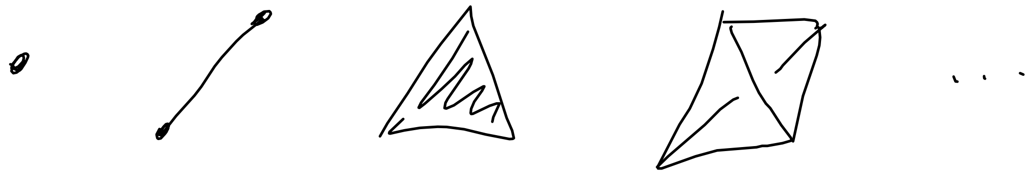
A *simplex* is an n-dimensional triangle/tetrahedron etc.



These have many incarnations:

1) topological

$$\Delta^n = \left\{ (x_0,\dots,x_n) \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{c} 0 \le x_i \le 1 \\ \sum_i x_i = 1 \end{array} \right\}$$

$$\cong \left\{ (x_1,\dots,x_n) \in \mathbb{R}^{n} \;\middle|\; \begin{array}{c} 0 \le x_i \le 1 \\ \sum x_i \le 1 \end{array} \right\}$$

$$\cong D^n$$

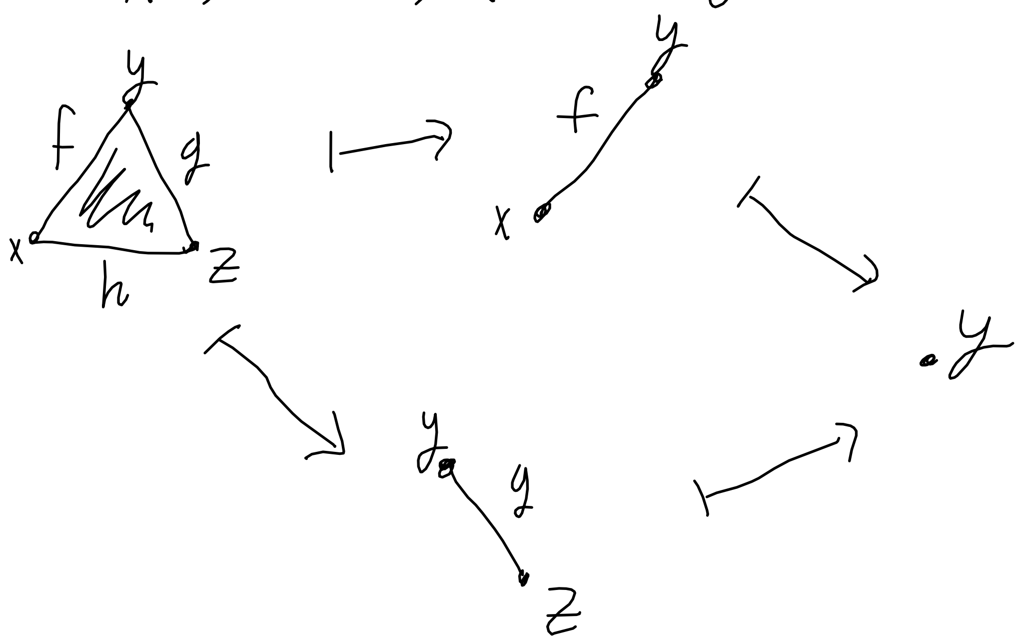
but Δ^n has a natural way to be glued together into cell cxs: the *faces* of any n-simplex are $(n-1)$-simplices.

A <u>semi-simplicial set</u> is a combinatorial presentation of a way to fit simplices together.

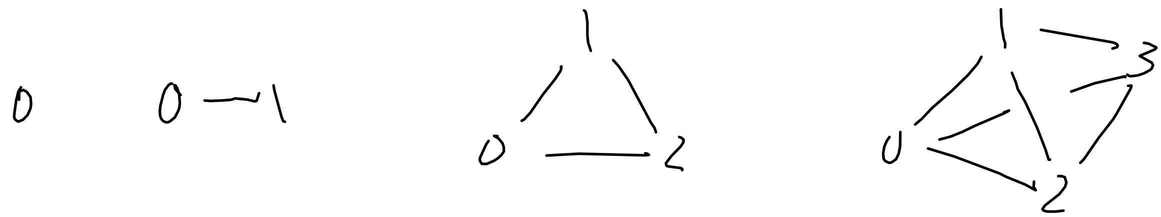
1) A set X_n of n-simplices $\forall n \in \mathbb{N}$

2) Face maps $d_k : X_{n+1} \longrightarrow X_n$ $\forall 0 \leq k \leq n+1$

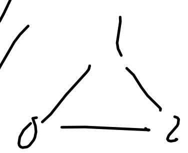
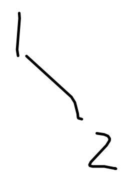
3) $d_k d_l = d_{l-1} d_k$ for $k < l$.



We number the <u>vertices</u> (= 0-simplices)



& d_k is the face that omits the k^{th} vertex. Maybe a little backwards at first

$d_0\Big($  $\Big) =$ 

$d_1 =$ 

$d_2 =$  .

Let Δ_m = the category presented by

- objects $[n]$ $\forall n \in \mathbb{N}$
- morphisms $\delta_k : [n] \to [n+1]$ $\forall 0 \leq k \leq n+1$
- equations $d_k d_\ell = d_{\ell-1} d_k$.

Then a sss is a functor $\Delta_m^{op} \to Set$.

$$sss = [\Delta_m^{op}, Set] = Set^{\Delta_m^{op}}.$$

Can think of $[n]$ as the "free-living n-simplex" and δ_k as the "inclusion".

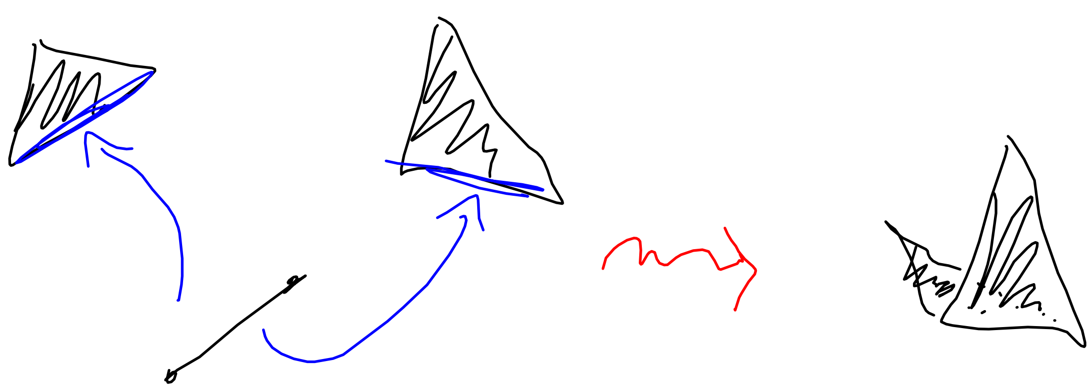
We can "realize" these simplices topologically, recall $\Delta^n \in Top$.

Have an actual inclusion $\delta_k : \Delta^n \hookrightarrow \Delta^{n+1}$

$\rightsquigarrow$ a functor $\Delta_m \to Top$.

A sss X then gives us a way to glue these topological simplices together. The <u>geometric realization</u>

$$|X| = \sum_n X_n \times \Delta^n \Big/ \begin{array}{c} x \in X_{n+1},\ p \in \Delta^n \\ (d_k x, p) = (x, \delta_k p) \end{array}.$$



Not every top space has a cell presentation w/ simplices, but we can ask for a "best approximation."

What are *all* the possible simplices in a space Y?

continuous maps $\Delta^n \to Y$.

$S_n Y = \{\Delta^n \to Y\}$ is a sss,

the *singular complex* of Y.

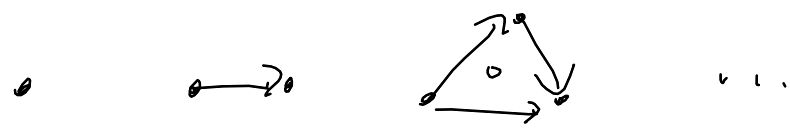
We have an adjunction

$$\begin{tikzcd} sss \arrow[r, shift left, "|\cdot|"] \arrow[r, phantom, "\perp"] & Top \arrow[l, shift left, "S_\cdot"] \end{tikzcd}$$

Much algebro-topological info is detected by $S_\cdot Y$.

We can do the same thing with Top replaced by any other category containing "concrete simplices".

eg Cat



$$SSS \underset{\text{Nerve}}{\overset{\tau_1}{\rightleftarrows}} Cat$$

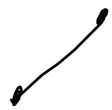
eg SSS itself!

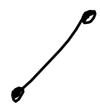
$(\Delta^k)_n = \Delta_m(n,k)$ the representable fn.

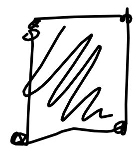
$$SSS \underset{Id}{\overset{Id}{\rightleftarrows}} SSS.$$

Density theorem – every presheaf is a colimit of representables.

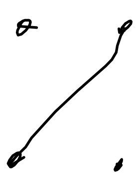
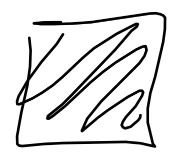
BUT These functors are not as good as we'd like.

eg the sSet Δ^1_{sSet} =  = $\left(\begin{array}{c} \vdots \\ 0 \\ \downarrow\downarrow\downarrow \\ 1 \\ \downarrow\downarrow \\ 2 \end{array}\right)$

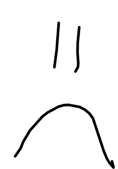
has $|\Delta^1_{sSet}| = \Delta^1_{top}$ = 

of course, $\Delta^1_{top} \times \Delta^1_{top}$ = 

but $\Delta^1_{sSet} \times \Delta^1_{sSet}$ = $\begin{array}{c} 0 \\ \downarrow\downarrow\downarrow \\ 1 \\ \downarrow\downarrow \\ 4 \end{array}$

 not much like 

ie $|-| : sSet \longrightarrow Top$ doesn't preserve products.



A <u>simplicial set</u> is a sss X w/

- <u>degeneracy maps</u> $s_k : X_n \to X_{n+1}$, $0 \le k \le n$
- $d_k s_\ell = s_{\ell-1} d_k \quad k < \ell$

  $d_k s_k = d_{k+1} s_k$

  $d_k s_\ell = s_\ell d_{k-1} \quad k > \ell + 1$

  $s_k s_\ell = s_{\ell+1} s_k \quad k \le \ell$

Not supposed to understand those deeply.

$\Delta = \Delta_m$ w/ σ_k's.

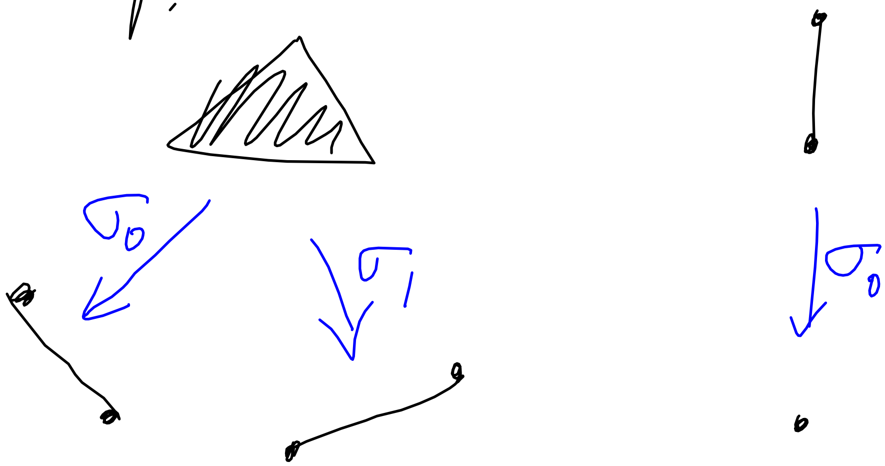
$sSet = [\Delta^{op}, Set]$.

(Cute fact: $\Delta \simeq$ finite nonempty linearly ordered sets)

Idea: $\sigma_k : [n+1] \longrightarrow [n]$ "squashes" an $(n+1)$-simplex down to an n-simplex

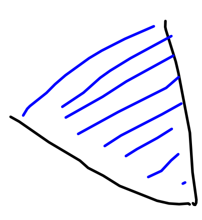
so $s_k : X_n \to X_{n+1}$ assigns squashed or degenerate simpl.

eg in Top.



so $s_0(x) =$ constant path at x
$s_0 Y$
$\in$

for $f : \Delta^1_{top} = [0,1] \longrightarrow Y$,

$s_0(f) =$  is const along the blue lines.

in <u>Cat</u>

$$s_0(\text{object } x) = x \xrightarrow{id_x} x$$

$$s_0(x \xrightarrow{f} y) = \begin{tikzcd} & x \arrow[dr, "f"] & \\ x \arrow[ur, equal, "id_x"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

$$s_1(x \xrightarrow{f} y) = \begin{tikzcd} & y \arrow[dr, equal] & \\ x \arrow[ur, "f"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

<u>Now</u> $|-| : sSet \to Top$, $\tau_1 : sSet \to Cat$ do preserve products.

$(\Delta^1_{sSet})_n = \Delta([n],[1])$ is more complicated

$$\left(\cdots \; 3 \; \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \; 2 \right)$$

$$\begin{pmatrix} x \\ \downarrow f \\ y \end{pmatrix} \times \left(z \xrightarrow{g} w \right) = \begin{tikzcd} xz \arrow[r, "{s_0x,\, g}"] \arrow[d, "{f,\, s_0z}"'] \arrow[dr, "{f,g}"] & xw \arrow[d, "{f,\, s_0w}"] \arrow[dl, phantom, "{s_0f,\, s_1g}", near end] \\ yz \arrow[r, "{s_0y,\, g}"'] \arrow[ur, phantom, "{s_1f,\, s_0g}", near end] & yw \end{tikzcd}$$

ie even if $a \in X_n$, $b \in Y_n$ are degen,
$(a,b) \in (X \times Y)_n$ need not be!

(This is not a proof, but it is true that $|-|$ preserves products... if you define "Top" correctly.)