathcal{U}$ for P, and $\mathcal{V}$ for Q, on which local sections are known to exist. By taking intersections, or any other way, we can get a mutual refinement on which both P and Q trivialise so we can assume $\mathcal{U} = \mathcal{V}$. We thus are looking at a morphism f and local sections $s : U \to P$, $t : U \to Q$, which (locally) determine isomorphisms to T_G over U. We thus have reduced the problem, at least initially, to showing that $f : T_G \to T_G$ is always an isomorphism, but

$$f(1_G) = g.1_G$$

for some $g \in G_B$, i.e., for some global element of G. Moreover g is uniquely determined by f. Now it is clear that the morphism sending 1_G to $g^{-1}.1_G$ is inverse to f. (Although it is probably an obvious comment, we should point out that saying where a single global element goes determines the morphism, and, within T_G, any (locally defined) element is given by multiplication of the global section, 1_G, by that element, but now regarded as an element of G itself.)

Back to our original $f : P \to Q$, on each U, we have $f_U : P_U \to Q_U$, its restriction to the parts of P and Q over U, is an isomorphism, so we construct the inverse locally and then glue it into a single f^{-1}. ■

Remark on descent of morphisms: Although we have not yet completed the proof, it is instructive to go into this in a bit more detail, since it introduces methods and intuitions that here should be more or less clear, but later, in more 'lax' or 'categorified' settings will need both good intuition and the ability to argue in detail with (generalisations of) local sections.

If we use s and t, then with respect to these local sections over U, every local element of P_U has the form $g_U.s_U$ for some unique locally defined $g_U : U \to G$ (or in sheaf theoretic notation $g_U \in G(U)$). Similarly in Q_U, local elements looks like $g_U.t_U$, but then

$$f(g_U.s_U) = g_U.f(s_U),$$

so we only need to look at $f(s_U)$. As $f(s_U) \in Q_U$, it determines some unique local element $h_U \in G(U)$ with

$$f(s_U) = h_U.t_U,$$

and checking for behaviour when composing morphisms, it is then clear that

$$f_U^{-1}(t_U) = h_U^{-1}.s_U$$

with continuity of f^{-1} handled by the continuity of inversion, that of t and of multiplication.

As the construction of f_U^{-1} is done using maps defined locally over U, f_U^{-1} is in Top/U (or alternatively, is a map of sheaves on U). We now have to check that this locally defined morphism 'descends' from $\bigsqcup\mathcal{U}$ to B.

Of course, it is 'clear' that it must do so! Each h_U is uniquely defined so ... . That *is* true, but when we go to higher dimensional situations we will often not have uniqueness, merely uniqueness up to isomorphism, or equivalence, so we will spell things out in all the 'gory detail'.

We need to check what happens on intersection $U_1 \cap U_2$ of local patches in our trivialising cover, $\mathcal{U}$. Write $f_i = f_{U_i}$, $i = 1, 2$, etc. for simplicity. The local sections s_1 and s_2 (resp. t_1 and t_2) will not, in general, agree on $U_1 \cap U_2$, so we have

$$f_1(s_1) = h_1.t_1,$$

$$f_2(s_2) = h_2.t_2,$$

but the key local elements $h_1|_{U_1\cap U_2}$ and $h_2|_{U_1\cap U_2}$ need not agree. A bit more notation will probably help. Let us denote by s_{12} the restriction of $s_1 : U_1 \to P$ to the intersection $U_1 \cap U_2$ and similarly $s_{21} = s_2|_{U_1\cap U_2}$, extending this convention to other maps when needed.

We then have some $g_{12} \in G_{U_1\cap U_2}$ for which

$$s_{21} = g_{12}.s_{12}, \text{ (and } s_{12} = g_{21}.s_{21}, \text{ so } g_{12} = g_{21}^{-1}),$$

but then, over $U_1 \cap U_2$,

$$f(s_{21}) = g_{12}.f(s_{12}).$$

We thus have

$$t_{21} = h_{21}^{-1} g_{12} h_{12} t_{12}.$$

Now turning to f^{-1}, defined locally by $f_i^{-1} : Q_{U_i} \to P_{U_i}$, $i = 1, 2$ with

$$f_i^{-1}(t_i) = h_i^{-1}.s_i,$$

then over $U_1 \cap U_2$, $f_{ij}^{-1}(t_{ij}) = h_{ij}^{-1} s_{ij}$, but we also have $f_j^{-1}(t_{ji}) = h_{ji}^{-1} s_{ji}$ and we have to check that on $Q_{U_i \cap U_j}$, $f_{ij}^{-1} = f_{ji}^{-1}$. To do this, it is sufficient to calculate $f_{ji}^{-1}(t_{ij})$ and to compare it with $f_{ij}^{-1}(t_{ij})$ as both are defined on the same generating local section and so extend via their G-equivariant nature. We have

$$\begin{aligned} f_{ji}^{-1}(t_{ij}) &= f_{ji}^{-1}(h_{ij}^{-1} g_{ji} h_{ji} t_{ji}) \\ &= h_{ij}^{-1} g_{ji} h_{ji} f_{ji}^{-1}(t_{ji}) \\ &= h_{ij}^{-1} g_{ji} h_{ji} h_{ji}^{-1}.s_{ji} \\ &= h_{ij}^{-1} g_{ji} g_{ij} s_{ij} \\ &= h_{ij}^{-1} s_{ij} \\ &= f_{ij}^{-1}(t_{ij}), \end{aligned}$$

so the two restrictions *do* agree over the intersection and hence *do* give a morphisms from Q to P inverse to f. (This last point is easy to check.) ■

If we denote the category of left G-torsors on B by $Tors(B,G)$ (or $Tors(G)$ if B is understood), then we have

Proposition 72 *$Tors(B,G)$ is a groupoid.* ■

6.4.2 Torsors and Cohomology

In the above discussion, we saw how a choice of local sections $s_i : U_i \to P$ gave rise to a map $g_{ij} : U_{ij} \to G$. (Here we will again abbreviate: $U_i \cap U_j = U_{ij}$. This notation will be extended to give $U_{ijk} = U_i \cap U_j \cap U_k$, etc.)

The maps g_{ij} are to satisfy

$$s_i = g_{ij} s_j$$

on U_{ij} and for all indices i, j. The map, g_{ij}, gives the translation from the description using s_i to that using s_j. Of course, as g_{ij} is invertible, it can also translate back again. These elements are uniquely determined by the sections, so over a triple intersection, U_{ijk}, we have the 1-cocycle equation,

$$g_{ij} g_{jk} = g_{ik}.$$

If we use different local sections, say s'_i, assumed to be on the same open cover, there will be local elements, $g_i : U_i \to G$, such that $s'_i = g_i.s_i$ for all $i \in I$. The corresponding cocycles g_{ij} and g'_{ij} will be related by a coboundary relation

$$g'_{ij} = g_i g_{ij} g_j^{-1}.$$

These equations will determine an equivalence relation on the set, $Z^1(\mathcal{U}, G)$, of 1-cocycles for $\mathcal{U}$, as before, the (fixed) open cover. The set of equivalence classes will be denoted $H^1(\mathcal{U}, G)$. To remove the dependence on the open cover, one passes to the limit on finer covers to get the Čech non-Abelian cohomology set, $\check{H}^1(B,G) = colim_{\mathcal{U}} H^1(\mathcal{U}, G)$ which, by its construction classifies isomorphism classes of G-torsors on B. The trivial left G-torsor, T_G, gives a natural distinguished element to $\check{H}^1(B,G)$.

This looks quite good. We have started with a torsor and seem to have classified it, up to isomorphism, by cocycles. The one deficiency is that we need to know that cocycles give torsors, i.e., a (re)construction process of P from the cocycle (g_{ij}), but without prior knowledge of P itself.

The method we will use will take the basic ingredients of the group bundle, G, and will twist them using the g_{ij}. First if we have $\gamma \in \check{H}^1(B, G)$, by the basic construction of colimits, we can pick an open cover $\mathcal{U}$ and a $g_{\mathcal{U}} = (g_{ij})$, whose cohomology class represents γ in the colimit. Next taking this $\mathcal{U} = \{U_i\}$, and g_{ij}, let

$$P = \bigsqcup_i G(U_i)/\sim.$$

As we are once again using a disjoint union, we will give our points an index, (g, i), and, of course,

$$(g, i) \sim (gg_{ij}, j).$$

We have a projection $P \to B$ induced from the bundle projections $G(U) \to B$. (For you to check that it works.) This is continuous if P is given the quotient topology. Moreover the multiplications

$$G(U) \times G(U) \to G(U)$$

give a left action

$$G \times P \to P$$

making P into a left G-torsor as hoped for.

To sum up:

Theorem 21 *The set, $\check{H}^1(B, G)$, is in one-one correspondence with the set of isomorphism classes of G-torsors on B, that is, with the set $\pi_0 Tors(B; G)$ of connected components of the groupoid, $Tors(B; G)$.* ■

The relationship for isomorphisms is **left for you to check**.

6.4.3 Change of base

This link with cohomology suggests that we should see what might happen if we changed the base space B in the above. As cohomology is about maps *out of* the space, we should expect that if $f : B \to B'$ is a continuous map then we would get an induced map going from $\check{H}^1(B', G)$ to $\check{H}^1(B, f^*(G))$, but what would this look like through the G-torsors perspective? Suppose we have a G-torsor, Q, over B', then Q is a sheaf on B', so we have an induced sheaf $f^*(Q)$ on B given by pullback, as above, page 266. Strictly speaking as G is a sheaf or bundle of groups on B', $f^*(Q)$ cannot be a G-torsor, but might be a $f^*(G)$-torsor.

We have checked some of what has to be examined before, in the simpler case of principal G-bundles. We will repeat some of the results, but with slightly more categorical proofs as the very element based approach we used is fine for that topological setting, but is here beginning to be less optimal with a sheaf of groups as coefficients. (We will not, however, go to a elegant, fully categorical proof as we have not treated geometric morphisms of toposes.)

First we need an action of $f^*(G)$ on $f^*(Q)$. We have the action of G on Q. There is a quick derivation of this which we will sketch. The functor f^* is a left adjoint and so preserves colimits ..., which is useless to us in this situation! It is also a right adjoint of another functor which we have not discussed. It therefore preserves products and thus actions. A way to see that

$f^*(G \times_{B'} Q) \cong f^*(G) \times_B f^*(Q)$, without producing the left adjoint of f^* is via the étale space description of sheaves. In that description, $f^*(G)$, etc., are all given by pullbacks. We draw a diagram:

$$\begin{tikzcd}
 & f^*(G \times_{B'} Q) \arrow[rrr] \arrow[ddl] \arrow[dr] & & & G \times_{B'} Q \arrow[dr] \arrow[ddl] & \\
 & & f^*(Q) \arrow[rrr] \arrow[ddl] & & & Q \arrow[ddl] \\
f^*(G) \arrow[rrr] \arrow[dr] & & & G \arrow[dr] & & \\
 & B \arrow[rrr] & & & B' &
\end{tikzcd}$$

Each face of the resulting cube is a pullback, as is the vertical square given by the diagonals of the two ends plus the top and bottom maps, but the same would be true of the equivalent diagram with $f^*(G \times_{B'} Q)$ replaced by $f^*(G) \times_B f^*(Q)$, so these two objects are isomorphic.

If we now look at what happens to the action then the original action of G on Q induces one of $f^*(G)$ on $f^*(Q)$ as hoped for. (The detailed verification is left to you as usual.) As the first condition of the definition of torsor again involves pullbacks, it is now fairly routine to check it for $f^*(Q)$. The other condition is the existence of local sections and we have to use a slightly different approach for this. We know that there is an open cover $\mathcal{U}$ of B' over which local sections exist, say, $s_i : U_i \to Q$, $U_i \in \mathcal{U}$. The obvious open cover for B is $f^{-1}(\mathcal{U})$, so we look for sections $f^{-1}(U_i) \to f^*(Q)$. As $f^*(Q)$ is given by a pullback, we will get such a map if we specify maps $f^{-1}(U_i) \to Q$ and $f^{-1}(U_i) \to B$ making the obvious square commute. The map $f^{-1}(U_i) \to B$ 'must' be the inclusion ... what else could it be, so we will try that. Composing that with f gives a map $f^{-1}(U_i) \to B'$, which can also be written as the composite of f restricted to $f^{-1}(U_i)$ followed by the inclusion of U_i into B', so we can compose that restriction of f with s_i to get a map to Q. Since s_i is a section over U_i of the map $Q \to B'$, it is now easy to check that the 'obvious square' commutes. (**Left to you**.) We have built a local section over $f^{-1}(U_i)$. We thus have

Proposition 73 *If Q is a G-torsor over B', then $f^*(Q)$ is a $f^*(G)$-torsor over B.* ■

The new torsor $f^*(Q)$ would here loosely be called the *induced torsor of Q along f*.

We have a cocycle description of torsors. If we have one for Q, what will be the one for $f^*(Q)$? In a sense, we know what the answer is without doing any calculation. The cocycle description of Q gives a class in $H^2(B', G)$ and the induced map from that to $H^2(B, f^*(G))$ must surely be given by composition with f. The fact that the coefficients change as well as the space should come out 'in the wash'. We would, from this perspective, also expect the maps induced from homotopic maps to be the same. We know what to expect but what about the details!

Suppose we pick local sections s_i for Q over the various U_i in a cover $\mathcal{U}$ of B', and we get the $g_{ij} \in G(U_{ij})$ as above. These satisfy

$$s_i = g_{ij} s_j.$$

We have just seen that suitable local sections over the $f^{-1}(U_i)$ are given by the pairs of maps $(s_i f, inc) : f^{-1}U_i \to Q \times_{B'} B$, but these are determined just be the first component. Likewise the sections g_{ij} over pairwise intersections of G, correspond by composition to the corresponding elements $g_{ij} f$ over the pairwise intersections of $f^{-1}(\mathcal{U})$, and, of course, these are the transition cocycles for the $s_i f$. That they are cocycles follows since the g_{ij} satisfy the cocycle condition.

To summarise: the cocycle data for $f^*(Q)$ can be derived from that for Q merely by precomposing by the relevant restrictions of f to the sets of the cover $f^{-1}(\mathcal{U})$ and their intersections. Just as we expected.

Having seen that homotopic maps induced isomorphic principal bundles in an earlier section, it is natural to expect the same thing to happen here. It does, but rather than explore that here we will put it aside for a little while until we have a simplicial description of torsors in sections 6.4.5 and 6.5.5. That will make life a lot easier.

We have changed the base, what about changing the 'coefficients'?

6.4.4 Contracted Product and 'Change of Groups'

In Abelian cohomology, one would expect the cohomology 'set' (there a group) to vary nicely with the coefficient sheaf of groups, G. Something like that occurs here as well and determines some essential structure on the torsors. Suppose $\varphi : G \to H$ is a homomorphism of sheaves of groups, then one expects there to be induced functors between $Tors(G)$ and $Tors(H)$ in one direction or the other. Thinking of the better known case of a ring homomorphism, $\varphi : R \to S$, and modules over R or S, then we could, for an S-module, M, form an R-module by restriction along φ. The analogue works for an H-set X as one gets a G-set by defining $g.x = \varphi(g).x$, but there is no reason to expect the resulting G-set to be principal, so this does not look so feasible for torsors. There is, however, another module construction. Suppose that N is a left R-module, and make S into a *right* R-module, S_R by $s.r = s\varphi(r)$, then we can form $S_R \otimes_R N$, and the left S-action by multiplication is nicely behaved. The point is that S is behaving here as a two sided module over itself, and also as a (S, R)-bimodule. The corresponding idea in torsor theory is that of a bitorsor, explored in depth by Breen in [31], which we will examine later in this chapter.

Before looking at this in a bit more detail, we will look at the contracted product, which replaces the tensor product here. Suppose we have a category, $\mathcal{C}$, and an internal group, G, in $\mathcal{C}$. Here we have various examples in mind. If $\mathcal{C} = Sh(B)$, G will be a sheaf of groups; if $\mathcal{C}$ is the category of groupoids, G will be an internal group in that category, i.e., a *(strict) gr-groupoid*, and will correspond to a crossed module, and, if we combine the two ideas, $\mathcal{C}$ is a category of sheaves of groupoids, so G is a sheaf of gr-groupoids, corresponding to a sheaf of crossed modules, and so on in various variants.

A left G-object in $\mathcal{C}$ is an object X together with a morphism, (left action),

$$\lambda : G \times X \to X,$$

satisfying obvious rules. Similarly a right G-object Y comes with a morphism, (right action),

$$\rho : Y \times G \to Y.$$

The *contracted product* of Y and X is, intuitively, formed from $Y \times X$ by dividing by an equivalence relation

$$(y.g, g^{-1}.x) \equiv (y, x).$$

The usual notation is $Y \wedge^G X$, *but* this is often inadequate as it assumes X, (resp. Y), stands for the object *and* the G-object, unambiguously, whilst, of course, X really stands for (X, λ) and Y for (Y, ρ). It is sometimes useful, therefore, to add the action into the notation, but only when confusion would occur otherwise, so $Y_\rho \wedge^G{}_\lambda X$ is the full notation, but variants such as $Y_\rho \wedge^G X$ would be used if it was clear what λ was, etc.

We gave an element based description of $Y \wedge^G X$, but how can we adapt this to work within our general $\mathcal{C}$? There are obvious maps

$$Y \times G \times X \underset{(Y,\lambda)}{\overset{(\rho,X)}{\rightrightarrows}} Y \times X\ ,$$

and we can form their coequaliser. (As usual, we assume that the category $\mathcal{C}$ has all limits and colimits that we need to make constructions, and to enable definitions to make sense, but we do not constantly remind the reader of these hidden conditions!) Of course, we met this construction earlier when considering a left principal G-bundle and a right G-space (fibre), F, forming the fibre bundle $X_F = F \wedge^G X$; it was also at the heart of the regular twisted Cartesian product construction from our discussion of simplicial twisting maps.

Example: Suppose $\varphi : G \to H$ is a morphism of group bundles on B, then we can give H a right G-action by

$$H \times_B G \stackrel{H\times\varphi}{\longrightarrow} H \times_B H \to H$$

where the second map is multiplication. If P is a G-object such as a G-torsor, we have a contracted product $H_\varphi \wedge^G P$.

Lemma 47 *If P is a G-torsor, then $H_\varphi \wedge^G P$ is an H-torsor.*

Proof: Writing $Q = H_\varphi \wedge^G P$, we check the usual map,

$$H \times_B Q \to Q \times_B Q,$$

is an isomorphism. This is merely checking that the 'obvious' fibrewise formula is well defined. This sends a pair $([h, p], [h_1, p])$ to $(hh_1^{-1}, [h_1, p])$. That verification is **left to the reader**. (That all elements in $Q \times_B Q$ can be written in this form follows from the fact that **P** is a **G**-torsor, and is again **left to the reader.**)

Local sections of P immediately yield local sections of Q, so Q *is* an H-torsor. ■

A group homomorphism

$$\varphi : G \to H$$

thereby gives us a functor

$$\varphi_* : Tors(G) \to Tors(H) \qquad\qquad \varphi_*(P) = H_\varphi \wedge^G P.$$

Of course, there are still some details (**for you**) to check, namely relating to behaviour on morphisms of G-torsors. (These are probably 'clear', but **do need checking**.)

Another point from this calculation is that we could work with 'elements' as if in a G-set. This can be thought of either as working, carefully, in each fibre of the torsor or using local sections or as a heuristic to obtain a formula that is then encoded purely in terms of the structural maps. All of these viewpoints are valid and all are useful.

Now suppose $\mu, \nu : G \to H$ are two group homomorphisms, thus giving us two functors,

$$\mu_*, \nu_* : Tors(G) \to Tors(H).$$

When is there a natural transformation $\eta : \mu_* \to \nu_*$? The answer is neat and very useful.

Lemma 48 *(cf. Breen, [33], Lemma 1.5)*
A natural transformation $\eta : \mu_* \to \nu_*$ is determined by a choice of a section h of H such that

$$\nu = h^{-1}\mu h.$$

Proof: Suppose that P is a G-torsor, then $\mu_*(P) = H_\mu \wedge^G P$, similarly for $\nu_*(P)$ and $\eta_P : H_\mu \wedge^G P \to H_\nu \wedge^G P$.

If we look locally

$$\eta_P([\mu(g), p]) = h.[\nu(g), p]$$

for some h, since $\eta_P(\mu(g), p)$ is of form $[h_1, p]$ for some h_1 and as $\nu_*(P)$ is an H-torsors, etc.

(Unfortunately we need to know h does not depend on g, and is defined globally, so this suggests looking at the special case where global sections *do* exist, i.e., $P = T_G$, the trivial G-torsor. There we can assume $g = 1_G$, so

$$\eta_{T_G}([1_H, p]) = h.[1_H, p],$$

giving us a possible h. We know that η_P is H-equivariant and natural as well as being 'well-defined'. We use these properties as follows:

If $g \in G$,

$$\begin{aligned} \eta_{T_G}[\mu(g), p] &= \eta_{T_G}[1_H, g.p] \\ &= h[1_H, g.p] \\ &= h[\nu(g), p] \\ &= h.\nu(g)[1_H, p], \end{aligned}$$

whilst also

$$\begin{aligned} \eta_{T_G}[\mu(g), p] &= \eta_{T_G}(\mu(g).[1_H, p]) \\ &= \mu(g)\eta_{T_G}[1_H, p] \\ &= \mu(g)h[1_H, p], \end{aligned}$$

using that η_{T_G} is H-equivariant. We thus have a globally defined h with

$$\mu(g)h = h\nu(g)$$

for all $g \in G$,

$$\text{or} \quad \mu = i_h \circ \nu \quad \text{or} \quad \nu = i'_h \circ \mu,$$

where i_h is inner automorphism by h and i'_h, that by h^{-1}.

Conversely given such an h, we can *define* η by our earlier formula, extending it by H-equivariance and naturality. Checking well definition is quite easy, but instructive, and so is left to you. ■

Recall from section 1.3.4 that for any groupoids G, H, the functor category H^G has groupoid morphisms as its objects and that the natural transformations can be seen to be 'conjugations'. In particular, if $G = H$ is a group, the full subcategory $\mathsf{Aut}(G)$ of G^G given by the automorphisms of G is an internal group object in the category of groupoids, so corresponds to a crossed module. What crossed module? What else, $i : G \to Aut(G)$.

Two automorphisms μ, ν are related by a natural transformation if and only if there is a g such the $\mu = i_g \circ \nu$, where i_g is inner automorphism by g. The similarity with our current setting is *not* coincidental and can be exploited!

Another fairly obvious result is that, if P is a G-torsor, then

$$G \wedge^G P \cong P,$$

since locally we have each representative (g, p) is equivalent to $(1_G, g.p)$. The details are **left as an almost trivial exercise**.

This notation is 'dangerous' however, as we pointed out earlier. We are using the right multiplication of G on itself to give us the contracted product, but we could also make G act on itself by conjugation *on the right*: for $g \in G$, $x \in G$, with G being considered as a bundle,

$$x.g = g^{-1}xg.$$

We will write this action as i', for 'inner', so have $G_{i'} \wedge^G P$ as well. This is, in fact, a very useful object. It is related to automorphisms of P in the following way:

Suppose that $\alpha : P \to P$ is a locally defined automorphism of G-torsors, i.e., a local section of $Aut_G(P)$. Continuing to work locally, pick a section (local element) p. As α is 'fibrewise',

$$\alpha(p) = g_p.p$$

for some local elements g_p of G, and as α is G-equivariant,

$$\alpha(g.p) = g\alpha(p) = gg_p.p.$$

Assigning, to each pair (g, p) in $G \times P$. the automorphism given by

$$\alpha(g_1, p) = g_1 g.p$$

gives a map

$$\lambda : G \times P \to Aut_G(P), \quad \lambda(g, p)(p) = g.p,$$

and this is an epimorphism by our previous argument. 'Obviously'

$$\lambda(g, p) = \lambda(gg', (g')^{-1}p),$$

so the map λ passes to the quotient $G \wedge^G P$ **-or does it**? We have not actually examined the definition of $\lambda(g, p)$ that closely.

Look at this from another direction. Examine $\lambda(g, g'p)$ as an automorphism of P. To work out $\lambda(g, g'p)(p)$, we have first to convert p:

$$\lambda(g, g'p)(p) = \lambda(g, g'p)((g')^{-1}g'.p),$$

as $\lambda(g, g'p)$ is specified by what it does to its basic P-part. Now

$$\lambda(g, g'p)((g')^{-1}g'.p) = (g')^{-1}\lambda(g, g'p)(g'.p)$$

by G-equivariance, and so equals

$$(g')^{-1}gg'.p,$$

which is $\lambda((g')^{-1}gg', p)(p)$.

Thus our initial impulse was hasty. We do have $Aut_G(P)$ as a contracted product, $G \wedge^G P$, but not with right multiplication as the action of G on itself, rather it uses right conjugation. We have proved

Lemma 49 *For any G-torsor P, there is an isomorphism*

$$\lambda : G_{i'} \wedge^G P \overset{\cong}{\to} Aut_G(P),$$

where $i' : G \to Aut(G)^o$, $i'(g)(g') = g^{-1}g'g$, yielding the right conjugation action of G on itself. ■

Perhaps something more needs to be said about $Aut_G(P)$ here. We are working with sheaves or bundles and so have an essentially Cartesian closed situation, in other words function objects exist. For each pair of sheaves, X, Y on B, $Hom(X, Y)$ is a sheaf. In particular $End(X)$ is a sheaf and $Aut(X)$ a subsheaf of it. It thus makes basic sense to have that $Aut_G(P)$ is a G-torsor. Of course, it is also a group object, since automorphisms (gauge transformations) of P are invertible. This group is sometimes written P^{ad}. It is the group (bundle) of G-equivariant fibre preserving automorphisms of P; it is also called the *gauge group* of P. (The precise origin in the thoughts of Hermann Weyl of the use of 'Gauge' are fun to look up, but they make me think that the term is very much over used in mathematical physics, as Weyl's use seems to have been beautifully simple and down to earth, whilst the mystique of the modern use by comparison may be tending to obscure the simple idea from a simple minded mathematician's viewpoint.)

In the isomorphic $G_{i'} \wedge^G P$ version, it is instructive to explore the group structure, but this is left for you to do. This group operates on the *right* of P, by the rule

$$p.\alpha = \alpha^{-1}(p),$$

and makes P into a right P^{ad}-torsor. (Exploration of these statements is well worth while and is **left as an exercise**. It, of course, presupposes that P^{ad} is seen as a bundle /sheaf of groups, which itself needs 'deconstructing' before you start. The overall intuition should be fairly clear *but* the technicalities, detailed verifications, etc., **do need mastering**.)

A cohomological perspective on change of groups. We have that $\check{H}^1(B, G)$ is the set of isomorphism classes of G-torsors on B, i.e., $\pi_0 Tors(G)$, the set of connected components of the groupoid $Tors(G)$. We have now seen that if $\varphi : G \to H$ is a homomorphism of group bundles and P is a G-torsor, then $H_\varphi \wedge^G P = \varphi_*(P)$ is an H-torsor and that this gives a functor $\varphi_* : G \to H$. This will, of course, induce a function on sets of connected components and hence, as one might expect, an induced function

$$\varphi : \check{H}^1(B, G) \to \check{H}^1(B, H).$$

There is another obvious way of inducing such a function, as the elements of $\check{H}^1(B, G)$ are classes of cocycles, (g_{ij}), and so composing with φ sends $[(g_{ij})]$ to $[\varphi(g_{ij})]$. It is standard to check that this does induce a function from $H^1(\mathcal{U}, G)$ to $H^1(\mathcal{U}, H)$ and, by its independence from $\mathcal{U}$, it is then routine to check that it induces a corresponding map on Čech non-Abelian cohomology.

It is easy to see that these two induced maps are the same. (It would be surprising if they were not!) Pick a set of local sections, $\{s_i\}$, for P over a trivialising cover, $\mathcal{U}$, and we get $\{[1, s_i]\}$ is a set of local sections for $H_\varphi \wedge^G P$. Changing patches, $s_i = g_{ij}s_j$, and so

$$[1, s_i] = [1, g_{ij}sj] = [\varphi(g_{ij}).1, s_j] = \varphi(g_{ij})[1, s_j],$$

and the transition functions for $\varphi_*(P)$ are exactly as expected. (The rest of the details are **left as an exercise**.) The important thing for later use is the identification of the cocycles for $\varphi_*(P)$. This will be especially important when discussing G-bitorsors in the next section.

6.4.5 Simplicial Description of Torsors

As usual we look at a sheaf or bundle of groups, G, on a space, B, and suppose P is a G-torsor. We then know there is an open cover, $\mathcal{U}$, of B and trivialising local sections, $s_i : U_i \to P$, over the various different open sets U_i of $\mathcal{U}$. We have seen that over the intersections U_{ij}, the restrictions of the two local sections s_i and s_j must be related and this gives us transition cocycles $g_{ij} : U_{ij} \to G$ such that

$$s_i = g_{ij}s_j,$$

where, over triple intersections, the 1-cocycle condition

$$g_{ij}g_{jk} = g_{ik}$$

must be satisfied.

The information on intersections in $\mathcal{U}$ is neatly organised in the simplicial sheaf, $N(\mathcal{U})$, (cf. page 263 in section 6.3.6). We also know that from a sheaf of groups we can construct various simplicial sheaves. Is there a way of viewing the cocycles g_{ij} from this simplicial perspective?

From a group, G, (no sheaves for the moment), we earlier saw the uses of models for the classifying space, BG, of G. We could use the nerve of G as a group or rather its nerve as a single object groupoid, $G[1]$. We could alternatively take the constant simplicial group, $K(G, 0)$ (so $K(G, 0)_n = G$ for all $n \geq 0$, with all face and degeneracies, being the identity isomorphism of G). If we then formed $\overline{W}(K(G, 0))$, we get $Ner(G[1])$ back.

These different approaches all yield a simplicial set (and if you really want a space, you just take its geometric realisation). This simplicial set will be denoted BG, even though that notation is often restricted to that corresponding space. We have to be a bit careful about the order of composition in the groupoid, $G[1]$, if it is to be consistent with the construction K, which was the nerve of an internal groupoid in the category of groups. We also have to be careful about our use of *left* actions and the assumption that that makes about the order of composition being 'functional' rather than algebraic (which latter order works best with right actions). That being said, we have

- BG_0 = a singleton set, $\{*\}$;
- $BG_1 = G$, as a set, and in general,
- $BG_n = \underbrace{G \times \ldots \times G}_{n}$

Writing $\mathbf{g} = (g_n, \ldots , g_1)$ for an n-simplex of BG, we have

$$\begin{aligned} d_0\mathbf{g} &= (g_n, \ldots , g_2), \\ d_i\mathbf{g} &= (g_n, \ldots , g_{i+1}g_i, \ldots , g_0), \quad 0 < i < n, \\ d_n\mathbf{g} &= (g_{n-1}, \ldots , g_1), \end{aligned}$$

with the degeneracy maps, s_j, given by insertion of 1_G in the j^{th} place, shifting later entries one place to the right. (Warning: multiple use of the label s_j here *may* cause some confusion, but each use is the natural one in that context!)

We have already seen this several times (but repetition *is* useful). The key diagram is usually that indicating a 2-simplex, $\mathbf{g} = (g_2, g_1)$, namely

$$\begin{tikzcd} & * \arrow[dr, "g_2"] & \\ * \arrow[ur, "g_1"] \arrow[rr, "g_2g_1"'] & & * \end{tikzcd}$$

Back to G being a sheaf of groups, and we get BG will be a sheaf of simplicial sets. We now have two simplicial sheaves, $N(\mathcal{U})$ and BG. Curiosity alone should suggest that we compare these via a simplicial morphism and, for our purposes, it should be a simplicial sheaf map, $f : N(\mathcal{U}) \to BG$.

Looking back at $N(\mathcal{U})$ and its construction (page 263), the zero simplices are formed by the open sets and as BG_0 is trivial, f_0 is not much of interest!

At the next level, $f_1 : N(\mathcal{U})_1 \to BG_1$, so consists - yes, of course, - of local sections over the intersections U_{ij}, hence g_{ij} in $G(U_{ij})$ or G_{ij}. Over triple intersections U_{ijk}, f_2 will give a 2-simplex, as above, so $g_{ij}g_{jk} = g_{ik}$, given by $f_2 : U_{ijk} \to G \times G$, $f_2 = (g_{jk}, g_{ij})$.

We thus have our 1-cocycle condition is automatic from the simplicial structure.

What about change of the choice of local sections of P, i.e., $s_i : U_i \to P$. If we change these, we get elements $g_i \in G_i$ such that $s' = g_i s_i$ and the new g'_{ij} are related to the old by a sort of conjugacy rule:

$$g'_{ij} = g_i g_{ij} g_j^{-1},$$

which can be visualised as a square

$$\begin{tikzcd} \cdot \arrow[r, "g_{ij}"] \arrow[d, "g_j"'] & \cdot \arrow[d, "g_i"] \\ \cdot \arrow[r, "g'_{ij}"'] & \cdot \end{tikzcd}$$

This is reminiscent of a homotopy, and, in fact, defines one from our f (relative to the $\{s_i\}$) to f' (relative to the $\{s'_i\}$). In other words, we are identifying isomorphism classes of G-torsors that trivialise over $\mathcal{U}$ with homotopy classes, i.e., elements of $[N(\mathcal{U}), BG]$. We will return to this later when we discuss passing to refinements of $\mathcal{U}$ to get a homotopy description of all G-torsors, so we will not give the details here.

Several questions should come to mind at this stage. Given our recent description of 'change of groups', an obvious thing to do is to view that from a simplicial perspective. Suppose $\varphi : G \to H$ is a homomorphism of sheaves of groups. It is easy to see that φ induces a map of simplicial sheaves, $B\varphi : BG \to BH$, so we get, for given $\mathcal{U}$, an induced map

$$[N(\mathcal{U}), B\varphi] : [N(\mathcal{U}), BG] \to [N(\mathcal{U}), BH].$$

If we start off with a G-torsor, P, and use our change of groups methods above, what is the link between $\varphi_*(P)$ and the image of the isomorphisms class of P as represented by some map from $N(\mathcal{U})$ to BG. Of course, we have just seen that if $\{g_{ij}\}$ represents P then $\{\varphi(g_{ij})\}$ represents $\varphi_*(P)$ - but this is exactly the image under $[N(\mathcal{U}), B\varphi]$. There is thus yet another good way of interpreting the change of groups functor from $Tors(G)$ to $Tors(H)$, namely as a simplicial induced map from BG to BH. (Later we will see that $Tors(G)$ is the stack completion of BG or equivalently of $G[1]$ and this yields a variant of this simplicial viewpoint.)

Picking up an earlier problem, what about change of base. If we have the above simplicial description of isomorphism classes of those G-torsors on a base B that trivialise over some open cover $\mathcal{U}$, in terms of homotopy classes of maps from $N(\mathcal{U})$ to BG, and then we change the base along a continuous map, how does this look from a simplicial viewpoint?

To start with we rename some objects to get things into line with our earlier discussion. We will consider two spaces B and B' and a continuous map $f : B \to B'$. We have a sheaf or bundle of groups G on B' and hence an induced pullback sheaf $f^*(G)$ on B. We assume given some open cover $\mathcal{U}$ of B', and hence an open cover $f^{-1}(\mathcal{U})$ of B, and will be interested in those $f^*(G)$-torsors that trivialise over $f^{-1}(\mathcal{U})$ and which are induced from G-torsors that trivialise over $\mathcal{U}$.

6.4.6 Torsors and exact sequences

One classical method of analysing the cohomology, and, in so doing, of providing interpretations of cohomology classes, is to vary the coefficients within an exact sequence. For instance, if

$$1 \to L \overset{u}{\to} M \overset{v}{\to} N \to 1$$

is an exact sequence of sheaves of groups, then one might try to relate torsors over L, M and N. The usual techniques would then be to see what is the likelihood of having something like a long exact sequence of the cohomology 'sets' or groups. Where should it start?

We will, to start with, look at the Abelian case, but will try not to use commutativity so as to get as general a result as possible. Sheaf cohomology with coefficients in sheaves of Abelian groups, etc., is considered as measuring the non-exactness of the global sections functor. Given a sheaf, L, of Abelian groups on B, $\Gamma_B(L)$ is one of several notations used for the Abelian group of global sections of L. Another is $L(B)$, of course. If the exact sequence above had been of Abelian sheaves, we would have had a long exact sequence

$$0 \to L(B) \to M(B) \to N(B) \to \check{H}^1(B, L) \to \check{H}^1(B, M) \to \check{H}^1(B, N) \to \check{H}^2(B, L) \to \dots,$$

and so on. It is to be noted that the induced map, $v_* : M(B) \to N(B)$, need not be onto, so $\check{H}^1(B, L)$ picks up the obstruction to 'lifting' a global section of N to one of M. This is particularly interesting to us here since we have linked $\check{H}^1(B, L)$ with L-torsors in the general situation - and, of course, that interpretation is also valid in the Abelian case.

To see how $\check{H}^1(B, L)$ arises naturally in this situation, suppose given a global section h of N. As our exact sequence above was of sheaves, we have to examine what that means. This can be viewed from several angles. An exact sequence of sheaves may not be exact as a sequence of presheaves. The functor that forgets that sheaves are sheaves has a left adjoint namely 'sheafification', so will itself be 'left exact', e.g., will preserve monomorphisms. (If you do not know of this type of result, try to prove it yourself.) It need not preserve epimorphisms. Sheafification itself will preserve epimorphisms, but not all epimorphisms need be the sheafification of an epimorphism at the presheaf level. An epimorphism of sheaves will give an epimorphism on stalks. (We are thinking here of sheaves on a space, B, rather than more general topos centred results.) This means epimorphisms are locally defined. Suppose we have a point $b \in B$, then if x is in the stalk of N above b, it means that x is representable as a pair (x_U, U), where $b \in U$, U is an open set and $x_U \in N(U)$, the group of local sections of N over U. (Recall, from page 259, section 6.3.3, that the stalk of a sheaf N at a point b is a colimit of the $N(U)$ for $b \in U$.) The morphism v being an epimorphism, there is an element y in the stalk of M at b, say $y = [(y_V, V)]$, such that over some open set $W \subseteq U \cap V$, $v(y_W) = x_W$.

Now start, not with an element in a stalk, but rather with a global section x of N. This does give an element in each stalk and we can find an open cover $\mathcal{U}$ such that over each U_i in $\mathcal{U}$, we can find a local section, y_i, mapping down to the restriction, x_i, of x to U_i, (but remember that different global sections will most likely need different covers, etc.). There is no reason these y_i should be compatible on intersections U_{ij}, so there will be (unique) elements, $\ell_{ij} \in L_{ij} = L(U_{ij})$, such that

$$y_i = u(\ell_{ij})y_j,$$

since both y_i and y_j map to x_{ij} over U_{ij}. As u is a monomorphism, these ℓ_{ij} will satisfy the cocycle condition,

$$\ell_{ij}\ell_{jk} = \ell_{ik}$$

and, as you no doubt now expect, if we change the local sections y_i within the L_i-coset of possible choices, then $y_i' = u(\ell_i)y_i$ and the ℓ_i define a coboundary.

In other words, there is an L-torsor, $P(x)$, which is constructed from the global section x of N, and which is trivial exactly when the y_i can be chosen compatibly, i.e., when there is a global section y mapping down to x. We can thus think of $P(x)$ as being the obstruction to lifting x to a global section of M. (Of course, the choices made have to be checked not to matter, up to isomorphism of $P(x)$ - but that can be safely '**left to the reader**'.)

There is thus an extension of the earlier sequence to

$$0 \to L(B) \to M(B) \to N(B) \to \pi_0(Tors(L)),$$

where the last term corresponds to $\check{H}^1(B, L)$. (The notation π_0 is, you may recall, to designate the set of connected components of a groupoid, simplicial set or space and $Tors(L)$ is a groupoid as we have seen.)

The next two terms in the long exact sequence, $\check{H}^1(B, M)$ and $\check{H}^1(B, N)$, are easy to handle geometrically. They give $\pi_0(Tors(M))$ and $\pi_0(Tors(N))$ respectively, and, of course, the induced maps are those given by the 'change of groups' along u and v. Exactness of the result is then routine to check, but

$$v_* : \pi_0(Tors(M)) \to \pi_0(Tors(N))$$

will not, in general, be onto. (You would not expect it to be as the standard homological machinery gives a $\check{H}^2(B, L)$ term.) Of course, none of the above depended on the sheaves involved being Abelian, but if they are not, $\check{H}^1(B, L)$ is not an Abelian group, it is just a pointed set. It is still given by $\pi_0(Tors(L))$, and $Tors(L)$ *is* always a groupoid, so there is a second layer that is hidden by the homological approach namely the automorphisms of the different objects in this groupoid.

6.5 Bitorsors

The fact that the left G-torsor is also a right P^{ad}-torsor suggests the notion of a bitorsor, the analogue of a left R-, right S-module for our non-Abelian setting. (Our basic reference for this will be Breen's Grothendieck Festschrift paper, [31] and his beautiful 'Notes on 1- and 2-gerbes', [34], based on his Minneapolis lectures.)

6.5.1 Bitorsors: definition and elementary properties

Definition: Let G, H be two bundles of groups on B or more generally two group objects in a topos, $\mathcal{E}$. A (G, H)-*bitorsor* on B is a space P over B together with fibre preserving left and right actions of G and H, respectively, on P, which commute with each other,

$$(g.p).h = g.(p.h),$$

and which define both a left G-torsor and a right H-torsor structure on P. If $G = H$, we say *G-bitorsor* rather than (G, G)-bitorsor.

There is an obvious extension of the notion to that of a (G, H)-bitorsor in a topos. We leave the exact formulation to you.

A family of local sections s_i of a (G, H)-bitorsor defines a local identification of P as *the* trivial left G-torsor and *the* trivial right H-torsor. It therefore determines a family of local isomorphisms $u_i : H_{U_i} \to G_{U_i}$, given by the rule $s_i h = u_i(h)s_i$, for $h \in H_{U_i}$. It is important to note that this does not mean that G and H are *globally* isomorphic.

Examples: a) The trivial (left) G-torsor T_G is also a right G-torsor (using right multiplication) and has a G-bitorsor structure.

b) Any left G-torsor, P, is a (G, P^{ad})-bitorsor, as above. Any G-torsor, P, is a (G, H)-bitorsor if and only if $H \cong P^{ad}$.

c) Let

$$1 \to G \xrightarrow{i} H \xrightarrow{j} K \to 1$$

be an exact sequence of bundles of groups on B. Form $G_K = G \times_B K$, which is again a bundle of groups, then H is a G_K-bitorsor over K. This needs a bit of working through. For a start, K is a bundle of groups so has a (hidden) structural projection, $K \to B$. Thinking of this as a cover as we have done previously, then G_K is the induced bundle of groups on K (as a space), so we have transferred attention from Top/B to Top/K or from $Sh(B)$ to $Sh(K)$. There are actions of G_K on H,

$$h \star (g, k) = hi(g),$$

(but note that requires us to use $H \xrightarrow{j} K$, as the structural projection of H over K, again, going to bundles on K,

$$(g, k).h = i(g).h,$$

but is only defined if $j(h) = k$, as we are 'over K,' in this equation).

This is somewhat simplified if we have $B = 1$, when it is simply an exact sequence of groups, G_K is $G \times K$ as a group over K, via projection, and so on.

There is an obvious notion of morphism of bitorsors and thus various categories, $Bitors(G, H)$, $Bitors(G) := Bitors(G, G)$, ... It should come as no surprise that if P is a (G, H)-bitorsor and Q is a (H, K)-bitorsor, both on B, then $P \wedge^H Q$ is a (G, K)-bitorsor. Moreover, P gives a (H, G)-bitorsor, P^o, (o for 'opposite') by reversing the two actions. (**For you to check out.**) We thus have that a (G, H)-bitorsor will induce a functor

$$Tors(H) \to Tors(G)$$

and that, for a given bundle of groups G, the category of G-bitorsors has a monoidal structure given by $P \wedge^G Q$ and with T_G as unit object. The opposite construction acts like an inverse,

$$P \wedge^G P^o \cong T_G \cong P^o \wedge^G P,$$

but note that these are isomorphisms *not* equality.

Lemma 50 *The category $Bitors(G)$ with contracted product is a group-like monoidal category, with the bitorsor T_G as unit and P^o, an inverse for P.*

Proof: This is **left as an exercise**, but here is a suggestion for the above isomorphisms: use local sections to send any $[p, p']$ in $P^o \wedge^G P$ to an element of G, now show independence of that element on the choice of local section. It is also necessary to check through the group-like monoidal category axioms, which are left for you to find in detail. ■

A group-like monoidal category is often called a *gr-category*. We have already (essentially introduced on page 25) seen that strict gr-categories are 'the same as' crossed modules, so once again that crossed structure is lurking around just beneath the surface. It is interesting and useful (i.e., an **exercise left to the reader!**) to examine the above structure when G is a sheaf of *Abelian* groups, for instance to show that the monoidal structure is symmetric.

A very useful result, akin to Lemma 49 above, gives a similar interpretation of $Isom_G(P, Q)$, where P is a (G, H)-bitorsor and Q a left G-torsor. As P is thus also a left G-torsor and $Tors(G)$ is a groupoid, $Isom_G(P, Q)$ is just the sheaf of G-equivariant torsor maps from P to Q, all of which are invertible. The following lemma identifies this as a contracted product.

Lemma 51 *Let P be a (G, H)-bitorsor and Q a left G-torsor, then there is an isomorphism*

$$Isom_G(P, Q) \stackrel{\cong}{\to} P^o \wedge^G Q.$$

Proof: We start by noting a morphism in the other direction. Suppose we take a local element in $P^o \wedge^G Q$ given by $(p, q) \in P^o \times Q$, defined over an open set U. We have

$$(p, q) \equiv (p.g^{-1}, g.q),$$

but as $p \in P^o$, $p.g^{-1} = q.p$ with the original left G-action on P. We assign to (p, q) the isomorphism, $\alpha_{(p,q)}$, from P to Q defined over U, which sends p to q. Of course, $\alpha_{(p,q)}$ is to be extended to a G-equivariant map, $\alpha_{(p,q)}(g.p) = g.q$, but we effectively knew that fact already since

$$\alpha_{(p,q)} = \alpha_{(p.g^{-1},g.q)},$$

so it sends $p.g^{-1}$ to $g.q$. Of course, if $\beta : P_U \to Q_U$ is a local morphism defined over some U, then we can assume P_U has a local section p and that $\beta(p) = q$ for some local section q of Q. (If not, refine U by an open cover on which P trivialises and work on the open sets of that finer open cover.) However then we can assign $[p, q]$ in $P^o \wedge^G Q$ to the morphism β. The rest of the details should now be easy to check. ■

6.5.2 Bitorsor form of Morita theory (First version):

Within the theory of modules and more generally of Abelian categories, there is a very important set of results known as Morita theory, describing equivalences between categories of modules. The idea is that if R and S are rings, then we can use a homomorphism as above to induce a right R, left S module structure on S itself and this is what induces, via tensor product, a functor from $Mod(S)$ to $Mod(R)$. We have seen the corresponding idea with torsors above. Not all functors between $Mod(R)$ and $Mod(S)$ are induced by morphisms at the ring level in this way however, but provided we look at equivalences between categories, this bimodule idea allows us to describe the equivalences precisely - and this does go across to the torsor context.

The first essential is to recall the definition of an equivalence of categories.

Definition: A functor $F : \mathcal{C} \to \mathcal{D}$ between two categories is an *equivalence* if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and two natural isomorphisms, $\eta : GF \Rightarrow Id_{\mathcal{C}}$ and $\eta' : FG \Rightarrow Id_{\mathcal{D}}$. We say G is *(quasi-)inverse* to F.

Proposition 74 *A (G, H)-bitorsor Q on B induces an equivalence*

$$Tors(H) \stackrel{\Phi_Q}{\to} Tors(G)$$

$$M \longmapsto Q \wedge^H M$$

between the corresponding categories of left torsors on B. In addition if P is a (H, K)-bitorsor on B, then there is a natural isomorphism of functors

$$\Phi_{Q\wedge^H P} \cong \Phi_Q \circ \Phi_P,$$

and, in particular, the equivalence Φ_{Q^o} is quasi-inverse to Φ_Q.

Proof: The last part follows from the statement on composites, which should be clear by construction and, of course, $T_H \wedge^H Q \cong Q$, as we saw earlier. This proof is thus just a compilation of earlier ideas - and so will be **left to the reader**! ■

In fact it is now easy to give a weak version of the torsor Morita theorem.

Proposition 75 *If*

$$\Phi : Tors(H) \to Tors(G)$$

is an equivalence of categories, then there is a (G, H)-bitorsor, Q, which itself induces such an equivalence.

Proof: We will limit ourselves to pointing out that we can take $Q = \Phi(T_H)$. This inherits its right H-action from the right action of H on T_H. (You should **check** that it is a right H-torsor for this action.) ■

It is, in fact, the case that Φ is equivalent to the equivalence induced by Q, but this is more relevant in a later context, so will be revisited then.

6.5.3 Twisted objects:

Continuing our study of torsors and bitorsors, as such, we should mention the analogue of fibre bundles in this context.

Let P be a left G-torsor on B and E a space over B on which G acts on the right. We can again use the contracted product construction to form $E^P := E \wedge^G P$ over B. In this context we call E^P the *P-twisted form* of E.

Choice of a local section s of P over an open set U determines an isomorphism $\varphi_P : E^P_{|U} \cong E_U$, so E^P is locally isomorphic to E. (Beware, especially if you are used to the case where E is a product space over B, so $E = F \times B$, say. In that case E^P is locally trivial in a very strong sense, but this need not be so in general).

Suppose E_1 is now a space over B and there is an open cover $\mathcal{U}$ of B over which E_1 is locally isomorphic to E, then the sheaf or bundle $Isom_B(E_1, E)$ is a left torsor on B for the action of the bundle of groups, $G := Aut_B(E)$. This gives us a G-torsor and a space, E, on which G acts on the right.

These two constructions are inverse to each other.

In particular, if we are given G and have a second bundle of groups, H, on B, which is locally isomorphic to G, then $P := Isom_B(H, G)$ is a $Aut_B(G)$-torsor. It is worth pausing to think out the components of this fact. The object $Isom_B(H, G)$ exists, as before, because of the Cartesian closed assumption about our categories of bundles over B, (e.g. if we are interpreting bundles as sheaves, $Isom_B(H, G)$ is a subsheaf of the function sheaf, $Sh(B)(H, G)$, but although it would always have an action of $Aut_B(G)$, we need the 'H is locally isomorphic to G' condition to ensure the existence of local sections and hence to ensure it is a $Aut_B(G)$-torsor).

Look now at $G \wedge^{Aut(G)} P$ and the map

$$G \wedge^{Aut(G)} P \to H$$

$$(g, u) \mapsto u^{-1}(g).$$

(We make $Aut_B(G)$ act on the right of G, via the obvious left action.) This map is an isomorphism and so H is the P-twisted form of G for this right $Aut_B(G)$-action.

On the other hand, if G is a bundle of groups on B and P is a left G-torsor, $H := G \wedge^{Aut(G)} P$ is a bundle of groups on B locally isomorphic to G and this identifies P with the left $Aut_B(G)$-torsor, $Isom_B(H, G)$.

This provides a torsor's-eye-view of our examples on fibre bundles given in section 6.1.3, (Case study, page 240). We will sketch in a few more details:

A vector bundle, V, of rank n on B is locally isomorphic to $\mathbb{R}^n_B := \mathbb{R}^n \times B$. The group of automorphisms of this is the trivial bundle of groups, $G\ell(n, \mathbb{R})_B := Gl(n, \mathbb{R}) \times B$. The left $G\ell(n, \mathbb{R})_B$-torsor on B associated to V is $Isom(V, \mathbb{R}^n_B)$ and this is just the *frame bundle*, P_V, of V. The vector bundle V is a bundle of groups, so the above discussion applies, showing it to be the P_V-twist of $\mathbb{R}^n_B$. Conversely for any $G\ell(n, \mathbb{R})_B$-torsor P on B, the twisted object $V = \mathbb{R}^n_B \wedge^{G\ell(n,\mathbb{R})_B} P$ is the rank n vector bundle associated to P and its frame bundle P_V is canonically isomorphic to P. (If you have not explored vector bundles and differential manifolds, a brief excursion into that area may be well worthwhile, as it reinforces the geometric origins and intuitions behind this area of cohomology.)

6.5.4 Cohomology and Bitorsors

Earlier, (page 271), we saw how local sections, s, of a torsor, P, over an open cover, $\mathcal{U}$, led to 'transition maps', or 'cocycles', $g_{ij} : U_{ij} \to G$, on the intersections. Changing local sections to $s'_i : U_i \to P$, $s'_i = g_i s_i$, we have that the corresponding cocycles g'_{ij} are related via the coboundary relation

$$g'_{ij} = g_i g_{ij} g_j^{-1},$$

to the earlier ones. This led to the set of equivalence classes, $H^1(\mathcal{U}, G)$, and eventually to the cohomology set $\check{H}^1(B, G)$, which classified isomorphism classes of G-torsors on B.

What would be the additional structure available if P was a (G,H)-bitorsor? The family of local sections $s_i : U_i \to P$ then would also determine a family of local isomorphisms $u_i : H_{U_i} \to G_{U_i}$, where

$$u_i(h)s_i = s_i.h.$$

Remark: This formula needs a bit of thought. That u_i *is* a bijection is clear, as it follows from the fact that P is a G-torsor, but that it *is* a homomorphism needs a bit more care. The defining equation is specifically using the local section s_i so, for instance, on a more general element $g.s_i$ we have to extend the formula using G-equivariance, (remember the two actions are independent), so $(g.s_i).h = g.u_i(h).s_i$. In particular, if h_1 and h_2 are two local section of H over U_i, then $s_i.(h_1h_2) = u_i(h_1).s_i.h_2 = u_i(h_1)u_i(h_2).s_i$, so $u_i(h_1h_2)$ does equal $u_i(h_1)u_i(h_2)$.

Over an intersection U_{ij} of the cover, $s_i = g_{ij}s_j$, so

$$u_i = i_{g_{ij}}u_j$$

with as usual, i the inner automorphism homomorphism from G to $Aut_B(G)$, sending g to i_g. The (u_i, g_{ij}) therefore satisfy the *cocycle conditions*

$$g_{ik} = g_{ij}g_{jk}$$

and

$$u_i = i_{g_{ij}}u_j.$$

Changing the local sections to $s'_i = g_is_i$ in the usual way determines coboundary relations

$$g'_{ij} = g_ig_{ij}g_j^{-1}$$

and

$$u'_i = i_{g_i}u_i.$$

Isomorphism classes of (G,H)-bitorsors on B with given local trivialisation over $\mathcal{U}$, thus are classified by the set of equivalence classes of such cocycle pairs (g_{ij}, u_i) modulo coboundaries. In the most important case of G-bitorsors, the u_i are locally defined automorphisms of the G_{U_i} and so are local sections of $Aut(G)$.

We thus have from a G-bitorsor, P, a fairly simple way to get a piece of descent data, $\{(g_{ij}, u_i)\}$, with the right sort of credentials to hope for a 'reconstruction' process. We needed P to trivialise over the open cover $\mathcal{U} = \{U_i\}$ and then to chose local sections, $s_i : U_i \to P$. This gave $\{g_{ij} : U_{ij} \to G\}$ and $\{u_i : U_i \to Aut(G)\}$, so let us start off with these and see how much of P's structure we can retrieve.

Putting aside the u_is for the moment, we have a G-valued cocycle, $\{g_{ij}\}$, and we already have seen how to build a G-torsor from that information. Recall we take

$$P = \bigsqcup_i G(U_i)/\sim,$$

where $(g, i) \sim (gg_{ij}, j)$. (The basic relation is really that $(1_{U_i}, i) \sim (g_{ij}, j)$ with the left translation $G(U_{ij})$-action giving the more general form.) We thus have a lot of the structure already available. We are left to obtain a right G-action, which has to be 'independent' of the left action, i.e., to commute with it as in the first definition of this section. (To avoid confusion between the two actions, we will pass to the (G,H)-bitorsor case so $u_i : U_i \to Isom(H,G)$, and will denote local elements that act on the right by h_i, whilst any acting on the left by g_i.)

In our 'reconstructed' P, there is clearly a natural choice for a local section over U_i, namely the equivalence class of the identity element $1_{U_i} \in G(U_i)$, or, more exactly of $(1_{U_i}, i)$, then we could define

$$[g,i].h := [g.u_i(h), i].$$

It is clear that this is a right action, since u_i is a homomorphism, and that it does not interfere with the left $G(U_i)$-action, which is $g'[g,i] = [g'g,i]$. Of course, we have to check compatibility with the equivalence relation, and that is exactly what is needed for checking that it works on adjacent patches / open sets of the cover. The key case is to work with a local section h of G over an open set, U, and examine what h does on patches U_i, U_j and their intersection. (Of course, this presupposes that we are intersecting U_i, etc., with U, i.e., that we are effectively working with an open cover of U itself.)

We know how the U_i are related over the different patches, namely

$$u_i = i_{g_{ij}} u_j,$$

which on our local element, h, gives

$$u_i(h) = g_{ij} u_j(h) g_{ij}^{-1}.$$

As h is defined on U, the restrictions to the various U_i form a compatible family, (i.e., we do not need to worry about transitions for h in formulae), so

$$[g,i].h = [gu_i(h), i] = [g.u_i(h)g_{ij}, j],$$

on the one hand, and also

$$[g.g_{ij}, j].h = [gg_{ij}u_j(h), j].$$

The earlier identity shows that

$$u_i(h)g_{ij} = g_{ij}u_j(h),$$

so these are the same local element of P over U_{ij}.

The u_i were introduced as the way to link local right and left actions,

$$u_i(h).s_i = s_i.h.$$

They also have an interpretation if we seek to study when a given left G-torsor, P, has an additional G-bitorsor, or more generally, a (G,H)-bitorsor structure. The cocycle rules linking the u_i with the g_{ij} involve the group homomorphism $i : G \to Aut(G)$. The g_{ij} part of the cocycle family only uses the left G-torsor structure on P. It is perhaps only because of 'natural curiosity', but it does seem natural to look at the $Aut(G)$-torsor, $i_*(P)$. Our earlier calculations show that suitable cocycles for this are given by $\{i(g_{ij})\} = \{i_{g_{ij}}\}$, but the u_i now look very like a coboundary! In fact that key equation, $u_i = i_{g_{ij}} u_j$, can obviously be rewritten as

$$i_{g_{ij}} = u_i u_j^{-1},$$

or

$$i_{g_{ij}} = u_i.1.u_j^{-1},$$

so the class of $\{i_{g_{ij}}\}$ is 'cohomologically null', i.e., equivalent to 1 modulo coboundaries. In other words, $i_*(P) \cong T_{Aut(G)}$.

Conversely, if we have P and hence its cocycle representation, *and* a 0-cocycle trivialising $i_*(P)$, so $\{i_{g_{ij}}\}$ is a coboundary,

$$\{i_{g_{ij}}\} = \alpha_i \alpha_j^{-1},$$

then taking $u_i = \alpha_i$, we have a cocycle pair, (g_{ij}, u_i), giving P a G-bitorsor structure.

We clearly should look at this from the viewpoint of contracted products as they have a clearer *geometric* interpretation. The $Aut(G)$-torsor, $i_*(P)$, has a description as $Aut(G)_i \wedge^G P$, thus, by quotienting $Aut(G) \times P$ by the equivalence relation

$$(\alpha.g.p) \sim (\alpha \circ i(g), p).$$

The fact that $i_*(P)$ is locally trivial was given by the local sections induced by those $s_i : U_i \to P$ for P, namely

$$[(1, s_i)] : U_i \to Aut(G)_i \wedge^G P.$$

(Note this formulation is slightly different from that in Breen, [31], as he uses the opposite group $Aut^o(G)$ and i', but we can avoid that extra complication for our purposes here, since we really only need $\alpha = 1$ in the above.)

We can compare these local sections on overlaps U_{ij},

$$(1, s_i) \sim (1, g_{ij}s_j) \sim (i_{g_{ij}}, s_j) \sim (u_i u_j^{-1}),$$

but now our local sections $[(1, s_i)]$ are equivalent to others $t_i = [(u_i^{-1}, s_i)]$, which *agree* on overlaps

$$t_i = [(u_i^{-1}, s_i)] = [(u_i^{-1} u_i u_j^{-1}, s_i)] = t_j$$

over U_{ij}. These t_i thus form a global section for $i_*(P)$, which is hence the trivial torsor, up to isomorphism.

Reversing the argument, a global section of $i_*(P)$, together with the structural cocycle $\{g_{ij}\}$ for P gives a G-bitorsor structure on P. (We will return to this in more generality a bit later.)

We thus have that a G-bitorsor is a *relative* $\mathsf{Aut}(G)$*-torsor*, where $\mathsf{Aut}(G) = (G, Aut(G), \iota)$. It corresponds to a G-torsor, P, together with a trivialisation of $\iota_*(P)$. Using the fact that morphisms from the induced torsor $\iota_*(P)$ to $T_{Aut(G)}$ corresponds to morphisms over ι from P to $T_{Aut(G)}$, we get a second description, which is very useful for further generalisation.

6.5.5 Bitorsors, a simplicial view.

Pausing in our development, let us return to the simplicial viewpoint that we adopted earlier. The cover $\mathcal{U}$ gives a sheaf / bundle,

$$p : E = \sqcup \mathcal{U} \to B$$

and by repeated pullbacks, we get a simplicial sheaf / bundle,

$$N(\mathcal{U}) \to B.$$

The cocycle $\{(u_i, g_{ij})\}$ consists of a family $\{u_i\}$ giving a morphism,

$$\mathbf{g}_0 : N(\mathcal{U})_0 = \sqcup\mathcal{U} \to Aut(G),$$

together with a second family

$$\mathbf{g}_1 : N(\mathcal{U})_1 \to G \rtimes Aut(G).$$

This second piece of data is not quite as obvious as it might seem. The earlier model of the crossed view of group extensions used the crossed module, $\mathsf{Aut}(G) = (G, Aut(G), \iota)$ directly. Here we are using the cat^1-group / gr-groupoid / 2-group analogue, which can also be thought of simplicially as in our discussion of algebraic 2-types, page 63. Recall the face maps

$$d_i : G \rtimes Aut(G) \to Aut(G), \quad i = 0, 1,$$

are given by

$$\begin{aligned} d_1(g, \alpha) &= \alpha, \\ d_0(g, \alpha) &= i_g \circ \alpha \end{aligned}$$

and the degeneracy is

$$s_0(\alpha) = (1_G, \alpha).$$

The maps $\mathbf{g}_0$, $\mathbf{g}_1$ are to be hoped to be a part of a simplicial map from the simplicial sheaf $N(\mathcal{U})$ to the sheaf of simplicial groups, $K(\mathsf{Aut}(G))$, and to check that this is indeed the case, we need to recall that 'bundle-wise' the elements of $\sqcup\mathcal{U} = N(\mathcal{U})_0$ can usefully be thought of as pairs (x, U), where $U \in \mathcal{U}$ and $x \in U$. Of course, the projection maps p sends (x, U) to x itself. The 1-simplices of $N(\mathcal{U})$ therefore are given by triples (x, U_0, U_1) with $x \in U_0 \cap U_1$, so the corresponding face and degeneracy maps are

$$\begin{aligned} d_1(x, U_0, U_1) &= (x, U_0), \\ d_0(x, U_0, U_1) &= (x, U_1), \\ s_0(x, U) &= (x, U, U). \end{aligned}$$

We can thus see what this $\mathbf{g}$ must satisfy. We write $\mathbf{g}_1 = (g, \alpha)$ as before, and will try to identify what g and α must be. We have, then,

- $d_1\mathbf{g}_1 = \mathbf{g}_0 d_1$ means $\alpha = u_{|U_0} =: u_0$;
- $d_0\mathbf{g}_1 = \mathbf{g}_0 d_0$ means $i_g u_0 = u_{|U_1} =: u_1$;
- $s_o\mathbf{g}_0 = \mathbf{g}_1 s_0$ is a normalisation condition, which will make more sense when the first two conditions have been explored in more detail.

The obvious way to build $\mathbf{g}_1$, i.e., g itself, is thus to take

$$\mathbf{g}(x, U_0, U_1) = (g_{10}(x), u_0(x)),$$

and to require that g_{ii} is 1_G restricted to $U_{ii} = U_i \cap U_i$ for the normalisation.

To continue our simplicial description, we should look at triple intersections, i.e., $N(\mathcal{U})_2$, and the corresponding $K(\mathsf{Aut}(G))_2$. The points of $N(\mathcal{U})_2$ are, of course, represented by symbols such as (x, U_0, U_1, U_2), whilst those of $K(\mathsf{Aut}(G))_2$ above the point x, are of form $(g_2, g_1, \alpha)(x)$. The face maps of $N(\mathcal{U})$ are the obvious ones, $d_2(x, U_0, U_1, U_2) = (x, U_0, U_1)$, and so on, whilst

$$\begin{array}{rcl} d_2(g_2, g_1, \alpha) & = & (g_1, \alpha), \\ d_1(g_2, g_1, \alpha) & = & (g_2 g_1, \alpha), \\ d_0(g_2, g_1, \alpha) & = & (g_2, i_{g_1}\alpha), \end{array}$$

with the s_i inserting an identity in the appropriate place. (Of course, all these g_i, etc., are 'local elements', so are really local sections, and our formulae would have, over a given x, the values $g_2(x)$, etc., as above.)

We want $\mathbf{g}$ to be a simplicial morphism, so on 2-simplices we expect, for (x, U_0, U_1, U_2),

$$d_2\mathbf{g}_2 = \mathbf{g}_1 d_2,$$

etc., i.e., if $\mathbf{g}_2(x, U_0, U_1, U_2) = (g_2, g_1, \alpha)(x)$, the d_2-face $(g_1, \alpha)(x) = (g_{10}(x), u_0(x))$, so $g_1, = g_{10}$, $\alpha = u_0$, and then the d_0 face gives $g_2 = g_{21}$. Finally the d_1-face gives

$$g_2 g_1 = g_{20},$$

so this gives us the cocycle condition

$$g_{21} g_{10} = g_{20}$$

over U_{012}.

The other simplicial morphism rules give compatibility with degeneracies, but using simplicial identities, these then give that $g_{01} = g_{10}^{-1}$, i.e., again a normalisation condition.

We thus have

(i) the bundle of crossed modules $\mathsf{Aut}(G)$ given by $(G, Aut(G), \iota)$;

(ii) the corresponding bundle of simplicial groups, $K(\mathsf{Aut}(G))$;

(iii) the bundle / sheaf of simplicial sets, $N(\mathcal{U})$;
and

(iv) our local cocycle description of our bitorsor, P,

giving, it would seem, a simplicial map

$$\mathbf{g} : N(\mathcal{U}) \to K(\mathsf{Aut}(G)).$$

Conversely such a simplicial map gives a cocycle (for **you to check**).

(Here we are abusing notation slightly, since the domain of $\mathbf{g}$ is a bundle of simplicial sets, whilst the right hand side is the underlying simplicial set bundle of the simplicial group bundle, not that simplicial group bundle itself, however we have not shown that in the notation. It is, however, an important point to note.)

Continuing with this quite detailed look at the 'cocycles for bitorsors' context, we clearly have next to look at the 'change of local sections' from this simplicial viewpoint.

Suppose we change to local sections, $s'_i = g_i s_i$, so, as before, get

$$g'_{ij} = g_i g_{ij} g_j^{-1}$$

and

$$u'_i = i_{g_i} u_i.$$

If we are describing cocycles as simplicial maps, then fairly naturally, we might hope that the equivalence relation coming from coboundaries, as here, was something like homotopy of simplicial maps. We can see immediately that this looks to be not that stupid an idea, by looking at the base of the corresponding simplicial objects.

$$\begin{tikzcd}
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & G^{(2)} \rtimes Aut(G) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G \rtimes Aut(G) \arrow[r, shift left] \arrow[r, shift right] & Aut(G) \\
\cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & N(\mathcal{U})_2 \arrow[u, shift left, "g_2"] \arrow[u, shift right, "g'_2"'] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & N(\mathcal{U})_1 \arrow[u, shift left, "g_1"] \arrow[u, shift right, "g'_1"'] \arrow[r, shift left] \arrow[r, shift right] & N(\mathcal{U})_0 \arrow[u, shift left, "g_0"] \arrow[u, shift right, "g'_0"']
\end{tikzcd}$$

then we would expect that a homotopy between $\mathbf{g}$ and $\mathbf{g}'$ would pick out, for each (x, U_0) in $N(\mathcal{U})_0$, an element $(g, \alpha) \in G \rtimes Aut(G)$ with $g = d_1(g, \alpha) = g_0$, $d_0(g, \alpha) = g'_0$, i.e., $\alpha = u_0$ and $g'_0 = u'_0 = i_{g_0} \circ u_0$, exactly as needed. To see if this works in higher dimensions, we need to glance at simplicial homotopies. We will take a fairly naïve view of them to start with. We have already met them in passing in our discussion of simplicial mapping spaces in Chapter 5.3, page 220.

Given $f, g : K \to L$, two morphisms of simplicial sets, a *simplicial homotopy* from f to g is, of course, a map

$$h : K \times \Delta[1] \to L$$

such that if $e_0 : \Delta[0] \to \Delta[1]$ is the 0-end of $\Delta[1]$, (so is actually represented by the d_1 face - beware of confusion) and $e_1 : \Delta[0] \to \Delta[1]$, gives the 1-end, then

$$f = h \circ (K \times e_0),$$

$$g = h \circ (K \times e_1).$$

(More on such cylinder based homotopies in abstract settings can be found in Kamps and Porter, [131]. In a general context, simplicial homotopy does *not* give an equivalence relation on the set of simplicial maps as although it gives a reflexive relation symmetry and transitivity depend on the existence of fillers in the simplicial set of morphisms.)

This is the neat geometric way of picturing simplicial homotopies. There is an alternative 'combinatorial' way that is also very useful (see [131], p.184-186, for a discussion - but not for the formulae which were left as an exercise!) This gives h being specified by a family of maps,

$$h_i^n : K_n \to L_{n+1},$$

indexed by $n = 0, 1, \ldots$, and i with $0 \leq i \leq n$, and satisfying some face and degeneracy relations that we will give later on. For the moment, we will only need to use these in low dimensions, so imagine the lowest dimension $h_0^0 : K_0 \to L_1$. For each vertex, k_0, we get an edge / 1-simplex in L_1 joining $f_0(k_0)$ and $g_0(k_0)$. Now if $k_1 \in K_1$, we expect a square in $K \times \Delta[1]$ looking like

$$\begin{tikzcd}
{} \arrow[r, "{(k_1, s_0 1)}"] & {} \\
{} \arrow[u, "{(s_0 d_1 k_1, \iota)}"] \arrow[r, "{(k_1, s_0 0)}"'] \arrow[ur] \arrow[ur, phantom, "\tau_0", shift left=3] \arrow[ur, phantom, "\tau_1"', shift right=3] & {} \arrow[u, "{(s_0 d_0 k_1, \iota)}"']
\end{tikzcd}$$

with $\iota \in \Delta[1]_1$, the unique non-degenerate 1-simplex, corresponding to $id : [1] \to [1]$. (Remember the product of simplicial sets, K and L, has $(K \times L)_q = K_q \times L_q$.) The homotopy h has to thus give two 2-simplices of L. These will be $h_0^1(k_1) := h(\tau_0)$ and $h_1^1(k_1) := h(\tau_1)$ respectively. We first note that $d_1\tau_0 = d_1\tau_1$, so

$$d_1 h_0^1 = d_1 h_1^1.$$

Likewise the geometric picture tells us that $d_2 h_1^1 = f_1$ and $d_0 h_0^1 = g_1$ and finally that $d_0 h_0^1 = h_0^0 d_0$, whilst $d_2 h_1^1 = h_0^0 d_1$.

In our special case of that general square, $k_1 = (x, U_0, U_1)$ with $d_0 k_1 = (x, U_1)$, $d_1 k_1 = (x, U_0)$, thus our earlier choices should mean the horizontal edges are mapped to

$$\begin{aligned} h((x, U_0, U_1), 0) &= (g_{10}(x), u_0(x)), \\ h((x, U_0, U_1), 1) &= (g'_{10}(x), u'_0(x)), \end{aligned}$$

and the vertical ones,

$$\begin{aligned} h((x, U_1), \iota) &= (g_1(x), u_1(x)), \\ h((x, U_0), \iota) &= (g_0(x), u_0(x)). \end{aligned}$$

They match up as required.

We need to work out h_0^1 and h_1^1. These will map (x, U_0, U_1) to 2-simplices of $K(\mathsf{Aut}(G))$, i.e., to triples $(\gamma_2, \gamma_1, \alpha)$, with $\gamma_i \in G$ and $\alpha \in Aut(G)$. First we look at $h_0^1(x, U_0, U_1)$ and the faces we know of it.

Let $h_0^1(x, U_0, U_1) = (\gamma_2, \gamma_1, \alpha)$, then the two descriptions of $d_2 h_0^1$ give

$$(g_{10}(x), u_0(x)) = (\gamma_1, \alpha),$$

whilst for $d_0 h_0^1$, we have

$$(g_1(x), u_1(x)) = (\gamma_2, i_{\gamma_1} \circ \alpha).$$

We thus have $\gamma_1 = g_{10}(x)$, $\alpha = u_0(x)$ and $\gamma_2 = g_1(x)$ and we can check back that $i_{g_{10}} u_0 = u_1$ from earlier calculations. We have h_1^1 completely specified as

$$h_1^1(x, U_0, U_1) = (g_1(x), g_{10}(x), u_0(x)).$$

This gives $d^1 h_1^1(x, U_0, U_1) = (g_1(x) g_{10}(x), u_0(x))$, which we will need shortly.

We next turn to $h_0^1(x, U_0, U_1)$ and reset the meaning of γ_i and α, so this is $(\gamma_2, \gamma_1, \alpha)$. We do a similar calculation and this gives

$$h_0^1(x, U_0, U_1) = (g'_{10}(x), g_0(x), u_0(x)).$$

This 'feels' right, but we have to check it matches h_0^1 on the diagonal:

$$d_1 h_0^1(x, U_0, U_1) = (g'_{10}(x) g_0(x), u_0(x)),$$

but $g'_{10}(x) = g_1(x) g_{10}(x) g_0(x)^{-1}$, so this equals $(g_1(x) g_{10}(x), u_0(x))$, as hoped.

We have laboriously checked through the calculations of (h_0^1, h_1^1) to show how well behaved things really are. It is reasonably easy to extend the calculation to all dimensions. What needs to be retained is that h was completely specified by the coboundary and cocycle data and, conversely, if we were given any homotopy h between $\mathbf{g}$ and $\mathbf{g}'$, then $\mathbf{g}$ and $\mathbf{g}'$ will be equivalent. This suggests that the simplicial mapping sheaf or bundle $\underline{\mathcal{S}Sh_B}(N(\mathcal{U}), K(\mathsf{Aut}(G)))$, is what is really encoding the data in a neat way. (If you are hazy about simplicial mapping spaces, recall that if K and L are simplicial sets, $\underline{\mathcal{S}}(K, L)$ is the simplicial set of simplicial maps and (higher) homotopies, so

$$\underline{\mathcal{S}}(K, L)_n = \mathcal{S}(K \times \Delta[n], L).$$

Using the constant simplicial sheaves, $\Delta[n]_B$, to replace the use of the $\Delta[n]$ gives a similar simplicial enrichment for the category of simplicial sheaves / bundles on B, but this can be localised to make $\underline{\mathcal{S}Sh_B}(K, L)$, a simplicial sheaf as well.)

Earlier we omitted the detailed description of homotopies as families of maps. To complete our picture here, that description will now be useful. We first give it for simplicial sets, so in the very classical setting.

Let K and L be simplicial sets, and $f, g : K \to L$ two simplicial maps, then a homotopy

$$h : K \times I \to L$$

between f and g can be specified by a family of functions

$$h_, = h_i^n : K_n \to L_{n+1},$$

satisfying various relations. To understand how these arise, we use some simple notation extending that which we used above.

The non-degenerate $(n+1)$-simplices of $\Delta[n] \times \Delta[1]$ are of form $(s_j \iota_n, s_{\hat{j}} \iota_1)$, where $\iota_n \in \Delta[n]_n$ is the unique non-degenerate n-dimensional simplex corresponding to $id_{[n]} : [n] \to [n]$ in the description of $\Delta[n]$ as $\mathbf{\Delta}(-, [n])$, ι_1 being similarly specified for $n = 1$, and where $s_{\hat{j}}$ is the multiple degeneracy corresponding to $\hat{j} = (0, \ldots, \hat{j}, \ldots, n)$, i.e., $s_n \ldots s_0$, but without s_j. (Any $(n+1)$ simplex of $\Delta[1]$ is given by an increasing map $[n+1] \to [1]$, so can be represented as a string $(0, \ldots, 0, 1, \ldots, 1)$, say with j zeroes. This will be $s_{\hat{j}} \iota_1$, since the first j degeneracies 'add in' 0s, whilst those after the $(j+1)^{st}$, that is, after the break, will add in 1s. The simplicial identities give $s_i s_j = s_j s_{i-1}$ if $i > j$, so $s_{\hat{j}}$ has a second useful description as $(s_{last})^{n-j}(s_0)^j$.)

For an n-simplex $k \in K$, we denote $(s_j k, s_{\hat{j}} \iota_1)$ by τ_j, or, more exactly, $\tau_j(k)$ if confusion might arise. We then encode our $h : K \times I \to L$ by $h_j^n(k) = h(\tau_j(k))$. The homotopy h is, of course, a simplicial map so, for any $0 \leq i \leq n+1$, we have $d_i h = h d_i$. These relations translate to give the following rules:

$$d_0 h_0 = g, \qquad\qquad d_{n+1} h_n = f,$$

$$\begin{cases} d_i h_j &= h_{j-1} d_i \qquad \text{for } i < j, \\ d_{j+1} h_{j+1} &= d_{j+1} h_j, \\ d_i h_j &= h_j d_{i-1} \qquad \text{for } i > j+1, \end{cases}$$

and the corresponding degeneracy rules are

$$s_i h_j = h_{j+1} s_i, \qquad i \leq j,$$

$$s_i h_j = h_j s_{i-1}, \qquad i > j.$$

Of course, these h_js etc. are further indexed by a dimension h_j^n, so, for instance, $d_i h_j^n = h_{j-1}^{n-1} d_i$ is the full form of the second line of these.

Aside on Tensors and Cotensors: It is often the case, when considering simplicial objects in a category, $\mathcal{A}$, that one can form a 'tensor', $X \otimes I$, using a coproduct in each dimension, then one defines a homotopy to be a morphism

$$h : X \otimes I \to Y.$$

The construction of this 'tensor' is : given any simplicial set K, and a simplicial object X in $\mathcal{A}$, (where $\mathcal{A}$ has the coproducts that we will be using below),

$$(X \otimes K)_n = \bigsqcup_{k \in K_n} X_n(k) \text{ with each } X_n(k) = X_n,$$

i.e., a K_n-indexed copower of X_n. Using an element based notation, the usual way of denoting the copy of $x \in X_n$, in the k-indexed copy of X_n would be $x \otimes k$ and then face and degeneracy maps are given, in $X \otimes K$, by $d_i(x \otimes k) = d_i x \otimes d_i k$, etc., i.e.,'component-wise'. In this setting again $h : X \otimes \Delta[1] \to Y$ can be decomposed to give a family $\{h^n_j : X_n \to Y_{n+1}\}$. The same description works if instead of a tensor, we have a cotensor.

The setting is that of $\mathcal{S}$-enriched categories having enough (finite) limits. Suppose now $\mathcal{C}$ is $\mathcal{S}$-enriched, so for objects $X, Y \in \mathcal{C}$, we can form a simplicial set $\underline{\mathcal{C}}(X, Y)$ of 'morphisms' from X to Y. A homotopy between $f, g \in \underline{\mathcal{C}}(X, Y)_0$ will, of course, be a 1-simplex $h \in \underline{\mathcal{C}}(X, Y)$ with $d_1 h = f$, $d_0 h = g$. If $\mathcal{C}$ is *cotensored* then, for any simplicial set K, there is a *cotensor*, $\overline{\mathcal{C}}(K, Y)$, for each Y in $\mathcal{C}$, such that

$$\mathcal{S}(K, \underline{\mathcal{C}}(X, Y)) \cong \mathcal{C}(X, \overline{\mathcal{C}}(K, Y)).$$

Of particular use is the case $K = \Delta[1]$, as a 1-simplex $h \in \underline{\mathcal{C}}(X, Y)$ can be represented by an element in $\mathcal{S}(\Delta[1], \underline{\mathcal{C}}(X, Y))$ and thus by an element of $\mathcal{C}(X, \overline{\mathcal{C}}(\Delta[1], Y))$. In other words, a homotopy is a morphism

$$h : X \to \overline{\mathcal{C}}(\Delta[1], Y),$$

so $\overline{\mathcal{C}}(\Delta[1], Y)$ behaves like a path-space object or cocylinder on Y. The construction of $\overline{\mathcal{C}}(K, Y))$ uses limits and can be 'deconstructed' to give a family based description of homotopies, just as before. The nice thing about that description is, however, that it makes sense whatever category $\mathcal{A}$ is as it is merely governed by some small list of identities between composite maps. (For any $\mathcal{A}$, *Simp*.$\mathcal{A}$ is $\mathcal{S}$-enriched, so can be taken to be the $\mathcal{C}$ above; see Kamps and Porter, [131] for a discussion of some of these ideas, in particular on cylinders and cocylinders as a basis for 'doing' homotopy theory in some seemingly unlikely places! We will examine simplicially enriched categories more fully later on, starting on page **??**.) A word of caution, however, is in order. As we mentioned earlier, homotopies are not always composable, nor reversible. If we have a homotopy, in this abstract setting, between morphisms f_0 and f_1 and another between f_1 and f_2, then there may not be one directly from f_0 to f_2. This is annoying! It depends on Kan filling conditions in the simplicial hom-sets. Luckily in many of the cases that we need, the composition of homotopies does work, however once or twice we will have to be careful in the wording. Of course, we could generate the equivalence relation defined by 'direct' homotopy, but, whilst this is very useful, it does often require a chain or 'zig-zag' of explicit 'direct' homotopies if it is to be of maximal use. Conditions on $\mathcal{A}$ can be found that imply that homotopy in *Simp*($\mathcal{A}$) is an equivalence relation, (but I do not know if optimal such conditions are known).

Remark: We are heading for a fairly simplicial description of cohomology. A very useful reference at this point is Jack Duskin's memoir, [83], although that emphasises the Abelian theory only, and also his outline of a higher dimensional descent theory, [85]. From this simplicially based theory, it is then a short journey to give a 'crossed' description of the bitorsor based, (and then gerbe based), non-Abelian cohomology.

Pause: At this point, it is a good idea to take stock of what we have shown. We have used local sections $\{s_i\}$ to get cocycles $\{(g_{ij}, u_i)\}$ and have constructed the beginnings of a simplicial morphism $\mathbf{g}$ from $N(\mathcal{U})$ to $K(\mathsf{Aut}(G))$. So far we have explicitly given $\mathbf{g}_n$ for $n \leq 2$ only, and so should check higher dimensions as well. (Intuitively it would be strange if something came adrift in higher dimensions, since $\mathsf{Aut}(G)$ '*is* a 2-type', but we should make certain!) We also have to check our interpretation of homotopies in higher dimensions.

Let us see what $\mathbf{g}_n : N(\mathcal{U}) \to K(\mathsf{Aut}(G))$ would have to satisfy. Let

$$\mathbf{g}_n(x, U_0, \ldots, U_n) = (g_n, \ldots, g_1, \alpha),$$

then

$$\begin{array}{rcl} d_n\mathbf{g}_n(x, U_0, \ldots, U_n) & = & (g_{n-1}, \ldots, g_1, \alpha), \\ d_0\mathbf{g}_n(x, U_0, \ldots, U_n) & = & (g_n, \ldots, g_2, i_{g_1} \circ \alpha), \\ d_i\mathbf{g}_n(x, U_0, \ldots, U_n) & = & (g_n, \ldots, g_{i+1}g_i, \ldots, g_1, \alpha), \end{array}$$

for $0 < i < n$, so we *can* thus read off $\mathbf{g}_n$ from a knowledge of its faces! In other words, our intuition was right and $\mathbf{g}_0$, $\mathbf{g}_1$ and $\mathbf{g}_2$ determined $\mathbf{g}_n$ in all dimensions.

A very similar calculation shows that $\mathbf{h} : N(\mathcal{U}) \times I \to K(\mathsf{Aut}(G))$ corresponds to the 1-cocycle $\{g_i\}$ and nothing more.

We thus have established a one-one correspondence between the set of isomorphism classes of G-bitorsors that trivialise over $\mathcal{U}$ and the set $[N(\mathcal{U}), K(\mathsf{Aut}(G))]$ of homotopy classes of simplicial sheaf maps from $N(\mathcal{U})$ to the underlying simplicial sheaf of the simplicial group, $K(\mathsf{Aut}(G))$.

We should continue our pause here and make some comments about the overall situation. This set can be interpreted as a type of zeroth non-Abelian hyper-cohomology of B relative to the cover $\mathcal{U}$. It is $H^0(N(\mathcal{U}), \mathsf{Aut}(G))$. But what is hyper-cohomology? We will have a look at its classical Abelian form below, but note that the coefficients, here, are in a sheaf of crossed modules, so will also need to look at that in more detail. We saw earlier a related situation (in section 5.1) where we replaces the crossed module $\mathsf{Aut}(G)$ by a general one $\mathsf{Q} = (K, Q, q)$, when discussing non-Abelian extensions of G by K 'of the type of Q'. We there obtained a cohomology set, there called $H^2(G, \mathsf{Q})$, identifiable as $[\mathsf{C}(G), \mathsf{Q}]$, and the correspondence was obtained by identifying the cocycles as maps of crossed complexes and, as $\mathsf{C}(G)$ is 'free', it sufficed to give them on the generating elements, in other words on the analogue of $N(\mathcal{U})$.

The reason given for introducing the notion of extension of type Q was to obtain functoriality in the coefficients. (Recall that if $\varphi : G \to H$ is a homomorphism of groups then it is not clear when there is a morphism of crossed modules from $\mathsf{Aut}(G)$ to $\mathsf{Aut}(H)$ which is φ on the 'top group'.) This also gave a good possibility of a finer classification of *all* extensions of G by H: some will be of the type of a particular Q, others not.

In our bitorsor situation, the functoriality is once again important, but the second aspect gains an additional geometric significance. A very important part of classical fibre bundle theory relates to the possibility of 'reducing the group'. For instance, suppose we have a n-dimensional real manifold, X, then its tangent bundle is a fibre bundle with each fibre a vector space of dimension n and with the transition functions taking their values in $G\ell(n,\mathbb{R})$, i.e., a n-dimensional vector bundle. (Its associated $G\ell(n,\mathbb{R})_X$-torsor is, as we saw, the frame bundle.) If X is at all 'nice', we can put a Riemannian metric on it, (i.e., additional structure of considerable geometric importance), and this corresponds to showing that our transition functions can be replaced by ones taking values in $O(n,\mathbb{R})$, the corresponding group of orthogonal matrices, as these are the ones that preserve the metric/inner product. Note that the tangent bundle naturally has an action by $Aut(F)$, that is the corresponding automorphism group of the fibre, F. (With our bitorsors, the corresponding acting object is a strict automorphism gr-groupoid, and we have used the corresponding crossed module, $\mathsf{Aut}(G)$.)

Other examples would correspond to other subgroups of general linear groups. Foliated structures, systems of partial differential equations, etc., correspond to sub-bundles of bundles of jets on X. These structures may be on X itself or on some given fibre bundle $E \to X$ over X. In each case, giving a G-structure on E, for a group, G, which is a subgroup of the natural group of automorphisms, corresponds to 'reducing' the $Aut(F)$-torsor to a G-torsor. Another type of structure corresponds to 'lifting' the transition functions from some given H to a G, where $\varphi : G \to H$ is a nice epimorphism. For instance, the special orthogonal group $SO(n,\mathbb{R})$ for $n \geq 2$, has a universal covering group, $Spin(n) \to SO(n,\mathbb{R})$, and extra structure of use for some applications, corresponds to *lifting* the $u_{ij} : U_{ij} \to SO(n,\mathbb{R})$ to take values in $Spin(n)$. Of course, this is not always possible. Obstructions may exist to doing it, depending in part on the topological structure of X.

All these examples were of Lie groups, i.e., groups in the category of differential manifolds, but a similar intuition was central to discussions in the 1960s and 1970s of the relationship between smooth and piecewise linear structures on topological manifolds, in which various *simplicial* groups of automorphisms were related and the obstructions to lifting transition functions of certain natural simplicial bundles were the key to the problem. Again analogous situations exist in algebraic geometry involving group schemes and their 'subgroups'. Here, as a group scheme over a fixed base $Spec(K)$ is in many ways a bundle of groups, the more general theory of group bundles and change of group bundles, rather than merely change of groups, as such, is what is important here.

It would almost be fair to say that, from a historical perspective, this is one modern interpretation of Klein's original intuition of what geometry is, i.e., the study of the automorphisms that preserve some 'structure'. What seems now to be emerging is the relationship between higher level 'automorphism gadgets' such as $\mathsf{Aut}(G)$ and classical invariants such as cohomology and consequently, some appreciation of higher level 'structure'. Many of the ingredients of the theory are still missing or are merely 'embryonic' in the crossed module / 2-group case as yet, but the plan of action is becoming clearer.

Returning to the detail, we therefore consider a sheaf or bundle of crossed modules, $\mathsf{M} = (C, P, \partial)$, and look at data of the form

$$\mathbf{g} : N(\mathcal{U}) \to K(\mathsf{M}),$$

so $g_0(x, U_i) = p_i(x)$ with $p_i : U_i \to P$, a local section of P over U_i and $g_1(x, U_i, U_j) = (c_{ji}(x), p_i(x))$, where $c_{ji} : U_{ji} \to C$ is a local section of C over the intersection U_{ji}. These local sections satisfy

$$\partial(c_{ji})p_i = p_j \text{ and } c_{kj}c_{ji} = c_{ki}$$

over the intersections. Corresponding to a change in local sections will be a coboundary rule of the form:

$$c'_{ij} = c_i c_{ij} c_j^{-1},$$

and

$$p'_i = \partial(c_i) p_i,$$

i.e., a homotopy between $\mathbf{g}$ and $\mathbf{g}'$. The equivalence classes will be in $H^0(N(\mathcal{U}), \mathsf{M})$ and, both in this general case and in the particular case of $\mathsf{M} = \mathsf{Aut}(G)$, it is natural to pass to the limit over coverings (or if working in a more general Grothendieck topos, over hypercoverings) to get the zeroth Čech hyper-cohomology set with values in M, denoted $\check{H}^0(B, \mathsf{M})$.

We have $H^0(N(\mathcal{U}), \mathsf{M}) = [N(\mathcal{U}), K(\mathsf{M})]$, and it is reasonably safe to think of $\check{H}^0(B, \mathsf{M})$ in these terms, but, in fact, one really needs to introduce the category $D(\mathcal{E}) = Ho(Simp(\mathcal{E}))$, obtained by taking the category of simplicial objects in the topos, $\mathcal{E}$, in our simplest case that of simplicial sheaves on B, and inverting the 'quasi-isomorphisms', i.e., those simplicial maps that induce isomorphisms on all homotopy groups. There are several detailed treatments of this type of construction in the literature - not all completely equivalent - so we will not give another one here!

We could, and later on will, go further. We could replace the crossed module M by a crossed complex, or, in general, could use a simplicial group, H, instead of $K(\mathsf{M})$. We will definitely keep this in mind, but just because it could be done, does not mean it needs doing *now*. The problem is that we, as yet, have only an embryonic understanding of the algebraic and geometric properties of the situation with M a crossed module or bundle / sheaf of such things. Past experience shows that the generalisation and abstraction *will be* worth doing, but we may not yet have the auxiliary concepts and intuitions to interpret what that theory will tell us, nor what are the *significant* new questions to ask and problems to solve. As yet, there are few signposts in that new land!

6.5.6 Cleaning up 'Change of Base'

Although we have considered change of base several times, we have not had available enough machinery to handle it really adequately. In particular, we have left the question of homotopic maps inducing 'isomorphic torsors' up in the air. Now we can give a reasonable treatment of that results and at the same time treat change of base for bitorsors, (and in such a way as to handle change of base for relative M-torsors as well, and we have not formally defined *them* yet).

One conceptual difficulty left over from earlier was that if f and f' were homotopic maps from B to B', and P was a G-torsor on B', we want to be able to say that somehow $f^*(P)$ and $(f')^*(P)$ are isomorphic, yet they are 'over' different groups bundles. The first is a $f^*(G)$-torsor, the second a $(f')^*(G)$-torsor. This problem did not arise with principal G-bundles as there the 'coefficient group' was just that, a group, corresponding to a constant sheaf of groups, so the two coefficient 'groups', $f^*(G)$ and $(f')^*(G)$ were the same. Both were trivial. Our first task is thus to look at a simplicial treatment of change of base, and once that is done, a lot of things will simplify!

Suppose that $f : B \to B'$ is a continuous map and $\mathbf{g} : N(\mathcal{U}) \to K(\mathsf{M})$ represents either a G-torsor, or a G-bitorsor or, looking forward to the next section, a relative M-torsor, for M a sheaf or bundle of crossed modules on B' and we assume that that object trivialises over the open cover $\mathcal{U}$. The continuous function f pulls back that cover to $f^{-1}(\mathcal{U})$. This can either be viewed as the result of pulling back each open set to get a cover, or, equivalently but perhaps better, by forming the sheaf / étale space, $\bigsqcup\mathcal{U}$ over B' and then pulling back that sheaf to $f^*(\bigsqcup\mathcal{U})$. The result is the same. In fact we saw earlier that f^* preserved pullbacks and so $N(f^*(\bigsqcup\mathcal{U}))$ is isomorphic to $f^*(N(\mathcal{U}))$. This isomorphism is given by examining local sections of the two simplicial sheaves, so local sections of $f^*(\bigsqcup\mathcal{U})$ are induced by composition of f with a local section of $\bigsqcup\mathcal{U}$. (A detailed treatment is not quite that simple. The map can better be examined at the level of germs of local sections as we did in our discussion of f^*, page 266.)

Remark: In situations where hypercoverings are needed to give an adequate cohomology theory, the functor f^* still works more or less as above. Of course, the detailed geometric nature of the construction is a bit different as ideas of germs of local sections, etc., have to be interpreted slightly differently, say, in a topos, however the intuition is much the same.

Viewed as a pullback construction, there is a canonical map from $f^*(N(\mathcal{U}))$ to $N(\mathcal{U})$, namely the projection, and this is 'over' f itself. At the level of elements, this sends $(x, f^{-1}U_0, \dots f^{-1}U_n)$ to $(x, U_0, \dots, U_n)$. Abusing notation we will call this f as well. The induced cocycle is then just the composite, $\mathbf{g}f : N(f^{-1}\mathcal{U}) \to K(\mathsf{M})$, and this gives the induced torsor, but that is a $f^*(G)$-torsor. Thus at the level of the simplicial description of the induced torsor, the work is done for us without too much pain! We just have composition with f, and that, of course, is what we expected.

The next thing to look at is the connection between the induced functors for homotopic maps. We will restrict to compact spaces to simplify the discussion. If $h : f \simeq f' : B \to B'$, and we are looking at a torsor on B' that trivialises over the open cover $\mathcal{U}$, then we can get an open cover $h^{-1}(\mathcal{U})$ on $B \times I$ and a torsor on that space just by thinking of h as a continuous map. Because of our simplifying assumption of compactness, it is possible to refine $h^{-1}(\mathcal{U})$ to a cover of the form, $\{U \times V \mid U \in \mathcal{U}', V \in \mathcal{V}\}$ for $\mathcal{U}'$ an open cover of B and $\mathcal{V}$ an open cover of the unit interval I. We will denote this cover by $\mathcal{U} \times \mathcal{V}$. We can assume that the nerve of $\mathcal{I}$ is a simplicial sheaf that is essentially a subdivision $\underline{Sd}(\Delta[1])$ of the constant simplicial sheaf on I with value $\Delta[1]$. (The cover $\mathcal{V}$ may need further refinement to get it to be of this form, and you should look at this point, but we also are using that I is contractible to get that we have a trivial sheaf.) The nerve of a product cover is isomorphic to the product of the nerves as can be seen by inspection. We thus have that $N(\mathcal{U}\times\mathcal{V})$ can be replaced by $N(\mathcal{U})\times\underline{Sd}(\Delta[1])$. The subdivided $\Delta[1]$ is a concatenation of a number of copies of $\Delta[1]$, end to end, so the map induced at the simplicial level from $N(\mathcal{U} \times \mathcal{V})$ to $K(\mathsf{M})$ gives us not only the two maps induced by f and f', but also a sequence of simplicial homotopies between intermediate maps. These can be composed to get a simplicial homotopy between the original induced maps. Notice none of this uses any information about the actual torsor involved except the initial assumption that it trivialises over $\mathcal{U}$. This does it! We have a description of isomorphism classes of torsors in terms of homotopic maps, we have homotopic maps so .... .

From this lots of good things flow. Homotopically equivalent spaces, say B and B', give equivalent categories of torsors over 'linked' sheaves of groups, and, in particular, if G is a constant sheaf of groups, or M a constant sheaf of crossed modules, then over the two spaces the induced sheaves are also constant, hence we can talk of G-torsors over B or over B' without fussing too much about the fact that we really mean $\underline{G}_B$- and $\underline{G}_{B'}$-torsors.

The situation for contractible spaces is then simple. All torsors over $\underline{G}_B$ are trivial, and as a consequence, if B is a space which has an open covering by contractible open sets, and such that all finite intersections of the open sets are also contractible, (i.e., a Leray cover), then we automatically have lots of local sections over that cover. As manifolds are examples of spaces with this property, this comes in to be very useful in applications of the torsors to geometry.

6.6 Relative M-torsors

(The basic references are Breen's paper [32], (but our conventions are different and so some of the results also look different), and also the papers of Jurčo, in particular, [130].)

6.6.1 Relative M-torsors: what are they?

What are the objects corresponding to a $\mathbf{g} : N(\mathcal{U}) \to K(\mathsf{M})$? We saw that this consisted of some local sections

$$p_i : U_i \to P$$

and others

$$c_{ij} : U_{ij} \to C$$

satisfying some evident relations, one of which was the cocycle condition

$$c_{kj}c_{ji} = c_{ki}.$$

These c_{ji} will give us a left C-torsor, E, say. We can examine the induced P-torsor, $\partial_*(E)$, and - surprise, surprise - the p_i part of the cocycle pair, $\{(c_{ij}, p_i)\}$, provides a trivialising coboundary, since

$$p_i = \partial(c_{ij})p_j$$

yields

$$\partial(c_{ij}) = p_i p_j^{-1} = p_i.1.p_j^{-1}.$$

Conversely suppose we have a C-torsor, E, and we know that $\partial_*(E)$ is trivial, then we can find p_is satisfying the above equations and making E into an M-torsor. If we look back to our motivating case with $\mathsf{M} = \mathsf{Aut}(G)$, then we can adapt the argument given there (page 289) to get an explicit global section of $\partial_*(E) = P_\partial \wedge^C E$, namely, for local sections e_i of E, define $\mathbf{t} = \{t_i\} = \{[p_i^{-1}, e_i]\}$ to get a compatible family and hence a global section, t, of $\partial_*(E)$. This process can be reversed, so from t and a choice of e_i, one can obtain p_i. We will see a neat way of doing this shortly.

What happens if we choose different local sections e_i' of E? These e_i' will give some c_is such that $e_i' = c_i e_i$, and also $p_i' = \partial(c_i)p_i$, but then

$$[(p_i')^{-1}, e_i'] = [p_i^{-1}\partial(c_i)^{-1}, c_i e_i] = [p_i^{-1}, e_i],$$

so the global section does not change.

We saw earlier that contracted product gave the category of G-bitorsors the structure of a group-like monoidal category with inverses, a gr-groupoid. (If P and Q are in $Bitors(G)$, then $P \wedge^G Q$ gave the 'product', whilst P^o was 'inverse' to P. Of course, the trivial bitorsor, T_G, was the identity object.) There is an obvious category of M-torsors, which we will denote by $\mathsf{M}{-}Tors$, (so $\mathsf{Aut}(G){-}Tors = Bitors(G)$), does this in general have any similar structure?

Before we attempt to answer that, we should give formal definitions of M-torsors, etc, as a base reference:

Definition: Let $\mathsf{M} = (C, P, \partial)$ be a bundle or sheaf of crossed modules over a space B, (or more generally a crossed module in a topos $\mathcal{E}$). By a *(relative)* M*-torsor*, or M*-relative torsor* we mean a left C-torsor together with a global section t of $\partial_*(E)$.

A morphism of M-torsors, $f : (E, t) \to (E', t')$, is a C-torsor morphism, $f : E \to E'$, such that

$$\begin{tikzcd} \partial_*(E) \arrow[rr, "\partial_*(f)"] & & \partial_*(E') \\ & B \arrow[ul, "t"] \arrow[ur, "t'"'] & \end{tikzcd}$$

commutes.

We will denote the category of M-torsors by $\mathsf{M}{-}Tors$.

Remark on terminology: The idea of a relative M-torsor lies between that of torsors and global sections and in the long exact sequences, the $\pi_0(\mathsf{M}{-}Tors)$-term is the transition from global section terms, $P(B)$, etc. to true torsor terms, $\pi_0(Tors(C))$. It is a Janus, looking back and forward. Various names have been applied to this. Breen in [31], following Deligne, used something of the form (C, P)-torsor, but that does not use the boundary map, ∂ and clearly various different crossed modules having the same C and P, but perhaps different actions or boundary maps might give differently behaved (C, P)-torsors. Aldrovandi, in conversation, favours a terminology that said, what we might write as $\pi_0(\mathsf{M}{-}Tors)$ or $\check{H}^0(B, K(\mathsf{M}))$, was a $\check{H}^0$-term so was the group of global sections of M. That is very good terminological reasoning, but it neglects the fact that the objects *are* C-torsors plus extra structure. It looks back in the sequence and neglects the future! Using the terminology of M-torsor, which I originally favoured, fails to look back and also hits the problem that the corresponding gr-groupoid $\mathcal{M} = \mathsf{M} - \mathsf{tors}$ is used later on to build $\mathcal{M}$-torsors, which are stacks with a nice action of $\mathcal{M}$, and these live at the next 'janus step' of the exact sequence. There seems no really good choice here. We have used 'relative M-torsor' or 'M-relative torsor' in the definition, but will continue to use 'M-torsor' later on as 'relative M-torsor' is quite tedious to type!

At this point, we need to revisit an old intuition that we have used several times before, but without which 'life' will seem unduly complicated! That intuition is that a principal G-set is a copy of G with an 'identity crisis'. In more detail, in situations such as that of universal covering spaces, E over a space B, the fibre is a copy of $\pi_1(B)$, but without a definite element being chosen as the identity. The natural path lifting property of covering spaces gives that any loop γ at a chosen base-point b_0 in B will lift uniquely to a path in the covering space, *once a start point* e_0 *above* b_0 *has been chosen.* If you choose a different start point e'_0, you, of course, get a different lifted path. The end point of the lifted path will give the image of e_0 under the action of the path class $[\gamma] \in \pi_1(B)$. Thus once e_0 is chosen $p^{-1}(b_0) = E_{b_0}$ can be mapped bijectively to $\pi_1(B)$. (Remember we did say E was a *universal* covering space.) Under this bijection, the identity element of $\pi_1(B)$ corresponds to e_0, but our alternative choice, e'_0, will give a bijection with e'_0 itself corresponding to $1_{\pi_1(B)}$. There is no canonical choice of start point in E_{b_0}, so no definitive identification of E_{b_0} with $\pi_1(B)$.

For a G-bitorsor, with a local section $e_i : U_i \to E$, we have essentially the same situation. The left and right G-actions are globally independent and yet are locally linked by the $u_i : G_{U_i} \to G_{U_i}$. To use these it *is* necessary to use the e_i to temporarily pick a 'start point' in each fibre of E. Thus the equation,

$$u_i(g).e_i = e_i.g,$$

interprets as both the definition of u_i given the right action *and* conversely, given the u_i, as a defining equation of a right action. This does need to be spelt out again: given any local element

x of E over U_i, it has the form $x = g'e_i$ for some local element g' of G. Suppose we now operate with g on the right of x, then we get

$$x.g = g'e_i.g = g'u_i(g)e_i.$$

(This is very analogous to defining a linear transformation between vector spaces by transforming the elements of a chosen basis and then 'extending linearly'. Here we extend G-equivariantly for the *left* action, having transformed the 'basic' element e_i to $e_i.g$.)

The key transition equation for the u_is was

$$u_i' = i_{g_i} \circ u_i,$$

which emphasises this viewpoint. We changed e_i to e_i' using g_i, so $e_i' = g_ie_i$, but then, for right action by g,

$$e_i'g = u_i'(g)e_i' = u_i'(g)g_ie_i,$$

whilst also

$$e_i'g = g_ie_ig = g_iu_i(g).e_i,$$

giving the transition equation in the form $g_iu_i(g) = u_i'(g)g_i$.

We now need to translate this into a tool that can be used for M-torsors. The plan of action is to show that any M-torsor, E, has a natural C-bitorsor structure and for this we have to use $t : B \to \partial_*(E)$ to obtain a right C-action on E. In Lemma 45, (page 268), we saw how to go from a global section of a torsor to an identification of it as an 'identity-less' copy of the group bundle. We thus have that t allows us to identify $\partial_*(E)$ with T_P, i.e., with P itself (as left P-torsor). We can unpack the recipe in Lemma 45, (but beware the change of notation, P is here the basic group of our crossed module M, but was the torsor in that earlier discussion). Any local element of $\partial_*(E)$ over some U_i is of form $[p, e]$, with p a local section of P over U_i and e a local section of E, again over U_i. We can get from t an expression $[p, e] = p'.t$ for some p' defined over U_i. Using the structural map of $\partial_*(E)$ as a P-torsor, we get

$$\partial_*(E) \overset{(t\pi,id)}{\to} \partial_*(E) \times \partial_*(E) \overset{\cong}{\to} P \times_B \partial_*(E) \overset{proj}{\to} P,$$

which, from $[p, e]$ gives the p'. (Recalling that, given e_i, the unadjusted choice of local sections is $[1, e_i]$, then this process picks out the corresponding p_i, so that $t = [p_i^{-1}, e_i]$.) Thus from t, we get a map from $\partial_*(E)$ to P.

In this 'game', it pays to go back-and-fore between the different descriptions and to revisit the special case, $\mathsf{M} = \mathsf{Aut}(G)$, for guidance, and, hopefully, inspiration. Our key equation defining the u_i was $u_i(g)e_i = e_i.g$. In our general case of $\mathsf{M} = (C, P, \partial)$, the rôle of the u_i is taken by the local elements p_i, which act on C (since, recall, that action is part of the crossed module structure) and the corresponding equation would be

$$^{p_i}c.e_i = e_i.c,$$

but $e_i.c$ is not defined, a least not yet! We will take this as its definition (and remember our earlier discussion of right actions, and what here would be the C-equivariant extension), then see if it works!

First let us underline what the equation actually says. An arbitrary local element of E_{U_i} has form $e = c_i.e_i$ and the expression for $e.c$ will be $c_i.{}^{p_i}c.e_i$ as the right action has to be left C-equivariant, now if $c_1, c_2 \in C_{U_i}$, then

$$(e_i.c_1).c_2 = {}^{p_i}c_1.e_i.c_2 = {}^{p_i}c_1.{}^{p_i}c_2.e_i = {}^{p_i}(c_1c_2).e_i = e_i.(c_1.c_2),$$

so it does define an action, at least locally. Next we have to check on intersections. Supposing that p_i on U_i and p_j on U_j satisfy $p_j = \partial(c_{ji})p_i$, where $e_j = c_{ji}e_i$, then over U_{ij},

$$e_j.c. = c_{ji}e_i.c = c_{ji}\,{}^{p_i}c.e_i = c_{ji}\,{}^{p_i}c_{ji}^{-1}.e_j$$

and also

$$e_j.c = {}^{p_j}c.e_j = {}^{\partial(c_{ji})p_i}c.e_j,$$

and the Peiffer rule for crossed modules gives

$${}^{\partial c}c' = cc'c^{-1},$$

so the two local actions patch together neatly. We thus have an action of C on the right of E. Is it giving us a right C-torsor structure on C? This amounts to asking if locally the equation $x = yc$ can be solved uniquely for c in (some) terms of x and y over U_i, but $x = c'.y$ for a unique c', since E is a left C-torsor. The obvious element to try out as our required solution, c, is ${}^{p_i^{-1}}c'$ - **try it**! It works. We have proved:

Lemma 52 *If (E,t) is a* M*-torsor, then E is a C-bitensor.* ■

From another perspective, this is quite clearly due to the natural map from M to Aut(C), given by the identity on C and the action map

$$\begin{tikzcd} C \arrow[r, "="] \arrow[d] & C \arrow[d] \\ P \arrow[r, "\alpha"'] & Aut(C) \end{tikzcd}$$

We would expect an M-torsor to be mapped to a Aut(C)-torsor, that is, a C-torsor, via this morphism of crossed modules, so from this viewpoint the lemma may not seem surprising.

A few pages ago, we set out to extend the contracted product to M-torsors. Now that we have this lemma, we can, at least, work with a contracted product of the associated C-bitorsors. In other words, if (E_1,t_1), (E_2,t_2) are M-torsors, then we might tentatively explore a definition of $(E_1,t_1) \wedge^{\mathsf{M}} (E_2,t_2)$ as being $(E_1 \wedge^C E_2, t)$ with t still to be described. Here is a suitable, almost heuristic, approach that tells us we are going in the right direction.

We have $\partial_*(E) = P_\partial \wedge^C E_1$, where P_∂ is the trivial (left) P-torsor with, in addition, a right C-action given by : if $x \in P_\partial$, $x = p.t$, where t is a global section (fixed for the duration of the calculation), then, for $c \in C$, $x.c = p.\partial(c).t$. Now if $\partial_*(E)$ is assumed to have a global section, it is easy to show that it is, itself, isomorphic to P_∂. Next look at (E_1,t_1), and (E_2,t_2) and let us examine $\partial_*(E_1 \wedge^C E_2)$. This is $P_\partial \wedge^C E_1 \wedge^C E_2 = (P_\partial \wedge^C E_1) \wedge^C E_2 \cong P_\partial \wedge^C E_2$ by the above calculation, using t_1 to trivialise $(P_\partial \wedge^C E_1)$, and finally this is trivial using t_2.

This argument, although valid, merely shows that t exists. It could be taken apart further to get an explicit formula, but we will, instead, approach that through cocycles. We pick local sections of E_1 and E_2 over the same open cover $\mathcal{U}$. These we will denote by $e^1_i : U_i \to E_1$, $e^2_i : U_i \to E_2$. Given t_1 and t_1, we get local elements of P, p^1_i and p^2_i, so that

$$t_1 = [(p^1_i)^{-1}, e^1_i],$$

and similarly for t_2. These p^1_is are those for the local cocycle description of E_1 as (c^1_{ij}, p^1_i), so are the parts of the contracting homotopy on $\partial_*(E_1)$, etc.

Now look at $E_1 \wedge^C E_2$. The obvious local sections of this would be $e_i = [e^1_i, e^2_i]$, and using these we want to work out the corresponding cocycle pair. We need to work out the relationship of e_i with $e_j = [e^1_j, e^2_j]$. We have $e^1_i = c^1_{ij}e^1_j$, $e^2_i = c^2_{ij}e^2_j$, so

$$\begin{aligned} (e^1_i, e^2_i) &= (c^1_{ij}e^1_j, c^2_{ij}e^2_j) \equiv c^1_{ij}(e^1_j, c^2_{ij}e^2_j) \\ &= c^1_{ij}({}^{p^1_j}c^2_{ij}.e^1_j, e^2_j) = c^1_{ij}\,{}^{p^1_j}c^2_{ij}(e^1_j, e^2_j), \end{aligned}$$

and we have $e_i = c^1_{ij}\,{}^{p^1_j}c^2_{ij}.e_j$. This C-coefficient may look familiar (or not), but before we identify it, we should look for the p_is. The obvious ones to try are $p_i = p^1_i p^2_i$, i.e., the product within P of the two values. We have a $c_{ij} = c^1_{ij}.{}^{p^1_j}c^2_{ij}$, so can see if this works for the equation $p_i = \partial(c_{ij})p_j$:

$$\begin{aligned} p_i = p^1_i p^2_i &= \partial(c^1_{ij})p^1_j.\partial(c^2_{ij})p^2_j \\ &= \partial(c^1_{ij})p^1_j.\partial(c^2_{ij})(p^1_j)^{-1}p^1_j p^2_j = \partial(c_{ij})p_j. \end{aligned}$$

The simplicial interpretation of the cocycles gave a map from $N(\mathcal{U})$ to $K(\mathsf{M})$, and in dimension 1, $K(\mathsf{M})$ is $C \rtimes P$. The multiplication in this semidirect product is

$$(c_1, p_1).(c_2, p_2) = (c_1\,{}^{p_1}c_2, p_1p_2).$$

In other words, if (E_1, t_1) corresponds to a simplicial map $\mathbf{g}_1 : N(\mathcal{U}) \to K(\mathsf{M})$ and similarly $\mathbf{g}_2$ corresponding (E_2, t_2), then $(E_1, t_1) \wedge^{\mathsf{M}} (E_2, t_2)$ is associated to the product $\mathbf{g}_1.\mathbf{g}_2$,

$$N(\mathcal{U}) \to K(\mathsf{M}) \times K(\mathsf{M}) \to K(\mathsf{M}),$$

using the multiplication map of the simplicial group $K(\mathsf{M})$ corresponding to the crossed module, M. Does this give us a gr-groupoid structure on $\mathsf{M}{-}Tors$? The above description of the multiplication as corresponding to contracted product tells us that we can use the inverse of that multiplication to construct an inverse for the contracted product. The detailed formula for the inverse of an M-torsor, (E, t), is **left as an exercise**.

Note that we have not checked certain necessary facts about the (c_{ij}, p_j), namely that $c_{ij}c_{jk} = c_{ik}$ and they transform correctly under change of local sections. The details of these are **left to the reader**. They use the crossed module axioms several times. We have proved the following:

Proposition 76 *Under the identification of $\pi_0(\mathsf{M}{-}Tors)$ and $\check{H}^0(B, \mathsf{M})$, the group structure on the first given by the contracted product coincides with that given on the second under the group structure of $K(\mathsf{M})$, the associated simplicial group bundle of the bundle of crossed modules, M.* ■

6.6.2 An alternative look at Change of Groups and relative M-torsors

When we discussed change of groups, we saw a neat induced torsor construction. Recall we had

$$\varphi : G \to H,$$

a morphism of sheaves of groups and a torsor E over G, we obtained $\varphi_*(E)$ by first forming H_φ, i.e.,the (H, G)-object with right G-action given via φ and then $\varphi_*(E) = H_\varphi \wedge^G E$.

This construction has various universal properties that we have not yet made explicit nor exploited, yet which are very useful. We will need to recall that if P and Q are two G-torsors, a morphism $f : P \to Q$ is a map over B such that $f(g.p) = g.f(p)$ for all $g \in G$ and $p \in P$. In other words, it is a sheaf map $f : P \to Q$, which is G-equivariant. We can represent this by a diagram:

$$\begin{tikzcd} G \times_B P \arrow[r, "G\times f"] \arrow[d] & G \times_B Q \arrow[d] \\ P \arrow[r, "f"'] & Q \end{tikzcd}$$

in which the vertical maps give the actions, and which is required to commute.

There is a neat notion from the theory of group actions (on sets), which adapts well to the torsor context. Suppose that $\varphi : G \to H$ is a homomorphism of ordinary groups, and (X, a_X) and (Y, a_Y) are a G-set and an H-set respectively, with $a_X : G \times X \to X$ and $a_Y : H \times Y \to Y$ being the actions. A map $f : X \to Y$ is said to be *over* φ if for all $x \in X$ and $g \in G$, we have $f(g.x) = \varphi(g).f(x)$. This is, of course easily represented by a similar commutative diagram:

$$\begin{tikzcd} G \times X \arrow[r, "\varphi\times f"] \arrow[d, "a_X"'] & H \times Y \arrow[d, "a_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

It thus follows that a G-map between G-sets is a slightly degenerate form of this notion.

Before we return to the situation of torsors, it will pay to note that φ makes H into a *right* G-set and that $\varphi_*(X)$ as being $H_\varphi \wedge^G X$, makes sense here as well. suppose $f : X \to Y$ is over φ in the above sense, then we look at f and see if it induces an H-map from $\varphi_*(X)$ to T. The elements of $\varphi_*(X)$ will be equivalence classes of pairs (h, x), where $(h, g.x) \equiv (h\varphi(g), x)$. We write $[(h, x)]$ for the equivalence class and try to guess what form an map induced from f might take. The obvious form to try would seem to be to set $\tilde{f}[(h, x)] = h.f(x)$ and to see if this works. Even though this is easy, let us do it explicitly:

$$h.f(g.x) = h.\varphi(g)f(x),$$

since f is over φ, but $\tilde{f}[(h\varphi(g), x)] = h.\varphi(g)f(x)$ as well, so we are done. We note, however, that this is really the only sensible way to define such a $\tilde{f}$. This is thus well defined as an H-map from $\varphi(X)$ to Y. (The fact that it is an H-map **should be clear**.)

We now have $f : X \to Y$ and $\tilde{f} : \varphi_*(X) \to Y$, so is there a possible factorisation of f as a composite of some map $X \to \varphi_*(X)$ over φ followed by $\tilde{f}$? There is an obvious map from X to $\varphi_*(X)$ namely that which sends x to $[(1_H, x)]]$. This then sends $g.x$ to $[(1_H, g.x)]$, which is the same as $[(\varphi(g), x)]$, which is $\varphi(g)[(1_H, x)]$, by the definition of the left H-action on $H_\varphi \wedge^G X$. This is thus a map over φ as expected and does not depend on f itself.

Going back to $\tilde{f}$, we hinted that this might be unique in some sense. What sense? First let us give a name to the map that we have just examined, say $\varphi_\sharp : X \to \varphi_*(X)$. We noted that $f = \tilde{f}\varphi_\sharp$ - but did not **check it**. That done, suppose we had some 'other' H-map $f' : \varphi_*(X) \to Y$, so that $f = f'\varphi_\sharp$, then $f'[(1, x)] = f(x)$, but f' is assumed to be an H-map, so $f'[(h, x)] = f'(h.[(1, x)]) = h.f(x)$ and $f' = \tilde{f}$.

If we write $Maps_\varphi(X, Y)$ for the set of maps from X to Y over φ, we have shown it to be isomorphic to $H - Sets(\varphi_*(X), Y)$. As both are functorial in Y, and (**for you to check**), the isomorphism is natural, we have shown that $Maps_\varphi(X, -)$ is a representable functor with $\varphi_*(X)$ as a representing object. There are still more things to work through and question here. What happens if we change X, for instance? But these can be **left to the reader**.

We did the above in the easy case of $Sets$, now transport the idea across to $Sh(B)$, or better still, to an arbitrary topos, $\mathcal{E}$. We have our original situation of a morphism, $\varphi : G \to H$, of sheaves of groups. We suppose E is a G-torsor and E' an H-torsor.

Definition: A sheaf map $f : E \to E'$ is said to be a *morphism of torsors over* φ if the diagram:

$$\begin{tikzcd} G \times E \arrow[r, "\varphi \times f"] \arrow[d, "a_E"'] & H \times E' \arrow[d, "a_{E'}"] \\ E \arrow[r, "f"'] & E' \end{tikzcd}$$

commutes, the vertical arrows representing the actions.

We can equally well state this in terms of 'local elements'. (The choice of the approach used is largely a question of taste and is left to you.It is advisable to be able to follow and use any of the different methods when handling such discussions - although you may prefer one, say the diagrammatic one, to some other.)

We will write $Sh(B)_\varphi(E, E')$ for the sheaf of morphisms over φ from E to E'. (This is sloppy as E and E' really have to have the actions included in their labeling, but this is fairly anodyne sloppiness.) It should now be easy to prove:

Proposition 77 *(i) For any E, E' as above, there is a natural isomorphism of sheaves*

$$Sh(B)_\varphi(E, E') \cong Tors(H)(\varphi_*(E), E').$$

(ii) The functor $Sh(B)_\varphi(E, -)$ is representable. ■

Although easy, there are quite a lot of things to **check** here!

We thus have a neat universal property for φ_* as a functor from $Tors(G)$ to $Tors(H)$. We can now apply it to the case of relative M, where $\mathsf{M} = (C, P, \partial)$ is a sheaf of crossed modules. We had a description of a relative M-torsors as a C-torsor, E, together with a specified trivialisation $t : \partial_*(E) \overset{\cong}{\to} T_P$.

Proposition 78 *Suppose E is a C-torsor and $t : E \to T_P$, a morphism over ∂, then $(E, \tilde{t})$ is an* M*-torsor. Conversely f $E, f)$ is a relative* M*-torsor, then E is a C-torsor and $f\partial_\sharp : E \to T_P$ is a morphism of torsors over ∂.*

Proof: this is mostly just a corollary of our earlier result. The one point is that $\tilde{t} : \partial_*(E) \to T_P$ is a morphism of H-torsors, and hence is an isomorphism, hence, also, $\tilde{t}^{-1}(1_P)$ is a global section of $\partial_*(E)$. ■

We can use this to get a separate description of the category of M-torsors, which incidentally justifies the choice of name 'relative M-torsors' as they are somehow 'relative to T_P in a controlled way. In this description a morphism of M-torsors is a C-torsor morphism, f, making

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "t"'] & & E' \arrow[dl, "t'"] \\ & T_P & \end{tikzcd}$$

commute. (Here f is a C-torsor map, but t and t' are maps over φ. This diagram thus 'lives' in the category of sheaves on B.)

We will categorify this description later replacing M by a lax gr-groupoid, and, in fact, in a particular case by $\mathsf{M}-Tors$ itself, but all that requires stacks for a thorough handling, so must wait.

6.6.3 Examples and special cases

Right at the start of our discussion of crossed modules, in section 1.1, we gave various different examples. One was the $(G, Aut(G), \partial)$ case, where ∂ sending g to the inner automorphism determined by g. Others were normal subgroups and P-modules. We based the definition of (relative) M-torsor on that of G-bitorsor and thus on the first of these. What about the others?

(i) To take an almost silly example, let $\mathsf{M} = (1, P, inc)$, that is, the case $C = 1$. If $\mathcal{C}$ is our open cover, then the cocycle description of M-torsors gives us a family of local sections of P, say, $u_i : U_i \to P$, satisfying $p_i = p_j$ on intersections, $U_i \cap U_j$, but that means that the family glues to a global section of P. Conversely any global section of P gives a morphism from $N(\mathcal{U})$ to M. (We leave to the reader the examination of how this corresponds to a 1-torsor that yields a trivial P-torsor on application of ∂_*.) Thus in this case, M-torsors are just global sections of P and $\check{H}^0(B, \mathsf{M}) \cong \check{H}^0(B, P)$. (There is no question of coboundaries or equivalent cocycles as there is nothing above dimension 0 in M.)

(ii) The other extreme case is when C is Abelian and P is trivial. (We will sometimes write this as $C[1] = (C \to 1)$. It is a 'suspended' or 'shifted' form of C.) Here we just have a C-torsor E, and, of course $\partial_*(E)$ is a 1-torsor! There is not much choice of trivialisation, so we just have that C-torsor. In this case, we have $\check{H}^0(B, \mathsf{M}) \cong \check{H}^1(B, C)$, that is, cohomology in the old sense of Abelian cohomology.

(iii) The next obvious case is 'inclusion crossed modules' or 'normal subgroup pairs'. In other words, suppose N is a normal subgroup of P and M is the corresponding crossed module. (We write ∂ for the inclusion of N into P.) We would expect that, writing G for P/N, an M-torsor would be more or less the same, up to equivalence perhaps, as a $(1 \to G)$-torsor, i.e., to a global section of G. The conditions on the local sections p_i over some cover $\mathcal{U}$, and the corresponding n_{ij} are now

$$p_i = n_{ij}p_j,$$

as well as $n_{kj}n_{ji} = n_{ki}$.

Remark: There is a morphism of crossed modules with kernel $(N, N, =)$ giving a short exact sequence,

$$\begin{tikzcd} N \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 1 \arrow[d] \\ N \arrow[r] & P \arrow[r, "\varphi"] & G \end{tikzcd}$$

we know that this will give a short exact sequence of simplicial groups and that M-torsors correspond to maps from $N(\mathcal{U})$ to $K(\mathsf{M})$ if they trivialise over the open cover $\mathcal{U}$. Our observation that M-torsors might lead to global sections of G relates to composition with the quotient map φ from M to $(1, G, inc)$. (This raises the question of maps of crossed modules inducing functors between the corresponding categories of torsors, in general. We will return to this shortly.)

Looking in more detail, suppose we have a M-torsor specified by a cocycle pair (p_i, n_{ij}) over some open cover $\mathcal{U}$, and we write g_i for $\varphi(p_i)$, then the g_is *do* form a global section of G, since they are compatible over the intersections. Conversely, given a global section g of G, we know that φ is an epimorphism of sheaves, so would like to lift g to something in P. This situation is one we have encountered before and will do so again later. An epimorphism of sheaves need not be an epimorphism of the underlying presheaves. In our spatial context, it will be an epimorphism on stalks, however. We thus do not know if there is a global section p of P satisfying $\varphi(p) = g$, but, thinking about the idea of stalk, for any $b \in B$, and any open set U containing b, there is a representative (g_U, U) of the element $g_b = g(b)$, which is in the stalk over b. As φ is an epimorphism on stalks, we can choose U such that there is a $p_u \in P(U)$ with $\varphi_U(p_U) = g_U$. This gives us an open cover $\mathcal{U}$ of B and a family of local section of P over $\mathcal{U}$. Next look at the intersections, $U \cap V$, of sets from $\mathcal{U}$. There the restrictions of p_U and p_V need not agree, but as their images are the same under φ, there is a $n_{U,V}$ in N over $U \cap V$, which satisfies $p_U = n_{U,V}p_V$, and the family of these ns satisfy the cocycle condition, so from our global section of G, we have constructed a cocycle pair for an M-torsor. Different liftings of g give local sections that agree up to a coboundary, n_u, (possibly on a joint refinement of the covers), so M-torsors do give global sections of G, and *vice versa.*

(iv) The last case is $\mathsf{M} = (M, G, 0)$, i.e., M is a sheaf of G-modules. Here we have that cocycle pairs, (g_i, m_{ij}), must satisfy

$$g_i = \partial(m_{ij})g_j,$$

but ∂ is trivial, so the g_is give a global section, whilst the m_{ij} give a M-torsor in the usual sense.

This example is good because it links M-torsors in this case with M-torsors *and* global sections, i.e., some sort of 'extension', $G(B) \to \mathsf{M}-Tors \to Tors(M)$, or perhaps in the other order? We have not analysed the effect of the action of G on M. Does this mean that we have some sort of 'G-equivariant' cohomology, or cohomology of the sheaf of groups G with coefficients in the G-module M, ... and what about the gr-category structure. The detailed examination of all the structures involved is interesting and useful to do, so is, once again, **left as an exercise**.

This class of examples is also very important as amongst the examples of this type are, of course, the G-bitorsors with G a sheaf of Abelian groups, since for such a G, we have that $\mathsf{Aut}(G)$ is of the form $(G, Aut(G), 0)$. The best known example is where G is $U(1)$ or, equivalently, $G\ell(1,\mathbb{C})$, the group of unit modulus complex numbers. We will return to this later.

The above discussion suggests some interesting areas to explore. Reaction of these M-torsors to 'change of M', short exact sequences of sheaves of crossed modules and their 'reflection' in the behaviour of the M-torsors, etc. One particular short exact sequence is

$$\begin{tikzcd} K \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & N \arrow[d] \\ 1 \arrow[r] & P \arrow[r] & P, \end{tikzcd}$$

where $K = Ker\,\partial$ and $N = Im\,\partial$. It suggests that $\mathsf{M} - Tors$ is an extension of $G(B)$ by a category of K-torsors for an Abelian group sheaf, K, somehow twisted by the G-action. After examining one or two related subjects, we will be able to give a bit more insight and precision about this idea.

6.6.4 Change of crossed module bundle for 'bitorsors'.

We now have a very thorough knowledge of G-bitorsors and the more general (relative) M-torsors, via the link with simplicial maps from $N(\mathcal{U})$ to $K(\mathsf{M})$, but, of course, that link makes change of 'coefficients' more or less obvious.

First it should be noted, once again that the identification of $\check{H}^0(B, \mathsf{Aut}(G))$ as a second non-Abelian cohomology group of B with coefficients in G, runs foul of non-functoriality in G, but that this is not due to some subtle deep property of non-Abelian cohomology, rather it is due to the banal failure of $Aut(G)$ to be functorial in G, in other words, to a low level group theoretic fact, low level but in fact fundamental. It is here group theoretic, but generally automorphism groups do not vary functorially - and that opens the way to crossed modules.

If $\varphi : G \to H$ is a morphism of group bundles, then there may, or may not, be a morphism $\varphi' : Aut(G) \to Aut(H)$ such that

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "i"'] & H \arrow[d, "i"] \\ Aut(G) \arrow[r, "\varphi'"'] & Aut(H) \end{tikzcd}$$

is a morphism of crossed modules.

There is an induced morphism on $\check{H}^0(B, \mathsf{Aut}(G))$ if such a φ' does exist, and, of course, in more generality, if we have that $\varphi : \mathsf{M} \to \mathsf{N}$ is a morphism of crossed modules, then there is an induced homomorphism of groups

$$\varphi_* : \check{H}^0(B, \mathsf{M}) \to \check{H}^0(B, \mathsf{N}).$$

(It could happen that two crossed modules of the form $\mathsf{Aut}(G)$ could be linked by a zig-zag of other crossed modules so that the morphisms in the reverse direction were weak equivalences / quasi-isomorphisms in our earlier sense, and then there would be an induced map between the two $\check{H}^0(B, \mathsf{Aut}(G))$ groups. We will explore this more fully later on, using the beautiful theory of 'butterflies' as developed by Noohi, [174, 175].)

Exploring the above at a gr-groupoid level, i.e., on $\mathsf{M}-Tors$ with contracted product, rather than at connected component / cohomology level, we get an induced gr-functor between $\mathsf{M}-Tors$ and $\mathsf{N}-Tors$, since it uses the functor K from crossed modules to simplicial groups. Explicitly $\varphi : \mathsf{M} \to \mathsf{N}$ induces $K(\varphi) : K(\mathsf{M}) \to K(\mathsf{N})$, a morphism of simplicial groups, but then our identification of the contracted product structure on $\mathsf{M}-Tors$ as being induced from the simplicial group structure of $K(\mathsf{M})$ immediately implies that $K(\varphi)$ induces a functor from M–$Tors$ to N–$Tors$ compatibly with the gr-groupoid structures.

6.6.5 Representations of crossed modules.

In the classical group based case, the naturally occurring vector bundles such as the tangent and normal bundles had the general linear group of some dimension as the basic G over which one worked. Extra structure corresponded to restricting to a subgroup or lifting to some 'covering group'. We recalled earlier, e.g., page 242, that the fibres of the bundles were vector spaces with an action of the chosen group, i.e., a matrix representation of that group. What is, or should be, the representation theory 'of crossed modules'? There are several tentative answers.

A representation of a (discrete) group G and thus an action of G on some object, can be thought of in different ways. For instance, as a group homomorphism $G \to H$, where H is some group of permutations or matrices in which we can use methods from outside group theory, perhaps combinatorics, perhaps linear algebra, to analyse more deeply the properties of the elements of G. We could also consider this as a functor from $G[1]$, the corresponding groupoid with one object, to *Sets* for the permutation representations, or to some category of vector spaces or modules in the linear case.

The generalisations are to 'categorify' this second description by taking $\mathcal{X}(\mathsf{M})$, the 2-groupoid with one object (i.e., the 2-group) of M, and looking for a nice category of '2-vector spaces' or '2-modules'. (The permutation version has not been that well explored yet, but we will see some ideas later on.) Some doubt exists as to what is the 'best' category of '2-vector spaces' to use, in fact the discussion is really about what that term should mean. We mention two possibilities here, but there are others and we will look at them later. The first is due independently to Forrester-Barker, [99], and to Baez and Crans, [12]. The second is based on an idea of Kapranov and Voevodsky, [133], using more monoidal category theory than we have been assuming so far.

Here we will adopt the simpler version, more as an illustration then as a claim that this is the 'correct' version. The motivation for the definition, used by Forrester-Barker and by Baez and Crans, is that, as crossed modules are category objects in the category of groups, for a linear representation theory of such things, it is natural to try category objects in the category of vector spaces, but such objects are equivalent to short complexes of vector spaces. The idea is also that some of the potential applications of the structures that we have been studying use chain complexes as coefficients. (We will see this more clearly in the later discussion of hyper-cohomology.) Keeping things simple, we look at chain complexes of vector spaces (or more generally of modules) of length 1. (Warning: for us here 'length 1' means *one morphism*, $C_1 \to C_0$, not 'one group' so our objects are linear transformation between vector spaces and our morphisms are commutative squares.) These are highly Abelian versions of crossed modules, so we will use similar notation such as C, D, etc., for them.)

We recall that chain complexes have a natural 'internal hom' construction, well known from classical homological algebra. (We will see this again in our discussion of hyper-cohomology so will treat it in more detail there.) The chain complex, $Ch(\mathsf{C}, \mathsf{D})$, has graded maps of degree n in dimension n, so, for instance, has chain homotopies in dimension 1. Putting $\mathsf{D} = \mathsf{C}$ and looking at the invertible maps gives an automorphism group, $\mathsf{Aut}(\mathsf{C})$, which is also a chain complex of groups, i.e.,we get a crossed module. If we have a general (discrete) crossed module M, we can consider a morphism $\mathsf{M} \to \mathsf{Aut}(\mathsf{C})$ as a representation of M, and can talk of M acting on C by 'linear maps'. We will not explore this further here, but note that we are very near the idea of representing a simplicial group as a simplicial group of simplicial automorphisms, somewhat as in section 5.3. At present, the available discussions of 2-group representations of this form include the thesis, [99], and papers, [12]. A more extensive use of monoidal category theory would allow us to consider a variant that considers 2-vector spaces to mean the 2-categorical version of the monoidal category of vector spaces. We will return to this later.

Chapter 7

Categorifying G-torsors and M-torsors

For G a sheaf of groups on B, we had the notion of a G-torsor. Although we should now know this very well, let us briefly recall that this was a sheaf, E, on B with a (left) G-action satisfying various properties. We also saw G-bitorsors and found they were a special case of a neat notion of M-torsor for $\mathsf{M} = (C, P, \partial)$, a sheaf of crossed modules. Again, recall an M-torsor was a pair, (E, t), with E a C-torsor and t, a trivialisation of $\partial_*(E)$, and that when M was $(G, Aut(G), i)$, an M-torsor was just a G-bitorsor.

Let's categorify! Our treatment of 'categorification' has been quite relaxed. We have thought of replacing sets by categories or groupoids, functions or morphisms by functors and adding an extra layer of natural transformations. This led to equality becoming suspect and being pushed aside in favour of isomorphism, which then categorified to equivalence.

Passing to the sheaf-like situation, perhaps over a space, B, or in some topos $\mathcal{E}$, sets became sheaves and categorified to stacks and gerbes - so what does G-torsor categorify to? This is not simply a fun question, as our understanding - what little we have of it as yet - is that categorification somehow corresponds to going up a level in cocycle descriptions of things. This then suggests that categorifying G-torsors might shed some other light on interpreting the next level of cohomology that we have been looking at.

The argument that crossed modules are (one of) the 'best' categorifications of groups, as they model (connected) 2-types, whilst groups model (connected) 1-types, amongst other things, applies equally to the equivalent strict gr-groupoids. This is a good point to review gr-groupoids from earlier in the book. If (C, P, ∂) was a crossed module, then the associated strict gr-groupoid was

$$C \rtimes P \underset{t}{\overset{s}{\rightrightarrows}} P.$$

Remember it from page 25. We also saw various non-strict gr-groupoids, for instance, $Bitors(G)$ with contracted product was a monoidal category as was $\mathsf{M}-Tors$ for a crossed module, M. These were (page **??**) not only gr-groupoids, but, as everything localised nicely, *gr-stacks*. Now *there* is a nice categorification of sheaf of groups! It categorifies 'group' to 'gr-groupoid' *and* 'sheaf' to 'stack', *and*, as a bonus, we have several natural examples of such things for the cohomological context. (As it is not that long since these examples were discussed, we will not 're-give' the definition of gr-stack here.)

The suggested categorification of G-torsors would therefore involve a gr-stack acting on a stack (of groupoids) satisfying some axioms, so we start with the first part of such a structure:

7.1 Torsors for a gr-stack

Definition: Let G be a gr-stack on B (or on a site, $\mathcal{E}$,) with a multiplication, denoted $C_1{\cdot}C_2$ when need be, but usually just by juxtaposition, a unit object, denoted I, and an associator (which will not usually be ascribed an explicit notation).

An *action of* G on a stack, Q, of groupoids is a morphism of stacks (on B)

$$\mathsf{m} : \mathsf{G} \times \mathsf{Q} \to \mathsf{Q}$$

together with a natural transformation, μ, the *action associator* filling the square

$$\begin{tikzcd}
\mathsf{G} \times \mathsf{G} \times \mathsf{Q} \arrow[r, "\mathsf{m}\times 1"] \arrow[d, "1\times\mathsf{m}"'] \arrow[dr, phantom, "\not\Downarrow\mu"] & \mathsf{G} \times \mathsf{Q} \arrow[d, "\mathsf{m}"] \\
\mathsf{G} \times \mathsf{Q} \arrow[r, "\mathsf{m}"'] & \mathsf{Q}
\end{tikzcd}$$

(so $\mu : \mathsf{m} \circ (\mathsf{m} \times 1) \Rightarrow \mathsf{m} \circ (1 \times \mathsf{m})$) and a unit transformation, $u_D : I{\cdot}D \to D$, at a local object D, satisfying some axioms; see below. (We will also write, for C, an object of G_U, and for D, an object of Q_U, $\mathsf{m}(C, D) = C{\cdot}D$, or just use concatenation if no confusion will arise. If C_1, C_2 are such (local) objects of G and D of Q, then μ will, of course, be given by morphisms

$$\mu_{C_1,C_2,D} : (C_1{\cdot}C_2){\cdot}D \to C_1{\cdot}(C_2{\cdot}D),$$

in Q, and these are *isomorphisms* because we have assumed Q is a stack of *groupoids*. We thus get no bother from the direction specified for μ.) The action associator and unit are required to satisfy some coherence conditions:

(i) **Pentagon condition:** For any object (C_1, C_2, C_3, D) of $\mathsf{G}^3 \times \mathsf{D}$, the following pentagonal diagram commutes

$$\begin{tikzcd}
 & ((C_1C_2)C_3)D \arrow[r] \arrow[dl] & (C_1(C_2C_3))D \arrow[dr] & \\
(C_1C_2)(C_3D) \arrow[drr] & & & C_1((C_2C_3)D) \arrow[dl] \\
 & & C_1(C_2(C_3D)) &
\end{tikzcd}$$

in which each arrow is constructed from the action associator, μ, or the associator for the monoidal structure of G in the obvious way (generalising the pentagonal diagram for monoidal structures);

(ii) **Unit condition:** For any local object (C, D) of $\mathsf{G} \times \mathsf{Q}$, the triangles

$$\begin{tikzcd}
I(CD) \arrow[r, "\mu"] \arrow[dr] & (IC)D \arrow[d] \\
 & CD
\end{tikzcd}$$

and

$$\begin{tikzcd} C(ID) \arrow[rr, "\mu"] \arrow[dr] & & (CI)D \arrow[dl] \\ & CD & \end{tikzcd}$$

commute, in which each sloping arrow is constructed from the unit transformation, u, of the action or the right or left unit isomorphisms of the monoidal structure of G.

That takes care of the 'categorified' action, but, in a torsor, one has additional conditions (cf. page 268). The action $\mathsf{m} : \mathsf{G} \times \mathsf{Q} \to \mathsf{Q}$ induces a morphism of stacks

$$(\mathsf{m}, \mathsf{pr}_2) : \mathsf{G} \times \mathsf{Q} \to \mathsf{Q} \times \mathsf{Q}$$

completely analogous to the morphism of sheaves

$$(m, pr_2) : G \times P \to P \times P$$

used in section 6.4.1 when we were defining a G-torsor. There we required this map to be an isomorphism. Following our rules of thumb for categorifying concepts, we should require the corresponding morphism of stacks to be an equivalence of stacks (on B or on a site $\mathcal{E}$, whichever is the context required). Breen, [31], p. 439, gives the corresponding concept the term 'pseudo-torsor'. More formally, suppose that G is a gr-stack, Q a stack (of groupoids) and $\mathsf{m} : \mathsf{G} \times \mathsf{Q} \to \mathsf{Q}$, an action of G on Q.

Definition: Given the above, we say Q is a *left pseudo-torsor* for G if the morphism of stacks

$$(\mathsf{m}, \mathsf{pr}_2) : \mathsf{G} \times \mathsf{Q} \to \mathsf{Q} \times \mathsf{Q}$$

is an equivalence.

The definition of torsor had yet one more requirement. To be able to get a hold on the structure, one needed enough local sections, i.e., for the spatial case, we needed an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of B such that $P(U_i)$ was a non-empty set for each $i \in I$.

Definition: If Q is a left pseudo-torsor for G, we say it is a G-*torsor* if there is an open cover, $\mathcal{U} = \{U_i\}_{i \in I}$,of B such that each of the groupoids $\mathsf{Q}(U_i)$ is non-empty, i.e., if Q is a locally non-empty stack in our earlier terminology.

Remark: Breen, in [31], mostly considers the case of a site $\mathcal{E}$, so the condition above takes on a slightly different form. There has to be a covering, R, of the terminal object of $\mathcal{E}$ such that for each object S of R, the fibre groupoid Q_S over S is non-empty; see Breen, [31], p. 439 for more details.

Before we look for examples and elementary properties of these G-torsors, we should also define morphisms between them. Morphisms between torsors were essential for the study of their properties and equivalences were the key to the link with cohomology. Clearly we need morphisms of G-torsors as well.. Such a morphism should be a morphism of the stacks concerned, but should also be 'compatible with the G-action'. You can probably guess the form that this compatibility takes.

Definition: Suppose Q and Q' are G-torsors. A *morphism of* G*-torsors* from Q to Q' consists of a pair (p, q), where $\mathsf{p} : \mathsf{Q} \to \mathsf{Q}'$ is a morphism of stacks on B, and q is a natural transformation

$$\begin{tikzcd}
\mathsf{G} \times \mathsf{Q} \arrow[r, "1\times \mathsf{p}"] \arrow[d, "\mathsf{m}"'] \arrow[dr, phantom, "\not\Downarrow q"] & \mathsf{G} \times \mathsf{Q}' \arrow[d, "\mathsf{m}'"] \\
\mathsf{Q} \arrow[r, "\mathsf{p}"'] & \mathsf{Q}'
\end{tikzcd}$$

between the two actions, thus, for C in G and D in Q, two local objects, we have a morphism $q_{C,D} : Cp(D) \to p(CD)$. This data is to satisfy a compatibility condition that, for C_1, C_2 local objects in G and D, one in Q, the diagram

$$\begin{tikzcd}
& (C_1C_2)p(D) \arrow[r, "\mu"] \arrow[dl] & C_1(C_2p(D)) \arrow[dr] & \\
p((C_1C_2)D) \arrow[drr, "p(\mu)"] & & & C_1p(C_2D) \arrow[dl] \\
& & p(C_1(C_2D)) &
\end{tikzcd}$$

commutes, where the unlabelled arrows are induced from q in a fairly obvious way.

As always, after a bunch of new concepts, it is a good idea to give an example or two.

Example: Let G be a gr-stack. The multiplication of G gives a left action of G on the stack G, i.e., on the underlying stack of the gr-stack. The associator of G is the action associator of that action with the pentagon condition just the usual pentagon for the monoidal category structure, and the unit being the left unit arrow of that structure. This action makes G into a pseudo-torsor over itself via the inverter for the gr-stack structure:

$$(\mathsf{m}, \mathsf{pr}_2) : (C_1, C_2) \mapsto (C_1C_2, C_2),$$

so, given (C, C_2), we send it to (CC_2^{-1}, C_2), where, of course, C_2^{-1} is the result of inverting C_2. (For instance, if G is the gr-stack of G-bitorsors for a sheaf of groups, G, then C_2^{-1} is merely the opposite bitorsor C_2^o of C_2 in the notation of page 283.)

One of the axioms for a gr-groupoid, or for a gr-stack, for that matter, is the existence of a unit object, I, and, in the case of gr-stacks on B or on a site $\mathcal{E}$, that must be a *global object.* We thus have that the underling stack of G is a G-torsor. Of course, it is called the *trivial* G*-torsor* and will be denoted T_{G}.

You will not be surprised at the following result.

Lemma 53 *Suppose* Q *is a* G*-torsor, for some gr-stack,* G*. If the stack* Q *has a global object, i.e.,* $\mathsf{Q}(B)$ *is a non-empty groupoid, then* Q *is equivalent to* T_{G} *as a* G*-torsor.*

This lemma, of course, is the categorified analogue of Lemma 45, (page 268). Before we prove it, we will reprove that earlier lemma, using essentially the same proof, but doing it slightly more abstractly and categorically within $Sh(B)$ (and thus effectively doing it in an arbitrary topos). This will make the proof of the current lemma more or less obvious.

Back on page 268, the input was a G-torsor, P, and we used $P \xrightarrow{\pi} B$, the projection of the étale space corresponding to the sheaf P. Of course, this is a continuous map and we need to consider it as, or perhaps replace it by, a map within the category $Sh(B)$. This is easy. The terminal object in $Sh(B)$ is the one point sheaf, corresponding to the étale space $B \xrightarrow{=} B$, so $P \xrightarrow{\pi} B$, within $Sh(B)$, is also $P \to 1$, where 1 is that terminal object. This is great! The lemma assumes the existence of a global section for P, i.e. an element, s in $P(B)$, but that is just a morphism of sheaves from 1 to P, $s : 1 \to P$, since 1 obviously has a unique global section!

We will change notation slightly to emphasise that 1 is terminal. If X is any sheaf on B, $t_X : X \to 1$ will be the unique map of sheaves to the terminal object. We will sometimes get lazy and just write t, whatever the domain is. We also note that in the notation within $Sh(B)$, $P \times_B P$ is just $P \times P$.

Reproof of Lemma 45: The condition that P is a G-torsor is that the map

$$\phi = (m, pr_2) : G \times P \to P \times P$$

is an isomorphism. It has an inverse

$$\theta : P \times P \to G \times P$$

and we write $\tau : P \times P \to G$ for the *translation morphism*, $\tau = pr_1 \circ \theta$ (as in our discussion of principal G-bundles, page 242 and for the corresponding idea for a torsor, page 268). Thus $\theta = (\tau, pr_2)$.

We assume $s : 1 \to P$ is a global section of P. We define a morphism

$$f : G \to P$$

using s, as follows: f is the composite

$$G \xrightarrow{\cong} G \times 1 \xrightarrow{G\times s} G \times P \xrightarrow{m} P\ ,$$

where the first arrow is the canonical isomorphism between G and $G \times 1$, i.e. its two components are the identity morphism and the morphism t_G. The essence of the proof is showing that f is an isomorphism by constructing the inverse of f.

The construction of this inverse, f', for f goes as follows:

$$f' = (\ P \xrightarrow{\cong} P \times 1 \xrightarrow{P\times s} P \times P \xrightarrow{\theta} G \times P \xrightarrow{pr_1} G).$$

Of course, $f' = \tau(id, s)$, since the first component of θ is τ as we noted above.

Now consider the diagram

$$P \xrightarrow{(id,st)} P \times P \xrightarrow{\theta} G \times P \underset{(pr_1, st_G pr_1)}{\overset{id}{\rightrightarrows}} G \times P$$

and note (i.e. for you to check) that the two composites are equal. We now have that ff' is the composite: $P \xrightarrow{(id,st)} P \times P \xrightarrow{\theta} G \times P \xrightarrow{pr_1} G \xrightarrow{\cong} G \times 1 \xrightarrow{G \times s} G \times P \xrightarrow{m} P$

$= P \xrightarrow{(id,st)} P \times P \xrightarrow{\theta} G \times P \xrightarrow{(pr_1, st.pr_1)} G \times P \xrightarrow{m} P$

$= P \xrightarrow{(id,st)} P \times P \xrightarrow{\theta} G \times P \xrightarrow{(pr_1, st.pr_1)} G \times P \xrightarrow{\phi} P \times P \xrightarrow{pr_1} P,$ by the above argument

$= P \xrightarrow{(id,st)} P \times P \xrightarrow{\theta} G \times P \xrightarrow{\phi} P \times P \xrightarrow{pr_1} P$

$= P \xrightarrow{(id,st)} P \times P \xrightarrow{pr_1} P,$ since $\phi\theta = id_{P \times P}$,

$= P \xrightarrow{=} P,$

as hoped for.

For the other composite, $f'f$, you need to prove a similar preliminary result on two composites. This is then followed by an almost identical argument - so will be **left to you to do**. ■

Proof of Lemma 53: The proof of the previous lemma is now easy. It follows exactly the same form as the above except that θ and ϕ are now equivalences of stacks, so $\phi\theta \overset{\cong}{\Rightarrow} Id_{\mathsf{Q}\times\mathsf{Q}}$ and similarly $\theta\phi \overset{\cong}{\Rightarrow} Id_{\mathsf{G}\times\mathsf{Q}}$, so $f'f \cong Id_{\mathsf{Q}}$ and $ff' \cong Id_{\mathsf{G}}$ as a result. ■

It almost goes without saying that a quick check by the reader of this is 'advisable', as, in fact, there are some complications, for instance, the analogue of θ does not have quite as nice a description in the categorified case. With a bit of extra reflection, you may guess that similar results that are even more 'categorified' could potentially be proved in like manner, perhaps with sheaves of simplicial groups, G and a 'hyper-torsor' condition requiring homotopy equivalence or better some notion of weak equivalence. (For an exploration of aspects of this, see early work by Duskin, [83], and Glenn, [104], followed by more recent extensive work by Cegarra and others in the Granada homological algebra group. Note in particular that many of the results on modelling n-types that we looked at earlier, have analogues in this setting.)

We can now look at an arbitrary G-torsor, Q and try to 'adapt', i.e., categorify, the analysis that we did for a G-torsor. (As the terminology can get confusing, it is sometimes convenient to use 1-torsor or G-1-torsor for "stack with action by a gr-stack" and 0-torsor or G-0-torsor for the earlier, more classical type of object.) We have the condition that there is an open cover $\mathcal{U}$ of B such that for each $U \in \mathcal{U}$, $\mathsf{Q}(U)$ is a non-empty groupoid, having a (global) object, X or X_U, if we need a bit more precision. We can pullback Q and G along the inclusion, thus restricting them to U and then the resulting Q_U is a G_U-torsor as can easily be checked. We thus have that Q_U is equivalent to $\mathsf{T}_{\mathsf{G}_U}$, the trivial G_U-torsor. Of course, an equivalence is given by

$$\begin{array}{rcl} \chi_U : \mathsf{T}_{\mathsf{G}_U} & \to & \mathsf{Q}_U \\ C & \mapsto & C.X_U, \end{array}$$

where X_i is the global object of Q_{U_i}.

Remark: The family of all these χ_i can be either handled as is, or can be thought of as an equivalence

$$\chi : \mathsf{T}_{G_U} \to \mathsf{Q}_U$$

between the 'pullbacks' of T_G and Q along the covering map $U \to B$, where $U = \coprod U_i$, or, in the notation, we have used several times $U \to 1$. Recall also that $U \times U$ corresponds to the family $\{U_{ij}\}$ of intersections. If you are in a context where hypercoverings are needed, the simplicial sheaf

$$\equiv\!\!\!\Rightarrow U \times U \rightrightarrows U$$

will need replacing by a hypercovering.

From a practical point of view, the indices involved in multiple intersections U_{1234}, etc., makes the simplicial based version of hypercovering style notation very attractive - as it avoids all that.

If $U_{ij} = U_i \cap U_j$, then we have two pullbacks of Q to U_{ij}, one via U_i, the other via U_j. (There is also a direct one along the inclusion of U_{ij} into B.) As we are considering stacks, the pullbacks are only defined up to equivalence. As usual explicit constructions of pullback can be made that can strengthen this to 'isomorphism'. In fact, either will do equally well.

We denote by $p_1 : U_{ij} \to U_i$, $p_2 : U_{ij} \to U_j$, since this notation, p, is clearly the right one in the topos, $Sh(B)$, since the intersections are parts of $U \times U$, where again $U = \coprod U_i$ as before and the ps *are* projections.

On U_{ij}, or, if you prefer on $U \times U$, there is a natural isomorphism, $\{f_{U_{ij}} : p_1^*(\mathsf{Q}_{U_{ij}}) \cong p_2^*(\mathsf{Q}_{U_{ij}})\}$ of $G_{U_{ij}}$-torsors ($G_{U\times U}$-torsors). We can pick a quasi-inverse for the family χ and thus get an auto-equivalence

$$\varphi_{12} := \chi^{-1} f_{U_{ij}} \chi : G_{U_{ij}} \to G_{U_{ij}},$$

which we can also write as

$$\varphi := \chi^{-1} f \chi : G_{U\times U} \to G_{U\times U}.$$

If we are on the diagonal, the intersection is equal to both, so p_1 and p_2 are the identity morphism and f is the identity as well, hence we have that

$$\begin{tikzcd} G_{|U} \arrow[r, "Id"] & G_{|U} \\ G_{|U\times U} \arrow[u, "\Delta"] \arrow[r, "\varphi"] & G_{|U\times U} \arrow[u, "\Delta"'] \end{tikzcd}$$

commutes.

If we write $p_{12} : U \times U \times U \to U \times U$, for the $(1,2)$-projection, (so we think $p_{12}(x, y, z) = (x, y)$) and similarly for p_{13} and p_{23}, then we have some 'simplicial'-like relations, for instance,

$$\begin{aligned} p_1 p_{12} &= p_1 p_{13} \\ p_1 p_{23} &= p_2 p_{12}, \end{aligned}$$

and so on. (These *are* projections and this *is* a good notation, *but* why 'simplicial'? Simply because, if we replace (x, y, z) by (x_0, x_1, x_2), the simplicial operator d_0 is given by $d_0(x_0, x_1, x_2) = (x_1, x_2)$, so *is* p_{23}. we thus are in the situation described by the simplicial descent setting that we looked at earlier (page **??**).)

If we look at $f : p_1^*(\mathsf{Q}_U) \cong p_2^*(\mathsf{Q}_U)$, and then restrict back to $U \times U \times U$, we get isomorphisms such as

$$f_{12} = p_{12}^*(f) : p_{12}^* p_1^*(\mathsf{Q}_U) \xrightarrow{\cong} p_{12}^* p_2^*(\mathsf{Q}_U)$$

and there will be a natural transformation between the corresponding automorphisms $\varphi_{ij} = p_{ij}^*(\varphi)$ of the trivial G_{U^3}-torsor:

$$\begin{tikzcd} G_{U^3} \arrow[r, "\varphi_{12}"] \arrow[dr, "\varphi_{13}"'] & G_{U^3} \arrow[d, "\varphi_{23}"] \arrow[dl, phantom, "\Swarrow\psi"] \\ & G_{U^3} \end{tikzcd}$$

(To be continued)

Chapter 8

Topological (Quantum) Field Theories

(As a basic reference for the initial parts of this chapter, you might look at Joachim Kock's book, [140], or Quinn's introductory lectures, [188].)

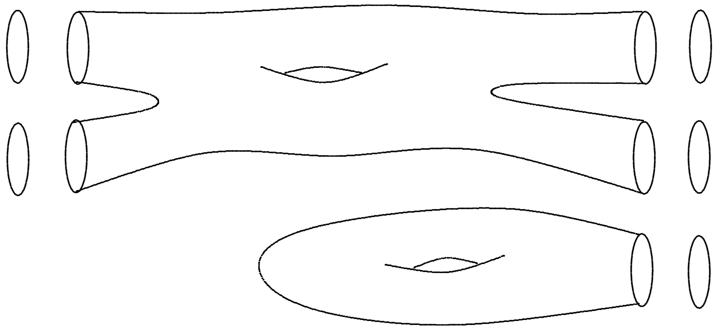
8.1 What is a topological quantum field theory?

Topological (Quantum) Field Theories form a relatively new area of mathematics, somewhere near the 'frontier', always a fuzzy one, between mathematics, and mathematical physics. To some extent it studies 'space-times', and uses them to look for possibly new invariants and properties of manifolds. It has interesting interactions with cohomology theory and we will look at some of these, (almost universally we will use the abbreviation 'TQFT' for 'topological quantum field theory'.)

There are good reasons to use the term 'Topological Field Theory' and the abbreviation 'TFT' instead of inserting the 'quantum' and the Q, and, in the literature, you will often see that use. Here we will, in general, use the TQFT form rather than the TFT, (and later HQFT rather than HFT), as, at the time of writing this, the Q form is slightly more common, still. (This may change, in which case I may have to do a search and replace on the source code of these notes!)

8.1.1 What is a TQFT?

In Topological Quantum Field Theory, one studies $(d-1)$-dimensional orientable smooth or piecewise linear manifolds and the d-dimensional (orientable) cobordisms between them, pictured, for $d=2$, as:



$X = S^1 \sqcup S^1$ $\quad$ $M : X \longrightarrow Y$ $\quad$ $Y = S^1 \sqcup S^1 \sqcup S^1$
but we may write $Y = S^1 \otimes S^1 \otimes S^1$,

For later use, we record the following:

Definition: A *d-cobordism*, $W : X_0 \to X_1$, is a compact oriented d-manifold, W, whose boundary is the disjoint union of pointed closed oriented $(d-1)$-manifolds, X_0 and X_1, such that the orientation of X_1 (resp. X_0) is induced by that on W (resp., is opposite to the one induced from that on W).

It is usual to call X_0, the *incoming* manifold and X_1 the *outgoing* one, as well as referring to them as the domain and codomain of the morphisms that the cobordism will give as follows:

After some technical difficulties, one shows these manifolds and cobordisms form a category, $d{-}Cob$, with $(d{-}1)$-manifolds as its objects and, more-or-less, the d-cobordisms as the morphisms. (The 'more-or-less' is that as far as their being morphisms is concerned, we have to consider two cobordisms to be 'the same' if they are isomorphic relative to the boundary, i.e., there is an isomorphism between them that preserves the boundaries. This is examined in more detail in the sources that have been indicated for the basic theory.) If and when it is necessary, we will add a suffix PL, $Diff$ or Top to distinguish the cases in which the manifolds are to be piecewise linear (PL), smooth ($Diff$) or merely topological.) This category has a monoidal category structure given by disjoint union, $\sqcup$, but which will often be written as a tensor, $\otimes$. The unit of the structure is the empty manifold, $\emptyset$. We thus formally have $d{-}Cob = (d{-}Cob, \sqcup, \emptyset)$, with the convention that we used in section **??**, page **??**.

We will often use the case $d = 2$ as an illustrative example. For instance, in the above picture, $M = M_1 \otimes M_2$, where $M_1 : X \to Y_1 = S^1 \otimes S^1$, $M_2 : \emptyset \to Y_2 = S^1$ and $Y = Y_1 \otimes Y_2$.

We will also need the monoidal category, $Vect^{\otimes} = (Vect_{\Bbbk}, \otimes, \Bbbk)$, of (finite dimensional) complex vector spaces with the usual tensor product. (We could use fields, $\Bbbk$, other than the usual one, $\mathbb{C}$, of complex numbers and the minimal one for things to be fairly simple, $\mathbb{Q}$. We may even use a commutative ring, R, with some restriction on the characteristic, although characteristic zero will always work.)

Definition: A TQFT is a monoidal functor, $Z : d - Cob \to Vect^{\otimes}$, or, more generally, to $R - Mod^{\otimes}$, so Z preserves $\otimes$ and $Z(\emptyset) = \mathbb{C}$, resp. R. For an object, X of $d - Cob$, $Z(X)$ is sometimes called the *state space* or *state module* of X.

Terminology: There is some disagreement as to terminology when considering $d-Cob$, and, as we will try to stay relatively close to sources, we will hit this full on! There are two basic conventions. In one the key dimension is that of the manifolds, whilst in the other it is that of the cobordisms. The above uses the second of these. (Later when discussing homotopy quantum field theories, the first convention tends to dominate the literature, so we will need to be careful.) There is also an intermediate situation in which one writes $(n+1)-Cob$, emphasising both dimensions, so $(2+1)-Cob$ is the monoidal category of 2-dimensional manifolds, and cobordisms that are 3 dimensional. In this convention, which is a very useful one, a $(2+1)$ dimensional TQFT is one defined on what we would denote as $3-Cob$.

We mentioned 'technical difficulties'. These relate mostly to composition of cobordisms and identification of identities. We will not go into the details, as this is well discussed in the main sources, using different ways of getting around the difficulties. The simplest way is to *think* of the morphisms as equivalence classes of cobordisms, under isomorphism (so diffeomorphism if in the smooth case), relative to the two ends. That gives sufficient detail to be going on with. We will discuss this some more later on, but the **reader is urged** to look at one or more of the sources to see the means used to get a monoidal category structure.

Definition: A *morphism*, $\varphi : Z \to Z'$, of TQFTs will be a monoidal (natural) transformation between them.

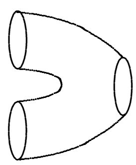
All such morphisms are, in fact, isomorphisms. (**You are left** to try to prove this or to look it up.)

Exploring briefly the simple case of $d = 1$, clearly any 1-manifold is a disjoint union, $X = (S^1)^{\otimes n}$, of n copies of S^1 for some $n \geq 0$, so $Z(X) = Z(S^1)^{\otimes n}$, and much of the structure of Z will be about the vector space $Z(S^1)$. This is not quite right. The point is that we have to take into account an orientation of the circle. Let us fix S^1 to have an anticlockwise orientation and write $-S^1$ for the opposite.

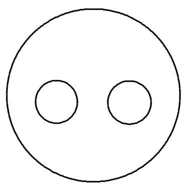
If we put $A = Z(S^1)$, we get an algebra structure on A given by a linear map

$$\mu : A \otimes A \to A,$$

that is, from $Z(S^1 \sqcup S^1)$ to $Z(S^1)$. (We will look at algebras in this way in more detail in the next section.) To get this we use the cobordism:

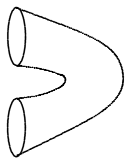


known also as the 'pair of pants'. It has a useful representation as a disc with two holes



with all three circles given an anti-clockwise orientation. The outer circle corresponds to the right hand 'output' end with two inner circles being 'inputs' at the left side of the previous picture.

The cobordism

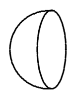


gives a bilinear form $A \otimes A \longrightarrow \mathbb{C}$, which is not hard to show is non-degenerate, so A must be finite dimensional (because the pairing gives an isomorphism between A and its dual). We thus do not really have to *impose* finite dimensionality on the vector spaces as, if they do form a TQFT, they will be finite dimensional. It also shows that $Z(-S^1) \cong A^*$, the dual space of A, since we can picture this cobordism as constructed from a cylinder, which is bent back on itself. This is quite general and does not just apply to this 1+1 dimensional case.

A lot of other structure can be visualised in similar ways. The algebra A has a unit

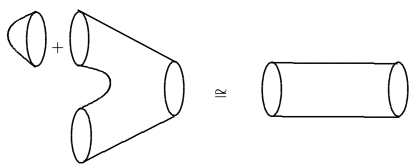
$$\mathbb{C} \to A$$

corresponding to



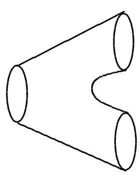
which is, of course a cobordism from the empty manifold to the circle.

The proof that this is a unit is simply:



i.e., $\mu(1, a) = a$, and so on.

We can read the diagrams 'in a mirror' to get a coalgebra structure, $A \to A \otimes A$, corresponding to



There are similarly a copairing, $\mathbb{C} \to A \otimes A$, and a counit, $A \to \mathbb{C}$, and these all satisfy a bunch of equations including a 'Frobenius equation' linking algebraic and coalgebraic structures that we will see shortly (in the next section).

Verification of axioms such as associativity for the algebra structure can all be done in a diagrammatic form. You compose the corresponding cobordisms, and they are evidently isomorphic / equivalent. As we will see, for the Frobenius equation is easier to understand in diagrammatic form than to write the equations.

What is the sort of algebra involved here. We explore this in the next section.

8.1.2 Frobenius algebras: an algebraic model of some of the structure

(These are discussed in some detail in Kock's book, [140], and occur, under a different name, in Quinn's notes, [188]. They are also discussed in a Wikipedia article, which you can safely be left to find and read. We will give a brief introduction to them strongly influenced by Joachim Kock's lectures at the Almería workshop on TQFTs.)

We will fix a field $\mathbb{k}$. We think of this usually as being $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Q}$, but others can be useful. As usual, certain situations may benefit from using a more general commutative ring, R, as a 'ground ring'.

We first look at algebras, as this introduces a way of thinking about algebraic structures in a monoidal category, here, $Vect^{\otimes}_{\mathbb{k}} = Vect^{\otimes} = (Vect_{\mathbb{k}}, \otimes, \mathbb{k})$, so the objects are vector spaces over $\mathbb{k}$, the 'multiplication' is the usual tensor product for which $\mathbb{k}$ is the unit. (We may sometimes omit the suffix $\mathbb{k}$ if it is not essential for the discussion.)

Definition: A $\mathbb{k}$-*algebra*, A, is a monoid in the monoidal category, $Vect^{\otimes}$.

Taking this apart, a monoid in the usual situation is a set, M, with a multiplication $\mu : M \times M \to M$, and a unit, satisfying associativity and unit axioms. Internalising this into a monoidal category we replace the product structure of Set with the $\otimes$ of the monoidal category, and, making several other slight adjustments,we have that A has a multiplication

$$\mu : A \otimes A \to A,$$

and a unit

$$\eta : \mathbb{k} \to A,$$

satisfying an associativity condition namely that

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes A"] \arrow[d, "A\otimes\mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd}$$

commutes,
and unit laws (left unit)

$$\begin{tikzcd} \mathbb{k} \otimes A \arrow[r, "\eta\otimes A"] \arrow[dr, "="'] & A \otimes A \arrow[d, "\mu"] \\ & A \end{tikzcd}$$

It is best if one requires also a right unit law, and, if we were being very strict with ourselves, we should allow for the fact that the monoidal category, $Vect^{\otimes}$, is not strict, but we will not handle this here.

Definition: A *Frobenius algebra* is a finite dimensional $\mathbb{k}$-algebra, A, equipped with a non-degenerate 'associative' pairing,

$$\beta : A \otimes A \to \mathbb{k}.$$

The pairing is sometimes called the *Frobenius form* of the algebra, A.

There are some terms here that need a bit more detail. First:

Definition: A pairing, as above, is said to be *associative* if for all $x, a, y \in A$, we have

$$\beta(x, ay) = \beta(xa, y).$$

There are two forms of non-degeneracy. Given a pairing, $\beta : V \otimes W \to \Bbbk$, we get an induced map

$$V \to W^* = Vect_{\Bbbk}(W, \Bbbk)$$

given by

$$v \mapsto \overline{v}$$

where $\overline{v}(w) = \beta(v \otimes w)$. There is a similarly defined one from W to V^*.

Definition: (i) A pairing, $\beta : V \otimes W \to \Bbbk$, is *weakly non-degenerate* if the induced maps, $V \to W^*$ and $W \to V^*$, are both injective.

(ii) A pairing, β, is *non-degenerate* if there is some $\gamma : \Bbbk \longrightarrow W \otimes V$ such that

$$V \xrightarrow{V\otimes\gamma} V \otimes W \otimes V \xrightarrow{\beta\otimes V} V$$

is the identity on V, whilst

$$W \xrightarrow{\gamma\otimes W} W \otimes V \otimes W \xrightarrow{W\otimes\beta} W$$

is that on W.

Here we are, once again, slightly abusing notation, since we should really write $V \xrightarrow{\cong} V \otimes \Bbbk \xrightarrow{V\otimes\gamma} V \otimes W \otimes V$, and so on, and even take note of the associativity isomorphisms $V \otimes (W \otimes V) \cong (V \otimes W) \otimes V$, as $Vect_{\Bbbk}^{\otimes}$ is not a strict monoidal category, however it usual to leave such details aside, unless strictly needed.

The weak form of degeneracy is equivalent to the strong one if V and W are finite dimensional.

Remark: There are two other forms of definition that can be given for Frobenius algebra. One is less categorical, the other is more so. We will look at the second of these in a bit of detail later, but we note that the less categorical one is sometimes very useful when verifying that an example algebra *is* a Frobenius algebra. It is as follows:

Proposition 79 *A $\Bbbk$-algebra, A, is Frobenius if and only if (i) it is finite dimensional and (ii) it has a linear functional, $\varepsilon : A \to \Bbbk$, such that $Null(\varepsilon)$ contains no non-zero left ideal.*

Sketch proof: Given ε, define $\beta(x \otimes y)$ to be $\varepsilon(xy)$, and, conversely, given a β, define $\varepsilon(x) = \beta(1 \otimes x) = \beta(x \otimes 1)$. The conditions can then be safely left as an exercise. ■

Examples of Frobenius algebras

1. Take $A = \Bbbk$ itself with ε any non-zero map.

2. Any finite field extension of $\Bbbk$ will yield a Frobenius algebra. As an example, consider $\mathbb{C}$ as an $\mathbb{R}$-algebra, taking $\varepsilon(x + iy) = x$.

3. Any matrix ring, $Mat_{n\times n}(\Bbbk)$, gives a Frobenius algebra on taking ε to be the trace.

4. Any finite dimensional semi-simple algebra is Frobenius.

5. For G a finite group, define as before $\Bbbk G$ to be the group algebra of G, and take $\varepsilon : \Bbbk G \to \Bbbk$ to be the usual augmentation (see page 38, adapted to have coefficients in $\Bbbk$).

6. For G again a finite group, and taking $\Bbbk = \mathbb{C}$, let

$$R(G) = \{\varphi : G \to \mathbb{C}^\times \mid \varphi \text{ is constant on conjugacy classes}\}$$

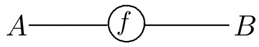
be the ring of class functions of G. (Here $\mathbb{C}^\times$ is the group of non-zero complex numbers.) We set

$$\beta(\varphi, \psi) = \frac{1}{|G|}\sum_{g\in G} \varphi(g)\psi(g^{-1}).$$

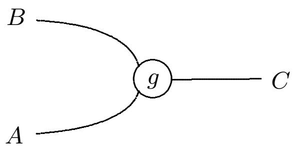
This again is a Frobenius algebra.

To handle Frobenius algebras, or more generally *Frobenius objects* in a more general monoidal category, it is useful to use a graphical calculus. (The treatment here is again strongly based on Joachim Kock's lectures in Almería.)

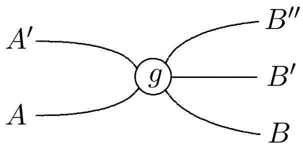
Objects are labelled A, B, etc., but maps are labelled by lines with circled function labels, or sometimes, for ease of typing, labelled bullet points, on them:



Tensors are represented by vertical juxtaposition, so a map $g : A \otimes B \to C$ becomes

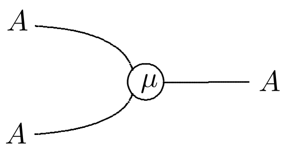


or more complicated, for $g : A \otimes A' \to B \otimes B' \otimes B''$,



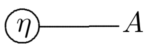
With this, we note the algebra structure: the multiplication

$$\mu : A \otimes A \to A$$



and the unit,

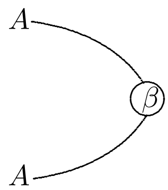
$$\eta : \Bbbk \to A$$



This perhaps takes a bit more thought. We have $\Bbbk = A^{\otimes 0}$, so, as for tensor powers such as $A^{\otimes n}$, we stack n copies of A, here we stack *no* copies of A!

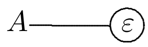
For the Frobenius form, we have

$$\beta : A \otimes A \to \Bbbk$$



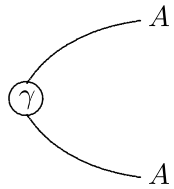
or, if we want to specify the structure in the other form, we can give a counit

$$\varepsilon : A \to \Bbbk$$



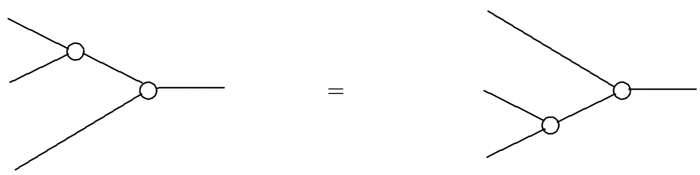
The γ that was used in the definition of 'non-degenerate' becomes:

$$\gamma : \Bbbk \to A \otimes A$$

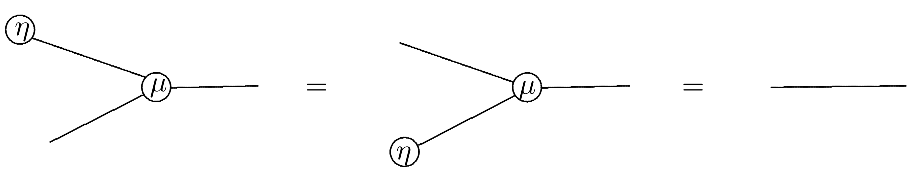


These diagrams are, more or less, the diagrams relating to 0+1 TQFTs and, again more or less, you can construct the 1+1 versions by taking a product of a diagram with S^1. We will return to this point a little later. For the moment, playing with these diagrams gives some neat pictorial versions of the axioms of Frobenius algebra. We will not give them all - they can be found in numerous sources in the literature - but give some as a taster.

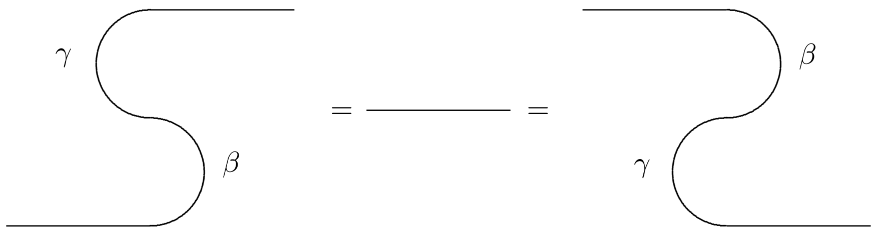
Associativity



Unit



Non-degeneracy



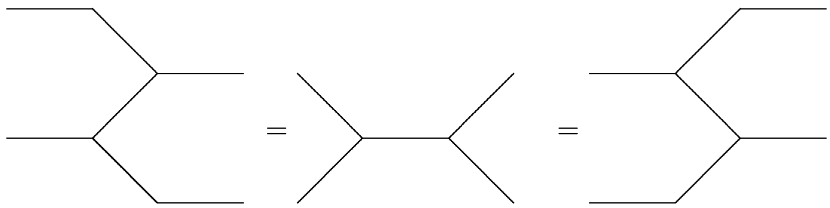
and **you are left** to do the relation between β and ε, etc.

We can now look at the more categorical version of the definition of Frobenius algebra. We state it as a definition, but, of course, it is a *re-definition* really.

Definition: A *Frobenius* $\Bbbk$*-algebra* is $\Bbbk$-vector space with maps

$$\begin{array}{ll} \mu : A \otimes A \to A, & \eta : \Bbbk \to A, \\ \delta : A \to A \otimes A, & \varepsilon : \Bbbk \to A, \end{array}$$

such that (i) unit rule, (ii) counit rule (i.e., mirror of unit), and (iii) the Frobenius rule:

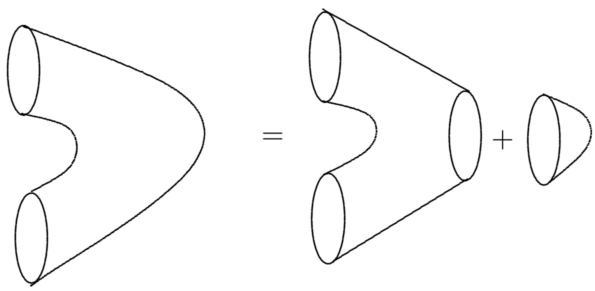


Remark: You can derive associativity from these.

Proposition 80 *This definition is equivalent to the earlier one.*

Sketch proof plan: First define δ using μ and γ, then check the axioms etc. In the other direction, we need β and this can be constructed from μ and ε. ■

It takes a bit of time to become proficient in manipulating these diagrams, but they are very often used in studies of 'tensor categories'. If you prefer to think of surface diagrams, just take any of the above and take its 'product with' S^1, the circle. The manipulations envisaged above are just homeomorphisms of the resulting cobordisms. As an example, the last one of the sketch proof (constructing β from μ and ε), is



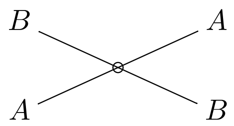
in the surface form, that is, sewing a disc on a pair of pants is isomorphism to a cylinder.

From the description of Frobenius algebras, it becomes more or less easy to prove that

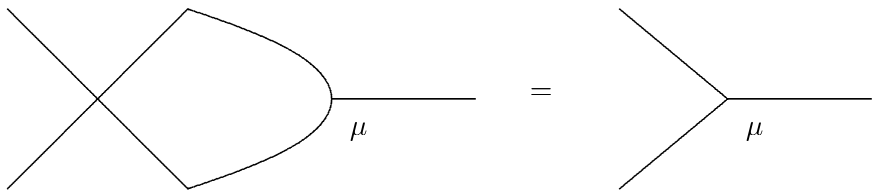
Theorem 22 *The category of 2d TQFTs is equivalent to that of commutative Frobenius algebras.*

Proof: The main idea is clear. We start with a 1+1 TQFT, Z, and then $Z(S^1)$ will be Frobenius algebra. Conversely given a Frobenius algebra, A, we define $Z(S^1) = A$ and then start generating the other structure. (For this, it is simplest to look at the generators and relations for $2-Cob$, and then to check each part in turn. You can find this in Joachim Kock's book, [140], amongst other places.)

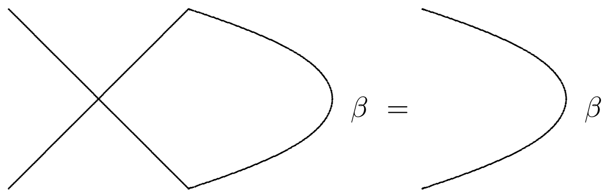
There are one or two points that need noting. We have, in the statement of the result, used the term 'commutative Frobenius algebra'. If we place ourselves in $(Vect_{\Bbbk}, \otimes, \Bbbk)$, there is a symmetry $\sigma : A \otimes B \to B \otimes A$, or as a diagram:



A *commutative Frobenius algebra* is a Frobenius algebra such that

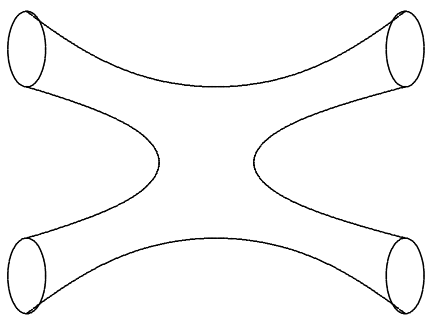


(There is a notion of *symmetric Frobenius algebra*, where one requires β to be a symmetric form:



but note $Mat_{n\times n}(\Bbbk)$ is symmetric, but is not commutative.)

The cobordism



shows that 2 dimensional TQFTs will be commutative.

The reader is left to collect up the pieces, check that functors, etc. work as hoped, and, if all else fails, to look up a neat proof in one of the sources mentioned earlier. (The first published full proof is by Lowell Abrams, [2].) ■

The above has a useful interpretation in terms of *Frobenius objects*. If we look at the definition above (page 329), it is easy to see how we can adapt it to give a *Frobenius object* in a suitably structured monoidal category. We will use a description derived from that given in Rodrigues' paper, [191].

8.1.3 Frobenius objects

Let $\mathcal{A}$ be a symmetric monoidal category with monoidal structure, denoted $\otimes$, and with $\Bbbk$ as unit. (That is, we will suspend our convention that $\Bbbk$ is a commutative ring for rest of this discussion! Of course, in one of the main examples, i.e., $Vect^{\otimes}_{\Bbbk}$, it still is as that is the unit in this case.)

Defintiion: We say $\mathcal{A}$ has a (left) *duality structure* if for each object A, there is an object, A^*, the dual of A, and morphisms

$$b_A : \Bbbk \to A \otimes A^*,$$

$$d_A : A^* \otimes A \to \Bbbk$$

such that

(i) $$(A \xrightarrow{\cong} \Bbbk \otimes A \xrightarrow{b_A \otimes A} A \otimes A^* \otimes A \xrightarrow{A \otimes d_A} A \otimes \Bbbk \xrightarrow{\cong} A) = Id_A$$

and

(ii) $$(A^* \xrightarrow{\cong} A^* \otimes \Bbbk \xrightarrow{A^* \otimes b_A} A^* \otimes A \otimes A^* \xrightarrow{d_A \otimes A^*} \Bbbk \otimes A^* \xrightarrow{\cong} A^*) = Id_{A^*},$$

where the unlabelled isomorphisms are the structural isomorphisms of $\mathcal{A}$ corresponding to $\Bbbk$ being a left and right unit for $\otimes$.

The assignment of A^* to A extends to give a functor from $\mathcal{A}$ to $\mathcal{A}^{op}$, the opposite category. If $f : A \to B$ is a morphism in $\mathcal{A}$, its *dual* or *adjoint morphism* $f^* : B^* \to A^*$ is given by the composition

$$B^* \xrightarrow{\cong} B^* \otimes \Bbbk \xrightarrow{B^* \otimes b_A} B^* \otimes A \otimes A^* \xrightarrow{B^* \otimes f \otimes A^*} B^* \otimes B \otimes A^* \xrightarrow{d_B \otimes A^*} \Bbbk \otimes A^* \xrightarrow{\cong} A^*.$$

Definition: If $\mathcal{A}$ has a duality structure as above, a *Frobenius object* in $\mathcal{A}$ consists of

- an object A of $\mathcal{A}$;
- a 'multiplication' morphism, $\mu : A \otimes A \to A$;
- a 'unit' morphism, $\eta : \Bbbk \to A$ such that (A, μ, η) is a monoid in $(\mathcal{A}, \otimes)$;

and

- a symmetric 'inner product' morphism,

$$\rho : A \otimes A \to \Bbbk,$$

such that (i)

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "A\otimes\mu"] \arrow[d, "\mu\otimes A"'] & A \otimes A \arrow[d, "\rho"] \\
A \otimes A \arrow[r, "\rho"'] & \Bbbk
\end{tikzcd}$$

commutes (so writing $\mu(a,b) = ab$, $\rho(ab,c) = \rho(a,bc)$),
and
(ii) ρ is non-degenerate, i.e., the following two induced maps from A to A^* are isomorphisms:

$$A \xrightarrow{\cong} A \otimes \Bbbk \xrightarrow{A\otimes b_A} A \otimes A \otimes A^* \xrightarrow{\rho\otimes A^*} \Bbbk \otimes A^* \xrightarrow{\cong} A^*$$

and

$$A \xrightarrow{\cong} \Bbbk \otimes A \xrightarrow{\rho^*\otimes A} A^* \otimes A^* \otimes A \xrightarrow{A^*\otimes d_A} A^* \otimes \Bbbk \xrightarrow{\cong} A^*.$$

(This second composite tacitly uses the isomorphisms $(A \otimes A)^* \cong A^* \otimes A^*$, and $\Bbbk^* \cong \Bbbk$ which hold since $\mathcal{A}$ is assumed to be symmetric monoidal.)

Examples: (i) Frobenius objects in the category, $(Vect_{\Bbbk}, \otimes)$, or, more generally, $(Mod\ R), \otimes)$ are Frobenius algebras in the usual sense.

One of the main reasons for mentioning Frobenius objects is the following result, which is really a very remarkable one.

Theorem 23 *The category $2-Cob$ is the free symmetric monoidal category on a commutative Frobenius object.* ■

We are not going to prove this. (A proof can be found in Joachim Kock's book, [140].) The Frobenius object is the circle. The meaning of the result is that if you have an assignment that sends the circle to a Frobenius object in another category, compatibly with the structural maps of Frobenius objects, then that assignment extends uniquely to a monoidal functor defined on $2-Cob$. We will see other similar results later on. The amazing thing about this is that we have a geometric situation involving cobordisms, etc., and yet have a universal property and a complete categorical characterisation of $2-Cob$.

8.2 How can one construct TQFTs?

8.2.1 Finite total homotopy TQFT (FTH theory)

As a first exercise in constructing a TQFT, we will look at the 'toy' example given by Quinn in his notes, [188]. We will not give all the details, but will sketch some of the main ideas. This not only gives an easily understood method, but in many ways is a precursor for the TQFTs and HQFTs that we will meet later on, when the link with earlier material in these notes will be more explicit.

The basic category is not as complicated as $d-Cob$, as we can get away with objects and 'cobordisms' merely being CW complexes.

The idea is that one fixes a space, B, and then the TQFT, Z_B, is to have, for a space, Y, the state module is

$$Z_B(Y) = \Bbbk[Y, B],$$

the $\mathbb{k}$-vector space with a basis corresponding to the homotopy classes of maps from Y to B. Here $\mathbb{k}$ needs to be of characteristic zero, so $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ will be enough to be going on with. We need $Z_B(Y)$ to be finite dimensional, so need $[Y, B]$ to be a finite set.

Definition: A space, B, is said to have *finite total homotopy, (FTH)* if it has finitely many components, and for any base point $b \in B$, and any i, $\pi_i(B, b)$ is finite, all but finitely many of these groups being trivial.

Example: (i) For any finite group, G, its classifying space, BG has finite total homotopy.

(ii) If $\mathsf{C} = (C, P, \partial)$ is a finite crossed module, then $B\mathsf{C}$, its classifying space, being the realisation of $\overline{W}K(\mathsf{C})$ has only π_1 and π_2 non-trivial, and these two groups are finite, being $Coker\,\partial$ and $Ker\,\partial$, respectively.

The importance of the idea is due to the following lemma:

Lemma 54 *If B has finite total homotopy and Y is a finite CW-complex, then the set, $[Y, B]$, is finite.* ■

The proof is by induction on the number of cells in Y, using a long exact sequence argument.

From now on in this section, we will assume that B has finite total homotopy.

The 'cobordism' role in this CW-complex context is a CW-triad, $(X; Y_1, Y_2)$, thought of as $X : Y_1 \to Y_2$. The two subcomplexes are *disjoint* subcomplexes of X. For such a $(X; Y_1, Y_2)$, we will need to define

$$Z_X := Z_{(X;Y_1,Y_2)} : Z_B(Y_1) \to Z_B(Y_2).$$

Suppose $[f_1]$ is a basis element of $Z_B(Y_1)$, so $f_1 : Y_1 \to B$ (and, as is customary, $[f_1]$ will denote the corresponding homotopy class). We must have

$$Z^B_X([f_1]) = \sum_{[f_2]} \mu_{X,f_1,f_2}.[f_2],$$

where the sum is over all $[f_2] \in [Y_2, B]$. The μ are just matrices over $\mathbb{k}$, expressing this linear transformation in terms of the given bases, but we do not know that much about them! Of course, they are constrained by the axioms of TQFTs (i.e., monoidal functoriality) and, thus, respect for the composition of cobordisms. It might be possible to find the most general form that they can take, but rather than that we will follow Quinn in his notes, [188], and give a solution, sketching parts of the verification of the axioms (adapted to the slightly wider context of this theory). We first need some terminology and notation.

Let X be a space with finite total homotopy.

Definition: The *homotopy order* of X is the rational number, $\sharp^\pi(X)$, defined, if X is connected, to be equal to

$$\sharp^\pi(X, x) := \prod_{i=1}^{\infty} \sharp(\pi_i(X, x))^{(-1)^i} = (\sharp\pi_1)^{-1}(\sharp\pi_2)(\sharp\pi_3)^{-1}...,$$

for any basepoint, $x \in X$, and, if X is not connected, as $\prod \sharp^\pi(X, x)$, the product of the homotopy orders of the connected components of X, based at a representative family of base points, thus one in each component.

Now let $f_1 : Y_1 \to B$, $f_2 : Y_2 \to B$, and set

$$Maps_{f_1}(X,B)_{[f_2]} = \{F : X \to B \mid F|_{Y_1} = f_1, F|_{Y_2} \simeq f_2\},$$

which is a subspace of the space of maps from X to B. An argument similar to that for the lemma above shows that this has finite total homotopy, provided $(X; Y_1, Y_2)$ is a finite CW-triad. (It is interesting that changing f_1 within its homotopy class does not change the homotopy order of this space of maps.)

We can now give Quinn's scaling factor matrix, μ:

$$\mu_{X,f_1,f_2} = \sharp^\pi(Map_{f_1}(X,B)_{[f_2]}).$$

We will also use the space of maps $Map_{f_1}(X,B)$, which is $\{F : X \to B \mid F|_{Y_1} = f_1\}$, then we have:

Lemma 55

$$Z_X^B([f_1]) = \sum_{[F]} \sharp^\pi(Map_{f_1}(X,B), F)[F|_{Y_2}],$$

where the sum is over all homotopy classes, rel Y_2, of maps $F : X \to B$, restricting to f_1 on Y_1. ■

The proof is by a repackaging of the previous expression, so is **left to you**.

We will look at the composition of 'cobordisms'. First another lemma, which will also be useful later on.

Lemma 56 *Suppose $F \to E \xrightarrow{p} B$ is a fibration of spaces with finite total homotopy and assume that B is connected, then*

$$\sharp^\pi(E) = \sharp^\pi(F)\sharp^\pi(B).$$

Proof: This should be clear from the long exact homotopy sequence of a fibration. ■

Now assume given triads, $(X_1; Y_1, Y_2)$, $(X_2; Y_2, Y_3)$, then so is $(X_1 \sqcup_{Y_1} X_2, Y_1, Y_2)$ and

$$X_1 \to X_1 \sqcup_{Y_1} X_2$$

is a cofibration, so

$$Map_{f_1}(X_1 \sqcup_{Y_1} X_2, B) \to Map_{f_1}(X_1, B)$$

is a fibration. To use the lemmas, we need to work out the fibre over a map, $F : X_1 \to B$, but we can identify this with $Map_{F|Y_2}(X_2, B)$ by the pushout property of $X_1 \sqcup_{Y_1} \sqcup X_2$. This gives, after a fairly obvious calculation involving the previous lemma:

Lemma 57

$$Z^B_{X_1 \sqcup_{Y_1} X_2} = Z^B_{X_2} Z^B_{X_1}.$$

■

The other properties are **left for you to investigate**.

The following is a closely related construction in which B is the classifying space of a finite group. This provides a first example of a lattice or triangulation based construction of a TQFT via a 'state sum' model.

8.2.2 How can we construct TQFTs ... from finite groups?

One method of generation of TQFTs which is frequently used is based on simplicial lattices or triangulations and we will use this. Although it is a bit more complicated than some of the other constructions, it generalises nicely to higher dimensions *and* has a nice interpretation.

(The version here, and in the next few sections, is based on constructions of Dave Yetter, [224, 225], see also the papers, [179, 180]. The original idea is discussed quite fully in the first of the two papers by Yetter. It is a version of a construction due to Dijkgraaf and Witten, [81].)

First we work with triangulations of the oriented manifolds and cobordisms. For our immediate use of 'triangulations', we will work with an intuitive idea of triangulation. (You can base that intuition informally on wire-grid models such as are used in computer graphics.) We will, later on, have to look at them in a bit more detail.

We will go into quite a lot of detail on this construction itself as the methods it uses are quite intuitively simple, but are also the basis for those that we will use later on, which are perhaps less so.

Fix a finite group, G, and let X be a space with triangulation, $\mathbf{T}$. It will be useful, but initially not essential, to have that $\mathbf{T}$ is an ordered triangulation, so will consist of a simplicial complex, T, a homeomorphism between $|T|$ and X, and, in addition, a *total order* on the set of vertices, $V(T) = T_0$, of T. (Sometimes it may be notationally useful to specify the vertices of T as being explicitly indexed by natural numbers in agreement with the ordering, so then $T_0 = \{v_0, v_1, \ldots, v_n\}$ where $\sharp(T_0) = n$.) The choice of the ordering is not crucial in any way. We will initially use it to help the construction along, and later will need it to turn simplicial complexes into simplicial sets using the construction that we saw near the start of these notes, page **??**.

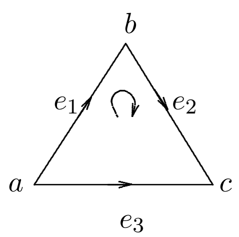
Definition: A *G-colouring* of $\mathbf{T}$ is a map,

$$\lambda : T_1 \to G,$$

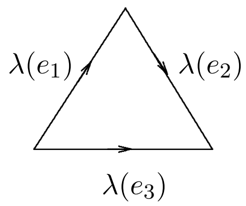
such that given $\sigma \in T_2$, $\lambda(e_1)^{\varepsilon_1}\lambda(e_2)^{\varepsilon_2}\lambda(e_3)^{\varepsilon_3} = 1$, where the boundary $\partial\sigma$ of *sigma* is given by $\partial\sigma = e_1^{\varepsilon_1} e_2^{\varepsilon_2} e_3^{\varepsilon_3}$.

To help understand the formula, we look at a very simple example.

Picture: To simplify, assume the orientation is given and, as above, the vertices of $\mathbf{T}$ are ordered. If we write $\sigma = (a, b, c)$, then $a < b < c$, and we assume the order is compatible with the orientation:



The boundary of σ is $e_1 e_2 e_3^{-1}$, so a coloring, λ gives



with $\lambda(e_1)\lambda(e_2)\lambda(e_3)^{-1} = 1$.

The intuition is: on looking at G-valued functions on edges, integrating around a triangle is to give you ‘nothing’, that is the identity element of G.

The G-valued functions concerned are typically those associated with transition functions of a bundle, usually of G-sets, i.e., a G-torsor or principal G-bundle. That intuition then corresponds to situations where a G-bundle on X is being specified by charts, and the elements, g, h, k, etc., are transition automorphisms of the fibre. (Because of this, the triangle condition above is a *cocycle condition.* It is often also termed a ‘flatness’ condition, as in the differential case, it corresponds to a condition on a ‘connection’ which say that it is ‘flat’.) The construction methods for the TQFT then use manipulations of the pictures as the triangulation is changed by subdivision, etc.

Another closely related view of this is to consider continuous functions, $f : X \to BG$, to the classifying space of G. We can assume that f is a cellular map, using a suitable cellular model of BG, and at the cost of replacing f by a homotopic map, and by subdividing the triangulation of X. From this perspective, the previous model is a combinatorial description of such a continuous *characteristic* map, f. The edges of the triangulation pick up group elements since the end points of each edge get mapped to the base point of BG, and $\pi_1 BG \cong G$, whilst the faces give a realisation of the cocycle condition. Likewise we can, and, later on will, use a labelled decomposition of the objects as regular CW-complexes.

So much for the moment on the G-colourings as such, ..., what do we do with them?

Since G is assumed to be finite, the set, $\Lambda_G(\mathbf{T})$, of all G-colourings of $\mathbf{T}$ is also finite. Let $Z_G(X, \mathbf{T})$ be the vector space having $\Lambda_G(\mathbf{T})$ as basis. (The vector space will usually be over $\mathbb{C}$, but any other field of suitable characteristic will do, provided the constructions used do not involve elements that ‘aren’t there. For instance, sometimes a formula will have the order of a group raised to a fractional power and, clearly, that is fine if we work over $\mathbb{C}$ or $\mathbb{R}$, but could be problematic over $\mathbb{Q}$. Because of this we will work over $\mathbb{C}$, rather than having to change the ground field or ring each time a new construction needs an extra condition. In general, of course, we could replace ‘vector space’ by free module over a commutative ring with a short list of conditions on the ring.)

We will need, later, to consider subdivisions of triangulations and their effect on these vector spaces, and, as $\mathbf{T}$ is being ordered and that structure is part of the structure needed for the construction, we need, in considering subdivisions of $\mathbf{T}$, to take the ordering into consideration. (As has been said before, here the ordering could be avoided, but it is great help in the exposition even in this simple case, and will be more or less essential in more complicated cases later on. It is another instant of introducing structure to help with a construction, although once the thing is constructed, we can show that it is independent of the extra structure.)

Definition: A *subdivision* of an ordered triangulation, $\mathbf{T}$, is an ordered triangulation, $\mathbf{T}'$, such that the underlying triangulation is a subdivision of the underlying triangulation of $\mathbf{T}$ and the inclusion of $V(T_0)$ into $V(T'_0)$ is a monotone function for the given orderings.

Comment: We are basing this definition on a fairly informal definition of triangulations and subdivisions, but it will suffice for the moment. Shortly we will make this a bit more formal.

Yetter uses a very simple form of subdivision, namely 'edge-stellar' subdivision. Although, in fact, we will also use other means and other types of subdivision, it is worth briefly noting the justification that he gives for his choice. We will need a result of Alexander's from [4]. To be able to state this, we first need the idea of a dimensionally homogeneous polyhedron. A polyhedron is *dimensionally homogeneous* if there is a dimension k such that every point is contained in some closed k-simplex.

Theorem 24 *(Alexander, 1930) If X is a dimensionally homogeneous polyhedron, then any two triangulations of X are related by a series of edge-stellar subdivisions and inverses of such.* ■

From this it is not hard to prove:

Corollary 13 *Any two ordered triangulations of an n-manifold are related by a sequence of edge-stellar subdivisions and their inverses.* ■

Remark: In fact, Yetter restricts to surfaces, so the full force of these results is not needed. In [224], he considers this case of a finite group, but later uses very similar methods for a finite crossed module / categorical group; see below and [225], but, in both, the case of surfaces, with 3-dimensional cobordisms between them, is what is considered in detail and this is quite reasonable as we will see.

Whichever type of triangulation you use, as it is extra structure beyond the basic manifolds, it is necessary to eliminate dependence on this. We will turn to this shortly, but we also need to consider cobordisms, so how are they studied?

Suppose $(X, \mathbf{T})$ and $(Y, \mathbf{S})$ are two triangulated *oriented* d-manifolds (and, as in [224], let us restrict to surfaces and thus to $d = 2$ for simplicity of exposition, although, for much of the time, this will make no difference). A triangulated cobordism, $(M, \mathcal{T})$, between them will be a cobordism, M, between X and Y, i.e., M will be an oriented manifold of dimension $(d + 1)$, (so here usually 3), with boundary $X \sqcup -Y$, that is, 'X disjoint union with Y, which is oriented with the opposite orientation', and $\mathcal{T}$ will be an (ordered) triangulation of M that restricts to $\mathbf{T}$ on X and to $\mathbf{S}$ on Y. We can consider G-colourings of $(M, \mathcal{T})$ as well, and we can define a linear map,

$$Z^!_G(M, \mathcal{T}) : Z_G(X, \mathbf{T}) \to Z_G(Y, \mathbf{S}),$$

by, for $\lambda \in \Lambda_G(\mathbf{T})$,

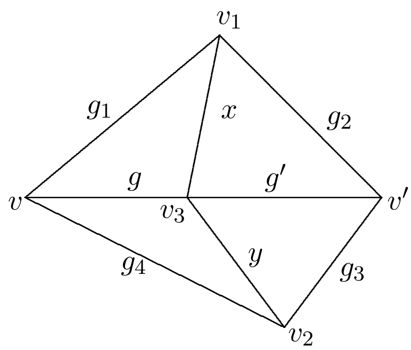
$$Z^!_G(M, \mathcal{T})(\lambda) = \sum_{\substack{\mu \in \Lambda_G(\mathcal{T}) \\ \mu|\mathbf{T} = \lambda}} \mu|\mathbf{S},$$

and extending linearly.

In other words, you have a basis colouring, λ, on the 'input' end and look at those colourings of $(M, \mathcal{T})$ that extend it, then see what they give you at the 'output' end, summing over all the possible answers.

This, at the same time, looks good and also slightly suspect. We started with a basis element of $Z_G(X, \mathbf{T})$ and ended up with lots of basis elements for $Z_G(Y, \mathbf{S})$. That is, somehow, too 'inflationary'. Even for a single basis element, λ', in $Z_G(Y, \mathbf{S})$, there would be triangulations, $\mathcal{T}$, of M, which would give a large number of copies of λ' in this sum. Perhaps a compensating factor, say depending on the size of G, is needed to correspond more to an 'average' value over these μs. We need to see 'how many' there are. For that, we need to look 'inside' the cobordism and how the colourings of $\mathcal{T}$ change with subdivision.

We will call the vertices, edges, etc., in $\mathcal{T}$, *interior* if they are away from the ends. Suppose that we subdivide *one* interior edge, e, of $\mathcal{T}$, giving a new triangulation, $\mathcal{T}'$, of M. For each old colouring, λ, of $\mathcal{T}$, we now have $\sharp(G)$ G-colourings of $\mathcal{T}'$, which are the same on all other edges, since on the subdivided e, we can assign any $g \in G$ to one half of it with $g' = \lambda(e)g^{-1}$ or $g^{-1}\lambda(e)$ given to the other, which one to use depending on the ordering. This does not disturb the face relations in the relevant 2-simplices:



keeping $g_1, \ldots, g_4$ fixed, it is easy to find unique values for x and y giving a new colouring, and not changing anything outside the immediate neighbourhood of the edge, e. There are **several cases to check** depending on the relative position of the new vertex, v_3, in the ordering, but they clearly are all easily handled. (We will in any case look at this in a bit more detail shortly, but it is a good idea to have **tried the calculations yourself**.) We have

$$Z^!_G(M, \mathcal{T}') = \sharp(G) Z^!_G(M, \mathcal{T}).$$

Now, for a general $\mathcal{T}$, set $n_{\mathcal{T}}$ to be the number of vertices in $\mathcal{T}$, but not in the two ends, so $n_{\mathcal{T}} = \sharp(\mathcal{T}_0 - (T_0 \cup S_0)) = \sharp(\mathcal{T}_0) - \sharp(T_0) - \sharp(S_0)$, as S_0 and T_0 are disjoint.

Lemma 58 *If $\mathcal{T}$ and $\mathcal{T}'$ are any two triangulations of M that agree with $\mathbf{T}$ and $\mathbf{S}$ on the two ends, then*

$$\sharp(G)^{-n_{\mathcal{T}}} Z^!_G(M, \mathcal{T}) = \sharp(G)^{-n_{\mathcal{T}'}} Z^!_G(M, \mathcal{T}').$$

Proof: You just consider a mutual subdivision, $\mathcal{T}''$, and compare the two linear maps of the statement of the result with $Z^!_G(M, \mathcal{T}'')$, using the earlier comment on the 'one-extra-vertex' case, and induction. The details are best **left to you**. ∎

Of course, this means that:

Corollary 14 *This common linear map is independent of the triangulation, $\mathcal{T}$.* ∎

This linear map is still dependent on the triangulations on the two ends, $\mathbf{T}$ and $\mathbf{S}$. We denote this common value, $Z^!_G(M, \mathbf{T}, \mathbf{S})$. We have thus, in this new linear map, included compensatory scaling factors to handle the subdivisions in the cobordism, but even if a big step in the right direction, they still do not give us the final linear map that we want. That has to be compatible with composition and preserve the monoidal structure.

It is easy to see that the above is all compatible with the monoidal structure, since within the cobordism setting $\otimes$ interprets as disjoint union, and so a G-colouring of $X \otimes Y$ will be given precisely by a G-colouring of X together with a G-colouring of Y; that is all. We thus have $Z_G(X \otimes Y, \mathbf{T} \otimes \mathbf{S}) \cong Z_G(X, \mathbf{T}) \otimes Z_G(Y, \mathbf{S})$, with the isomorphism originating on the given bases. (Hopefully, the above notation is more or less self explanatory.)

There remains the question of compatibility of the above with composition. There is an obvious composition of triangulated cobordisms. Suppose $(M, \mathcal{T})$ is a triangulated cobordism from $(X, \mathbf{T})$ to $(Y, \mathbf{S})$, and $(N, \mathcal{S})$ another from $(Y, \mathbf{S})$ to $(Z, \mathbf{R})$. We can form a cobordism, $M +_Y N$, from X to Z by gluing the two given cobordisms along the copies of Y (i.e., by forming a pushout). This clearly comes with a triangulation, which could, sensibly, be denoted $\mathcal{T} +_{\mathbf{S}} \mathcal{S}$. This is not, however, an arbitrary triangulation of $M +_Y N$ as, on the copy of Y 'in its middle', the triangulation agrees with $\mathbf{S}$. This affects the compensating factors:

Lemma 59 *In the above situation*

$$Z^!_G(N, \mathbf{S}, \mathbf{R}).Z^!_G(M, \mathbf{T}, \mathbf{S}) = \sharp(G)^{\sharp(S_0)} Z^!_G(M +_Y N, \mathbf{T}, \mathbf{R}).$$

Proof: The compensating factor for the term $Z^!_G(M +_Y N, \mathbf{T}, \mathbf{R})$ uses

$$\begin{aligned} n^{\mathcal{T} +_{\mathbf{S}} \mathcal{S}} &= \sharp((\mathcal{T} +_{\mathbf{S}} \mathcal{S})_0 - \mathbf{T}_0 - \mathbf{R}_0) \\ &= n^{\mathcal{T}} + n^{\mathcal{S}} + \sharp(S_0). \end{aligned}$$

Now look at what happens in the composite on the left. ■

We thus let $Z_G(M, \mathbf{T}, \mathbf{S}) = \sharp(G)^{-\frac{1}{2}(\sharp(T_0)+\sharp(S_0))} Z^!_G(M, \mathbf{T}, \mathbf{S})$, dividing the effects of the two ends between them to compensate for this. (**Think about this**. It is a nice way of getting this to work, although we should also consider, and from several different angles, **why** does it get it to work! It looks like a factor introduced to make things work and, in fact, is, but it *should* have some other more 'elegant' description, but what should that be? One extra point to note is that this does potentially need $\sqrt{\sharp}(G)$ to be an element of the ground field, so, as mentioned before, for convenience we take $\Bbbk$ to be $\mathbb{R}$ or $\mathbb{C}$, although with a bit of attention in any particular case, we do not need all the extra elements that this provides.)

We thus obtain:

Corollary 15 *For M, $\mathbf{T}$, $\mathbf{S}$, etc. as above:*

$$Z_G(N, \mathbf{S}, \mathbf{R}).Z_G(M, \mathbf{T}, \mathbf{S}) = Z_G(M +_Y N, \mathbf{T}, \mathbf{R}).$$

■

Note: For simplicity, we have assumed that the field for the vector spaces has characteristic 0, but, in fact, for the above, we only need $\sharp(G)$ to be invertible in it. (This may, perhaps, remind you of parts of group representation theory, and that is not just coincidence.)

We thus have a monoidal ‘functor’ from the ‘category’ of triangulated surface and cobordisms to that of vector spaces. Although this looks good, we have left ‘functor’ and ‘category’ in inverted commas, because the so-called ‘functor’ is not going to preserve identities, and, worse than that, it is not so clear what the identities should be in this case of triangulated surfaces. Oh dear! But there is another point left outstanding, namely that we have manifolds with *triangulations*, and that $Z_G(X, \mathbf{T})$, etc., depend on the choice of triangulation. On handling *that* point, we will actually end up close to managing the ‘identity’ problem.

We are thus back with subdivisions. Suppose we are given X and $\mathbf{T}$, as before, and let $\mathbf{T}'$ be obtained by subdividing $\mathbf{T}$ at a single edge, e, divided into two parts, e_1 and e_2, for instance, if $v < v'$,

$$v \xrightarrow{e} v' \quad \text{goes to} \quad v \xrightarrow{e_1} . \xrightarrow{e_2} v',$$

(in the case where the new vertex is between v and v' in the ordering on T_0'). We define a function

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}} : \Lambda_G(\mathbf{T}') \to \Lambda_G(\mathbf{T})$$

by multiplying the values of a colouring on the subdivided bits of the edge, (taking into account signs to handle the vertex ordering), so, in this simple case: for λ, a G-colouring of $\mathbf{T}'$,

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}}(\lambda)(e) = \lambda(e_1)\lambda(e_2),$$

with $\mathrm{res}_{\mathbf{T}',\mathbf{T}}(\lambda)$ taking the same value as λ on all other edges of $\mathbf{T}$, $\mathrm{res}_{\mathbf{T}',\mathbf{T}}(\lambda)(e') = \lambda(e')$ if $e' \in T_1$, $e' \neq e$.

It is really *necessary* to draw some diagrams, as above, to check that, if λ is a colouring of $\mathbf{T}'$, then $\mathrm{res}_{\mathbf{T}',\mathbf{T}}(\lambda)$ is one of $\mathbf{T}$. We work with the diagram given earlier (page 338) with g corresponding to $\lambda(e_1)$, and g' to $\lambda(e_2)$. (There are several cases to check corresponding to the placement of the ‘new vertex’ in the order on the vertices of $\mathbf{T}$ and the form of that ordering. We will just treat the case where in $\mathbf{T}$, $v < v_1 < v_2 < v'$ and the new vertex, v_3, lies between v_2 and v' in the order on $\mathbf{T}'$. **You are left to think about the other cases**, but you should quickly realise that reversing the order on a pair of vertices just replaces $\lambda(e)$ by its inverse, so the other cases are easy once one is done. Of course, this is just another instance of the argument we sketched when looking at subdivisions of the triangulations of the cobordisms.

We have four equations, with, from the top two triangles,

$$\begin{aligned} g_1 x &= \lambda(e_1) \\ x\lambda(e_2) &= g_2 \end{aligned}$$

and similarly for the lower two triangles. The value, $\mathrm{res}_{\mathbf{T}',\mathbf{T}}(\lambda)(e)$ is $\lambda(e_1)\lambda(e_2)$, and it is immediate that $g_1 g_2 = \mathrm{res}_{\mathbf{T}',\mathbf{T}}(\lambda)(e)$, as required.

Remark: That was easy, and, of course, is the basic calculation in all (co)homological situations. We ‘integrate’ along the edge labelled x first in one direction, then later in the opposite one and adding up the contributions cancels that x out. The only slightly subtle point is that as G may be non-commutative, we need to be careful about the multiplication order. (Here we have used ‘algebraic’ concatenation order as that is what is used ‘traditionally’ in this area.) The actual formula for $\mathrm{res}_{\mathbf{T}',\mathbf{T}}$ that we gave above only works (without adjustment that is) if the new vertex is between v and v' in the order on T_0'. As commented above, if another order occurs, say $v < v' < v_3$, then e_2 may be, as here, reversed in direction and, in that case, we would have $\lambda(e_1)\lambda(e_2)^{-1}$ in the expression. This is easy to do in this case of G being a group, and of being in low dimensions, but still looks as if numerous similar cases would be needed in general. It one replaces the group, G, by a crossed module, as we will shortly, the complications would look to be getting out of hand, so we *do need to keep our eyes open* for a neater way of handling things, that is, other than a proof by exhaustion! Case-by-case analysis is always there as a backup, but begins to look unfeasible for later on. We will continue to use it for the moment as it *does* emphasise the combinatorics of what is going on and thus when we do go on to slicker methods, we will some background intuition of how these methods encode this combinatorial analysis.

Going back to subdivisions, if $\mathbf{T}'$ is obtained by subdividing edges in $\mathbf{T}$, then it can be obtained inductively by doing the above simple case repeatedly. (It is clear the order in which this is done is immaterial to the end result.) We thus have

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}} : \Lambda_G(\mathbf{T}') \to \Lambda_G(\mathbf{T}),$$

in general, together with a resulting linear extension from $Z_G(X, \mathbf{T}')$ to $Z_G(X, \mathbf{T})$.

What happens to the cobordisms? We have, say, M from X to Y, as before, and hence $Z_G(X, \mathbf{T}', \mathbf{S})$ and $Z_G(X, \mathbf{T}, \mathbf{S})$. If $\eta \in \Lambda_G(\mathbf{T}')$, a similar argument to before shows

$$Z_G(X, \mathbf{T}', \mathbf{S}) = \sharp(G)^{-\frac{1}{2}(\sharp(T_0')-\sharp(T_0))} Z_G(X, \mathbf{T}, \mathbf{S}) \circ \mathrm{res}_{\mathbf{T}',\mathbf{T}},$$

in other words, the obvious diagram does not commute. (Note: if you are cross-referencing, or consulting, the original source, the relevant diagram on page 5 of [225] has an arrow going in the wrong direction.) A similar calculation would apply to a subdivision of $\mathbf{S}$. To obtain better compatibility of the 'res' maps with the cobordism induced ones, we thus scale the restriction map, defining a new

$$res_{\mathbf{T}',\mathbf{T}} : Z_G(X, \mathbf{T}') \to Z_G(X, \mathbf{T}),$$

by $res_{\mathbf{T}',\mathbf{T}} = \sharp(G)^{-\frac{1}{2}(\sharp(T_0')-\sharp(T_0))}\mathrm{res}_{\mathbf{T}',\mathbf{T}}$, and, thus adjusted, we will then have the desired compatibility between the cobordism structure and the restriction maps coming from subdivision.

Before we look at that however, we note another essential feature of the restriction maps, namely that they are epimorphisms. It is clear that, even at the 'basic' non-linear level

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}} : \Lambda_G(\mathbf{T}') \to \Lambda_G(\mathbf{T}),$$

is 'onto', as, given a colouring, λ, of $\mathbf{T}$, we can even work out a colouring of $\mathbf{T}'$ that maps down to it. We just look at a simple case to give the idea. Suppose $\mathbf{T}'$ contains just one more vertex subdividing the edge e as before. We define a colouring of $\mathbf{T}'$ by assigning to most edges the same value as for λ, but, using the same notation as before, to e_1 we assign 1, and, to e_2, assign $\lambda(e)$. The other edges in $\mathbf{T}'$ now can be assigned values so that the result is a colouring. (Just **try it out** on the diagram we saw earlier.)

We thus have a diagram of finite dimensional vector spaces and epimorphisms, indexed by the (directed) category of triangulations of X. To eliminate the dependence of $Z_G(X, \mathbf{T})$ on the triangulations, we just take the colimit of this diagram.

Let $Z_G(X) = colim_{\mathbf{T}} Z_G(X, \mathbf{T})$. This vector space is finite dimensional. Although fairly simple to prove, this fact is important, so we will spend a moment examining it. It is important, since otherwise Z_G would not give us a TQFT, since $Vect$ is the category of *finite* dimensional vector spaces. (The study of infinite dimensional vector spaces, of course, usually uses different tools for their study than does that of their finite dimensional counterparts and that machinery, norms, completeness, etc., is not really what is being used in this theory.) It is also important since we can, in the process of proving this, show that $Z_G(X)$ is a quotient of $Z_G(X, \mathbf{T})$, in fact, the natural linear maps,

$$r^X_{\mathbf{T}} : Z_G(X, \mathbf{T}) \to Z_G(X),$$

are epimorphisms, so it suffices to examine their kernels to obtain a neat and useful representation of the elements of $Z_G(X)$, ..., but we are going too fast and must backtrack.

We note

(i) for any polyhedron, X, the category of triangulations of X is directed. More exactly, for any simplicial complex, K, and subdivisions, K', K'', of K, there is a subdivision, K''', finer than both K' and K''. There are subtleties here, but this works fine for X a piecewise linear (PL) manifold, which is sufficient for us. Some of the subtle points are discussed in [5], but there is also a correction note available, adjusting one or two of the statements. We have already referred to Alexander's paper, [4] and this will suffice for us. The link between triangulations and coverings that we will look at shortly, is also relevant here.

(ii) If $\mathcal{V} : \mathcal{D} \to Vect$ is a functor, where $\mathcal{D}$ is a directed set (thought of as a directed category), and, for each $d' < d$, the corresponding $V^{d'}_d : \mathcal{V}(d') \to \mathcal{V}(d)$ is an epimorphism, then, for any d, the canonical map

$$r_d : \mathcal{V}(d) \to colim\mathcal{V}$$

is an epimorphism. To see this, note how $V = colim\mathcal{V}$ can be constructed. You first form the direct sum, $\bigoplus_{d\in\mathcal{D}} \mathcal{V}(d)$, and then divide out by the equivalence relation:

$v_{d_1} \equiv v_{d_2}$ if there is a v_{d_3}, for d_3 such that
(i) $d_3 < d_1$, and (ii) $d_3 < d_2$,
with
(iii) $V^{d_3}_{v_1}(v_{d_3}) = v_{d_1}$ and (iv) $V^{d_3}_{v_3}(v_{d_3}) = v_{d_2}$,

so two elements are to be equivalent if they are both images of some third element 'further back' in the diagram. If we write $[v]$ for the equivalence class determined by v, then $r_d(v_d) = [v_d]$, where, notationally, we do not distinguish between $v_d \in \mathcal{V}(d)$ and its image in the direct sum $\bigoplus \mathcal{V}(d)$. To check the statement that r_d is an epimorphism, we need only show that any $[v]$ has a representative in $\mathcal{V}(d)$, but $[v]$ will be a finite sum of elements of the form $[v_{d'}]$ for a finite family of indices d'. Using that $\mathcal{D}$ is directed, we can find a d'', finer than d and also finer than *all* of the d's, and then using the fact that all the $V^{d''}_{d'}$ are epimorphisms, pick elements $v'' \in \mathcal{V}(d'')$, each mapping down to the corresponding $v_{d'}$, finally replace the $v_{d'}$ in the sum by the equivalent $V^{d''}_{d'}(v'')$ to get an element, equivalent to v, but which is just in the image of r_d. As $[v]$ was arbitrary, this shows that r_d is itself an epimorphism.

This verification is standard, and elementary, but it shows why $Z_G(X)$ is finite dimensional in a very concrete and effective way. It also helps identify the kernel of $r^X_{\mathbf{T}}$, or, in general, r_d, since if $v \in Ker\, r_d$, there must be some d' and d'' with d'' less than both d and d', and an element $v'' \in \mathcal{V}(d'')$ such that $V_d^{d''}(v'') = v$ and $v'' \in Ker\, V_{d'}^{d''}$.

This, thus, gives a good description of the elements of $Z_G(X)$. You just take a $Z_G(X, \mathbf{T})$ and work out the kernel of $r_{\mathbf{T}}^X$. We next return to the cobordisms.

As we know that the linear maps, $Z_G(X, \mathbf{T}) \to Z_G(X, \mathbf{S})$, are compatible with the restriction maps, $res_{\mathbf{T}',\mathbf{T}}$ (and $res_{\mathbf{S}',\mathbf{S}}$), it is now easy to check that a cobordism M from X to Y induces a linear map,

$$Z_G(M) : Z_G(X) \to Z_G(Y).$$

(This does require a bit of care as the domain is a colimit over the category of triangulations of X, whilst the codomain is over that to triangulations of Y. This is, however, quite easy to handle, so the details are **left to you**.)

Theorem 25 *(Yetter, [224]) The assignment, above, defines a monoidal functor,* $Z_G : d - Cob \to Vect^{\otimes}$. ■

We will not prove this here. We will discuss generalisations of it later on and will indicate why they work.

Of more immediate interest than a straightforward direct proof is an interpretation of the resulting TQFT. The compensatory factors looks mysterious. What are they 'really'? Although the initial idea may be clear, the process of finding those compensatory factors does cloud the view a bit. The theory left like this looks a bit like cohomology when cocycles were the only way of looking at things. Cocycles are useful and have a geometric interpretation provided, for instance, we think of them as transitions between structure over some open cover. (We will turn towards that in the next section linking triangulations and open coverings.)

Yetter's proof of the above result uses an important observation, which also tells us a lot more about this TQFT. First we note that, as X is a triangulable manifold, we can work out the fundamental groupoid, ΠX, of X, up to equivalence, using the classical *edge path groupoid* construction. (For details of the origins of this construction, see below.) We have really met this several times earlier in these notes, but not explicitly, so here it is.

Given a simplicial complex, K, we form the free groupoid, $F_{Grpd}(K^{(1)})$, on the 1-skeleton, $K^{(1)}$ of K, (i.e., the 1-dimensional subcomplex, and hence 'graph', made up of the edges and vertices of K), then we divide by relations corresponding to the 2-simplices. This is worthwhile making explicit, as it make what follows more or less trivial.

We introduce some notation. We will put an order on the vertices of K, for convenience. The 1-simplices of K will be denoted $\langle v_0, v_1\rangle$, and the elements of $F_{Grpd}(K^{(1)})$ will be composable chains of these such as $\langle v_0, v_1\rangle\langle v_1, v_2\rangle$, where the reverse of the given order corresponds to inverting the element so, if $v_1 < v_0$, then $\langle v_1, v_0\rangle$ will be the same as the 'virtual' element, $\langle v_0, v_1\rangle^{-1}$. (We have essentially got rid of the imposed order, already, at this step. We can thus assume that we only take those edges, $\langle v_0, v_1\rangle$, with $v_0 < v_1$.) For each 2-simplex, $\langle v_0, v_1, v_2\rangle$, of K, we then introduce the relation

$$\langle v_0, v_1\rangle\langle v_1, v_2\rangle \equiv \langle v_0, v_2\rangle.$$

(Here we can again assume $v_0 < v_1 < v_2$.) The resulting quotient groupoid, denoted $\Pi_{edge}K$, is the edge path groupoid of K.

Remark: Of course, this construction is discussed, classically, in Spanier, [198], p.136, and, according to Brown, [40, 43], was already essentially in Reidemeister's book, [189].

We note that, if we order the vertices of K, then we have a simplicial set that corresponds to it, (see page **??**), and hence the simplicially enriched groupoid, $G(K)$, (cf. page 213). We considered taking π_0 of each of the simplicial sets $G(K)(v, v')$ to get the fundamental groupoid of K as a quotient of $G(K)$, (again review page 213, and the discussion on the pages that follow that one). This is, of course, the same construction as the edge path groupoid, but has the advantage of having the higher n-types of K available in the $\mathcal{S}$-groupoid, $G(K)$. We will return to this point in a later section, when we replace the finite group, G, by a finite n-type, or similar.

Suppose that λ is a G-colouring of $(X, \mathbf{T})$, then considering the group, G, as a single object groupoid, $G[1]$, the assignment, λ, extends to a functor (or, if you prefer, a morphism of groupoids) from $F_{Grpd}(T^{(1)})$ to $G[1]$. This would be the case just with 'any-old' assignment of elements of G to edges of T, i.e., even without the cocycle-like condition for the values around each 2-simplex, $\sigma \in T_2$. That extra family of conditions means that actually λ induces $\lambda : \Pi_{edge}T \to G[1]$.

As $\Pi_{edge}T$ is equivalent to ΠX, this gives a representation, $\lambda : \Pi X \to G[1]$. This will not be uniquely determined by the previous one since there will always be many different equivalences between $\Pi_{edge}T$ and ΠX. We have that $\Pi_{edge}T$ is more or less the same as $\Pi X|T_0|$, the fundamental groupoid of X, based at the vertices of T_0. The usual retraction of ΠX onto this subgroupoid involves choices of paths and different choices give different, but conjugate, morphisms, λ. This is not conclusive, but may give sufficient intuition to see the feasibility of proving Yetter's main representation theorem:

Theorem 26 *(Yetter, [224], p.7) The vector space, $Z_G(X)$, is isomorphic to the vector space whose base is the set of conjugacy classes of representations from ΠX to $G[1]$.* ■

Of course, a *representation* in this context is just a groupoid morphism and hence is just a functor. Two groupoid morphisms, $f_0, f_1 : \mathcal{G} \to \mathcal{H}$, are conjugate if and only if they are homotopic and if and only if, as functors, there is a natural transformation between them.

There is another 'take' on G-colourings that is worth mentioning here. It is somehow 'adjacent' to that which we have just given. We made the link between $\Pi_{edge}T$ and $G(T)$ above. This link extends to a very simply defined one between $G(T)$ and G-colourings. This then gives a bit more substance to our hint of link with G-torsors, etc.

Let $\mathbf{T}$ be an ordered triangulation of a space, X, and as usual, let $K(G,0)$ denote the constant simplicial group of value G, (i.e., $K(G,0)_n = G$ for all n, and all the face and degeneracy maps in $K(G,0)$ being identity isomorphisms). (On a niggling notational point, perhaps we should really write $K(G,0)[1]$, or $K(G[1],0)$, to include the information that we are thinking of the group, G, as a one object groupoid. This extra precision needs to be kept in mind, but will not be used except if it turns out to be useful at a particular place in our discussions.)

Proposition 81 *Suppose that λ is a G colouring of T, then λ defines an $\mathcal{S}$-groupoid morphism*

$$\lambda' : G(T) \to K(G,0)$$

given by
$$\lambda_0\langle a,b\rangle = \lambda(a,b) \in G = K(G,0)_0;$$
and, if $\sigma = \langle a_0,\ldots,a_{n+1}\rangle \in T_{n+1}$, $n \geq 1$,
$$\lambda'_n\sigma = s_0^n\lambda(a_0,a_1).$$

Proof: First note that we use the ordering to convert the simplicial complex, T, to a simplicial set. (Here we will not recall this again, but when we have different coefficients than just G, a bit later in this chapter, then we will need to be more precise at the corresponding point in the discussion.)

Remembering that $G(T)$ is free in each dimension, we only have to see what λ' does to non-degenerate simplices, (as the values on degenerate ones will be determined by the fact the morphism is simplicial). We then have to check that the simplicial identities work for this choice of λ'_ns. Most of this is routine and inconsequential, so is left **to the reader**, but it is worth noting what happens in dimension 1 with the d_0-face relation.

We will sometimes use an overbar to give the generating element, $\overline{\sigma}$, of $G(T)$ that corresponds to a simplex σ in T. This provides a bit more precision that is sometimes useful (although it gets awkward if the convention is slavishly followed).

Let $\langle a_0,a_1,a_2\rangle$ be a non-degenerate 2-simplex of T, so $\overline{\langle a_0,a_1,a_2\rangle}$ will be a generator of $G(T)_1$ (check back for the definition of all the structure of $G(K)$, for K a simplicial set, page 218).

$$\begin{aligned}\lambda'_1\overline{\langle a_0,a_1,a_2\rangle} &= s_0\lambda_0(a_0,a_1)\\ &= s_0\lambda(a_0,a_2)s_0\lambda(a_1,a_2)^{-1},\end{aligned}$$

since λ satisfies the flatness / cocycle condition on 2-simplices, so $\lambda(a_0,a_1)\lambda(a_1,a_2)\lambda(a_0,a_2)^{-1} = 1$. Now clearly $d_0\lambda'_1 = \lambda_0 d'_0$, as required. ■

This correspondence is clearly bijective. If $\lambda' : G(T) \to K(G,0)$ is a $\mathcal{S}$-groupoid morphism, then it gives back a G-colouring of T and everything matches.

Proposition 82 *There is a bijection*

$$\Lambda_G(T) \leftrightarrow \mathcal{S}-Gpds(G(T),G[1]).$$

■

We have, on passing to Moore complexes, that $N(K(G,0))_n = G$ if $n = 0$ and is trivial otherwise, so $N(G(T))_n$ is 'killed off' by λ' in all dimensions except 0, and, of course, there we get an induced map from $\pi_0(G(T))$ to G, but $\pi_0(G(T))$ is, as we noted, just $\Pi_{edge}(T)$ again.

The advantage of this simplicial viewpoint will be clearer when we pass to generalisations, but we note also that λ' corresponds to a morphism of simplicial sets,

$$\lambda' : T \to \overline{W}(K(G,0)) = Ner(G[1]) = BG,$$

by the adjointness of G and $\overline{W}$ that we have used several times. It thus corresponds to an isomorphism class of simplicial principal G-bundles on T (or G-torsors, if you prefer). As G is a finite group, this bundle also can be thought of as a finite covering space on T, or as a twisted Cartesian product, as in section 5.5, starting page 229.

As we said above, we will not give a proof of the result of Yetter here, but will look at generalisations later on. The proof in [224] is interesting and quite neat, so may be worth looking at anyway. It is also useful since aspects of it are used by Yetter in his paper on TQFTs associated to categorical groups, [225]. We will look at this shortly, but in the next section must 'backfill' on some of the ideas on triangulations, etc.

To finish up this section, we note a consequence of Lemma 59 / Corollary 15. Suppose M is a closed $(d+1)$-dimensional manifold and consider it as a cobordism from the empty d-manifold to itself. Next pick any triangulation, $\mathcal{T}$ of M. We have the domain and codomain of M in $(d+1)-Cob$ are empty, so there is only one G-colouring of them, namely the unique empty function from the empty set of edges of the empty triangulation of the empty manifold. (This, of course, corresponds to $Z_G(\emptyset) \cong \mathbb{C}$, which is the unit of the monoidal structure of $Vect^{\otimes}$.)

Our next task is to work out $Z^!_G(M,\mathcal{T})(\lambda)$, for this unique G-colouring λ. The formula gives

$$Z^!_G(M,\mathcal{T})(\lambda) = \sum_{\substack{\mu\in\Lambda_G(\mathcal{T})\\ \mu|\mathbf{T}=\lambda}} \mu|\mathbf{S},$$

but λ corresponded to the unique basis element of the vector space, $Z_G(\emptyset)$ and hence is 1, as is each $\mu|_S$, since $\mathbf{S}$ is also empty. This means that we have a contribution of 1 for each $\mu \in \Lambda_G(\mathcal{T})$, that is,

$$Z^!_G(M,\mathcal{T}) = \sharp(\Lambda_G(\mathcal{T})),$$

the number of G-colourings of $\mathcal{T}$.

Now we apply Lemma 59 or Corollary 15. Recall $n_{\mathcal{T}}$ is the number of vertices of $\mathcal{T}$.

Proposition 83 *The number*

$$I_G(M) = (G)^{-n_{\mathcal{T}}}\sharp(\Lambda_G(\mathcal{T})),$$

is independent of the triangulation. ∎

This is just a special case of Corollary 15.

Remarks: (i) We note, almost just for fun, that in this situation, the scaling factor to get from $Z^!_G(M,\mathcal{T})$ to $Z_G(M,\mathcal{T})$ has value 1, since both T_0 and S_0 are empty.

(ii) This invariant, $I_G(M)$ is the simplest case of the *Yetter invariants* of M. When M is 3-dimensional, then it is an 'untwisted' version of the Dijkgraaf-Witten invariant. We will see this construction in other instances when we have generalised the construction of Z_G to having 'target' a finite crossed module.

This invariant has a homotopy theoretic description, which shows that it is a homotopy invariant of M. This description is

$$I_G(M) = \frac{[M,BG]}{\sharp(G)}.$$

8.2.3 Triangulations and coverings

We mentioned that the intuition behind the finite group case was linked to the transition functions of a G-torsor or principal G-bundle. With such 'transition functions', (cf. page 268), one has an open cover over which the bundle / torsor is assumed to trivialise. Recall that by this we mean that we have a cover, $\mathcal{U} = \{U_\alpha : \alpha \in A\}$, say, of a space X and a 'bundle', $p : Y \to X$, such that if we restrict to a U_α, $p^{-1}(U_\alpha) \to U_\alpha$ is just the projection of a product $U_\alpha \times F \to U_\alpha$ for some nice 'fibre' F. In other words, the bundle is locally trivial. How does this correspond to the triangulation approach? For this we need to look more closely at triangulations. (For convenience, we will repeat some material on simplicial complexes from earlier in the notes, but sometimes from a slightly different perspective. This may seem slightly strange sometimes, as we have been using the associated definitions, notation, etc., for quite some time on a partially informal basis.)

The history, and some of complications, of the fact that manifolds can be triangulated, and thus can be represented by simplicial complexes, is well discussed in Stillwell's book, [200], at a fairly non-technical level and in Anderson and Mnev, [5], for more technicalities. (One of the claims mentioned later in that paper has been withdrawn, and a correction made available. The paper still is well worth consulting, as it is quite short, well written and to the point.) We will, in fact, be using other more technical aspects of that general area within examples later on and will then need to introduce more detail than given in [200].

Here we retain the formal definition of a triangulation.

Definition: A *triangulation*, $\mathbf{T} = (K, f)$, of a space, X, consists of a simplicial complex, K, and a homeomorphism, $f : |K| \to X$.

We will usually confuse $|K|$ with X, and so will call X, itself, a *polyhedron* in this case.

We put a formal definition of ordered triangulation here as well for convenience of reference. We will leave you to adjust the definition of ordered subdivision that we gave above.

Definition: A finite *ordered triangulation* of a space, X, is a triangulation, $\mathbf{T} = (K, f)$, together with a bijection, $K_0 \leftrightarrow \{0, 1, \ldots, n\}$, for some n. (Note that the bijection is part of the structure.)

We will also need a more formal definition of *subdivision*. Firstly it may help to look up the canonical geometric realisation of an (abstract) simplicial complex, (see page **??**). The following definition of subdivision is from Spanier, [198], p. 121. (It is not independent of the use of geometric realisations, so to some extent seems 'external' to the theory of abstract simplicial complexes. This is relevant when considering the observational viewpoint for this area, (see below). It is however a useful definition and *is* the usual one!)

Definition: If K is a simplicial complex, a *subdivision* of K is a simplicial complex, K', such that
a) the vertices of K' are (identified with) points of $|K|$;
b) if s' is a simplex of K', there is a simplex, s, of K such that $s' \subset |s|$; and
c) the mapping from $|K'|$ to $|K|$, that extends the mapping of vertices of K' to the corresponding points of $|K|$, is a homeomorphism (thus continuous with a continuous inverse),

. . . , and the corresponding notion for triangulations:

Definition: If $\mathbf{T} = (K, f)$ is a triangulation of a space, X, a subdivision of $\mathbf{T}$ is a triangulation $\mathbf{T}' = (K', f')$, where K' is a subdivision of K, and $f' : |K'| \to X$ compatible with f, i.e., f' is equal to $|K'| \to |K| \xrightarrow{f} X$.

An important idea for us will be that of the star of a vertex in a triangulation. (Now is a good time to briefly look back at the construction of the geometric realisation of an abstract simplicial complex given in the first chapter, (page **??**), if you did not do it above. The important point to hold on to is the idea of a point in $|K|$ as being a function from the vertex set, $V(K)$, of K to the unit interval, $[0, 1]$. Recall the condition on such a function, $\alpha : V(K) \to [0, 1]$, to be a 'point' in $|K|$ was that its *support*, $\{v \in V(K) \mid \alpha(v) \neq 0\}$, forms one of the simplices in K.) We have

Definition: If K is a simplicial complex and $\sigma \in S(K)$ is a simplex of K, the subspace,

$$|\sigma| = \{\alpha \in K \mid \alpha(v) \neq 0 \Rightarrow v \in \sigma\}$$

is called the *closed simplex* corresponding to σ.

We can now define the *star of a vertex, v,* by:

Definition: The *star of a vertex*, v, in a simplicial complex K is the open subset of $|K|$ given by

$$st(v) = \bigcup \{Int|s| \mid v \text{ is a vertex of } s\} \cup v,$$

the union of the interiors of those closed simplices that have v as a vertex together with that vertex itself.

Alternatively, and equivalently, given any vertex v of K, its *star* is defined by

$$st(v) = \{\alpha \in |K| \mid \alpha(v) \neq 0\}.$$

Remark: More generally than needed when discussing triangulations, sometimes it is useful to think of a simplicial complex, K, as encoding combinatorial information on an 'observed space', X, with a continuous map, $f : |K| \to X$ or $f : X \to |K|$, giving the translation between the two contexts. We might be observing a physical object (thought of as the space, X). The vertices are observations of 'points' in X. (We will briefly explore this a bit more below.) The case of a triangulation then corresponds to the simplicial complex, K, being, somehow, a correct encoding of the structure as it allows a complete reconstruction of the 'space', X, itself to be made.

In such an 'observational interpretation', for each vertex, $v \in V(K)$, the set, $st(v)$, is an open set in $|K|$ and, if $\alpha : V(K) \to [0, 1]$ is loosely interpreted as a 'fuzzy superposition of observations', then $st(v)$ consists of those such 'observations' that 'observe' the notional point, v.

The other ingredient for our comparison between triangulations and coverings is the formal definition of the nerve of an open covering. We have been using this idea in another more structured form when we have considered an open cover as specifying a simplicial sheaf, but here we will need the older, non-sheaf theoretic version, due to Čech and Alexandrov back in the 1930s, that we briefly mentioned in section 6.2.4, where we introduced the idea in discussion the descent aspect of simplicial fibre bundles. The link between the two ideas is by taking connected components of the spatial part of the simplicial étale space version of the simplicial sheaf. (This is certainly not 'optimal', but makes the connection through to the study of triangulations more clear and 'classical'.)

We assume given a space, X, and an open covering, $\mathcal{U} = \{U_\alpha : \alpha \in A\}$, of X.

Definition: The *Čech complex*, *Čech nerve* or simply, *nerve*, of the open covering, $\mathcal{U}$, is the simplicial complex, $N(\mathcal{U})$, specified by:

- Vertex set : the collection of open sets in $\mathcal{U}$ (alternatively, the set, A, of labels or indices of $\mathcal{U}$);
- Simplices : the set of vertices, $\sigma = \langle \alpha_0, \alpha_1, ..., \alpha_p \rangle$, belongs to $N(\mathcal{U})$ if and only if the open sets, U_{α_j}, $j = 0, 1, \ldots, p$, have non-empty common intersection.

You will probably have remembered that we have already used the notation, $N(\mathcal{U})$, for another thing that (cf. section **??**) we called the nerve of the open cover $\mathcal{U}$. That was the simplicial sheaf version of this. The connection between the two is very close, so any confusion that may arise is not that serious. If, as in the case of manifolds, we can refine covers so that each U_α is a contractible open set, then one can use either the sheaf of the simplicial complex version of the nerve with no great advantage to either. In that sort of situation the Yetter construction we saw earlier can be adapted to give one using simplicial sheaves, and it seems feasible that it can be extended to one with a sheaf of finite groups as the 'background' coefficients. (There are technicalities here that we will not go into for the moment.)

Remark (continued): In various parts of mathematics and mathematical physics, as we said above, it is sometimes useful to think of an open set as the support of an 'observation'. Physically, one cannot make measurements *at a point*, rather one uses the abstract idea of value *at a point* as a convenience for the average measurement 'locally' near the *point* in some space. One can thus replace 'point' by 'observed open set' and then see how overlapping 'observations' fit together. The nerve then serves as the combinatorial gadget that 'organises' the observations.

Similarly, in the relatively new area of *topological data analysis*, the information on a spatial model of some phenomenon is given by a point cloud of sample values, so the sample points are thought of as being small regions, and so are replaced by small discs. A similar idea comes in when considering Voronoi patches and the related Delaunay triangulations, (for which area it is suggested that you use a websearch to find a summary). Variants of the nerve construction are then use to help in the construction of geometric and topological models of the phenomena.

The idea of a triangulation is physically slightly problematic as the observer 'imposes' a triangulation on the space or space-time being observed. If 'points' are suspect then perhaps imposed triangulations are even more so!

If the space, X, is a polyhedron, then we can easily obtain a link between nerves and triangulations, so as to connect up this 'observational' idea with the 'imposition' of a triangulation.

The vertex stars give an open covering of $|K|$ and the following classical result tells us that the nerve of this covering is K itself (up to isomorphism):

Proposition 84 *(cf. Spanier [198], p. 114) Let X be a polyhedron and let $\mathcal{U} = \{st(v) \mid v \in V(K)\}$ be the open cover of X by vertex stars. The vertex map, φ, from K to $N(\mathcal{U})$, defined by*

$$\varphi(v) = \langle st(v) \rangle,$$

is a simplicial isomorphism,

$$\varphi : K \cong N(\mathcal{U}).$$

■

As an example, suppose a triangle, as simplicial complex, has vertices

$$V(K) = \{v_0, v_1, v_2\}$$

and simplices $\{v_0\}, \{v_1\}, \{v_2\}, \{v_0, v_1\}, \{v_0, v_2\}, \{v_1, v_2\}$. (This is the *triangle* not the *2-simplex*, so there is no 2-dimensional face.) This obviously provides a triangulation of the circle, S^1, which you are left to 'draw'.

The above result, and the example, illustrate that for polyhedra (and thus for triangulated manifolds), an approach via open coverings is at least as strong as that via triangulations. Triangulations give open coverings that themselves give back the triangulation. If we, on the other hand, start with an open covering of a polyhedron, can we always find a triangulation that is finer than it in the sense that any open star of a vertex is completely within some open set of the covering? The following classical result (for instance, in Spanier, [198], p.125) tells us that we can, and hence that, for polyhedra, the two approaches, triangulations and open coverings are, in fact, of equal strength:

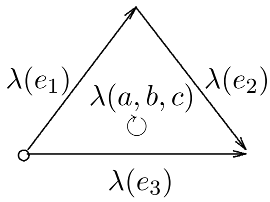
Theorem 27 *Let $\mathcal{U}$ be any open covering of a (compact) polyhedron X, then X has triangulations finer than $\mathcal{U}$.* ■

We thus have a method of going between a triangulations based approach to an open coverings based one, both theoretically and intuitively. This works extremely well for spaces that are triangulable, however many spaces encountered in diverse areas of mathematics are not manifolds or even polyhedra, and then, evidently, triangulation based ideas cannot be directly used. The open covering based ones have no such restriction, but that advantage will not concern us in this chapter as we mainly deal with polyhedra and manifolds, (which will often be PL ones, and hence have well understood triangulations).

8.2.4 How can we construct TQFTs ... from a finite crossed module?

In Yetter's second construction of a TQFT in [225], he replaced the finite group, G, by a finite crossed module, $\mathsf{C} = (C, P, \partial)$. It should be fairly clear, given the route we have taken so far, how we can treat this from our perspective. We look at C-colourings as being an assignment of elements of P to edges of a triangulation, elements of C to the 2-simplexes with a boundary condition, and with any tetrahedrons giving some cocycle condition.

In pictures:



where $e_1 = \langle a, b\rangle$, $e_2 = \langle b, c\rangle$ and $e_3 = \langle a, c\rangle$, $\lambda(a, b, c) \in C$, and the various $\lambda(e_i) \in P$.

The boundary condition, as given by Yetter, is then that

$$\lambda(e_1)\lambda(e_2)\lambda(e_3)^{-1} = \partial\lambda(a, b, c)^{-1}.$$

If we think of the categorical group or 2-group, $\mathcal{X}(\mathsf{C})$, instead of C itself, this means that, for $\sigma = \langle a, b, c\rangle$,

$$\tilde{\lambda}(\sigma) : \lambda(e_1)\lambda(e_2) \Rightarrow \partial\lambda(a, b, c)\lambda(e_1)\lambda(e_2),$$

and, of course, $\lambda(e_3)$ is the expression on the right here. (Here $\tilde{\lambda}(\sigma) = (\lambda(\sigma), \lambda(e_1)\lambda(e_2)) \in \mathcal{X}(\mathsf{C})_1 = C \rtimes P$.)

It will also be necessary to have a 2-flatness / cocycle condition for a tetrahedron. This will say that the diagram in the 2-category / 2-group, $\mathcal{X}(\mathsf{C})$, corresponding to the faces of a tetrahedron, $\langle a_0, a_1, a_2, a_3\rangle$, must commute. We will first look at this condition within the 2-group, $\mathcal{X}(\mathsf{C})$, using the two compositions, $\sharp_0$ and $\sharp_1$, then we will convert from 2-categorical to simplicial notation (in fact, in two different ways) to get this condition in more simplicial language.

We assume $\mathbf{T}$ is an ordered simplicial complex and that we have λ as above. The faces of a tetrahedron, $\langle a_0, a_1, a_2, a_3\rangle$, form a square diagram (as we have seen several times before). In this case, the vertices of that square correspond to the objects of $\mathcal{X}(\mathsf{C})$, or, if you prefer, to elements of P. We have

$$\begin{tikzcd}
\lambda(a_0a_1)\lambda(a_1a_2)\lambda(a_2a_3) \arrow[r, "①"] \arrow[d, "②"'] & \lambda(a_0a_2)\lambda(a_2a_3) \arrow[d, "\tilde{\lambda}(a_0,a_2,a_3)"] \\
\lambda(a_0a_1)\lambda(a_1a_3) \arrow[r, "\tilde{\lambda}(a_0,a_1,a_3)"'] & \lambda(a_0a_3)
\end{tikzcd}$$

where ① $= \tilde{\lambda}(a_0, a_1, a_2)\sharp_0\lambda(a_2, a_3)$ and ② $= \lambda(a_0, a_1)\sharp_0\tilde{\lambda}(a_1, a_2, a_3)$, (cf. section **??**, page **??**). The cocycle condition is that this commutes in $\mathcal{X}(\mathsf{C})$, where the composition used is the category composition in $\mathcal{X}(\mathsf{C})$, that is, $\sharp_1$. (You can easily convert this to a condition in the crossed module, C, if you like. The above references more or less tell you what form things should take. You can also refer to the paper by Faria Martins and Porter, [98], but the best course of action is to **work it out for yourself**.)

As the categorical composition, $\sharp_1$, is determined by the other structure and the group multiplication (which is $\sharp_0$), we could rewrite this condition solely using these. In fact, as T is 'simplicial', it seems better to translate the conditions into simplicial ones. (This may seem a bit arbitrary, but we have seen the efficiency of simplicial methods when handling coherence in earlier chapters, and cocycle conditions are coherence conditions, Never fear, the translation works and is worth it, ...)

To help with this, we replace $\mathcal{X}(\mathsf{C})$ by $K(\mathsf{C})$. Recall that this is the simplicial group obtained by taking the (internal) nerve of the (internal) category structure of $\mathcal{X}(\mathsf{C})$, everything being done 'internally' in the category of groups, cf. page **??**. This functor is also one part of the Dold-Kan equivalence between crossed complexes and simplicial T-complexes, see page 216. If you want yet another glimpse of $K(\mathsf{C})$ in its many and varied manifestations, think of $\mathcal{X}(\mathsf{C})$ as a one-object

2-category, $\mathcal{X}(\mathsf{C})[1]$, and similarly think of the simplicial group, $K(\mathsf{C})$, as an $\mathcal{S}$-groupoid (with one-object); that really should be $K(\mathsf{C})[1]$, of course. In section **??**, (page **??**), we saw that any 2-category gives a simplicially enriched category by using the nerve functor on each hom-category. As the nerve functor embeds Cat in $\mathcal{S}$, the resulting simplicial enriched category is really the same as the 2-category. In our case here, that $\mathcal{S}$-category is $K(\mathsf{C})[1]$.

In the above square diagram, the vertices are vertices of $K(\mathsf{C})$ and the edges are 1-simplices of $K(\mathsf{C})$, so we need to use the simplicial form of the $\sharp_1$-composition so as to work out the diagonal of the square in two different ways. (The results of these calculations must be equal as the square has to be commutative.)

We have actually used this simplicial form several times already, but perhaps not always with an explicit mention! That means we should give it 'for convenience'. Given a pair of 1-simplices, g_0, g_2, in a simplicial group, G, with $d_0g_2 = d_1g_0$ (so the picture is

$$\cdot \xrightarrow{g_2} \cdot \xrightarrow{g_0} \cdot$$

within G), we clearly can form a $(2,1)$-horn (cf. page **??** if you have forgotten what that means). Using the algorithm given in Proposition **??**, we can fill it. This gives an explicit element, $x = s_1g_2.s_1s_0d_0(g_2)^{-1}.s_0(g_0)$, in G_2 and its d_1 face is $g_2.s_0d_0(g_2)^{-1}.g_0$. This can be thought of as the composite of g_0 and g_2, and, as x is thin, *is* that composite if $NG_2 \cap D_2 = 1$. In our square diagram, this composite will be a diagonal arrow (pointing SE). The formula for the top composed with the right-hand side is

$$\begin{aligned}\tilde{\lambda}(a_0,a_1,a_2)s_0\lambda(a_2,a_3).(s_0\lambda(a_1,a_2)s_0\lambda(a_2,a_3))^{-1}\tilde{\lambda}(a_0,a_2,a_3)\\ = \tilde{\lambda}(a_0,a_1,a_2)s_0\lambda(a_1,a_2)^{-1}\tilde{\lambda}(a_0,a_2,a_3)\end{aligned}$$

and the composite '(left side) $\sharp_1$ (bottom)' is

$$s_0\lambda(a_0,a_1)\tilde{\lambda}(a_1,a_2,a_3)s_0\lambda(a_1,a_3)^{-1}s_0\lambda(a_0,a_1)\tilde{\lambda}(a_0,a_1,a_3),$$

and these two must be equal, since the square is to commute in $\mathcal{X}(\mathsf{C})$.

Although seeming moderately complex, this cocycle condition is very like others that we have seen before, at least in its general form. We will also see how, when encoded simplicially, it becomes just the condition corresponding to the *existence* of a simplicial morphism.

Let us now look at the assignment λ again in a new light:

- λ assigns vertices in $K(\mathsf{C})$ to edges, i.e., to elements in T_1;
- λ assigns edges in $K(\mathsf{C})$ to elements in T_2.

This suggests another construction that we have seen before. We have, for a simplicial set, K, the loop groupoid, $G(K)$, (see page 213). This was an $\mathcal{S}$-groupoid, and the vertices were generated by the edges / 1-simplices of K, the 1-simplices were generated by K_2, and so on. This 'hints' that a C-colouring might correspond to, perhaps, an $\mathcal{S}$-groupoid map from $G(T)$ to $K(\mathsf{C})$, so we should 'check this out' as an idea. (This was the case when C was just a group; see page 344 and Proposition 81.)

First note that, as T is an *ordered* triangulation of X, T can be replaced by its associated simplicial set. This, as was mentioned back in section **??** (page **??**), has as its n-simplices those totally ordered sets, $\langle a_0, a_1, \ldots, a_n\rangle$, of vertices of T (so $a_0 \leq a_1 \leq \ldots \leq a_n$) such that $\{a_0, a_1, \ldots, a_n\}$ is a simplex of T, after deletion of any repeats. The face maps omit the corresponding element, so, for instance, $d_0\langle a_0, a_1, a_2\rangle = \langle a_1, a_2\rangle$, and the degeneracies repeat, in an obvious way, a vertex in the list, so, for example, $s_1\langle a_0, a_1\rangle = \langle a_0, a_1, a_1\rangle$.

If we look at a C-colouring, λ, it is fairly clear that there is possible way to define a simplicial morphism,

$$\lambda' : G(T) \to K(\mathsf{C}),$$

using λ, and encoding the same information, so let us try it.

- the simplicially enriched groupoid, $G(T)$, has T_0 as its set of objects, whilst $K(\mathsf{C})$ is a simplicial group. (We, perhaps, should write $K(\mathsf{C})[1]$ or $K(\mathsf{C}[1])$ here, but will often omit the [1], unless in a situation where that little bit of extra precision seems to be useful or needed.) We thus have a unique 'object map' underlying λ'.

- We want $\lambda'_0 : G(T)_0 \to K(\mathsf{C})_0 = P$, and $G(T)_0$ is the free groupoid on the directed graph given by the 1-skeleton of T, so this suggests $\lambda'_0\langle a_0, a_1\rangle = \lambda(a_0, a_1)$, where, as when we discussed $G(K)$, we will tacitly confuse a simplex in K_{n+1} with the corresponding generator in $G(K)_n$. Note that the degenerate 1-simplices of form, $\langle a, a\rangle$ are not covered by this definition, but, since, according to the defining relations in $G(T)$, the elements that they generate have been discarded or, rather, equated to identities, the corresponding value for $\lambda'_0\langle a, a\rangle$ will be the identity element, $\langle a\rangle$, of the group P.

- For $\lambda'_1 : G(T)_1 \to K(\mathsf{C})_1$, we want $\lambda'_1\langle a_0, a_1, a_2\rangle \in K(\mathsf{C})_1$. Because of the twist in the d_0-face of $G(T)$, (cf. page 213), we need to take a bit of care. (**It may help just to draw a triangle**, label it using λ, as above, and see the difference between the $G(T)$ '2-cell' and that in the given λ.) We take
  - $\lambda'_1\langle a_0, a_1, a_2\rangle = \tilde{\lambda}(a_0, a_1, a_2).s_0\lambda(a_1, a_2)^{-1}$, provided $a_0 < a_1 < a_2$. (You should check this works for both d_0 and d_1, for instance that $d_1\lambda'_1 = \lambda'_0 d_1$.)
  - On $\langle a_0, a_0, a_1\rangle = \langle s_0(a_0, a_1)\rangle = id_{\langle a_0\rangle}$, (since the relations $\langle s_0^{G(T)}x\rangle = id$, for each $x \in T_n$ hold in $G(T)$), there is no problem as λ' is to preserve identities.
  - As $\langle a_0, a_1, a_1\rangle = s_1^T\langle a_0, a_1\rangle = s_0^{G(T)}\langle a_0, a_1\rangle$, λ'_1, on this, must be $s_0\lambda'_0\langle a_0, a_1\rangle$, since it is a simplicial morphism, so again 'no problem'.

  In fact, this already determines λ'. We investigate why by looking at λ'_2:

- For $\lambda'_2 : G(T)_2 \to K(\mathsf{C})_2$, we use the fact that the simplicial group, $K(\mathsf{C})$, is a group T-complex to find out what $y = \lambda'_2\langle a_0, a_1, a_2, a_3\rangle$ must be. We use the face operators, d_1 and d_2, to get values on a (2,0)-horn in $K(\mathsf{C})$. Filling that horn 'thinly', we get a unique thin candidate for what y must be and then check that it works for d_0. (It must work, because $K(\mathsf{C})$ has no homotopy above level 1. That was the significance of the 2-flatness / cocycle / tetrahedron condition on λ itself.) This gives a new form of the cocycle condition.

There is, in fact, a small complication here. The earlier form of the cocycle condition was written in terms of 2-commutativity of a tetrahedron in the 2-group, $\mathcal{X}(\mathsf{C})$. It thus uses the geometric or Duskin nerve of $\mathcal{X}(\mathsf{C})$. (More exactly, it uses the lax, rather than the op-lax form of this nerve.) You may recall that we looked at this in section **??**. It was the 2-categorical form of the homotopy coherent nerve of $\mathcal{X}(\mathsf{C})$, considered as the $\mathcal{S}$-enriched category, $\mathcal{X}(\mathsf{C})[1]$. Of course, $\mathcal{X}(\mathsf{C})$ corresponded to the simplicial group, $K(\mathsf{C})$, and is really the 'same thing', viewed from a slightly different angle (and, of course, with care taken on the convention for compositional order).

On the other hand, here we are using the loop-groupoid, $G(T)$, so a morphism from there to $K(\mathsf{C})$ corresponds to a simplicial map from T to $\overline{W}(K(\mathsf{C})$, and that is the *other* model for the nerve. We had (Proposition **??** in Chapter **??**) that $Ner(\mathcal{X}(\mathsf{C}))$ and $\overline{W}(K(\mathsf{C}))$ were isomorphic. We have that our cocycle condition translated to the commutativity of that square, and thus to the fact that the assignment, λ, extends to a simplicial map from T to $Ner(\mathcal{X}(\mathsf{C}))$, whilst the construction above states that λ' extends to one from $G(T)$ to $K(\mathsf{C})$, i.e., from T to $\overline{W}(K(\mathsf{C}))$. As the two nerves are isomorphic, we have:

Proposition 85 *A* C*-colouring,* λ*, of* $\mathbf{T}$ *corresponds uniquely to*

(i) a simplicial map $\lambda : T \to Ner(\mathcal{X}(\mathsf{C}))$*; also to*

(ii) a $\mathcal{S}$*-groupoid morphism,* $\lambda' : G(T) \to K(\mathsf{C})[1]$*, and thus to*

(iii) a simplicial map, $\lambda' : T \to \overline{W}(K(\mathsf{C}))$*.* ■

This is really just a corollary of the Bullejos-Cegarra result, ([56]), that we mentioned in Chapter **??**, and proved in its conjugate form. It will be very useful, because of other results that we have looked at on the coskeletal properties of $Ner(\mathcal{X}(\mathsf{C}))$, and of $\overline{W}(K(\mathsf{C}))$. It means that we can use either monoidal categorical, simplicial set or T-complex / simplicial group methods as convenient, and can, up to a point, choose which context to work with so as to optimise our chances of obtaining results (and hopefully with fairly intuitive proofs). From an expositional viewpoint, we can simply take as a definition:

Definition: A C-colouring of $\mathbf{T}$ will be defined to be a simplicial map from the associated simplicial set, T, to $\overline{W}(K(\mathsf{C}))$.

This will replace the equivalent earlier one, but either of the other two formulations could also be used. This will enable us shortly to generalise in two distinct but related directions, not only to colourings with values in a more general 'finite' simplicial group, but also to more general monoidal categories, provided that suitable finiteness conditions are satisfied (corresponding to some extent to the fact that C is a finite 'categorical group' here), - but, as usual, we are getting ahead of ourselves! We need to construct Z_{C} from C.

To do this, we follow the same basic path as we did when considering G-colourings in section 8.2.2. We let $\Lambda_{\mathsf{C}}(\mathbf{T})$ be the set of C-colourings of $\mathbf{T}$, and then $Z_{\mathsf{C}}(X, \mathbf{T})$ be the (complex) vector space with basis labelled by $\Lambda_{\mathsf{C}}(\mathbf{T})$, We then turn to cobordisms. Let $(M, \mathcal{T})$ be an ordered triangulated cobordism from $(X, \mathbf{T})$ to $(Y, \mathbf{S})$ and define, as before

$$Z^!_{\mathsf{C}}(M, \mathcal{T}) : Z_{\mathsf{C}}(X, \mathbf{T}) \to Z_{\mathsf{C}}(Y, \mathbf{S}),$$

by, for $\lambda \in \Lambda_{\mathsf{C}}(\mathbf{T})$,

$$Z^!_{\mathsf{C}}(M, \mathcal{T})(\lambda) = \sum_{\substack{\mu \in \Lambda_{\mathsf{C}}(\mathcal{T}) \\ \mu|\mathbf{T} = \lambda}} \mu|\mathbf{S},$$

on the basis elements, then extending linearly.

We, as before, need to normalise this with respect to independence from the triangulation $\mathcal{T}$ (keeping, of course, $\mathbf{T}$ and $\mathbf{S}$ fixed), and also for composition. What will the compensating scaling factor be in this case?

To answer this, it will help to re-examine subdivision, from a slightly different point of view. We will use the form of the definition of C-colouring corresponding to a morphism

$$\mu : \mathcal{T} \to \overline{W}(K(\mathsf{C})).$$

We will need to use the proof that $\overline{W}(K(\mathsf{C}))$ is Kan, which uses the explicit algorithms for filling horns in a simplicial group to get explicit algorithms for filling horns in $\overline{W}G$; see section **??** and the arguments there. This leads to the lemma. (Remember $\Lambda^i[n]$ is obtained from $\Delta[n]$ by omitting the unique non-degenerate n-simplex, and its i^{th} face.)

In the following, by a *finite simplicial group*, we mean one in which each G_n is a finite group and all but a finite number of terms in its Moore complex are trivial, so the Moore complex has finite length. We note that a finite simplicial group represents a homotopy type with finite total homotopy in the sense that we met earlier in section 8.2.1.

Lemma 60 *Suppose $\lambda : \Lambda^0[2] \to \overline{W}G$ is a (2,0)-horn, for a finite simplicial group, G, then the number of fillers of λ is $\sharp(NG_1)$.*

Proof: We know that $\overline{W}G$ is Kan, so we have a filler for λ. Any two such fillers share the same d_1 and d_2-faces, but, then in the first place of their expressions, they must differ only by multiplication by an element of NG_1. To see this, remember that an element in $\overline{W}G_2$ has form (g_1, g_0) with $g_i \in G_i$, $i = 0, 1$, then

$$\begin{aligned} d_1(g_1, g_0) &= d_0 g_1.g_0 \\ d_2(g_1, g_0) &= d_1 g_1. \end{aligned}$$

If (g_1, g_0) and (g'_1, g'_0) are two fillers for the same $(2,0)$-horn, then $d_1(g'_1 g_1^{-1}) = 1$, so $g'_1 g_1^{-1} \in NG_1 = Ker\, d_1$. The expressions for the faces then give that the relationship between g_0 and g'_0 is determined by that between g_1 and g'_1. The number of such fillers is thus exactly $\sharp(NG_1)$. ■

Generally most of the results that we will see in this section hold either 'as they are' or with slight adaptation if we replace $K(\mathsf{C})$ by a general finite simplicial group, G, and we will often, though not always, state and prove results in that extra generality.

There are higher dimensional versions of this lemma as we will see. Note that similar results hold for (2,1)- and (2,2)-horns. In each case, there are $\sharp NG_1$ different fillers. **You are left to prove** these two alternative forms. For that you will need results on $\sharp(Ker\, d_0 \cap Ker\, d_2)$ and $\sharp(Ker\, d_0 \cap Ker\, d_1)$, namely that they are the same as $\sharp NG_1$. More generally, for a simplicial group G, some $n \geq 0$ and some $0 \leq r \leq n$, let

$$NG_n^{(r)} = \bigcap \{Ker\, d_i \mid i \neq r\},$$

so, for example, $NG_n^{(0)} = NG_n$, and $Ker\, d_0 \cap Ker\, d_2$, above, is $NG_2^{(1)}$, and so on.

Lemma 61 *There is a bijection*

$$NG_n^{(r)} \leftrightarrow NG_n.$$

■

The proof of this is **left to you**. Note that it is not claimed to be an isomorphism, nor to be compatible with face or degeneracies in any way. It is just a bijection, but explicit formulae can be given. (The result is used by Cegarra and Carasco, [63] and [62], to reduce the group T-complex condition, $T3$, of Ashley to the single one, $D_n \cap NG_n = 1$, as we mentioned on page **??**.)

Of course, for our purposes here, we have $\mathsf{C} = (C, P, \partial)$ is a finite crossed module and $G = K(\mathsf{C})$, so $\sharp NG_1 = \sharp(C)$. A similar calculation to the one we gave shows that any $(3, i)$-horn in $\overline{W}K(\mathsf{C})$ has a unique filler, since $\sharp NG_2 = 1$. Although this is a consequence of later more general lemmas, **it is easy to check directly** and is quite fun!

Now let us concentrate on the simplest case. We assume $\mathcal{T}'$ is formed from $\mathcal{T}$ by taking an interior edge, e, and subdividing it. We will look at the case where $e = \langle a_0, a_1\rangle$, so $a_0 < a_1$, and will subdivide e with a new vertex, v, which is greater than all the vertices of all simplices incident to e. (It is easy to adapt the argument to other relative positions of the new vertex.)

We assume given a C-colouring, $\mu : \mathcal{T} \to \overline{W}K(\mathsf{C})$, which agrees with some given $\lambda : T \to \overline{W}K(\mathsf{C})$ on the 'input end'. We want to see what colourings, $\mu' : \mathcal{T}' \to \overline{W}K(\mathsf{C})$, there are which agree with μ on simplices which are not subdivided in the passage from $\mathcal{T}$ to $\mathcal{T}'$ and which 'sum' to the value of μ on subdivided ones. (The meaning of this last condition will emerge as we go along.)

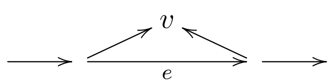
The edge, e, of $\mathcal{T}$ has been replaced, in $\mathcal{T}'$, by $e_0 = \langle a_0, v\rangle$ and $e_1 = \langle a_1, v\rangle$. (**It is a good idea** to glance back at the way this was handled for G-colourings, page 338.) We need to assign a value to $\mu'(e_0)$ and any value in P will do. We then assign a value to $\mu'(e_1)$, so as to retain the overall value of $\mu(e)$ on the original edge, i.e., $\mu(e) = \mu'(e_0)\mu'(e_1)^{-1}$, with the inverse resulting from the change in direction along e_1, of course. (We, so far, have had a possibility of making one out of a possible $\sharp(P)$ choices.)

Next look at a 2-simplex, σ, incident to e in $\mathcal{T}$. Again there are several cases to consider, but they are all similar, so we assume $\sigma = \langle a_0, a_1, a_2\rangle$, i.e., $a_2 > a_1$, (and we also have assumed previously that $a_2 < v$). Let $\sigma_0 = \langle a_0, a_2, v\rangle$, $\sigma_1 = \langle a_1, a_2, v\rangle$ and we look at possible values of $\mu'(\sigma_0)$. We know $\mu'\langle a_0, v\rangle$ as this is $\mu'(e_0)$. We also know $\mu'\langle a_0, a_2\rangle$. We thus have a $(2, 0)$-horn and can fill that in exactly $\sharp(C)$ ways by our lemma above.

From any fixed filler, we obtain $\mu'\langle a_2, v\rangle$. We now have no choice for $\mu'\langle a_1, a_2, v\rangle$, since it and $\mu'\langle a_0, a_2, v\rangle$ must combine to give $\mu\langle a_0, a_1, a_2\rangle$. You may, quite rightly, ask what 'combine' must mean. In the original paper by Yetter, [225], the 12 different possibilities are given corresponding to the relative position of the new vertex amongst the three vertices of $\langle a_0, a_1, a_2\rangle$, thus an explicit answer can be given. Perhaps, however, a slightly different perspective is worth taking, one that can be pushed further later on, as it is not so much combinatorial as 'geometric' or 'topological'. A purely combinatorial form might become increasingly difficult with increasing dimension, but a suitable geometric construction should mean 'the same' whatever dimension it is in.

We will think of the process of subdivision in a slightly different way and here a certain amount of informal imagery may help. We will make it more formal slightly later on.

Given $\mathcal{T}$ and e, we think of the new vertex, v, as being slightly off e and form the cone, $e * v$, on e with vertex v:



The diagram contains both e and the subdivided e (by going over the top of the triangular 'lump').

More generally, if σ is a simplex of $\mathcal{T}$ incident to e, then the join, $\sigma * v$, of v with σ will contain a copy of the subdivision of σ, resulting from subdividing e. (It may help to look at the classical treatment of subdivision as described, say, in Spanier, [198], p. 123.) Perhaps you can imagine the process of subdividing e as being like the formation of a little 'lump' on the union of the simplices incident to e. The original level corresponds to $\mathcal{T}$, the level going 'over the lump' corresponds to $\mathcal{T}'$, but we have both $\mathcal{T}$ and $\mathcal{T}'$ in the same simplicial set, which can be very useful. If you do not like the 'lump' picture, then instead you can make a cylinder on M, triangulate one end using $\mathcal{T}$ and the other end using $\mathcal{T}'$, then, if a simplex σ is not incident to e, triangulate $\sigma \times I$ in the standard way (as both ends of that part of the cylinder are the same), that is, using the product triangulation, whilst those simplices σ incident to e, (which are therefore subdivided in passing to $\mathcal{T}'$), you use a triangulation generalising the one given below for e itself:

$$\begin{tikzcd}
\cdot \arrow[r, "e_0"] & \cdot & \cdot \arrow[l, "e_1"'] \\
\cdot \arrow[u] \arrow[ur] \arrow[rr, "e"'] & & \cdot \arrow[u] \arrow[ul]
\end{tikzcd}$$

The arguments that we will use can be adapted to either picture. (Of course, the 'lump' is a subcomplex of the cylinder.)

Now, retaining that imagery, look at σ, σ_0, and σ_1, as part of a join $\sigma * v$, which will be a 3-simplex. (Remember we are in the case $a_0 < a_1 < a_2 < v$, and this 3-simplex is $\langle a_0, a_1, a_2, v\rangle$ in this new simplicial set.) We will look at the faces of this and where they are mapped to in $\overline{W}K(\mathsf{C})$, but, before we can really do that, we look at the cone, $v * e$, i.e., the new 2-simplex, $\langle a_0, a_1, v\rangle$. We have

- $d_0\langle a_0, a_1, v\rangle = \langle a_1, v\rangle$, so this can be assigned $\mu'(e_1)$;
- $d_1\langle a_0, a_1, v\rangle = \langle a_0, v\rangle$, so, similarly, corresponds to $\mu'(e_0)$;
- $d_2\langle a_0, a_1, v\rangle = \langle a_0, a_1\rangle$, and we use $\mu(e)$ on this.

From our discussion of composition above, we have that this gives us a thin filler for the $(2, 1)$-horn, corresponding to the d_0 and d_2 values and the d_1 of that filler will be $\mu(e)\mu'(e_1)$. We can see that this gives us *exactly* what we need to be able to say that $\mu'(e_0)\mu'(e_1)^{-1} = \mu(e)$, i.e., the thin filler *is* this equation, or perhaps slightly more exactly, *is the reason* for this equation. (This works perfectly for the case of G-colourings that we looked at earlier, and so is highly relevant here. We take it as the 'definition' of that equality.)

This suggests looking at the faces of $\sigma * v = \langle a_0, a_1, a_2, v\rangle$. We list them with a μ or μ' image where available.

- $d_0(\sigma * v) = \langle a_1, a_2, v\rangle = \sigma_1$, so use $\mu'(\sigma_1)$;
- $d_1(\sigma * v) = \langle a_0, a_2, v\rangle = \sigma_0$, so use $\mu'(\sigma_0)$;
- $d_2(\sigma * v) = \langle a_0, a_1, v\rangle$, use the thin filler above;
- $d_3(\sigma * v) = \langle a_0 a_1, a_2\rangle = \sigma$, so use $\mu(\sigma)$.

Taking the (3,1)-horn, we find a thin filler in $\overline{W}K(\mathsf{C})$, and this will, in fact, be *unique*, since $NG_2 = 1$. We thus get that $\mu'(e_1)$ is *uniquely determined* by the values of $\mu(\sigma)$ and $\mu'(e_0)$, as claimed.

We have to take account of all such contributions, from each simplex, σ, incident to e. We introduce a measure of the complexity of the triangulation that will help us handle this. Let K be an arbitrary (finite) simplicial complex and let $\chi_k(K) = (-1)^k\chi(sk_kK)$, where χ is the Euler characteristic, thus

$$\begin{aligned} \chi_0(K) &= \sharp(K_0), \\ \chi_1(K) &= \sharp(K_1) - \chi_0(K), \\ \chi_2(K) &= \sharp(K_2) - \chi_1(K), \end{aligned}$$

and so on.

Now, if $\mathcal{T}$ is a triangulation of the cobordism, M, as before, we set

$$\begin{aligned} \chi_0^{int}(\mathcal{T}) &= \sharp(\mathcal{T}_0 - T_0 - S_0), \\ \chi_1^{int}(\mathcal{T}) &= \sharp(\mathcal{T}_1 - T_1 - S_1) - \chi_0^{int}(\mathcal{T}), \\ \chi_k^{int}(\mathcal{T}) &= \sharp(\mathcal{T}_k - T_k - S_k) - \chi_{k-1}^{int}(\mathcal{T}). \end{aligned}$$

Later on, we will also need, $\chi_0^{\partial}(\mathcal{T}) = \sharp(T_0 \cup S_0)$, and then inductively,

$$\chi_k^{\partial}(\mathcal{T}) = \sharp(T_k \cup S_k) - \chi_{k-1}^{\partial}\mathcal{T},$$

the count of the boundary k-simplices of $(M, \mathcal{T})$.

The usefulness of these modified Euler characteristics is because of the following in which e is an interior 1-simplex of the triangulation, $\mathcal{T}$:

Lemma 62 *(Yetter, [225], for the case $k = 2$.) The number, $s_{k+1}(e)$, of $(k+1)$-simplices in $M - \partial M$ that are incident to e is*

$$\chi_k^{int}(\mathcal{T}') - \chi_k^{int}(\mathcal{T}),$$

where $\mathcal{T}'$ is obtained from $\mathcal{T}$ by edge stellar subdivision of the edge e.

Proof: The subdivision takes each $(k+1)$-simplex, σ, that is incident to e and replaces it by two such, incident to σ_0 and σ_1, respectively, (in the notation we used earlier). Typically, if the new vertex, v is considered greater than all other vertices of $\sigma = \langle a_0, \ldots, a_{k+1}\rangle$, the two replacement $(k+1)$-simplices will be $\langle a_0, a_2 \ldots, a_{k+1}, v\rangle$ and $\langle a_1, a_2 \ldots, a_{k+1}, v\rangle$. These meet in the face $\langle a_2 \ldots, a_{k+1}, v\rangle$ of dimension k. Other relative positions of v in the ordering yield similar results.

The net gain in χ_k^{int} in passing from $\mathcal{T}$ to $\mathcal{T}'$ is thus equal to $s_{k+1}(e)$. ∎

Now let

$$Z^!_{\mathsf{C}}(M, \mathbf{T}, \mathbf{S}) = \sharp(P)^{-\chi_0^{int}(\mathcal{T})}\sharp(C)^{-\chi_1^{int}(\mathcal{T})}Z_{\mathsf{C}}(M, \mathcal{T}).$$

Proposition 86 *The linear mapping, $Z^!_{\mathsf{C}}(M, \mathbf{T}, \mathbf{S})$, is independent of the choice of the triangulation, $\mathcal{T}$, extending $\mathbf{T}$, and $\mathbf{S}$ to the cobordism M.* ∎

The proof has essentially the same form as the earlier case, page 338, in which the place of the crossed module, C, was taken by the group, G, so this will be **left to you**.

We next need to 'fix' the problem of lack of 'compositionality'. The argument will be more or less the same as that for the G-colouring case. We refer you back to page 339 for the notation, etc. We get

Lemma 63

$$Z^!_G(N,\mathbf{S},\mathbf{R}).Z^!_G(M,\mathbf{T},\mathbf{S}) = \sharp(P)^{\chi_0(\mathbf{S})}\sharp(C)^{\chi_1(\mathbf{S})}Z^!_G(M+_Y N,\mathbf{T},\mathbf{R}).$$

Proof: The adjustment $\sharp(P)^{\chi_0(\mathbf{S})}\sharp(C)^{\chi_1(\mathbf{S})}$ corresponds to the different ways of colouring the $\mathbf{S}$, which is common to both cobordisms, the left hand term has a 'compensating factor' without any contribution from $\mathbf{S}$. The compensatory scaling factor on the right hand term has this adjustment factor in the denominator, so the equation balances. ∎

We again distribute this adjustment between the two ends of M to get a new linear map:

$$Z_{\mathsf{C}}(M,\mathbf{T},\mathbf{S}) = \sharp(P)^{-\frac{1}{2}\chi_0^{\partial}(\mathcal{T})}\sharp(C)^{-\frac{1}{2}\chi_1^{\partial}(\mathcal{T})}Z^!_{\mathsf{C}}(M,\mathbf{T},\mathbf{S}).$$

Note that $\chi_k^{\partial}(\mathcal{T})$ only depends on $\mathbf{T}$ and $\mathbf{S}$, so is, in fact, independent of $\mathcal{T}$. As an evident corollary, we obtain, with similar notation to before:

Corollary 16 *For M, $\mathbf{T}$, $\mathbf{S}$, etc., as above:*

$$Z_{\mathsf{C}}(N,\mathbf{S},\mathbf{R}).Z_{\mathsf{C}}(M,\mathbf{T},\mathbf{S}) = Z_{\mathsf{C}}(M+_Y N,\mathbf{T},\mathbf{R}).$$

∎

As we are following a parallel trail, more or less, to that that we used for Z_G, the next step is to go for the restriction maps,

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}} : \Lambda_{\mathsf{C}}(\mathbf{T}') \to \Lambda_{\mathsf{C}}(\mathbf{T}),$$

corresponding to subdividing $\mathbf{T}$. We already have partially met this when discussing the compensating terms for $\mathcal{T}'$ and $\mathcal{T}$, but that 'lump' imagery can, and should, be made more precise. In fact, this means that we will be able to treat arbitrary subdivisions, not just edge-stellar ones, if we want to.

As we mentioned before, one way to view a subdivision of a simplex is to see it as a cone on a subdivision of the boundary of the simplex. Suppose $s = \langle a_0,\ldots,a_n\rangle$ is an n-simplex in T, and take some point $v \in |s|$, the realisation of s. (We have $v \in \langle s'\rangle = \langle\{a \in \{a_0,\ldots,a_n\} \mid v(a) \neq 0\}\rangle$, often called the *carrier* of v . We will assume $s' = s$, so v is in the interor of $|s|$; if this is not the case, then replace s by s'.) Take ∂s to be the boundary of s and form $\partial s * v$, the join of ∂s with the 'new' vertex. The 'lump' comes from thinking of $|s|$ in the canonical realisation (i.e., in $\mathbb{R}^{V(T)}$, cf. page **??**). Subdividing adds a new vertex, which will increase the dimension of the ambient space. The 'new vertex', v, is not in $\mathbb{R}^{V(T)}$. We take s and form the cone $s*v$. This is the 'lump'! If s has dimension n, this has dimension $n+1$. It retains s itself, so, as a simplicial complex, it has T in it as a subcomplex, but it also has T' as one as well. (Of course, we have simplified things a bit here as, if $dim\, T > n$, then T' may have more than one simplex subdivided as that subdivision will need to be constructed 'up the skeletons'.) If needed, this 'ambient complex' provides a setting for various constructions. Most importantly, it contains both T and T' as deformation retracts, but we can do better than this algebraically.

We again will assume that $dim\, T = n$, so we are just handling one simplex and, again, for simplicity, assume that v, the new vertex is placed after all the a_i in the ordering on $V(T')$. (The other possibilities are not really any more difficult, but are a bit more 'messy'.) Within $G(T')$, we have a (n, n)-horn with i^{th} face the generator corresponding to $\langle a_0, \ldots, \widehat{a_i}, \ldots, a_n, v\rangle$, where the $\widehat{\ }$, of course, means that this term is omitted. Using the filling algorithm, we can fill this and obtain d_n of the filler, representing the original $s \in G(T)$. This filler, then, gives strong deformation retraction data:

$$r^{\mathbf{T}'}_{\mathbf{T}} : G(\mathbf{T}) \to G(\mathbf{T}'),$$
$$s^{\mathbf{T}}_{\mathbf{T}'} : G(\mathbf{T}') \to G(\mathbf{T}),$$

with $s^{\mathbf{T}}_{\mathbf{T}'} r^{\mathbf{T}'}_{\mathbf{T}} = id_{G(\mathbf{T})}$, and $r^{\mathbf{T}'}_{\mathbf{T}} s^{\mathbf{T}}_{\mathbf{T}'} \simeq id_{G(\mathbf{T}')}$, essentially using the description of the filler. Here $r^{\mathbf{T}'}_{\mathbf{T}}$ assigns to the generator corresponding to s, the algebraic composite obtained from d_n of the filler. For $s^{\mathbf{T}}_{\mathbf{T}'}$, choices have to be made. It can be chosen to be induced by a simplicial approximation to the identity on $X = |T| = |T'|$, or, more explicitly, by mapping all but one of the n-simplices generating the subdivided simplex to the identity, or, rather, to a degeneracy, with the remaining one mapping to s itself, thus collapsing $G(T')$ back to $G(T)$. (In fact, we will not use any explicit form of $s^{\mathbf{T}}_{\mathbf{T}'}$, just that it exists.) There are many different ways of constructing $s^{\mathbf{T}}_{\mathbf{T}'}$, but they are all homotopic. (If constructed simplicially, they are even *contiguous*.)

We note that $r^{\mathbf{T}'}_{\mathbf{T}}$ expresses the generator, s, *algebraically* as a pasting of the subdivided s. If s is not a top dimensional simplex, then the subdivision will need propagating up the skeletons of T, but this again can be done by a filling argument within $G(T')$ to get the $r^{\mathbf{T}'}_{\mathbf{T}}$ expressing s as a composite of its subdivided parts. If a subdivision, T', of T is obtained by subdividing several times, then we just iterate the above construction to get the appropriate $r^{\mathbf{T}'}_{\mathbf{T}}$. In terms of our 'lump', this algebraic construction inserts T as a subcomplex of the lumpy complex, L, and there is an *elementary collapse* from L to T, similarly there is one from L to T'. The maps $r^{\mathbf{T}'}_{\mathbf{T}}$ and $s^{\mathbf{T}}_{\mathbf{T}'}$ are algebraic models of the composites obtained from $T \overset{insert}{\to} L \overset{collapse}{\to} T'$, and the other way around. The construction of L, together with using fillers, gives the homotopy, $r^{\mathbf{T}'}_{\mathbf{T}} s^{\mathbf{T}}_{\mathbf{T}'} \simeq id_{G(\mathbf{T}')}$. (For 'elementary collapses' in general, look up sources on Simple Homotopy Theory.)

We define our restriction map, $\mathrm{res}_{\mathbf{T}',\mathbf{T}}$, by:
if $\lambda : G(T') \to K(\mathsf{C})$ is a C-colouring of $\mathbf{T}'$, then

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}} : \Lambda_{\mathsf{C}}(\mathbf{T}') \to \Lambda_{\mathsf{C}}(\mathbf{T})$$

sends λ to $\lambda \circ r^{\mathbf{T}'}_{\mathbf{T}}$ and then extend linearly to get it defined on $Z_{\mathsf{C}}(X, \mathbf{T}')$.

We immediately have that $\mathrm{res}_{\mathbf{T}',\mathbf{T}}$ is an epimorphism, since it is split by sending λ' to $\lambda' \circ s^{\mathbf{T}}_{\mathbf{T}'}$. We repeat that none of this requires that $\mathbf{T}'$ is obtained by edge-stellar subdivisions from $\mathbf{T}$, so we will usually quietly drop that assumption, except if needed later for some specific argument. We also note for use later that this definition did not require other than a (finite) simplicial group, G as 'coefficients' to make the definition work. No specific properties of $K(\mathsf{C})$ are used as everything important happens back in $G(T)$.

It will pay to examine the strong deformation retraction data for the 'lump' slightly more closely. Suppose that $\mathbf{T}'$ is obtained from $\mathbf{T}$ by forming $\partial s * v$ for some $s \in T_n$ and $v \in \langle s\rangle$, the carrier of s. This gave us a (n, n)-horn in $G(T')$, which we used to get the *algebraic* analogue of the 'lump' and thus the composite corresponding to $s \in G(T)$. That horn's parts all were n-simplices, so gave $(n-1)$-simplices of $G(T')$. The filling algorithm *only* uses these and their faces and so everything happens in $sk_{n-1}G(T')$, including the homotopy $r^{\mathbf{T}'}_{\mathbf{T}} s^{\mathbf{T}}_{\mathbf{T}'} \simeq id_{G(\mathbf{T}')}$. Intuitively, this is a 'thin' homotopy, i.e., if $n \geq 1$, the images of all the $(n-1)$-simplices use only $sk_{n-1}G(T')$. We will return to this slightly later on, formalising things a bit more.

As we had earlier for the G-colourings, we now define:

$$Z_{\mathsf{C}}(X) = colim_{\mathbf{T}} Z_{\mathsf{C}}(X, \mathbf{T}),$$

and can lift from that earlier discussion that the natural linear maps,

$$r^X_{\mathbf{T}} : Z_{\mathsf{C}}(X, \mathbf{T}) \to Z_{\mathsf{C}}(X),$$

are epimorphisms. We therefore have that $Z_{\mathsf{C}}(X)$ is a finite dimensional vector space. One point is that the $\mathrm{res}_{\mathbf{T}',\mathbf{T}}$ maps need to be scaled so as to obtain compatibility with the maps coming from the cobordisms. The scaling, in this case, gives

$$res_{\mathbf{T}',\mathbf{T}} : Z_{\mathsf{C}}(X, \mathbf{T}') \to Z_{\mathsf{C}}(X, \mathbf{T}),$$

by

$$res_{\mathbf{T}',\mathbf{T}} = \sharp(P)^{-\frac{1}{2}(\chi_0(T')-\chi_0(T))}\sharp(C)^{-\frac{1}{2}(\chi_1(T')-\chi_1(T))}\mathrm{res}_{\mathbf{T}',\mathbf{T}}.$$

This rescaling of the $\mathrm{res}_{\mathbf{T}',\mathbf{T}}$ does not change the end result, $Z_{\mathsf{C}}(X)$, at least, up to isomorphism. It does, however, change the quotient maps $r^X_{\mathbf{T}}$ by a scalar factor, and this *is* important, especially when doing calculations, as we will see slightly later.

We have

$$\begin{tikzcd}
Z_{\mathsf{C}}(X, \mathbf{T}) \arrow[dr, "r^X_{\mathbf{T}}"] & \\
& Z_{\mathsf{C}}(X) \\
Z_{\mathsf{C}}(X, \mathbf{T}') \arrow[uu, "\mathrm{res}_{\mathbf{T}',\mathbf{T}}"] \arrow[ur, "r^X_{\mathbf{T}'}"'] &
\end{tikzcd}$$

commutes where $\mathrm{res}_{\mathbf{T}',\mathbf{T}}(\lambda) = [\lambda]$, etc., but replacing $\mathrm{res}_{\mathbf{T}',\mathbf{T}}$ by its scaled version, $res_{\mathbf{T}',\mathbf{T}}$ leads to a diagram that does not commute. This can be fixed by scaling the quotienting maps, so replacing each $r^X_{\mathbf{T}}$ by $\rho^X_{\mathbf{T}} := \sharp(P)^{-\frac{1}{2}\chi_0(T))}\sharp(C)^{-\frac{1}{2}\chi_1(T))} r^X_{\mathbf{T}}$ gives us back commutativity, yet does not disturb the universal property of the codomain.

With that the following result should come as no surprise.

Theorem 28 *(Yetter, [225]) For any dimension d, the construction above applied to $(d+1)$–Cob_{PL} gives rise to a $(d+1)$-dimensional TQFT, Z_{C}.*

Proof: The 'proof' sketched above *is* incomplete. It is clear that although we 'thought' $d = 2$, there was no point at which we actually used that. The arguments did use tetrahedra, but these ended up mapped across to $\overline{W}K(\mathsf{C})$, so ended as being filled with thin elements, leading to commutativity and the 2-flatness conditions.

The other more major point we have left out of the discussion is that of identities. We will **leave this to you to check up** in the sources (although they are not all that explicit on this).■

Remarks: We have skated over the actual construction of $(d+1)-Cob$ as a category. This is well discussed in many convenient sources, so we will not go into it in all its gory detail, however some more comments probably are necessary.

The objects are d-manifolds, the morphisms are ...? What? We think of them as being cobordisms, but cobordisms compose *only up to homeomorphism* and similarly for identities. (It is the same old problem that is at the heart of a lot of what we have been doing.) For instance, $M := X \times [0,1]$ will be a cobordism between X and itself, so may serve as an identity morphism, but if $N : X \to Y$ is another cobordism, $M +_X N$ is not equal to N. (The only 'cobordism' that would work to give us equality would be to think of the d-manifold X as a $(d+1)$-cobordism, and that would open a 'can of worms'!) Of course, $M +_X N$ *is* homeomorphic to N, so really we should use either homeomorphism classes of cobordisms as the morphisms or encode the homeomorphisms somehow into the structure of the 'category'. This latter idea is, in the end, the better one, but means that the use of 'weak categories', or better, some form of quasicategory, is the way out of the difficulty. For more on this, see Lurie's summary, [148]. This involves as well the notion of extended TQFT, but we will not explore that here.

The epimorphism,

$$r^X_{\mathbf{T}} : Z_{\mathsf{C}}(X, \mathbf{T}) \to Z_{\mathsf{C}}(X),$$

means that we 'merely' have to understand when two C-colourings yield the same element of $Z_{\mathsf{C}}(X)$ and we will understand $Z_{\mathsf{C}}(X)$. We have analogues of the results of our earlier analysis of the G-colourings.

Proposition 87 *Suppose $\lambda : T \to \overline{W}K(\mathsf{C})$, and $\lambda' : T' \to \overline{W}K(\mathsf{C})$ are two C-colourings such that $r^X_{\mathbf{T}}(\lambda) = r^X_{\mathbf{T}'}(\lambda')$, then there is a joint subdivision, $\mathbf{T}''$, of $\mathbf{T}$ and $\mathbf{T}'$ and simplicial approximations to the identity, $\overline{s} : \mathbf{T}'' \to \mathbf{T}$ and $\overline{s'} : \mathbf{T}'' \to \mathbf{T}'$, such that $\lambda \circ s$ and $\lambda' \circ s'$ are thinly homotopic. (If $T_0 \cap T'_0$ is non-empty, the homotopy can be chosen to be constant on the vertices of this intersection.)*

Proof: We know that $\mathbf{T}''$ and a λ'' exist such that $res_{\mathbf{T}'',\mathbf{T}}(\lambda'') = \lambda$ and $res_{\mathbf{T}'',\mathbf{T}'}(\lambda'') = \lambda'$ from our discussion of the G-colouring case. We thus have

$$\lambda = \lambda'' \circ \overline{r},$$

where $\overline{r} : \mathbf{T} \to \overline{W}G(\mathbf{T}'')$ is adjoint to $r : G(\mathbf{T}) \to G(\mathbf{T}'')$. We noted that $rs : G(\mathbf{T}) \to G(\mathbf{T})$ is thinly homotopic to the identity, where s is part of the strong deformation retraction for $G(\mathbf{T})$ and $G(\mathbf{T}'')$, chosen by collapsing the 'lump' or, equivalently, as a simplicial approximation to the identity. (It can be chosen so as to fix old vertices if that is needed.) We thus have

$$\lambda \circ \overline{s} = \lambda'' \circ \overline{r} \circ \overline{s} \simeq_{thin} \lambda'' \circ \eta_{\mathbf{T}''} : \mathbf{T}'' \to \overline{W}G(T'') \to K(\mathsf{C}),$$

where $\eta_{T''}$ is the unit of the adjunction between $\overline{W}$ and G. To understand this enough to proceed, we will need to make precise the notion of thin homotopy here.

Definition: Suppose that G and H are two $\mathcal{S}$-groupoids, and $f_0, f_1 : G \to H$ are two morphisms of $\mathcal{S}$, which are homotopic by a homotopy h. We say h is a *thin* homotopy if for each $x \in G_n$, the homotopy restricted to x takes values in $sk_n H$, the n-skeleton of H.

Of course, we can use the description of homotopies as families of maps, $\{h_i^n : G_n \to H_{n+1}\}$, to be slightly more explicit; (refer back to page 294 if you need that description). We would have that h is thin if for all n and appropriate i, $h_i^n : G_n \to D(H_{n+1})$, the subgroupoid in H_{n+1} generated by the degenerate elements. As we are handling $\mathcal{S}$-groupoids, we have a map from $Ob(G)$ to H_0 related to the f_0 on the objects in the evident way. This means that a thin homotopy can move the objects along a 'path' in the codomain. **You are left to examine** the way in which 'thinness' of homotopies between maps from $G(T)$ to a simplicial group, G, transforms into special homotopies between maps from T to $\overline{W}G$. Again 'thin' seems a good term to use for this type of homotopy, but **you are left to formalise it**.

In words, a thin homotopy deforms the images of a n-simplex, x, from $f_0(x)$ to $f_1(x)$, within the n-skeleton, never using any non-degenerate parts of H_{n+1}. (We will briefly discuss the terminology after the end of the proof.) It is clear that the homotopy, $h : rs \simeq id_{G(T)}$, is thin in the sense above, as it uses just composites of degenerate elements. It will correspond to a thin homotopy from $\overline{rs}$ to the unit of the adjunction. (Recall that there are *explicit* formulae for the adjointness relationship between G and $\overline{W}$.)

We thus have thin homotopies

$$\lambda \circ \overline{s} \simeq_{thin} \lambda'' \circ \eta_{T''},$$

and, similarly,

$$\lambda' \circ \overline{s'} = \lambda'' \circ \overline{r} \circ \overline{s'} \simeq_{thin} \lambda'' \circ \eta_{T''},$$

which we can glue to get one as required. ■

Remark: The notion of 'thin homotopy' goes right back to the heart of the link between simplicial sets and weak infinity categories, and thus to the link between geometric or topological cohomology and infinity category theory.

When forming the fundamental group or groupoid of a space, X, the homotopies used are rather special. Take for instance, when proving that composition of (normalised) path classes is associative. We have paths, $a, b, c, : I \to X$ such that $a * (b * c)$ is defined, (so end points match in the usual way). Here $a * b$ is the usual normalised concatenated path,

$$a * b(t) = \begin{cases} a(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ b(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Its essence is a map from $I\,{}_{d_0}\sqcup_{d_1} I \cong [0,2]$ to I sending t to $\frac{1}{2}t$. This composition starts as a co-operation in the 'models', i.e., in the intervals used to 'probe' the space. Proving that $a * (b * c)$ is homotopic to $(a * b) * c$ uses a homotopy defined *in the interval*, so the image of that homotopy in X happens completely within the trace of the composite path. It does not sweep out any 'surface'. Its image is a very 'thin' region of X. The homotopy has domain I^2 of dimension 2, but its image, somehow, has 'dimension 1'.

The singular complex of X gives us another instance of such thinness. The simplicial set, $Sing(X)$ is a Kan complex. The fillers for horns can be chosen to be 'thin', since if $x : \Lambda^i[n] \to Sing(X)$ is a horn, then we can find a filler which just uses the information in the horn. That is clearly of dimension $n-1$. We can use a retraction of Δ^n to the geometric realisation of $\Lambda^i[n]$, which is the union of all but the i^{th} face of Δ^n. We then have the filler $\overline{x} \in Sing(X)_n$ is the composite

$$\Delta^n \to |\Lambda^i[n]| \xrightarrow{x} X.$$

this, of course, only uses the parts of $Sing(X)$ of dimension $(n-1)$ or less. It is a 'thin' filler in an intuitive sense (although it does not give a T-complex structure to $Sing(X)$, since that would require uniqueness of such a thin filler; see the discussion back on page **??**.)

When defining the fundamental 2-groupoid of a (Hausdorff) space, X, Hardie, Kamps and Kieboom use thin homotopies to prevent a collapse of the 2-dimensional information on X (see [44, 112] and also [113]), and, of course, in the work of Brown and Higgins, filtered spaces and filtered homotopies are used for the same purpose, [49].

A similar idea was used by John Roberts explicitly for cohomology and, via Street's study, [201], this led to the work of Verity on complicial sets, [212] and their weakened form, [213–215], that we saw earlier, page **??**.

In [180], the naturally occurring need for thin homotopy was filled by what was, there, called 'filtered homotopy'. Here we used 'thin' rather than 'filtered' as that term really fits the bill better.

Finally. in the differential geometric setting, a related notion of thinness has become quite common, (cf. [19, 61, 153]). Here we need X to be some sort of smooth space such as a smooth manifold, and $a, b : I \to X$ smooth paths with the same end-points (often with the property that they are constant in a neighbourhood of the end points of the interval). A (fixed end-point) homotopy, $h : I^2 \to X$, between them is thin if the rank of the differential, $Dh : T_{\underline{t}}I^2 \to T_{h(\underline{t})}X$, is everywhere smaller than 2. Intuitively that means that there is no 'transversal' deformation of the path and, again, we get the idea that the homotopy does not 'sweep out any surface' in X.

In our situation, the clear connection is with the notion of thin elements in a group T-complex (page **??**). The thin elements there are products of degenerate elements, but to get a T-complex, we needed the $NG \cap D = 1$ condition, which corresponded to uniqueness of thin fillers.

The above result has a partial converse:

Proposition 88 *Suppose that T is an ordered triangulation of X and $\lambda, \lambda' : T \to \overline{W}K(\mathsf{C})$ are C-colourings. If there is a thin homotopy, $h : \lambda \simeq \lambda'$, then $r_T^X(\lambda) = r_T^X(\lambda')$.* ■

We will not prove this here, so **look at the proof in** [179]. The method is probably clear. You have to look for a subdivision of T and a C-colouring, λ'', which restricts to the two given colourings. The first thing to do is to look at h at the level of $\langle v \rangle \times I$ for the various $v \in T_0$ and to build a subdivision bit-by-bit, together with the C-colouring. (To some extent you can think of this as taking two copies of T, displacing one a bit then forming the joint subdivision in a fairly standard way. That *is* incorrect, but gives a bit of intuition that might help.)

As a result of this, we obtain

Theorem 29 *The vector space, $Z_{\mathsf{C}}(X)$, has a basis which is in bijective correspondence with the set, $[G(T), K(\mathsf{C})]_{thin}$, of thin homotopy classes of morphisms from $G(T)$ to $K(\mathsf{C})$, for any triangulation* $\mathbf{T}$ *of X.* ■

Any simplicial map, $\lambda : G(T) \to K(\mathsf{C})$ must kill all the higher dimensional Moore complex terms of $G(T)$, since the Moore complex of $K(\mathsf{C})$ is precisely C. The Moore complex, $N(G(T))$, will be mapped to $M(G(T), 2)$, which is the crossed module (of groupoids over T_0 as its set of objects),

$$\frac{NG(T)_1}{d_0(NG(T)_2} \xrightarrow{\partial} G(T)_0,$$

and there will be an induced map of crossed modules,

$$\lambda : M(G(T), 2) \to \mathsf{C}.$$

As we have a crossed module of groupoids as the domain and a single object one as codomain, a homotopy between two such C-colourings can assign an element of $C_0 = P$ to each object $\langle a_0 \rangle$ of $G(T)$. (This is the conjugation that is apparent in the simple G-colouring case.) Usually, in an arbitrary homotopy of crossed modules, there would be a derivation from $G(T)_0 = M(G(T), 2)_0$ to $C_1 = C$, but, if that homotopy is thin, this derivation must be trivial. As there is nothing in C above dimension 1, we have that thin homotopies are just conjugations.

It is worth experimenting with what happens if, instead of C being a crossed *module*, we had that it was a crossed *complex*. We would have (after adapting earlier results) the crossed complex $\pi(T)$ and a C-colouring would be a morphism of crossed complexes, $\lambda : \pi(T) \to \mathsf{C}$. **You are left to investigate** what 'thin' means in this case. It may help to consider $\mathrm{C{\small RS}}(\pi(T), \mathsf{C})$, the mapping crossed complex. We will look at an even more general setting in the next section. (For the above, it may be useful to look at Faria Martins and Porter, [98] and others of Faria Martins' papers, [96, 97], for some methods. Some of the suggested investigation is not detailed anywhere in the published literature and the outcomes of some of the questions that may occur to you may not be explicitly known.)

We now turn to the version of the Yetter invariant of a closed manifold, M, with target a finite crossed module. This is the value taken by $Z_{\mathsf{C}}(M)$ on the empty C-colouring.

As in our earlier discussion, M is a $(d+1)$ dimensional closed manifold (usually assumed to be PL, since we use triangulations). It is thought of as being a cobordism from the empty d-manifold to itself. We have, almost as before (page 346), that, for $\mathcal{T}$, a triangulation of M,

$$Z_{\mathsf{C}}(M, \mathcal{T}) = \sharp(\Lambda_{\mathsf{C}}(\mathcal{T})),$$

and we set

$$I_{\mathsf{C}}(M) = \sharp(P)^{-\chi_0(\mathcal{T})} \sharp(C)^{-\chi_1(\mathcal{T})} \sharp(\Lambda_{\mathsf{C}}(\mathcal{T})),$$

Corollary 17 *(of Proposition 93, page 377) The quantity, $I_{\mathsf{C}}(M)$, is independent of the choice of triangulation.* ■

We can do some trivial manipulations of this expression. We have $\chi_1(\mathcal{T}) = \sharp(\mathcal{T}_1) - \sharp(\mathcal{T}_0)$, so, to ease notation, writing $n_i = \sharp(\mathcal{T}_i)$, then

$$I_{\mathsf{C}}(M) = \frac{\sharp(C)^{n_0}}{\sharp(P)^{n_0} \sharp(C)^{n_1}} \sharp(\Lambda_{\mathsf{C}}(\mathcal{T})).$$

If we write $N = \partial C$, $\pi_0 = P/N$ and $\pi_1 = Ker\,\partial$, then clearly

$$\frac{\sharp(C)^{n_0}}{\sharp(P)^{n_0}} = \Big(\frac{\sharp(\pi_1)}{\sharp(\pi_0)}\Big)^{n_0},$$

and the expression begins to be similar to the scaling factors for Quinn's FTH theory, (see page 334). The precise link is explored in Faria-Martins and Porter, [98], using a bit more of the theory of crossed complexes than we have assumed here, so you are left to look that up. (Remember to check on the conventions as right actions are being used there.) The description uses the space $Map(M, B\mathsf{C})$ and is essentially

$$I_{\mathsf{C}}(M) = \sharp^{\pi}(Map(M, B\mathsf{C}))$$

in the notation that we introduced earlier (page 333).

This quantity, $I_{\mathsf{C}}(M)$, is referred to as the *Yetter Invariant* for M (with target, C).

Example: Although we will investigate this more fully in the next section, it is interesting to note that, if ∂ is a monomorphism, so that C is isomorphic to an inclusion crossed module, $N \stackrel{\triangleleft}{\to} P$ (with $N = \partial C$), then

$$I_{\mathsf{C}}(M) = I_{\pi_0}(M),$$

as one would expect.

This follows because π_1 is trivial and, in this situation,

$$\sharp(\Lambda_{\mathsf{C}}(\mathcal{T})) = \sharp(N)^{n_1}\sharp(\Lambda_{\pi_0}(\mathcal{T})).$$

8.3 Examples, calculations, etc.

The above example suggests that there should be links between Z_G and Z_{C}, for C, an inclusion crossed module with $\pi_0 \cong G$. More generally, we might suggest there to be a relationship between Z_{C} and the two related TQFTs, Z_{π_0} and $Z_{\pi_1[1]}$, where $\pi_1[1]$ is to be the crossed module, $\pi_1 \to 1$. That link is much harder to investigate and we will *not* be giving any deep general results on, just scratching the surface, as little is known about the general situation.

To start on this open problem in a semi-systematic way, let us look at some examples of Z_{C} for different types of crossed module, C.

We retain our previous notation for spaces, (ordered) triangulations, etc.

8.3.1 Example 0: The trivial example

As a first, almost silly, example, we can consider Z_G, when G is the trivial group. **You are left to check** that the resulting TQFT is the trivial one, which is constant with value the ground field.

8.3.2 Example 1: $\mathsf{C} = (1, G, inc) = K(G, 0)$

We would expect that, in this case, Z_{C} would reduce to just Z_G, and, of course, it does, but, as usual, it is useful to examine even this simplest case although it is 'clear'.

A C-colouring of a triangulation T of a manifold, X, is an assignment of elements of G to the edges on T satisfying the commutativity condition (of page 335), since the only value that we can assign to a triangle is 1, i.e., a C-colouring *is* just a G-colouring. Next glance at each of the scaling factors in turn. In each we set $\sharp(C) = 1$, and, of course, retrieve the corresponding values used in the construction of Z_G.

That was 'obvious'. We will have to work harder when $\mathsf{C} = (N, P, inc)$ with $P/N \cong G$, but that is for later.

8.3.3 Example 2: $\mathsf{C} = (A, 1, \partial) = K(A, 1)$

Here we take A to be a finite Abelian group, so C has $P = 1$, the trivial group. A C-colouring, λ of T assigns an element, $\lambda(a, b, c)$, in A to each triangle, $\langle a, b, c\rangle$, in T. These are to satisfy: for $a < b < c < d$, (so $\langle a, b, c, d\rangle$ is a 3-simplex of T),

$$\lambda(a, b, c)\lambda(a, c, d) = \lambda(b, c, d)\lambda(a, b, d).$$

This is just the simplified form of the colouring cocycle condition (page 352) that results since A is Abelian allowing several terms to cancel in the original general formula.

Writing things additively, this is just

$$\lambda(b, c, d) - \lambda(a, c, d) + \lambda(a, b, d) - \lambda(a, b, c) = 0,$$

so C-colourings are just 2-cocycles in the classical sense of simplicial cohomology theory (cf. section **??**) applied to the simplicial complex, T. We have, in fact, a bijection

$$\Lambda_{\mathsf{C}}(T) \leftrightarrow Z^2(T, A),$$

where we follow the traditional notation, as used, for instance, on page **??**, in denoting by $Z^2(T, A)$, the group of 2-cocycles of T with coefficients in A.

Looking at the restriction maps, we find that $res_{\mathbf{T}',\mathbf{T}}$ sends a 2-cocycle on $\mathbf{T}'$ to, at worst, a multiple of a cocycle on $\mathbf{T}$, never anything more complex in the way of linear combinations. Moreover, it is part of a chain homotopy equivalence between $C(\mathbf{T}', A)$ and $C(\mathbf{T}, A)$, the simplicial chain complexes on $\mathbf{T}'$ and $\mathbf{T}$ respectively, since it is induced by part of a strong deformation retraction on the corresponding loop $\mathcal{S}$-groupoid level. If we now check through the quotient maps from $Z_{\mathsf{C}}(X, T)$ to $Z_{\mathsf{C}}(X)$, we find they correspond exactly to the quotienting from $\mathbb{C}[Z^2(T, A)]$ to $\mathbb{C}[H^2(X, A)]$. (The argument has a 'classical' feel to it and **you should be able to build it fairly easily**. It can be found in Yetter's paper, [225].) In other words, $Z_{\mathsf{C}}(X)$ is the vector space $\mathbb{C}[H^2(X, A)]$ generated by $H^2(X, A)$.

Turning to the cobordisms, we have $M : X \to Y$ and a triangulation, $\mathcal{T}$, of M extending T on X and S on Y. The analysis of the scaling factors uses simple counting arguments based on the identification of $\sharp(A)^{\sharp(\mathcal{T}_i)}$ as being

$$\frac{\sharp(C^i(\mathcal{T}, A))}{\sharp(C^i(T, A))\sharp(C^i(S, A))},$$

followed by use of the exact sequences

$$0 \to Z^i(\mathcal{T}, A) \to C^i(\mathcal{T}, A) \to B^i(\mathcal{T}, A) \to 0,$$

etc. Again, it is worth playing with these before turning to Yetter's paper for the analysis in the case of surfaces. (His argument crucially uses that each edge in T is incident to two 2-simplices, i.e., that he is working with $3 - Cob$.) The situation for other dimensions than $d = 2$ **does not seem to be known**.

We will not be using Yetter's result, so merely record the value of the Yetter invariant in this case, for M, a 3-manifold. It is

$$I_{\mathsf{C}}(M) = \frac{\sharp(H^0(M,A))\sharp(H^2(M,A))}{\sharp(H^1(M,A))},$$

which certainly suggests a good candidate for it for higher dimensional manifolds.

8.3.4 Example 3: C, a contractible crossed module

The above two examples seem to suggest that Z_{C}, in general, should depend on the homotopy of C, i.e. at least on its π_0 and π_1, (or π_1 and π_2, depending on whether algebraic or topological grading is being used). This idea then suggests that we look at a very simple test case, namely a crossed module for which both π_0 and π_1 are trivial.

For a group, P, we can form $\mathsf{P} = (P, P, =)$. Of course, the obvious morphism, $\mathsf{P} \to 1$, is a weak homotopy equivalence and $K(\mathsf{P})$ is, in fact, contractible as a simplicial group, as is easily verified. We might guess Z_{P} was, therefore, the same as Z_1, the trivial TQFT.

To verify that our guess is correct, we can use the plan sketched out earlier for determining $Z_{\mathsf{C}}(X)$, namely first looking at $Z_{\mathsf{C}}(X,T)$, then analyse the kernel of r^X_T using Propositions 87 and 88, or, alternatively, by showing that any $\lambda : G(T) \to K(\mathsf{P})$ is thinly homotopic to the trivial P-colouring, so by Theorem 29, $Z_{\mathsf{P}}(X) \cong \Bbbk$.

We will take this chance to take apart the morphisms from $G(T)$ to this crossed module, P, as this provides a good illustration of various features of colourings in an elementary example, that we, hopefully, can put to good use later.

We first recall that if $a < b$ in T_0, then $\langle a, b\rangle$ will define a 0-simplex in $G(T)(a,b)$. In the next dimension, if $a < b < c$, the 1-simplex, $\langle a,b,c\rangle \in G(T)(a,b)_1$ goes from $\langle a,b\rangle$ to $\langle a,c\rangle.\langle b,c\rangle^{-1}$. Now assume we have a P-colouring,

$$\lambda : G(T) \to K(\mathsf{P}),$$

then $\lambda\langle a,b,c\rangle$ is of form $(p(a,b,c), \lambda(z,b))$ and, as usual, we have $\partial p(a,b,c)\lambda(a,b) = \lambda(a,c)\lambda(b,c)^{-1}$. Let us abbreviate $\lambda(a,b) = x_2$, etc, (so $x_i = \lambda d_i\langle a,b,c\rangle$), and $\lambda\langle a,b,c\rangle = x$, and we translate the condition to give

$$x : x_2 \Rightarrow x_1.x_0^{-1},$$

and, hence, $\partial x = x_1.x_0^{-1}.x_2^{-1}$, (cf. the definition of C-colouring on page 351). That is true for any crossed modules, but in P, ∂ is the identity morphism, so $x = x_1x_0^{-1}x_2^{-1}$. We have:

Lemma 64 *Any assignment, $\lambda : T_1 \to P$, extends uniquely to a P-coloring of T.* ■

We have, within $K(\mathsf{P})$, the 1-simplex

$$x_2 \stackrel{(x,x_2)}{\longrightarrow} x_1x_0^{-1},$$

corresponding to λ, and we want to show λ is equivalent to 1, the trivial P-colouring. We can shift the vertices to 1 along paths such as

$$x_2 \stackrel{(x_2^{-1},x_2)}{\longrightarrow} 1,$$

so need next to see how to use thin elements to produce a thin homotopy extending this. On (x, x_2), we can build a 'local homotopy':

$$\begin{tikzcd}
1 \arrow[r, "1"] & 1 \\
x_2 \arrow[u, "{(x_2^{-1},x_2)}"] \arrow[ur] \arrow[r, "{(x,x_2)}"'] & x_1x_0^{-1} \arrow[u, "{(x_0x_1^{-1},x_1x_0^{-1})}"']
\end{tikzcd}$$

in which the diagonal is also (x_2^{-1}, x_2). The top-left 2-simplex is degenerate, being just $s_1(x_2^{-1}, x_2)$. The bottom right one is thin. It is the thin filler of the $(2, 1)$-horn given by the non-diagonal edges. (Recall (page 352) that the composite 'g_2 then g_0' is obtained by forming the filler,

$$s_1g_2.s_1s_0d_0(g_2)^{-1}.s_0(g_0),$$

and then taking its d_1-face to get $g_2.s_0d_0(g_2)^{-1}.g_0$.) The composite on the diagonal is thus the product $(x, x_2).(1, x_0x_1^{-1}).(x_0x_1^{-1}.x_1x_0^{-1})$, calculated within $K(\mathsf{P})_1 = P \rtimes P$, where the right hand P acts by conjugation on the left hand one. It is **routine to check** that this is (x_2^{-1}, x_2).

As $K(\mathsf{P})$ has no non-thin n-simplexes for $n > 1$, this is automatically going to give a thin homotopy, but it *is* useful to see exactly its form and the manner in which it depends explicitly on λ. The fact that the simplicial group, $K(\mathsf{P})$, has no non-thin elements in higher dimensions also tells us that we do not have to do anything more! Our local thin homotopy extends without problem to a thin homotopy between λ and 1, and hence the P-colouring, λ, is equivalent to the constant colouring, 1, but, as it was arbitrary, we have that $Z_{\mathsf{P}}(X) \cong \mathbb{C}$, as expected.

The transformations corresponding to the cobordisms also need looking at. To do this, we examine a cobordism, $M : X \to Y$, as usual, and obtain the commutative diagram:

$$\begin{tikzcd}
Z_{\mathsf{P}}(X,T) \arrow[r, "{Z_{\mathsf{P}}(M,T,S)}"] \arrow[d, "\rho_T^X"'] & Z_{\mathsf{P}}(Y,S) \arrow[d, "\rho_S^Y"] \\
Z_{\mathsf{P}}(X) \arrow[r, "Z_{\mathsf{P}}(M)"'] & Z_{\mathsf{P}}(Y)
\end{tikzcd}$$

We know $Z_{\mathsf{P}}(X)$ and $Z_{\mathsf{P}}(Y)$ are copies of $\mathbb{C}$, so have to work out $Z_{\mathsf{P}}(M)[1]$ as a number. (We really only need to show it to be constant, i.e., independent of M, as that would suffice. Actually we can calculate it exactly, so do not need that safety net!)

As the information that we will need is a bit scattered through the previous section, we will recall each fact that we need and will give it in general (we will reuse some of this slightly later) as well as in the form for P. As usual $\mathsf{C} = (C, P, \partial)$ will be our general crossed module.

- If $\lambda \in \Lambda_{\mathsf{C}}(T)$ and, as before, $\mathcal{T}$ triangulates M, extending T on X and S on Y, then

$$\begin{aligned} Z^!_{\mathsf{C}}(M, \mathcal{T})(\lambda) &= \sum\{\mu|_S \mid \mu \in \Lambda_{\mathsf{C}}(\mathcal{T}), \mu|_T = \lambda\} \\ &= \sum\{\mu|_S \mid \mu \in \Lambda_{\mathsf{C}}(\mathcal{T})_\lambda\}, \end{aligned}$$

  thereby introducing $\Lambda_{\mathsf{C}}(\mathcal{T})_\lambda$ as a shorthand for the set of μ extending λ on T. (This does not change for $\mathsf{C} = \mathsf{P}$.)

- $$Z^!_{\mathsf{C}}(M,T,S)(\lambda) = \sharp(P)^{-\chi_0^{int}(\mathcal{T})}\sharp(C)^{-\chi_1^{int}(\mathcal{T})}Z^!_{\mathsf{C}}(M,\mathcal{T})(\lambda),$$

  which, in the case $\mathsf{C} = \mathsf{P}$, gives

  $$Z^!_{\mathsf{C}}(M,T,S)(\lambda) = \frac{\sharp(P)^{\sharp(T_1)}\sharp(P)^{\sharp(S_1)}}{\sharp(P)^{\sharp(\mathcal{T}_1)}}\sum\{\mu|_S \mid \mu \in \Lambda_{\mathsf{C}}(\mathcal{T})_\lambda\}$$

  using the definition of $\chi_1^{int}(\mathcal{T})$.

- $$Z_{\mathsf{C}}(M,T,S)(\lambda) = \sharp(P)^{-\frac{1}{2}\chi_0^{\partial}(\mathcal{T})}\sharp(C)^{-\frac{1}{2}\chi_1^{\partial}(\mathcal{T})}Z^!_{\mathsf{C}}(M,T,S)(\lambda),$$

  and, when $\mathsf{C} = \mathsf{P}$,

  $$Z_{\mathsf{C}}(M,T,S)(\lambda) = \frac{\sharp(P)^{\frac{1}{2}\sharp(T_1)}\sharp(P)^{\frac{1}{2}\sharp(S_1)}}{\sharp(P)^{\sharp(\mathcal{T}_1)}}\sum\{\mu|_S \mid \mu \in \Lambda_{\mathsf{C}}(\mathcal{T})_\lambda\}.$$

- $$\rho^X_T(\lambda) = \sharp(P)^{-\frac{1}{2}\chi_0(T)}\sharp(C)^{-\frac{1}{2}\chi_1(T)}[\lambda],$$

  so, for the case, $\mathsf{C} = \mathsf{P}$, in which $[\lambda] = [1]$,

  $$\rho^X_T(\lambda) = \sharp(P)^{-\frac{1}{2}\sharp(T_1)}[1]$$

  and similarly for $\rho^X_T(\mu|_S) = \sharp(P)^{-\frac{1}{2}\sharp(S_1)}[1]$.

We now can track λ, or, more exactly, the basis element labelled by λ, around the commutative square in the two possible ways.

- Down-then-right:
  $$Z_{\mathsf{P}}(M)\rho^X_T(\lambda) = \sharp(P)^{-\frac{1}{2}\sharp(T_1)}Z_{\mathsf{P}}(M)[1].$$

- Right-then-down:

$$\begin{aligned}\rho^Y_S Z_{\mathsf{P}}(M,T,S)(\lambda) &= \sharp(P)^{-\frac{1}{2}\sharp(S_1)}\frac{\sharp(P)^{\frac{1}{2}\sharp(T_1)}\sharp(P)^{\frac{1}{2}\sharp(S_1)}}{\sharp(P)^{\sharp(\mathcal{T}_1)}}\sharp(\Lambda_{\mathsf{P}}(\mathcal{T})_\lambda)[1] \\ &= \frac{\sharp(P)^{\frac{1}{2}\sharp(T_1)}}{\sharp(P)^{\sharp(\mathcal{T}_1)}}\sharp(\Lambda_{\mathsf{P}}(\mathcal{T})_\lambda)[1].\end{aligned}$$

We therefore have that

$$Z_{\mathsf{P}}(M)[1] = \frac{\sharp(P)^{\sharp(T_1)}}{\sharp(P)^{\sharp(\mathcal{T}_1)}}\sharp(\Lambda_{\mathsf{P}}(\mathcal{T})_\lambda)[1],$$

so we have to count the P-colourings of $\mathcal{T}$, which have value λ on T. We note that any assignment, $\mu : \mathcal{T}_1 \to P$, extends *uniquely* to a P-colouring, and conversely. We thus have $\sharp(\Lambda_{\mathsf{P}}(\mathcal{T}) = \sharp(P)^{\sharp(\mathcal{T}_1)}$, and so

$$\sharp(\Lambda_{\mathsf{P}}(\mathcal{T})_\lambda) = \frac{\sharp(P)^{\sharp(\mathcal{T}_1)}}{\sharp(P)^{\sharp(T_1)}}.$$

We thus have

Lemma 65

$$Z_{\mathsf{P}}(M) = 1.$$

■

and

Proposition 89 *For the crossed module,* $\mathsf{P} = (P, P, =)$*,* $Z_{\mathsf{P}} \cong Z_1$*, the trivial TQFT with constant value,* $\mathbb{C}$*.* ■

8.3.5 Example 4: C, an inclusion crossed module

The next most obvious case to look at is that of inclusion crossed modules, i.e., when $\mathsf{C} = (C, P, \partial)$ has ∂ a monomorphism. We write $N = \partial C$, so N is just a normal subgroup of P, set $G = P/N$, and assume that ∂ is an actual inclusion, so $\mathsf{C} = (N, P, inc)$.

We can think of this as giving a morphism of crossed modules, $\mathsf{p} : \mathsf{C} \to \mathsf{G}$:

$$\begin{tikzcd} N \arrow[r, "p_1"] \arrow[d] & 1 \arrow[d] \\ P \arrow[r, "p_0"] & G \end{tikzcd}$$

that is, from C to what we will denote by G, or $K(G, 0)$, the 'crossed module', $(1, G, inc)$. We saw (page 366) that $I_{\mathsf{C}}(M)$ and $I_G(M)$ are the same. We would expect there to be a 'very close' relationship between Z_{C} and Z_G as well, and, of course, the previous example showed this in the case where G is trivial.

Given any C-colouring of a triangulation, T, we clearly get a G-colouring by composing with p. The commutativity cocycle condition is easy to check. We thus get a function

$$\mathsf{p}_* : \Lambda_{\mathsf{C}}(T) \to \Lambda_G(T).$$

Is this a bijection? This is unlikely, except in very exceptional cases.

Is it a surjection? Can we reverse the process and build a C-colouring from a G-colouring?

We pick a transversal, $t : G \to P$, so $p_0 t(g) = g$ for all $g \in G$. We can assume that it is 'normalised', i.e., that $t(1_G) = 1_P$, but, of course, it need not be a homomorphism. Given any G-colouring, λ of T, we use t to try to build a C-colouring of T. If $\langle a, b\rangle \in T_1$, we try $\lambda'\langle a, b\rangle = t\lambda\langle a, b\rangle$ and, if $\langle a, b, c\rangle \in T_2$, we set

$$\lambda'\langle a, b, c\rangle = t\lambda\langle a, c\rangle.(t\lambda\langle b, c\rangle)^{-1}.(t\lambda\langle a, b\rangle)^{-1} \in N,$$

which automatically satisfies the boundary condition.

Lemma 66 *The assignment,* λ'*, satisfies the cocycle condition.*

Proof: The proof is by direct calculation, so that is **left to you**. For a 3-simplex, $\langle a, b, c, d\rangle$, both composites in the cocycle 'square' come to $t\lambda\langle a, b\rangle.t\lambda\langle b, c\rangle.t\lambda\langle c, d\rangle.t\lambda\langle a, d\rangle^{-1}$. ■

We thus have that λ' is a C-colouring and, as $\mathsf{p}_*(\lambda') = \lambda$, we have:

Proposition 90 *The function,*

$$\mathsf{p}_* : \Lambda_{\mathsf{C}}(T) \to \Lambda_G(T),$$

is a surjection. ■

It is easy to see that each λ, in fact, gives rise to $\sharp(N)^{\sharp(T_1)}$ different C-colourings, since if we pick any function from T_1 to N, say, $\{n(a,b) \in N \mid \langle a,b\rangle \in T_1\}$, then we get a new C-colouring, $\{t\lambda(a,b)n(a,b) \mid \langle a,b\rangle \in T_1\}$. This effectively changes the chosen transversal, t, to a new one, so the result is a C-colouring. Conversely, if λ_1 and λ_2 are two C-colourings with $p\lambda_1 = p\lambda_2$, then $p(\lambda_1\lambda_2^{-1})$ is trivial and setting $n(a,b) = \lambda_1(a,b).\lambda_2(a,b)^{-1}$ gives a function from T_1 to N, as before.

The idea of our attack will be to adapt techniques from our previous example almost, but not quite, as if attacking the $n(a,b)$s that link a given $\lambda \in \Lambda_{\mathsf{C}}(T)$ to one of the form $tp_*(\lambda)$. To see why this might work, we note that p induces an epimorphism

$$Z_{\mathsf{C}}(X,T) \to Z_G(X,T),$$

and also one,

$$Z_{\mathsf{C}}(X) \to Z_G(X),$$

compatibly with the projections to the colimits. (To see that the second of these is an epimorphism, it suffices to see how it is defined. Given an element of its domain, you pick a linear combination of colourings mapping to it, than map that across to a combination of G-colourings and finally back down to $Z_G(X)$. That will be well defined, as we will show.) As these maps p_* and t_*, are defined by mapping basis elements, it is simple to find a basis for $Ker\,\mathsf{p}_*$, namely all the $\lambda - tp(\lambda)$ that are not zero.

We have some observations which seem to be useful:

(i) if, for each λ, $\lambda \simeq_{thin} tp(\lambda)$, then all elements of $Ker\,\mathsf{p}_*$ will vanish in the quotienting process. Put more simply and precisely: $Ker\,\mathsf{p}_* \subseteq Ker\,\rho_T^X$.

(ii) If we prove that $Ker\,\mathsf{p}_* \subseteq Ker\,\rho_T^X$, then the map, $Z_{\mathsf{C}}(X) \to Z_G(X)$, will be well defined and epimorphic.

We therefore need to look closely at the situation for (i). We have already seen this in the case of trivial G, as that is just $\lambda \simeq_{thin} 1$ in our previous example. The obvious approach is, as suggested above, to try to adapt and extend the idea behind that proof to this more general context.

Suppose $\lambda \in \Lambda_{\mathsf{C}}(X)$, so $\lambda : G(T) \to K(\mathsf{C})$. For each generating arrow $\langle a,b\rangle$ in $G(T)_0$, we get $\lambda(a,b) \in P$ and need a 1-simplex joining it to $tp\lambda(a,b)$.

We will be considering a 2-simplex, $\langle a,b,c\rangle$, of T shortly, so as $\langle a,b\rangle$ is that simplex's d_2-face, we will denote $p\lambda(a,b)$ by g_2, and later on use $g_1 = p\lambda(a,c)$ and $g_0 = p\lambda(b,c)$, as notation for the images, down in G, of the other faces.

There is a 1-simplex,

$$(t(g_2)\lambda(a,b)^{-1}, \lambda(a,b)) : \lambda(a,b) \to t(g_2),$$

and this looks good. This suggests that, for any $x \in P$, we use

$$(tp(x)x^{-1}, x) : x \to tp(x),$$

and this is almost what we want, however the homotopy is not just happening within $K(\mathsf{C})$, but also should involve the structure of $G(T)$, so, for instance, on the element $\lambda(a,c)\lambda(b,c)^{-1} \in P$ that we will use shortly, we will need to deform it along $(t(g_1)t(g_0)^{-1}\lambda(b,c)\lambda(a,c)^{-1}, \lambda(a,c)\lambda(b,c)^{-1})$ and not along the edge, $(t(g_1.g_0^{-1})\lambda(b,c)\lambda(a,c)^{-1}, \lambda(a,c)\lambda(b,c)^{-1})$. These will coincide if t is a splitting, but not necessarily otherwise. The first of these ends up where we want, not the second. We will return to this point shortly.

The 2-simplex, $\langle a, b, c\rangle$, of T gives rise to $\langle a, b, c\rangle \in G(T)_1$, (we will omit the overline that we have sometimes used here, as that notation gets burdensome in the formulae and diagrams below), and is mapped by λ to a 1-simplex of $K(\mathsf{C})$, that is, to an element of $N \rtimes P$. This, thus, has form $x, \lambda(a, b))$, and, as ∂ is a monomorphism, we can work out that $x = \lambda(a, c,)\lambda(b, c)^{-1}\lambda(a, b)^{-1} \in N$.

We have an embryonic 'local homotopy' as in the previous example:

$$\begin{tikzcd}
t(g_2) \arrow[r, "?"] & t(g_1)t(g_0)^{-1} \\
\lambda(a,b) \arrow[u, "(t(g_2)\lambda(a,b)^{-1},\lambda(a,b))"] \arrow[ur, "?"] \arrow[r, "(x,\lambda(a,b))"'] & \lambda(a,c)\lambda(b,c)^{-1} \arrow[u, "(y,\lambda(a,c)\lambda(b,c)^{-1})"']
\end{tikzcd}$$

where $y = t(g_1)t(g_0)^{-1}\lambda(b, c)\lambda(a, c)^{-1}$. As we hope to build the homotopy 'thinly', we use the unique thin filler for the $(2, 1)$-horn made up of the bottom and right-hand edges. (This is the composition 2-simplex in the (internal) nerve of the groupoid part of $\mathcal{X}(\mathsf{C})$, as before, and so we have, but will not need, explicit formulae.) This gives a thin filler and a resulting diagonal arrow given by the product (within $N \rtimes P$)

$$(x, \lambda(a, b)).(1, \lambda(b, c)\lambda(a, c)^{-1}).(y, \lambda(a, c)\lambda(b, c)^{-1}),$$

and **we can check** that this gives $(yx, \lambda(a.b))$. calculating yx then gives $(t(g_1)t(g_0)^{-1}\lambda(a, b)^{-1}$ as we might have guessed and hoped.

Across the top of the square, the natural candidate for the 1-simplex is $t(g_1)t(g_0)^{-1}t(g_2)^{-1}$, which is, incidentally, the factor set of the extension, $N \to P \to G$, evaluated on (g_2, g_0).

We now need to find a 2-simplex fitting into the top left of the square. We again have a (2,1)-horn, this time made up of the left edge and the top. We fill this thinly (to see if that will work!) and then **evaluate its d_1-face**. This gives the same value as our previous label on the diagonal, so we have our thin homotopy on this edge.

This does not handle all the cases we will need, but is a good start. We will see other cases later.

This looks, good, but we should glance at the cocycle condition for λ, that is, the relationship between the faces of the image of a tetrahedron, $\langle a, b, c, d\rangle$, considered as a generator in $G(T)_2$. The resulting 2-simplex in $K(\mathsf{C})$ is $\lambda(a, b, c, d)$, which looks like :

$$\begin{tikzcd}
 & \lambda(a,c)\lambda(b,c)^{-1} \arrow[dr, "(y,\lambda(a,c)\lambda(b,c)^{-1})"] & \\
\lambda(a,b) \arrow[ur, "(x,\lambda(a,b))"] \arrow[rr, "(z,\lambda(a,b))"'] & & \lambda(a,d)\lambda(b,d)^{-1}
\end{tikzcd}$$

where $x = \lambda(a, c)\lambda(b, c)^{-1}\lambda(a, b)^{-1}$, $y = \lambda(a, d)\lambda(b, d)^{-1}\lambda(b, c)\lambda(a, c)^{-1}$ and $z = \lambda(a, d)\lambda(b, d)^{-1}\lambda(a, b)$.

In some ways, each side is exactly what we need to get from the source to the target, but you would be justified in wanting more indication of how these are obtained. This will also indicate why, just now, we used $(t(g_1)t(g_0)^{-1}\lambda(b, c)\lambda(a, c)^{-1}, \lambda(a, c)\lambda(b, c)^{-1})$. The point is that we need homomorphisms wherever possible. Here λ is a *morphism* of $\mathcal{S}$-groupoids. Earlier we need the homotopy to consist of morphisms. To see the effect this has on the calculation look at the edge from $\lambda(a,c)\lambda(b,c)^{-1}$ to $\lambda(a,d)\lambda(b,d)^{-1}$. We have a 1-simplex

$$(\lambda(a,d)\lambda(c,d)^{-1}\lambda(a,c)^{-1}, \lambda(a,c)) : \lambda(a,c) \to \lambda(a,d)\lambda(c,d)^{-1},$$

and another

$$(\lambda(b,d)\lambda(c,d)^{-1}\lambda(b,c)^{-1}, \lambda(b,c)) : \lambda(b,c) \to \lambda(b,d)\lambda(c,d)^{-1}.$$

Taking the group theoretic inverse of the second (not the groupoid one, and remember we have both in the group groupoid $\mathcal{X}(\mathsf{C})$), we get

$$(\lambda(b,d)\lambda(c,d)^{-1}\lambda(b,c)^{-1}, \lambda(b,c))^{-1} : \lambda(b,c)^{-1} \to \lambda(c,d)\lambda(b,d)^{-1},$$

and hence multiplying the two expressions term by term:

$$(\lambda(b,d)\lambda(c,d)^{-1}\lambda(b,c)^{-1}, \lambda(b,c)).(\lambda(b,d)\lambda(c,d)^{-1}\lambda(b,c)^{-1}, \lambda(b,c))^{-1}$$

from $\lambda(a,c)\lambda(b,c)^{-1}$ to $\lambda(a,d)\lambda(c,d)^{-1}.\lambda(c,d)\lambda(b,d)^{-1} = \lambda(a,d).\lambda(b,d)^{-1}$. Using the multiplication in $K(\mathsf{C})_1$, which is that of the semi-direct product $N \rtimes P$, we can easily **check** that this is $(\lambda(a,d)\lambda(b,d)^{-1}\lambda(b,c)\lambda(a,c)^{-1}, \lambda(a,c)\lambda(b,c)^{-1}$ as on the above figure.

Within the internal category structure of $\mathcal{X}(\mathsf{C})$, this diagram commutes. In terms of thin fillers, we can take the $(2,1)$-horn, form the thin 'composition' filler as we have done several time before, and then take its d_1-face, and, yes, this *is* $(z, \lambda(a,b))$ as **you can easily check**. In fact, as $K(\mathsf{C})$ is a T-complex, it just has thin elements in dimension 2, so this conclusion was 'obvious' for other reasons.

There is thus nothing to stop the construction of a thin homotopy between λ and $tp(\lambda)$, first locally and then extended up the skeletons. (There are some things you may want to check, so as to convince yourself that there are no problems. For instance, if we take the above 2-simplex as base and build a thin homotopy over it, how do we know it has $tp(\lambda(a,b,c,d))$ at the top? It is a good idea to think about this sort of question, and from several different angles, as the answers use various features of the T-complex structure of $K(\mathsf{C})$ in a beautiful way.)

To sum up, we have proved:

Lemma 67 *For any $\lambda \in \Lambda_{\mathsf{C}}$, $\lambda \simeq_{thin} tp(\lambda)$.* ■

and thus have

Corollary 18

$$Ker\, \mathsf{p}_* \subseteq Ker\, \rho_T^X$$

■

It now only needs a simple bit of diagram chasing to prove:

Proposition 91 *If $\mathsf{C} = (N, P, inc)$ is an inclusion crossed module and $G = P/N$, then the projection, p, induces a natural isomorphism*

$$p_* : Z_{\mathsf{C}}(X) \to Z_G(X)$$

■

We thus have what we suspected, at least on objects. What about on the cobordisms?

We can use the general formulae that we recalled in the previous example, and the experience gained there will be useful, but we need to 'keep alert' as well!

We have the linear transformation,

$$Z^!_{\mathsf{C}}(M,T,S) : Z_{\mathsf{C}}(X,T) \to Z_{\mathsf{C}}(Y,S),$$

and also the geometrically significant bases given by the colourings. We therefore have a matrix representing $Z^!_{\mathsf{C}}(M,T,S)$ and will calculate the matrix entries. We thus pick $\lambda_T \in \Lambda_{\mathsf{C}}(X,T)$ and $\lambda_S \in \Lambda_{\mathsf{C}}(Y,S)$ and work out the corresponding entry. Referring back to the previous example, we get

$$Z^!_{\mathsf{C}}(M,T,S)_{\lambda_T,\lambda_S} = \frac{\sharp(G)^{\sharp(T_0)}\sharp(G)^{\sharp(S_0)}}{\sharp(G)^{\sharp(\mathcal{T}_0)}}.\frac{\sharp(N)^{\sharp(T_1)}\sharp(N)^{\sharp(S_1)}}{\sharp(N)^{\sharp(\mathcal{T}_1)}}.\sharp(\Lambda_{\mathsf{C}}(\mathcal{T})_{\lambda_T,\lambda_S}).$$

(Yes, we have used that $G = P/N$, so $\sharp(G) = \sharp(P)/\sharp(N)$, which simplifies things a lot!) We have set

$$\Lambda_{\mathsf{C}}(\mathcal{T})_{\lambda_T,\lambda_S} = \{\mu \in \Lambda_{\mathsf{C}}(\mathcal{T}) \mid \mu|_T = \lambda_T, \mu|_S = \lambda_S\}.$$

Our hope is to compare the result with that for Z_G, initially in the triangulated version, then, passing to the quotient, to compare $Z_{\mathsf{C}}(M)$ and $Z_G(M)$.

We have for $p(\lambda_T)$ and $p(\lambda_S)$,

$$Z_G(M,T,S)_{p(\lambda_T),p(\lambda_S)} = \frac{\sharp(G)^{\sharp(T_0)}\sharp(G)^{\sharp(S_0)}}{\sharp(G)^{\sharp(\mathcal{T}_0)}}.\sharp(\Lambda_G(\mathcal{T})_{p(\lambda_T),p(\lambda_S)})).$$

This looks very hopeful as the first term is the same. Of course, life is not quite as simple as it would seem, as the quotient maps are not without a certain amount of complication and the above refers to $Z^!_{\mathsf{C}}$ and $Z^!_G$, not to the final versions Z_{C} and Z_G. This being said, it still seems worth calculating $\sharp(\Lambda_{\mathsf{C}}(\mathcal{T})_{\lambda_T,\lambda_S})$ as explicitly as possible, and, for that there is a surjection,

$$\mathsf{p}_* : \Lambda_{\mathsf{C}}(\mathcal{T})_{\lambda_T,\lambda_S} \to \Lambda_G(\mathcal{T})_{p(\lambda_T),p(\lambda_S)},$$

induced by composition with $\mathsf{p} : \mathsf{C} \to \mathsf{G}$. At the start of the discussion of this example, we pointed out that, if two C-colourings have the same image after composition with p, then they differed by an element of $N^{\mathcal{T}_1}$. If the two colourings had the same values on the ends, then the corresponding element of $N^{\mathcal{T}_1}$ will be constant on T_1 and S_1, having value 1. The surjection, p_*, thus has all its fibres having the same size, namely

$$\frac{\sharp(N)^{\sharp(\mathcal{T}_1)}}{\sharp(N)^{\sharp(T_1)}\sharp(N)^{\sharp(S_1)}},$$

which is good! This means

$$\sharp(\Lambda_{\mathsf{C}}(\mathcal{T})_{\lambda_T,\lambda_S}) = \frac{\sharp(N)^{\sharp(\mathcal{T}_1)}}{\sharp(N)^{\sharp(T_1)}\sharp(N)^{\sharp(S_1)}}.\sharp(\Lambda_G(\mathcal{T})_{p(\lambda_T),p(\lambda_S)}).$$

This almost does the trick. It 'almost' proves that $Z_{\mathsf{C}}(M)$ and $Z_G(M)$ are 'the same', that is, after identification of $Z_{\mathsf{C}}(X)$ with $Z_G(X)$ and of $Z_{\mathsf{C}}(Y)$ with $Z_G(Y)$. 'Almost', but not quite... . The actual map from $Z_{\mathsf{C}}(X)$ to $Z_{\mathsf{C}}(Y)$ is defined using $Z_{\mathsf{C}}(M,T,S)$ and the quotients ρ^X_T and ρ^Y_S. The above uses $Z^!_{\mathsf{C}}(M,T,S)$ and r^X_T, r^Y_S, however if you **check** the scaling factors involved it becomes clear that they in fact cancel out. In other words, we have:

Proposition 92 *If* $\mathsf{C} = (N, P, inc)$ *and* $G = P/N$*, then* $Z_C \cong Z_G$*.* ■

(You **should check** the last point in detail as it needs a certain amount of care.)

(To be continued)

8.4 How can one construct TQFTs (continued)?

From these examples we can see what to expect and how to proceed with a general construction.

8.4.1 TQFTs from a finite simplicial group

It is natural to try to extend the methods of the above sections to a more general setting in which C is replaced by a finite simplicial group or 'finite crossed n-cube' or similar. How general would this be? Would it be useful?

Definition: A simplicial group, G, is said to be *finite* if each G_m is a finite group *and* there is some n such that NG_k is trivial for all $k > n$.

Clearly, for such a simplicial group, its homotopy groups are all trivial above some level. Also clearly, any finite simplicial group models an n-type for some n, since everything is generated by its group theoretic n-skeleton, by Conduché's decomposition result, Proposition 63. Any finite simplicial group will have NG of finite length and consisting of finite groups, so all the homotopy groups of G will be finite. Ellis, [93], proved a converse:

Theorem 30 *(Ellis, [93]) Suppose that* $\pi_k(K)$ *is trivial for all* $k \geq c+1$*, and that each of the homotopy groups,* $\pi_k(K)$*, is finite for* $k \leq c$*, then the homotopy type of* K *is faithfully represented by a simplicial group whose Moore complex is of length at most* $c-1$ *and whose group of* n*-simplices is finite for each* $n \geq 0$*.* ■

We thus have that these finite simplicial groups are quite abundant!

The discussion in the previous section was given in such a way that the majority of the results and proofs made little or no use of the fact that $K(\mathsf{C})$ was other than just a finite simplicial group. Well, that is not quite true, as the compensating and scaling factors would presumably have needed more terms in general - just look at the extra terms involving $\sharp(C)$ in the case of C-colourings rather than the simpler G-colourings for a finite group G. That, however, does suggest what to do for a generalisation to G being a finite simplicial group. We would need more terms involving $\sharp(NG_n)$ for k more than just 0 or 1. The full details can be found in Porter, [180], but are not hard to derive, so generally will be **left to you**.

We thus assume that G is a finite simplicial group and construct a $Z_G : (d+1)-Cob_{PL} \to Vect$, that is, a TQFT of dimension $(d+1)$. The set-up is as before, with $(X, \mathbf{T})$, a $d-$manifold with ordered triangulation.

Definition: A G-*colouring* of $\mathbf{T}$ with values in a (finite) simplicial group, G, is a morphism,

$$\lambda : G(T) \to G,$$

of simplicial groupoids, or, equivalently, a simplicial map

$$\lambda : T \to \overline{W}G.$$

We write $\Lambda_G(\mathbf{T})$ for the set of such G-colourings and $Z_G(X, \mathbf{T})$ for the vector space with basis labelled by $\Lambda_G(\mathbf{T})$, as before. We go through the same process as previously:

(i) If $\mathbf{T}'$ is a subdivision of $\mathbf{T}$, composition with the map $r_{\mathbf{T}}^{\mathbf{T}'}$, coming from the strong deformation retraction data, induces a function,

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}} : \Lambda_G(\mathbf{T}') \to \Lambda_G(\mathbf{T}),$$

as above, and hence extends to a linear map from $Z_G(X, \mathbf{T}')$ to $Z_G(X, \mathbf{T})$.

(ii) If $(M, \mathcal{T})$ is a triangulated cobordism from $(X, \mathbf{T})$ to $(Y, \mathcal{S})$, then define a linear map, as before, by: for $\lambda \in \Lambda_G(\mathbf{T})$,

$$Z_G^!(M, \mathcal{T})(\lambda) = \sum_{\substack{\mu \in \Lambda_G(\mathcal{T}) \\ \mu|\mathbf{T}=\lambda}} \mu|\mathbf{S}.$$

We set $g_i = \sharp NG_i$, the size of the i^{th} Moore complex term.

Let $\mathcal{T}'$ be obtained from $\mathcal{T}$ by edge stellar subdivision of an interior edge, e.

Lemma 68 *For any colouring μ of $\mathcal{T}$ fixed to be λ on $\mathcal{T}$ and λ' on $\mathbf{S}$, there are exactly $g_0 g_1^{s_2(e)} g_2^{s_3(e)} \dots g_d^{s_d(e)}$ colourings of $\mathcal{T}'$ restricting to μ, where $s_k(e)$ is the number of k-simplices of $\mathcal{T}$ incident to e.* ∎

The proof is just a question of counting possible fillers.

(iii) Let $Z_G^!(M, \mathbf{T}, \mathbf{S}) = \prod_k g_k^{-\chi^{int}(\mathcal{T})} Z_G^!(M, \mathcal{T})$.

Proposition 93 *The linear map,*

$$Z_G^!(M, \mathbf{T}, \mathbf{S}) : Z_G(X, \mathbf{T}) \to Z_G(Y, \mathbf{S}),$$

is independent of the triangulation, $\mathcal{T}$, extending $\mathbf{T}$ and $\mathbf{S}$ to the cobordism. ∎

The proof follows the same lines as earlier results with some obvious replacements for lemmas valid in those cases for which generalisations are needed (as that above).

(iv) Now assume that we have $(Z, \mathbf{R})$ as another triangulated manifold and cobordisms M and N, as earlier. With the previous notation, we have:

Lemma 69

$$Z_G^!(N, \mathbf{S}, \mathbf{R}).Z_G^!(M, \mathbf{T}, \mathbf{S}) = \prod g_k^{\chi_k(S)} Z_G^!(M +_Y N, \mathbf{T}, \mathbf{R}).$$

∎

This gives that the linear maps

$$Z_G(M, \mathbf{T}, \mathbf{S}) = \prod g_k^{-\frac{1}{2}\chi_k^\partial(\mathcal{T})} Z_G(M, \mathbf{T}, \mathbf{S})$$

satisfy

Corollary 19

$$Z_G(N, \mathbf{S}, \mathbf{R}).Z_G(M, \mathbf{T}, \mathbf{S}) = Z_G(M +_Y N, \mathbf{T}, \mathbf{R}).$$

■

(v) Following our now customary route, we look at rescaling the restriction maps. If $\mathbf{T}'$ is a subdivision of $\mathbf{T}$, both being triangulations of X, let

$$\mathrm{res}_{\mathbf{T}',\mathbf{T}} = \prod g_k^{\frac{1}{2}(\chi_k(T')-\chi_k(T))} res_{\mathbf{T}',\mathbf{T}}.$$

These adjusted maps are compatible with the cobordisms.

(vi) Finally let

$$Z_G(X) = colim_{\mathbf{T}} Z_G(X, \mathbf{T}),$$

using the adjusted restriction maps, and as expected, we get:

Theorem 31 *(Porter, [180]) For any dimension d, the construction above applied to $(d+1)-Cob_{PL}$, gives a $(d+1)$-dimensional TQFT.* ■

The proof has the same form as for the low dimensional cases, using the above adjustments.

Remark: In our earlier discussions, we left the question of compatibility with the 'identities', i.e., the cobordisms $X \times [0, 1]$, for you to investigate. In case you need a hint, here is an idea to follow up on. We said cobordisms were considered 'up to homeomorphism', but if $M : X \to Y$ is a cobordism, then

$$(X \times I) +_X M \cong M$$

and, if $\mathbf{T}$ is a triangulation of X, the usual product triangulation of $X \times I$ glues to $(M, \mathcal{T}) : (X, \mathbf{T}) \to (Y, \mathbf{S})$ to give a triangulation of $(X \times I) +_X M \cong M$, so gives a new triangulation of M, but, from (iv), the induced map, $Z_G(M, \mathbf{T}, \mathbf{S})$, is independent of the triangulation. It is now easy to check that $Z_G(X \times I, \mathbf{T}, \mathbf{T})$ must be the identity. Alternatively, look at its construction in detail and do some 'sums'!

(To be continued)

.

Chapter 9

Relative TQFTs: some motivation and some distractions

Before we introduce Turaev's Homotopy Quantum Field Theories, we will look at the motivation for wanting such things, and will examine two 'Case Studies' with that aim

9.1 Beyond TQFTs

One disadvantage of standard TQFTs is that the basic categories of form $d - Cob$ consist of orientable manifolds and cobordisms *without any extra structure beyond being 'Top', 'PL' or 'Diff'*. In many geometric situations, there is often a lot more structure around, for instance, if the basic situation is that of smooth manifolds and cobordisms, then each object, X, naturally has a tangent bundle, TX, and we mentioned, when we looked at bundles earlier (page 297), this will have as basic structure group, $G\ell(d-1,\mathbb{R})$, assuming, of course, that we are working with *real* $d-1$-dimensional manifolds. As the manifolds are orientable, their tangent bundles will be orientable, i.e., the transition functions can be assumed to lie in $G\ell^+(d-1,\mathbb{R})$, the subgroup of $G\ell(d-1,\mathbb{R})$ consisting of the invertible matrices of positive determinant. If, now, we ask for extra geometric structure such as a Riemannian metric on our manifolds, then the group in which the transition functions live can be chosen to be the orthogonal group, $O(d-1)$. If our manifolds are foliated in some way then the structure groups will correspond to some block decomposition of $G\ell^+(d-1,\mathbb{R})$, and so on. In general, this leads to the theory of *G-structures*.

9.1.1 Manifolds 'with extra structure'

Recall, from page 286, that a vector bundle, V, of rank n on a space, B, is locally isomorphic to $\mathbb{R}^n_U := \mathbb{R}^n \times U$ for some open set U of B. The group of automorphisms of $\mathbb{R}^n_B$ is, of course, the trivial bundle of groups, $G\ell(n,\mathbb{R})_B := G\ell(n,\mathbb{R}) \times B$. The left $G\ell(n,\mathbb{R})_B$-torsor on B associated to V is $Isom(V,\mathbb{R}^n_B)$ and this is just the *frame bundle*, P_V, of V. Of particular note in our context of a $d-1$-manifold is the *tangent frame bundle*. This is the associated $G\ell(d-1,\mathbb{R})$-torsor of TX for X, one of our manifolds. Now assume given a potential structure group G, so we will assume it comes with a homomorphism $G \to G\ell(d-1,\mathbb{R})$, which may or may not be an inclusion / monomorphism.

Definition: A *weak G-structure* on X is a principal sub-G-bundle of the tangent frame bundle.

Note reducibility of the structure group to G is usually coupled with an integrability condition when considering geometrically significant structures. The result is then a *G-structure.* This integrability condition is sometimes called the *solder form* of the G-structure. (We will not go into this in any more detail as this is only intended to motivate the constructions we will be considering. If you need this, check the literature, for instance, Ehresmann's original work, [90], or look at Kobayashi and Nomizu, [139].) We will be sloppy in our terminology and refer to weak G-structures as being just 'G-structures'.

The notion of G-structure is thus a general notion of geometric structure. It does not handle all geometric structures, but is a good start. (We could ask if it 'categorifies' nicely, and to some extent we will see some categorifications of it in the coming pages.) For the moment, we just need to accept that it is a step towards looking at manifolds with extra structure. That leaves the question of what to do about the cobordisms. This is a bit delicate as the structure on an $M : X \to Y$ between manifolds with G-structures will need to reflect the structure on its two ends. The general situation is too complex for us to handle here, but we can give an abstract version of this sort of set-up, which encompasses a fair number of the suggested cases, and, in fact, also includes a lot of more categorified geometric contexts, and yet is very easy to describe given the sort of discussion we have above.

Let us gather up things a bit. Almost all of this type of structure involves a group, G, and a reduction of the structural maps of some natural bundle on X (possibly analogous to, or linked to, TX), together with a principal G-bundle / G-torsor associated with that natural bundle.

We know that the G-torsor corresponds to a characteristic map from X to BG. We can extend this is in what should seem an obvious way. The torsor is 'really' given by transitions on an open cover and, given the link between open covers and triangulations, our characteristic map could be equally well specified by a simplicial map $T \to Ner\, G[1] = BG$. This assumes that G is a discrete group, but we will see how that restriction can be got around in a moment. (Here we are really using the simplicial approximation theorem and we are not going to give details - so **you should check them.** There *is* a slight 'fudge' here, but it is not that crucial as with more work we could get around it, but we would have to divert from our central themes.) We can extend this idea to one where G is a simplicial group and BG is $\overline{W}G$ and that is well known territory for us.

We thus might start with basic objects being $(d-1)$-manifolds with a structure map, $g : T \to \overline{W}(G)$, given locally on some open covering or triangulation of X. For the cobordisms, some of the motivating examples gives a slight problem. For instance, if G is $O_{d-1}(\mathbb{R})$, or rather a simplicial model for it such as its singular complex, then do we take cobordisms with the same G or with $O_d(\mathbb{R})$ as group, in which case what does the boundary condition of the cobordism look like? We take the 'cowards way out and keep the same G for all manifolds and cobordisms. The complications of the other situation look 'interesting', but without a good knowledge of the simpler case, they look 'too interesting' for the moment. In any case the cobordism should correspond intuitively to the way in which X *evolves* into Y, so perhaps having intermediate states which are G-structured manifolds is what is best.

Note: If here, as in previous sections, we are looking at simplicial groups, however we now make no restriction of finiteness on them as this would be not appropriate in this context.

9.1.2 Interpretation of simplicial groups from a geometric perspective

To concentrate attention on the interpretation of some of the simplicial groups that might be of interest, suppose G is a topological group of (linear) automorphisms of $\mathbb{R}^n$, so is essentially a subgroup of $G\ell_n(\mathbb{R})$. Typically, G might be asked to preserve some structure such as a specified quadratic form on $\mathbb{R}^n$. (We will look at this again in more detail in section **??**.)

If we 'take apart' $Sing(G)$ in such a case, (cf. section 5.6), then each $\sigma \in Sing(G)_m$ is a continuous map,

$$\sigma : \Delta^m \to G,$$

but then σ is a continuously varying family of $n \times n$ invertible matrices over $\mathbb{R}$. (As a simple example, look at when $m = 1$, so that σ is just a path in G, corresponding to a t-indexed family, $\sigma(t)$ or σ_t, of invertible matrices for $0 \leq t \leq 1$, or equivalently, but perhaps more vividly, an $n \times n$ matrix of paths in $\mathbb{R}$ such that, for each t, $\sigma(t)$ is invertible. Such paths are often met in studies of the interaction of topological and algebraic properties of matrix groups, (cf. [75]). (If G is a Lie group, the singular simplex, σ, would usually be restricted to being smooth, or, at least, piecewise smooth, although the exact sense of smoothness at the endpoints 0 and 1 may vary in different contexts.)

For each $\underline{t} \in \Delta^m$, we have an automorphism $\sigma_{\underline{t}}$ of $\mathbb{R}^n$. Each automorphism is assumed to be continuous. (Our assumption earlier was that they were linear, and continuity is certainly true in that context. 'Linearity' is mainly for expositional reasons.) As a result, we can think of G as being a subset of $Top(\mathbb{R}^n, \mathbb{R}^n)$, and, usually, as a subspace, depending on what topology is given to that space of continuous maps. We thus have that σ can be recast as a map

$$\sigma : \Delta^m \times \mathbb{R}^n \to \mathbb{R}^n,$$

(which is, of course, reminiscent of the definition of the $\mathcal{S}$-enrichment of Top; see page **??**.) The 'Δ^m-indexed family of automorphisms' viewpoint then enters here, as we can build a map

$$\begin{tikzcd} \Delta^m \times \mathbb{R}^n \arrow[rr, "\tilde{\sigma}"] \arrow[dr, "proj"'] & & \Delta^m \times \mathbb{R}^n \arrow[dl, "proj"] \\ & \Delta^m & \end{tikzcd}$$

using $\tilde{\sigma}(\underline{t}, \underline{x}) = (\underline{t}, \sigma(\underline{t}, \underline{x}))$, so $\tilde{\sigma}$ will be a homeomorphism (over Δ^m). We have already visited this construction, briefly, on page 233 and a related idea on page 221. It would be a **good idea** to look back at those discussions now, although we will 'revise' that material below. We thus met this sort of construction first when discussing simplicial automorphisms in section 5.3. We supposed that Y was a simplicial set, and considered $\underline{\mathcal{S}}(Y, Y)$. In dimension m, this has simplicial maps

$$\sigma : \Delta[m] \times Y \to Y,$$

and, for composition of two such, σ and τ,

$$\tau \cdot \sigma := (\Delta[m] \times Y \xrightarrow{diag \times Y} \Delta[m] \times \Delta[m] \times Y \xrightarrow{\Delta[m] \times \sigma} \Delta[m] \times Y \xrightarrow{\tau} Y).$$

(This is $\tau \cdot \sigma$ or $\sigma \cdot \tau$ depending on your choice of composition convention.) The identity mapping from Y to Y, of course, lives in dimension zero, as the projection, $\Delta[0] \times Y \to Y$, but, of course, has unique degenerate copies of itself in all dimensions and as maps from $\Delta[m] \times Y$ to Y, this degenerate copy of the identity is just the projection onto Y.

This composition makes life slightly awkward for deciding what it means for σ to be an automorphism, and, thus, to describe the simplicial group, $\mathsf{aut}(Y)$. We therefore rethink things a bit. We do not change the composition, that would be silly, but we do change our perspective on it by using the trick that we used above. (This is, in some ways, a *reprise* of earlier discussions, but given its importance, it does seem useful to review it here.) We will place ourselves in a more general context, as that will lead to greater simplicity and, hopefully, clarity.

We will assume that X, Y, and Z are simplicial sets and

$$f : \Delta[m] \times X \to Y,$$

and

$$g : \Delta[m] \times Y \to Z,$$

are m-simplices in $\underline{\mathcal{S}}(X, Y)$ and $\underline{\mathcal{S}}(Y, Z)$, respectively. We replace them by

$$\tilde{f} : \Delta[m] \times X \to \Delta[m] \times Y,$$

and

$$\tilde{g} : \Delta[m] \times Y \to \Delta[m] \times Z,$$

where $\tilde{f} = (p_1, f)$, etc., with p_1 being the first projection of the product, $p_1 : \Delta[m] \times X \to \Delta[m]$, (and we will not bother with any indicator of X, using p_1 indiscriminately for the first projection of the product regardless of the other factor). It is now easy to check

Lemma 70

$$\tilde{g} \circ \tilde{f} = (p_1, g \cdot f),$$

where $g \cdot f$ is the composite of g and f in the usual $\mathcal{S}$-category structure on $\mathcal{S}$. ∎

Remark: This is probably simplest to see using a (simplicial) set theoretic representation of $\tilde{f}$ as being given by a formula, $\tilde{f}_n(t, x) = (t, f_n(t, x))$ and so on, but it can also be seen using a diagrammatic argument and is valid in settings other than that of simplicial sets, in which elements are problematic. One such is that of simplicial maps in a (Grothendieck) topos.

Note that any simplicial map, $\tilde{f} : \Delta[m] \times X \to \Delta[m] \times Y$, over $\Delta[m]$, (so $p_1\tilde{f} = p_1$), corresponds to a simplicial map, $f : \Delta[m] \times X \to Y$, in this way. One just sets $f = p_2(\tilde{f})$.

It is clear that this second description of the simplices in $\underline{\mathcal{S}}(X, Y)$, gives an easy solution to what it means for a $\sigma \in \underline{\mathcal{S}}(Y, Y)$ to be an automorphism.

Geometrically, this second perspective on the maps in $\underline{\mathcal{S}}(X, Y)$ is very bundle theoretic (as we saw when we first met it back in section 5.3). The object, $\Delta[m] \times X$, is an embryonic (trivial) simplicial bundle, a 'bundle patch', and if we have a 'base' simplicial set, B, we can form simplicial fibre bundles over B by gluing such patches together by automorphisms defined on faces, that is, on the overlaps between neighbouring patches. That is the whole point of the twisted Cartesian product construction and is thus at the heart of the $\overline{W}$ construction from this geometric point of view.

We thus have close links between geometric structure of a certain type and simplicial groups. In fact, the simplicial theory is the discrete analogue of the smooth theory of G-structures, (but without the integrality conditions), and as discrete analogues of physical theory are quite sought after and are difficult to do, simplicial techniques are one of several areas that seem well equipped to be exploited in developing such a 'discrete differential geometry'. Other ares include forms of n-category theory, and, as we have seen, that is very close to this one.

We can summarise the above as saying that one possible version of 'manifold with structure' would be a manifold, X, together with a structural map, $g : X \to B$, where B is a 'classifying space' for some sort of geometric structure on X. As we will be needing triangulations of X, we can be a bit more concrete and model this by a $g : T \to \overline{W}G$ for G, here, a simplicial group. This we can work with, and is so near to the G-colouring technology that we have been using that it is clearly worth exploring more thoroughly.

9.1.3 Case Study 1: Spin structures

To return to more specific examples, a metric on an n-dimensional vector bundle, $p : E \to X$, on X, is a bundle map $g : E \times_X E \to X \times \mathbb{R}$, such that the restriction of g to each fibre is a non-degenerate bilinear map, thus making each fibre, E_x, into an inner-product space. In this context, we will look in a bit more detail at the reduction of the structure group to the special linear group. We assume E is oriented, so already the transition functions of E could be given as having values in the subgroup $G\ell^+(n, \mathbb{R})$ of $G\ell(n, \mathbb{R})$. The oriented *orthonormal* frames in such an E form a principal $SO(n)$-bundle, $P_{SO}(E)$, since here the transition functions must preserve the orthonormality so must have determinant 1. An $SO(n)$-structure on X, an n-dimensional manifold, can thus be specified by an orientation together with a metric.

A related structure is a $Spin(n)$-structure. This has significant applications and interpretations in theoretical physics. The spin group, $Spin(n)$, is the double cover of the special orthogonal group, $SO(n)$. There is a short exact sequence:

$$1 \to C_2 \to Spin(n) \xrightarrow{\rho} SO(n) \to 1,$$

where C_2 is the cyclic group of order 2. (If $n > 2$, $Spin(n)$ is, in fact, simply connected and so coincides with the universal cover of $SO(n)$. The usual construction of the universal cover then gives $Spin(n)$ in terms of a quotient of the space of based paths in $SO(n)$. This idea is worth retaining for higher homotopy dimension analogues, later on.)

As before, E will be an oriented n-dimensional vector bundle on X, often, but not necessarily, a tangent bundle.

Definition: A *spin structure on* E is a lift of $P_{SO}(E)$ to a principal $Spin(n)$-bundle, $P_{Spin}(E)$.

This means that there is a ρ-equivariant bundle map and over X, that is, a map

$$\phi : P_{Spin}(E) \to P_{SO}(E)$$

of bundles over X such that, for $p \in P_{Spin}(E)$ and $\gamma \in Spin(n)$, $\phi(\gamma.p) = \rho(\gamma)\phi(p)$.

Definition (continued): If $E = TX$, the tangent bundle on X, then a spin structure on E is called a *spin structure on* X and X is said to be a *spin manifold*

We will not be using, nor proving, the following result, but it is an indication of some close links that we will not follow up on.

Proposition 94 *If T is a triangulation of a manifold X, a spin structure on X can be specified by a homotopy class of trivialisations of $TX|_{sk_1T}$, that is, TX restricted to the 1-skeleton of the triangulation, that extends over the 2-skeleton.* ■

As we said, we will not explore this point further here.

You should be thinking:

$$\begin{tikzcd} & BSpin(n) \arrow[d, "B\rho"] \\ X \arrow[ur, dashed, "?"] \arrow[r] & BSO(n). \end{tikzcd}$$

You have a map classifying one type of structure and want to ask if that structure lifts to another 'finer' type of structure, and if so, to classify those 'finer' structures using some classification othe lifts.

You may have noticed the similarity between the ideas here and those discussed back in section **??** and, in particular, page **??**, where a somewhat similar problem was examined using the language of *bundle gerbes*. There is a connection, but also a slight, but significant, difference. In handling $Spin(n)$, we are using matrices over the real numbers and hence are considering *real* vector bundles, and the kernel of the extension is C_2, in the bundle gerbe case the kernel is $U(1)$, that is, the circle. The connection can be made stronger, but this needs the intermediate situation of $spin^c$-structures, for which look initially in Wikipedia (under spin-structure), and then at the work of Murray and his coauthors, see the bibliography, and in particular, [165]. Another link that is worth following up is that with Stiefel-Whitney classes, as the vanishing of these in low dimensions is related to the existence (or otherwise) of the lifts to $Spin(n)$, and hence to that of spin structures. You should note that, in both cases, that of the classical theory of characteristic classes and that using bundle-gerbe ideas, the exact sequence of the above extension plays a central role.

What *we* need to take from this discussion is that in this type of context, there will be a fibration of simplicial groups

$$p : H \to G$$

hopefully, as here, with finite fibre. (Remember that, for simplicial groups, fibrations are just the same as epimorphisms, so the fibre is just the kernel of p.) To end this initial discussion on 'motivation' we can sketch what a 'relative TQFT' might look like and how Yetter's construction might be adapted to give instances of this. We postpone a detailed look, until we discuss homotopy quantum field theories in later sections, but it is good to have the idea and motivation in front of us when introducing that new notion.

We could take a $p : H \to G$, as above, which would have finite fibre / kernel, K, (at least for the moment). The objects of study would be manifolds with structure maps $g_X : X \to BG$ and we would need triangulated simplicial versions of these, $g_T : T \to \overline{W}G$. Between these, we would have G-cobordisms, so, if $M : X \to Y$, with g_X and g_Y being the structure maps of X and Y, we would want $g_M : M \to BG$, restricting to them on the input and output ends.

A p-colouring of (T, g_T) would then be a simplicial map $\gamma_T : T \to \overline{W}H$, so that $\overline{W}p.\gamma_T = g_T$, in other words, a lift of the 'G-structure' to an 'H-structure'. The finite kernel assumption will mean that there are only finitely many such lifts, so we can form $\Lambda_p(T, g_T)$ and $Z_p(T, g_T)$. The route to check that G-cobordisms work then looks fairly clear (if perhaps slightly tortuous and long).

Will this give a TQFT? Well, that is the wrong question, as a TQFT is defined on $d{-}Cob$, and we have here some $d{-}Cob_G$, as the source. The better question is: what sort of structure does this construction give - if it 'works' at all and does not hit any 'snags'? We will return to this later.

9.1.4 Case Study 2: Microbundles, background

The discussion of this second case study will occupy quite a few sections and will be quite long, as we have to sketch in some background before we can really attack the structures involved.

The theory of microbundles is, in some ways, an aside from the main development of this chapter, yet it illustrates several of the themes that have been intertwining their way through the chapter, and more generally throughout these notes, and so can serve as a useful case study helping in building intuition, in providing examples of the general processes involved and also in highlighting some of those places at which occur the various inadequacies of the current theory.

The most accessible (and stable) recent reference for the theory is probably in Buoncristiano, '*Fragments of geometric topology from the sixties*', [59], but another source that gives a slightly different perspective is Lurie's '*Topics in Geometric Topology (18.937)*', [147].

Some of the discussion in previous sections depended on the manifolds involved being smooth as they depend on the existence of a tangent bundle. What could replace the role of the tangent bundle if the manifolds are only specified as being 'topological' or, perhaps, piecewise-linear (PL). For that matter, can PL manifolds always be smoothed? Can topological manifolds always be triangulated, so as to become PL, and is there only one way of doing it if it is possible? These are some of the fascinating questions that were 'centre stage' in Geometric Topology in the 1960s. In the development of a comprehensive theory for handling them, there were many tools of basic geometric topology, of course, but also a rich interaction with other areas such as simplicial homotopy theory, and other parts of algebraic topology. Amongst them, we find Milnor's introduction of microbundles. These were a very pretty geometric idea that linked many of the above questions in a neat and very fruitful way.

The treatment here, loosely follows that in [59], but **you may like to look at the original papers** listed in the bibliography of those 'fragments', especially at Milnor's paper, [158], and his Princeton notes, [157]. We are going to leave out a lot of the topological detail when that uses ideas whose detailed development would lead us far from our main themes.

9.1.5 Microbundles: topological and PL

Initially, we will work in a category, Top, of topological spaces and continuous maps.

Recall that early on we defined a bundle on a space, B, in its most general form to be just a map $E \to B$. Consistently with previous use, in section 6.1.1, we define:

Definition: A *split bundle*, ξ, with *base*, B, and *total space*, E, consists of two spaces and two maps, (E, B, i, p), with $B \xrightarrow{i} E \xrightarrow{p} B$, such that $p \circ i = id_B$. The splitting, i, is usually referred to as the *zero section*, whilst p is the *projection* of ξ.

Of course, a split bundle is just a split epimorphism of spaces. A *microbundle* will be a split bundle satisfying a local triviality condition that we will meet shortly.

Notation: If we need to, we will use notation such as $E(\xi)$, $B(\xi)$, p_ξ, etc. to distinguish the structural data of ξ from those of another split bundle. It is often the case that we think of i as an inclusion, and thus may 'innocently' confuse B and $i(B)$.

You can probably think of several examples of split bundles, but to get the local triviality condition that we need, and it is *not* quite the one we have met before, we will need some examples for terminology *and* motivation.

Examples: (a) The *product split bundle*, ε_B^m, with fibre $\mathbb{R}^m$ and base B is given by

$$\varepsilon_B^m = (B \xrightarrow{i} B \times \mathbb{R}^m \xrightarrow{p} B),$$

with $i(b) = (b, 0)$ and $p = p_1$, the first projection of the product.

(b) More generally, any vector bundle, $\mathsf{V} = (V, B, p)$, will naturally give a split bundle, since the map $i : B \to V$, that picks out the zero of the fibre V_b over b, as b varies in the base, is a continuous map.

(c) Let $B = M$, a topological manifold. (Aside: that B is a manifold is only needed for this to be a microbundle, it clearly gives a split bundle without that assumption.) The *tangent split bundle*, TM, of M is given by taking $E = M \times M$, $B = M$, $i = \Delta$ being the diagonal,

$$\Delta : M \to M \times M,$$

given by $\Delta(x) = (x, x)$, and p, the projection (onto the first factor). (This will eventually be the basis for the *tangent microbundle* of M.)

Remark: It is *very convenient* to write TM for this tangent split or micro-bundle, but, of course, if M is a smooth manifold, there is already a contender for that notation, namely the classical tangent vector bundle, and, what might be more annoying, by (b), that will define another microbundle. Oh dear, will there be confusion? We will shortly see that there is nothing to fear.

The key to the definition of microbundle as against just split bundle is the specific form of local triviality that is imposed. 'Local triviality' is easy to define. It usually means 'locally isomorphic' to a product of some sort, so the key must be the meaning of 'locally isomorphic' and even more basically, that of 'isomorphic'. In other words, the important thing in this, as in many other contexts, is not the objects but the morphisms between them.

Morphisms, and thus, in particular, isomorphisms, of split bundles are defined in an obvious way (**left to you**), but a more useful notion is what we will call '*micro-isomorphism*'.

Definition: Given $\xi_1 = (E_1, B, i_1, p_1)$, and $\xi_1 = (E_2, B, i_2, p_2)$, two split bundles on the same base, a *micro-isomorphism* from ξ_1 to ξ_2 consists of open neighbourhoods, V_i, of $i_i(B)$ in E_i, for

$i = 1, 2$, and a homeomorphism, $\varphi : V_1 \to V_2$, such that the diagram

$$\begin{tikzcd} & V_1 \arrow[dd, "\varphi"] \arrow[dr, "p_1"] & \\ B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \\ & V_2 \arrow[ur, "p_2"'] & \end{tikzcd}$$

commutes. We say ξ_1 and ξ_2 are *micro-isomorphic* in this case.

Properly speaking, the open neighbourhoods, V_1 and V_2, are part of the specification of the micro-isomorphism and so need recording in the notation, however we will usually be interested in the homeomorphism, φ, in a 'micro-neighbourhood' of the zero sections, i.e., in a *germ* of φ, just as in section 6.3.3, in comparing the étale space approach to sheaves with that coming from presheaves, we used *germs* of local sections. Here, if we have a micro-isomorphism as specified above, and a smaller open neighbourhood, say U_1, of the zero section in E_1, restricting φ to U_1 (and 'co-restricting' it to the image of U_1 in E_2) will give us a *new* micro-isomorphism, but one that could be considered more or less equivalent to the original one. We will frequently use a certain amount of 'sloppiness', at least to start with, not mentioning V_1 or V_2, as this does not tend to lead to any problem, but as the extra information sometimes needs to be 'recalled' for safety, we will also discuss germs of micro-morphisms more formally a bit later.

The following more formal statement of the relationship between a split bundle and a restriction to a neighbourhood illustrates the use of this 'micro-sloppiness'! The result is more or less immediate without any sloppiness, but the wording is quite carefully chosen!

Proposition 95 *If $\xi = (E, B, i, p)$ is a split bundle on B, and U is an open neighbourhood of $i(B)$ in E, then $\xi_U = (E|_U, B, i, p|_U)$ is a split bundle on B, which is micro-isomorphic to ξ.* ■

The point here is that the inclusion of U into E gives a micro-isomorphism from ξ_U to ξ. Of course, there is also a micro-isomorphism *from* ξ to ξ_U, as we can just take $V_1 = U$ and $V_2 = U$, ...

Proposition 96 *If M is a (paracompact) smooth manifold, its tangent vector bundle and its tangent split bundle are micro-isomorphic.*

Sketch Proof: (First, the condition of paracompactness is there merely to guarantee that M has a metric tensor with good properties. We used this earlier. That result is, for instance, in Husemoller, [123], Chapter 5, section 7.)

If two points, x and y, of M are near enough, then there will be a *unique* geodesic from x to y and hence a tangent vector v at x along that curve. The neighbourhood of the diagonal of $M \times M$ given by such pair is mapped to a neighbourhood of the zero section in the tangent vector bundle, $T(M)$, by sending (x, y) to the corresponding (x, v). (The details are **left to you**, although we will also be examining the tangent microbundle in a bit more detail shortly and the discussion then may help you to 'spruce' this sketch up to give a proof.) ■

This, of course, means that we do not need to worry about using TM for the tangent microbundle provided we are working with constructs 'up to micro-isomorphism'.

It should be clear that micro-isomorphisms are particular *micro-morphisms.*

Definition: If $\xi_\alpha = (E_\alpha, B, i_\alpha, p_\alpha)$, $\alpha = 1, 2$, are split bundles over possibly different bases, a *micro-morphism*, $\varphi = (f_B, f_V)$, from ξ_1 to ξ_2, is a commutative diagram:

$$\begin{tikzcd} B_1 \arrow[r] \arrow[d, "f_B"'] & V_1 \arrow[r] \arrow[d, "f_V"'] & B_1 \arrow[d, "f_B"] \\ B_2 \arrow[r] & V_2 \arrow[r] & B_2 \end{tikzcd}$$

where V_α is an open neighbourhood of $i_\alpha(B_\alpha)$ in E_α.

Note that as we have the possibility of using different bases, we can take $\xi = (E, B, i, p)$ and a subspace, A, of B and can restrict ξ to A. This we could have done before but now we have a morphism from $\xi|_A$ to ξ itself.

Although we will not be exploring this much, we should note that the resulting $\xi|_A$ is really only determined up to isomorphism of split bundles. (Here we do mean 'isomorphism' rather than 'micro-isomorphism'.)

It is probably becoming clear what the local triviality condition should be.

Definition: A *topological microbundle*, ξ, of fibre dimension n, *base*, $B = B(\xi)$, and *total space*, $E = E(\xi)$, is a split bundle,

$$B \xrightarrow{i} E \xrightarrow{p} B,$$

satisfying the *local triviality condition*:

• for each $b \in B$, there exist open neighbourhoods U of b in B, and V of $i(b)$ in E with $i(U) \subset V$, together with a *local trivialising homeomorphism*,

$$h : V \to U \times \mathbb{R}^n,$$

such that h defines a micro-isomorphism between $\xi|_U$ and $U \times \mathbb{R}^n$, thus we have $hi(b) = (b, 0)$ for $b \in U$ and $p_1 h(v) = p(v)$ for $v \in V$.

It is sometimes useful in the above situation to say, briefly, that ξ is an *m-microbundle* on B

We note that being a microbundle is a 'condition' not a 'structure', i.e., the basic morphisms between microbundles are simply morphisms between the split bundles and are not required to respect the trivialising homeomorphisms in any mysterious way. One can also deal with 'atlases' of trivialising data in a fairly obvious way.

That being said, it is not the morphisms between microbundles that are really the useful notion, rather it is the (germs of) micromorphisms that are of note geometrically.

Examples of microbundles: Clearly (a), above, gives a microbundle, ε^m_B, called the *standard trivial microbundle on B of fibre dimension m.* Any microbundle isomorphic to such is called *trivial.*

As for (b), as vector bundles are locally trivial fibre bundles, one can adapt that given local trivialisation to get a microbundle version, so any vector bundle, V, on B yields a microbundle, known as its *underlying microbundle.*

For (c), we have:

Lemma 71 *For a topological manifold, M, its tangent split bundle, TM, is a microbundle (called, of course, the* tangent microbundle *of M).*

Proof: We just have to check 'local triviality'. We assume M to be of dimension m. (The tangent microbundle will then have fibre dimension m.) Given $x \in M$, choose a neighbourhood, U_x, of x, homeomorphism to $\mathbb{R}^n$ under some homeomorphism, f. Next set $V = U_x \times U_x$, and construct a homeomorphism $h : V \to U \times \mathbb{R}^n$ by $h(u_1, u_2) = (u_1, f(u_2) - f(u_1))$. ■

The tangent microbundle construction is easily seen to be functorial:

Proposition 97 *If $f : M \to N$ is a map between topological manifolds then*

$$\begin{array}{ccccc} M & \xrightarrow{\Delta} & M \times M & \longrightarrow & M \\ {\scriptstyle f}\downarrow & & {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle f} \\ N & \xrightarrow{\Delta} & N \times N & \longrightarrow & N \end{array}$$

defines a microbundle micromorphism, $df : TM \to TN$.

Proof: The diagram certainly commutes, so gives a micromorphism. ■

The morphism, df, is called the *differential* of f. (Note that df really needs to be considered as a germ.)

More generally, there is a construction of an induced microbundle.

Definition: Let $\xi = (E(\xi), B, i_\xi, p_\xi)$ be a microbundle over B and suppose $f : A \to B$ is a continuous map. The *induced microbundle*, $f^*(\xi)$, on A is constructed by taking $E(f^*(\xi))$ to be the pullback,

$$\begin{array}{ccc} E(f^*(\xi)) & \xrightarrow{f'} & E(\xi) \\ {\scriptstyle p_{f^*(\xi)}}\downarrow & & \downarrow{\scriptstyle p_\xi} \\ A & \xrightarrow[f]{} & B \end{array}$$

which, then, also gives us $p_{f^*(\xi)}$. The zero section, $i_{f^*(\xi)}$, is the map given, by the universal property of pullbacks, by the pair of maps, $(id_A : A \to A, i_\xi f : A \to E(\xi))$.

This definition will, of course, work when ξ is merely a split bundle on B, but we have:

Lemma 72 *If ξ satisfies 'local triviality', then so does $f^*(\xi)$.* ■

The proof should be fairly clear, so is **left to you**.

Remarks: (i) The definition only specifies $f^*(\xi)$ up to isomorphism, as it involves a pullback, but there is a 'elementary' representative of the isomorphism class given by using the 'usual' construction of a pullback of spaces in terms of elements. From this 'elementary' point of view, $E(f^*(\xi)) = \{(a, y) \mid a \in A, y \in p_\xi^{-1}(f(a))\} \subset A \times E(\xi)$, and $p_{f^*(\xi)} : E(f^*(\xi)) \to A$ is the obvious projection onto A, whilst $i_{f^*(\xi)}(a) = (a, i_\xi f(a))$.

(ii) Of course, if we have $f : A \to B$ and $g : B \to C$, and some microbundle, η, on C, then $f^*g^*(\eta) \cong (gf)^*(\eta)$, but it is not 'equal' for the usual reasons.

(iii) For any $f : A \to B$ and ξ is a microbundle (or more generally any split bundle) on B, the pair, (f, f'), defined by the horizontal maps in the above pullback square, defines a microbundle map from $f^*(\xi)$ to ξ. This has the property that any map of form $(f, f_V) : \zeta \to \xi$ will factor via (f, f') and a map of microbundles over B. (This follows from the universal property of pullbacks. You **may wish to check up on any fibred category properties of this**. We will return to this shortly, as it is useful for understanding an important construction and concept later on.)

The induced microbundle construction works especially well for tangent bundles.

Proposition 98 *If $f : M \to N$ is a map of topological manifolds, then $f^*(TN)$ has a representative of form,*

$$M \xrightarrow{i} M \times N \xrightarrow{p_1} M,$$

where $i(x) = (x, f(x))$.

Proof: This is fairly straightforward, but is quite neat so we will give it in a little detail.

Look at the 'elementary' description of $E(f^*(TN))$. We have $E(TN) = N\times N$ and $p_{TN}(n_1, n_2) = n_1$, so, if $m \in M$, $p_{TN}^{-1}(f(m)) = \{(f(m), n) : n \in N\} = \{f(m)\} \times N$. This now gives $E(f^*(TN)) = \{(m, (f(m), n)) \mid m \in M, n \in N\} \cong M \times N$. It thus remains to check that the descriptions of i and p are as claimed, modulo this isomorphism.

The form of $p_{f^*(TN)}$ is projection onto the first factor, so that is easily seen to translate to the first projection from $M \times N$ to M. For the zero section, $i_{f^*(TN)}(m) = (m, i_{TN}f(m)) = (m, (f(m), f(m))$, and, tracking that across to $M \times N$, this is $i(m) = (m, f(m))$. ■

We thus have a neat induced microbundle construction that gives a sensible, and manageable, result on the tangent microbundle.

We now come back to the question of those *germs*.

Definition: Two microbundle maps, $\varphi = (f_B, f_V) : \xi_1 \to \xi_2$ and $\gamma = (g_B, g_V) : \xi_1 \to \xi_2$, are said to be *germ equivalent* if f_B and g_B are equal, whilst f_V and g_V agree on some neighbourhood of the zero section, $i_1(B_1)$, in E_1. More precisely, if we have $f_V : U_1 \to U_2$ and $g_V : V_1 \to V_2$, we require that there is a neighbourhood $W_1 \subset U_1 \cap V_1$ containing $i_1(B_1)$, such that, for all $x \in W_1$, $f_V(x) = g_V(x)$.

The following are easy to prove, so .... .

Lemma 73 *(i) Germ equivalence is an equivalence relation on the set of microbundle morphisms from ξ_1 to ξ_2.*

(ii) Germ equivalence is stable under composition. ■

Definition: A *germ of a morphism* from ξ_1 to ξ_2 is a germ equivalence class of some micromorphism from ξ_1 to ξ_2.

The following should now almost be routine. You just put together the pieces from the above:

Proposition 99 *Germs of morphisms can be composed.* ■

This means that the following is more or less a formality.

Proposition 100 *There is a category, $Microbundles^{Top}$, whose objects are topological microbundles and whose morphisms are the germs of (micro)morphisms between them.*

The proof is **left to you to assemble**. There is one additional point to do and that is the question of identities, but they are easy to handle.

In a formalisation of the 'micro-sloppiness' from earlier, we tend to make no distinction between a germ and any of its representatives.

It is worth thinking about the fact that if $\xi = (E, B, p, i)$ is a microbundle and V is an open neighbourhood of $i(B)$ in E, then (V, B, p, i) is a microbundle, which is isomorphic to ξ in $Microbundles^{Top}$, so whether one uses 'germs of morphism' or 'germs of micromorphisms' is largely a question of taste, not of substance.

So far we have restricted attention to the notion of topological microbundle and the corresponding morphisms. The category, $Microbundles^{PL}$, is the piecewise linear analogue of $Microbundles^{Top}$. The definitions look formally the same except that the spaces are all assumed to be polyhedra and the maps to be piecewise linear. Any PL manifold, M, has a *tangent PL-microbundle*, which will, if needed, be denoted TM, or possibly TM_{PL} if greater precision is required in any given context. A PL-map, $f : M \to N$, between m-dimensional PL-manifolds gives a differential, $df : TM \to TN$, which is a PL-map of PL-microbundles. The induced microbundle, $f^*(\xi)$, of a PL-microbundle, ξ, along a PL-map, f, has a natural PL-microbundle structure. The corresponding maps and factorisations are PL as well.

As any PL map is continuous, there is a functor from $Microbundles^{PL}$ to $Microbundles^{Top}$, forgetting the 'PL'. The study of this will occupy us later on.

There is also a smooth, or, more exactly, a piecewise smooth version of these constructions.

It is natural to ask what is the relationship between microbundles and more general fibre bundles. There is one obvious block to too naive a hypothesis, - general fibre bundles need not have global sections. The simplest example is probably the 2-fold covering of the circle. Any vector bundle yields a microbundle, as we have seen, and in that context there *are*, of course, zero sections! Looking back, however, the construction of a microbundle from a vector bundle did not actually use the linear structure of the fibres.

Clearly, as microbundles are locally of the form $U \times \mathbb{R}^m$, for some m, if there is to be a more general construction, it has to involve fibre bundles with fibre some $\mathbb{R}^m$ and with a specified zero section. We are thus led to consider bundles with fibre $(\mathbb{R}^m, 0)$, but in which the linear structure of $\mathbb{R}^m$ is not being considered, so the transitions functions needed to define them, need not be linear, but they do, however, need to preserve 0. We will be more precise below. These $(\mathbb{R}^m, 0)$-bundles can be considered in an obvious way, as split bundles of a particular kind, or as a special case of a natural generalisation of fibre bundles where the fibre has (a little) more structure, namely a distinguished point. This latter viewpoint seems a good one as later on we will want to consider 2-vector bundles and there are similarities between some approaches to them and to what we call here $(\mathbb{R}^m, \mathbb{R}^k)$-bundles. In these the the transition functions will preserve a subspace. (A reference for some of this is Holm, [120].) We give the definitions for such bundles over a general space, X, but in practice, we will need them when X is a polyhedron, or, at very least, is paracompact.

Definition: An $(\mathbb{R}^m, 0)$-*bundle*, or, simply, an $\mathbb{R}^m$-*bundle*, on a space X is a split bundle,

$$X \xrightarrow{i} E \xrightarrow{p} X$$

which

(i) satisfies the *local triviality condition*:

there is an open cover $\mathcal{U}$ of X and, for each U in $\mathcal{U}$, a homeomorphism (called a *local trivialisation*),

$$\Phi : p^{-1}(U) \xrightarrow{\cong} U \times \mathbb{R}^m,$$

such that the map

$$U \times \mathbb{R}^m \xrightarrow{\Phi^{-1}} p^{-1}(U) \xrightarrow{p} U$$

is the projection onto the first factor, whilst

$$U \xrightarrow{i|_U} p^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^m,$$

is the zero section of $U \times \mathbb{R}^m$, and

(ii) for which there is a partition of unity subordinate to $\mathcal{U}$, i.e., whose supports form an open cover refining $\mathcal{U}$.

The second condition will not concern us as, in practice, our examples of X will usually be *compact* manifolds or, at 'worst', polyhedrons, and partitions of unity are, then, no problem. As we said before, any $\mathbb{R}^m$-bundle clearly defines a microbundle using a slight adaptation of the process that we outlined that gave one from a vector bundle.

As usual, we can compare trivialisations over an intersection, $U_i \cap U_j$, and then, on that, $\Phi_i^{-1}\Phi_j$ will be a homeomorphism of $(U_i \cap U_j) \times \mathbb{R}^m$ over $U_i \cap U_j$, *which fixes the zero section of that space*. We therefore consider the group, $Homeo(\mathbb{R}^m, 0)$, of homeomorphisms of $\mathbb{R}^m$ that leave the origin, 0, fixed. The transition maps, $g_{ij} : U_i \cap U_j \to Homeo(\mathbb{R}^m, 0)$ are given so that $\Phi_i^{-1}\Phi_j(x, v) = (x, g_{ij}(x)(v))$. If $Homeo(\mathbb{R}^m, 0)$ is given the compact open topology, then not only does it become a topological group, but the g_{ij} become continuous.

A relative form of this involves an $\mathbb{R}^m$-bundle and an $\mathbb{R}^k$-bundle subbundle. More exactly, suppose $0 \leq k \leq m$ and consider $\mathbb{R}^k$ as the subspace of $\mathbb{R}^m$ defined by $\mathbb{R}^k = \{x_1, \ldots, x_k, 0, \ldots, 0) \in \mathbb{R}^m\}$.

Definition: An $(\mathbb{R}^m, \mathbb{R}^k)$-*bundle* on X is an $\mathbb{R}^m$-bundle, ξ, such that all the transition maps, g_{ij}, actually take their values in the subgroup, $Homeo(\mathbb{R}^m, \mathbb{R}^k, 0)$, of $Homeo(\mathbb{R}^m, 0)$, consisting of those homeomorphisms of $\mathbb{R}^m$ that map $\mathbb{R}^k$ onto itself.

Remarks: (i) We note that there is a $\mathbb{R}^k$-bundle here as a subbundle of ξ. For some of discussion of this idea in later sections, it is useful to think of $(\mathbb{R}^m, \mathbb{R}^k)$ as a linear transformation

$inc^k_m : \mathbb{R}^k \to \mathbb{R}^m$ or even as an inclusion crossed module / 2-group of vector spaces, that is, some sort of '2-vector space'.

(ii) In a similar vein, a homeomorphism h in $Homeo(\mathbb{R}^m, \mathbb{R}^k, 0)$ can usefully be thought of as a pair of homeomorphisms $h = (h_m : \mathbb{R}^m \to \mathbb{R}^m, h_k : \mathbb{R}^k \to \mathbb{R}^k)$ such that the diagram

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "h_m"] & \mathbb{R}^m \\ \mathbb{R}^k \arrow[u, "inc^k_m"] \arrow[r, "h_k"] & \mathbb{R}^k \arrow[u, "inc^k_m"'] \end{tikzcd}$$

commutes. In other words h_k is the restriction of h_m to the subspace, $\mathbb{R}^k$.

We can also apply the analysis that we made earlier to 'translate' the topological groups, $Homeo(\mathbb{R}^m, \mathbb{R}^k, 0)$ into simplicial 'format'. We can take the singular complex of these topological groups and then we know that the n-simplices of the resulting simplicial group can be thought of as homeomorphisms,

$$h : \Delta^n \times \mathbb{R}^m \to \Delta^n \times \mathbb{R}^m,$$

over Δ^n, that restrict to $\Delta^n \times \mathbb{R}^k$, or more precisely, they are pairs,

$$h = (h_m : \Delta^n \times \mathbb{R}^m \to \Delta^n \times \mathbb{R}^m, h_k : \Delta^n \times \mathbb{R}^k \to \Delta^n \times \mathbb{R}^k)$$

such that

$$\begin{tikzcd} \Delta^n \times \mathbb{R}^m \arrow[r, "h_m"] & \Delta^n \times \mathbb{R}^m \\ \Delta^n \times \mathbb{R}^k \arrow[u, "\Delta^n \times inc^k_m"] \arrow[r, "h_k"] & \Delta^n \times \mathbb{R}^k \arrow[u, "\Delta^n \times inc^k_m"'] \end{tikzcd}$$

commutes.

We will use a very similar formulation in section 9.3.2 to define the simplicial groups that provide the classifying spaces for microbundles. Before we do that we should look at the Kister-Mazur theorem that gives the link between microbundles and $\mathbb{R}^m$-bundles. We will also take time off to briefly look at some aspects of K-theory that are relevant to the overall understanding of the cohomology / TQFT context.

Definition: Let $\xi = (E, B, p.i)$ be an m-microbundle, then ξ is said to *admit*, or to *contain* a bundle if there is an open neighbourhood, E_1, of $i(B)$ in E such that $p : E_1 \to B$ is an $\mathbb{R}^m$-bundle.

The term *admissable* has been used for such microbundles.

Theorem 32 *(Kister-Mazur, 1964) If ξ is an m-microbundle whose base is a polyhedron, then it admits a bundle, unique up to isomorphism.* ■

The proof is to be found in Kister's paper, [138]. The PL version is due to Kuiper and Lashof, [141, 142]. This shows that, to a certain degree, we could replace microbundles by the seemingly simpler $\mathbb{R}^m$-bundles that they contain, however the tangent bundle is a functorially defined object, whilst the corresponding $\mathbb{R}^m$-bundle is only defined up to isomorphism, so is more 'lax-functorial'. That would be a minor inconvenience if the notion of microbundle had not the geometric naturality that it has. It seems best to retain both notions.

9.2 Vector bundles and various forms of topological K-theory

The following section is a bit of an aside, albeit a lengthy one. Parts of it could be put in various other chapters of this work, but they also fit quite well here. They bring in various other aspects of the theory of bundles that are very useful, but have not been treated elsewhere. In particular, some of the notions of (topological) K-theory are useful to have available, although we will not go deeply into that large area of study. (The vector bundle side of this can be found in the many good sources on topological K-theory. In the main the sources used here are Husemoller, [123] and its 'sequel', [124]. Other sources may be mentioned later.)

In a later chapter, we will need to attack what '2-vector bundle' might mean, so we will need at least a modicum of intuition on how vector bundles are handled to see what 'categorifies' well and to make comparisons between potential choices.

It is often the case that one needs properties of a microbundle or vector bundle that become apparent after 'stabilisation', that is, after the addition of a standard trivial bundle. This needs, in each context, the idea of the *Whitney sum* of two bundles. We will look at this, and its consequences, in the case of vector bundles before going to the microbundle case in a later section.

9.2.1 Vector bundles: Whitney sum, and K-theory

Notation and terminology: It is sometimes convenient, when referring to a vector bundle, ξ, in which all the fibres have the same dimension n (for instance, if the base space is connected), to note this by a suffix, ξ^n, and to refer n as the *rank* of ξ.

Suppose we have vector bundles, $\xi_j := (E_j \stackrel{p_j}{\to} X)$, $j = 1, 2$, then we can form a vector bundle $\xi_1 \oplus \xi_2$ as the pullback, $\xi_1 \times_X \xi_2$, which is easily seen to be a vector bundle over X, whose fibre over a point b is the direct sum of the fibres over b of the two bundles.

Definition: The vector bundle, $\xi_1 \oplus \xi_2$, defined in this way is called the *Whitney sum* of ξ_1 and ξ_2.

There is a second way of getting the same bundle. First form the vector bundle, $\xi_1 \times \xi_2 = (E(\xi_1) \times E(\xi_2), X \times X, p_1 \times p_2)$. (**Check this is a vector bundle**.) There is a diagonal map $\Delta : X \to X \times X$, and $\xi_1 \oplus \xi_2 \cong \Delta^*(\xi_1 \times \xi_2)$, the pullback of $\xi_1 \times \xi_2$ along that diagonal.

We have not made precise whether we are considering real or complex vector bundles, and it does not really matter. (In fact we could be working in a sheaf theoretic / algebraic geometric setting and many of the ideas would be 'the same', in essence at least.) We will write $\mathbb{k}$ for the field being used. We will give a brief *résumé* of the construction of the topological K-theory groups of a space, X. We give few details, but would recommend that the reader who needs more information look at Chapter 8 of the book by Husemoller, [123], on fibre bundle theory in general, and / or a text, such as Karoubi's [134], more specifically on the area.

We assume that we have a topological space, X, which will usually have the homotopy type of a CW-complex in what follows. We let $Vect_{\mathbb{k}}(X)$ denote the category of (finite dimensional) $\mathbb{k}$-vector bundles over X. This is a monoidal category, using tensor products and also has an additive structure coming from the Whitney sum. The tensor product is defined using the tensor products of corresponding fibres.

Let $[Vect_{\mathbb{k}}(X)]$ denote the set of isomorphism classes in $Vect_{\mathbb{k}}(X)$. This inherits from $Vect_{\mathbb{k}}(X)$ the tensor and Whitney sum structures and so is easily checked to be a semi-ring. (It will not have inverses for addition because Whitney sum cannot decrease the dimension.)

Remarks: The above is the classical viewpoint as exemplified by, for instance, Husemoller, [123], but we should not leave an aspect of this that is dear to the various themes of these notes to pass by without comment! Later on we will be 'categorifying' $Vect_{\mathbb{k}}(X)$, the category of vector spaces (or, if you like, of vector bundles over a singleton space) when we discuss 2-vector spaces in a later chapter, so we should not leave this construct without noting that it has itself a 'decategorification' side to it.

In any category $\mathcal{C}$, we can form the groupoid consisting of *all* the objects of $\mathcal{C}$ and all the *isomorphisms* between them. (In other words, you throw out of $\mathcal{C}$ all the non-isomorphisms.) This groupoid is sometimes referred to as the *core* of $\mathcal{C}$, although that is not that evocative a term. If we denote it by $Core(\mathcal{C})$, then, for $\mathcal{C} = Vect_{\mathbb{k}}$, $Core(\mathcal{C})$ is the category of all finite dimensional $\mathbb{k}$-vector spaces and *isomorphisms* between them. It is a groupoid which has a connected component corresponding to each non-negative integer, and the natural rank function, $rk : Core(Vect_{\mathbb{k}}) \to \mathbb{N}$, to $\mathbb{N}$, thought of as a discrete category, induces an isomorphism of categories if we apply π_0, the 'connected components' functor.

Of course, direct sum and tensor product of vector spaces make $Vect_{\mathbb{k}}$ into a category with the categorified analogue of a semiring structure and this structure restricts to $Core(Vect_{\mathbb{k}})$, and then is preserved by rk with $\oplus$ being sent to $+$ and $\otimes$ to $\times$.

If we now look at the vector bundle case, $\pi_0(Core(Vect_{\mathbb{k}}(X)))$ is just the same as $[Vect_{\mathbb{k}}(X)]$ and, as we noted slightly earlier, the obvious 'rank of a vector bundle' makes sense as long as X is connected. We can still obtain a rank of a vector bundle V, in general, as a function $rk_X : \pi_0(X) \to \mathbb{N}$.

We have said that $[Vect_{\mathbb{k}}(X)]$ is a *semi-ring*, but that concept is not that well known, so let us be a bit careful and make precise the basic structures involved.

Definition: A *semiring*, (S, a, m), consists of a set together with two binary operations, addition, a, and multiplication, m, (which are usually written $x + y := a(x, y)$ and $x.y := m(x, y)$). These are required to satisfy the usual axioms of a ring except for that concerning the existence of an inverse for addition.

A *semiring morphism*, $\theta : S \to R$, is a function preserving addition and multiplication. The category of semirings and their morphisms will be denoted *SemiRings*.

Examples: (i) Any ring, by default, is a semiring. There is an inclusion functor from the category, *Rings*, to *SemiRings*. This is full and faithful. It forgets the property of having an inverse.

(ii) The classic example is $\mathbb{N} = \{0, 1, 2, \ldots\}$.

In that last classical case, one can 'complete' $\mathbb{N}$ to get $\mathbb{Z}$, loosely by adding in additive inverses for everything and asking that they behave themselves! This is an elementary example of a ring completion. There are various more-or-less equivalent ways to think of the completion process. The one that is most convenient for the generalisation of that simple case is to think of $\mathbb{Z}$ as consisting of equivalence classes of pairs of elements of $\mathbb{N}$. We think of the pair (a,b) as being (eventually) $a-b$, so that the new negative elements that we will be adding have form $(0,b)$, or rather equivalence classes with one representative of that form. From that idea, we clearly have

$$(a,b) \sim (a',b')$$

if and only if there is some $c \in \mathbb{N}$ such that $a + b' + c = a' + b + c \in \mathbb{N}$. This process works in general and gives us a general *ring completion* functor.

Definition: The *ring completion* of a semiring, S, is a pair, (S^*, θ), where S^* is a ring (thought of as a semiring) and $\theta : S \to S^*$ is a semiring morphism, with the universal property that, if $f : S \to R$ is a semiring morphism with R a ring, then there is a unique ring homomorphism, $f^* : S^* \to R$, such that $f = f^*\theta$.

In other word, and as we will make slightly more precise shortly, the ring completion is left adjoint to the inclusion functor from *Rings* to *SemiRings*.

Remark: We later will have situations in which we merely have a semi-group, albeit an Abelian one. There will be an addition, but no multiplication. In that case, there is a completely analogous notion of a *group completion*. The theory and constructions are so similar that we will not give a separate treatment.

Does a ring completion exist for any semiring? Yes, that would follow from general categorical principles, and the Kan extension process would, in fact, give a construction. More immediately, the elementary construction of $\mathbb{Z}$ from $\mathbb{N}$ that we gave above generalises. We consider a semiring, S, and form $S \times S$ (thinking, as before, of (a,b) as $a-b$ in the ring completion). We define

$$(a,b) \sim (a',b')$$

if and only if there is some $c \in S$ such that $a + b' + c = a' + b + c$. As **you can check**, this is an equivalence relation and the set, S^*, of equivalence classes inherits an obvious addition and a multiplication from S. (The age old trick for explaining the formula for multiplication goes as follows: if we write $\langle a,b\rangle$ temporarily for $[(a,b)]_\sim$, the equivalence class of (a,b) in S^*, we need to define $\langle a,b\rangle.\langle c,d\rangle$, so we think of $\langle a,b\rangle$ as being $a-b$, etc. and work out $(a-b).(c-d)$ in a general associative ring - the formula is then obvious.) The operations have, of course, to be **checked** to be well-defined and to have the right properties such as associativity, distributivity of multiplication over addition, etc. The new semiring, S^*, is a ring, since the negative of $\langle a,b\rangle$ is $\langle b,a\rangle$. The obvious morphism, $\theta : S \to S^*$, sends a to $\langle a,0\rangle$.

If we write $i : \mathit{Rings} \to \mathit{SemiRings}$ for the 'inclusion', then this completion gives a functor $(-)^* : \mathit{SemiRings} \to \mathit{Rings}$. The following is just a formal statement of the situation. The proof is **left to you**.

Proposition 101 *$(-)^*$ is a functor, which is left adjoint to i.* ∎

In some contexts, and, in particular, in that of microbundles, there is no tensor product, so we would not have a semiring, merely an Abelian semigroup, that is, there is an addition, but no multiplication. As was hinted at above, the same construction works. (In fact, it works for any semigroup, not just the Abelian ones.)

Proposition 102 *If i : Groups $\to$ SemiGroups is the inclusion, then $(-)^*$ defines a functor, which is left adjoint to it.* ■

You are left to fill in the detailed definitions, etc. as they are so similar to what we have already given.

Definition: The ring, $K_{\Bbbk}(X)$, is the ring completion of the semiring, $[Vect_{\Bbbk}(X)]$. It is known as the *topological K-theory ring* of X.

For $\Bbbk = \mathbb{R}$, we would speak of 'real K-theory' and for it being $\mathbb{C}$, 'complex K-theory'. In general we will usually miss out any indication over which field is being considered.

Example: As a useful, but simple, example, we note, if X is a singleton space, then $Vect(X)$ is isomorphic to $Vect$. It is then easy to see that, in this case, $K(X)$ is the ring of integers. (It is suggested that **you should check this statement** as it is relatively easy to see, but checks the use of the definitions above.)

This ring, $K(X)$, varies *contravariantly* with the space. More precisely, if $f : X \to Y$ is a continuous map, and ξ is a vector bundle on Y, we saw, (section 6.1.4) that the pullback of a bundle on Y along f is also a bundle with the same fibre. The pullback of a trivial (product) bundle is again a trivial bundle, and it is easy to see that the same holds for locally trivial ones as well. It thus is fairly **routine to check** that $f^*(\xi)$ will be a vector bundle on X. The next step is to note that morphisms between vector bundles get transformed to morphisms between their pullbacks, so as to get an induced functor,

$$f^* : Vect(Y) \to Vect(X),$$

and that this preserves Whitney sum (and tensor product if it is available). This then implies that f^* induces a semiring morphism,

$$f^* : [Vect(Y)] \to [Vect(X)],$$

and then you just invoke the universal property of the ring completion process to get a unique extension,

$$K(f) = f^* : K(Y) \to K(X).$$

We will see an alternative route to this later, but record what we have sketched out so far:

Proposition 103 *The K-group construction, $K(-)$, gives a contravariant functor from the category of CW-complexes to that of rings.* ■

Note that the pullback construction is only functorial up to isomorphism, but by passing to isomorphism classes, that slight nuisance is eliminated.

We can turn back, once more, to section 6.1.4 and lift, from there, further information on the pullback construction. We note, in particular, that if f and g are homotopic maps, (and we need the spaces, X and Y, to be compact), then $f^*(\xi)$ and $g^*(\xi)$ will be isomorphic. Because of this, we can add a bit more detail to the above proposition:

Proposition 104 *The K-group construction, $K(-)$, gives a contravariant functor from the homotopy category of compact CW-complexes to that of rings.* ■

The 'compactness' requirement is not 'best possible' and can be replaced by 'paracompactness'. It will however serve for what we need it for, which is more 'illustrative' then for detailed exposition. In any case, for the proof, we refer the reader to [123] or [134], or numerous other sources in the literature.

9.2.2 Vector bundles: reduced K-theory and stable equivalence

Getting back to generalities of the ring, $K(X)$, we have that for a non-connected space, X, $K(X)$ is the direct sum of the $K(X_i)$ for the various components, X_i, of X. It is therefore usual to consider connected spaces most of the time, since the extension to the general case is then routine.

In the case of a connected space, we note that the *rank function*, $rk := rk_X : [Vect(X)] \to \mathbb{N}$, defined earlier, gives an extension, $rk_X : K(X) \to \mathbb{Z}$, which is a ring morphism. This is a split epimorphism, since the multiplicative identity in $K(X)$ is represented by a trivial line bundle and $rk(1) = 1$. As a consequence, we can define a ring morphism,

$$\varepsilon : \mathbb{Z} \to K(X),$$

where $\varepsilon(n)$ 'is' the class of the trivial n-dimensional vector bundle, ε^n_X, on X if $n \geq 0$ and is $\langle 0, \varepsilon^{-n}_X\rangle$ if $n \leq 0$. Of course, $rk(\varepsilon(n) = n$ giving us the splitting.

Another way to think of the splitting is to pick a point, x_0, in X. The inclusion of $\{x_0\}$ into X induce a ring morphism from $K(X)$ to $K(\{x_0\}) \cong \mathbb{Z}$. (**You are left to show that this will be** rk up to the identification of $K(\{x_0\})$ with $\mathbb{Z}$.) There is also a (unique) continuous map from X to $\{x_0\}$, and its composite with the previous one is the identity. There is, thus, an induced morphism from $\mathbb{Z}$ to $K(X)$, and, again, **you are left to see why this is** ε. The splitting is just the above fact about the composite.

We record this for convenience:

Lemma 74 *The rank map, $rk_X : K(X) \to \mathbb{Z}$, is a split epimorphism.* ■

It is important to note that this split epimorphism is natural. It is compatible with maps induced from continuous maps of the spaces concerned, in other words, it is a natural transformation from K to the constant functor with value $\mathbb{Z}$.

Definition: The *reduced K-theory*, $\tilde{K}_{\Bbbk}(X)$, of a space X is the kernel of the rank map, $rk_X : K(X) \to \mathbb{Z}$.

The reduced K-functor *is* a functor, as rk_X is natural in X. Our next task is to give alternative descriptions of $\tilde{K}$. First we get the idea of 'stable equivalence' or 's-equivalence'.

Definition: Two vector bundles, ξ_1 and ξ_2, over a space, X, are *s-equivalent*, or *stably equivalent*, if there are (non-negative) integers, n_1 and n_2, such that $\xi_1 \oplus \varepsilon^{n_1}_X \cong \xi_2 \oplus \varepsilon^{n_2}_X$. We write $\xi_1 \sim_s \xi_2$.

A bundle, ξ, that is s-equivalent to ε^1_X is said to be *s-trivial* or *stably trivial*

The following are easy to see:

Lemma 75 *(i) Stable equivalence is an equivalence relation on the vector bundles over a given space, X.*

(ii) Isomorphic bundles are stably equivalent.

(iii) All trivial bundles are s-trivial. ■

For certain categories of space, X, there exist complementary bundles, so, for a vector bundle, ξ, over such a space, X, there is a second vector bundle, η, such that $\xi \oplus \eta$ is stably trivial. This is discussed in Husemoller, for instance, [123], who calls this property (S) or in [124], section 2.5, where the vector bundle is called *finitely generated*, or to be *of finite type* if there is a complementary bundle in this sense. Any vector bundle over a compact space is finitely generated in this sense (see [124], page 31), and so all compact spaces have property (S). (It is important to note that property (S) is a property of a base space, whilst being 'finitely generated' is a property of the bundle.) This will suffice for what we need here and now, although we will revisit this later.

Theorem 33 *If X satisfies (S), then $\tilde{K}(X)$ is isomorphic to the ring of s-equivalence classes of vector bundles on X.*

Before we launch into a proof of this, let us sketch why this property (S) is the key to this description. Remember $K(X)$ is obtained by adding 'negatives' to $[Vect(X)]$, but if we have $\xi \oplus \eta \cong \varepsilon_X^m$, then η is behaving a bit like an inverse *up to s-equivalence*, since ε_X^m is s-trivial. The complementary summand of $\tilde{K}(X)$ in $K(X)$ just consists of classes of trivial bundles, so dividing out by that summand corresponds to ... s-equivalence. We thus might suspect that, if $\xi \oplus \eta$ is s-trivial, then in $\tilde{K}(X)$, '$\eta = -\xi$'. That gives some idea of 'why', now we prove it more formally.

Proof: We define a function,

$$\alpha : [Vect(X)] \to \tilde{K}(X),$$

by $\alpha(\xi) = \xi - \varepsilon_X^{rk(\xi)}$. (This is the projection of $K(X)$ onto $\tilde{K}(X)$, restricted to the original semiring $[Vect(X)]$.) This, it is claimed, is a surjection.

To show this, note that any element in $K(X)$ has form $x = \xi - \xi'$. (We here use a lighter notation than $\langle \xi, \xi' \rangle$, which was our earlier one, when looking more formally at the ring completion process.) If $x \in \tilde{K}(X)$, then $0 = rk(x) = rk(\xi) - rk(\xi')$, so ξ and ξ' have equal rank. Now by property (S), there is a bundle η' such that $\xi' \oplus \eta' \cong \varepsilon_X^m$ for some m. In $\tilde{K}(X)$, we have

$$\begin{aligned} \xi - \xi' &= \xi \oplus \eta' - \xi' \oplus \eta' \\ &= \xi \oplus \eta' - \varepsilon_X^m. \end{aligned}$$

We have $rk(\xi \oplus \eta') = rk(\xi) + rk(\eta')$, but $rk(\xi) = rk(\xi')$, so this equals $rk(\xi') + rk(\eta') = rk(\xi' \oplus \eta') = m$. We thus have $\xi - \xi' = \alpha(\xi \oplus \eta')$ and α is surjective.

The next claim is that $\alpha(\xi) = \alpha(\eta)$ *if and only if ξ and η are s-equivalent.*

Suppose, therefore, that ξ and η are s-equivalent, so there are p, q such that $\xi \oplus \varepsilon_X^p \cong \eta \oplus \varepsilon_X^q$, but then $\alpha(\xi \oplus \varepsilon_X^p) = \alpha(\eta \oplus \varepsilon_X^q)$. Now $\alpha(\xi \oplus \varepsilon_X^p) = \xi \oplus \varepsilon_X^p - \varepsilon_X^{rk(\xi)} \oplus \varepsilon_X^p = \xi - \varepsilon_X^{rk(\xi)} = \alpha(\xi)$, etc., so we can conclude that $\alpha(\xi) = \alpha(\eta)$.

Conversely, suppose that $\alpha(\xi) = \alpha(\eta)$, then $\xi - \varepsilon_X^{rk(\xi)} = \eta - \varepsilon_X^{rk(\eta)}$ in $K(X)$. There is thus a bundle ζ such that

$$\xi \oplus \varepsilon_X^{rk(\eta)} \oplus \zeta = \eta \oplus \varepsilon_X^{rk(\xi)} \oplus \zeta,$$

by the definition of the equivalence relation in the ring completion (cf. page 396). Let ζ' be a 'complementary bundle' for ζ, so $\zeta \oplus \zeta'$ is trivial, say is isomorphic to ε^q_X, but then

$$\xi \oplus \varepsilon_X^{rk(\eta)} \oplus \varepsilon^q_X = \eta \oplus \varepsilon_X^{rk(\xi)} \oplus \varepsilon^q_X,$$

so ξ and η are s-equivalent. ∎

Remark: There is another approach that is worth thinking about to get reduced K-theory. If we restrict to finitely generated vector bundles in the sense mentioned above, then, as the above proof is purely algebraic and is based on the existence of complementary bundles, you can *define* a suitable $\tilde{K}(X)$ by taking it to be the set of s-equivalence classes of finitely generated vector bundles on the space X. Of course, then you have to check the existence of extra structure such as the addition and multiplication. Following this line idea leads in the direction of coherent sheaves. We leave that for you to check up on.

9.2.3 Vector bundles: classifying spaces

To get useful 'representations' of $\tilde{K}(X)$, we need to work our way through some classification results for vector bundles. Generally we will not give proofs as these are well known, and easily available in the literature, but we will wherever possible try to give some idea of why a result is true, and, again, where possible to hint at links with some results we have already seen, and others that we will see later on, so as to try to weave a fuller view of the theory. We will need to retrieve and adapt results from several earlier sections. (We will state things for real vector bundles only.)

First note that any n-dimensional vector bundle, $\xi = (V, X, p)$, on a base space X is locally trivial, so gives an open cover, $\mathcal{U}$, a local trivialisation over $\mathcal{U}$, and thus transition functions, $g_{ij} : U_i \cap U_j \to G\ell_n(\mathbb{R})$. These latter are continuous. If, instead of the topological group $G\ell_n(\mathbb{R})$, we had a discrete group, G, we would already have several interpretations of this at hand. These transition functions would define a G-torsor and we would classify that by a map from $N(\mathcal{U})$ to BG, the classifying space of G (or, if you prefer to think simplicially, $\overline{W}G$). We thus would expect, and that expectation would be correct, that there was a '$G\ell_n(\mathbb{R})$-torsor' / principal $G\ell_n(\mathbb{R})$-bundle over X and a classifying space for such things.

If we next recall the simplicial theory (sketched in section 5.5), we would also expect a universal bundle. (There is $WG \to \overline{W}G$, where G was a simplicial group.) In our current vector bundle context, we have, in fact, already met a universal bundle for the vector bundles of a given fixed rank and that was way back in section 6.1.3. What should be expected of such a 'universal' bundle? We should have some 'special' n-dimensional vector bundle, $\gamma = (E, B, p)$, such that, if $\xi = (V, X, \alpha)$ is any vector bundle of the same rank, there would be a continuous map, $f : X \to B$, such that the induced bundle $f^*(\gamma)$ will be isomorphic to ξ. That would not do for all vector bundles, as it involves a restriction on the rank, but that difficulty can be got around.

What is this 'miraculous' universal n-dimensional vector bundle? We have met it earlier in section 6.1.3. We recall its construction, repeating some of the definitions and structure (with minor adjustments to notation). Firstly, the *Stiefel variety* of n-frames in $\mathbb{R}^m$, denoted $V_n(\mathbb{R}^m)$, is the subspace of $(S^{m-1})^n$ such that $(v_1, \ldots, v_n) \in V_n(\mathbb{R}^m)$ if and only if each $\langle v_i \mid v_j\rangle = \delta_{i,j}$, so are pairwise orthogonal. We use this to put a topology on the Grassmann variety.

The *Grassmann variety* of n-dimensional subspaces of $\mathbb{R}^m$, denoted $G_n(\mathbb{R}^m)$, is the set of n-dimensional subspaces of $\mathbb{R}^m$. There is an obvious function,

$$\alpha : V_n(\mathbb{R}^m) \to G_n(\mathbb{R}^m),$$

mapping $(v_1, \ldots, v_n)$ to $span_{\mathbb{R}}\langle v_1, \ldots, v_n\rangle \subseteq \mathbb{R}^m$ and we give $G_n(\mathbb{R}^m)$ the quotient topology defined by α. This space, $G_n(\mathbb{R}^m)$, is compact as it is a quotient of a closed subspace of a finite power of a compact space.

There is a natural inclusion of $G_n(\mathbb{R}^m)$ into $G_n(\mathbb{R}^{m+1})$, just by including $\mathbb{R}^m$ into $\mathbb{R}^{m+1}$ as $\mathbb{R}^m \oplus \{0\}$, that is, the 'first m-coordinates', plus 0 in the final place. We can form the union,

$$G_n(\mathbb{R}^\infty) = \bigcup_{n\leq m} G_n(\mathbb{R}^m),$$

giving it the induced topology. In the following, m can be finite or ∞.

Definition: The *canonical n-dimension vector bundle*, $\gamma_n^m = (E(\gamma_n^m), G_n(\mathbb{R}^m), p)$, on $G_n(\mathbb{R}^m)$ is the subbundle of $(G_n(\mathbb{R}^m) \times \mathbb{R}^m, G_n(\mathbb{R}^m), proj)$, whose total space is given by those (V, x) with $x \in V$ and where the projection, p, is given by $p(V, x) = V$.

A neat example of this is the simplest case, that is, when $n = 1$, then $G_1(\mathbb{R}^m) = RP_{m-1}$, the $(m-1)$-dimensional real projective space, and then γ_1^m is the *canonical line bundle* on RP_{m-1}.

It is clear that, in γ_n^m, the fibres are n-dimensional vector spaces. In fact, we have the neat calculation (that looks almost ridiculous), $p^{-1}(V) = \{(V, x) \mid x \in V\} \cong V$. Of course, one could agonise about the multiple use of V here, but each sense of V is clear, it is be hoped. The only thing to check is local triviality and for that you have to track back through the topology of $G_n(\mathbb{R}^m)$, and why is a manifold. (You are **left to find this** in the literature, if you need it in further studies, or for your own research.)

To show the properties of these canonical bundles as far as classification is concerned, we need to consider what Husemoller, [123], calls a *Gauss map*. For *full* details, as usual, we refer you to the texts, [123, 124], which are very clear, but we will sketch in a bit of background and will then give a fairly detailed overview of this area.

We start by noting that there is another projection, $q : E(\gamma_n^m) \to \mathbb{R}^m$, since $E(\gamma_n^m) \subseteq G_n(\mathbb{R}^m) \times \mathbb{R}^m$, so we can take $q(V, x) = x$. We will think of Gauss maps in general as being generalisations or abstractions of these second projections, but it is instructive (and quite fun) to **do a search** back in differential geometry books to see the original sense of the term, 'Gauss map' and to find the important uses it has there. (The original intuitions are often rich geometrically and can be a help in guiding new directions of research.)

Definition: A *Gauss map* from an n-dimensional vector bundle, ξ^n, to $\mathbb{R}^m$ (for $n \leq m \leq \infty$) is a continuous map

$$g : E(\xi^n) \to \mathbb{R}^m$$

such that g gives a linear monomorphism when restricted to any fibre of ξ.

Example: (i) The above map q is a Gauss map.

(ii) If $(u, f) : \xi^n \to \gamma_n^m$ is a vector bundle morphism which is fibrewise an isomorphism, then $qu : E(\xi^n) \to \mathbb{R}^m$ is a Gauss map.

This second example hides a very useful converse.

Lemma 76 *If $g : E(\xi^n) \to \mathbb{R}^m$ is a Gauss map, then there is a vector bundle morphism, $(u, f) : \xi^n \to \gamma_n^m$, which is fibrewise an isomorphism,*

Proof: We first want to construct the map, $f : B \to G_n(\mathbb{R}^m)$, on the bases, but, for $b \in B$, we know $g(p^{-1}(b))$ is a subspace of $\mathbb{R}^m$ of dimension n, so is an element of $G_n(\mathbb{R}^m)$, which we will take to be $f(b)$. This is tailor made to define $u(x) = ((f(p(x)), g(x))$, so that it ends up in $E(\xi^n)$. It remains to check that f and u are continuous, and for that we pick local coordinates for ξ and track through what f does, then u is clearly also continuous. ■

We next examine further the properties of such a map, (u, f), which is fibrewise an isomorphism, but in a bit more generality.

Proposition 105 *Suppose $(u, f) : \eta \to \xi$ is a vector bundle morphism, so that, for all $b \in B(\eta)$, $u_b : E(\eta)_b \to E(\xi)_{f(b)}$ is an isomorphism, then (u, f) induces an isomorphism from η to $f^*(\xi)$ (over $B(\eta)$.*

Proof: We first note that, for any map $u : \xi \to \xi'$ of vector bundles over a space, B, u is an isomorphism if and only if it is an isomorphism on each fibre. To see this you take local trivialisations of both ξ and ξ', where it can be assumed that the open cover of B involved is the same in both. Next you examine the situation for product bundles. The only point to check is continuity of the inverse. The details can be **left to the reader.**

(The converse of this is clearly true as well.)

Returning to the main result, the induced map from η to $f^*(\xi)$ is easily checked to be a fibrewise isomorphism (over $B(\eta)$), so is as claimed. ■

We get a consequence that the existence of a Gauss map, $g : E(\xi^n) \to \mathbb{R}^m$, implies that $\xi^n \cong f^*(\xi)$ for some map, $f : B(\xi) \to G_n(\mathbb{R}^m)$, and conversely, thus, to get a classifying map is equivalent to getting a Gauss map. The next result (given in Husemoller, [123], chapter 3 section 5) depends for its proof on a knowledge of partitions of unity, which we have not discussed other than 'in passing', we therefore state it without proof.

Theorem 34 *(i) For a vector bundle, ξ^n, over a paracompact space, B, there is a Gauss map, $g : E(\xi^n) \to \mathbb{R}^\infty$.*

(ii) If B has a finite trivialising cover, $\mathcal{U} = \{U_i \mid 1 \leq i \leq k\}$, for ξ, then a Gauss map, $g : E(\xi^n) \to \mathbb{R}^{kn}$, exists. ■

It is useful to note that $G_n(\mathbb{R}^\infty)$ is an increasing union of compact spaces, and, if $f : B \to G_n(\mathbb{R}^\infty)$, where B is itself compact, we would expect its image to be in some $G_n(\mathbb{R}^m)$, for large enough m. If B is compact, an open cover as in (ii) clearly exists, and so (ii) gives extra information on the m concerned in the image of f.

Corollary 20 *Every n-dimensional vector bundle over a paracompact space, B, is isomorphic, over B, to some $f^*(\gamma_n^\infty)$.* ■

We now give more formally the definition of a term used a few pages back.

Definition: A vector bundle, ξ, is said to be *finitely generated*, or *of finite type*, over its base, B, if there is an open covering, $\mathcal{U} = \{U_i \mid 1 \leq i \leq k\}$, of B such that $\xi|_{U_i}$ is a trivial bundle for each i, for all $1 \leq i \leq k$.

We have seen, before, some of the links in the statement of the following result. (As usual, we state it for real vector bundles, but it is true more generally.)

Proposition 106 *For a vector bundle, ξ, over B, the following are equivalent:*
(1) ξ is of finite type;
(2) There is a map, $f : B \to G_n(\mathbb{R}^m)$ for some finite m such that $\xi \cong f^*(\gamma_n^m)$;
(3) There is a vector bundle, η, over B such that $\xi \oplus \eta$ is trivial.

Before sketching out those parts of the proof that we have not already seen, principally those involving (3), we need another bundle from our initial discussion of γ_n^m, page 241. There we also introduced the *orthogonal complement bundle*, again over $G_n(\mathbb{R}^m)$. Denoted ${}^*\gamma_n^m$, this was the subbundle of $\varepsilon^m_{G_n(\mathbb{R}^m)} = (G_n(\mathbb{R}^m) \times \mathbb{R}^m, G_n(\mathbb{R}^m), proj)$, whose total space consists of those (V, x) with x perpendicular to V, i.e., $\langle v \mid x\rangle = 0$ for all $v \in V$. As $\mathbb{R}^m = V \oplus {}^\perp V$, (remember m is nice and finite), we have

$$\gamma_n^m \oplus {}^*\gamma_n^m \cong \varepsilon^m_{G_n(\mathbb{R}^m)},$$

over $G_n(\mathbb{R}^m)$.

Proof: We will concentrate on the relationship between (2) and (3), as we have not looked at this yet.

Suppose that $f : B \to G_n(\mathbb{R}^m)$ and $\xi \cong f^*(\gamma_n^m)$. As f^* preserves both direct / Whitney sum and triviality (**for you to check**),

$$\xi \oplus f^*({}^*\gamma_n^m) \cong f^*(\gamma_n^m) \oplus f^*({}^*\gamma_n^m) \cong f^*(\gamma_n^m \oplus {}^*\gamma_n^m) \cong \varepsilon^m_B,$$

which shows that (2) implies (3).

Conversely, if (3) holds, there is an inclusion of ξ into $\xi \oplus \eta$, and so a composite map,

$$E(\xi) \to E(\xi \oplus \eta) \cong B \times \mathbb{R}^m \to \mathbb{R}^m,$$

which is a Gauss map. ■

A detailed proof is to be found in [123]. In addition, in the corresponding discussion in [124], several other equivalent statements are given. One of these is very important for understanding how vector bundles fit into the overall picture of cohomology. To explore this, we need to go back to the origins of K-theory and the links with algebraic geometry.

9.2.4 Vector bundles: the Serre - Swan theorem

The original development of K-theory was by Grothendieck. It formed part of the machinery for his work on the Riemann-Roch theorem in algebraic geometry. In that context, you do not work with vector bundles on a space, X, rather with sheaves of various types, on a *scheme*, X. A scheme is one analogue of a manifold within the algebraic geometric context and it will be useful to briefly review its definition and those of some related ideas. It consists of a space, X, together with a sheaf of rings, $\mathcal{O}_X$, on X, with certain 'local' properties.

Definition: A *ringed space*, $\mathsf{X} = (X, \mathcal{O}_X)$, is a pair consisting of a space, X, and a sheaf of rings, $\mathcal{O}_X$, on X, called the *structure sheaf* of X.

A *morphism of ringed spaces*, $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$, is a pair $\mathsf{f} = (f, f^\sharp)$ consisting of a continuous map $f : X \to Y$, and a morphism $f^\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ of sheaves of rings on Y. Here $f_*(\mathcal{O}_X)$ is the 'direct image sheaf' defined as a presheaf (see page 265) by $*f_*(\mathcal{O}_X)(V) = \mathcal{O}_X(f^{-1}(V))$, for V, open in Y.

Examples: (i) Let X be a topological space and $\mathcal{O}_X$ be the sheaf of continous real valued functions on X, thus, for U and open set in X, $\mathcal{O}_X(U) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\}$.

(ii) If A is a commutative ring, then $Spec(A)$ is the set of prime ideals in A with the Zariski topology. There is a sheaf, $\mathcal{O}_{Spec(A)}$, of rings on $Spec(A)$, and, the pair forms a ringed space. This is the 'prime' example of a ringed space from the point of view or algebraic geometry. It is the *spectrum on* A. Any ringed space isomorphic to such an example is said to be *affine* and called an *affine scheme* Each stalk of $\mathcal{O}_{Spec(A)}$ is a local ring, (i.e., it has a unique maximal ideal).

If $\theta : A \to B$ is a morphism of commutative rings, then it induces a morphism of ringed spaces from $(Spec(B), \mathcal{O}_{Spec(B)})$ to $(Spec(A), \mathcal{O}_{Spec(A)})$. In fact the category of commutative rings is equivalent to that of affine ringed spaces.

A particular example corresponds to the case $A = \Bbbk[x_1, \ldots, x_n]$, which yields an affine n-dimensional space over $\Bbbk$, but note it is an *affine* analogue of $\Bbbk^n$, not a linear one.

(iii) A *scheme* is a locally affine ringed space. It therefore has that all the stalks of its structure sheaf, $\mathcal{O}$, are local rings. Just as with our mentions of *differential* geometry, it is not our intention to do more than indicate where there are connections. We do not intend to provide an introduction to algebraic geometry within these notes, *but* many of the ideas that we have been exploring have partial origins in that area, and at least some acquaintance with it seems advisory. (A possible place to start is a standard text such as Hartshorne, [114], but there are numerous alternatives.)

We return to the ringed space, $(X, \mathcal{O}_X)$, of the first of the examples. If we have a vector bundle, $\xi = (S, X, p)$ over X, then we can look at its sheaf of local sections, Γ_E, or, perhaps more accurately, Γ_ξ. We saw this first in section 6.3.1 for a general map, $\alpha : A \to B$, so recall that, for an open set, U, in X, $\Gamma_\xi = \{s : U \to E \mid ps(x) = x \text{ for all } x \in U\}$. These local sections, in our case here, can be added: $(s_1 + s_2)(x) = s_1(x) + s_2(x)$, using the vector space structure in the fibre. We thus have that Γ_ξ is a sheaf of Abelian groups. More is true. If we use the vector space structure, we can multiply any local section, s, by any scalar (real number, in this case), so Γ_ξ is a sheaf of vector spaces. Still more structure is around. If $f \in \mathcal{O}_X(U)$, $f : U \to \mathbb{R}$, so we can define $(f \cdot s)(x) = f(x)s(x)$, and $\Gamma_\xi(U)$ becomes a module over $\mathcal{O}_X(U)$. This corresponds to a bilinear *action*: for each $U \in Open(X)$,

$$\mathcal{O}_X(U) \times \Gamma_\xi(U) \to \Gamma_\xi(U),$$

and, if $V \in U$, then these actions are compatible with restrictions. We have Γ_ξ is a sheaf of modules over $\mathcal{O}_X$, - but note that we should more accurately say that it is a sheaf of $\mathcal{O}_X$-modules, since $\mathcal{O}_X$ is a sheaf of rings, not just a ring.

Here we can glimpse an important insight that has already been hinted at before. If we work in the topos, $Sh(X)$, of sheaves on X (cf. section 6.1.3), $\mathcal{O}_X$ *is* a ring object and Γ_ξ is a module object over that ring object. We could 'pretend' that $\mathcal{O}_X$ is a ring and Γ_ξ, a module, and use standard ideas of module theory, (Abelian) homological algebra, etc. to give ideas for results and methods for proving them. This works beautifully, but with one proviso - we would have to be careful with the logic. There is an internal logic in $Sh(X)$ and it is not a 'classical' two valued logic. Instead of 'truth values' 0 and 1 (standing for 'false' and 'true'), we would use truth values in the lattice, or more accurately, the Heyting algebra, of open sets of X. This means that the truth value of a statement such as '$f \neq 0$', where f is an 'element' of $\mathcal{O}_X$, (which is a ring), will be an open set of X, and, in fact, this is beautifully 'honest' and geometrically self evident. What is that open set? It is the set of points $x \in X$ such that $f(x)$ is defined and is non-zero. That encodes so much more than just f being either zero or not, but at a small price, namely you have no longer the *Law of Excluded Middle* and, often not the *Axiom of Choice*, as tools for proofs. (If you have not met this logic before, **think about** what a statement 'p or not p' should have as its truth value. That raises the question as to how to handle 'or' , which is easy, and also how to handle negation. The truth value of 'not p, will not be the complement of the truth value of p, as the complement of an open set is not usually an open set, so how should this logic all work out?) We will not be following this up except to note that the sheaf of continuous real valued functions on X is not a field object in $Sh(X)$. (In fact, how could you encode 'field' in a language where negation is as it is there?) It is a local ring object, however. (You **are also left to think** about the formulation of that as well.)

It would be surprising if Γ_ξ was just any old $\mathcal{O}_X$-module, and it is not. We have not used yet in our analysis the local triviality of ξ. If ξ is actually trivial, $\xi = (X \times \mathbb{R}^m, X, proj_1)$, then it is evident that $\Gamma_\xi \cong (\mathcal{O}_X)^m$, the free $\mathcal{O}_X$-module on m-generators, a direct sum of m copies of the $\mathcal{O}_X$-module, $\mathcal{O}_X$ itself. Next, we assume that we have a trivialising open cover, $\mathcal{U}$, so, for each U in $\mathcal{U}$, there is given an isomorphism between each $\xi|_U$, the restriction of ξ to U, and $(U \times \mathbb{R}^m, U, proj_1)$, the corresponding product bundle. (For simplicity, we will assume that X is connected, so that ξ has a well defined rank, m.) We can again use the trick of considering the open cover, $\mathcal{U}$, as a sheaf of sets, U, and so as an object of $Sh(X)$. We pullback the sheaf, Γ_ξ, to this object. Thinking of the unique map, $\alpha : \mathsf{U} \to 1$ in $Sh(X)$, and forming $\alpha^*(\Gamma_\xi)$, this is equivalently $(U) \times \Gamma_\xi$. We can think of this in two related ways. Firstly as an object in $Sh(X)/\mathsf{U}$, the 'slice topos' of objects over U, and secondly, as an object of $Sh(\mathsf{U})$, where now, with no real risk of confusion, we write U for the disjoint union of $\mathcal{U}$, considered as a space. The second of these uses the induced 'geometric' morphism of toposes from $Sh(X)$ to $Sh(\mathsf{U})$, induced by α. The point of all this is that $\alpha^*(\Gamma_\xi) \cong \alpha^*(\mathcal{O}_X)^m$ for some m. (Remember that, for simplicity of exposition, we did assume X was connected.)

Definition: A sheaf, F, of $\mathcal{O}_X$-modules is *locally free* if there is a covering $\mathsf{U} \to 1$ such that $\alpha^*(F)$ is a free $\mathcal{O}_X$-module.

Remark: It is worth noting, if you are needing ideas from algebraic geometry, that there is a neat notion of vector bundle in that area, which incorporates more of the algebraic geometric structure. It is discussed, for example, in Hartshorne's book, [114], p. 128, and is closely linked to locally free sheaves on a scheme. We do not give details here.

Topological vector bundles on a compact space, X, give (finitely generated) locally free sheaves of $\mathcal{O}_X$-modules and conversely, where $\mathcal{O}_X$ is the sheaf of real valued continuous functions on X. Suppose X is connected, so that rank makes sense in the simplest sense, and , for simplicity, assume also that X is compact (Hausdorff, being assumed). If, as always here, ξ is a vector bundle over X, then we had a 'complementary' η such that $\xi \oplus \eta \cong \varepsilon^m_X$ for some m. This means that there is a projection, $p_1 : \varepsilon^m_X \to \xi$, but the sheaf of sections of ε^m_X is simply $(\mathcal{O}_X)^m$, so we have an epimorphism from $(\mathcal{O}_X)^m$ to Γ_ξ . In $(\mathcal{O}_X)^m$, there is a family of m independent *global* sections, corresponding to the 'basis vectors', e_i, in the free $(\mathcal{O}_X)$-module. If we project these down to Γ_ξ, we get m global sections, $s_1, \ldots, s_n$, say, so that, for any point $x \in X$, the values $s_i(x)$ generate the fibre $E(\xi)_x$. As usually m will be (much) bigger than $rk(\xi)$, this generating set will be far from being a basis.

We can interpret these various pieces of information in various different ways. For instance, the category of modules over a ring has various linearity and categorical properties, and, within that setting, we have the definition of a projective object. We will give some definitions, partially to stress the 'Abelianness' of this situation in contrast with the 'non-Abelian' character of other part of these notes. This is not the place to develop the standard theory of Abelian categories, homological algebra, etc., as there are good introductions to this readily available, and starting from varying initial positions, having differing aims, enough for most tastes and uses - provided you do not need the non-Abelian phenomena, that is!

We adopt definitions below more for their accessibility than their elegance, not that they are inelegant, and to some extent for compatibility with some of the principal sources that have been mentioned earlier. We have already mentioned, and even used, Abelian categories at several places in these notes, without yet giving a formal definition. Here is one.

We let $\mathcal{Ab} = (Ab, \otimes_{\mathbb{Z}})$ be the symmetric monoidal category of Abelian groups with the usual tensor product of such.

Definition: An *Abelian category* is a category, $\mathbb{A}$, such that

- for each A, B, objects of $\mathbb{A}$, $\mathbb{A}(A, B)$ is an Abelian group, and composition is bilinear, (so $\mathbb{A}$ is an $\mathcal{Ab}$-enriched category);
- for any objects, A, B, in $\mathbb{A}$, their direct sum, $A \oplus B$, exists, (so $A \oplus B$ is, at the same time, the product and the coproduct of A and B in a compatible way), and there is a zero object;
- every morphism has a kernel and a cokernel;
- every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel;
  and finally
- every morphism can be factored as an epimorphism followed by a monomorphism.

The prime examples of Abelian categories are categories of modules over a ring, but any categories of modules over a ringed space, as above, also form an Abelian category. The category of vector bundles over a space is not usually an Abelian category, however, since there can be monomorphisms that need not be the kernel of any morphism.

We earlier, in section 2.8.3, defined a projective module. We recall it here.

Definition: A module P is *projective* if, given any epimorphism, $f : B \to C$, the induced map $Hom(P, f) : Hom(P, B) \to Hom(P, C)$ is onto.

In other words any map from P to C can be lifted to one from P to B.

Examples: (i) Any free module is projective.

(ii) Any direct summand of a free module is projective and conversely. (**For you to prove, or look up**.)

Our earlier results imply that Γ_ξ is a direct summand of a free module over $\mathcal{O}_X$, so should be 'projective'. This leads to a statement of the Serre-Swan theorem. We let, as before, X be a compact (Hausdorff) space and $\mathcal{P}(C(X))$ be the category of finitely generated projective modules over the ring, $C(X)$, of continuous real valued functions on X, which is also the ring of global sections of $\mathcal{O}_X$. As before, we will write $Vect_{\mathbb{R}}(X)$ for the category of (real) vector bundles on X.

Theorem 35 *(Serre-Swan) The global sections functor,* $\Gamma(X,-) : Vect_{\mathbb{R}}(X) \to \mathcal{P}(C(X))$*, is an equivalence of categories.* ∎

Although many parts of the proof have already been touched on, and so are readily accessible, we direct the interested reader to the literature for a full proof.

9.3 Case Study 2: (continued)

9.3.1 Microbundle K-theory

Having briefly reviewed the vector bundle scene, we return to microbundles. (As before, we will tend to treat topological microbundles in some detail, leaving the adaptation to the PL case to you, but once or twice the theories diverge, so then the PL case will be treated more thoroughly. We will tend not to handle the $Diff$-case as much, as it is well treated in the literature and to handle all would take is further from our 'themes' in what is, after all, just a motivating case study!)

There is a very close parallel, and even a close relationship, between vector bundles and microbundles, so it is clear that we might attempt to produce a K-theory type construction for microbundles, as well. Significantly, this idea goes right back to the original paper of Milnor, [158] and the unpublished notes, [157], and was central to the intended geometric use of microbundles.

We have already seen the construction of induced microbundles (page 389). For later use, we would add one more result to that, analogous to the vector bundle case, (cf. section 6.1.4).

Proposition 107 *Suppose that X is paracompact, and f, $g : X \to Y$ are homotopic maps. For any microbundle, ξ, on Y, $f^*(\xi) \cong g^*(\xi)$.* ∎

The need for paracompactness is in the use of partitions of uinity in the proof. Most of the spaces we consider are compact, so that condition causes no problem.

Definition: Let $\xi_k := (E_k, B_k, i_k, p_k)$, $k = 1, 2$, be two microbundles on spaces B_1 and B_2. The *product microbundle*, $\xi_1 \times \xi_2$, is given by $(E_1 \times E_2, B_1 \times B_2, i_1 \times i_2, p_1 \times p_2)$.

It is easily **checked** that this *is* a microbundle.

We can now define the Whitney sum of microbundles on the same base. As before, we denote by $\Delta : B \to B \times B$, the diagonal map.

Definition: If ξ_1 and ξ_2 are two microbundles on B, their *Whitney sum*, $\xi_1 \oplus \xi_2$, is defined by $\xi_1 \oplus \xi_2 := \Delta^*(\xi_1 \times \xi_2)$.

Lemma 77 *The microbundle, $\xi_1 \oplus \xi_2$, can be specified by $E(\xi_1 \oplus \xi_2) \cong E(\xi_1) \times_B E(\xi_2)$, with $i_{\xi_1 \oplus \xi_2}(b) = (i_1(b), i_2(b))$ and $p_{\xi_1 \oplus \xi_2}(e_1, e_2) = p_1(e_1)$.*

Proof: Left to you. ■

This gives a semi-group structure to the set of isomorphism classes of microbundles on B. To define the K-theory corresponding to this, we use 's-equivalence' or, alternatively, the group completion of $[Core(Microbundles^{Top})]$.

Definition: Microbundles ξ and ξ' over B are said to be *stably equivalent* or *s-equivalent* if there are non-negative integers q, r such that

$$\xi \oplus \varepsilon^q_B \cong \xi' \oplus \varepsilon^r_B.$$

Of course, isomorphic microbundles are s-equivalent.

The key result that links the semigroup completion of the set of isomorphism classes and the notion of s-equivalence is:

Theorem 36 *(Milnor, [158]) If B is the realisation of a finite dimensional simplicial complex, and ξ is an microbundle on B, then there exists a microbundle, η, so that $\xi \oplus \eta \cong \varepsilon^n_B m$.* ■

We refer the reader to Milnor's paper, [158], for the proof, or to the Princeton notes, [157].

Theorem 37 *If B is the realisation of a finite dimensional simplicial complex, then the set, $K_{Top}(B)$, of s-equivalence classes of microbundles on B forms an Abelian group under the operation of Whitney sum:*

$$[\xi]_s + [\xi']_s := [\xi \oplus \xi']_s.$$

Proof: Most of this is routine. The Whitney sum is well defined, and is associative up to isomorphism as it is given by a pullback, so gives an associative operation on the s-equivalence classes. The class of the trivial bundles is clearly the identity element, and Milnor's theorem gives an inverse. Finally $\xi \oplus \xi' \cong \xi' \oplus \xi$, so this does give $K_{Top}(B)$ an Abelian group structure. ■

Remark: The corresponding K-groups for PL and $Diff$-microbundles are defined similarly and may be denoted K_{PL} and K_{Diff}, respectively. They all give contravariant functors from the corresponding categories of manifolds.

Given any vector bundle, $\xi = (E, B, p)$ on B, we have pointed out that there is a corresponding (topological) microbundle, (E, B, i, p), where i is the zero section. It should be clear (and is **easy to check**) that this gives a functor from $Vect(B)$ to $Microbundles^{Top}_B$, as it sends a vector bundle morphism (over B) to a (micro)morphism of the corresponding microbundles. The functor, again **clearly**, preserves Whitney sum, as that is just the pullback, and is compatible with the induced bundle constructions in the two cases, for the same reason. (This is all fairly routine, but **do** check that it *is* all clear.)

If ξ_1 and ξ_2 are s-equivalent as vector bundles, it is easy to see that their 'images' will be s-equivalent in $Microbundles_B^{Top}$. We thus get:

Theorem 38 *There is a natural transformation, $K \to K_{Top}$, of group valued functors.* ■

Milnor, in his original paper, [158] and also in the Princeton notes, [157], shows that this natural transformation does not always lead to a surjection, although one might have thought it would. Explicitly, from real K-theory, one has that $K(S^{4n})$, the K-group of the $4n$-sphere is an infinite cyclic group. Let γ be a generator of $K(S^{4n})$, then its image in $K_{Top}(S^{4n})$ is divisible by some non-unit integer (the value is not important here). For instance, for $n = 2$, $K_{Top}(S^8) \cong C_\infty$, and the class of the image of γ is divisible by 7. This shows the extent to which subtle geometric structures can be tracked by microbundles, whilst unnoticed by the more standard vector bundle methods. The geometric significance of this integer is briefly discussed in [157]. We will extract some other consequences, which are really why we have mentioned microbundles.

Theorem 39 *(i) A PL-manifold is smoothable if and only if the s-class of τ_M lies in the image of the homomorphism $K_{Diff}(M) \to K_{PL}(M)$.*

(ii) There is a topological manifold that cannot be smoothed.

Proof: (Sketch) For (i), we direct the interested reader to Milnor's notes, [157].

For (ii), choose an open set $U \subseteq \mathbb{R}^q$, homotopic to S^8. Let ξ be a microbundle over U_1, whose s-class does not belong to the image of the morphism $K(U) \to K_{Top}(U)$, which we know not to be a surjection. The s-class of the micobundle ξ then has a representative whose total space is a topological manifold that cannot be smoothed. ■

Recall: We *intend* using these results mostly to indicate and motivate our relative TQFT and to suggest possible methodology for exploitation of both that tentative relative theory and also for the slightly differently defined HQFT theory that we will examine shortly. Because of that, we will be **leaving lots of 'stuff' for you to check up on**, if you need it. If you do not need is, you need only retain the central point that questions of the existence of structure are often related to 'lifting problems'. We will see this again shortly. To do that we will need to start indicating how this area of smoothing, triangulation, etc., fits into the simplicial framework. (With regard to more general G-structures and their link with microbundles, have a look at article by Siebenmann,

9.3.2 Microbundles: the simplicial groups, Top, and PL

We saw how the Grassmannians were the classifying spaces of vector bundles of a fixed finite dimension. If ξ^n was a vector bundle of dimension n on a paracompact space, B, then we had (page 401), the canonical n-dimensional bundle, γ_n^m on $G_n^m(\mathbb{R})$, and a map $f : B \to G_n^m(\mathbb{R})$, cf. Proposition 106, such that $\xi^n \cong f^*(\gamma_n^m)$. We also had vector bundles of dimension n correspond to a principal $G\ell_n(\mathbb{R})$-bundle together with a representation of $G\ell_n(\mathbb{R})$. (That latter structure is slightly hidden, because we always used the natural identity representation of $G\ell_n(\mathbb{R})$ on itself, that is, its natural action on $\mathbb{R}^n$.) To handle vector bundles of arbitrary dimension, one needs to 'pass to the limit' of the $G\ell_n(\mathbb{R})$ as $n \to \infty$. We saw that through Grassmannian 'eyes', by considering γ^∞, and $G\ell_\infty(\mathbb{R})$.

We can give analogous constructions for microbundles. As before, we will concentrate on the structures for *topological* microbundles. The ideas go across to the PL case easily enough (but remember, we have not made precise what a PL-structure is, relying on 'intuition' for the descriptions, and other source material for the details.)

Let m be a fixed positive integer.

Definitions: (i) The simplicial group, Top_m, has typical k-simplex given by a microbundle isomorphism-germ over Δ^k,

$$\varphi : \Delta^k \times \mathbb{R}^m \to \Delta^k \times \mathbb{R}^m.$$

Face and degeneracy maps are defined by composition with the evident affine coface and codegeneracy maps.

(ii) The simplicial group, PL_m, has typical k-simplex given by a PL-microbundle isomorphism-germ over Δ^k,

$$\varphi : \Delta^k \times \mathbb{R}^m \to \Delta^k \times \mathbb{R}^m,$$

with face and degeneracy maps as in (i).

9.3.3 Microbundles: the classifying spaces, $BTop$, and BPL

Following the usual conventions for notation, we will write $BTop_m$ and BPL_m, for the result of applying first $\overline{W}$, and then geometric realisation to the corresponding simplicial groups.

Definition: The spaces $BTop_m$ and BPL_m are called the *classifying spaces* for Top_m and PL_m, respectively.

The name is justified by the following.

For convenience, denote by $Micro_m(B)$, the set of isomorphism classes of microbundles on B of fibre dimension m, and by $PL{-}Micro_m(B)$, the corresponding set for PL-microbundles. (We could 'manufacture' a more complicated notation using π_0 and $Core$, but it is not necessary for the limited use here, so we will refrain!)

Theorem 40 *(i) $BTop_m$ classifies topological m-microbundles with base a Euclidean neighbourood retract, that is, there is a natural bijection*

$$Micro_m(B) \leftrightarrow [B, BTop_m].$$

(ii) BPL_m classifies PL m-microbundles with base a polyhedron, that is, there is a natural bijection

$$PL{-}Micro_m(B) \leftrightarrow [B, BPL_m].$$

■

The proof is given in various places in the literature. Milnor's Princeton Notes, [157], from 1961, Buonocristiano's, [59], and Lurie's *Lecture Notes for Course, 937, (lecture 12)*, [147], all have versions of this. We will limit our discussion to a sketch proof which brings out certain useful notions which, hopefully, we can adapt for later use.

For the classification of m-microbundles, we can now call on the work that we did earlier on classification of simplicial bundles, and, in particular, the classification of principal simplicial bundles (spread over several sections of several chapters, but we mostly need ideas from sections 6.2.2 and 6.2.3).

Let K be a (locally) finite simplicial complex. (We will really only need the results for the compact, finite case, but they also work for the locally finite case.) As we have noted several times, putting an order on the vertices of K allows us to consider its associated simplicial set, which we will also denote K. We note that the 'geometric realisation of K' is unambiguous. In other words, we can take $|K|$ to mean either the realisation of the simplicial complex, K, or of the simplicial set, K. (We will be concerned, often, with K being a simplicial complex triangulating some manifold, B, and we thus need $B \cong |K|$ to make unambiguous sense ... and it does!)

We will denote by $Princ_{PL_m}(K)$, the set of isomorphism classes of principal PL_m bundles on K. We write $B = |K|$ to keep notation fairly consistent, but in general B may not be a manifold.

Theorem 41 *There is a natural bijection*

$$PL - Microm(B) \leftrightarrow Princ_{PL_m}(K).$$

Proof: Suppose ξ is a PL m-microbundle on $|K|$. We define its *associated principal bundle*, $Princ(\xi)$, as follows:

The total space, E, of $Princ(\xi)$, has q-simplices given by pairs $\mathbf{h} = (f, h)$, where $f \in K_q$ and h is a micro-isomorphism

$$h : \Delta^n \times \mathbb{R}^m \to |f|^*(\xi).$$

(Here we have $f : \Delta[q] \to K$, so $|f| : \Delta^q \to B$, and we pullback ξ along it. The result will be a trivial m-microbundle since Δ^q is contractible, so such an h will exist.) If $\mu : [r] \to [q]$, then

$$E_\mu : E_q \to E_r$$

sends (f, h) to $(K_\mu(f), \mu^*(h))$. Here $\mu^*(h)$ is the unique micro-isomorphism making the diagram

$$\begin{tikzcd}
\Delta^q \times \mathbb{R}^m \arrow[r, "h"] & {|f|^*(\xi)} \\
\Delta^r \times \mathbb{R}^m \arrow[u] \arrow[r, "\mu^*(h)"'] & {|K_\mu(f)|^*(\xi)} \arrow[u]
\end{tikzcd}$$

commute, (so is 'h restricted along μ).

The projection $p : E \to K$ is just given by $p(f, h) = f$. (This is clearly simplicial.)

The action of PL_m is by precomposition. This is free and the simplicial set of orbits can naturally be identified with K itself, so $Princ(\xi)$ is a principal PL_m-bundle.

Conversely, given a principal PL_m-bundle, $\eta = (E(\eta), K, p_\eta)$ on K we construct a m-microbundle as follows:

We have $\eta \cong K \times_t PL_m$ for some twisting function t (alternatively viewed following our earlier discussion as t_0 of a normalised regular family, $\{t_i\}$, of transitions, cf. sections 6.2.2 and 6.2.3, as before). Next consider $\coprod \varepsilon^m_\sigma$ with $\sigma \in K$. We form a quotient, in the time honoured way, (a coend!), by identifying $\varepsilon^m_{d_i\sigma}$ with $\varepsilon^m_\sigma|_{d_i\sigma}$ through the micro-isomorphism, $t_i(\sigma)$. (Recall that this can be chosen to be the identity for all $i \neq 0$, but is t itself for $i = 0$.)

You are left to show that this produces an inverse (up to isomorphism) for the associated principal bundle construction. ■

There are considerable details 'left to you' in the above sketch and some extra words are needed, perhaps, to help. A helpful notion is that of a microbundle over a simplicial set, K. If we were just handling the topological case, this could be taken to be a microbundle over $|K|$, but in the PL case, it would then not be clear how to endow $|K|$ with a PL-structure. To get around this we can use the category associated to K that was introduced back in section 6.2.4. This was the comma category, (Yon, K), so, effectively, has simplices of K as objects, then took the $\mu : [n] \to [m]$, making a diagram over K commute, as the morphisms, so, if $\sigma \in K_m$ and $\tau \in K_n$, we required $K_\mu(\sigma) = \tau$.

We assign to each $\sigma \in K_m$, a PL-microbundle ξ_σ on Δ^m and to each $\mu : ([n], \tau) \to ([m], \sigma)$, as above, an isomorphism

$$\xi_\tau \cong (\Delta^\mu)^*(\xi_\sigma),$$

i.e., compatibility with faces and degeneracies.

We thus have that a PL-microbundle, ξ, on K is just a functor, ξ, from (Yon, K) to the category of PL-microbundles and bundle maps, such that for each $\sigma \in K_q$, $\xi(\sigma)$ (or ξ_σ in our previous notation, is a microbundle with base, Δ^q. This idea is useful because we can talk about microbundles over a classifying space and that is not so easy without this extended notion.

Following up that idea, and using the classification of principal simplicial bundles, we have a universal PL_m-bundles on $\overline{W}(PL_m)$, and the induced bundle construction just gives the simplicial analogue of earlier bijections:

$$PL - Micro_m(K) \leftrightarrow [K, \overline{W}(PL_m)].$$

The construction takes a simplicial map from K to $\overline{W}(PL_m)$ and pulls back a universal PL_m microbundle which is over $\overline{W}(PL_m)$. Of course, we can think of this universal bundle either purely in simplicial terms or as a PL m-microbundle on the simplicial set $\overline{W}(PL_m)$. That second idea is a very good one to retain for future intuition.

There is a natural inclusion of $\mathbb{R}^m$ into $\mathbb{R}^{m+1}$ that we used before (page 401) This leads to an inclusion, or, more exactly, a simplicial monomorphism, from Top_m to Top_{m+1}, and, similarly, from PL_m to PL_{m+1}. We get natural 'inclusions' of $\overline{W}Top_m$ into $\overline{W}Top_{m+1}$ and of $\overline{W}Pl_m$ into $\overline{W}PL_{m+1}$, and we set $BTop = colimBTop_m$ and $BPL = colimBPL_m$.

Theorem 42 *For X a polyhedron, there are bijections*

$$K_{Top}(X) \leftrightarrow [X, BTop],$$

and

$$K_{PL}(X) \leftrightarrow [X, BPL].$$

■

Remark: The structure of $K_{Top}(X)$ as a group can be given in terms of structure on $BTop$. (**You are left** to see how to do this. It can be constructed by taking two elements, one in $BTop_m$, the other in $BTop_n$, say, and trying to construct one, the right one, in $BTop_{m+n}$. Similar remarks apply to BPL.)

We clearly have a simplicial map

$$\overline{W}PL_m \to \overline{W}Top_m,$$

induced by the 'inclusion' of PL_m into Top_m, which forgets PL-ness of the isomorphisms . It is useful to have a 'geometric' description of its homotopy fibre. (If you need to recall what the homotopy fibre is in general, look back first at page 17 and section **??**.) Note that there is an *inclusion* of simplicial groups, but it has a *homotopy fibre* (and thus a 'homotopy kernel').

We first note that $WTop_m$ is contractible and has a Top_m-action. (This is not something peculiar to Top_m as we saw that, for any simplicial group, G, WG is always contractible, and has a G-action with 'orbit space' $\overline{W}(G)$, cf. section 5.5.) If we restrict to the subgroup PL_m, we get an action of that on $WTop_m$, and the orbit space, BPL_m, will have the same homotopy type as $\overline{W}PL_m$. (To see this, look at the fibrations

$$\begin{tikzcd}
PL_m \arrow[r] \arrow[d, dash, "="'] & WPL_m \arrow[r] \arrow[d, dash, "\simeq"'] & \overline{W}PL_m \arrow[d] \\
PL_m \arrow[r] & WTop_m \arrow[r] & BPL_m
\end{tikzcd}$$

and then at the long exact sequences. Remember that, although $\overline{W}PL_m$ is the 'classifying simplicial set' of PL_m, any other Kan complex of the same homotopy type will do just as well. Here is will be convenient to abuse, or rather 'reuse', notation a bit and write BG for any such 'classifying space'.)

The great thing about this model of BPL_m is that there is an induced map from it to $BTop_m$, which is a fibration, and we take Top_m/PL_m to be its fibre:

$$Top_m/PL_m \to BPL_m \to BTop_m.$$

If we look back at the long exact fibration sequence, we would 'expect' that we would also have a Kan fibration

$$PL_m \to Top_m \to Top_m/PL_m,$$

since 'clearly' the fibre in the earlier fibration will be Top_m divided out by the action of PL_m. (It is 'instructive' to see this using the W and $\overline{W}$ models. We had, back in section 5.5, that $W(G) := G \times_t \overline{W}(G)$, and the G-action was on the fibre G in the obvious way. If H is a subsimplicial group of G then the construction we have given above corresponds to forming $(G/H) \times_t \overline{W}(G)$, with, of course, a different twisting function. The conclusion above is then a simple question of **checking everything** fits as it seems to.)

9.3.4 PL-structures on topological microbundles

To give a *PL-structure on a topological m-manifold*, M, we have to find a PL-manifold, K and a homeomorphism, $f : M \to K$, to the underlying topological manifold of K. Two such structures

(K, f) and (K', f') are considered *equivalent* if they are *concordant*, i.e., there is a PL structure on $M \times I$ agreeing with (K, f) on one end and with (K', f') on the other.

If ξ is a topological microbundle on a polyhedron, B a *PL-structure on* ξ is a PL-microbundle, η, over B such that ξ and η are isomorphic 'as topological microbundles'. (Really it is better to encode the isomorphism, f, as part of the structure, in which case we would write (η, f). In fact for the moment we will not do this, holding it 'in reserve'.) Two PL-structures, η_0 and η_1 on ξ are *equivalent* if they are concordant, that is, if there is a PL-structure on $\xi \times I$, agreeing with η_0 on one end and with η_1 on the other.

Now consider a polyhedron, X, and a topologica $m$microbundle, ξ, on it, classified by $f : X \to BTop_m$, or, more excatly by the homotopy class of f.

Theorem 43 *There is a bijection between concordance classes of PL-microbundle structures on ξ and homotopy classes of lifts, $\tilde{f}$ of f:*

$$\begin{tikzcd} & BPL_m \arrow[d] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & BTop_m \end{tikzcd}$$

■

Similarly, we can also consider f as having codomain $BTop$ by including $BTop_m$ into it. If $\tilde{f} : X \to BPL$ lifts $f : X \to BTop$, then we have a PL-microbundle structure on $\xi \oplus \varepsilon^s_X$ for some $s \geq 0$, and conversely.

We would like to use microbundles to help determine the existence of PL-structures on a topological manifold, M, or, at least on $M \times \mathbb{R}^q$ for some $q \geq 0$, this latter because we might be testing using s-equivalence classes. The basic idea is simple, although to get all the bits together is not!

- If K is a PL-manifold, then the tangent microbundle, TK, will be a PL-microbundle;
- if M is a topological manifold, then each PL-structure on M should give a PL-structure on its *topological tangent microbundle*, TM, hence
- search for PL-structures on TM, and if one finds one, try to rebuild a PL-structure on M,
- if there are no PL-structures on TM, then you know there can be none on M.

If we try to use the s-class of TM, we could think of looking for a PL-structure on microbundles in that class. This would not immediately give us a PL-structure on M itself, but would on some $M \times \mathbb{R}^s$, which would be useful information. Still there is a problem, PL-microbundles are defined over simplicial complexes / polyhedra, or, slightly less 'geometrically', over simplicial sets, but M is not yet given any explicit structure of a polyhedron, so we embed M in a high dimensional Euclidean space, $\mathbb{R}^n$, n potentially much bigger than m, the dimension of M, and take a small open neighbourhood, V, of that embedded M. As V is an open subsets of $\mathbb{R}^n$, it is a polyhedron, and we can choose it small enough to retract onto to M, since this latter space is an Euclidean neighbourhood retract (ENR). We let $r : V \to M$ be such a retraction. This gives an induced microdbundle r^*TM on V. We hope for a PL-microbundle structure on r^*TM, or on a PL-microbundle in the same s-class.

We have a map, $\widehat{t_M} : M \to BTop$ that classifies the stable tangent microbundle on M and $\widehat{r^*t_M} = \widehat{t_M} \circ r$, classifying the stable microbundle r^*TM. If we have a lift of $\widehat{t_M}$ to BPL, it will provide a lift, $r^*\tilde{t}_M$, also, for $\widehat{r^*t_M}$. This will give a PL-structure on $r^*TM \oplus \varepsilon_V^s$ (on V) for some sufficiently large s:

$$\begin{tikzcd}
 & & BPL \arrow[d] \\
V \arrow[r, "r"] \arrow[urr, dashed, "\widetilde{r^*t_M}"'] & M \arrow[r, "\widehat{t_M}"'] \arrow[ur, dashed] & BTop
\end{tikzcd}$$

We do not gibve the details as we will not be using these results other than as motivation, and the details involve topological results that would need more preparation than sees reasonable to include here. Details can be reconstructed from the notes of Buoncristiano, [59], or from Lurie's, [147], that were mentioned earlier. We will quote the following:

Theorem 44 *If M^m is a topological manifold with tangent microbundle TM, classified by $\widehat{t_M}$: $M \to BTop$, then a lift $\widetilde{t_M}$ of $\hat{t_M}$ to BPL determines a PL-structure on $M \times \mathbb{R}^q$ for some sufficiently large q.* ■

There are converses to this and there is a bijection between homotopy classes of such lifts and concordance classes of PL-structures on $M \times \mathbb{R}^q$. We refer to the above references for the continuation of this story.

9.3.5 The homotopy type of Top/PL

Simply stated, we have the fundamental result of Kirby and Siebenmann, (1969):

Theorem 45 *The space Top/PL has the homotopy type of a $K(C_2, 3)$, that is,*

$$\pi_n(Top/PL) = \begin{cases} C_2 & \textit{for } n = 3 \\ 1 & \textit{otherwise.} \end{cases}$$

■

Greater clarity can be achieved in the above by introducing the simplicial set $PL(M)$ of PL-structures on M, basically as a simplical mapping space. The path components of this then correspond to isotopy classes of PL-structures on M. The PL-structures on a compact m-dimensional manifold, M are in bijective correspondence with lifts of $\widehat{t_M}$ to $\widetilde{t_M} : M \to BPL$ and hence with $[M, Top/PL]$ and this gives another result of Kirby and Siebenmann from the same period:

Theorem 46 *If M is a PL-manifold of dimension $m \geq 5$, then $PL(M) \simeq K(C_2, 3)^M$, the space of maps from M to an Eilenberg-MacLane space, $K(C_2, 3)$.* ■

Note this gives 'isotopy' not 'concordance' as the relation, but that there are results linking the two (that we will not go into here).

We will leave this case study to return to the more central TQFT area, but first:

9.3.6 Summary of Case Studies

We launched into the case studies to emphasise several point

- the usefulness of classifying spaces for encoding structure;
- lifts of structural maps correspond to classification of the 'finer' structure with respect to the 'coarser' one;
- simplicial techniques seem almost essential to encode these ideas, as they provide a link between geometric aspects and some, as yet ill perceived, ∞-categorical aspect, glimpsed via Kan complex and quasi-category structures.

The keys to these lifting and classification problems were in the (homotopy)fibres of maps between classifying spaces. It is exactly that aspect that our idea of a relative TQFT is meant to capture, but although, in the case of $Spin$-structures, the fibre was finite, in the Top/PL case, it is only 'homotopy finite', i.e., has finite total homotopy (recall the definition back in section 8.2.1, page 333). At present, our TQFT methods do not generalise to this second case, although where are certainly indication that they should do so.

Chapter 10

Homotopy Quantum Field Theories: Introduction and general properties

In these 'Case studies', we have seen that one possible approach to handling 'manifolds with structure' might be using a 'classifying space', B, and, taking a 'manifold with B-structure' to be a map, or perhaps a homotopy class of maps, from the manifold to this space, B. We also suggested a situation that would potentially generate some TQFT-like structure from a fibration of simplicial groups having finite fibres / kernel, what we called a 'relative TQFT'.

In 1999-2000, Turaev, in two papers, [209, 210] came up with exactly this sort of theory, (but that has only recently been formally published in full, see [211]). He used the term *Homotopy Quantum Field Theory*. Everything is done 'over B' and the setup is such that in the case where B is a singleton space, the HQFTs are simply TQFTs. (Subsequently Turaev has published several papers using that theory, or rather a slightly modified form of it.)

We thus want to study 'manifolds with extra structure' and that extra structure will be given by a 'characteristic map' from the manifold to the *target background* space, B. These 'B-manifolds' and 'B-cobordisms' are then studied using tools similar to those of Topological Quantum Field Theories. In those initial papers by Turaev, one axiom in the theory was unnecessarily strong and resulted in the elimination of any influence of the homotopy structure in B above its d-type, when the manifolds concerned were of dimension d. A modified version with change to one axiom (see below) was introduced by Rodrigues, [191]. This gave dependence of $(d+1)$-HQFTs over B on the $(d+1)$-type of B. This idea in a slightly different formulation was used by Brightwell and Turner, [37], and Bunke, Turner and Willerton, [58], to look at (1+1)-HQFTs with background space a simply connected space.

The initial results of [209] classified (1+1)-HQFTs with background spaces which were 1-types and the later results handled simply connected spaces, classification results there being in terms of the second homotopy group of B. It is therefore natural to try to classify such HQFTs for which the background space is a 2-type, a situation that would include both the previous cases. This was done in Porter-Turaev, [183]. We will summarise this in a later chapter, once some of the earlier ideas have been explored in depth.

Turaev in [210] looked at 2+1 dimensional HQFTs and showed some neat structural classifications there as well. We will describe some of these ideas later as well as looking at interpretations, links with other theories and generalisations of the ideas and results to all dimensions.

We will start with a general development of HQFTs, explaining the main ideas of Turaev's work, before looking at (1+1)-HQFTs as these are the analogues of the TQFT situations that we studied earlier. The two cases of B being a $K(G, 1)$ or a $K(A, 2)$, will be looked at but as they will be special cases of the general 2-type situation, the detailed cases can be handled via that route. This will take several chapters.

It is clear that (i) a good part of the basic theory works with little change if we did not restrict to (1+1)-HQFTs, allowing d-dimensional B-manifolds, and (ii) for $(d+1)$-HQFTs, we can assume B is, at least, the classifying space of a crossed complex, in the sense [48]. Some of the methods work in even greater generality namely when B is the classifying space of a $(d+1)$-truncated simplicial group, and thus was a general $(d+1)$-type. This leads to a concept of *simplicial formal map*, which provides an algebraic / combinatorial model for the characteristic map, $g : M \to B$, that specifies the basic background structure for the manifold, M. We will see that this idea is something that we have met before.

We will be using as initial sources the two papers by Turaev, [209, 210], his book, [211] and Rodrigues paper, [191]. Here we will be looking at aspects that fit in with our general themes in this 'Menagerie', in particular, to crossed things, cohomological constructions, stacks, gerbes, bundle gerbes, etc. This means that there will be many aspects that we do mention, or only touch on lightly, and so direct the reader wishing to follow things up in more detail to those sources.

10.1 The category of B-manifolds and B-cobordisms

The basic objects on which an $(n+1)$-homotopy quantum field theory is built are compact, oriented n-manifolds together with maps to a 'background' or 'target' space, B. This space, B, will be path connected with a fixed base-point, $*$.

Definition: A *B-manifold* is a pair, (X, g), where X is a closed oriented n-manifold (with a choice of base-point, m_i, in each connected component X_i of X), and g is a continuous map $g : X \to B$, called the *characteristic map*, such that $g(m_i) = *$ for each base-point m_i.

A *B-isomorphism* between B-manifolds, $\varphi : (X, g) \to (Y, h)$, is an isomorphism, $\varphi : X \to Y$, of the manifolds, preserving the orientation, taking base-points into base-points and such that $h\varphi = g$.

Remark: The manifolds under consideration will often be differentiable and then 'isomorphism' is interpreted as 'diffeomorphism', but equally well we might position the theory in the category of PL-manifolds, or topological manifolds, with the obvious changes. In fact, for some of the time, we could develop constructions for simplicial complexes rather than manifolds, since, as for our earlier look at TQFTs, it is triangulations that provide the basis for the combinatorial descriptions of the structures that we will be using.

We will denote by $\mathbf{Man}(n, B)$, the category of n-dimensional B-manifolds and B-isomorphisms. We define a 'sum' operation on this category using disjoint union. The disjoint union of B-manifolds is defined by

$$(X, g) \amalg (Y, h) := (X \amalg Y, g \amalg h),$$

with the obvious characteristic map, $g \amalg h : X \amalg Y \to B$. With this 'sum' operation, $\mathbf{Man}(n, B)$

becomes a symmetric monoidal category with the unit being given by the empty B-manifold, $\emptyset$, with the empty characteristic map. Of course, this is an n-manifold by default.

It is important to note that $(X, g) \amalg \emptyset$ is not the same as (X, g), but merely isomorphic to it via the obvious B-isomorphism

$$l_{(X,g)} : (X, g) \amalg \emptyset \to (X, g).$$

Of course, there is a similar B-isomorphism, $r_{(X,g)} : \emptyset \amalg (X, g) \to (X, g)$. Likewise $(X, g) \amalg (Y, h)$ is a categorical coproduct, so is only determined up to natural (and universal) isomorphism. There are, of course, similar problems in most naturally arising monoidal structures such as the monoidal category, $(Vect, \otimes)$, of finite dimensional vector spaces with tensor product as the monoidal structure.

For convenience, we recall that a ($(n + 1)$-dimensional) *cobordism*, $W : X_0 \to X_1$, is a compact oriented $(n + 1)$-manifold, W, whose boundary is the disjoint union of pointed closed oriented n-manifolds, X_0 and X_1, such that the orientation of X_1 (resp. X_0) is induced by that on W (resp., is opposite to the one induced from that on W). (The manifold, W, is not considered as being pointed.) It may be convenient to write $\partial W = -X_0 \amalg X_1$ and also $\partial_- W = X_0$ and $\partial_+ W = X_1$.

Definition: A *B-cobordism*, (W, F), from (X_0, g) to (X_1, h), is a cobordism, $W : X_0 \to X_1$, endowed with a homotopy class of maps, $F : W \to B$, relative to the boundary, such that $F|_{X_0} = g$ and $F|_{X_1} = h$.

Generally, i.e., unless some confusion would ensue otherwise, we will not make a notational distinction between the homotopy class, F, and any of its representatives.

Finally, a *B-isomorphism of B-cobordisms*, $\psi : (W, F) \to (W', F')$, is an isomorphism, $\psi : W \to W'$, such that

$$\psi(\partial_+ W) = \partial_+ W',$$
$$\psi(\partial_- W) = \partial_- W',$$

and $F'\psi = F$, in the obvious sense of homotopy classes relative to the boundary.

We can glue B-cobordisms along their boundaries, or, more generally, along a B-isomorphism between their boundaries, in the usual way, see Turaev's [209], or Rodrigues, [191]. For each B-manifold, (X, g), there is a B-cobordism, $(I \times X, 1_g) : (X, g) \to (X, g)$, with $1_g(t, x) = g(x)$ and where, as usual, I denotes the unit interval. This cobordism will be called the *identity B-cobordism* on (X, g) and will be denoted $1_{(X,g)}$.

As for disjoint union of B-manifolds, we can define a disjoint union of B-cobordisms, in the obvious way.

Remarks: (i) The detailed structure of B-cobordisms and the resulting category, $\mathbf{HCobord}(n, B)$, is given in the Appendix to [191], at least in the important case of differentiable B-manifolds. This category is a monoidal category with strict duals.

(ii) Again it is worth remarking that the HQFT sources tend to index things by the dimension of the objects, whilst the conventions for TQFTs is to index by the dimensions of the cobordisms.

This slight inconvenience is, for the moment, best kept as it seems likely that for specific instances, the notational conventions of *extended* TQFTs and HQFTs may prevail. If they do, the terminology may change so that the term '012-TQFT' will refer to one in which the objects are 0-manifolds, the 1-morphisms are 1-manifolds and there are 2-morphisms that will be isomorphism classes of 2-dimensional cobordisms, and so on, whilst a 123-TQFT would correspond to having objects that were 1-manifold, morphisms that are 2-manidfolds and cobordisms being isomorphism classes of 3-D cobordisms, and so on. In the meantime, clearly it is essential to check the conventions of any source, before launching into its use in detail.

10.2 The definitions of HQFTs

We will give two forms:

10.2.1 Categorical form

Definition (categorical form): A $(n+1)$-*dimensional homotopy quantum field theory with background, B,* is a symmetric monoidal functor, τ, from $\mathbf{HCobord}(n, B)$ to the monoidal category, $Vect_{\Bbbk}^{\otimes}$, of finite dimensional vector spaces over the field $\Bbbk$.

We may abbreviate the terminology in various ways, for instance, such a τ may be called a $(n+1)$-*dimensional HQFT with background, B* or a $(n+1)$-*dimensional B-HQFT* or even a $(n+1)$ *B-HQFT* The exact meaning of the abbreviation should usually be clear from the context and so it is hoped will cause no problems. We may, abusively, also drop specification of the dimension or of the background from the terminology.

It is useful also to give here a more 'elementary' structural definition of a homotopy quantum field theory.

10.2.2 Structural form

Definition (structural form):
A $(n+1)$-*dimensional homotopy quantum field theory, τ, with background B* assigns

- to any n-dimensional B-manifold, (X, g), a vector space, $\tau(X, g)$;
- to any B-isomorphism, $\varphi : (X, g) \to (Y, h)$, of n-dimensional B-manifolds, a $\Bbbk$-linear isomorphism, $\tau(\varphi) : \tau(X, g) \to \tau(Y, h)$,

and

- to any B-cobordism, $(W, F) : (X_0, g_0) \to (X_1, g_1)$, a $\Bbbk$-linear transformation, $\tau(W) : \tau(X_0, g_0) \to \tau(X_1, g_1)$.

These assignments are to satisfy the following axioms:

(1) τ is functorial in $\mathbf{Man}(n, B)$, i.e., for two B-isomorphisms, $\psi : (X, g) \to (Y, h)$ and $\varphi : (Y, h) \to (P, j)$, we have
$$\tau(\varphi\psi) = \tau(\varphi)\tau(\psi),$$
and if $1_{(X,g)}$ is the identity B-isomorphism on (X, g), then $\tau(1_{(X,g)}) = 1_{\tau(X,g)}$.

(2) There are natural isomorphisms,

$$c_{(X,g),(Y,h)} : \tau((X,g) \amalg (Y,h)) \cong \tau(X,g) \otimes \tau(Y,h),$$

and an isomorphism, $u : \tau(\emptyset) \cong \Bbbk$, that satisfy the usual axioms for a symmetric monoidal functor.

(3) For B-cobordisms, $(W,F) : (X,g) \to (Y,h)$ and $(V,G) : (Y',h') \to (P,j)$ glued along a B-isomorphism, $\psi : (Y,h) \to Y',h')$, we have

$$\tau((W,F) \amalg_\psi (V,G)) = \tau(V,G)\tau(\psi)\tau(W,F).$$

(4) For the identity B-cobordism, $1_{(X,g)} = (I \times X, 1_g)$, we have

$$\tau(1_{(X,g)}) = 1_{\tau(X,g)}.$$

(5) For B-cobordisms, $(W,F) : (X,g) \to (Y,h)$, $(V,G) : (X',g') \to (Y',h')$ and $(P,J) : \emptyset \to \emptyset$, the following diagrams are commutative:

$$\begin{tikzcd} \tau((X,g) \amalg (X',g')) \arrow[r, "c"] \arrow[d, "\tau((W,F)\amalg(V,G))"'] & \tau(X,g) \otimes \tau(X',g') \arrow[d, "\tau(W,F)\otimes\tau(V,G)"] \\ \tau((Y,h) \amalg (Y',h')) \arrow[r, "c"] & \tau(Y,h) \otimes \tau(Y',h') \end{tikzcd} \qquad \begin{tikzcd} \tau\emptyset \arrow[r, "u"] \arrow[d, "\tau(P,J)"'] & \Bbbk \\ \tau\emptyset \arrow[ur, "u"'] & \end{tikzcd}.$$

Remark: These axioms are slightly different from those given in the original paper, [209]. The really significant difference is in axiom 4 which is weaker than as originally formulated, where any B-cobordism structure on $I \times X$ was considered as trivial. The effect of this change is important for us in as much as it is now the case that the HQFT is determined by the $(n+1)$-type of B, cf. Rodrigues, [191]. Because of this, it is feasible to attempt a full classification of all $(1+1)$-HQFTs as there are simple algebraic models for 2-types, namely crossed modules. We will return to this later on.

10.2.3 Morphisms of HQFTs

To be able to discuss classification of HQFTs, it is first necessary to discuss some notion of map between different such theories.

Definition: Let τ and ρ be two $(n+1)$-HQFTs with background B, then a map, $\theta : \tau \to \rho$, is a family of maps, $\theta(X,g) : \tau(X,g) \to \rho(X,g)$, indexed by the B-manifolds, (X,g), such that for every B-isomorphism, $\psi : (X,g) \to (Y,h)$, and every B-cobordism, $(W,F) : (X,g) \to (Y,h)$, the maps $\theta(X,g)$ and $\theta(Y,h)$ satisfy the obvious naturality and conditions for compatibility with the structure maps, r, l, etc.

Using this, we can define a category, $\mathbf{HQFT}(n,B)$, with obvious objects and maps. Change of background space induces a functor between the corresponding categories and, extending a result of Turaev (for the initial form of HQFT), Rodrigues proved in [191] that the equivalence class of $\mathbf{HQFT}(n,B)$ depended only on the homotopy $(n+1)$-type of B. (We will examine this in more detail later.) One form of the classification problem is thus to start with an algebraic model of the $(n+1)$-type of B and to find an algebraic description of the category, $\mathbf{HQFT}(n, B)$. For instance, if B is a $K(G, 1)$, then Turaev showed that there is a bijective correspondence between the isomorphism classes of $(1+1)$-dimensional HQFTs with background $K(G, 1)$ and isomorphism classes of crossed G-algebras (see [209] and below). Brightwell and Turner, [37], for B a $K(A, 2)$ with, of course, A Abelian, showed that $(1+1)$-dimensional HQFTs, with such a background, form a category equivalent to that of A-Frobenius algebras, i.e., Frobenius algebras with a specified A-action.

Before we pass to consideration of examples, we note several consequences of the definition of a homotopy quantum field theory. One of the most important is that if τ is a $(n+1)$-HQFT and (X, g) and (X, h) are two B-manifolds with the same underlying manifold, X, and the two characteristic maps, g and h are freely homotopic, then a choice of homotopy, $F : g \simeq h$, gives a B-cobordism, $(I \times X, F)$, which induces an isomorphism between $\tau(X, g)$ and $\tau(X, h)$. (This is an easy exercise, but is also a consequence of Rodrigues, [191], Proposition 1.2.). Because of this, one can expect that some of the essential features of $\tau(X, g)$ can be gleaned from the homotopy class of g.

10.3 Examples of HQFTs

We will start with summarising some of the examples and constructions given by Turaev in his original paper and his monograph, [211].

10.3.1 Primitive cohomological HQFTs

For this, B is an Eilenberg-MacLane space, $K(G, 1)$, so it has fundamental group isomorphic to a group G, and all other homotopy groups trivial. The description of this HQFT will need a few facts, which we have not yet met. We will give a description based on Turaev's paper, [209], but will also translate this to one involving triangulations, more akin to our treatment of TQFTs.

We will, as suggested before, work over a fixed field, $\Bbbk$, and, as usual in such situations, $\Bbbk^*$ will denote the group of invertible elements in $\Bbbk$. (We will usually be most interested in the case $\Bbbk = \mathbb{C}$, and in that case $\Bbbk^*$ can usually be replaced by $U(1)$, the circle group, thought of as the group of unit modulus complex numbers.)

The original cohomological approach to the cohomology of groups was via the 'spatial' cohomology of a $K(G, 1)$, so $H^n(G, A)$ was thought of as $H^n(K(G, 1), A)$ and, although now almost purely algebraic approaches to group cohomology are often given, the link with that 'spatial' origin is still strong, (see, for instance, K. Brown's book, [39], that we have mentioned before). Here, if we are considering d-dimensional oriented manifolds for our HQFT, then we will specify a cohomology class, $\theta \in H^{d+1}(G, \Bbbk^*) \cong H^{d+1}(B, \Bbbk^*)$. Such a θ will enable us to define of a $(d+1)$-HQFT, with target, B, having each $\tau(X, g)$ of dimension 1.

The next ingredient that we need is the notion of a *fundamental class* of a d-manifold. This is well known, standard material and uou may have already met it, but, just in case, we will 'recall' it in brief. For the detailed background theory, we refer to standard books on algebraic topology.

If X is a d-dimensional connected orientable manifold without boundary, then its d^{th} homology group, $H_d(X) = H_d(X, \mathbb{Z})$ (with integer coefficients), is an infinite cyclic group. There are, of course, two choices of generator, the second being the inverse of the first, corresponding to a choice of orientation, or of the reverse orientation. In fact, the *definition* of *orientation* is exactly that, a choice of generator for $H_d(X)$. If we look into this a bit more geometrically, we can intuitively think of X as being triangulated by some simplicial complex, T, and, form $C_d(T)$, the Abelian group of formal sums (over $\mathbb{Z}$) of the d-dimensional oriented simplices. (Think of a surface, which is orientable, and 'add up' all the simplices.) As there are no $(d+1)$-simplices around, $C_{d+1}(T)$ is the trivial group, and in $C_d(T)$, the sum of all the d-simplices has trivial boundary, as each boundary bit of a simplex will be matched, exactly, by another, of an adjacent simplex, which will have the opposite orientation and hence the opposite sign. (To get a feel for this, if you have not met it before, go to your surface picture and try it out!) Here, we need that X itself has no boundary, so there is nothing 'left over' from that sum.

Definition: A *fundamental class*, $[X]$, of a d-dimensional connected orientable closed manifold, X, is a generator of $H_d(X)$. If a choice of such a class is made, it specifies an *orientation* of X, which is then referred to as an *oriented*, rather than just an orientable, manifold and $[X]$ is then *the* fundamental class of X.

Remarks: (i) We will not need it, but there is a notion of fundamental class for a non-orientable manifold, made by working over the 'integers mod 2', i.e., $\mathbb{Z}_2$. There the difference between +1 and -1 has been eliminated, so the oriented simplices of a triangulation always fit together correctly.

(ii) If X is not connected, but is orientable, a *fundamental class* for X is a choice of fundamental class for each component of X, and hence an element of $H_d(X)$, which is a free Abelian group of rank the number of components of X. This is important for us as cobordisms, of course, usually have disconnected boundaries.

We will also need fundamental classes for $(d+1)$-cobordisms between d-dimensional manifolds, and, of course, if $W : X_0 \to X_1$ is such a thing, "$\partial W = X_1 - X_0$", that is, the boundary of W has an inward part, X_0, and an outward part, X_1, both of which may be disconnected, so to link the fundamental classes of X_0 and X_1, we need to have a fundamental class for the probably *non-closed* $(d+1)$-dimensional manifold, W. For this, we need the $(d+1)$-dimensional relative homology group, $H_{d+1}(W, \partial W)$, so we will briefly handle this next.

Suppose we have a space, X, and a subspace, A. Considering either both to be simplicial complexes, or using singular simplices, we get, as usual, a chain complex, $C(X)$, of X, and a corresponding one consisting of the chains within A, $C(A)$. We have a short exact sequence of chain complexes,

$$0 \to C(A) \to C(X) \to C(X)/C(A) \to 0.$$

The n^{th} relative homology group, $H_n(X, A)$, of the pair, (X, A), is the n^{th} homology of $C(X)/C(A)$, i.e., the quotient $Ker\, \partial_n / Im\, \partial_{n+1}$.

If we now go to our $(d+1)$-cobordism, $W : X_0 \to X_1$, (so $\partial W = (-X_0) \coprod X_1$, and think of $H_{d+1}(W, \partial W)$, our earlier 'handwave' suggests that, if we take the sum of the oriented $(d+1)$-simplices, the bits of the boundary that will not cancel out with other parts of the expression will be those that lie in ∂W, *but* in $C(X)/C(\partial W)$, we have 'killed' those pieces off. This makes it feasible that $H_{d+1}(W, \partial W)$ will be infinite cyclic as well - of course, we will need W to be a *connected* $(d+1)$-manifold for this to work, but the extension to the non-connected case is as before. We write $[W]$ for its (chosen) generator.

The relative homology forms part of a long exact sequence giving in the critical dimensions

$$0 \to H_{d+1}(W, \partial W) \xrightarrow{\delta} H_d(\partial W) \to H_d(W) \to \ldots,$$

and the linking map, δ, joining the different dimensions, sending $[W]$ to $[X_1] - [X_0]$. More exactly, the conditions on the orientations that we imposed on W relative to X_0 and X_1, were (recalled from pages 322 and 419) that 'the orientation of X_1 (resp. X_0) is induced by that on W (resp., is opposite to the one induced from that on W.' This translates, more precisely, to:

$$\delta[W] = [X_1] - [X_0].$$

Note that here $H_d(\partial W)$ will consist of a direct sum of infinite cyclic groups, one for each component of ∂W, and $[X_1]$ and $[X_0]$ will be the sums of the chosen fundamental / orientation classes.

Remark: One additional point to note is that there is a choice that can be made here, one that was hinted at above. If we do not want to assume that we have triangulations of all manifolds and cobordisms, we can use singular complexes throughout. This avoids some of the work later that is used to eliminate the dependence on the triangulations. On the other hand, it comes at a slight price as there will be non-trivial chains in dimensions greater than the dimension of the manifolds or cobordism. In fact, in many places, this enables the singular complex based theory to be 'sleaker' than the simplicial complex / triangulation based one. The two approaches yield the same result as they both give HQFTs based around the cohomology theory of the manifolds and cobordisms and, of course, singular and simplicial based cohomology theories give isomorphic cohomology groups. The singular complex has a side which could be exploited here and will be later on. For a space, X, $Sing(X)$ is a Kan complex and behaves like an ∞-groupoid. It is the fundamental (weak) ∞-groupoid of X. The 'quasi-algebraic' structure allows analogues of many algebraic constructions to be given; see the nLab, [173].

The input into a primitive cohomological HQFT is a space B. (This is often given by a group, G, and then B is a corresponding Eilenberg-MacLane space, $B = K(G, 1)$.) This will act as the 'target' space, and we assume given a $(d+1)$-dimensional cohomology class, $\theta \in H^{d+1}(B, \Bbbk^*)$. To each B-manifold, (X, g), of dimension d, we will assign a 1-dimensional $\Bbbk$-vector space, which we will often write as $A_{(X,g)}$, and sometimes as $\tau^\theta(X, g)$, generated by a vector $\langle a \rangle$, corresponding to a singular cycle, $a \in C_d(X)$, which represents the fundamental class $[X]$. We will loosely say that a is a *fundamental d-cycle*. Different choices of a within the fundamental class, $[X]$, will give related basis elements, thus involving the homotopy type of the space, B, and the cocycle, θ. Suppose $c \in C_{d+1}(X)$ is such that $\partial c = a - b$ for $b \in C_d(X)$, therefore, b is another choice of 'fundamental d-cocycle' for X. We require

$$\langle a \rangle = g^*(\theta)(c)\langle b \rangle.$$

This formula needs some 'deconstruction'. We have $\theta \in H^{d+1}(B, \Bbbk^*)$, (which, as was mentioned above, is the same as $H^{d+1}(G, \Bbbk^*)$), but we have $g : X \to B$, so this gives $g^*(\theta) \in H^{d+1}(X, \Bbbk^*)$, and this is represented by some homomorphism, which we also call $g^*(\theta)$, from $C_{d+1}(X)$ to $\Bbbk^*$. This gives $g^*(\theta)(c) \in \Bbbk^*$, thus a non-zero element of the field (or, if you need $\Bbbk$ to be a commutative ring and are replacing 'vector space' by finite rank free module, then it will be a unit of $\Bbbk$). It is worth noting that this 'scalar' does not depend on the choice of c, merely on the properties that c has, as, if c' was another such element of $C_{d+1}(X)$ satisfying $\partial c' = a - b$, then, as the homology of X is trivial in dimension $d+1$, c and c' are homologous, i.e., there is some $e \in C_{d+2}(X)$ with $\partial e = c - c'$. **You are left to check** that this implies $g^*(\theta)(c) = g^*(\theta)(c')$. (Alternatively there is a proof (one equation) in [209] or in Chapter 1, section 2.1. of [211].)

Because of the above, we can think of $A_{(X,g)}$ as having as basis element, the fundamental class of X, 'twisted' by the characteristic map $g : X \to B$ and 'weighted' by the element θ. That is just 'words', as the 'twisting' is subtle and to understand it, we do need to see its interaction with the other structure.

Now, let $f : (X, g) \to (Y, h)$ be a B-homomorphism, then we can obtain an isomorphism, f_*, from $A_{(X,g)}$ to $A_{(Y,h)}$, by mapping the basis element $\langle a\rangle$ to $\langle f_*(a)\rangle \in A_{(Y,h)}$. Here $f_* : H_d(X) \to H_d(Y)$, of course, and **you should check** that the resulting $\langle f_*(a)\rangle$ is independent of the choice of representing $a \in C_d(X)$.

If (X, g) is a disjoint union of the B-manifolds, (X_1, g_1) and (X_2, g_2), then $a \in C_d(X)$ can be written as the sum of the images of fundamental cycles of X_1 and X_2 under the induced maps, $i_1 : H_d(X_1) \to H_d(X)$, etc., thus we can assume $a = i_{1,*}a_1 + i_{2,*}a_2$. Clearly, as all of the vector spaces, $A_{(X,g)}$, $A_{(X_1,g_1)}$ and $A_{(X_2,g_2)}$, are of dimension 1, they are just copies of the 'vector space', $\Bbbk^1$, and the tensor product, $A_{(X_1,g_1)} \otimes A_{(X_2,g_2)}$, will be isomorphic to $A_{(X,g)}$, but what is important is the description and specification of that isomorphism. It does need to be checked that matching $\langle a_1\rangle \otimes \langle a_2\rangle$ with $\langle a\rangle$, is the 'right' isomorphism, compatible with the 'twisting', etc. (This will again be **left to you**. The **only problem** is to work out exactly what has to be checked, so, in some sense, what 'right' means! Clearly we need the isomorphism to be 'well defined', so, if $\langle a_1\rangle = g^*(\theta)(c_1)\langle b_1\rangle$, etc., we need it to match $\langle b_1\rangle \otimes \langle b_2\rangle$ with $\langle b\rangle$. This needs some fairly routine calculations that are better done **by the reader**.)

Formally we take $\tau^\theta(X, g)$ to be this $A_{(X,g)}$, and we will often omit θ from the notation.

The next point is to see how to define the hoped for HQFT, τ^θ, on B-cobordisms. Here again the 'twisting' comes in to play again. Let (W, F) be a B-cobordism from (X_0, g_0) to (X_1, g_1). Pick a fundamental $(d+1)$-cycle, $b \in C_{d+1}(W, \partial W)$, so that $\delta[b] = [a_1] - [a_0]$, the difference of fundamental cocycles for X_0 and X_1, where we use square brackets, here, to denote homology classes. The B-cobordism defines a map $\tau(W, F) : \tau(X_0, g_0) \to \tau(X_1, g_1)$ by mapping the basis element, $\langle a_0\rangle$ to $(F^*(\theta)(b))^{-1}\langle a_1\rangle$. This looks neat and loosely corresponds to manipulations that we have seen in our earlier discussions on how to define a TQFT starting with a finite group, etc., but again we *do* need to take it apart, as there is a lot happening in a short space. We first need to back-track to $\delta[b] = [a_1] - [a_0]$, and to recall how δ is constructed.

The construction of the connecting map in the homology long exact sequence is well known, but is worth recalling. We look at the general case in a homological algebra situation first. It is a particular case of the Puppe type sequence argument.

Suppose

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

is a short exact sequence of chain complexes of modules over some commutative ring, then we want the connecting homomorphism, δ, from $H_{n+1}(C)$ to $H_n(A)$. We start with some $(n+1)$-cycle, $c \in C_{n+1}$ (so $\partial c = 0$). As B_{n+1} maps down to C_{n+1}, we can find a $b \in B_{n+1}$ with $\beta b = c$. (In our case of chains on X, we could split the exact sequence,

$$0 \to A_{n+1} \xrightarrow{\alpha_{n+1}} B_{n+1} \xrightarrow{\beta_{n+1}} C_{n+1} \to 0,$$

as we have just vector spaces, but this would obscure the use of choices and a certain indeterminacy within the construction.) We know β is a chain, so $\beta\partial b = \partial\beta b = \partial c = 0$ and $\partial b \in Ker\, \beta_n = Im\, \alpha_n$. We can thus find a unique $a \in A_n$ with $\alpha a = \partial b$. We, tentatively, define $\delta[c] = [a]$. Checking that this gives a well defined homomorphism is then **left to you**. (Even if you have met this within homological algebra, **do check** you recall how that verification goes.) Interpreting this, now, in our situation, it is clear that the connecting homomorphism sends a fundamental cycle of W to the difference, $[X_1] - [X_0]$, in $C_d(\partial W)$.

We next bring in F, the characteristic map of the cobordism. It induces

$$F^* : H^{d+1}(G, \Bbbk^*) \to H^{d+1}(W, \Bbbk^*),$$

so $F^*(\theta)$ is a cohomology class in this latter group. We will also repeat our earlier (ab)use of notation and write the same thing for a (choice of) $(d+1)$-cocycle representing that class, so $F^*(\theta) : C_{d+1}(W) \to \Bbbk^*$. This means that we can form $F^*(\theta)(b)$, an invertible element of $\Bbbk$. We divide $\langle a_1 \rangle$ by this element and that will define

$$\tau^\theta(W, F) : \tau^\theta(X_0, g_0) \to \tau^\theta(X_1, g_1),$$

as in our formula

$$\tau^\theta(W, F)\langle a_0 \rangle = (F^*(\theta)(b))^{-1} \langle a_1 \rangle.$$

Of course, it really needs to be checked that this is independent of the choices of representative made. (This is examined in Turaev's paper, [209] and his book, [211], but it is a good idea to **look at it yourself** first.)

Remark: We will look in more detail a bit later on, at the relationship between this construction and the 'labelled triangulation' approach to TQFTs that we sketched out earlier, however it is worth looking at this briefly now. We had the hint that a fundamental class, whether of a closed d-manifold or a $(d+1)$-cobordism (hence in a relative homology group) was, more-or-less, the sum of the top dimensional oriented simplices. We thus can think of this $(F^*(\theta)(b))^{-1}$ as dividing by a "θ-weighted" sum of $\Bbbk^*$-valued cochains, the sum being over all the $(d+1)$-simplices of W. This is analogous to the weighting factors we used in constructing our TQFTs earlier.

It is worth noting that Turaev, in his introduction of this construction states; "This construction is inspired by the work of Freed and Quinn, [101], on TQFTs associated with finite groups." This work of Freed and Quinn was also the inspiration for Yetter's work on TQFTs from homotopy 2-types.

Much of the checking that τ^θ gives a HQFT can be **left to the reader**, either to work out themselves or to check up in the sources referred to above. We will, however, glance at the construction's interaction with the composition of B-cobordisms as this is a little more tricky than some of the other parts. It uses the Mayer-Vietoris sequence, which we now recall, but refer to standard texts such as Spanier, [198], or Hatcher, [115], for detailed proofs.

We suppose given a space, X, and two subspaces, A and B, whose interiors cover X, then there is a long exact sequence

$$\ldots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \ldots$$

The idea of the proof is to form

$$0 \to C(A \cap B) \xrightarrow{\varphi} C(A) \oplus C(B) \xrightarrow{\psi} C(A+B) \to 0,$$

where $C(A+B)$ is shorthand for the subcomplex of $C(X)$ consisting of chains that are sums of chains wholly in A and chains wholly in B. (If, as here, we use singular chains then the terms of an element of $C_n(A+B)$ are in A or in B (or in both, of course). This is clearly the image of the obvious map from $C_n(A) \oplus C_n(B)$ to $C_n(X)$.) The chain maps, φ and ψ, are given by

$$\varphi(x) = (x, -x)$$

for x, a chain in $C(A \cap B)$, and

$$\psi(x, y) = x + y.$$

The exactness of the Mayer-Vietoris sequence then follows from the induced long exact sequence for the homology of these chain complexes, together with a proof that the inclusion of $C(A+B)$ into $C(X)$ induces an isomorphism on homology.

We next go to our B-cobordisms:

$$(W_0, F_0) : (X_0, g_0) \to (X_1, g_1),$$

$$(W_1, F_1) : (X_1, g_1) \to (X_2, g_2),$$

and their composite

$$(W, F) = (W_0 \amalg_{X_0} W_1, F_0 \amalg F_1) : (X_0, g_0) \to (X_2, g_2).$$

(Turaev, [209], does consider a slightly more complex situation, namely that the input, (X_1, g_1), of the second cobordism is replaced by another B-manifold, (X_1', g_1'), with a specified B-isomorphism, $f : (X_1, g_1) \to (X_1', g_1')$, which is then used in the 'gluing together' of the two B-cobordisms. The simpler case that we will consider can be shown to be equivalent to this, but, in any case, will suffice for our exposition here. The argument we will give adapts easily to cover his case.)

In our situation, what are A and B? Clearly X should be $W = W_0 \amalg_{X_0} W_1$, and 'obviously' we should take $A = W_0$, and $B = W_1$, but that will not work as their interiors do not cover W. Here we use a slightly technical detail that was not explicitly mentioned earlier. In the cobordisms, the boundary does need to have a cylindrical neighbourhood nicely embedded in the larger manifold. For instance, this is needed to ensure a reasonable smooth structure on composites if we are in the smooth case, or to ensure a nice triangulation if we are in the PL case, etc. It is not difficult to obtain, but is usually assumed as it aids the definition and construction of the gluing / composition. Here we need to take a small open neighbourhood, N_0, of X_1 in W_0, and another one, N_1, of the copy of X_1 in W_1. These must be retractable into X_1 in each case. We then take $A = N_1 \cup_{X_1} W_0$ and $B = N_0 \cup_{X_1} W_1$, so A retracts to W_0 and B retracts to W_1. Now we can use the Mayer-Vietoris sequence, together with these retractions, to get what we need, namely a composition of the fundamental classes of W_0 and W_1 and the right sort of behaviour on the boundary. The resulting long exact sequence is then:

$$\ldots \to H_n(X_1) \to H_n(W_0) \oplus H_n(W_1) \to H_n(W) \to H_{n-1}(X_1) \to \ldots$$

Now let b_0 be a fundamental $(d+1)$-cycle on W_0, so that

$$\delta[b_0] = [a_1] - [a_0],$$

where a_i is, as before, a fundamental cycle on X_i, $i = 0, 1$. Similarly, suppose $b_1 \in C_{p+1}(X_1)$ is a fundamental $(d+1)$-cycle on W_1, with

$$\delta[b_1] = [a_2] - [a_1].$$

In the Mayer-Vietoris sequence, $\psi([b_0], [b_1]) = [b_0 + b_1] \in H_{d+1}(W, \partial W)$, and $\delta[b_0 + b_1]$ is what we would expect, namely $[a_2] - [a_0]$. (A bit of easy element chasing around various interlocking exact sequences will **check this** for you.) Moreover, that $b = b_0 + b_1$ is a fundamental $(d+1)$-cycle for W easily drops out of the same sequence in the top dimension.

The composite,

$$\tau(W_1, F_1)\sharp\tau(W_0, F_0) : \tau(X_0, g_0) \to \tau(X_2, g_2),$$

sends $\langle a_0\rangle$ to $(F_1^*(\theta)(b_1)^{-1}F_0^*(\theta)(b_0)^{-1})\langle a_2\rangle$.

By definition, $\tau(W, F) : \tau(X_0, g_0) \to \tau(X_2, g_2)$ sends $\langle a_0\rangle$ to $(F^*(\theta)(b))^{-1}\langle a_2\rangle$. The operation in $\Bbbk^*$ is multiplication, so another diagram chase shows that

$$F_1^*(\theta)(b_1)F_0^*(\theta)(b_0) = F^*(\theta)(b),$$

and hence that

$$\tau(W_1, F_1)\sharp\tau(W_0, F_0) = \tau(W, F).$$

For verification of the other conditions necessary for τ^θ to be an HQFT, we refer to the original source, [209], the monograph, [211], or **to your own resourcefulness!**

10.3.2 Operations on HQFTs

These primitive cohomological HQFTs provide a very basic set of examples, but these can be used as building blocks for more complex ones, provided we have some operations for combining arbitrary HQFTs and we turn to these next.

Duals: We suppose τ is a HQFT with background, B. Its dual, $\overline{\tau}$, is given by:

- for a B-manifold, (X, g), $\overline{\tau}(X, g) = Hom_\Bbbk(\tau(X, g), \Bbbk) = \tau(X, g)^*$, the dual space of $\tau(X, g)$;
- for any B-isomorphism, $\varphi : (X, g) \to (Y, h)$, $\overline{\tau}(\varphi)$ is the transpose of $\tau(\varphi)$;
- for $(W, F) : (X_0, g_0) \to (X_1, g_1)$, a B-cobordism, we first take the opposite B-cobordism, $(-W, X_1, X_0, -F) : (X_1, g_1) \to (X_0, g_0)$, and then take $\overline{\tau}(W, F)$ to be the transpose of τ of this.

Verification of the axioms is straightforward.

Tensor product: If τ and τ' are two $(n+1)$ B-HQFTs, then $\tau \otimes \tau'$, defined in the obvious way (so, for instance, $(\tau \otimes \tau')(X, g) = \tau(X, g) \otimes \tau'(X, g)$), defines a $(n+1)$ B-HQFT.

Direct sum: A similar construction gives $\tau \oplus \tau'$, starting from the idea of using the direct sum of the vector spaces, etc. More precisely, on connected B-manifolds, (X, g), one does exactly that, defining $(\tau \oplus \tau')(X, g)$ to be $\tau(X, g) \oplus \tau'(X, g)$, then using axiom 2 one extends to non-connected B-manifolds[1].

Rescaling: HQFTs can be *rescaled* using numerical invariants of B-cobordisms. Suppose we have a $\mathbb{Z}$-va;ued assignment, ρ, sending a B-cobordism, $\underline{W}$, (a shorthand for (W, X_0, X_1, F)), to an integer $\rho(\underline{W})$. We say it is an *invariant* if it respects B-isomorphisms and homotopies of the characteristic map, F, relative to the boundary. The invariant, ρ, is said to be *additive* if

(i) it sends disjoint union to addition:

$$\rho(\underline{W} \amalg \underline{W'}) = \rho(\underline{W}) + \rho(\underline{W'}),$$

and

(ii) sends 'gluing' to addition, as well, so if $\underline{W} = (W, X_0, X_1, F)$ and $\underline{W'} = (W', X_1, X_2, F')$, then on forming $\underline{W} +_{X_1} \underline{W'}$, the B-cobordism obtained by gluing the two given B-cobordisms along X_1, we have

$$\rho(\underline{W} +_{X_1} \underline{W'}) = \rho(\underline{W}) + \rho(\underline{W'}).$$

An example of such an additive invariant is the *relative Euler characteristic*, $\chi(W, X_0) = \chi(\underline{W}) = \chi(W) - \chi(X_0)$. (It is clear that this is additive under disjoint union and to see that it is also additive under gluing, think of it as the alternating sum of the numbers of vertices, edges, faces, etc., of a triangulation of W, but in which the contributions from simplices in X_0 are not counted. In $W +_{X_1} W'$, the contribution of X_1, which is in $\chi(W) + \chi(W')$ twice, is correctly counted. (This is, of course, a similar point to one we saw in Lemma 59 on page 339, when handling Yetter's construction.)

Now suppose that ρ is such an invariant, and $a \in \Bbbk^*$ is an invertible element of $\Bbbk$. If τ is a $(n + 1)$-dimensional B-HQFT, then we can form the a^ρ-scaled HQFT, $a^\rho\tau$, which has the same vector spaces as its 'values' on B-manifolds as does τ itself, but if $(W, F) : (X_0, g_0) \to (X_1, g_1)$ is a B-cobordism, then

$$a^\rho\tau(W, F) = a^{\rho(\underline{W})}\tau(W, F),$$

that is, the images of $\tau(W, F)$ are *scaled* by a factor $a^{\rho(\underline{W})}$.

10.3.3 Geometric transfer

'Transfer' is a process well known in group representation theory and group cohomology; see, for instance, Chapter II.9 of Ken Brown's book, [39]. The connection with covering spaces is well known in those situations.

Suppose that we have a finite sheeted covering space, $p : E \to B$, of B. We collapse the fibre over the base-point, $* \in B$, to a point getting a new space, E'. Suppose τ is a E'-HQFT. Let $q : E \to E'$ be the projection and $(X, g : X \to B)$ be a B-manifold. Looking at all lifts, $\overline{g} : X \to E$, we have $(M, q\overline{g})$ is a E'-manifold, so we can set

$$A_{(M,g)} = \bigoplus\{\tau(M, q\overline{g}) \mid \overline{g} : X \to E, p\overline{g} = g\}.$$

It is then not hard to extend this assignment to give a B-HQFT structure.

[1] see Turaev's monograph, [211], for more details.

10.4 Change of background

The operation of transfer, given above, uses a 'transfer' of background. There was an HQFT with background E', the result of collapsing the fibre of the covering, $p : E \to B$, over the base-point of B. The method used the induced map from E' to B to build an HQFT on B. We will see this process and related ones several times in the following sections. it is clearly related to the relative TQFT construction sketched earlier. To examine it in detail, we will need to study the general problem of change of background under a pointed continuous map, and, in fact, that will give us very valuable information about B-HQFTs and their dependence on B.

10.4.1 The induced functor on HCobord(n, B)

Suppose $f : B \to B'$ is a base-point preserving continuous map between two 'background' spaces, by which we mean that they satisfy the conditions given earlier, (page 418).

Let $(X, g : X \to B)$ be an n-dimensional B-manifold, then it is clear that (X, fg) is an n-dimensional B'-manifold. This process clearly respects B-isomorphisms, sending them to the B'-isomorphisms.

If $(W, F : W \to B)$ is a B-cobordism, then, similarly, (W, fF) is a B'-cobordism, and, as the process of gluing takes place without any use of the characteristic maps, this 'induction' or 'post-composition' process will respect composition. (This deserves a bit more detail, but can be left at that on a first read. The point is composition of morphisms in $HCobord(n, B)$ is *derived* from gluing of B-cobordisms, but the B-cobordisms themselves are not 'officially' the morphisms as those are B-isomorphism classes of B-cobordisms (relative to their boundaries). **Writing down all that in detail** and then post-composing with f, gives the proof of the claim just made.) We thus have that 'post-composition' with f gives a functor, f_*, from $HCobord(n, B)$ to $HCobord(n, B')$.

Proposition 108 *The functor, f_*, is monoidal.*

Proof: The tensor product in $HCobord(n, B)$ is given by disjoint union / coproduct of manifolds and uses the universal property of coproduct, so it is easy to check that

$$f_*((X, g) \amalg (X', g')) \cong f_*(X, g) \amalg f_*(X', g'),$$

and that these isomorphisms are compatible with composition. ∎

In fact, the functor, f_*, can be considered to be strictly monoidal provided the disjoint union is suitably rigidly specified.

As the categorical form of the definition of an $(n + 1)$-dimensional homotopy quantum field theory with background, B, is very simply a symmetric monoidal functor, τ, from $HCobord(n, B)$ to $Vect_{\mathbb{k}}^{\otimes}$, and a morphism between two such is a monoidal transformation, we obtain the following by applying just a bit of simple monoidal category theory.

Proposition 109 *A continuous map, $f : B \to B'$, of backgrounds induces a functor,*

$$f^\sharp : HQFT(n, B') \to HQFT(n, B).$$

Proof: This is simply defined by $f^\sharp(\tau) = \tau f_*$. ■

We next consider what happens if $H : f_0 \simeq f_1 : B \to B'$, so we have two homotopic maps between B and B'.

If (X, g) is a B-manifold, then $(X \times I, Hg) : (X, f_0 g) \to (X, f_1 g)$ is a B'-cobordism. In the following lemma, we look at a slightly more general result:

Lemma 78 *Let (X, g) and (X, h) be two B-manifolds such that there is a (free) homotopy between g and h, then there is an isomorphism $(X, g) \overset{\cong}{\to} (X, h)$ in $HCobord(n, B)$.*

Proof: Let$F : X \times I \to B$ be a homotopy between g and h, and $(X \times I, F)$ the corresponding cobordism. We also have the reverse homotopy, which we denote by $\overline{F} : h \simeq g$, so $\overline{F}(x, t) = f(x, 1 - t)$ for $t \in I$, and $(X, \overline{F})$ is also a B-cobordism. We know that the composite homotopies $F \circ \overline{F}$ and $\overline{F} \circ F$ are homotopic to the constant 'identity' homotopy on the respective maps, h or g. The corresponding glued B-cobordisms are therefore B-isomorphic to the identities. Hence $(X \times I, F)$ is an isomorphism in $HCobord(n, B)$. (We have skimmed over some of the details here. They can safely be **left to you** and, in the smooth case, they are given by Rodrigues, ([191], Proposition 1.2.). ■

Using this lemma, we have that in our previous setting, if (X, g) is a B-manifold, then the homotopy yields a natural isomorphism between $(X, f_0 g)$ and $(X, f_1 g)$. It is easy, then, to see that the following holds:

Corollary 21 *If $H : f_0 \simeq f_1 : B \to B'$, then H induces (i) a monoidal natural isomorphism $f_{0*} \overset{\cong}{\to} f_{1*}$, and (ii) a natural isomorphism $f_0^\sharp \overset{\cong}{\to} f_{1*}^\sharp$* ■

Corollary 22 *If $f : B \to B'$ is a homotopy equivalence, then the induced functor, f_*, is an equivalence of categories, as also is $f^\sharp$.* ■

This basically says that the properties of a HQFT only depend on the homotopy type of its background. We will shortly see how it depends on the dimension, n, of the manifolds involved. Before that we will use the above to explore the relationship between HQFTs and TQFTs.

10.4.2 HQFTs and TQFTs

Any manifold, X, has a trivial B-manifold structure, since we can always take the constant characteristic map, $g : X \to B$, $g(x) = *$, the base point of B. The same goes for unadorned cobordisms between manifolds. We thus, after **doing a bit of checking**, have a monoidal functor from $(n+1)-Cob$ to $HCobord(n, B)$, and this functor is an inclusion on the objects. It is not full, in general, the obvious case being a closed $(n+1)$-manifold thought of as a cobordism from $\emptyset$ to $\emptyset$. This could have other than a trivial characteristic map to B. Of course, if B is contractible, then this could not happen and $HCobord(n, B)$ will be equivalent to $(n+1)-Cob$.

In general, we have a unique pointed change of background from the one point pointed space, $*$, to B and, similarly, from B to $*$, with the composite in one sense giving the identity on $*$. We thus get, on identifying $HCobord(n, *)$ and $(n+1)-Cob$, monoidal functors

$$(n+1)-Cob \to HCobord(n, *) \to (n+1)-Cob,$$

with composite the identity functor. (The left hand one is the inclusion we gave earlier.)

Similarly, at the level of HQFTs, we have

$$(n+1)-TQFT \to HQFT(n,B) \to (n+1)-TQFT,$$

so every TQFT extends, trivially, to an HQFT. This is useful, as it gives us information about models for HQFTs. We have, in certain cases, characterisations, even classifications, in terms of algebraic models, of the TQFTs and the above relationship strengthens the intuition that introducing B into the picture may not perturb the basic constructions of those models too much.

The discussion suggests a means of attack on B-HQFTs, so as to analyse them and interpret what they tell us, both about B and about the structured B-manifolds. If we can decompose the homotopy type of B in a sensible way, we might be able to build up a picture of how a B-HQFT depended on related, hopefully simpler, ones. We saw this to some extent with transfer, where we could use E'-HQFTs to construct certain B-HQFTs. The next ingredient in the discussion will thus be one such decomposition, namely that involving n-types,

10.4.3 Change along an $(n+1)$-equivalence

Our next aim is to see what an $(n+1)$-dimensional HQFT records about B by seeing how changing the background along a $(n+1)$-equivalence influences things.

Suppose that $f : B \to B'$ is an $(n+1)$-equivalence, and look at the induced functor,

$$f_* : HCobord(n,B) \to HCobord(n,B').$$

If f is an $(n+1)$-equivalence, then we know that for any n-manifold, X,

$$[X,f] : [X,B] \to [X,B']$$

is a bijection (cf. section 4.1.1, page 157).

Lemma 79 *For any B'-manifold structure, $g' : X \to B'$ on X, there is a B-manifold structure, (X,g), so that $f_*(X,g) \cong (X,g')$.*

Proof: This is clear, since there is a $g : X \to B$ such that $[X,f][g] = [g']$, whilst by Lemmas 78, this means that $(X,g') \cong (X,fg)$ in $HCobord(n,B')$. ■

We thus have that f_* is essentially surjective on objects. This suggests that we check if f_* is full and faithful.

Suppose W is a cobordism between X_0 and X_1 and we look at the set, $[W,B]_{rel\,\partial W}$, of homotopy classes (relative to the boundary) of maps from W to B. The induced map, given by composition with f, does give

$$[W,f]_{rel\,\partial W} : [W,B]_{rel\,\partial W} \to [W,B']_{rel\,\partial W},$$

and we can adapt the arguments of section 4.1.1, page 157, to show that $[W,f]_{rel\,\partial W}$ is also a bijection.

As a morphism in $HCobord(n,B)$ is a B-isomorphism class of B-cobordisms, (W,F), where F 'is' a homotopy class $rel\,\partial W$ of maps $F : W \to B$, it is now a simple matter to check that f_* is full and faithful, and we have proved:

Theorem 47 *(Rodrigues, [191]) If f is an $(n+1)$-equivalence, then the induced functor, f_*, is an equivalence of categories.* ■

The consequences of this result for HQFTs is clear.

Corollary 23 *If f is an $(n+1)$-equivalence, then the induced functor, $f^\sharp$, is an equivalence of categories.* ■

We can, thus, restrict attention to background spaces which are m-types for $m \leq n+1$, as any n-dimensional HQFT is isomorphic to one with a $(n+1)$-type as background; you simply replace B by $P_{n+1}B$, a Postnikov $(n+1)$-section of B.

As an instance of this, for any simply connected space, B, there is equivalence of categories:

$$HCobord(1, B) \stackrel{\simeq}{\to} HCobord(1, K(A, 2)),$$

where $A \cong \pi_2(B)$.

This raises the possibility of studying, for instance, $HCobord(1, K(A, 2))$ directly obtaining algebraic and categorical classifications of it. Rodrigues, in [191], uses this to give a description of $HCobord(1, K(A, 2))$, up to equivalence, as *the free symmetric monoidal category with strict duals on an 'A-Frobenius object'.* (An A-Frobenius object is a Frobenius object (cf. page 331), a, together with an action $A \to End(a)$, satisfying some compatibility conditions. We will consider these in more detail a bit later on.) This, in turn, implies that $HCobord(1, K(A, 2))$ is equivalent to the category of A-Frobenius algebras (over $\Bbbk$). This result was first discovered by Brightwell and Turner, [37], and extends the classification of TQFTs that we mentioned briefly on page 330.

This is just one of the potential instances of the theorem, but its link with a categorical characterisation of $HCobord(1, B)$, in this case, suggests a host of generalisations and potential applications / interpretations of HQFTs. (For this beyond to where we will discuss in these notes, see the discussion the entries on the nLab, [173], on the 'Cobordism Hypothesis', and the preprint of Lurie, [148].)

If we look at the simple case of $HCobord(1, B)$, but with no additional constraint on B, then we know that, up to equivalence, this is the same as $HCobord(1, P_2B)$, so we may assume $\pi_i B = 0$ for $i > 2$. In other words, we can suppose that B is a 2-type, hence is the classifying space of some crossed module, C. If we know *algebraic* information abnout C, can we glean categorical information, and perhaps, eventually, geometric information about $HCobord(1, B\mathsf{C})$. Of course, that question is just the start, as we can ask similar questions about $HCobord(n, B\mathsf{C})$, where C is a model for a homotopy $(n+1)$-type, something like a crossed n-cube, or an $(n+1)$-hypergroupoid. How does $HCobord(n, B\mathsf{C})$ reflect properties of C, for instance, if C is a $(n+1)$-truncated k-crossed complex? In each case, there are also two excellent questions to ask. Firstly, are there 'interesting' examples (for various values of 'interesting')? and secondly, what does it all mean?

10.5 Simplicial approaches to HQFTs.

Before we ask for answers to these questions in more detail, we will look at some of the simplicial aspects of all this. We can adapt ideas from our treatment of (relative) simplicially generated TQFTs to give a common generalisation of the Brightwell-Turner classification theorem and the complementary classification of $HCobord(1, K(G, 1))$ in terms of G-graded crossed algebras, which we have not yet seen in any detail, so, in the next few sections, we will look at some of the ways in which the intuitions gained from the Yetter approach to TQFTs can be applied to HQFTs, initially just in low dimensions. Later we will discuss the general cases. (The treatment given is adapted from [182, 183]. We may repeat some of the points from our treatment of the Yetter construction for convenience.)

10.5.1 Background

As we have seen, in the construction of models for topological quantum field theories, one can use a (finite) group G, and a triangulation of the manifolds, Σ, etc., involved, and one assigns labels from G to each (oriented) edge of each (oriented) triangle, for example in the diagram below:

$$\begin{tikzcd} & \cdot \arrow[dr, "h"] & \\ \circ \arrow[ur, "g"] \arrow[rr, "k"'] \arrow[r, phantom, shift left=3ex, "\circlearrowleft"] & & \cdot \end{tikzcd}$$

with the boundary/cocycle condition that $kh^{-1}g^{-1} = 1$, so $k = gh$.

(Here the orientation is given as anticlockwise, which may seem unnatural given the ordering, but this is necessary as we are using the 'path order convention' on composition of labels on edges. The other convention also leads to some inelegance at times.)

The geometric intuition behind this is that 'integrating' the labels around the triangle yields the identity. This intuition corresponds to problems in which a G-bundle on Σ is specified by charts and the elements g, h, k, etc. are transition automorphisms of the fibre. The methods then use manipulations of the pictures as the triangulation is changed by subdivision, etc.

Another closely related view of this is to consider continuous functions, $f : \Sigma \to BG$, to the classifying space of G. If we triangulate Σ, we can assume that f is a cellular map using a suitable cellular model of BG and at the cost of replacing f by a homotopic map and perhaps subdividing the triangulation. From this perspective, the previous model is a combinatorial description of such a continuous 'characteristic' map, f. The edges of the triangulation pick up group elements since the end points of each edge get mapped to the base point of BG, and $\pi_1 BG \cong G$, whilst the faces give a realisation of the cocycle condition. Likewise we can use a labelled decomposition of the objects as CW-complexes, cf. [145, 209].

Let B be a CW-complex model for a 2-type (so $\pi_k B$ is trivial for $k > 2$). Assume it is reduced, so has a single vertex, then, denoting by B_1, the 1-skeleton of B, the crossed module, $(\pi_2(B, B_1), \pi_1(B_1), \partial)$, will represent the 2-type of B . For any B-manifold, the characteristic map, $g : \Sigma \to B$, or for a B-cobordism, $F : M \to B$, can be replaced, up to homotopy, by a cellular map, so, in general, we can think of a combinatorial model for the B-manifolds and B-cobordisms, in terms of combining labelled triangles with $g, h, k \in \pi_1(B_1)$ and $c \in \pi_2(B, B_1)$, and where the

$$\begin{tikzcd} & \cdot \arrow[dr, "h"] & \\ \circ \arrow[ur, "g"] \arrow[rr, "k"'] \arrow[r, phantom, shift left=3ex, "c\ \circlearrowleft"] & & \cdot \end{tikzcd}$$

cocycle condition is replaced by a boundary condition

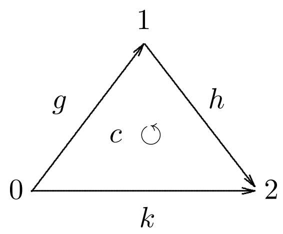
$$\partial c = kh^{-1}g^{-1}.$$

Usually $\pi_1(B_1)$ will be free, but for our discussion, it will be useful to replace this particular type of crossed module by a general one.

The approach that we will explore is via 'formal' HQFTs. These can be seen as being analogous to combinatorial or lattice approaches to TQFTs, and thus, via some duality, to state-sum approaches. They rely initially on triangulations, but subsequently on cell decompositions of the manifolds and cobordisms, which, of course, then have to be shown not to influence the theories unduly.

10.5.2 Formal C maps, circuits and cobordisms

From now on, we fix a crossed module, $\mathsf{C} = (C, P, \partial)$, as given. Our formal C-maps will initially be introduced via C-labelled / coloured triangles, as above, but will then be replaced by a cellular version as soon as the basic results are established confirming some basic intuitions. The labelled triangles, tetrahedra, etc., will all need a base point as a 'start vertex'. The need for this can be seen in an elementary way as follows: if we have the situation below, we get the boundary



condition $\partial c = kh^{-1}g^{-1}$, which was read off starting at vertex 0: first along k, back along h giving h^{-1}, then the same for g giving g^{-1}. The element c is assigned to this 2-simplex with this ordering / orientation, but if we tried to read off the boundary starting at vertex 1, we would get $g^{-1}kh^{-1}$, which is not ∂c, but is $\partial({}^{g^{-1}}c)$. We thus have that the P-action on C is precisely encoding the change of starting vertex.

Remark: Our simplices will have a marked vertex to enable the boundary condition, and later on a cocycle condition, to be read off unambiguously. We could equally well work with a pair of marked vertices corresponding to 'start' and 'finish' or 'source' and 'target'. For triangles this would give, for instance, the above with start at 0 and finish at 2, and would give a boundary condition read off as $k = \partial c \cdot gh$. This can lead to a 2-categorical formulation of formal C-maps, which is connected with the way in which a crossed module, C, is equivalent to a strict 2-group.

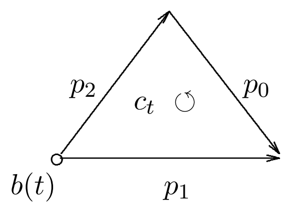
Formal C-maps are a combinatorial and algebraic model of the characteristic maps, mirroring in many ways the use of colourings in the Yetter approach to generation of TQFTs, and as our main initial use of formal C-maps will be in low dimensions, we will first describe them for closed 1-manifolds, then for surfaces, etc.

Let C_n denote an oriented n-circuit, that is, a triangulated oriented circle with n-edges and a choice of start-vertex. A *formal* C-*map* on C_n is a sequence of elements of P, $\mathbf{g} = (g_1, \ldots, g_n)$, thought of as labelling the edges in turn. We will also call this a *formal* C-*circuit*. Two formal C-circuits will be isomorphic if there is a simplicial isomorphism between the underlying circuits preserving the orientation and labelling.

If S is a closed 1-manifold, it will be a k-fold disjoint union of circles and an oriented triangulation of S gives a family of n-circuits for varying n. A *formal* C-*map* on S will be a family of formal C-maps on the various C_ns. (This includes the empty family as an instance, where S is the empty 1-manifold.) As with the construction of TQFTs, it will be technically useful to have chosen an ordering of the vertices in any 1-manifold or, later, cobordism / triangulated surface. This ordering may be a total order, in which case it can be used to replace the orientation, but a partial order in which the vertices of each simplex and equally the base points of components, are totally ordered, will suffice. With such an order on the vertices of a 1-manifold, we have that a formal C-map on it is able to be written as an *ordered* family of formal C-circuits, that is, a list of lists of elements of P. Of course, the end result depends on that order and care must be taken with this, just as care needs to be taken with the order of the constituent spaces in a vector product decomposition - and for the same reasons.

Given two formal C-maps, $\mathbf{g}$ on S_1, $\mathbf{h}$ on S_2, we can take their disjoint union to obtain a C-map $\mathbf{g} \sqcup \mathbf{h}$ on $S_1 \sqcup S_2$. We note that $\mathbf{g} \sqcup \mathbf{h}$ and $\mathbf{h} \sqcup \mathbf{g}$ are not identical, merely 'isomorphic', via an action of the symmetric group of suitable order, but, of course, this can be handled in the usual ways, depending to some extent on taste, for instance, via the standard technical machinery of symmetric monoidal categories.

Now let M be an oriented (triangulated) cobordism between two such 1-manifolds, S_0 and S_1, and suppose given formal C-maps, $\mathbf{g}_0$, $\mathbf{g}_1$, on S_0 and S_1 respectively. A *formal* C-*map*, $\mathbf{F}$, on M consists of a family of elements $\{c_t\}$ of C, indexed by the triangles t of M, a family, $\{p_e\}$ of elements of P indexed by the edges of M and for each t, a choice of base vertex, $b(t)$, such that the boundary condition below is satisfied:
in any triangle t,



we have

$$\partial c_t = p_1 p_0^{-1} p_2^{-1}.$$

We call such a formal C-map on M a *formal* C-*cobordism* from $(S_0, \mathbf{g}_0)$ to $(S_1, \mathbf{g}_1)$ if it restricts to these formal C-maps on the boundary 1-manifolds. We will denote it $(M, \mathbf{F})$.

To be able to handle manipulation of formal C-cobordisms 'up to equivalence', so as to be able to absorb choices of triangulation, base vertices, etc. and eventually to pass to regular cellular decompositions, we need to consider triangulations of 3-dimensional simplicial complexes and formal C-maps on these. We, in fact, can use a common generalisation to all simplicial complexes.

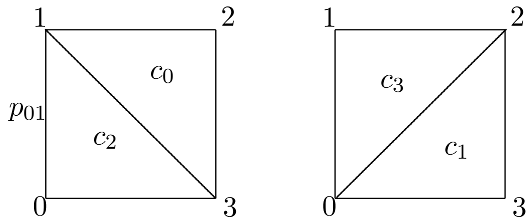
10.5.3 Simplicial formal maps and cobordisms

We make things a bit more abstract (and 'formal'!)

Definition: Let K be a simplicial complex. A *(simplicial) formal* C*-map*, λ, on K consists of families of elements
(i) $\{c_t\}$ of C, indexed by the set, K_2, of 2-simplices of K,
(ii) $\{p_e\}$ of P, indexed by the set of 1-simplices, K_1, of K,
and a partial order on the vertices of K, so that each simplex is totally ordered. The assignments of c_t and p_e, etc., are to satisfy
(a) the boundary condition,

$$\partial c_t = p_1 p_0^{-1} p_2^{-1},$$

where the vertices of t, labelled v_0, v_1, v_2 in order, determine the numbering of the opposite edges, e.g., the edge, e_0, is between v_1 and v_2, and p_{e_i} is abbreviated to p_i;
and
(b) the cocycle condition:
in a tetrahedron yielding two composite faces, as below,



we have

$$c_2 \cdot {}^{p_{01}}c_0 = c_1 \cdot c_3.$$

Explanation of the cocycle condition: The left hand and right hand sides of the cocycle condition have the same boundary, namely the boundary of the square, so $c_2 \cdot {}^{p_{01}}c_0(c_1c_3)^{-1}$ is a cycle. There is no reason for this to be non trivial, so we should expect it to be 1. A crossed module can be considered, in this context, as having elements in dimensions 1 and 2, but nothing in dimension 3, therefore, just as the case in which $B = BG$ for G, a group, led to a cocycle condition in dimension 2, so when labelling with elements of a crossed module, we should expect the cocycle condition to be a 'tetrahedral equation', hence in dimension 3.

When 'integrating' a labelling over a surface corresponding to three faces of a tetrahedron, the composite label is on the remaining face, so given a formal C-map on the tetrahedron, and a specification of p_{01}, any one of $c_0, \ldots, c_3$ is determined by the others. (For example, if all but c_0 are given, then

$${}^{p_{01}}c_0 = c_2^{-1} c_1 c_3,$$

and then acting throughout with p_{01}^{-1} yields c_0.)

A third related view is that coming from the homotopy addition lemma, [49], which loosely says that any one face of an n-simplex is a (suitably defined) composite of the others.

Remark: For the moment, we will restrict attention to 1+1 HQFTs and to formal C-maps on 1-manifolds, surfaces and 3-manifolds. If a higher dimensional theory was being considered based on B-manifolds of dimension d, the cocycle condition would naturally occur in dimension $d+2$. In that case, the natural coefficients would be in one of the higher dimensional analogues of a crossed module such as crossed complexes, suitably truncated hypercrossed complexes (or, equivalently, truncated simplicial groups). We will look at this later; see chapter **??**.

10.5.4 Equivalence of formal C-maps

Suppose X is a polyhedron with a given family of base points, $\mathbf{m} = \{m_i\}$, and K_0, K_1 are two triangulations of X, i.e., K_0 and K_1 are simplicial complexes with geometric realisations homeomorphic to X (by specified homeomorphisms) with the given base points among the vertices of the triangulation.

Definition: Given two formal C-maps, (K_0, λ_0), (K_1, λ_1), then we say they are *equivalent* if there is a triangulation, T, of $X \times I$ extending K_0 and K_1 on $X \times \{0\}$ and $X \times \{1\}$ respectively, and a formal C-map, Λ, on T extending the given ones on the two ends and respecting the base points, in the sense that T contains a subdivided $\{m_i\} \times I$ for each basepoint m_i and Λ assigns the identity element 1_P of P to each 1-simplex of $\{m_i\} \times I$.

We will use the term 'ordered simplicial complex' for a simplicial complex, K, together with a partial order on its set of vertices such that the vertices in any simplex of K form a totally ordered set. If we give the unit interval, I, the obvious structure of an ordered simplicial complex with $0 < 1$, then the cylinder, $|K| \times I$, has a canonical triangulation as an ordered simplicial complex and we will write $K \times I$ for this.

If we are given two formal C-maps defined on the same ordered K, (K, λ_0), and (K, λ_1), we say they are *simplicially homotopic* if there is a formal C-map defined on the ordered simplicial complex, $K \times I$, extending them both.

The following is fairly easy to prove.

Lemma 80 *Equivalence is an equivalence relation.* ∎

Equivalence combines the intuition of the geometry of triangulating a (topological) homotopy, where the triangulations of the two ends may differ, with some idea of a combinatorially defined simplicial homotopy of formal maps. We have:

Lemma 81 *If (K, λ_0), and (K, λ_1) are two formal maps, which are simplicially homotopic as formal C-maps, then they are equivalent.* ∎

There are several possible proofs of the following result. The one in [183] is amongst the longer ones as it illustrates the processes of combination of labellings of simplices given by a formal C-map by explicitly constructing the required extension. (As it is quite long, we will **leave it to you** to check up on, after hopefully **attempting to give a sketch proof yourself**.)

Proposition 110 *Given a simplicial complex, K, with geometric realisation $X = |K|$, and a subdivision, K', of K.*
(a) Suppose λ is a formal C-map on K, then there is a formal C-map, λ', on K' equivalent to λ.
(b) Suppose λ' is a formal C-map on K', then there is a formal C-map, λ, on K equivalent to λ'. ∎

Remarks: (i) To help with the understanding of what needs to be done, we can see it as a series of nested inductions 'up the skeleton' of various parts of the structure. To handle higher dimensions, we continue that process only handling $\sigma \in K_n$ when all its faces have been done, then using inverse induction and a join formulation of the triangulation, which is easy to see for the case $n = 2$.

(ii) There is a simplicial set formulation of the above in terms of the Kan complex condition on the simplicial nerve of C. This is useful for the extensions of this theory to higher dimensions and we will develop them later on.

(iii) Remember that the idea of a formal C-map is to represent, combinatorially, the characteristic map of a B-manifold or B-cobordism, and from this perspective, equivalent formal maps will correspond to homotopic characteristic maps.

Proposition 111 *A change of partial order on the vertices of K, or a change in choice of start vertices for simplices, generates an equivalent formal C-map.*

Proof: More formally, let K_0 be K with the given order and K_1 the same simplicial complex with a new ordering. Construct a triangulation T of $|K| \times I$ having K_0 and K_1 on the two ends. (Inductively, we can suppose just one pair of elements has been transposed in the order.) It is now easy to extend any given λ_0 on K_0 over T and then to restrict to get an equivalent λ_1 on K_1. ■

Note if $\langle v_0, v_1\rangle$ is an ordered edge of K_0 and, with the reordering, $\langle v_1, v_0\rangle$ is the corresponding one in K_1, then if λ_0 assigns p to $\langle v_0, v_1\rangle$, λ_1 assigns p^{-1} to $\langle v_1, v_0\rangle$ as is clear for the simplest assignment scheme:

$$\begin{tikzcd}
(v_0,1) & (v_1,1) \arrow[l, "p^{-1}"'] \arrow[dl, phantom] \\
(v_0,0) \arrow[u, "1_P"'] \arrow[r, "p"'] & (v_1,0) \arrow[u, "1_P"'] \arrow[ul, "p^{-1}"']
\end{tikzcd}$$

(The triangulation T assumes here that vertices of $|K| \times \{1\}$ are always listed after those of $|K| \times \{0\}$.) A similar, but more complex, observation is valid for higher dimensional simplices. Once the use of the boundary and cocycle conditions is understood, the choice of local ordering within the triangulation easily determines the simplest choice of extension. That extension can be perturbed or deformed by changing the choice of fillers for the 2-simplices in the faces of the prisms however.

10.5.5 Cellular formal C-maps

We can use the cocycle condition to combine formal C-data given locally on simplices into cellular blocks, up to equivalence. Combining simplices provides a simplification process which allows us to replace triangulated manifolds by manifolds with a given regular cellular decomposition. These are much easier to handle and, given what we have discussed before, the definitions and some of the proofs just 'fall out'. We still will need base points in each 1-manifold and start vertices in each cell.

Assume given a regular CW-complex, X, having, for each cell, a specified 'start 0-cell' among which are a set of distinguished base points. Assume, further, that each cell has a specified orientation.

Definition: A *cellular formal* C*-map*, λ, on X consists of families of elements
(i) $\{c_f\}$ of C indexed by the 2-cells, f, of X, and
(ii) $\{p_e\}$ of P indexed by the 1-cells, e, of X such that

a) the boundary condition,

$$\partial c_f = \text{the ordered product of the edge labels of } f$$

is satisfied;
and

b) the cocycle condition is satisfied for each 3-cell.

(In words, b) gives, for each 3-cell, σ, that the product of the labels on the boundary cells of σ is trivial.)

For a connected 1-manifold, S, decomposed as a CW-complex, (thus a subdivided circle), there is no difference from the simplicial description we had before. We have notions of formal C-circuit given by a sequence of elements of P and, more generally, if S is not connected, we have a list of such formal C-circuits.

A *cellular formal* C*-cobordism* between cellular formal C-maps is the obvious thing. It is a cellular cobordism between the underlying 1-manifolds endowed with a formal C-map that agrees with the two given C-maps on the two ends of the cobordism. Here the important ingredient is the cocycle condition and, before going further, we will say something more about both this and the boundary condition.

The algebraic-combinatorial description of the cellular version formal C-map is less explicitly given above than for the simplicial version as a full description would require the introduction of some additional detail, but this is not essential for the *intuitive* development of the ideas. We will, however, briefly sketch this extra theory.

Recall the following ideas from earlier in the notes:

- *Crossed complex*: (cf. Section 2.1) The main example for us is the crossed complex of X, a CW-complex as above. This has $C_n = \pi_n(X_n, X_{n-1}, \mathbf{x})$, X_n, being as usual, the n-skeleton of X and with ∂ the usual boundary map. Here we really need the many-object /groupoid version working with the multiple base points $\mathbf{x}$, but we will omit the detailed changes to the basic idea. We write $\pi(\mathbf{X})$ for this crossed complex.

- *Free crossed module*: (cf. Section 1.2.2 for the single vertex case.) The case of a 2-dimensional CW-complex, X, is of some importance for our theory as the B-cobordisms will be surfaces and hence 2-dimensional regular CW-complexes, once a decomposition is given. Any such 2-dimensional CW-complex yields a *free* crossed module,

$$\pi_2(X_2, X_1, X_0) \to \pi_1(X_1, X_0)$$

with $\pi_1(X_1, X_0)$, the fundamental groupoid of the 1-skeleton, X_1, of X based at the set of vertices X_0 of X. Each 2-cell of X gives a generating element in $\pi_2(X_2, X_1, X_0)$ and the assignment of the data for a cellular formal C-map satisfying the boundary condition, is equivalent to specifying a morphism, λ, of crossed modules,

$$\begin{tikzcd} \pi_2(X_2, X_1, X_0) \arrow[r, "\partial"] \arrow[d, "\lambda_2"'] & \pi_1(X_1, X_0) \arrow[d, "\lambda_1"] \\ C \arrow[r, "\partial"'] & P \end{tikzcd} .$$

The boundary condition just states $\lambda_1 \partial = \partial \lambda_2$.

- *Free crossed complex*: (See Brown-Higgins-Sivera, [49] for a detailed discussion of the concept.) The idea of free crossed complex is an extension of the above and $\pi(\mathbf{X})$ is free on the cells of X. (In particular, $C_3 = \pi_3(X_3, X_2, \mathbf{x})$ is a collection of free $\pi_1(X)$-modules over the various basepoints. The generating set is the set of 3-cells of X.)

A formal C-map, λ, is equivalent to a morphism of crossed complexes,

$$\lambda : \pi(\mathbf{X}) \to \mathsf{C},$$

or, expanding this, to

$$\begin{tikzcd} {} \arrow[r] & \pi_3(X_3, X_2, X_0) \arrow[r, "\partial"] \arrow[d, "\lambda_3"'] & \pi_2(X_2, X_1, X_0) \arrow[r, "\partial"] \arrow[d, "\lambda_2"'] & \pi_1(X_1, X_0) \arrow[d, "\lambda_1"'] \\ {} \arrow[r] & 1 \arrow[r, "\partial"'] & C \arrow[r, "\partial"'] & P \end{tikzcd} .$$

Each 3-cell gives an element of $\pi_3(X_3, X_2, X_0)$. More exactly, if σ is a 3-cell of X, then it can be specified by a characteristic map, $\varphi_\sigma : (B^3, S^2, \mathbf{s}) \to (X_3, X_2, X_0)$, and thus we get an induced crossed complex morphism, which in the crucial dimensions gives

$$\begin{tikzcd} {} \arrow[r] & \pi_3(B^3, S^2, \mathbf{s}) \arrow[r, "\partial"] \arrow[d, "\varphi_{\sigma,3}"'] & \pi_2(S^2, \varphi_\sigma^{-1}(X_1), \mathbf{s}) \arrow[r, "\partial"] \arrow[d, "\varphi_{\sigma,2}"'] & \pi_1(\varphi_\sigma^{-1}(X_1), \mathbf{s}) \arrow[d, "\varphi_{\sigma,1}"'] \\ {} \arrow[r] & \pi_3(X_3, X_2, X_0) \arrow[r, "\partial"] \arrow[d, "\lambda_3"'] & \pi_2(X_2, X_1, X_0) \arrow[r, "\partial"] \arrow[d, "\lambda_2"'] & \pi_1(X_1, X_0) \arrow[d, "\lambda_1"'] \\ {} \arrow[r] & 1 \arrow[r, "\partial"'] & C \arrow[r, "\partial"'] & P \end{tikzcd} .$$

We have $\pi_3(B^3, S^2, \mathbf{s})$ is generated by the class of the 3-cell, $\langle e^3 \rangle$ and $\varphi_{\sigma,3}(\langle e^3 \rangle) = \langle \sigma \rangle$. The cocycle condition is then explicitly given by $\lambda_2 \partial \langle \sigma \rangle = 1$.

The explicit *combinatorial* form of the cocycle condition for σ will depend on the decomposition of the boundary, S^2, given by $\varphi_\sigma^{-1}(X_1)$. (This type of argument was first introduced in the original paper by J. H. C. Whitehead, [221]. It can also be found in the forthcoming book by Brown, Higgins, and Sivera, [49], work by Brown and Higgins, [46, 48] and by Baues, [22, 23], where, however, crossed complexes are called *crossed chain complexes*.) Our use of this cocycle condition does not require such a detailed description so we will not attempt to give one here.

The next ingredient is to cellularise 'equivalence'. We can do this for arbitrary formal C-maps specialising to 1- or 2-dimensions (cobordisms) afterwards. We use a regular cellular decomposition of the space, $X \times I$, with possibly different regular CW-complex decompositions on the two ends, but with the base points 'fixed' so that $\mathbf{x} \times I$ is a subcomplex of $X \times I$.

Definition: Given cellular formal C-maps, λ_i, on $X_i = X \times \{i\}$, for $i = 0, 1$, they will be *equivalent* if there is a cellular formal C-map, Λ, on a cellular decomposition of $X \times I$ extending λ_0 and λ_1 and assigning 1_p to each edge in $\mathbf{x} \times I$.

Again equivalence is an equivalence relation. It allows the combination and collection processes examined in the previous subsection to be made precise. In other words:

- *if we triangulate each cell of a CW-complex, X, in such a way that the result gives a triangulation, K, of the space, then a formal* C*-map, λ, on K determines a cellular formal* C*-map on X;*
- *equivalent simplicial formal* C*-maps on (possibly different) such triangulations yield equivalent formal* C*-maps on X;*
- *given any cellular formal* C*-map, μ, on X and a triangulation, K, of X subdividing the cells of X, there is a simplicial formal* C*-map on K that combines to give μ.*

Remarks: (i) Full proofs of these would use cellular and simplicial decompositions of $X \times I$, but would also need the introduction of far more of the theory of crossed modules, crossed complexes and their classifying spaces than we have available in these notes. Because of that, the proofs are omitted here in order to make this introduction to formal C-maps easier to approach. (It goes almost without saying that is is worth **trying for yourselves** to construct proofs, as they do indicate the link between the simplicial and cellular approaches.)

(ii) Any simplicial formal C-map on K is, of course, a cellular one for the obvious regular CW-structure on $|K|$.

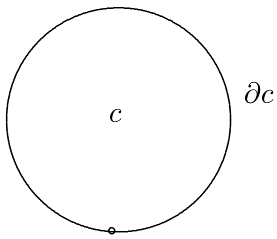
The notion of equivalent cellular formal C-cobordisms can now be formulated. Given the obvious set-up with $\mathbf{F}$ and $\mathbf{G}$, two such cobordisms between $\mathbf{g_1}$ and $\mathbf{g_2}$, they will be equivalent if they are equivalent as formal C-maps by an equivalence that is constant on the two 'ends'.

10.5.6 2-dimensional formal C-maps.

Now that we have cellular descriptions, it is easy to describe a set of 'building blocks' for all cellular formal C-maps on orientable surfaces and thus all cobordisms between 1-dimensional formal C-maps. Again we want to emphasise the fact that these models provide *formal combinatorial models* for the characteristic maps with target a 2-type.

We will give the formal version of 1+1 HQFTs with a 'background' crossed module, C, which is a model for a 2-type, B, represented by that crossed module. As the basic manifolds are 1-dimensional, they are just disjoint unions of pointed oriented circles, and so a formal C-map on a 1-manifold, as we saw earlier (page 436), is specified by a list of lists of elements in P, one list for each connected component. Cellularly, we can assume that the lists have just one element in them, obtained from the simplicial case by multiplying the elements in the list together in order. The corresponding cellular cobordisms are then compact oriented surfaces, W, with pointed oriented boundary endowed with a formal C-map, Λ, as above. Since such surfaces can be built up from three basic models, the disc, annulus and disc with two holes (pair of trousers), we need only examine what formal C-maps look like on these basic example spaces and how they compose and combine, as any formal 1 + 1 'C-HQFT' will be determined completely by its behaviour on the formal maps on these basic surfaces. This is the analogue, here, of the generators and relations approach to $1+1$- dimensional TQFTs that we sketched earlier.

Formal C-Discs: The only formal C-maps that makes sense on the disc must have an element $c \in C$ assigned to the interior 2-cell with the boundary, ∂c, assigned to the single 1-cell, i.e.,



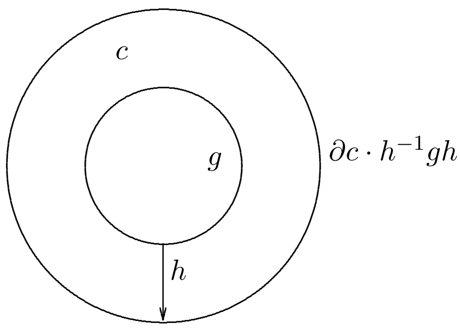
$Disc(c) : \emptyset \to \partial \mathbf{c}$

Later we will see that these give the crucial difference between the formal C-theory and that, in $1+1$ dimensions, for a $K(G,1)$, as sketched earlier, and explored thoroughly in [209, 211].

Formal C-Annuli: Let Cyl denote the cylinder / annulus, $S^1 \times [0,1]$. We fix an orientation of Cyl once and for all, and set $Cyl^0 = S^1 \times (0) \subset \partial Cyl$ and $Cyl^1 = S^1 \times (1) \subset \partial Cyl$. We provide Cyl^0 and Cyl^1 with base points, $z^0 = (s,0)$, $z^1 = (s,1)$, respectively, where $s \in S^1$. As in [209, 211], let $\varepsilon, \mu = \pm$, and denote by $Cyl_{\varepsilon,\mu}$ the triple $(Cyl, Cyl^0_\varepsilon, Cyl^1_\mu)$. This is an annulus with oriented pointed boundary,

$$\partial Cyl_{\varepsilon,\mu} = (\varepsilon Cyl^0_\varepsilon) \cup (\mu Cyl^1_\mu),$$

where, by $-X$, we mean X with opposite orientation. A formal C-map, Λ, on Cyl can be drawn diagrammatically as:



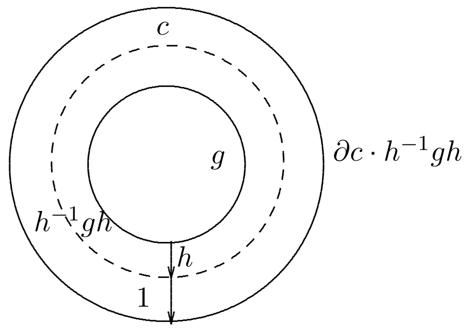
$Cyl_{\varepsilon,\mu}(c,g,h)$

(with, for instance, initial vertex, s, for the 2-cell at the end of h and with clockwise orientation, given by appropriate conventions on the ε, and μ . This diagram will represent the various cobordisms that we will denote $Cyl_{\varepsilon,\mu}(c,g,h)$. Similar notation may be used in other contexts without further comment. (We omit the orientations on the boundary circles, so as to avoid the need to repeat more or less the same diagram several times.)

The loop, $\Lambda|_{Cyl^1_\mu}$, represents $(\partial c \cdot h^{-1}gh)$ or its inverse depending on the sign of μ. There are two special cases that generate all the others:

(i) $c = 1$, which corresponds to the case of a $K(P, 1)$, and

(ii) $h = 1$, where the base point s does not move during the cobordism. The general case, illustrated in the figure, is the composite of particular instances of the two cases.

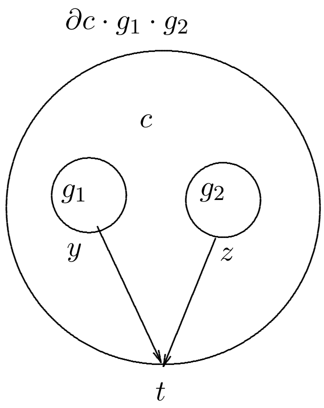


Combination of cobordisms is more or less obvious, so we will not give details, but should perhaps note that it is, here, the analogue of the vertical $\sharp_1$-composition in the (strict) 2-group $\mathcal{X}(\mathsf{C})$, corresponding to C. It thus almost reduces to the multiplication of the C-label with evident adjustment of the boundaries. This makes pleasing geometric sense as $\sharp_1$ corresponds to two 2-cells with a common 1-dimensional 'interaction'.

Formal C-Disc with 2 holes (Formal 'pair of pants'): Let D be an oriented 2-disc with two holes. We will denote the boundary components of D for convenience by Y, Z, and T and provide them with base points y, z and t respectively. For any choice of signs, ε, μ, $\nu = \pm$, we denote by $D_{\varepsilon,\mu,\nu}$ the tuple $(D, Y_\varepsilon, Z_\mu, T_\nu)$. This is a 2-disc with two holes with oriented pointed boundary. By definition,

$$\partial D_{\varepsilon,\mu,\nu} = (\varepsilon Y_\varepsilon) \cup (\mu Z_\mu) \cup (\nu T_\nu).$$

Finally we fix two proper embedded arcs, yt and zt, in D leading from y and z to t. A formal C-map, λ, on $D_{\varepsilon,\mu,\nu}$ will, in general, assign elements of P to each boundary component and to each arc. As for the annulus, we may assume that the formal map assigns 1_P to both yt and zt, as the general case can be generated by this one together with cylinders. In addition, the single 2-cell will be assigned an element, c, of C.



$D_{\varepsilon,\mu,\nu}(c, g_1, g_2)$

This situation leads to an interesting relation. If we have a formal C-map on $D_{\varepsilon,\mu,\nu}$ in which, for simplicity, we assume that the 2-cell is assigned the element 1_C and then add suitable cylinders, labelled with c_1 and c_2 respectively, to the boundary components, Y and Z, then the resulting cobordism can be rearranged to give a labelling with the 2-cell coloured $c_1 \cdot {}^{g_1}c_2$, as shown in the following diagram:

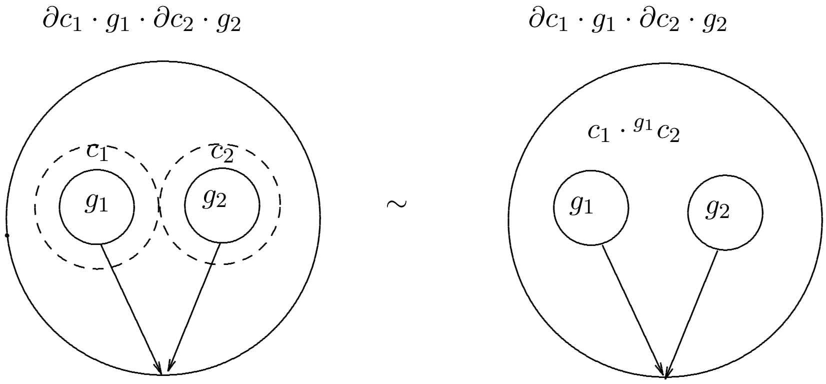


Figure 10.5.6 : From the case $c = 1$ to the general one.

The importance of this element, $c_1 \cdot {}^{g_1}c_2$, is that it is the C-part of the product of the two cylinder labels in the semidirect product, $C \rtimes P$, and thus in the 2-group; more exactly, the elements (c_1, g_1) and $(c_2, g_2) \in C \rtimes P$ correspond to the two added cylinders and within that semi-direct product $(c_1, g_1) \cdot (c_2, g_2) = (c_1 \cdot {}^{g_1}c_2, g_1 g_2)$, so here again the structure of the 2-group is in action, but, this time, it is the horizontal $\sharp_0$-composition.

10.6 Formal HQFTs

We will restrict attention to modelling 1+1 HQFTs and so, here, will give a definition of a formal HQFT only for that case. First we introduce some notation.

If we have formal C-cobordisms,

$$\mathbf{F} : \mathbf{g}_0 \to \mathbf{g}_1, \quad \mathbf{G} : \mathbf{g}_1 \to \mathbf{g}_2,$$

then we will denote the composite C-cobordism by $\mathbf{F}\#_{\mathbf{g}_1}\mathbf{G}$.

For $\mathbf{g}$ as before, the trivial identity C-cobordism on $\mathbf{g}$ will be denoted $1_{\mathbf{g}}$.

10.6.1 The definition

Fix, as before, a crossed module, $\mathsf{C} = (C, P, \partial)$, and also fix a ground field, $\mathbb{k}$.

Definition: A *formal HQFT* with background, C, assigns

- to each formal C-circuit, $\mathbf{g} = (g_1, \ldots, g_n)$, a $\mathbb{k}$-vector space, $\tau(\mathbf{g})$, and by extension, to each formal C-map on a 1-manifold, S, given by a list, $\mathbf{g} = \{\mathbf{g}_i \mid i = 1, 2, \ldots, m\}$ of formal C-circuits, a vector space, $\tau(\mathbf{g})$, and an identification,

$$\tau(\mathbf{g}) = \bigotimes_{i=1,\ldots,m} \tau(\mathbf{g}_i),$$

  giving $\tau(\mathbf{g})$ as a tensor product;

- to any formal C-cobordism, $(M, \mathbf{F})$ between $(S_0, \mathbf{g}_0)$ and $(S_1, \mathbf{g}_1)$, a $\mathbb{k}$-linear transformation

$$\tau(\mathbf{F}) : \tau(\mathbf{g}_0) \to \tau(\mathbf{g}_1).$$

These assignments are to satisfy the following axioms:

(i) Disjoint union of formal C-maps corresponds to tensor product of the corresponding vector spaces via specified isomorphisms:

$$\tau(\mathbf{g} \sqcup \mathbf{h}) \stackrel{\cong}{\to} \tau(\mathbf{g}) \otimes \tau(\mathbf{h}),$$

$$\tau(\emptyset) \stackrel{\cong}{\to} \mathbb{k}$$

for the ground field, $\mathbb{k}$, so that a) the diagram of specified isomorphisms

$$\begin{tikzcd} \tau(\mathbf{g}) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \tau(\mathbf{g} \sqcup \emptyset) \arrow[d, "\cong"] \\ \tau(\mathbf{g}) \otimes \mathbb{k} & \tau(\mathbf{g}) \otimes \tau(\emptyset) \arrow[l, "\cong"] \end{tikzcd}$$

for $\mathbf{g} \to \emptyset \sqcup \mathbf{g}$, commutes and similarly for $\mathbf{g} \to \mathbf{g} \sqcup \emptyset$, and b) the assignments are compatible with the associativity isomorphisms for $\sqcup$ and $\otimes$, (so that τ satisfies the usual axioms for a symmetric monoidal functor).

(ii) For formal C-cobordisms,

$$\mathbf{F} : \mathbf{g}_0 \to \mathbf{g}_1, \quad \mathbf{G} : \mathbf{g}_1 \to \mathbf{g}_2,$$

with composite $\mathbf{F}\#_{\mathbf{g}_1}\mathbf{G}$, we have

$$\tau(\mathbf{F}\#_{\mathbf{g}_1}\mathbf{G}) = \tau(\mathbf{G})\tau(\mathbf{F}) : \tau(\mathbf{g}_0) \to \tau(\mathbf{g}_2).$$

(iii) For the identity formal C-cobordism on $\mathbf{g}$,

$$\tau(1_{\mathbf{g}}) = 1_{\tau(\mathbf{g})}.$$

(iv) Interaction of cobordisms and disjoint union is transformed correctly by τ, i.e., given formal C-cobordisms,

$$\mathbf{F} : \mathbf{g}_0 \to \mathbf{g}_1, \quad \mathbf{G} : \mathbf{h}_0 \to \mathbf{h}_1$$

the diagram,

$$\begin{tikzcd}
\tau(\mathbf{g}_0 \sqcup \mathbf{h}_0) \arrow[r, "\cong"] \arrow[d, "\tau(\mathbf{F}\sqcup\mathbf{G})"'] & \tau(\mathbf{g}_0) \otimes \tau(\mathbf{h}_0) \arrow[d, "\tau(\mathbf{F})\otimes\tau(\mathbf{G})"] \\
\tau(\mathbf{g}_1 \sqcup \mathbf{h}_1) \arrow[r, "\cong"'] & \tau(\mathbf{g}_1) \otimes \tau(\mathbf{h}_1)
\end{tikzcd}$$

commutes, compatibly with the associativity structure.

10.6.2 Basic Structure

Formal C-maps can be specified by composing / combining the basic building blocks outlined in section 10.5.6. As a formal 1+1 HQFT transforms the formal C-maps to vector space structure, to specify a formal HQFT, we need only give it on the connected 1-manifolds, thus on formal C-circuits, and on the building-block cobordisms mentioned before. We assume τ is a formal HQFT with C as before.

On a formal C-circuit, $\mathbf{g} = (g_1, \ldots, g_n)$, we can assume $n = 1$, since the obvious formal C-cobordism between $\mathbf{g}$ and $\{(g_1 \ldots g_n)\}$, based on the cylinder yields, an isomorphism

$$\tau(\mathbf{g}) \overset{\cong}{\to} \tau(g_1 \ldots g_n),$$

and, consequently, a decomposition of $\tau(g_1 \ldots g_n)$ as a tensor product. For any element $g \in P$, we thus have the formal C-circuit, $\{(g)\}$, and a vector space, $\tau(g)$. In fact, for later use, it will be convenient to change notation to L_g (or, for the more general case, $L_{\mathbf{g}}$).

For a general $\mathbf{g} = (g_1, \ldots, g_n)$, we now have

$$L_{\mathbf{g}} = \bigotimes_{i=1}^{n} L_{g_i}.$$

The special case when $\mathbf{g}$ is empty gives $L_\emptyset = \Bbbk$ and the isomorphisms, here, are compatible with these assignments.

The basic formal C-cobordisms give us various structural maps:

- the formal C-disc with $c \in C$ gives

$$\tau(Disc(c)) : \tau(\emptyset) \to \tau(\partial c),$$

  that is, a linear map, which we will write as

$$\ell_c : \Bbbk \to L_{\partial c}.$$

  (We write $\tilde{c} := \ell_c(1) \in L_{\partial c}$.)

- the formal C-annuli of the two basic types yield
  (a) $Cyl_{\varepsilon,\mu}(1, g, h) : \{(g)\} \to \{(h^{-1}gh)\}$, and hence a linear isomorphism,

$$L_g \to L_{h^{-1}gh},$$

  or a related one, depending on the sign of μ;
  or
  (b) $Cyl_{\varepsilon,\mu}(c, g, 1) : \{(g)\} \to \{(\partial c \cdot g)\}$ and a linear isomorphism,

$$L_g \to L_{\partial c \cdot g},$$

  again with variants for other signs.

- the formal C-disc with 2 holes,

$$(D_{\varepsilon,\mu,\nu}(c, g_1, g_2) : \{(g_1), (g_2)\} \to \{(\partial c \cdot g_1 \cdot g_2)\},$$

  giving a bilinear map,

$$L_{g_1} \otimes L_{g_2} \to L_{\partial c \cdot g_1 \cdot g_2}.$$

  Again, the key case is $c = 1$, and, consequently,

$$L_{g_1} \otimes L_{g_2} \to L_{g_1 g_2}.$$

  The general case can be obtained from that and a suitable formal C-annulus,

$$L_{g_1} \otimes L_{g_2} \to L_{g_1 g_2} \to L_{\partial c \cdot g_1 g_2}.$$

  This can be done, as here, by adding the annulus after the 'pair of pants' or adding it on the first component. The two formal C-cobordisms are equivalent.

  We can, of course, reverse the orientation to get

$$L_{g_1 g_2} \to L_{g_1} \otimes L_{g_2},$$

  a 'comultiplication'. It is fairly standard that this comultiplication is 'redundant', as it can be recovered from the annuli and a suitable 'positive pair of pants', see, for instance, the brief argument given in section 5.1 of [209].

The treatment here is designed to mirror Turaev's discussion in [209], which handles the case when B is a $K(G, 1)$. (We will give a sketch of this shortly, but will as usual refer you to Turaev's paper or his book for a full detailed version.) That way, we can see that the passage from that case with 'background' a 1-type, $B = K(P, 1)$ to the model for a general 2-type, $B = B\mathsf{C}$, merely requires the addition of extra linear isomorphisms, $L_g \to L_{\partial c \cdot g}$ for c in the top group of the crossed module, C.

Chapter 11

Homotopy Quantum Field Theories, II: Algebraic models for 2 dimensional HQFTs

We have seen that 2 dimensional TQFTs correspond to commutative Frobenius algebras. In this chapter we will examine that corresponding algebraic models for the corresponding HQFTs. The B-manifolds will be 1-dimensional whilst the B-cobordisms will be surfaces.

11.1 Crossed C-algebras: first steps

In [209], Turaev classified (1+1)-HQFTs with background a $K(G,1)$ in terms of *crossed group-algebras.* These were generalisations of classical group algebras with many of the same features, but 'twisted' by an action. In [37], M. Brightwell and P. Turner examined the analogous case when the background is a $K(A,2)$ for A, an Abelian group, and classified them in terms of A-Frobenius algebras, that is, Frobenius algebras with an A-action. In this section, and those that follow, we will look in some detail at both these types of algebra before introducing the more general type, 'crossed C-algebras', which combines features of both and which will classify formal HQFTs as above.

11.1.1 Crossed G-algebras

Here G will be a group corresponding to $\pi_1(B)$ if B is a 1-type.

Definition: A *G-graded algebra* or *G-algebra* over a field $\Bbbk$ is an associative algebra, L, over $\Bbbk$ with a decomposition,

$$L = \bigoplus_{g\in G} L_g,$$

as a direct sum of projective $\Bbbk$-modules of *finite* type such that
(i) $L_gL_h \subseteq L_{gh}$ for any $g, h \in G$ (so, if ℓ_1 is graded g, and ℓ_2 is graded h, then $\ell_1\ell_2$ is graded gh), and
(ii) L has a unit $1 = 1_L \in L_1$ for 1, the identity element of G.

In a G graded algebra, the degree of a homogeneous element, ℓ, is the index of the direct summand of which it is an element.

Example: (i) The group algebra, $\Bbbk[G]$, has an obvious G-algebra structure in which each summand of the decomposition is free of dimension 1.

(ii) For any associatve $\Bbbk$-algebra, A, the algebra, $A[G] = A \otimes_{\Bbbk} \Bbbk[G]$, is also G-algebra. Multiplication in $A[G]$ is given by $(ag)(bh) = (ab)(gh)$ for $a, b \in A$, $g, h \in G$, in the obvious notation.

(iii) If G is the trivial group, then a G-graded algebra is just an algebra (of finite type), of course.

Definition: A *Frobenius G-algebra* is a G-algebra, L, together with a symmetric $\Bbbk$-bilinear form,

$$\rho : L \otimes L \to \Bbbk$$

such that
(i) $\rho(L_g \otimes L_h) = 0$ if $h \neq g^{-1}$;
(ii) the restriction of ρ to $L_g \otimes L_{g^{-1}}$ is non-degenerate for each $g \in G$;
and
(iii) $\rho(ab, c) = \rho(a, bc)$ for any $a, b, c \in L$.

We note that (ii) implies that $L_{g^{-1}} \cong L_g^*$, the dual of L_g.

Remark: It is sometimes useful to think of ρ as a G-indexed family of bilinear maps, $\rho : L_g \otimes L_{g^{-1}} \to \Bbbk$. This in turn can be specified by a *trace* or *co-unit*, $\theta : L_1 \to \Bbbk$, as, together with the multiplication, $L_g \otimes L_{g^{-1}} \to L_1$, this induces a suitable ρ. (This is the approach taken by Moore and Segal in [163].)

Examples continued: (i) The group algebra, $L = \Bbbk[G]$, is a Frobenius G-algebra with $\rho(g, h) = 1$ if $gh = 1$, and 0 otherwise, and then extending linearly. (Here we write g both for the element of G labelling the summand L_g, and the basis element generating that summand.)

(iii) For G trivial, a Frobenius 1-algebra is a Frobenius algebra.

Finally the notion of crossed G-algebra combines the above with an action of G on L, explicitly:

Definition: A *crossed G-algebra* over $\Bbbk$ is a Frobenius G-algebra, L, over $\Bbbk$ together with a group homomorphism,

$$\varphi : G \to Aut(L)$$

satisfying:
(i) if $g \in G$ and we write $\varphi_g = \varphi(g)$ for the corresponding automorphism of L, then φ_g preserves ρ, (i.e., $\rho(\varphi_g a, \varphi_g b) = \rho(a, b)$) and

$$\varphi_g(L_h) \subseteq L_{ghg^{-1}}$$

for all $h \in G$;
(ii) $\varphi_g|_{L_g} = id$ for all $g \in G$;
(iii) (twisted or crossed commutativity) for any $g, h \in G$, $a \in L_g$, $b \in L_h$, $\varphi_h(a)b = ba$;

(iv) for any $g, h \in G$ and $c \in L_{ghg^{-1}h^{-1}}$,

$$Tr(c\varphi_h : L_g \to L_g) = Tr(\varphi_{g^{-1}}c : L_h \to L_h),$$

where Tr denotes the $\Bbbk$-valued trace of the endomorphism. (The homomorphism $c\varphi_h$ sends $a \in L_g$ to $c\varphi_h(a) \in L_g$, whilst $(\varphi_{g^{-1}}c)(b) = \varphi_{g^{-1}}(cb)$ for $c \in L_h$. This is sometimes called the 'torus condition'.)

Note: We note that the usage of terms differs between [211] and here, as we have taken 'crossed G-algebra' to include the Frobenius condition. We thus follow Turaev's original convention in this; cf. [209]. The useful terminology 'twisted sector' in a crossed G-algebra refers to a summand, L_g, for and index, g, which is not the identity element of G, of course, then L_1 is called the 'untwisted sector'.

Examples: concluded. (i) The group algebra, $L = \Bbbk[G]$, is a crossed G-algebra with $\varphi_g(h) = hgh^{-1}$, and extended linearly.

(iii) The notion of crossed 1-algebra, (that is the case when $G = 1$), is equivalent to a commutative Frobenius algebra. This comes from the 'twisted commutativity' axiom. More generally we have:

If L is a crossed G-algebra, then by the 'twisted / crossed commutativity' axiom,

$$\varphi_h(a)b = ba,$$

and, since the G-action, $\varphi : G \to Aut(L)$, is a group homomorphism, if $b \in L_1$, (so $h = 1$), then b is in the centre of L: $ab = ba$ for all $a \in L$. This gives us the following (after a minute amount of extra checking):

Proposition 112 *If L is a crossed G-algebra, then L_1 is a central sub-algebra, and is a commutative Frobenius algebra. Each of the 'twisted sectors', L_g have the structure of a L_1-module.* ■

11.1.2 Morphisms of crossed G-algebras

We clearly need to have a notion of morphism of crossed G-algebras. We start with a fixed group, G.

Definition: Suppose L and L' are two crossed G-algebras. A $\Bbbk$-algebra morphism, $\theta : L \to L'$, is a *morphism of crossed G-algebras* if it is compatible with the extra structure. Explicitly:

$$\begin{array}{rcl} \theta(L_g) & \subseteq & L'_g \\ \rho'(\theta a, \theta b) & = & \rho(a, b), \\ \varphi'_g(\theta a) & = & \theta(\varphi_g(a)), \end{array}$$

for all $a, b \in L$, $g \in G$, where, when necessary, primes indicate the structures in L'.

We will also need a version of this relative to a 'change of groups'. Suppose that $f : G \to H$ is a homomorphism of groups, that L is a crossed G-algebra, and L', a crossed H-algebra and let $\theta : L \to L'$ be a $\Bbbk$-algebra homomorphism.

Definition: We say that θ is *compatible* with f, or is a *morphism of crossed algebras over* f, if

$$\begin{array}{ll} \theta(L_g) & \subseteq L'_{f(g)} \\ \rho'(\theta a, \theta b) & = \rho(a,b), \\ \varphi'_{f(g)}(\theta a) & = \theta(\varphi_g(a)), \end{array}$$

for all $a, b \in L$, $g \in G$, where primes indicate the structure in L'.

We will use this definition shortly when looking at pullbacks and related construction. For the moment, we just record that there will be category, $Crossed.G{-}Alg$, of crossed G-algebras and the corresponding morphisms and a larger category consisting of all crossed algebras (over any group), denoted $Crossed.Alg$, and morphisms over group homomorphisms. As might be expected, there is a functor from $Crossed.Alg$ to the category of groups, and we would also expect this to form a fibered category in such a way that the fibre over a group G would, of course, be $Crossed.G{-}Alg$. We will investigate this slightly later.

Returning to the category $Crossed.G{-}Alg$ itself, this category is a groupoid as all morphisms are isomorphisms. (If **you looked at the proof** of the corresponding fact for TQFTs then the proof is more or less obvious, as it is 'the same'.) This fact is very easy to forget, but is crucial at certain stages of the development. We record it for future reference:

Proposition 113 *Any morphism of crossed G algebras is an isomorphism, so* $Crossed.G{-}Alg$ *is a groupoid.* ∎

11.2 From 1+1 HQFTs over $K(G,1)$s to crossed G-algebras

How should one go from a HQFT based on $K(G,1)$-manifolds of dimension 1 and cobordisms of dimension 2, to a crossed G-algebra? Since we already have sketched how to go from a 2 (or, if you prefer 1+1) dimensional TQFT to a commutative Frobenius algebra (cf. Theorem 22), the intuitions are mostly already in place.

To set things up, let $B = K(G,1)$ and τ be a 1+1 HQFT with background B. For each $g \in G$, we have a 1-dimensional B-manifold given by (S^1, g), where, abusing notation, $g : S^1 \to B$ is some (chosen) representative for the element $g \in G \cong \pi_1(B)$.

Remark: If you are screaming that that *is* abusing notation, good! It, in fact, is a justifiable abuse. Apart from great;y reducing notation complexity and clutter, it is intuitively 'correct', being, almost, the passage from the topological to the formal / combinatorial model. You are **left to check** that, if you write $\gamma : S^1 \to B$ such that $[\gamma] \in \pi_1(B)$ corresponds to $g \in G$, under the chosen isomorphism between $\pi_1(B)$ and G, then none of the choices makes any significant change to the end result - up to isomorphism or equivalence, whichever is appropriate.

Although this abuse *is* anodyne, it is important to note it and to take it seriously, as 'slip shod' abuse can crush a lot of important structure when isomorphisms linking objects and the automorphisms of the objects are of great significance to the end result.

We thus have (S^1, g) for each $g \in G$, and hence a vector space $\tau(S^1, g)$, which we will denote by L_g.

Note: The direct sum of these L_g for $g \in G$ will give us the object, L which will be the corresponding crossed G-algebra, but it is important to be able to retrieve the 'value' of τ on (S^1, g) from L and this is the significance of the use of G-graded algebras, as then the direct sum decomposition of L, in terms of a G-indexed family of subspaces, is part of the structure that is encoded.

We now proceed very much as we would expect. We need a multiplication on L, and so need $L \otimes L \cong \oplus_{g,h} L_g \otimes L_h \to L$, and this should be some family of linear maps

$$L_g \otimes L_h \to L_{gh},$$

indexed by $G \times G$. We therefore need B-cobordisms from $(S^1, g) \sqcup (S^1, h)$ to (S^1, gh). We can think back to the TQFT case, or look at the more general case of a formal 'pair of pants' (cf. page 444) but in which $c = 1$ (as we are in the simpler case of $B = K(G,1)$ and not the full $B = B\mathsf{C}$.) We refer to that page, (that is, page 444), for notation, and have that the relevant B-cobordism is $D_{\varepsilon,\mu,\nu}(1, g, h)$, for a suitable choice of ε, μ, ν.

Lemma 82 *(Turaev, [209, 211]) With this multiplication, L becomes a graded G-algebra.*

Sketch proof: The proof of associativity is a G-labelled adaptation of the one for the ungraded case (i.e., as here on page 328). That the existence of the $1 \in L_1$ needs the use of a disc B-cobordism from $\emptyset$ to $(S^1, 1)$, (but, again, the argument is as for the TQFT case (with frills attached), so you are **left to check details**), ■

In these proofs, a good policy is to **write a sketch** (using suitable diagrams) first, then to check in Kock's book, [140], or Quinn's introductory TQFT lectures, [188], followed by a thorough read of Turaev's description, in [209, 211], as sometimes there are subtleties that are easy to miss.

The next bit of structure to find will be the Frobenius form / inner product pairing, that is, a symmetric bilinear form,

$$\rho : L \otimes L \to \Bbbk.$$

The first condition (from page 450) that we need, does a lot of our work for us. We *define*

$$\rho : L_g \otimes L_h \to \Bbbk$$

to be zero unless $h = g^{-1}$, so are left to handle the

$$\rho : L_g \otimes L_{g^{-1}} \to \Bbbk.$$

This should come from a B-cobordism from $(S^1, g) \sqcup (S^1, g^{-1})$ to $\emptyset$, and the cylinder cobordism does this. (Which values to take for the ε and μ for $Cyl_{\varepsilon,\mu}(1, g, 1)$ are needed is, as usual, **left to you**.) The verification that

$$\rho(ab, c) = \rho(a, bc)$$

is the just the labelled version of the diagrams for the unlabelled result on TQFTs.

We thus can boost our previous lemma, (with **a bit more work**), to get:

Lemma 83 *(Turaev, [209, 211]) The G-graded algebra, L, has a natural Frobenius G-algebra structure.* ■

We finally need to get a crossed G-algebra structure, so want

$$\varphi : G \to Aut(L),$$

as on page 450. To start with, we need for each $g \in G$, and then $h \in G$

$$\varphi_g : L_h \to L_{ghg^{-1}},$$

and so look fpr a B-cobordism to do the job. It just needs a bit of adjustment on the basic $Cyl_{\varepsilon,\mu}(1, h, g^{-1})$ to get what is needed. (Again, you are invited to **work out the values** of ε and μ, here, and to **check that the result gives a φ as required**.)

We thus have sketched a proof of the forward direction of:

Theorem 48 *The 1+1 HQFT, τ, on background $B = K(G, 1)$ yields a crossed G-algebra, $L = \bigoplus_{g\in G} L_g$, where $L_g = \tau(S^1, g)$, for $g \in G$. Converely, any such crossed G-algebra gives a 1+1 HQFT. This correspondence sets up an equivalence of categories between that of 1+1 HQFTs on B and $Crossed.G-Alg$.* ■

(We will not give a detailed proof here, directing you to sources [209] and [211] for much of the detail. The key theorem in [211] is Theorem 3.1 on page 52. An alternative source for a proof is the survey paper of Moore and Segal, [163]. The proof starts in their section 7. We will return to this paper and some of their results in a later chapter when we discuss interpretations of the idea of HQFTs.)

11.3 Constructions on crossed G-algebras

Before looking at the G-Frobenius algebras that model the case of $1 + 1$-dimensional HQFTs where the background is a $K(A, 2)$, (and, therefore, where A must be Abelian), we should look at analogues of some of the constructions on HQFTs that we saw earlier, but this time on the crossed algebras that we have just introduced.

11.3.1 Cohomological crossed G-algebras

(The sources for this material are, once again, Turaev's book, [211], and his earlier paper, [209].)

The cohomological crossed algebras mentioned in the title of this section are the analogues of the cohomological HQFTs that we met in section 10.3.1.

We let $\theta = \{\theta_{g,h} \in \Bbbk^*\}$ be a normalised 2-cocycle for G, representing a cohomology class $[\theta] \in H^2(G, \Bbbk^*\}$. Recall that this means that $\theta : G \times G \to \Bbbk^*$, and we are writing $\theta_{g,h}$ for $\theta(g, h)$. That θ is a 2-cocycle means

$$\theta_{g,h}\theta_{gh,k} = \theta_{g,hk}\theta_{h,k}$$

for all triples, g, h, k, of elements of G and that θ is normalised means that $\theta_{1,1} = 1$.

Define a G-graded algebra, $L = L^\theta$, as follows:
for $g \in G$, L_g is a free $\Bbbk$-module of rank 1, with a generating vector denoted ℓ_g, so $L_g = \Bbbk\ell_g$.

This resembles the group algebra, $\Bbbk[G]$, in which the multiplication would be given on generators by $\ell_g\ell_h = \ell_{gh}$ and extended linearly. Here we twist this, using the 2-cocycle. In L^θ, we take the product to be defined on basis elements by

$$\ell_g\ell_h = \theta_{g,h}\ell_{gh}$$

and then, of course, extend linearly.

Associativity of the multiplication is exactly the cocycle condition. It is easy to check also that ℓ_1 is the 1 of L^θ.

Lemma 84 *Cohomologous 2-cocycles determine isomorphic graded G-algebras.* ∎

The proof is a routine manipulation.

This lemma says that the isomorphism class of L^θ only depends on $[\theta] \in H^2(G, \Bbbk^*)$.

Proposition 114 *The graded G-algebra, L^θ, is a Frobenius G-algebra.*

Proof: We define $\rho : L \otimes L \to \Bbbk$ by $\rho(\ell_g, \ell_h) = 0$ unless $g = h^{-1}$, and $\rho(\ell_g, \ell_{g^{-1}}) = \theta_{g,g^{-1}}$. The verification that this satisfies (ii) and (iii) of the definition of a Frobenius G-algebra is **left as an exercise**. (It can be found in [211].) ∎

Finally we want to show that L^θ is a crossed G-algebra, so we need to give, or find, a $\varphi : G \to Aut(L^\theta)$ satisfying the axioms. We first note that the multiplication,

$$L_h \otimes L_g \to L_{hg},$$

is clearly an isomorphism of $\Bbbk$-modules and, of course, equally well $L_{hgh^{-1}} \otimes L_h \to L_{hg}$ is one. This means that $\ell_h\ell_g$ must also be $\varphi_h(\ell_g)\ell_h$ for some *unique* $\varphi_h(\ell_g) \in L_{hgh^{-1}}$, and we use this to define φ_h, extending linearly as usual. This φ_h is an automorphism of L^θ.

Proposition 115 *$L^\theta = (L, \eta, \rho)$ is a crossed G-algebra.* ∎

The proof is quite long, as it has to verify a fair number of conditions, but it is not that difficult. It is therefore omitted. It can be found in Turaev, [211], p. 26 - 27.

Again the isomorphism class of the crossed G-algebra, L^θ, only depends on the cohomology class of θ in $H^2(G, \Bbbk^*)$. We also note that $L^{\theta_1+\theta_2} \cong L^{\theta_1} \otimes L^{\theta_2}$ and that if $\theta = 0$ then the resulting crossed algebra is the group algebra, $\Bbbk[G]$.

11.3.2 Change of groups: just for modules

This section is both a 'recall' and an 'aside'. We will shortly be looking at the way that crossed G-algebras react to changing the group, G, along some homomorphism, $f : G \to H$. This corresponds to changing the background in a HQFT and sometimes enables structural information on the algebras to be extracted by a functorial approach. The importance of this basic idea can be seen also in the theory of group representations and the cohomology of groups and algebras, in a simpler form, and we have seen related constructions earlier (cf. section 1.1.3). We want to use that representation theoretic case as a guide to what to expect and also a motivation for the adjoint functors that we will be looking for, so we need to 'recall' that classical theory. On the other hand, none of this is really to do with crossed algebras, at least not directly, so it is an aside. It could have been introduced earlier at several points, as it is conceptually and technically easier than other material in this chapter, but we had not needed the ideas before, so ... .

We will give this for change of groups. It could equally be given for change of rings, or more generally.

The set up is that we have a homomorphism of groups, $f : G \to H$, and want to see what this does to the module categories, $G-Mod$, $H-Mod$, etc., and how conditions on f may potentially influence properties of any induced functors we obtain.

Restriction of scalars: The simplest functor induced by f is denoted

$$f^* : H-Mod \to G-Mod,$$

and is called *restriction of scalars*. It is defined simply by considering H-modules as G-modules using f. More precisely, let N be an H-module, then we can give the underlying Abelian group of N a G-action by defining: for $n \in N$ and $g \in G$,

$$g \cdot n := f(g) \cdot n,$$

where the dot on the right-hand side is the action of H on N. This is a G-module structure which will be denoted $f^*(N)$.

If we think of the groups, G and H, as single object groupoids, denoted, as usual, $G[1]$ and $H[1]$, then N is defined by the functor

$$N : H[1] \to Ab,$$

(having N as the image of the single object of $H[1]$, etc.), and then f^*N is just the composite of this with $f[1] : G[1] \to H[1]$.

Of course, f^* is a functor from $H-Mod$ to $G-Mod$. This functor has adjoints on both sides, (in this situation, but beware, in similar situations, and, in particular, for crossed algebras, extra conditions, typically of some sort of finiteness, will be needed on f to get the adjoints of the analogous functors). This can be analysed in terms of (additive) Kan extensions, (cf. section **??**), in this context

$$\begin{tikzcd} G[1] \arrow[r, "f{[1]}"] \arrow[d, "M"'] & H[1] \arrow[dl, dashed] \\ Ab & \end{tikzcd}$$

(and this is worth noting and **checking up on**), but we will merely give descriptions of these adjoints in a fairly classical way (following the treatment found, for instance, in Brown, [39]).

Extension of scalars: Given $f : G \to H$, we get $\mathbb{Z}[f] : \mathbb{Z}[G] \to \mathbb{Z}[H]$, and hence $\mathbb{Z}[H]$ becomes a right G-module: $h \cdot g := h \cdot f(g)$. If we have a G-module, M, we can form $\mathbb{Z}[H] \otimes_{\mathbb{Z}[G]} M$, which is an H-module. We denote this module by $f_!(M)$ and the resulting functor,

$$f_! : G-Mod \to H-Mod,$$

is *extension of scalars* along f.

Suppose that we have a H-module morphism,

$$\alpha : f_!(M) \to N,$$

then we can form a composite

$$M \xrightarrow{i_M} f^*(\mathbb{Z}[H] \otimes_{\mathbb{Z}[G]} M) \xrightarrow{f^*(\alpha)} f^*(N).$$

The first morphism, i_M, sends m to $1 \otimes m$. This is a G-module morphism, since $i_M(g \cdot m) = 1 \otimes g \cdot m = (1 \cdot g) \otimes m = f(g) i_M(m)$. (Of course, i is a natural transformation $1 \to f^* f_!$, which is the unit of an adjunction (**for you to check**.))

This gives a homomorphism,

$$H\!-\!Mod(f_!(M), N) \to G\!-\!Mod(M, f^*(N)).$$

This is an isomorphism. To check this, take some $\beta : M \to f^*(N)$ and form $f_!(\beta) : f_!(M) \to f_! f^*(N)$. We examine $f_! f^*(N)$ hoping to see a natural morphism to N (which should be the counit of the 'hoped-for' adjunction). We have $f_! f^*(N) = \mathbb{Z}[H] \otimes_{\mathbb{Z}[G]} f^*(N)$, and a typical generating element would be something of the form $h \otimes n$, where $hf(g) \otimes n = h \otimes (g \cdot n) = h \otimes f(g)n$, so we can send $h \otimes n$ to $h \cdot n$ quite safely to get a morphism,

$$\varepsilon(N) : f_! f^*(N) \to N.$$

The obvious way to show that the homomorphism, above, is an isomorphism is to use $\beta \mapsto \varepsilon(N) f_!(\beta)$, to come in the opposite direction then **to check** (if you have not seen this before) that everything fits.

The whole of this is classical homological algebra or, as we are handling groups and modules over them, representation theory, so the details can safely be **'left to you'** to check, **find in your favourite source**, or for you to think 'I know that!'. We summarise this with the classical result:

Proposition 116 *For $f : G \to H$, a group homomorphism, the induced functor, $f^* : H\!-\!Mod \to G\!-\!Mod$, has as left adjoint the functor, $f_!$, given above.* ■

Co-extension of scalars: Again we have $f : G \to H$, and, hence, $f^* : H\!-\!Mod \to G\!-\!Mod$, but this time, we want to find a right adjoint. This is again well known, but it will be useful having it *here*.

We note that $\mathbb{Z}[f] : \mathbb{Z}[G] \to \mathbb{Z}[H]$, also makes $\mathbb{Z}[H]$ into a *left* G-module, (which is, in fact, $f^*(\mathbb{Z}[H])$. This is also a *right* H-module, so we look at the Abelian group $f_*(M) = G\!-\!Mod(f^*(\mathbb{Z}[H]), M)$ for, as before, M a (left) G-module. The right H-module structure on $f^*(\mathbb{Z}[H])$, now gives a *left* H-module structure on $f_*(M)$, given, explicitly, by: if $\mu : f^*(\mathbb{Z}[H]) \to M) \in f_*(M)$ and $h \in H$, $h \cdot \mu$ sends $h' \in \mathbb{Z}[H]$ to $\mu(h'h) \in M$. This f_* construction is clearly functorial in M and so gives

$$f_* : G\!-\!Mod \to H\!-\!Mod,$$

which is known as *co-extension of scalars* along f.

Proposition 117 *The functor f_* is right adjoint to f^*.*

Proof (sketch): We will just sketch this as before leaving **details for you to provide**. We need to produce a natural isomorphism

$$G-Mod(f^*(N), M) \cong H-Mod(N, f_*(M)),$$

and the obvious thing to do, starting with $\alpha : f^*(N) \to M$ is to apply f_* and thus get

$$N \xrightarrow{\eta(N)} f_*f^*(N) \xrightarrow{f_*(\alpha)} f_*(M)$$

for some as yet mysterious (natural) morphism $\eta(N)$. we examine

$$f_*f^*(N) = G-Mod(f^*(\mathbb{Z}[H]), f^*(N)),$$

and note that, if $n \in N$, there is a corresponding H-module morphism, $\ulcorner n \urcorner : \mathbb{Z}[H] \to N$, sending $1 \in \mathbb{Z}[H]$ to n and extending H-linearly. We can apply f^* to this to get $f^*\ulcorner n \urcorner \in f_*f^*(N)$ and our unit, $\eta(N)$ sends n to this element.

'Coming back the other way': if $\beta : N \to f_*(M)$, then $f^*(\beta) : f^*(N) \to f^*f_*(M)$, and we have to look at possible 'counits', $\varepsilon(M) : f^*f_*(M) \to M$. There is a natural morphism, exactly like this, since $f^*f_*(M)$ is the *G-module* $f^*(G-Mod(f^*(\mathbb{Z}[H]), M)$, and, if $\mu : f^*(\mathbb{Z}[H]) \to M$ is an element in here, then the obvious element of M to assign to it is $\mu(1)$. This *does* give a G-module morphism, $\varepsilon(M) : f^*f_*(M) \to M$ (with **you being left to check the details**), and the assignment to β of $\varepsilon(M)f^*(\beta)$ gives, (and again **this should be checked**), the inverse of the earlier one. ■

Before we go further, let us interpret these constructions in a simple case, namely when $H = 1$ the trivial group.

Coinvariants: Let G be a group and M a G-module.

Definition: The group of *coinvariants* of M, denoted M_G, is defined to be the quotient of M by the subgroup generated by the elements of the form $g \cdot m - m$ for $g \in G$ and $m \in M$.

Remarks: (i) M_G is the largest quotient of M on which G acts trivially. It is isomorphic to $M/I(G)M$, where, as always, $I(G)$ is the augmentation ideal of $\mathbb{Z}[G]$, (cf. page 38.) It is convenient to think of it as being obtained by 'killing' the G-action.

(ii) If $f : G \to 1$ is the unique terminal homomorphism to the trivial group, then $\mathbb{Z}[f] : \mathbb{Z}[G] \to \mathbb{Z}[1] \cong \mathbb{Z}$ is exactly the augmentation map, usually denoted ε - but this notation has been already 'spoken for' in this section.

(iii) The lack of the left exactness of the functor $(-)_G$ is the start of group homology, (see Brown, [39] p.34 and §II.3, starting p. 35).

Now look at $f^* : 1-Mod \to G-Mod$. Of course, $1-Mod$ is just *Ab*, the category of Abelian groups, whilst, for an Abelian group, A, $f^*(A)$ is A given the trivial G-action.

What about $f_!(M)$ for M, a G-module. We have the unit of the adjunction is

$$M \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} M,$$

$$m \mapsto 1 \otimes m.$$

Look at $g \cdot m - m$. This is sent to $1 \otimes g \cdot m - 1 \otimes m$, which is zero, since $1 \otimes g \cdot m = 1 \cdot g \otimes m = 1 \otimes m$ as $\mathbb{Z}$ has trivial right G-action. We thus have a morphism $M_G \to \mathbb{Z} \otimes_{\mathbb{Z}[G]} M$, but there is a morphism in the other direction given by : $a \otimes m \mapsto a\overline{m}$, where $\overline{m}$ is the image of m in M_G. (**Check this works**, i.e., is well defined and provides the inverse of the other morphism.)

We thus have that M_G is isomorphic, as an Abelian group, to $\mathbb{Z} \otimes_{\mathbb{Z}[G]} M$. As it has trivial G-action, this is 'really' an isomorphism of G-modules, but, and this is a slightly pedantic point that often escapes mention, M_G is a G-module and $f_!(M)$ is an Abelian group, so they are not isomorphic if we are dotting is and crossing ts. A more correct statement is:

Lemma 85 *There is a G-module isomorphism*

$$f^* f_!(M) \cong M_G,$$

compatibly with the natural morphisms from M to each of them. ■

Again pedantically, as a quotient by a submodule, M_G should be thought of as being only defined up to isomorphism, so this may be the best we can get. A similar point will be made for invariants, to which we now turn.

Invariants: We have already met the module of invariants of a G-module, M. It was (cf. page **??**)

$$M^G = \{m \mid gm = m \text{ for all } g \in G\}.$$

As we are trying to be (fairly categorically) precise, it should be stated that as it is the 'fixed point' set of a group action, M^G has the properties of a limit, and from that point of view is only really defined up to isomorphism, although this manifestation of the limit is, of course, much easier to work with than any other!

'Limit' tends to suggest right adjoint situation, and, again of course, we look at $f : G \to 1$ and $f_* : G{-}Mod \to 1{-}Mod = Ab$. We have

$$f_*(M) = G{-}Mod(f^*(\mathbb{Z}), M),$$

so an element of $f_*(M)$ *is*, from this representation of that module, a G-module morphism, μ, from $\mathbb{Z}$ with the trivial G-action, to M. The element $\mu(1)$ completely determines this, and, of course, $\mu(1) \in M^G$, and conversely. The pedantic point is that M^G is a G-module, whilst $f_*(M)$ is an Abelian group. We have:

Lemma 86 *There is a G-module isomorphism,*

$$f^* f_*(M) \cong M^G,$$

which is compatible with the counit $f^* f_*(M) \to M$ and the inclusion of M^G, as a submodule, into M. ■

These are natural isomorphisms in M and give the universal nature of the construction of the invariants. The point of the 'pedantic point' is that, in some categories other than that of modules, it may not be so appropriate to think of the object of invariants as being a subobject of some given G-object. The object of invariants is a limit and so may be usefully thought of as an object in the 'base' category, in this case, Ab.

Induced and Coinduced Modules: Another useful case to examine of the restriction of scalars is for $f : G \to H$ being a monomorphism, so 'inclusion of a subgroup'. In this case, $f^* : H-Mod \to G-Mod$ is exactly restriction of the action to the subgroup. The corresponding $f_!$ is then usually denoted something like Ind_G^H, whilst f_* is $Coind_G^H$. These are called *induction* and *coinduction* from G to H, and, if M is in $G-Mod$, $Ind_G^H M$ is the *induced H-module* and $Coind_G^H M$, the *coinduced* one.

These cases are important in representation theory and the cohomology of groups, (see Brown, [39], §III.5, for instance). They are worth exploring a little here as they show special features that also influence the crossed algebra situation.

As we have $G < H$, $f^*(\mathbb{Z}[H])$ is a free $\mathbb{Z}[G]$-module on the set H/G of cosets. Let E be a set of coset representatives for G in H, then

$$f^*(\mathbb{Z}[H]) \cong \bigoplus_{h_i \in E} \mathbb{Z}[G]h_i,$$

(**for you to check**). This gives neat descriptions of $f_!(M)$ and $f_*(M)$ in this case:

Proposition 118 *For M in $G-Mod$, there are Abelian group isomorphisms*
a) $Ind_G^H M \cong \bigoplus\{h_i M \mid h_i \in E\}$;
b) $Coind_G^H M \cong \prod\{h_i M \mid h_i \in E\}$. ■

Here, the notation 'hM' represents a subgroup isomorphic to M (so a 'copy of M') indexed by h. The H-action uses the natural action on the cosets, so, if $h_i m \in h_i M$ and $h \in H$, we write $hh_i = h_j g$ for $h_j \in E$ and $g \in G$, and then

$$h \cdot (h_i m) = h_j(g \cdot m),$$

where $g \cdot m$ uses the given G-action on M.

We refer to standard sources for further properties and application of these constructions. We note that, if $[H : G] < \infty$, then, as finite products of modules are also direct sums, the two constructions give isomorphic modules. (This can be confusing!)

We next look at analogues of some of these constructions in the context of crossed algebras.

11.3.3 Pulling back a crossed G-algebra

A morphism, as above in section 11.1.2, over a group morphism, $f : G \to H$, can be replaced by a morphism of crossed G-algebras, $L \to f^*(L')$, where $f^*(L')$ is obtained by pulling back L' along f.

If $f : G \to H$ is a group homomorphism, given a crossed H-algebra, L, we can obtain a crossed G-algebra, $f^*(L)$, by pulling back using f. The structure of $f^*(L)$ is given by:

- $(f^*(L))_g$ is $L_{f(g)}$, by which we mean that $(f^*(L))_g$ is a copy of $L_{f(g)}$ with grade g, and we note that, if $x \in L_{f(g)}$, it can be useful to write it $x_{f(g)}$ with 'x_g' denoting the corresponding element of $(f^*(L))_g$;

- if x and y have matching grades, g and g^{-1}, respectively, so $x \in (f^*(L))_g$, then $\rho(x, y)$ is the same as in $L_{f(g)}$, but if x and y have non-matching grades, then $\rho(x, y) = 0$;

- $\varphi_{g'}(x_p) := \varphi_{f(g')}(x_{f(g)})$.

The following is then easy to prove:

Proposition 119 *The algebra, $f^*(L)$, has a crossed G-algebra structure given by the above.* ■

The construction of $f^*(L)$, then, makes it clear that, if $f : G \to H$ is a homomorphism, L is a crossed G-algebra, and L', a crossed H-algebra, we have:

Proposition 120 *There is a bijection between the set of crossed algebra morphisms from L to L' over f and the set of crossed G-algebra morphisms from L to $f^*(L')$.* ■

(These proofs are fairly routine, so **just check up that they make sense and try to sketch some details**.) As with most such operations, this pullback construction gives a functor from the category of crossed H-algebras to that of crossed G-algebras.

Example: If $f : G \to H$ is the inclusion of a subgroup, then the operation of pulling back along f corresponds to restricting the crossed H-algebra to G.

11.3.4 Pushing forward a crossed G-algebra

This is analogous to the *extension of scalars* construction, $f_!$, for modules that we examined a short while ago, (in section 11.3.2, page 456).

We have shown that, given $f : G \to H$ and a $\theta : L \to L'$ over f, we can pull back L' over G to get a map from L to $f^*(L')$ that encodes almost the same information as L'. (The exception to this is if f is not an epimorphism, as then outside the image of f, we do not retain information on the 'fibres' of L'.) The obvious question to ask is whether there is some 'adjoint' push-forward construction with θ corresponding to some morphism from $f_*(L)$ to L' over H.

Cautionary note: Quite often in this sort of construction, there may have to be finiteness conditions imposed. This is usually due to the finite type conditions on the summands of the type of graded algebras being considered. Other conditions may also be needed on f. We are not guaranteeing that this adjoint exists for 'any old' f!

To start with we will look at the overall situation, with f and θ, as set out above, but we note that, if there is an adjoint $f_!$ construction, we should expect θ to factorise via $f_!(L)$ and to have an image in L'.

Given this context, first set $N = Ker f$, then we have that, for $n \in N$, and $a \in L_g$,

$$\varphi_n(a) - a \in Ker\,\theta,$$

as $\theta(\varphi_n(a)) = \varphi'_{f(n)}(\theta(a)) = \varphi'_1(\theta(a)) = \theta(a)$. We therefore form the ideal, K, generated by elements of this form, $\varphi_n(a) - a$, and note that it will be in the kernel of *any* θ. This is not a G-graded ideal, but that, in fact, is exactly what we need. We have that L/K is an associative algebra, and we will have to give it a *H-graded algebra structure*, and not a G-graded one. We note that, in order to get an H-grading and an action, we have to kill off the action of N on L and, of course, this is exactly what quotienting by K does.

If our adjoint exists, there has to be a universal morphism (over f) from L to the conjectured $f_!(L)$, and this must be compatible with the grading. This more or less forces one to look at the following:

For each $h \in H$, let

$$\mathsf{L}_h = \oplus\{L_g \mid g \in G, f(g) = h\},$$

and

$$\mathsf{L} = \oplus_{h\in H}\mathsf{L}_h.$$

This makes sense from the grading point of view, but we should note that it only works if N is finite, or more exactly, if the interaction of L with N only involves finitely many non-zero summands of L, as otherwise the vector space, or $\mathbb{k}$-module, L_h, may not be of finite type. Of course, for a given h, there may be no g satisfying $f(g) = h$, in which case that L_h will be trivial.

Let, now,

$$\mathsf{K}_h = \mathsf{L}_h \cap K.$$

The underlying H-graded vector space of $f_!(L)$ will be

$$f_!(L) = \oplus_{h\in H}\mathsf{L}_h/\mathsf{K}_h.$$

This is an associative algebra, as it is exactly L/K, but we have to check that this grading will be compatible with that multiplication.

Suppose $a + K \in f_!(L)_{h_1}$, and $b + K \in f_!(L)_{h_2}$, then $a \in L_{g_1}$ and $b \in L_{g_2}$ for some $g_1, g_2 \in G$ with $f(g_i) = h_i$, for $i = 1, 2$, but then $ab + K \in f_!(L)_{h_1h_2}$ as required.

We next define the bilinear form giving the inner product. Clearly, with the same notation,

$$\rho(a + K, b + K) := 0 \quad \text{if } h_1 \neq h_2^{-1}.$$

If $h_1 = h_2^{-1}$, then we can assume that $g_1 = g_2^{-1}$, and, after changing the element b representing $b + K$ if necessary, that $b \in L_{g_2}$. Finally we set

$$\rho(a + K, b + K) := \rho(a, b).$$

This is easily seen to be independent of the choices of a and b, since, once we have a suitable pair (a, b) with $a \in L_{g_1}$ and $b \in L_{g_1^{-1}}$, any other will be related by isometries induced by composites of φs. Clearly ρ, thus defined, is a symmetric bilinear form and, on restricting to $f_!(L)_{h_1} \otimes f_!(L)_{h_2}$, it is essentially the original inner product restricted to $L_{g_1} \otimes L_{g_2}$, so is non-degenerate and satisfies

$$\rho(ab + K, c + K) = \rho(a + K, bc + K).$$

The next structure to check is the crossed H-algebra action,

$$\varphi : H \to Aut(f_!(L)).$$

The obvious formula to try is

$$\varphi_h(a + K) := \varphi_g(a) + K$$

where $f(g) = h$. It is easy to reduce the proof that this is well defined to checking independence of the choice of g, but, if g' is another element of $f^{-1}(h)$, then $g' = ng$ for some $n \in N$ and $\varphi_{g'}(a) = \varphi_n\varphi_g(a) \equiv_K \varphi_g(a)$, so the action is well defined. Of course, this definition will give us immediately that the $\varphi_h(a)b = ba$ axiom holds and that $\varphi_h|_{f_!(L)_h} = id$, etc.

The trace axiom follows somewhat similarly via a fairly routine calculation.

Proposition 121 *With the above structure, $f_!(L)$ is a crossed H-algebra.* ∎

Now return to the morphism, θ over f. It is clear that as θ killed K, then θ induces a unique morphism, $\overline{\theta} : f_!(L) \to L'$, over H. We thus have

Proposition 122 *If f has finite kernel, there is a bijection between the set of crossed algebra morphisms from L to L' over f and the set of crossed H-algebra morphisms from $f_!(L)$ to L'.* ∎

Of course,

Proposition 123 *If f has finite kernel, $f_!$ gives a functor, $f_! : Crossed.G-Alg \to Crossed.H-Alg$.* ∎

As a corollary of these propositions and the earlier result on f^*, we have:

Corollary 24 *If f has finite kernel, then $f_!$ is left adjoint to f^*.* ∎

Examples: (i) A neat example of this construction occurs when, in addition to being finite, N is central. Taking $L = \Bbbk[G]$, then we have seen that L is a crossed G-algebra with L_g generated by some single element also labelled g. With this, for $n \in N$, φ_n is the identity automorphism, and L becomes a crossed H-algebra without any further bother. (The construction in this case is given in Chapter II of [211].)

(ii) If G is finite, then the unique homomorphism, $f : G \to 1$, from G to the trivial group, 1, satisfies the conditions for the existence of $f_!$. Here if L is a crossed G-algebra, then $f_!(L)$ will be a crossed 1-algebra, and, hence, just a commutative Frobenius algebra. The ideal, K, is generated by the $\varphi_g(\ell) - \ell$, and, of course, the induced action is an action of the trivial group, so is trivial! This is clearly the algebra of coinvariants of L.

11.3.5 Invariants of a crossed G-algebra

Since we have seen an algebra of coinvariants it is natural to ask about a module of invariants. Such an object has been considered by Kaufmann, [135], in a slightly more general context, and used by Moore and Segal in [163].

We are given G, as usual, and a crossed G-algebra, L, then G acts on L via φ. Here we must be a bit careful. The homomorphism, $\varphi : G \to Aut(L)$, is such that $\varphi(g) : L_x \to L_{gxg^{-1}}$, for each direct summand, L_x, of L, so $\varphi(g)$ is *not* an automorphism of *graded* algebras. (In fact, this 'categorical mismatch' causes quite a lot of problems and indicates that the notion of crossed G-algebra probably hides some alternative formulation that will be categorically neater. In fact, later we will give one alternative way of thinking of a crossed G-algebra and of φ, which is very neat and which shows the naturality of this, which here may seem slightly strange.) In particular, forgetting the additional structure (multiplication, inner product, etc.), the $\Bbbk$-vector space, L, is a G-module, so we can look at L^G, the *module* of invariants of that action. Does it have an algebra structure, and if 'yes', then what sort of structure is it? As we have discussed the formation of the module of invariants earlier, from both an elementary and a categorical viewpoint, we can attempt to explore L^G in a similar way.

The elements of L^G will be those $\ell \in L$ that satisfy $\varphi_g(\ell) = \ell$ for all $g \in G$. These clearly form a subspace (submodule) of L. Exploring a bit further, suppose $\ell_x \in L_x$ is a homogeneous such invariant, - there always will be some even if they are trivial - then $\varphi_g(\ell_x) \in L_{gxg^{-1}}$, so we must have $gxg^{-1} = x$, and, as here g is arbitrary, x must be central in G, and, conversely, if $x \in Z(G)$, the centre of G, then $L_x \cap L^G = L_x^G$, i.e., the G action restricts to one on the subspace, L_x, and the subspace of L^G over x will be L_x^G for that restricted action.

If ℓ is not homogeneous, life is a bit more complicated, - and a bit more interesting! If $\ell \in L$, then we can write it as a *finite* sum, $\sum \ell_x$, and, of course, $\varphi_g(\ell) = \sum \varphi_g(\ell_x)$. If ℓ is invariant, then clearly the support, $supp(\ell)$, of ℓ, that is, the set of $x \in G$ such that $\ell_x \neq 0$, is a finite union of conjugacy classes of G, since $supp(\varphi_g(\ell)) = g \cdot supp(\ell) \cdot g^{-1}$, and this has to be the same as $supp(\ell)$ if ℓ is invariant. Grouping terms in the sum in terms of conjugacy classes of the index, x, we have $\ell = \sum \ell_c$, where c is a 'generic' conjugacy class and $\ell_c \in \sum_{x\in c} L_x$. Again it is clear that $\varphi_g(\ell_c) = \ell_c$ for all g if ℓ is invariant, so we have only to look at these ℓ_c.

Remark: In many applications of these ideas, there is made the assumption that G is finite. In fact, it is easy to see that whatever G is, we only need the size of the conjugacy classes to be finite to given an allowable construction.

11.3.6 Algebraic transfer

As we said earlier, 'transfer' is a frequently used technique related to group representations and cohomology. In the representation theoretic situation, it related to 'induction'. (There is a good discussion of some aspects of it in Brown's book on *Cohomology of Groups*, [39], starting on page 80.) Here we look at the analogue for crossed algebras, where some more subtle behaviour seems to involved, partially, perhaps, because induction and coinduction do not coincide in this case.

We set up the crossed algebraic analogue of transfer, as follows. A detailed discussion is given in Turaev's [211].

Let H be a group and $G < H$, a subgroup of finite index, $n = [H : G]$. We write $f : G \to H$ for the inclusion morphism. We suppose L is a crossed G-algebra and we seek to build, from it, a crossed H-algebra, which will be $tr_G^H(L)$, for our putative algebraic transfer map, tr_G^H.

We pick a set, $w_1 = 1, w_2, \ldots, w_n$, of right coset representatives for G in H, and may sometimes use i as a shorthand for Gw_i. The idea of the construction is that the eventual action of H on $tr_G^H(L)$ can be linked to the action of H on the coset representatives, so that information encoded in those summands graded by elements of the subgroup, G, can be 'spread' around to build $tr_G^H(L)$.

For $h \in H$, set $N(h) = \{i \mid w_i h w_i^{-1} \in G\}$, and then

$$tr_G^H(L)_h = \oplus\{L_{w_i h w_i^{-1}} \mid i \in N(h)\}.$$

(Note that $tr_G^H(L)_h = 0$ if h is not conjugate to any element of G.)

Take $tr_G^H(L) = \oplus_{h\in H} tr_G^H(L)_h$ and give it the multiplication induced, componentwise, from L. If $a \in tr_G^H(L)_h$, think of it as a 'vector' with n coordinates (so a_k must be 0 if $k \notin N(h)$). In a product with $a \in tr_G^H(L)_h$ and $b \in tr_G^H(L)_{h'}$, the resulting ab will have $(ab)_k = a_k b_k$. This makes sense, since $a_k \in L_{w_k h w_k^{-1}}$, (it may be zero, of course), and similarly for b_k, so $a_k b_k$ can be defined using the multiplication in L.

Lemma 87 *$tr_G^H(L)$ is an H-graded associative $\Bbbk$-algebra.* ∎

The proof is **left to you**.

An important point to note is that, for $g \in G$, $tr_G^H(L)_g$ is *not* usually just L_g. Taking an extreme case, $1 \in G$ and $N(1) = \{w_1, \dots, w_n\}$. As a consequence, $tr_G^H(L)_1$ is a direct sum of n copies of L_1. (There is, of course, an inclusion of L_g into $tr_G^H(L)_g$.)

Similarly there is an inner product on $tr_G^H(L)$,

$$\tilde{\rho} : tr_G^H(L) \otimes tr_G^H(L) \to \mathbb{k},$$

whose restriction to $tr_G^H(L)_h \otimes tr_G^H(L)_{h'}$ is trivial unless $h^{-1} = h'$, and then uses the fact that $N(h) = N(h^{-1})$ and the inner product of L componentwise:

$$L_{w_i h w_i^{-1}} \otimes L_{w_i h^{-1} w_i^{-1}} \to \mathbb{k}.$$

We next need to consider H, and a possible action of it on $tr_G^H(L)$. We need a homomorphism

$$\tilde{\varphi} : H \to Aut(tr_G^H(L))$$

such that it restricts to isomorphisms,

$$\tilde{\varphi}_h : tr_G^H(L)_{h'} \to tr_G^H(L)_{hh'h^{-1}}.$$

We have a direct sum decomposition of $tr_G^H(L)_{h'}$ in terms of $N(h')$, and so we have to examine possible links between $N(h')$ and $N(hh'h^{-1})$.

There is a natural action of H *on the right* on the set, $G\backslash H$, of right cosets, since, of course, if Gw_i is a coset, so is $Gw_i h^{-1}$. We define a bijection

$$h(-) : G\backslash H \to G\backslash H$$

by

$$Gw_{h(i)} = Gw_i h^{-1}.$$

We note that, if $h(i) = i$, then $Gw_i h^{-1} = Gw_i$, so $w_i h^{-1} w_i^{-1} \in G$, i.e., $i \in N(h)$, and conversely.

If, on the other hand, $i \in N(h')$, then it is clear that $h(i) \in N(hh'h^{-1})$, and conversely. We set $h_i = w_{h(i)} h w_i^{-1}$ and note that $h_i \in G$. (Since $w_{h(i)} = g w_i h^{-1}$ for some $g \in G$, this is obvious.) We will use

$$\varphi_{h_i} : L_{w_i h' w_i^{-1}} \to L_{(h_i w_i) h' (h_i w_i)^{-1}},$$

but note that $(h_i w_i) h' (h_i w_i)^{-1} = w_{h(i)} (hh'h^{-1}) w_{h(i)}^{-1}$, and that $h(i) \in N(hh'h^{-1})$.

We now define $\tilde{\varphi}_{h'} : tr_G^H(L)_{h'} \to tr_G^H(L)_{hh'h^{-1}}$ to be the direct sum of these isomorphisms over all $i \in N(h')$. The following is then routine:

Lemma 88

$$\tilde{\varphi} : H \to Aut(tr_G^H(L))$$

gives $tr_G^H(L)$ the structure of a crossed H-algebra. ■

A detailed proof can be found in Turaev's book, [211], p. 29 - 30, (but note that the roles of G and H are swapped there).

If L is a Frobenius G-algebra, then $tr_G^H(L)$ is a Frobenius H-algebra, as is fairly easily checked.

Definition: If L is a crossed G-algebra, where $G < H$ is of finite index, then the *transfer* of L to H is the crossed H-algebra, $tr_G^H(L)$, described above.

Remark: The construction of $tr_G^H(L)$ is quite complicated and it is a little bit difficult to appreciate what 'makes it tick', so it is useful to note that under the correspondence between 2-dimensional HQFTs with background a $K(G,1)$ and Frobenius crossed G-algebras, geometric transfer of HQFTs, as we considered in section 10.3.3, corresponds to transfer in the above algebraic sense, and the construction can be analysed in that way. (This is suggested as a **useful thing to do**!)

The properties of the transfer, like its construction, seem to be quite subtle, and, in particular, trying to understand the construction categorically would seem to be quite hard. (As far as I know, no clear categorical interpretation of it known as yet.)

11.4 A-Frobenius algebras

The prime sources for this are the two papers, [37, 191]. In what follows in this section, A will denote an Abelian group. We have already defined a Frobenius object in a symmetric monoidal category, $\mathcal{A}$, (see page 331).

Definition: An *A-Frobenius object* in $\mathcal{A}$ is a Frobenius object, L, together with a homomorphism,

$$A \to End(L).$$

Definition: When $\mathcal{A} = (Vect, \otimes)$, or $(ModR, \otimes)$, the corresponding concept is that of a *A-Frobenius algebra*.

Examination of the action shows that, if we write $g \cdot a$ for the action of g on an element $a \in L$,

$$a(g \cdot b) = g \cdot (ab) = (g \cdot a)b$$

and

$$\rho(a, g \cdot b) = \rho(g \cdot a, b).$$

As L is a unital algebra,

$$g \cdot v = g \cdot 1v = (g \cdot 1)v,$$

so the action actually comes from a morphism of monoids

$$A \to L$$

$$g \to g \cdot 1,$$

and $g \cdot 1$ is in the center of L.

Recall that this is the variant of a Frobenuis algebra adapted for the classification and characterisation of 2-dimensional HQFTs with background that is a simply connected space.

We will look at this in a bit more detail. First recall the result of Theorem 22 that the category of 2d TQFTs is equivalent to that of commutative Frobenius algebras. We had that in a 2-dimensional TQFT, Z, the key object was the image, $Z(S^1)$, of the circle. This became a Frobenius algebra using the 'pair of pants' cobordism to get the multiplication,

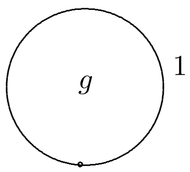
$$Z(S^1) \otimes Z(S^1) \to Z(S^1),$$

then the cylinder gave the inner product and the disc gave both a cobordism from $\emptyset$ to S^1, and one from S^1 to $\emptyset$, giving the unit $\Bbbk \to Z(S^1)$ and the counit $Z(S^1) \to \Bbbk$ structural maps, and so on, with axioms such as associativity coming from isomorphisms of cobordisms. Most neatly, S^1 can be seen as a Frobenius object in $2-Cob$, (for the monoidal structure that we discussed earlier).

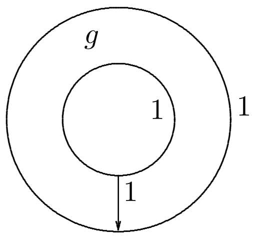
If we replace $2-Cob$ by $HCobord(1, B)$ with B being a $K(A, 2)$ (and A, Abelian), then the basic objects are the same, since any $g : \bigsqcup S^1 \to B$ will be homotopic to the constant map to the basepoint, but the cobordisms available will be richer. For the simplest example, we have a disc, $D^2 : \emptyset \to S^1$, but also have some $g : D^2 \to B$, and, of course, as g can be assumed to be cellular, this gives an element $g \in \pi_2(B) \cong A$. In fact, we can either look back at the general arguments that we saw in section 10.5.5, or, working at a more intuitive level, look at the diagrams for formal C-maps (for instance, page 443) replacing labels on the 1-skeleton by the identity element, 1.

There are several key examples here. The simplest is probably, for an element $g \in A$, the labelled disc,

$$Disc(g) : \emptyset \to (S^1, 1) :$$



whilst $Cyl_{\varepsilon,\mu}(g, 1, 1)$, (in the notation we used earlier), is a g-labelled cylinder:



This can be derived form a $Disc(g)$ and a 'pair of pants' using the disc to close off one 'trouser leg' hole. Conversely, we can obtain $Disc(g)$ from the cylinder by using a $Disc(1)$, which is already there (in disguise) in $2-Cob$, to fill one end of the cylinder.

If we now note that S^1 or, more exactly, $(S^1, 1)$, is still a Frobenius object in $HCobord(1, B)$ and that the g-indexed cylinder gives an automorphism of that object (**just check up on composition** of B-cobordisms and its relation to the multiplication of labels), we get that, in this case, there will be an equivalence of categories between $HQFT(1,B)$ and the category of A-Frobenius algebras. Explicily:

Theorem 49 *Let A be an Abelian group and $B = K(A,2)$, the corresponding Eilenberg-Mac Lane space. The assignment, to a (1+1)-dimensional HQFT, τ, with background B, of the vector space, $\tau(S^1,1)$, yields an equivalence of categories between $HQFT(1,B)$ and the category of A-Frobenius algebras.* ■

We refer to Brightwell and Turner's paper, [37], for a detailed proof, but, in fact, our discussion above provides quite a full sketch of it.

As mentioned earlier, a related result was proved by Rodrigues, [191]. We refer to that paper for the proof, as we will be examining related results a bit later on.

Theorem 50 *The monoidal category,$HCobord(1,K(A,2))$ is equivalent to the free symmetric monoidal category with strict duals on one A-Frobenius object.* ■

11.5 Crossed C-algebras: the general case

We now turn to the general case with $\mathsf{C} = (C,P,\partial)$, as earlier; see section 10.5.6. The additional structure is thus that given by the annuli or cylinders $(Cyl_{\varepsilon,\mu}; c, g_1, g_2)$. We saw earlier that this collection of operations could be reduced further to the case $g_2 = 1$ and $c \neq 1$, and, in fact, the only ones that we actually need are with $g_1 = 1$ as well, as the general case is a composite of this with the unit on the left and the 'pair of pants' multiplication. (The general case gives an isomorphism

$$\theta_{(c,g)} : L_g \to L_{\partial c \cdot g}$$

and we can build this up, for instance, by

$$L_g \to \Bbbk \otimes L_g \to L_1 \otimes L_g \overset{\theta_{(c,1)}\otimes L_g}{\to} L_{\partial c} \otimes L_g \overset{\mu}{\to} L_{\partial c \cdot g},$$

where the third morphism is that given by that special case $g = 1$. We say that $\theta_{(c,g)}$ is obtained by 'translation' from $\theta_{(c,1)}$.)

The extra structure can be thought of a collection of isomorphisms,

$$\Theta_{\mathsf{C}} = \{\theta_{(c,1)} : L_1 \to L_{\partial c}\}.$$

It is worth noting that if $C = \{1\}$, the resulting structure reduces to that of a crossed P-algebra and, if $P = 1$ and C is just an Abelian group, then the $\theta_{(c,1)} : L_1 \to L_1$ are just automorphisms of L_1, which is itself just a Frobenius algebra, ansd we return to the structure of section 11.4.

This structure of extra specified automorphisms does not immediately tell us how to retrieve the structure given by the C-discs. Those gave linear maps,

$$\ell_c : \Bbbk \to L_{\partial c}.$$

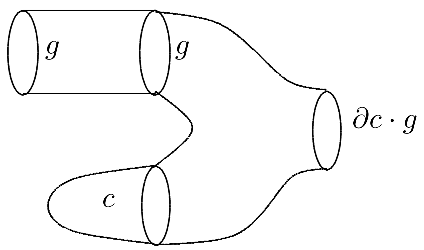
We can, however, recover them from $\ell_1 : \Bbbk \to L_1$, which was part of the crossed P-algebra structure, together with $\theta_{(c,1)} : L_1 \to L_{\partial c}$, but, conversely, given the ℓ_c, we can recover the $\theta_{(c,g)}$:

Proposition 124
The composite

$$L_g \stackrel{\cong}{\to} \Bbbk \otimes L_g \stackrel{\ell_c \otimes L_g}{\to} L_{\partial c} \otimes L_g \stackrel{\mu}{\to} L_{\partial c \cdot g}$$

is equal to $\theta_{(c,g)}$.

Proof:
We can realise this composite by a C-cobordism



but this is equivalent to the C-annulus that gives us $\theta_{(c,g)}$. ■

As before, we will write $\tilde{c} = \ell_c(1) \in L_{\partial c}$.

Corollary 25
For any $c \in C$, $g \in P$ *and for* $x \in L_g$,

$$\theta_{(c,g)}(x) = \tilde{c} \cdot x,$$

where $\cdot$ *denotes the product in the algebra structure of* $L = \bigoplus_{h\in P} L_h$. ■

Abstracting this extra structure, we get:

Definition: Let $\mathsf{C} = (C, P, \partial)$ be a crossed module. A *crossed* C*-algebra* consists of a crossed P-algebra, $L = \bigoplus_{g\in P} L_g$, together with elements $\tilde{c} \in L_{\partial c}$, for $c \in C$, such that

(a) $\tilde{1} = 1 \in L_1$;

(b) for $c, c' \in C$, $\widetilde{(c'c)} = \tilde{c'} \cdot \tilde{c}$;

(c) for any $h \in P$, $\varphi_h(\tilde{c}) = \widetilde{{}^h c}$.

We note for future use that the first two conditions say that 'tilderisation' is a group homomorphism $(\tilde{\ }) : C \to U(L)$, the group of units of the algebra, L.

The two special cases with (i) $C = 1$ and, (ii) for Abelian C, with $P = 1$, correspond, of course, to crossed P-algebras and C-Frobenius algebras respectively. An interesting special case of the general form is when C is a P-module and ∂ sends every element in C to the identity of P. In this case, we have an object that could be described as a C-crossed P-algebra! It consists of a crossed P-algebra together with a C-action by multiplication by central elements. This results in a very weak mixing of the two structures. The important thing to note is that the general form is more highly structured as the twisting in the crossed modules, in general, can result in non-central elements amongst the $\tilde{c}$s.

11.6 A classification theorem for formal C-HQFTs

11.6.1 FHQFTs and crossed C-algebras

The following extends the two types of result that we have already seen that classify subclasses of HQFTs with special types of background.

Theorem 51 *There is a canonical bijection between isomorphism classes of formal 2-dimensional HQFTs based on a crossed module, C, and isomorphism classes of crossed C-algebras.*

More explicitly:

Theorem 52 *a) For any formal 2-dimensional HQFT, τ, based on C, the crossed P-algebra, $L = \bigoplus_{g\in P} L_g$, having $L_g = \tau(g)$, is a crossed C-algebra, where for $c \in C$, $\tilde{c} = \ell_c(1)$ (notation as above).*
b) Given any crossed C-algebra, $L = \bigoplus_{g\in P} L_g$, there is a formal 2-dimensional HQFT, τ, based on C yielding L as its crossed C-algebra, up to isomorphism.

11.6.2 Proof of Theorem 51

We start by identifying the geometric behaviour of the isomorphisms, $\theta_{(c,g)}$. We know, from the special case of $C = 1$, that L is a crossed P-algebra, so we need to look at the extra structure. We have already done this in passing from a TQFT to a B-HQFT with B an $K(A, 2)$, but, although that gives the essence of what needs doing there,*are* some additional features that need attention.

- **Vertical composition of C-cobordisms.**

  The basic condition is that the composite,

  $$L_g \xrightarrow{\theta_{(c,g)}} L_{\partial c\cdot g} \xrightarrow{\theta_{(c',\partial c\cdot g)}} L_{\partial(c'c)\cdot g},$$

  is $\theta_{(c'c,g)}$:

  $$\theta_{(c'c,g)} = \theta_{(c',\partial c\cdot g)} \circ \theta_{(c,g)} : L_g \to L_{\partial(c'c)\cdot g},$$

  since τ must be compatible with the 'vertical composition' of C-cobordisms. Evaluating this on an element gives

  $$\widetilde{(c'c)} = \tilde{c'} \cdot \tilde{c},$$

  where $\tilde{c'} \cdot \tilde{c} = \mu(\tilde{c'}, \tilde{c})$. Similarly $\tilde{1} = 1$.

- **'Horizontal' composition of C-cobordisms.** Using the interchange law / Peiffer rule and a 'labelled pair of pants' to give the multiplication, we have

  $$(1, g_1)\#_0(c, g_2) = ({}^{g_1}c, g_1g_2) = ({}^{g_1}c, g_1)\#_0(1, g_2).$$

  (Here the useful notation $\#_0$ corresponds to the horizontal composition in the associated strict 2-group of C.) We thus have two composite C-cobordisms giving the same result, and hence,

  $$L_{g_1} \otimes L_{g_2} \xrightarrow{L_{g_1}\otimes\theta_{(c,g_2)}} L_{g_1} \otimes L_{\partial c\cdot g_2} \xrightarrow{\mu} L_{g_1\cdot\partial c\cdot g_2} = L_{g_1} \otimes L_{g_2} \xrightarrow{\mu} L_{g_1g_2} \xrightarrow{\theta_{({}^{g_1}c,g_2)}} L_{g_1\cdot\partial c\cdot g_2}.$$

In general, for $d \in C$, the second type of composite will be

$$L_{g_1} \otimes L_{g_2} \xrightarrow{\theta_{(d,g_1)} \otimes L_{g_2}} L_{\partial d \cdot g_1} \otimes L_{g_2} \xrightarrow{\mu} L_{\partial d \cdot g_1 \cdot g_2} = L_{g_1} \otimes L_{g_2} \xrightarrow{\mu} L_{g_1 g_2} \xrightarrow{\theta_{(d,g_1 g_2)}} L_{\partial d \cdot g_1 \cdot g_2},$$

and we need this for $d = {}^{g_1}c$, for which the corresponding composite cobordisms are equal. Geometrically these rules correspond to a pair of pants with g_1, g_2 on the trouser cuffs and the 2-cell colored c. We can push c onto either leg, but in so doing may have to conjugate by g_1.

Summarising, for given $c \in C$, $g_1, g_2 \in G$,

$$\mu(id_{L_{g_1}} \otimes \theta_{(c,g_2)}) = \theta_{({}^{g_1}c, g_1 g_2)} \circ \mu = \mu(\theta_{({}^{g_1}c, g_1)} \otimes id_{L_{g2}}).$$

As we have reduced $\#_0$ to 'whiskering' and vertical composition, $\#_1$, and have already checked the interpretation of $\#_1$, we might expect this pair of equations to follow from our earlier calculations, however we have invoked here the interchange law and that was not used earlier. The above equations reduce to

$$x \cdot \tilde{c} = \widetilde{{}^{g}c} \cdot x.$$

This is implied by axiom c) of a crossed C-algebra, since we have

$$\widetilde{{}^{g}c} \cdot x = \varphi_h(\tilde{c})x = x \cdot \tilde{c},$$

using the third axiom (page 450) of the crossed P-algebra structure on L. Thus the combination of these two rules corresponds, in part, to the Interchange Law. Conversely this rule in either form is clearly implied by the axioms for a formal HQFT.)

To complete the proof, we would have to check that the inner product structure of L and action of P via φ are compatible with the new structure. The compatibility of the isomorphisms $\theta_{(c,g)}$ defined via the $\tilde{c}$ will follow both from the geometry of the HQFT and from the axioms of crossed C-algebras. These parts of the proof are similar to the parts we have already given, so are **left to you, either to try to prove yourselves**.

The formal details of the reconstruction of τ from L follow the same pattern as for the case $C = 1$ and, for the most part, are exactly the same as the only extra feature is the 'tilde' operation. The details are, once again, **left to you**. ■

Remark: It is sometimes useful to have the extra rules of the $\tilde{c}$s written in the intermediate language of the family of isomorphisms,

$$\theta_{(c,g)} : L_g \to L_{\partial c \cdot g}.$$

The first two conditions are easily so interpreted and the last corresponds to the compositions given earlier and also to the equality of

$$L_g \xrightarrow{\theta_{(c,g)}} L_{\partial c \cdot g} \xrightarrow{\varphi_h} L_{h \cdot \partial c \cdot g \cdot h^{-1}},$$

and

$$L_g \xrightarrow{\varphi_h} L_{hgh^{-1}} \xrightarrow{\theta_{({}^{h}c, {}^{h}g)}} L_{h \partial c \cdot g \cdot h^{-1}},$$

and thus to

$$\varphi_h \circ \theta_{(c,g)} = \theta_{({}^h c, {}^h g)} \circ \varphi_h.$$

Put together, these boxed equations in their various forms give compatibility conditions for the various structures.

Actions of C on L, an algebraic interpretation of crossed C-algebras: There is a very neat algebraic interpretation of these conditions. Let L be an associative algebra and $U(L)$ be its group of units. There is a homomorphism of groups $\delta = \delta_L : U(L) \to Aut(L)$ given by $\delta(u)(x) = u \cdot x \cdot u^{-1}$.

Lemma 89 *With the obvious action of $Aut(L)$ on the group of units, $(U(L), Aut(L), \delta)$ is a crossed module.* ■

The proof is simple, although quite instructive, and will be **left to the reader**. We will denote this crossed module by $\mathfrak{Aut}(L)$. If L has extra structure such as being a Frobenius algebra or being graded, the result generalises to have the automorphisms respecting that structure.

Proposition 125 *Suppose that L is a crossed C-algebra. The diagram*

$$\begin{tikzcd} C \arrow[r, "(\tilde{\ })"] \arrow[d, "\partial"'] & U(L) \arrow[d, "\delta"] \\ P \arrow[r, "\varphi"'] & Aut(L) \end{tikzcd}$$

is a morphism of crossed modules from C to $\mathfrak{Aut}(L)$.

Proof: First, we check commutativity of the square above. Let $c \in C$, going around clockwise gives $\delta(\tilde{c})$ and on an element $x \in L$, this gives $\tilde{c} \cdot x \cdot \tilde{c}^{-1}$. We compare this with the other composite, again acting on $x \in L$. If we multiply $\varphi_{\partial c}(x)$ by $\tilde{c}$, then we get $\varphi_{\partial c}(x)\tilde{c} = \tilde{c} \cdot x$, but therefore $\varphi_{\partial c}(x) = \tilde{c} \cdot x \cdot \tilde{c}^{-1}$ as well.

The other thing to check is that the maps are compatible with the actions of the bottom groups on the top ones, but this is exactly what the third condition on the 'tilde' gives. ■

Remark: Given the categorisation of $2-Cob$ and of $HCobord(2, K(A, 2))$ as suitable types of monoidal category with a special object, the classification theorem for FHQFTs suggests that there should be a categorisation of $HCobord(2, B\mathsf{C})$ in more generality. The exact form this might take is not clear at the time of writing. That such a result should hold is extremely likely given the results on extended TQFTs that one finds in Lurie's paper, [148].

11.7 Constructions on formal HQFTs and crossed C-algebras.

As formal HQFTs correspond to crossed C-algebras by our main result above, the category of crossed C-algebras needs to be understood better if we are to understand the relationships between formal HQFTs. We clearly also need some examples of crossed C-algebras.

First we note that the usual constructions of direct sum and tensor product of graded algebras extend to crossed C-algebras in the obvious way, and also that our earlier results on crossed G-algebras will feed valuable information into this slightly more general case.

11.7.1 Examples of crossed C-algebras

As usual we will fix a crossed module $\mathsf{C} = (C, P, \partial)$. We assume, for convenience, that $Ker\,\partial$ is a finite group, although this may not always be strictly necessary.

The group algebra, $\Bbbk(C)$, as a crossed C-algebra: We take $L = \Bbbk(C)$ and will denote the generator corresponding to $c \in C$ by e_c rather than merely using the symbol c itself, as we will need a fair amount of precision when specifying various types of related elements in different settings. Define $L_p = \Bbbk\langle\{e_c : \partial c = p\}\rangle$, so, if $p \in P - \partial C$, this is the zero dimensional $\Bbbk$-vector space, otherwise it has dimension the order of $Ker\,\partial$.

Lemma 90 *With this grading structure, L is a crossed P-algebra.*

Proof: It is fairly obvious that

- L is P-graded: this follows since $e_c \cdot e_{c'} = e_{cc'}$, ∂ is a group homomorphism and $e_1 \in L_1$.
- There is an inner product:
$$\rho : L \otimes L \to \Bbbk,$$
given by
$$\rho(e_c \otimes e_{c'}) = \begin{cases} 0 & \text{if } c^{-1} \neq c' \\ 1 & \text{otherwise} \end{cases}$$
which is clearly non-degenerate. Moreover
$$\rho(e_{c_1} e_{c_2} \otimes e_{c_3}) = \rho(e_{c_1 c_2} \otimes e_{c_3}) = 0$$
unless $c_3 = c_2^{-1} c_1^{-1}$ when it is 1, whilst
$$\rho(e_{c_1} \otimes e_{c_2 c_3}) = 0$$
unless $c_1^{-1} = c_2 c_3$, etc., so the inner product satisfies the third condition for a Frobenius P-algebra.
- Finally, there is a group homomorphism,
$$\varphi : P \to Aut(L),$$
given by $\varphi_g(e_c) = e_{{}^g c}$, which permutes the basis, compatibly with the multiplication and innerproduct structures.

As $\partial({}^g c) = g \cdot \partial c \cdot g^{-1}$, φ clearly satisfies $\varphi_g(L_h) \subseteq L_{ghg^{-1}}$, and the Peiffer identity implies ${}^{\partial c}c = c$, so $\varphi_g|L_g$ is the identity. The Peiffer identity in general gives
$${}^{\partial c}c' = cc'c^{-1},$$
so $e_c e_{c'} = e_{{}^{\partial c}c'} e_c$, i.e., $\varphi_h(a)b = ba$ if $a \in L_g, b \in L_h$. ■

We want this to be a crossed C-algebra, so the remaining structure that we have to specify is the 'tildefication',
$$\tilde{} : C \to \Bbbk C.$$

The obvious mapping gives $\tilde{c} = e_c$, and, of course,

$$\delta(\tilde{c})(e_{c'}) = e_c e_{c'} e_{c^{-1}} = e_{cc'c^{-1}} = \varphi_{\partial c}(e_{c'}),$$

as above. We thus have

Proposition 126 *With the above structure, $\Bbbk(C)$ is a crossed C-algebra.* ■

By its construction, $\Bbbk(C)$ records little of the structure of P itself, only the way the P-action permutes the elements of C, but, of course, it records C faithfully. The next example give the other extreme.

The group algebra, $\Bbbk(P)$, as a crossed C-algebra: Recall

Lemma 91 *The group algebra, $\Bbbk(P)$, has the structure of a crossed P-algebra with $(\Bbbk(P))_p = \Bbbk e_p$, the subspace generated by a basis element labelled by $p \in P$.* ■

The one thing to note is that the axiom

$$\varphi_h(a)b = ba$$

for any $g, h \in P$, $a \in L_g, b \in L_h$ implies that

$$\varphi_h(e_g) = e_h e_g e_{h^{-1}} = e_{hgh^{-1}},$$

since e_h is a unit of $\Bbbk(P)$ with inverse $e_{h^{-1}}$.

Proposition 127
For $c \in C$, defining $\tilde{c} = e_{\partial c}$, gives $\Bbbk(P)$ the additional structure of a crossed C-algebra.

Proof: The grading is as expected and $\delta(\tilde{c}) = \varphi_{\partial c}$, by construction. ■

Of course, $\Bbbk(P)$ does not encode anything about the kernel of $\partial : C \to P$. In fact, it basically remains a crossed P-algebra as the extra crossed C-structure is derived from that underlying algebra.

We will give further examples of crossed C-algebras shortly.

11.7.2 Morphisms of crossed algebras, for a crossed module background

We clearly need to have a notion of morphism of crossed C-algebras, extending that for crossed G-algebra. the form of the definition is, therefore, designed to mirror the earlier ones. We start with a fixed crossed module $\mathsf{C} = (C, P, \partial)$.

Definition: Suppose L and L' are two crossed C-algebras. A $\Bbbk$-algebra morphism $\theta : L \to L'$ is a *morphism of crossed C-algebras* if it is compatible with the extra structure. Explicitly:

$$\begin{aligned} \theta(L_p) &\subseteq L'_p \\ \rho'(\theta a, \theta b) &= \rho(a, b), \\ \varphi'_h(\theta a) &= \theta(\varphi_h(a)), \\ \theta(\tilde{c}) &= \tilde{c} \end{aligned}$$

for all $a, b \in L$, $h \in P$, $c \in C$, where, when necessary, primes indicate the structure in L'.

We denote by $Crossed.\mathsf{C}-Alg$, the resulting category.

The version of morphism with 'change of base' is slightly more complicated, but in an obvious and 'handleable' way. First note that, if $f:\mathsf{C}\to\mathsf{D}$ is a morphism of crossed modules, the morphism, f, gives a commutative square of group homomorphisms,

$$\begin{tikzcd} C \arrow[r, "f_1"] \arrow[d, "\partial"'] & D \arrow[d, "\partial'"] \\ P \arrow[r, "f_0"'] & Q. \end{tikzcd}$$

We want to define a morphism of crossed algebras over f, i.e., an algebra morphism, $\theta : L \to L'$, where L is a crossed C-algebra and L', a crossed D-algebra.

Definition: Suppose L and L' are two crossed algebras over C and D, respectively. A $\Bbbk$-algebra morphism, $\theta : L \to L'$, is a *morphism of crossed algebras over* f if it is compatible with the extra structure. Explicitly:

$$\begin{aligned} \theta(L_p) &\subseteq L'_{f_0(p)} \\ \rho'(\theta a, \theta b) &= \rho(a,b), \\ \varphi'_{f_0(h)}(\theta a) &= \theta(\varphi_h(a)), \\ \theta(\tilde{c}) &= \widetilde{f_1(c)} \end{aligned}$$

for all $a,b\in L$, $h\in P$, $c\in C$, where, when necessary, primes indicate the structure in L'.

This is, thus, the same as for the simple case $f_0 : P\to Q$, except for the involvement of f_1 with the tilde operation.

11.7.3 Pulling back a crossed C-algebra

A morphism, as above, over f can be replaced by a morphism of crossed C-algebras, $L\to f_0^*(L')$, where $f_0^*(L')$ is obtained by pulling back L' along f. We will consider this construction independently of any particular θ.

If $P\to Q$ is a group homomorphism, we know that, given a crossed Q-algebra, L, we obtain a crossed P-algebra $f_0^*(L)$, by pulling back using f_0. If, in addition, we consider the crossed C-structure, assuming that L' is a crossed D-algebra, then we define $\tilde{c} := \widetilde{f_1(c)}_{\partial c}$ and this gives:

Proposition 128 *The crossed P-algebra $f_0^*(L)$ is a crossed* C*-algebra.* ■

It is clear that, as expected:

Proposition 129 *There is a bijection between the set of crossed algebra morphisms from L to L' over f and the set of crossed* C*-algebra morphisms from L to $f_0^*(L')$.* ■

This pullback construction gives a functor from the category of crossed D-algebras to that of crossed C-algebras (up to isomorphism in the usual way).

11.7.4 Applications of pulling back

There are now important uses for 'pulling back' that were not there when the question was jut of 1-types, i.e., of groups.

Consider our crossed module $\mathsf{C} = (C, P, \partial)$ and let $G = P/\partial C$. We can realise this as a morphism of crossed modules:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\partial"'] & 1 \arrow[d] \\ P \arrow[r, "q"'] & G. \end{tikzcd}$$

If ∂ was an inclusion then this would be a weak equivalence of crossed modules as then both the kernel and cokernels of the crossed modules would be mapped isomorphically by the induced maps. In that case, thinking back to our original motivations for introducing formal C-maps, we would really be in a situation corresponding to a HQFT with background a $K(G, 1)$ and we know such theories are classified by crossed G-algebras, thus, it is of interest to see what the pullback algebra of a crossed G-algebra along this morphism will be. We will look at the obvious example of $\Bbbk(G)$, the group algebra of G with its usual crossed G-algebra structure. We will assume that the crossed module, C, is finite.

Writing $N = \partial C$, for convenience, we have an extension

$$N \longrightarrow P \xrightarrow{q} G\ .$$

Pick a section s for q and define the corresponding cocycle, $f(g, h) = s(g)s(h)s(gh)^{-1}$, so that $f : G \times G \to N$ is naturally normalised, $f(1, h) = f(g, 1) = 1$ and satisfies the cocycle condition:

$$f(g, h)f(gh, k) = {}^{s(g)}f(h, k)f(g, hk).$$

Take $L = \Bbbk(G)$, the group algebra of G considered with its crossed G-algebra structure and form the crossed P-algebra, $q^*(L)$.

Proposition 130
The two crossed C-algebras, $\Bbbk(P)$ and $q^*(\Bbbk(G))$, are isomorphic.

Proof:
We first note that

$$q^*(L)_p = L_{q(p)} = \Bbbk e_{q(p)}.$$

We will write $g = q(p)$, so $p \in P$ has the form $p = ns(g)$. (We will need to keep check of which $e_{q(p)}$ is which and will later introduce notation which will handle this.)

Recall the description of the product in P in terms of the cocycle and the section:

$$\begin{aligned} n_1 s(g_1) \cdot n_2 s(g_2) &= n_1{}^{s(g_1)}n_2 s(g_1)s(g_2) \\ &= (n_1{}^{s(g_1)}n_2 f(g_1, g_2))s(g_1 g_2). \end{aligned}$$

Each unit e_g of $\Bbbk(G)$ gives $\#(N)$ copies in $q^*(L)$. Write $(e_g)_n$ for the copy of e_g in $q^*(L)_{ns(g)}$ and examine the multiplication in $q^*(L)$ in this notation:

$$(e_{g_1})_{n_1} \cdot (e_{g_2})_{n_2} = (e_{g_1 g_2})_{(n_1{}^{s(g_1)}n_2 f(g_1, g_2))}.$$

(That this gives an associative multiplication corresponds to the cocycle condition.)

We next have to ask: what is φ_p? Of course, as $p = ns(g)$, we can restrict to examining φ_n and $\varphi_{s(g)}$.

- φ_n links the two copies $q^*(L)_p$ and $q^*(L)_{npn^{-1}}$ of $L_{q(p)}$ via what is essentially the identity map between the two copies;
- $\varphi_{s(g)}$ restricts to $\varphi_{s(g)} : q^*(L)_p \to q^*(L)_{s(g)ps(g)^{-1}}$, but on identifying these two subspaces as $L_{q(p)}$ and $L_{gq(p)g^{-1}}$, this is just φ_g.

In fact, we can be more explicit if we look at the basic units and, as these do form a basis, behaviour on them determines the automorphisms:

$$\varphi_n((e_{g_1})_{n_1})(e_1)_n = (e_1)_n(e_{g_1})_{n_1},$$

so

$$\begin{aligned}\varphi_n((e_{g_1})_{n_1}) &= (e_{g_1})_{nn_1}(e_1)_n^{-1}\\ &= (e_{g_1})_{nn_1}(e_1)_{n^{-1}}\\ &= (e_{g_1})_{nn_1{}^{s(g_1)}n^{-1}}\end{aligned}$$

that is, conjugation by $(e_1)_n$.

This leads naturally on to noting that $\tilde{c} = (e_1)_{\partial c}$, so we have explicitly given the crossed C-algebra structure on $q^*(L)$. Sending e_p to $(e_{q(p)})_n$ (using the same notation as before) establishes the isomorphism of the statement without difficulty. ■

Remark: In this identification of $q^*(\Bbbk(G))$ as $\Bbbk(P)$, it is worth noting that

$$q^*(L)_1 = L_1 = \Bbbk e_1 \cong \Bbbk,$$

as a vector space, but also that $q^*(L)_n \cong \Bbbk$ for each $n \in N$. The notation $(e_g)_n$ used and the behaviour of these basis elements suggests that $q^*(\Bbbk(P))$ behaves like some sort of twisted tensor product with basis $e_n \otimes e_g$, with that element corresponding to $(e_g)_n$, and with multiplication

$$(e_{n_1} \otimes e_{g_1})(e_{n_2} \otimes e_{g_2}) = (e_{n_1{}^{s(g_1)}n_2 f(g_1,g_2)} \otimes e_{g_1g_2}).$$

11.7.5 Pushing forward

We have shown that, given $f : \mathsf{C} \to \mathsf{D}$ and a $\theta : L \to L'$ over f, we can pull L' back over C to get a map from L to $f^*(L')$ that encodes the same information as L' (provided f is an epimorphism and all crossed modules are finite). The obvious question to ask is whether the adjoint that we saw at the P-algebra level is also a crossed C-algebra, i.e., does f^* have an adjoint and is it $(f_0)_*$, with some extra structure defined on it? This is what we turn to next keeping the same assumptions of finiteness, etc., that we used earlier, (see section 11.3.4).

We will repeat, briefly, some of the construction from that earlier section. Of course, you can look back there for more discussion, if need be.

Given such a morphism, $f : \mathsf{C} \to \mathsf{D}$, and setting $N = Ker f_0$, $B = Ker f_1$, we have

$$\varphi_n(a) - a \in Ker\theta.$$

Similarly, since

$$\theta(\tilde{c}) = \widetilde{f_1(c)},$$

if $b \in B = Ker f_1$,

$$\theta(\tilde{b}) = \tilde{1},$$

so

$$\tilde{b} - 1 \in Ker\theta.$$

We form the ideal K generated by elements of these forms. We have that L/K is an associative algebra and we give it a Q-graded algebra structure as follows.

For each $q \in Q$, let

$$\mathsf{L}_q = \oplus_p \{L_p \mid f_0(p) = q\},$$

and

$$\mathsf{K}_q = \mathsf{L}_q \cap K.$$

The underlying Q-graded vector space of $f^*(L)$ is

$$f^*(L) = \oplus_{q \in Q} \mathsf{L}_q / \mathsf{K}_q.$$

This grading is compatible with that multiplication.

We next define the bilinear form giving the inner product. Clearly, with the same notation,

$$\rho(a + K, b + K) := 0 \quad \text{ if } q_1 \neq q_2^{-1}.$$

If $q_1 = q_2^{-1}$, then we can assume that $p_1 = p_2^{-1}$, and we set

$$\rho(a + K, b + K) := \rho(a, b).$$

This is easily seen to be independent of the choices of a and b and ρ thus defined is a symmetric bilinear form and restricting to $f^*(L)_{q_1} \otimes f^*(L)_{q_2}$, it is essentially the original inner product restricted to $L_{p_1} \otimes L_{p_2}$, so is non-degenerate and satisfies

$$\rho(ab + K, c + K) = \rho(a + K, bc + K).$$

The crossed Q-algebra action

$$\varphi : Q \to Aut(f^*(L))$$

is given by

$$\varphi_q(a + K) := \varphi_p(a) + K$$

where $f_0(p) = q$.

Proposition 131 *With the above structure, $f^*(L)$ is a crossed D-algebra.*

Proof: The above construction and our earlier results (in section 11.3.4) show it is a crossed Q-algebra, so we only have to define the tilde. The obvious definition is

$$\tilde{d} := \tilde{c} + K,$$

where $f_1(c) = d$. This works. It is well defined as each $\tilde{b} - 1$ is in K, and the equation

$$\varphi_q(\tilde{d}) = \widetilde{{}^q d}$$

follows from the corresponding one in L. ■

Proposition 132 *There is a natural bijection between the set of crossed algebra morphisms from L to L' over f and the set of crossed D-algebra morphisms from $f^*(L)$ to L'.* ■

The proof is obvious given our construction of $f^*(L)$. We note that this, with its companion result on pulling back, give a pairs of adjoint functors determined by $f : \mathsf{C} \to \mathsf{D}$ between the categories of crossed C-algebras and crossed D-algebras.

Bibliography

[1] H. Abels and S. Holz, *Higher generation by subgroups*, J. Alg, 160, (1993), 311– 341. 89, 91, 100, 101, 109, 116, 117, 118

[2] L. Abrams, *Two dimensional topological quantum field theories and Frobenius algebras*, J. Knot Theory and its Ramifications, 5, (1996), 569 –587. 331

[3] E. Aldrovandi, 2005, *2-Gerbes bound by complexes of gr-stacks, and cohomology*, URL http://www.arXiv.org:math/0512453. 212

[4] J. W. Alexander, *The combinatorial theory of complexes*, Annals of Math, 31, (1930), 294 –322. 337, 342

[5] L. Anderson and N. Mnev, *Triangulations of Manifolds and Combinatorial Bundle Theory: an Announcement*, Journal of Mathematical Sciences, 113, (2000), 755 – 758. 342, 347

[6] M. André, 1967, *Méthode simpliciale en algèbre homologique et algèbre commutative*, number 32 in Lecture Notes in Maths, Springer-Verlag, Berlin. 56, 57, 58, 59, 60

[7] M. André, 1970, *Homology of simplicial objects*, in *Proc. Symp. on Categorical Algebra*, 15 – 36, American Math. Soc. 56, 57

[8] M. Artin, A. Grothendieck and J. L. Verdier, 1973, *Théorie des Topos et Cohomologie Etale des Schémas*, volume 269, 270, 305, of *Springer Lecture Notes in Mathematics*, Springer, available from: http://www.math.mcgill.ca/∼archibal/SGA/SGA.html. 56, 162

[9] M. Artin and B. Mazur, *On the van Kampen Theorem*, Topology, 5, (1966), 179 – 189. 198

[10] M. Artin and B. Mazur, 1969, *Étale homotopy*, number 100 in Lecture Notes in Maths., Springer-Verlag, Berlin. 162, 179

[11] N. Ashley, *Simplicial T-Complexes: a non abelian version of a theorem of Dold-Kan*, Dissertations Math., 165, (1989), 11 – 58. 215, 218, 220

[12] J. C. Baez and A. S. Crans, *Higher-Dimensional Algebra VI: 2-Lie Algebras*, Theory and Applications of Categories, 12, (2004), 492 – 528. 310, 311

[13] J. C. Baez and J. Dolan, 1998, *Categorification*, URL http://www.arXiv.org:math/9802029. 25

[14] A. Bak, *Global Actions: The algebraic counterpart of a topological space*, Uspeki Mat. Nauk., English Translation: Russian Math. Surveys, 525, (1997), 955 –996. 93, 113

[15] A. Bak, 1998, *Topological Methods in Algebra*, in S. Caenepeel and A. Verschoren, eds., *Rings, Hopf Algebras and Brauer Groups*, number 197 in Lect. Notes in Pure and Applied Math, M. Dekker, New York. 93, 113

[16] A. Bak, R. Brown, G. Minian and T. Porter, *Global Actions, Groupoid Atlases and Application*, Journal of Homotopy and Related Structures, 1, (2006), 101 – 167. 91, 93

[17] M. Barr and J. Beck, 1969, *Homology and Standard Constructions*, in *Seminar on triples and categorical homology*, number 80 in Lecture Notes in Maths., Springer-Verlag, Berlin. 55, 56

[18] M. G. Barratt, V. Gugenheim and J. Moore, *On semisimplicial fibre-bundles*, American Journal of Mathematics, 81, (1959), 639–657. 245

[19] J. Barrett, *Holonomy and path structures in General Relativity and Yang-Mills theory*, Int. J. Theor. Phys., 30, (1991), 1171 – 1215. 364

[20] H. Bass, *K-Theory and stable algebra*, Publications Mathématiques de l'IHÉS, 22, (1964), 5 – 60. 82

[21] H. Bass, *Covering theory for graphs of groups*, Journal of Pure and Applied Algebra, 89, (1993), 3–47. 127, 132

[22] H. J. Baues, 1989, *Algebraic Homotopy*, volume 15 of *Cambridge Studies in Advanced Mathematics*, Cambridge Univ. Press. 33, 441

[23] H. J. Baues, 1991, *Combinatorial homotopy and 4-dimensional complexes*, Walter de Gruyter. 202, 441

[24] H. J. Baues, 1995, *Homotopy Types*, in I. M. James, ed., *Handbook of Algebraic Topology*, 1 – 72, Elsevier. 33, 156, 157

[25] H. Behr, *Explizite Präsentation von Chevalley-gruppen über Z*, Math. Z., 141, (1975), 235 – 241. 89

[26] T. Beke, *Higher Cech Theory*, K Theory, 32, (2004), 293 – 322, University of Michigan, preprint. 162, 167, 168

[27] G. Biedermann, *On the homotopy theory of n-types*, Homology, Homotopy Appl., 10, (2008), 305 – 325. 173

[28] V. Blanco, M. Bullejos and E. Faro, *Categorical non-abelian cohomology and the Schreier theory of groupoids*, Math. Z., 251, (2005), 41 – 59. 210

[29] F. Borceux and G. Janelidze, 2001, *Galois theories*, volume 72 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press. 244, 258, 260, 262, 264, 267

[30] G. Bredon, 1972, *Introduction to compact transformation groups*, Pure and applied mathematics, Academic Press, URL `http://books.google.co.uk/books?id=qXEl_AIeSBUC`. 127

[31] L. Breen, 1990, *Bitorseurs et cohomologie non abélienne*, in *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, 401 – 476, Birkhäuser Boston, Boston, MA. 212, 267, 274, 282, 289, 301, 315

[32] L. Breen, *Théorie de Schreier supérieure*, Ann. Sci. École Norm. Sup. (4), 25, (1992), 465 – 514. 267, 300

[33] L. Breen, *On the classification of* 2*-gerbes and* 2*-stacks*, Astérisque, 225, (1994), 160, ISSN 0303-1179. 212, 267, 275

[34] L. Breen, 2009, *Notes on 1-and 2-gerbes*, in J. Baez and J. May, eds., *Towards Higher Categories*, volume 152 of *The IMA Volumes in Mathematics and its Applications*, 193–235, Springer. 267, 282

[35] S. G. Brick and J. M. Corson, *Dehn functions and complexes of groups*, Glasgow Mathematical Journal, 40, (1998), 33–46. 106

[36] M. Bridson and A. Haefliger, 1999, *Metric Spaces of Non-Positive Curvature*, number 31 in Grundlehren der Math. Wiss, Springer. 101, 106, 117, 127, 128, 129, 141, 143, 144, 145, 147, 152

[37] M. Brightwell and P. Turner, *Representations of the homotopy surface category of a simply connected space*, J. Knot Theory and its Ramifications, 9, (2000), 855 – 864. 417, 422, 433, 449, 466, 468

[38] K. Brown, *Abstract Homotopy Theory and Generalized Sheaf Cohomology*, Trans. Amer. Math. Soc, 186, (1973), 419 – 458. 22

[39] K. Brown, 1982, *Cohomology of groups*, number 87 in Graduate Texts in Mathematics, Springer-Verlag. 137, 422, 429, 456, 458, 460, 464

[40] R. Brown, *From groups to groupoids: a brief survey*, Bull. London Math. Soc., 19, (1987), 113 –134. 344

[41] R. Brown, 1988, *Topology, a geometric account of general topology, homotopy types and the fundamental groupoid*, Ellis Horwood. 144, 481

[42] R. Brown, 2006, *Topology and Groupoids*, BookSurge, (This is an extended version of [41]). 114, 144, 148

[43] R. Brown, 2009, *Groupoids in Mathematics*, http://www.groupoids.org.uk/gpdsweb.html. 344

[44] R. Brown, K. A. Hardie, K. H. Kamps and T. Porter, *A Homotopy Double Groupoid of a Hausdorff Space*, Theory Appl. Categ, 10, (2002), 71–93. 364

[45] R. Brown and P. J. Higgins, 1982, *Crossed complexes and non-abelian extensions*, in *Int. Conf. Cat. Theory, Gummersbach*, volume 962 of *Lecture Notes in Maths*, Springer-Verlag. 35

[46] R. Brown and P. J. Higgins, *Tensor products and homotopies for ω-groupoids and crossed complexes,*, J. Pure Appl. Alg., 47, (1987), 1 – 33. 33, 441

[47] R. Brown and P. J. Higgins, *Crossed complexes and chain complexes with operators*, Math. Proc. Camb. Phil. Soc., 107, (1990), 33 – 57. 33, 36, 49

[48] R. Brown and P. J. Higgins, *The classifying space of a crossed complex*, Math. Proc. Cambridge Philos. Soc., 110, (1991), 95 – 120. 214, 220, 418, 441

[49] R. Brown, P. J. Higgins and R. Sivera, 2010, *Nonabelian Algebraic Topology: Filtered spaces, crossed complexes, cubical homotopy groupoids*, volume 15 of *EMS Tracts in Mathematics*, European Mathematical Society. 16, 33, 34, 36, 115, 187, 364, 437, 441

[50] R. Brown and J. Huebschmann, 1982, *Identities among relations*, in R. Brown and T. L. Thickstun, eds., *Low Dimensional Topology*, London Math. Soc Lecture Notes, Cambridge University Press. 18, 20, 21, 48, 52, 53

[51] R. Brown and J.-L. Loday, *Homotopical excision, and Hurewicz theorems, for n-cubes of spaces*, Proc. London Math. Soc., (3)54, (1987), 176 – 192. 194, 201

[52] R. Brown and J.-L. Loday, *Van Kampen Theorems for diagrams of spaces*, Topology, 26, (1987), 311 – 337. 176, 194, 201

[53] R. Brown, E. J. Moore, T. Porter and C. D. Wensley, *Crossed complexes, and free crossed resolutions for amalgamated sums and HNN-extensions of groups*, Georgian Math. J., 9, (2002), 623 – 644, ISSN 1072-947X, dedicated to Professor Hvedri Inassaridze on the occasion of his 70th birthday. 115

[54] R. Brown and T. Porter, *On the Schreier theory of non-abelian extensions: generalisations and computations*, Proc. Roy. Irish Acad. Sect. A, 96, (1996), 213 – 227. 211

[55] R. Brown and C. Spencer, *G-groupoids, crossed modules and the fundamental groupoid of a topological group*, Proc. Kon. Ned. Akad. v. Wet, 79, (1976), 296 – 302. 24

[56] M. Bullejos and A. Cegarra, *On the Geometry of 2-Categories and their Classifying Spaces*, K-Theory, 29, (2003), 211 – 229. 354

[57] M. Bullejos, A. M. Cegarra. and J. Duskin, *On cat^n-groups and homotopy types*, J. Pure Appl. Alg., 86, (1993), 135 – 154. 198

[58] U. Bunke, P. Turner and S. Willerton, *Gerbes and homotopy quantum field theories*, Algebr. Geom. Topol., 4, (2004), 407 – 437. 417

[59] S. Buoncristiano, 2003, *Fragments of geometric topology from the sixties*, http://www.emis.de/journals/GT//gtmcontents6.html. 233, 385, 410, 415

[60] S. Buoncristiano and C. Rourke, 2003, *Fragments of geometric topology from the sixties*, volume 6 of *Geometry & Topology Monographs*, Mathematical Sciences Publishers, University of California, Berkeley. 233

[61] A. Caetano and R. Picken, *An axiomatic definition of holonomy*, Int. J. Math., 5, (1994), 835 – 848. 364

[62] P. Carrasco, 1987, *Complejos Hipercruzados, Cohomologia y Extensiones*, Ph.D. thesis, Universidad de Granada. 180, 356

[63] P. Carrasco and A. M. Cegarra, *Group-theoretic Algebraic Models for Homotopy Types*, J. Pure Appl. Alg., 75, (1991), 195 – 235. 180, 215, 356

[64] A. Cegarra and J. Remedios, *The relationship between the diagonal and the bar constructions on a bisimplicial set*, Topology and its Applications, 153, (2005), 21 – 51. 198

[65] M. M. Cohen, 1973, *A course in simple-homotopy theory*, number 10 in Graduate Texts in Math., Springer-Verlag, URL http://books.google.co.uk/books?id=MFrvAAAAMAAJ. 80, 81

[66] D. Conduché, *Modules croisés généralisés de longueur 2*, J. Pure Appl. Alg., 34, (1984), 155 – 178. 179, 180, 214, 215

[67] D. Conduché, *Simplicial Crossed Modules and Mapping Cones*, Georgian Math. J., 10, (2003), 623 – 636. 185, 198

[68] J. M. Corson, *Complexes of Groups*, Proc. London Math. Soc., 65, (1992), 199–224. 106, 116, 127, 132, 134

[69] J. M. Corson, 1995, *Groups acting on Complexes and Complexes of Groups*, in R. Charney, M. Davis and M. Shapiro, eds., *Geometric Group Theory*, 79 – 97, de Gruyter, Berlin.

[70] J. M. Corson, *Howie Diagrams and Complexes of Groups*, Comm. Algebra, 23, (1995), 5221 – 5242. 116, 127

[71] J. M. Corson and B. S. Spieler, *The qsf property and group actions via complexes of spaces*, Mathematische Zeitschrift, 228, (1998), 589 – 603. 106, 117

[72] R. H. Crowell, *The derived module of a homomorphism*, Advances in Math., 5, (1971), 210 – 238. 38, 42, 44

[73] R. H. Crowell and R. H. Fox, 1977, *Introduction to Knot Theory*, number 57 in Graduate Texts, Springer. 45, 114

[74] E. Curtis, *Simplicial Homotopy Theory*, Advances in Math., 6, (1971), 107 – 209. 216, 232, 245, 253

[75] M. L. Curtis, 1984, *Matrix Groups*, Universitext, Springer-Verlag. 381

[76] R. Debremaeker, 1976, *Cohomologie met Waarden in een Gekruiste Groepenschoof op ein Situs*, Ph.D. thesis, Katholieke Universiteit te Leuven. 212

[77] R. Debremaeker, *Cohomologie à valeurs dans un faisceau de groupes croisés sur un site. I*, Acad. Roy. Belg. Bull. Cl. Sci. (5), 63, (1977), 758 – 764.

[78] R. Debremaeker, *Cohomologie à valeurs dans un faisceau de groupes croisés sur un site. II*, Acad. Roy. Belg. Bull. Cl. Sci. (5), 63, (1977), 765 – 772.

[79] R. Debremaeker, *Non abelian cohomology*, Bull. Soc. Math. Belg., 29, (1977), 57 – 72. 212

[80] P. Dedecker, *Les foncteurs* Ext_Π, ${H_\Pi}^2$ *et* ${H_\Pi}^2$ *non abéliens*, C. R. Acad. Sci. Paris, 258, (1964), 4891 – 4894. 211

[81] R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology*, Commun. Math. Phys., 129. 335

[82] C. H. Dowker, *Homology Groups of Relations*, Annals of Maths, 56, (1952), 84 – 95. 97, 99

[83] J. Duskin, 1975, *Simplicial methods and the interpretation of "triple" cohomology*, number 163 in Mem. Amer. Math. Soc., 3, Amer. Math. Soc. 56, 57, 162, 164, 179, 296, 318

[84] J. Duskin, 1979, *Higher-dimensional torsors and the cohomology of topoi: the abelian theory*, in *Applications of Sheaves*, number 753 in Lecture Notes in Maths., Springer-Verlag. 167

[85] J. Duskin, *An outline of a theory of higher dimensional descent*, Bull. de la Soc. Math. de Belgique, 41, (1989), 249 – 277. 296

[86] J. Duskin, *A simplicial matrix approach to higher dimensional category theory I: nerves of bicategories*, Theory and Applications of Categories, 9, (2002), 198 – 308, URL http://www.tac.mta.ca/tac/volumes/9/n10/9-10abs.html. 162, 173

[87] W. G. Dwyer and D. M. Kan, *Homotopy theory and simplicial groupoids*, Nederl. Akad. Wetensch. Indag. Math., 46, (1984), 379 – 385. 213

[88] P. Ehlers and T. Porter, *Varieties of simplicial groupoids I: Crossed complexes*, J. Pure Appl. Alg., 120, (1997), 221 – 233. 215

[89] P. Ehlers and T. Porter, *Erratum to "Varieties of simplicial groupoids I: Crossed complexes"*, J. Pure Appl. Alg., 134, (1999), 207 – 209. 215

[90] C. Ehresmann, 1995, *Les connexions infinitésimales dans un espace fibré différentiable*, in *Séminaire Bourbaki, Vol. 1*, Exp. No. 24, 153–168, Soc. Math. France, Paris. 380

[91] S. Eilenberg and S. M. Lane, *Cohomology theory in abstract groups, II*, Annals of Math., 48, (1957), 326 – 341. 62

[92] G. J. Ellis, *Crossed Squares and Combinatorial Homotopy*, Math. Z., 214, (1993), 93 – 110. 200

[93] G. J. Ellis, *Spaces with finitely many non-trivial homotopy groups all of which are finite*, Topology, 36, (1997), 501 – 504. 376

[94] G. J. Ellis and R. Steiner, *Higher dimensional crossed modules and the homotopy groups of $(n+1)$-ads*, J. Pure Appl. Alg., 46, (1987), 117–136. 194, 195, 198

[95] T. Everaert, M. Gran and T. Van der Linden, *Higher Hopf formulae for homology via Galois Theory*, Advances in Math., 217, (2008), 2231 – 2267. 193

[96] J. Faria Martins, *On the homotopy type and the fundamental crossed complex of the skeletal filtration of a CW-complex*, Homology, Homotopy Appl., 9, (2007), 295–329 (electronic), ISSN 1532-0073. 365

[97] J. Faria Martins, *The fundamental crossed module of the complement of a knotted surface*, Trans. Amer. math. Soc., 361, (2009), 4593–4360. 365

[98] J. Faria Martins and T. Porter, *On Yetter's Invariant and an Extension of the Dijkgraaf-Witten Invariant to Categorical Groups*, Theor. Appl. Categor., 18, (2007), 118–150. 351, 365

[99] M. E. Forrester-Barker, 2003, *Representations of crossed modules and cat^1-groups*, Ph.D. thesis, University of Wales Bangor, URL www.math.bangor.ac.uk/research/preprints/04/algtop04.html. 310, 311

[100] R. H. Fox, *Free differential Calculus, I: Derivation in the free group ring*, Ann. of Maths, 57, (1953), 547 – 560. 45

[101] D. S. Freed and F. Quinn, *Chern-Simons theory with finite gauge group*, Commun. Math. Phys., 156, (1993), 435 – 472. 426

[102] P. Gabriel and M. Zisman, 1967, *Calculus of Fractions and Homotopy Theory*, number 35 in Ergebnisse der Math. und ihrer Grenzgebeite, Springer. 148, 149, 150, 214

[103] N. D. Gilbert and T. Porter, 1996, *Knots and Surfaces*, Oxford University Press. 79

[104] P. Glenn, *Realization of cohomology classes in arbitrary exact categories*, J. Pure. Appl. Alg., 25, (1982), 33 – 105. 167, 318

[105] P. G. Goerss and J. F. Jardine, 1999, *Simplicial Homotopy Theory*, number 174 in Progress in Mathematics, Birkhauser. 173, 245

[106] M. Gran and V. Rossi, *Galois Theory and Double Central Extensions*, Homology, Homotopy and Applications, 6, (2004), 283 – 298. 193

[107] A. Grothendieck, 1983, *Pursuing Stacks*, manuscript, 600 + pages. 239

[108] A. Grothendieck, 2003, *Revêtements étales et groupe fondamental (SGA1)*, Documents Mathématiques (Paris), 3, Société Mathématique de France, Paris, séminaire de géométrie algébrique du Bois Marie 1960–61. Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; (50 #7129)]. 149, 243

[109] D. Guin-Walery and J.-L. Loday, 1981, *Obstructions à l'excision en K-théorie algèbrique*, in *Evanston Conference on Algebraic K-theory, 1980*, volume 854 of *Lecture Notes in Maths.*, 179 – 216, Springer. 176, 189

[110] A. Haefliger, 1991, *Complexes of Groups and Orbihedra*, in *Group Theory from a Geometric viewpoint*, 504 – 540, ICTP, Trieste, 26 March- 6 April 1990, World Scientific. 106, 116, 117, 118, 127, 128, 129

[111] A. Haefliger, *Extensions of Complexes of Groups*, Annales Inst.Fourier, 42, (1992), 275 – 311. 106, 116, 127, 129

[112] K. A. Hardie, K. H. Kamps and R. W. Kieboom, *A Homotopy 2-Groupoid of a Hausdorff Space*, Applied Categorical Structures, 8, (2000), 209–234. 364

[113] K. A. Hardie, K. H. Kamps and R. W. Kieboom, *A Homotopy Bigroupoid of a Topological Space*, Applied Categorical Structures, 9, (2001), 311–327. 364

[114] R. Hartshorne, 2000, *Algebraic Geometry*, number 52 in Graduate Texts in Mathematics, Springer. 404, 405

[115] A. Hatcher, 2002, *Algebraic topology*, Cambridge University Press. 158, 169, 170, 173, 426

[116] P. J. Higgins, *The fundamental groupoid of a graph of groups*, Journal of the London Mathematical Society, 2, (1976), 145–149. 133

[117] P. J. Higgins, 2005, *Categories and groupoids*, number 7 in Reprints in Theory and Applications of Categories, TAC, reprint of the 1971 original [*Notes on categories and groupoids*, Van Nostrand Reinhold, London] with a new preface by the author. 144, 145

[118] P. Hilton and U. Stammbach, 1971, *A course in Homological Algebra*, Graduate Texts in Mathematics, Springer Verlag. 70

[119] P. J. Hilton, 1953, *An Introduction to Homotopy Theory*, Cambridge University Press. 16

[120] P. Holm, *The microbundle representation theorem*, Acta mathematica, 117, (1967), 191 – 213. 392

[121] S. Holz, 1985, *Endliche Identifizier zwishen Relationen*, Ph.D. thesis, Univeristät Bielefeld. 116

[122] J. Huebschmann, *Crossed n-fold extensions of groups and cohomology*, Comm. Math. Helv., 55, (1980), 302 – 313. 35, 61

[123] D. Husemoller, 1975, *Fibre bundles*, volume 20 of *Graduate Texts in Mathematics*, Springer-Verlag, New York-Heidelberg, second edition. 243, 245, 387, 394, 395, 398, 399, 401, 402, 403

[124] D. Husemöller, M. Joachim, B. Jurco, S. Echterhoff, S. Fredenhagen, B. Krötz and M. Schottenloher, 2008, *Basic bundle theory and K-cohomology invariants*, number 726 in Lecture Notes in Physics, Springer Verlag. 394, 399, 401, 403

[125] K. Igusa, *The generalised Grassmann invariant*, preprint, (unpublished). 76

[126] K. Igusa, 1979, *The $Wh_3(\pi)$ obstruction for pseudo-isotopy*, Ph.D. thesis, Princeton University. 76

[127] L. Illusie, 1971, *Complexe cotangent et déformations I*, volume 239 of *Lecture Notes in Maths*, Springer-Verlag. 177, 178

[128] L. Illusie, 1972, *Complexe cotangent et déformations II*, volume 283 of *Lecture Notes in Maths*, Springer-Verlag. 177

[129] G. Janelidze, *What is a double central extension ? (the question was asked by Ronald Brown)*, Cahiers de Topologie et Géométrie Différentielle Catégoriques, 32, (1991), 191 – 201. 193

[130] B. Jurčo, 2005, *Crossed Module Bundle Gerbes; Classification, String Group and Differential Geometry*. 175, 300

[131] K. H. Kamps and T. Porter, 1997, *Abstract Homotopy and Simple Homotopy Theory*, World Scientific Publishing Co. Inc., River Edge, NJ. 174, 213, 292, 295

[132] M. Kapranov and M. Saito, 1999, *Hidden Stasheff polytopes in algebraic K-theory and in the space of Morse functions*, in *Higher homotopy structure in topology and mathematical physics (Poughkeepsie, N.Y. 1996)*, volume 227 of *Contemporary Mathematics*, 191–225, AMS. 75, 77, 78, 110

[133] M. M. Kapranov and V. A. Voevodsky, 1991, *2-categories and Zamolodchikov tetrahedra equations*, in *Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991)*, volume 56 of *Proc. Sympos. Pure Math.* 310

[134] M. Karoubi, 1978, *K-theory*, volume 226 of *Grundlehren der math. Wiss. N°*, Springer-Verlag Berlin. 245, 394, 398

[135] R. M. Kaufmann, *Orbifolding Frobenius algebras*, Int. J. Math., 14, (2003), 573–617. 463

[136] M. Kervaire, 1972, *Multiplicateurs de Schur et K-théorie*, in A. Haefliger and R. Narasimhan, eds., *Essays in Topology and Related Topics memoires dedies à Georges de Rham*, 212–225, Springer-Verlag. 111

[137] F. Keune, 1972, *Homotopical Algebra and Algebraic K-theory*, Ph.D. thesis, Universiteit van Amsterdam. 58, 60

[138] J. Kister, *Microbundles are bundles*, Ann. of Math, 80, (1964), 190–199. 393

[139] S. Kobayashi and K. Nomizu, 1996, *Foundations of differential geometry. Vol. II*, Wiley Classics Library, John Wiley & Sons Inc., New York, reprint of the 1969 original, A Wiley-Interscience Publication. 380

[140] J. Kock, 2003, *Frobenius Algebras and 2-D Topological Quantum Field Theories*, number 59 in London Mathematical Society Student Texts, Cambridge U.P., Cambridge. 321, 325, 330, 332, 453

[141] N. H. Kuiper and R. K. Lashof, *Microbundles and Bundles I. Elementary Theory*, Invent. Math., 1, (1966), 1 – 17. 233, 393

[142] N. H. Kuiper and R. K. Lashof, *Microbundles and Bundles II. Semisimplicial Theory*, Invent. Math., 1, (1966), 243 – 259. 233, 393

[143] J.-L. Loday, *Spaces with finitely many homotopy groups*, J. Pure Appl. Alg., 24, (1982), 179 – 202. 30, 33, 189, 190

[144] J.-L. Loday, 2000, *Homotopical Syzygies*, in *Une dégustation topologique: Homotopy theory in the Swiss Alps*, volume 265 of *Contemporary Mathematics*, 99 – 127, AMS. 20, 21, 49, 75, 76

[145] A. Lundell and S. Weingram, 1969, *The topology of CW complexes*, New York, Van Nostrand Reinhold Co. 434

[146] E. Lupercio and B. Uribe, *An introduction to gerbes on orbifolds*, Annales mathématiques Blaise Pascal, 11, (2004), 155 – 180. 263

[147] J. Lurie, Spring 2009, *Topics in Geometric Topology (18.937)*, notes for course 18.937. 385, 410, 415

[148] J. Lurie et al., *On the classification of topological field theories*, Current developments in mathematics, 2008, (2009), 129–280. 362, 433, 472

[149] S. Mac Lane, 1967, *Homology*, number 114 in Grundlehren, Springer. 22, 55, 56, 61, 70

[150] S. Mac Lane, 1978, *Categories for the Working Mathematician*, number 5 in Graduate Texts, Springer. 56, 57, 163

[151] S. Mac Lane, *Historical Note*, J. Alg., 60, (1979), 319 – 320, appendix to D. F. Holt, An interpretation of the cohomology groups, $H^n(G, M)$, J. Alg. 60 (1979) 307-318. 22

[152] S. Mac Lane and J. H. C. Whitehead, *On the 3-type of a complex*, Proc. Nat. Acad. Sci. U.S.A., 36, (1950), 41 – 48. 18, 65, 161

[153] M. Mackaay and R. Picken, *Holonomy and parallel transport for abelian gerbes,*, Adv. Math., 170, (2002), 287 – 339. 364

[154] W. S. Massey, 1967, *Algebraic Topology, an introduction*, Harcourt, Brace & World. 114

[155] J. P. May, 1967, *Simplicial objects in algebraic topology*, number 11 in Math. Studies, van Nostrand, Princeton. 223, 232, 245, 251, 253

[156] J. S. Milne, 2003, *Gerbes and abelian motives*, URL `http://www.arXiv.org:math/0301304`. 212

[157] J. W. Milnor, 1961, *Microbundles and Differentiable Structures*, (mimeographed notes),. Princeton Univ., Princeton, N. J. 385, 407, 408, 409, 410

[158] J. W. Milnor, *Microbundles. I*, Topology, 3, (1964), 53 – 80. 385, 407, 408, 409

[159] J. W. Milnor, *Whitehead torsion*, Bull. Amer. Math. Soc., 72, (1966), 358 – 426. 80, 81

[160] J. W. Milnor, 1971, *Introduction to Algebraic K-theory*, Annals of Math. Studies, Princeton University Press. 15

[161] I. Moerdijk, 2002, *Introduction to the Language of Stacks and Gerbes*, URL `http://arxiv.org/abs/math/0212266`. 256

[162] E. J. Moore, 2001, *Graphs of groups: word computations and free crossed resolutions.*, Ph.D. thesis, University of Wales Bangor. 133, 136

[163] G. W. Moore and G. Segal, 2006, *D-branes and K-theory in 2D topological field theory*, URL `http://arxiv.org/abs/hep-th/0609042`. 450, 454, 463

[164] J. C. Moore, 1956, *Seminar in Algebraic Homotopy*, Princeton. 58

[165] M. Murray and M. A. Singer, *Gerbes, Clifford Modules and the Index Theorem*, Annals of Global Analysis and Geometry, 26, (2004), 355–367. 384

[166] A. Mutlu and T. Porter, *Applications of Peiffer pairings in the Moore complex of a simplicial group*, Theory and Applications of Categories, 4, (1998), 148 – 173. 180

[167] A. Mutlu and T. Porter, *Freeness Conditions for 2-Crossed Modules and Complexes*, Theory and Applications of Categories, 4, (1998), 174 – 194.

[168] A. Mutlu and T. Porter, *Free crossed resolutions from simplicial resolutions with given CW-basis*, Cahiers Top. Géom. Diff. catégoriques, 50, (1999), 261 – 283. 58

[169] A. Mutlu and T. Porter, *Freeness Conditions for Crossed Squares and Squared Complexes*, K-Theory, 20, (2000), 345 – 368.

[170] A. Mutlu and T. Porter, *Iterated Peiffer pairings in the Moore complex of a simplicial group*, Applied Categorical Structures, 9, (2001), 111 – 130. 180

[171] G. Nan Tie, *A Dold-Kan theorem for crossed complexes*, J. Pure Appl. Alg., 56, (1989.), 177 – 194. 220

[172] G. Nan Tie, *Iterated W and T-groupoids*, J. Pure Appl. Alg., 56, (1989), 195 – 209. 220

[173] nLab team, *nLab*, URL `http://ncatlab.org/nlab/show/HomePage`. 3, 162, 163, 173, 174, 424, 433

[174] B. Noohi, 2005, *On weak maps between 2-groups*, URL `https://arxiv.org/abs/math/0506313`. 309

[175] B. Noohi, *Notes on 2-groupoids, 2-groups and crossed-modules*, Homotopy, Homology, and Applications, 9, (2007), 75 – 106, URL `www.intlpress.com/HHA/v9/n1/contv9n1.pdf`. 309

[176] S. Paoli, *Weakly globular cat^n-groups and Tamsamani's model*, Advances in Mathematics, 222, (2009), 621–727. 200

[177] T. Porter, $>$ 2019, *Profinite Algebraic Homotopy*, (uncertain as yet as to publisher). 41, 51, 149, 187, 194, 198, 200, 201, 205, 218

[178] T. Porter, *n-types of simplicial groups and crossed n-cubes*, Topology, 32, (1993), 5 – 24. 198, 199

[179] T. Porter, *Interpretations of Yetter's notion of G-coloring : simplicial fibre bundles and non-abelian cohomology*, J. Knot Theory and its Ramifications, 5, (1996), 687 – 720. 335, 364

[180] T. Porter, *TQFTs from Homotopy n-types*, J. London Math. Soc., 58, (1998), 723 – 732. 335, 364, 376, 378

[181] T. Porter, 2004, *$\mathcal{S}$-categories, $\mathcal{S}$-groupoids, Segal categories and quasicategories*, URL `http://arxiv.org/abs/math/0401274`. 213

[182] T. Porter, *Formal Homotopy Quantum Field Theories, II: Simplicial Formal maps*, Contemporary Math., 431, (2007), 375 – 403. 434

[183] T. Porter and V. Turaev, *Formal Homotopy Quantum Field Theories, I: Formal Maps and Crossed $\mathcal{C}$-algebras.*, Journal of Homotopy and Related Structures, 3, (2008), 113 –159. 417, 434, 438

[184] V. Prasolov, 2007, *Elements of homology theory*, volume 81 of *Graduate Studies in Mathematics*, Amer Mathematical Society. 127

[185] D. Quillen, 1973, *Higher algebraic K-theory, I*, in *Higher K-theories.*, volume 341 of *Lecture Notes in Math.*, 85 – 147. 101

[186] D. G. Quillen, 1967, *Homotopical algebra*, number 53 in Lecture Notes in Maths., Springer-Verlag. 56

[187] D. G. Quillen, 1970, *On the (co-)homology of commutative rings*, in *Proc. Symp. on Categorical Algebra*, 65 – 87, American Math. Soc. 56

[188] F. Quinn, 1995, *Lectures on axiomatic topological quantum field theory*, in D. Freed and K. Uhlenbeck, eds., *Geometry and Quantum Field Theory*, volume 1 of *IAS/Park City Mathematics Series*, AMS/IAS,. 321, 325, 332, 333, 453

[189] K. Reidemeister, 1950, *Einführung die kombinatorische Topologie*, Braunschweig, Berlin (1932); reprint Chelsea, New York. 344

[190] D. Rodelo and T. van der Linden, 2008, *The Third Cohomology group classifies double central extensions*, Pré-Publicacões do Departamento de Matematica 08-55, Universidade de Coimbra. 193

[191] G. Rodrigues, *Homotopy Quantum Field Theories and the Homotopy Cobordism Category in Dimension 1 + 1*, J. Knot Theory and its Ramifications, 12, (2003), 287 – 317. 331, 417, 418, 419, 421, 422, 431, 433, 466, 468

[192] C. Schommer-Pries, *Central extensions of smooth 2-groups and a finite dimensional string 2-group*, Available as arXiv 0911.2483v3. 175

[193] P. Scott and T. Wall, 1979, *Topological methods in group theory*, in *Homological group theory: proceedings of a symposium, held at Durham in September 1977, on" Homological and combinatorial techniques in group theory"*, volume 36 of *London Mathematical Society Lecture Note Series*, 137, Cambridge Univ Press. 134, 136

[194] J.-P. Serre, 1977, *Arbres, amalgames, SL_2*, volume 46 of *Astérisque*, Société mathématique de France. 106, 120, 127, 132

[195] J.-P. Serre, 2003, *Trees*, Springer Monographs in Mathematics, Springer-Verlag Berlin. 106, 120, 127, 132

[196] Y. Song, *The equivalence of two algebraic K-theories*, Kangweon-Kyungki Math. Jour., 5, (1997), 107 – 112. 109

[197] C. Soulé, *Groupes opérants sur un complexe simplicial avec domain fondamental*, C.R. Acad. Sci. Paris, Sér A 276, (1973), 607–609. 89

[198] E. H. Spanier, 1966, *Algebraic Topology*, McGraw Hill. 107, 264, 344, 347, 350, 357, 426

[199] N. Steenrod, 1974, *The topology of fibre bundles*, Princeton Univ. Press, Princeton, ninth edition. 243

[200] J. Stillwell, 1980, *Classical Topology and Combinatorial Group Theory*, number 72 in Graduate Texts in Mathematics, Springer Verlag. 347

[201] R. Street, *The algebra of oriented simplexes*, J. Pure Appl. Algebra, 49, (1987), 283 – 335. 364

[202] A. A. Suslin and M. Wodzicki, *Excision in algebraic K-theory*, The Annals of Mathematics, 136, (1992), 51 – 122. 94, 96, 109, 121

[203] Z. Tamsamani, *Sur des notions de n-catégorie et n-groupoïde non-strictes via des ensembles multi-simpliciaux*, K. Theory, 16, (1999), 51 – 99. 200

[204] H. Tietze, *Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten*, Monatshefte für Mathematik, 19, (1908), 1–118. 79

[205] J. Tits, 1974, *Buildings of spherical type and finite BN-pairs*, number 386 in Lecture Notes in Mathematics, Springer. 89

[206] J. Tits, 1981, *A local approach to buildings*, in *The geometric vein. The Coxeter Festshrift*, Springer-Verlag Berlin.

[207] J. Tits, *Buildings and group amalgamations*, Proceedings of groups: St. Andrew's 1985, 121, (1986), 110. 89

[208] A. P. Tonks, 1993, *Theory and applications of crossed complexes*, Ph.D. thesis, University of Wales, Bangor. 73

[209] V. G. Turaev, 1999, *Homotopy field theory in dimension 2 and group-algebras*, arXiv.org:math/9910010. 417, 418, 419, 421, 422, 425, 426, 427, 428, 434, 443, 448, 449, 451, 453, 454

[210] V. G. Turaev, 2000, *Homotopy field theory in dimension 3 and crossed group-categories*, arXiv.org:math/0005291. 417, 418

[211] V. G. Turaev, 2010, *Homotopy Quantum Field Theory (with Appendices by Michael Muger and Alexis Virelizier)*, number 10 in Tracts in Mathematics, European Math. Society. 417, 418, 422, 425, 426, 428, 429, 443, 451, 453, 454, 455, 463, 464, 466

[212] D. R. Verity, 2005, *Complicial Sets*, URL `http://arxiv.org/abs/math.CT/0410412`. 364

[213] D. R. Verity, 2006, *Weak complicial sets I: basic homotopy theory*, available from : arXiv.org:math/0604414. 364

[214] D. R. Verity, 2006, *Weak complicial sets. III. Enriched and internal quasi-category theory*, (in preparation).

[215] D. R. Verity, 2007, *Weak complicial sets. II. Nerves of complicial Gray-categories*, in *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, 441 – 467, Amer. Math. Soc., Providence, RI. 364

[216] I. Volodin, *Algebraic K-theory as extraordinary homology theory on the category of associative rings with unity*, Izv. Akad. Nauk. SSSR, 35, (Translation: Math. USSR Izvestija Vol. 5 (1971) No. 4, 859-887). 89, 93, 111, 113

[217] J. Wagoner, 1973, *Buildings, stratifications, and higher K-theory*, in H. Bass, ed., *Higher K-Theories*, volume 341 of *Lecture Notes in Mathematics*, 148–165, Springer Berlin / Heidelberg, URL http://dx.doi.org/10.1007/BFb0067054, 10.1007/BFb0067054. 93

[218] C. A. Weibel, 1995, *An Introduction to Homological Algebra*, number 38 in Cambridge Studies in Advanced Mathematics, Cambridge University Press. 70

[219] C. A. Weibel, 1999, *The development of algebraic K-theory before 1980*, in *Algebra, K-theory, groups, and education (New York, 1997)*, volume 243 of *Contemp. Math.*, 211 – 238, Amer. Math. Soc., Providence, RI. 88

[220] J. H. C. Whitehead, *Combinatorial homotopy I*, Bull. Amer. Math. Soc, 55, (1949), 213–245. 79

[221] J. H. C. Whitehead, *Combinatorial Homotopy II*, Bull. Amer. Math. Soc., 55, (1949), 453–496. 33, 79, 441

[222] J. H. C. Whitehead, 1950, *Algebraic Homotopy Theory*, in *Proc. Int. Cong. of Mathematics, Harvard*, volume 2, 354 – 357. 79

[223] J. H. C. Whitehead, *Simple Homotopy Type*, Amer. J. Math., 72, (1950), 1–57. 80

[224] D. N. Yetter, *Topological Quantum Field Theories Associated to Finite Groups and Crossed G-Sets*, J. Knot Theory Ramifications, 1, (1992), 1 – 20. 335, 337, 343, 344, 346

[225] D. N. Yetter, *TQFT's from Homotopy* 2*-Types*, J. Knot Theory Ramifications, 2, (1993), 113 – 123. 335, 337, 341, 346, 350, 356, 358, 361, 367

Index